Random walk in a dynamic environment of traps

by

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Preface

This thesis marks the culmination of my Master's journey at the Delft University of Technology, a journey that has been as challenging as it has been enlightening. A journey that I will remember with the warmest feelings.

The choice of the thesis topic was made by me quite randomly. With a touch of irony, the choice fell on the study of random walks in dynamic environments of traps — a decision that, though made somewhat whimsically, has proven to be richly rewarding and intellectually stimulating.

I would like to express my sincerest gratitude to my thesis supervisor, Prof. Dr. Frank Redig, for the immense support at every step of this research, for the countless hours spent answering my questions at the blackboard, and for his numerous kind and firm pieces of advice. I would also like to thank Dr. Alathea Barbaro for the feedback she provided and for her encouragement.

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But most importantly, I would like to thank my closest and dearest one, Daria. You were always there for me; your unwavering support and understanding have given me the strength to overcome everything.

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1 Introduction

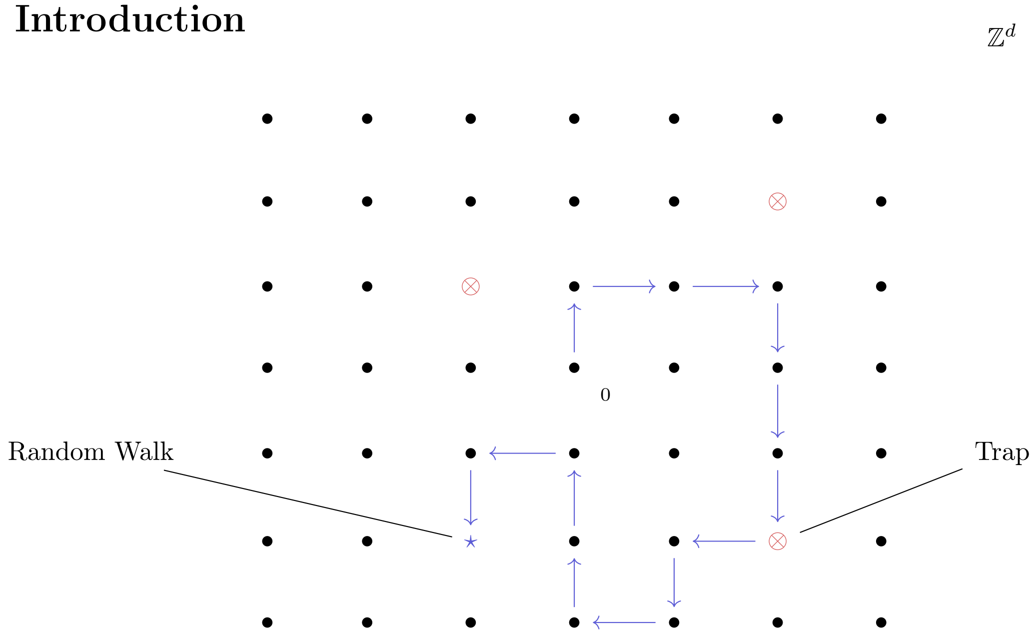

Figure 1: Visualization of the environment with random walker

Various problems related to Random Walk in Random Environment have been reseached intensively in the past 40 years by both mathematics and physics community. It still remains an active research topic in the field of Interacting Particle Systems. The setting of the problem is quite simple, one defines a Random Walk of the choice and lets it traverse the Random Environment that influences the Random Walk in some predetermined way. In this thesis, we study the survival time of a Random Walk in different Random Environments of traps. The Random Environments of traps are separated in two distinct classes: static and dynamic. In the static environment the trap configuration is chosen randomly at a time zero and remains fixed thereafter. While in dynamic case, we let the trap configuration evolve over time by some predefined dynamics. The problem in the static setting has a long story and the picture is understood fairly well.

The survival probability of the Random Walk in a static Random Environment is derived for many different static environments, and it happens to be sub-exponential. This is due to a celebrated result of Donsker and Varadhan [1], which connects to the large deviations of the range of random walk, or in the continuum to the Wiener sausage. Furthermore, many unusual phenomena has been obtained: recurrence criteria, laws of large numbers, invariance principles and refined large deviation estimates.

In the dynamic case, the state of the art is less developed. In recent years, much more has been understood about the empirical speeds of random walks in dynamic environments, the laws of large numbers have been proved under certain space-time mixing conditions, and large deviation principles under different assumptions. Many results yield exponential decay of survival probabilities of a random walk in a dynamic environment, but the literature suggests that for now there is still no unified approach in the derivation of survival times for the models with different environments. Finally, almost nothing is known about the interpolation between the models of random walk in static environments and dynamic environments. The interpolation model should yield an exponential survival time for a fast enough environment while showing a sub-exponential decay of survival probability whenever the environment is slow enough.

One of the aims of the thesis is to collect different results of survival probabilities in different dynamic environments and to create a set-up where one can interpolate between exponential and sub-exponential behavior.

In the second and third sections of this paper we introduce the reader with the needed mathematical background and discuss the problem in a static setting. We give an overview of the Markov Processes and Random Walks and give examples of the Trapping Processes that are used as trapping potentials for the models in future sections. Then we state the Feynman-Kac formula and give its applications by using the theory of semigroups. Then, we study the survival time of a random walk in a static environment of traps in a quite general setting.

The fourth section begins with the mathematical formalization of the problem in a dynamic setting. The rest of the sections is then dedicated to the studies of the random walk in various time-continuous models. More specifically we give the results of varying strength for the following environments: Independent Spin-Flip, Simple Symmetric Exclusion Process, Random Walking Traps, and Attractive Spin-Flip.

The last section of this paper is dedicated to the study of discrete-time random walk in a one-dimensional discrete-time dynamic random environment. In a discrete-time, the dynamics of the environment come from the build-up of the so-called "refreshing times", which dictate at which discrete times the environment changes. By controlling the distribution of the difference of these refreshing times, we show the interpolation between the slow regime, where the survival time decays sub-exponentially, and the fast regime, where the survival time decays exponentially. We prove that it suffices to have the renewal times difference to be distributed sub-exponentially with the rate $t^{d/(d+2)}$, for the survival time to have a sub-exponential lower bound.

2 Background

In this section, we provide some theoretical background such as definitions of Markov processes, semigroups, and generators corresponding to the Markov processes. Additionally, we provide three essential tools that are extensively used in the studies of Interacting Particle Systems, namely the Feynman-Kac formula and the large deviations theory.

2.1 Markov Processes

A Markov process is a such stochastic process that the distribution of a future state only depends on the current state and does not depend on the past history. Essentially a Markov process is a memoryless process. Let us define it more formally.

Definition 2.1 (Markov Process) *A stochastic process $\{X_t, t \geq 0\}$ on state space Ω with values in measurable space $(\Omega, \mathcal{F})$ is called Markov Process (or markovian), if for $\forall t \geq 0, n \in \mathbb{N}, 0 \leq t_1 \leq t_2, ..., t_n \leq 0$ and for $\forall f : \Omega \to \mathbb{R}$ bounded and $\mathcal{F}$-measurable, the following holds:*

$$\mathbb{E}(f(X_t)|X_{t_1}, X_{t_2}, ..., X_{t_n}) = \mathbb{E}(f(X_t)|X_{t_n}), \tag{1}$$

Or in more measure theoretic notation:

$$\mathbb{E}(f(X_t)|\mathcal{F}_s) = \mathbb{E}(f(X_t)|X_s), \tag{2}$$

where $\mathcal{F}_t = \sigma(X_r : r \leq t)$ and $0 \leq s \leq t$.
Note that further into this thesis we will have state spaces $\Omega = \mathbb{Z}^d$ and $\Omega = E$, where E will be a space of all possible configurations of traps (better definition will be discussed in future)

It is not always convenient, nor easy to define the Markov process and prove its existence directly. Thus, the semigroup of linear operators on functions spaces is used, which is quite a natural way to define the corresponding processes. We assume the state space Ω (in future notations we will denote it as E -the space of all configurations of environment) to be a compact metric space

Definition 2.2 (Semigroup) *Given a Markov process on a state space Ω we define semigroup as:*

$$S_t f(x) = \mathbb{E}(f(X_t)|X_0 = x) = \mathbb{E}_x(f(X_t)) \tag{3}$$

for $f \in \mathcal{C}_0(\Omega)$, where $\mathcal{C}_0(\Omega)$ is a Banach space of continuous functions defined on Ω vanishing at infinity, equipped with a supremum norm $\|\cdot\|$

The whole idea is to take the perspective from the probabilistic setting to the spaces of the linear operators which is a deterministic setting.

The probability semigroups must satisfy the following properties.

Lemma 2.1 (Semigroup Properties) *The semigroup S_t, $t \geq 0$ satisfies the following properties.*

1. *Identity at time zero: $S_0 = I$, i.e., $S_0 f = f$ for $\forall f$.*
2. *Right-continuity: the map $t \to S_t f$ is right continuous.*
3. *Semigroup property: for all $t, s > 0$, $f : S_{t+s} f = S_t(S_s f) = S_s(S_t f))$.*
4. *Positivity: $f \geq 0$ implies $S_t f \geq 0$.*
5. *Normalization: $S_t 1 = 1$.*
6. *Contraction: $\max_x |(S_t f)(x)| \leq \max_x |f(x)|$*

Note that, given a certain semigroup there is always a unique Markov process corresponding to that semigroup. Thus, to show the existence of the process, it suffices to show that the semigroup exists.
The semigroup in turn might be constructed from the generator. The generator is defined to be a sort of a time-derivative of a semigroup.

Definition 2.3 (Generator) *Generator L is an infinitesimal operator of the following form:*

$$Lf = \lim_{t \to 0} \frac{S_t f - f}{t} \tag{4}$$

defined on a domain

$$D(L) = \left\{ f \in \mathcal{C}_0(\Omega) : \exists g \in \mathcal{C}_0(\Omega) : \lim_{t \to 0} \left\| \frac{S_t f - f}{t} - g \right\| = 0 \right\} \tag{5}$$

where the $\|\cdot\|$ is a supremum norm and $\mathcal{C}_0(\Omega)$ is a space of functions on Ω decaying to zero at infinity. Note that for function f inside of the domain, the limit $\lim_{t \to 0} \frac{S_t f - f}{t}$ converges uniformly to g

In the studies of interacting particle systems, the underlying stochastic processes are usually given in terms of their generators. The logic of defining the process via its generator comes from the fact that by the Hille-Yoside Theorem there is a one-to-one correspondence between the generator and its underlying semigroup.

Theorem 2.2 *There is a one-to-one correspondence between a Markov generator L and a Markov semigroup $\{S_t, t \geq 0\}$ through*

a) The domain $D(L)$ is given by

$$D(L) = \left\{ f \in \mathcal{C}_0(\Omega) : \exists g \in \mathcal{C}_0(\Omega) : \lim_{t \to 0} \frac{S_t f - f}{t} \text{ converges uniformly to } g \right\}$$

and for $f \in D(L)$

$$Lf = \lim_{t \to 0} \frac{S_t f - f}{t}$$

b) The semigroup is given by

$$S_t = \lim_{n \to \infty} \left(I - \frac{t}{n} L \right)^{-n} =: e^{tL}$$

where I is an identity operator $If = f$

c) For $f \in D(L), S_t f \in D(L)$

$$\frac{d}{dt} S_t f = S_t L f = L S_t f$$

Moreover, $S_t f$ is the unique solution of the differential equation

$$\frac{d\psi_t}{dt} = L\psi_t$$

with initial condition $\psi_0 = f$.

One should note that generator L is closed, but not necessarily bounded (which means that the definition of a semigroup from item b) generally cannot be expanded as a Taylor series), and the domain $D(L)$ is dense in $\mathcal{C}_0(\Omega)$

2.2 Trap Processes

Usually, the object of interest in the IPS studies is a collection of particles that form some environment η, which evolves over time $\{\eta_t : t \geq 0\}$ in a Markovian way. In some models, researchers focus on the behavior of single or multiple particles, that are not part of the environment, traversing the said environment and how the behavior of the particle is affected by the environment. In some other models, the focus is given to the behavior of the environment itself.

Here we give a few examples of generators of the Markov processes, that are used as random dynamic trapping potential fields for the models in the subsequent sections of the paper.

For instance, let the environment η be a configuration of traps on a lattice $\mathbb{Z}^d$ and let E be a set of all possible configurations of traps. Thus, we can formalize the environment $\eta \in E = \{0, 1\}^{\mathbb{Z}^d}$ as a collection of the spins situated on the lattice nodes, where by spin 1 we can denote the existence of the trap at some site of the lattice $\mathbb{Z}^d$ and by spin 0 we mean the absence of the trap. Now, if we let all of these individual nodes to spin at a time-constant and environment-independent rates c we get the definition of the Independent Spin Flip Process.

Definition 2.4 (Independent Spin Flip Process Generator) *Let trap process be independent-flips process $\{\eta_t : t \geq 0\}$ and denote by $\eta_t(x) \in \{0, 1\}$ the value of environment at site $x \in \mathbb{Z}^d$, with the process defined on probability space $(E, \mathcal{F}, \mathbb{P})$ with following generator L*

$$(Lf)(\eta) = \sum_x c(f(\eta^x) - f(\eta)) \tag{6}$$

where $\eta \in E$ is "frozen" configuration of traps on $\mathbb{Z}^d$ and η^x is a configuration with point $x \in \mathbb{Z}^d$ flipped, meaning following

$$\eta^x = \begin{cases} 1 - \eta(x), & \text{if } y = x \\ \eta(y), & \text{if } y \neq x \end{cases} \tag{7}$$

It is also possible to have a system where multiple particles can occupy the same site x. For example we can have multiple random walking particles (or traps in the future) moving along the nodes.

Now the environment takes form $E = \mathbb{N}^{\mathbb{Z}^d}$ where we count how many particles per node there are. In this paper we would like the nodes that have more traps in them to update quicker, thus we can set the updating rate to be dependent on $\eta(x)$. Thus, we choose the following generator of the second studied process. The collection of Random Walking traps can be formalised in the following way

Definition 2.5 (Independent Random walking traps) *For $\eta \in E$ the trap process is a collection of independent random walks,*

$$\left((\{X_i^x(t) : t \geq 0\})_{i=1}^{\eta(x)} \right)_{x \in \mathbb{Z}^d} \tag{8}$$

where $\eta(x)$ is a number of traps at site x and X_i^x means random walking trap starting from x. The number $\eta(x)$ of the random walks at site x is is defined to be Poisson distributed with mean ρ and random walking traps are defined to have jumping rate ν.

Now the generator of such process is of the following form

Definition 2.6 (Independent Random walking traps generator) *Let trap process be a process of independent random walking traps defined as above, defined on a probability space $(E, \mathcal{F}, \mathbb{P})$ with following generator L,*

$$Lf(\eta) = \sum_{x \in \mathbb{Z}^d} \sum_{e \in \mathbb{Z}^d : |e|=1} \frac{\eta(x)}{2d} \left(f(\eta^{x,x+e}) - f(\eta) \right), \tag{9}$$

where $\eta^{x,x+e}$ stands for removing trap at x and putting it at $x + e$

$$\eta^{x,x+e}(y) = \eta(y) - \delta_{y,x} + \delta_{y,x+e} \tag{10}$$

To give another example we go back to the process with the environment $E = \{0, 1\}^{\mathbb{Z}^d}$ of singleton particles situated on a lattice, where we let each particle make a jump to one neighboring unoccupied node at an exponential rate. If the neighbor node is already occupied by another particle, the original particle has to wait another exponential time before jumping.

Definition 2.7 (Symmetric Exclusion Process Generator) *The generator of the exclusion process is given by:*

$$Lf(\eta) = \sum_{x \in \mathbb{Z}^d} \sum_{e \in \mathbb{Z}^d : |e|=1} \frac{1}{2d} \left(f(\eta^{x,x+e}) - f(\eta) \right), \tag{11}$$

where $\eta^{x,x+e}$ stands for exchanging occupations of points x and $x + e$

$$\eta^{x,x+e}(y) = \begin{cases} \eta(y) & y \notin \{x, x+e\} \\ \eta(x) & y = x + e \\ \eta(x+e) & y = x \end{cases} \tag{12}$$

The last example of the trap process that we are going to outline in this section is the Spin Flip Process which is the generalization of the Independent Spin Flip Process, as we let the rates $c(i,\eta)$ to be dependent on the state of the other particles in the system.

Definition 2.8 (Spin Flip Process Generator) *The generator of the trapping process*

$$Lf(\eta) = \sum_{i\in\mathbb{Z}^d} c(i,\eta)(f(\eta^i) - f(\eta)) \tag{13}$$

where $f(\eta)$ depends on η only through some finite subset of $\mathbb{Z}^d$.

Informally speaking, one can "see" how the Markov Process evolves infinitesimally over time by looking at the system's generator. For instance, in the case of the Spin Flip Process the $-f(\eta)$ means that the whole environment from the previous instance is removed and $f(\eta^i)$ means that the η^i environment is added (where η^i is the previous instance environment with the site i having its spin flipped). And this update from η to η^i happens at a rate $c(i,\eta)$ for each point i.

To not give a reader the impression that all generators of all Markov Processes take the same form of a sum over working space (like a $\mathbb{Z}^d$), we would like to give a few more examples of possible generators

Definition 2.9 (Generator of the Brownian Motion) *Let X_t be a Brownian motion on $\mathbb{R}$, then for $f \in \mathcal{C}_0^2(\mathbb{R})$*

$$\mathbb{E}_x f(X_t) = \mathbb{E}f(x + N(0,t)) = f(x) + f'(x)\mathbb{E}(\mathcal{N}(0,t)) + \frac{1}{2}f''(x)\mathbb{E}(\mathcal{N}(0,t))^2) + o(t)$$

Now the generator of the Brownian motion is the following

$$Lf(x) = \lim_{t\to 0} \frac{\mathbb{E}_x f(X_t) - f(x)}{t} = \frac{1}{2}\frac{d^2}{dx^2}f(x)$$

Definition 2.10 (Generator of the Diffusion process on $\mathbb{R}$) *Let X_t be diffusion process on $\mathbb{R}$ solving the following stochastic differential equation*

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t$$

By the Itô's formula we have

$$f(X_t) - f(X_0) = \int_0^t b(X_s)f'(X_s)ds + \int_0^t \frac{1}{2}\sigma^2(X_s)f''(X_s)ds + \int_0^t \sigma(X_s)f'(X_s)dW_s$$

Which results in the generator of X_t taking the following form

$$Lf(x) = b(x)\frac{d}{dx}f(x) + \frac{1}{2}\sigma^2(x)\frac{d^2}{dx^2}f(x)$$

for $f \in \mathcal{C}_0^2(\mathbb{R})$.

One should note that the generators of Trap processes are all informal in a sense that they are not bounded generators, compared to the generators of the Brownian Motion and Diffusion Process.

Now that the reader is getting acquainted with the generators, we will give the definitions and some results regarding Random Walks. Then we would like to work with the generators for more generalized processes. For example, we will compute the generator of the process where we let a certain Random Walk traverse a random dynamic environment with one of the previously stated trapping processes as underlying.

2.3 Random Walks

One of the examples of Markov Processes is a simple random walk, which also happens to be one of the central objects in our study. Let us define the discrete-time random walk and two types of continuous-time random walks on $\mathbb{Z}^d$.

Definition 2.11 (Discrete-time Nearest Neighbor Random Walk) *Continuous-time simple symmetric random walk on $\mathbb{Z}^d$ is a process that equiprobably jumps across the nearest neighbours after waiting a mean one exponential time. Each point has 2d neighbours, so at the i-th jump the probability to jump to a certain neighbour is $\mathbb{P}(\epsilon_i = e) = 1/2d$, where e is canonical vector of $\mathbb{Z}^d$ that corresponds to path to the neighbour. Denote N_t as number of jumps until time t, which is Poisson counting process (since times are exponential). The random walk is defined as*

$$X_t = X_0 + S_{N_t}, \qquad S_{N_t} = \sum_{i=1}^{N_t} \epsilon_i \tag{14}$$

The defined continuous time random walk is also called nearest neighbor random walk, as we constrained random walk to jump only to its neighbors. The generator of this RW takes the form,

Definition 2.12 (Generator of the Continuous-time Nearest Neighbor Random Walk) *Let $X := (X_t)_{t\geq 0}$ be a continuous time random walk on $\mathbb{Z}^d$ defined as previously, on probability space $(\Omega^X, \mathcal{F}^X, \mathbb{P}_0^X)$. Its generator L*

$$(Lf)(x) = \sum_{y:|x-y|=1} p(x,y)(f(y) - f(x)), \tag{15}$$

where $p(x,y) = \frac{1}{2d}$

In the literature, more generalized Random walks are studied, for instance, without constricting the processes' jumps to the neighboring states. In this thesis, we are mostly going to use Nearest Neighbor types of random walks, but a wide range of different types of RW's can be generalized to the setting where the rates are translational invariant $p(x,y) = p(0, x+y)$ (where $p(0,z) = 0$ for all $|z| > R$ where R is some predefined radius). If a random walk exhibits such translational invariance of jumping rates on a lattice, then the generator can be defined as follows

Definition 2.13 (Generator of the Random Walk) *Let $X := (X_t)_{t\geq 0}$ be a continuous time random walk on $\mathbb{Z}^d$ whose jumping rates are translationally invarian as above and whose jumping rates for points farther than $R > 0$ are zero. Then the generator L of such random walk takes the following form,*

$$(Lf)(x) = \sum_{y} p(0,y)(f(y+x) - f(0)), \tag{16}$$

where $p(0,z) = 0$ if $|z| > R$ for some predefined radius $R > 0$.

2.4 Basic Properties of the Random Walks

Here we present the properties of random walks. More specifically, we write down the results regarding the ranges of a simple random walk, mention facts regarding the transience/recurrence of simple random walks, and state invariance principle.

2.4.1 Donsker Invariance Principle

The random walk has many interesting properties, one of which is the Invariance principle. Essentially it means that the scaling limit of one-dimensional continuous time simple random walk is a Brownian Motion. Let us define a brownian motion and then state Donsker's Invariance Principle.

Definition 2.14 *Brownian motion $\{W_t : t \geq 0\}$ is a process defined to satisfy the following properties.*

- *Starting from 0: $W_0 = 0$.*
- *Independent normally distributed increments: for $0 < t_1 < t_2 < \ldots < t_n$*

  $$W_{t_i} - W_{t_{i-1}}$$

  are independent and normally distributed with expectation zero and variance $t_i - t_{i-1}$.

Note that the d-dimensional Brownian motion is then can be defined combining d independent Brownian motions: $(W_1(t), \ldots, W_d(t))$. The Markov property for this generalization holds. Then by the central limit theorem, the process $\epsilon X_{\epsilon^{-2}dt}$ converges to W_t for any $t > 0$, as $\epsilon \to 0$. This means that the scaling limit of random walk is Brownian motion. The more general result is

Lemma 2.3 (Donsker Invariance Principle) *The process $\{\epsilon X_{\epsilon^{-2}dt} : t \geq 0\}$ converges in distribution to the process $\{W_t : t \geq 0\}$ as $\epsilon \to 0$.*

2.4.2 Transience and Recurrence of the Random Walk

A very natural question that arises in the literature, would the Random Walk visit a certain site in a state space given finite or infinite time. Let us begin by defining the notion of recurrence and transience sites and then continue to define what it means for the Random Walk to be transient or recurrent.

Definition 2.15 (Recurrent State) *The state $i \in \Omega$ is called recurrent (or persistent), if $\mathbb{P}(T_i < \infty) = 1$ where T_i is the first hitting time of i (the first time the X_t reaches the state i)*

The state being recurrent naturally means that given an infinite time or steps, a process will eventually visit this state. The definition of the transient state is stated in an opposite manner. A state $i \in \Omega$ is said to be transient, if, starting from i, there is a non-zero probability that the chain will never return to i. This also means that the process will occupy the transient state quite rarely.

Now we can give definitions of recurrent/transient processes through their state spaces being a collection of recurrent/transient states.

Definition 2.16 (Recurrent Process) *If state space Ω is connected (every state is connected via some path to every other state) and every state $i \in \Omega$ is recurrent, then we say that the process X_t is recurrent.*

The transient processes are thus defined in an analogous way.

A natural subsequent question is whether Random Walks on $\mathbb{Z}^d$ are always recurrent or transient, and if not, what does this property depend on? Surprisingly, G.Pòlya proved in [2] that the transience or recurrence of the simple Random Walk depends on the dimensionality of state space.

Theorem 2.4 ([2]) *A simple symmetric nearest neighbor random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$, and transient in higher dimensions $d \geq 3$.*

This theorem is best characterized by the following quote of Shizuo Kakutani: "A drunk man will find his way home, but a drunk bird may get lost forever". In fact, if we denote by $p(d)$ to be a probability of a simple random walk on $\mathbb{Z}^d$ returning to origin, we may find that at dimensions $d = 1$ and 2 the probability of returning to origin is $p(1) = p(2) = 1$ and, then, at the dimension $d = 3$ there is sudden abruption with $p(3) \approx 0.34 < 1$ making random walk to become transient. The $p(d)$'s are called Pólya's random walk constants, for which the closed-form expression was not known up until the recent paper by Robert E. Gaunt and et.al. [3].

2.4.3 The Range of the Random Walk

A number of the results were derived using the notion of the Range of the Random walk, both in static and dynamic trapping cases. For instance, the proof of the exponential upper and lower bounds on the asymptotics of the survival time of a Random Walk among Random Walking Traps reduces the situation towards looking at the range of the difference between two random walks. The range of the random walk is defined as follows

Definition 2.17 (Random Walk Range) *Range of the random walk X_t on $\mathbb{Z}^d$ is defined to be*

$$R(X, t) = \sum_x \mathbb{1}_{\{X_s = x, \exists s \in [0,t)\}} \tag{17}$$

It is also beneficial to know how the expectation of the random walk range decays in time. The paper by A. Dvoretzky and P. Erdös [4] shows that the following asymptotics holds.

Theorem 2.5 (Expectation of Range of RW) *The expectation of Range of a Random Walk X_t on $\mathbb{Z}^d$ is*

$$\mathbb{E}(R(X,t)) = \begin{cases} \sqrt{t}, & \text{if } d = 1 \\ \frac{\pi t}{\log t} + \mathcal{O}\left(t \frac{\log \log t}{\log^2 t}\right), & \text{if } d = 2 \\ t\upsilon_d + \mathcal{O}(t^{2-d/2}), & \text{if } d \geq 3 \end{cases} \tag{18}$$

where υ_d is the probability of never returning to the origin

The asymptotics of this form, where we get a dependency on $\sqrt{t}$ for the one-dimensional case, on $t/\log t$ for the two-dimensional case, and on t in the higher-dimensional cases, are quite frequent along the studies of the Random Walks and pop up almost everywhere in the literature.

2.5 Feynman-Kac formula

Let L be a Markov generator for some Markov process X_t, then the corresponding semigroup $S_t f(x) = \mathbb{E}_x f(X_t)$ solves the following system of PDE's

$$\begin{cases} \frac{d}{dt}\psi(t,x) = L\psi(t,x) \\ \psi(0,x) = f(x) \end{cases}$$

Throughout this section we will be interested in the solutions of a more general "Schrödinger type" system of PDEs

$$\begin{cases} \frac{d}{dt}\psi(t,x) = L\psi(t,x) + V(t,x)\psi(t,x) \\ \psi(0,x) = f(x) \end{cases}$$

It happens to be that the solution expressed in terms of the Markov process X_t takes the following functional form

$$\psi(t,x) = \mathbb{E}_x(e^{\int_0^t V(t-s,X_s)ds}\psi(0,X_t))$$

This result is called the Feynman-Kac formula and is vastly useful in analyzing this type of exponential functionals.

Theorem 2.6 *[Feynman–Kac formula] The following system of PDEs*

$$\begin{cases} \frac{d}{dt}\psi(t,x) = L\psi(t,x) + V(t,x)\psi(t,x) \\ \psi(0,x) = f(x) \end{cases}$$

accepts solution of the form:

$$\psi(t,x) = \mathbb{E}_x(e^{\int_0^t V(t-s,X_s)ds}f(X_t))$$

Note that $\psi(t,x)$ forms a so called Feynman-Kac semigroup. Furthermore, if $V \equiv 0$, then the solution reduces to the semigroup S_t of a Markov process X_t.

$$\psi(t,x) = \mathbb{E}_x(e^0\psi(0,X_t)) = \mathbb{E}_x(\psi(0,X_t)) = S_t\psi(0,X_t)$$

In the following sections, we are interested in the case when the V is not dependent on time as it happens with several types of trap models. The Feynman-Kac semigroup with time-independent component V:

$$\psi(t,x) = \mathbb{E}_x(e^{\int_0^t V(X_s)ds}\psi(0,X_t))$$

happens to have very nice properties. In particular, we will discover that the Feynman-Kac semigroup can be expressed as an exponential operator that depends on L and V

2.5.1 Feynman-Kac semigroup

In this section, we prove important result that will help us apply the Feynman-Kac formula in the section dedicated to the Independent Spin Flip trap model. More specifically, we will derive an expression for the Feynman-Kac formula solution functional in terms of the generators.

Let us define the Feynman-Kac semigroup as

$$T(t)u(\eta) = \mathbb{E}_\eta\left[\exp\left\{\int_0^t V(\eta(s))ds\right\}u(\eta(t))\right] \tag{19}$$

Let us prove that this semigroup takes the form of some exponential operator.

Theorem 2.7 *The Feynman-Kac semigroup $T(t)$ defined in (19) that solves PDE from Theorem 2.6 with the generators L and V, can be expressed as follows*

$$T(t)u(\eta) = \mathbb{E}_\eta\left[\exp\left\{\int_0^t V(\eta(s))ds\right\}u(\eta(t))\right] = e^{t(L+V)}(u(\eta)) \tag{20}$$

where L is defined to be

$$Lu(\eta) = \lim_{t\to 0}\frac{\mathbb{E}^\eta\left[u(\eta(t))\right] - u(\eta)}{t} \tag{21}$$

Proof:

To prove that $T(t)u(\eta) = \exp(t(L+V))(u(\eta))$ is suffices to show that both

- $T(t+s)u = T(t)(T(s)u)$
- $\frac{d}{dt}T(t)\Big|_{t=0} = L+V$

hold for the semigroup $T(t)$.

Let us prove the first point directly

$$T(t+s)u(\eta) = \mathbb{E}_\eta\left[\exp\left\{\int_0^{t+s} V(\eta(r))dr\right\} u(\eta(t+s))\right] = \tag{22}$$

$$= \mathbb{E}_\eta\left[\mathbb{E}_\eta\left(\exp\left\{\int_0^{t+s} V(\eta(r))dr\right\} u(\eta(t+s)) \;\Big|\; \mathcal{F}_s\right)\right] = \tag{23}$$

$$= \mathbb{E}_\eta\left[\mathbb{E}_\eta\left(\exp\left\{\int_0^{s} V(\eta(r))dr + \int_s^{t+s} V(\eta(r))dr\right\} u(\eta(t+s)) \;\Big|\; \mathcal{F}_s\right)\right] \tag{24}$$

$$= \mathbb{E}_\eta\left[\mathbb{E}_\eta\left(\exp\left\{\int_0^{s} V(\eta(r))dr + \int_s^{t+s} V(\eta(r))dr\right\} u(\eta(t+s)) \;\Big|\; \mathcal{F}_s\right)\right] = \tag{25}$$

$$= \mathbb{E}_\eta\left[\exp\left\{\int_0^{s} V(\eta(r))dr\right\} \mathbb{E}_\eta\left(\exp\left\{\int_s^{t+s} V(\eta(r))dr\right\} u(\eta(t+s)) \;\Big|\; \mathcal{F}_s\right)\right] = \tag{26}$$

$$\tag{27}$$

Now we note that the right inner expectation is conditioned on starting the process from time s (conditioning on $\mathcal{F}_s$) and due to translational time invariance we can translate the distributions to start from time 0, thus we get,

$$T(t)u(\eta) = \mathbb{E}_\eta\left[\exp\left\{\int_0^{s} V(\eta(r))dr\right\} \mathbb{E}_\eta\left(\exp\left\{\int_0^{t} V(\eta(r))dr\right\} u(\eta(t))\right)\right] = \tag{28}$$

$$= \mathbb{E}_\eta\left[\exp\left\{\int_0^{s} V(\eta(r))dr\right\} (T(t)u(\eta))\right] = (T(s)(T(t)u)(\eta) \tag{29}$$

which proves the first point.

To prove the second statement, we need to expand $T(t)u$ into Taylor series around the zero and observe the leading term of order t, which will be exactly the derivative around the zero.

$$T(t)u(\eta) = \mathbb{E}_\eta\left[\exp\left\{\int_0^{t} V(\eta(s))ds\right\} u(\eta(t))\right] = \mathbb{E}_\eta\left[\left(1 + \int_0^t V(\eta(s))ds + \mathcal{O}(t)\right) u(\eta(t))\right] = \tag{30}$$

$$= \mathbb{E}_\eta\left[u(\eta(t))\right] + \mathbb{E}_\eta\left[u(\eta(t)) \int_0^t V(\eta(s))ds\right] + \mathcal{O}(t) \tag{31}$$

We can find the first expectation from the definition of the generator L:

$$\mathbb{E}_\eta\left[u(\eta(t))\right] = u(\eta) + tLu(\eta(t)) \tag{32}$$

The second term can be found by noticing that we have convergence in the distribution of the following things:

$$u(\eta(t)) \longrightarrow u(\eta) \;\; as\; t \to 0 \tag{33}$$

$$V(\eta(t)) \longrightarrow V(\eta) \;\; as\; t \to 0 \tag{34}$$

Thus the second term can be expressed as,

$$\mathbb{E}_\eta\left[u(\eta(t)) \int_0^t V(\eta(s))ds\right] = \mathbb{E}_\eta\left[u(\eta) \int_0^t V(\eta)ds\right] = \mathbb{E}_\eta\left[V(\eta)u(\eta) \int_0^t ds\right] = tV(\eta)u(\eta) \tag{35}$$

Combining two previous results yields,

$$T(t)u(\eta) = u(\eta) + tLu(\eta(t)) + tV(\eta)u(\eta) + \mathcal{O}(t) \tag{36}$$

Thus we the time derivative of T around $t = 0$ is

$$\frac{d}{dt}T(t)\Big|_{t=0} = L + V \tag{37}$$

Which combined with the first statement, indeed means that the semigroup takes the form of the following exponential operator

$$T(t)u(\eta) = e^{t(L+V)}(u(\eta)) \tag{38}$$

□

This result will allow us, in the future sections, to compute bounds on the exponential functionals involving Feynman-Kac semigroup by the biggest eigenvalues of the operator $e^{t(L+V)}$.

2.5.2 Applications in Interacting Particle Systems

In the studies of the Interacting Particle Systems the most widely used result is the Feynman-Kac formula. The reason is that the Feynman-Kac formula connects the exponential functionals, that arise everywhere in the IPS field, to the solutions for certain systems of partial differential equations.

For, instance the functional of interest in this thesis $Z_{\gamma,t}$ can be viewed as an averaged Feynman-Kac solution

$$Z_{\gamma,t} := \mathbb{E}^{\eta}\mathbb{E}_0^X \left(\exp\left[-\gamma \int_0^t \eta(s, X(s))ds\right]\right) = \tag{39}$$

$$= \mathbb{E}^{\eta}\mathbb{E}_0^X \left(\exp\left[-\gamma \int_0^t \eta(t-s, X(s))ds\right]\right) = \mathbb{E}^{\eta}\mathbb{E}_0^X \left(\exp\left[\int_0^t V(t-s, X(s))ds\right]\right) \tag{40}$$

if stochastic process $\eta(s,\cdot)$ (the definition of which will be discussed in future sections) has time-reversibility property (time reversibility is given by $\eta(s,\cdot)$ and $\eta(t-s,\cdot)$ having same distribution).

2.6 Large Deviation Theory

Another frequently used tool, that has proved its successfulness in the analysis of the exponential functionals that arise in the studies IPS, is the Large Deviation Principle.

Large Deviation theory deals with deviations of empirical processes from the law of large numbers. In the simplest setting with $S_n = \sum_{i=1}^n X_i$ and i.i.d. X_i's we want to estimate

$$\mathbb{P}\left(\frac{S_n}{n} \approx x\right),$$

where $x > \mu$ with μ being the expectation $\mathbb{E}X_i$.

Under the assumption that for $\lambda > 0$ the exponential moments $\mathbb{E}(e^{\lambda X_i})$ exist, we can find in fact

$$\mathbb{P}\left(\frac{S_n}{n} \approx x\right) \simeq e^{-nI(x)}$$

More formally speaking, the previous expression means the existence of the following limit

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left(\frac{S_n}{n} \in (x-\delta, x+\delta)\right) = -I(x)$$

where $I(x) = \sup_\lambda(\lambda x - F(\lambda))$ is the so called rate function, $F(\lambda) = \ln \mathbb{E}e^{\lambda X}$ and some $\delta > 0$.

In other words, the Large Deviation Theory quantifies the exponentially small probabilities and their decaying order behavior (on the exponential scale). We will give a few examples and basic results from this theory, adapted to the context of this thesis.

2.6.1 Deviations of empirical mean in Coin Tossing

Let us give an example demonstrating one of the simplest applications of a large deviation theory for the reader to get acquainted with the theory's formalism. We would like to work with the coin-tossing experiment and then formalize the ideas in more general setting.

Let us compute the decay of the probability $\mathbb{P}\left\{\frac{1}{n}S_n \geq x\right\}$. Let $X_1, X_2, \ldots$ be independent random variables taking values in $\{0,1\}$ with probabilities $p = q = \frac{1}{2}$. Denote $S_n = X_1 + \cdots + X_n$ as their partial sum. Let us prove that

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left\{\frac{1}{n}S_n \geq x\right\} = -I(x) \tag{41}$$

for all $x > \mu = \frac{1}{2}$, for some function $I(x)$.
Note that since X takes values $\{0,1\}$, for $x > 1$ we have the following

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left\{\frac{1}{n}S_n \geq x\right\} = -\infty$$

For the $x \in \left[\frac{1}{2}, 1\right]$ we can expand the probability of deviation as follows:

$$\mathbb{P}\{S_n \geq xn\} = \frac{1}{2^n} \sum_{k \geq xn} C_n^k$$

where C_n^k are binomial coefficients.
Now let us bound the object of our interest

$$\frac{1}{2^n} \max_{k\geq xn} C_n^k \leq \mathbb{P}\{S_n \geq xn\} \leq \frac{n+1}{2^n} \max_{k\geq xn} C_n^k.$$

The maximum is attained at the smallest integer $\geq xn$ which is $k = \lceil xn \rceil$. Denote $l := \lceil xn \rceil$.
Let us state the Stirling's formula

$$n! = n^n e^{-n} \sqrt{2\pi n} \left(1 + O\left(\frac{1}{n}\right)\right)$$

Since we are looking for the limit of n as in (41), let us take a limit of the derived expression and use the Stirling's formula

$$\lim_{n\to\infty} \frac{1}{n} \ln \max_{k\geq xn} C_n^k = -x\ln x - (1-x)\ln(1-x),$$

since $\frac{l}{n} = \frac{\lceil xn \rceil}{n} \to x$ as $n \to +\infty$.
This result together with the fact that $1/n \ln(2^{-n}) = -ln(2)$ and that $\lim_{n\to\infty} ln(n+1)/n = 0$ gives us

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left\{\frac{1}{n}S_n \geq x\right\} = -\ln 2 - x\ln x - (1-x)\ln(1-x)$$

for all $x \in \left[\frac{1}{2}, 1\right]$.

For the $x \in [0, \frac{1}{2}]$ we need to note the following symmetry

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left\{\frac{1}{n}S_n \leq x\right\} = -I(1-x) = -I(x)$$

because X_k and $1 - X_k$ have the same distribution and because of the fact that
$f(x) = \ln 2 + x\ln x + (1-x)\ln(1-x)$ is symmetric around $1/2$.

So, in the end

$$I(x) = \begin{cases} \ln 2 + x\ln x + (1-x)\ln(1-x) & \text{if } x \in [0,1] \\ +\infty & \text{otherwise} \end{cases}$$

It turns out that this function gives better information about the decay of such unlikely probabilities, compared to Central Limit theorem

2.6.2 Cramer Theory

Now let us formalize the ideas regarding probabilities of large deviations of empirical means that we have wrote previously

Let $X_1, X_2, ..., X_n, ...$ be i.i.d. distributed random variables. Let the $\mathbb{E}(X_i) = \mu \in \mathbb{R}$ and the $Var(X_i) = \sigma^2 \in \mathbb{R}^+$. Take the partial sum $S_n = X_1 + X_2 + ... + X_n$.

The Strong Law of Large Numbers tells us that the empirical mean converges almost surely to the mean of X:

$$\frac{1}{n}S_n \xrightarrow[n\to\infty]{a.s.} \mu \tag{42}$$

The Central Limit Theorem, on the other hand, quantifies the probability that the S_n deviates from μn by an amount $\sqrt{n}$

$$\frac{S_n - \mu n}{\sigma\sqrt{n}} \xrightarrow[n\to\infty]{\mathbb{P}} \mathcal{N}(0,1) \tag{43}$$

Let us think that the n is large but finite. In essence, these two results say the following. The SLLN tells us just that S_n/n converges to μ as finite n gets larger. The CLT tells us that S_n/n is very close to μ as finite n gets larger optimally quantifying the probability that the S_n/n deviates from μ by an amount of order $\sqrt{n}$.

But if one were to quantify the probability of a "large" deviation of S_n/n from μ by an amount of order n using CLT, it would give a non-optimal rate of decay of such an unlikely event.

For further derivations, let us define the Fenchel-Legendre transform, and note a few of its properties.

Definition 2.18 *The following function is called the Fenchel-Legendre transform*

$$f^*(x) = \sup_{\lambda\in\mathbb{R}}\{\lambda x - f(\lambda)\} \tag{44}$$

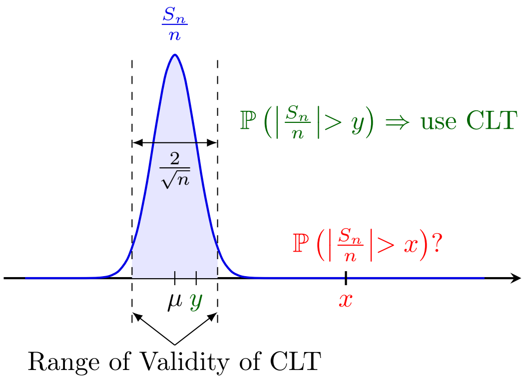

Figure 2: The visuals how CLT fails to capture the rate of decay of very unlikely events like $\left\{\left|\frac{S_n}{n} - \mu\right| > x\right\}$

Note that the Fenchel-Legendre transform of a convex function is once again convex, as the supremum of a convex function is also convex and $\lambda x - f(\lambda)$ is convex given f is. Also note that the function inside supremum that we often use in this study $f = \varphi = \ln \mathbb{E} e^{\lambda X}$ is convex in λ.

Now, let us derive the Cramer Theory. Let $x > \mu$ and $\forall \lambda \geq 0$.

$$\mathbb{P}\left(\frac{S_n}{n} \geq x\right) = \mathbb{P}\left(S_n \geq xn\right) = \mathbb{P}\left(e^{\lambda S_n} \geq e^{\lambda xn}\right) \leq \frac{1}{e^{\lambda xn}}\mathbb{E}\left(e^{\lambda S_n}\right) \overset{ind}{=} \tag{45}$$

$$\overset{ind}{=} e^{-\lambda xn} \prod_{i=1}^{n} \mathbb{E}\left(e^{\lambda X_i}\right) \overset{i.i.d.}{=} e^{-\lambda xn}\mathbb{E}\left(e^{\lambda X_1}\right)^n = e^{-\lambda xn} e^{n \ln \mathbb{E} \exp(\lambda X)} = \tag{46}$$

$$= e^{-n(\lambda x - \varphi(\lambda))} \tag{47}$$

where $\varphi(\lambda) = \ln \mathbb{E}\left(e^{\lambda X_1}\right)$ (which is also called cumulant generating function). If we take the supremum over all λ of the expression inside the exponent we would get a sharp upper bound

$$\mathbb{P}\left(\frac{S_n}{n} \geq x\right) \leq e^{-nI(x)} \tag{48}$$

where the $I(x)$ is of the following form

$$I(x) = \sup_{\lambda \in \mathbb{R}} \{\lambda x - \varphi(\lambda)\} \tag{49}$$

Now let us derive the lower bound with the same rate function $I(x)$. Let $\delta \geq 0$ be some tunable constant.

$$\mathbb{P}\left(\frac{S_n}{n} \geq x\right) \geq \mathbb{P}\left(\frac{S_n}{n} \in (x-\delta, x+\delta)\right) = \mathbb{E}\left[I\left(\frac{S_n}{n} \in (x-\delta, x+\delta)\right) \frac{e^{\lambda S_n}}{e^{n\varphi(\lambda)}} e^{-\lambda S_n}\right] e^{n\varphi(\lambda)} \tag{50}$$

The term $e^{\lambda S_n}/e^{n\varphi(\lambda)}$ realizes the change of measure with respect to λ thus we get

$$\mathbb{P}\left(\frac{S_n}{n} \geq x\right) \geq \mathbb{E}_\lambda\left[I\left(\frac{S_n}{n} \in (x-\delta, x+\delta)\right) e^{-\lambda S_n}\right] e^{n\varphi(\lambda)} \tag{51}$$

Let us remember the definition of $\varphi(\lambda)$ and find its derivative

$$\varphi(\lambda) = \ln \mathbb{E}[e^{\lambda X}] \qquad \frac{d}{d\lambda}\varphi(\lambda) = \frac{\mathbb{E}(Xe^{\lambda X})}{\mathbb{E}(e^{\lambda X})} =: x \tag{52}$$

Which also means that λ achieves the supremum in

$$\sup_{\lambda \in \mathbb{R}} \{\lambda x - \varphi(\lambda)\} = I(x) \tag{53}$$

since $\varphi(\lambda)$ is convex and a Fenchel-Legendre transform of a convex function is again convex. Furthermore, we get a unique optimizer λ through this procedure.

Now let us tune the λ such that $\mathbb{E}_\lambda(X) = x > \mu$. Now we would get the following lower bound. Let $\epsilon < \delta$,

$$\mathbb{P}\left(\frac{S_n}{n} \geq x\right) \geq \mathbb{E}_\lambda\left[I\left(\frac{\sum X_i}{n} \in (x-\delta, x+\delta)\right)e^{-\lambda S_n}\right]e^{n\varphi(\lambda)} \tag{54}$$

$$\geq \mathbb{E}_\lambda\left[I\left(\frac{\sum X_i}{n} \in (x-\epsilon, x+\epsilon)\right)e^{-\lambda S_n}\right]e^{n\varphi(\lambda)} \tag{55}$$

$$\geq e^{-\lambda n(x+\epsilon)}\mathbb{P}\left(\frac{\sum X_i}{n} \in (x-\epsilon, x+\epsilon)\right)e^{n\varphi(\lambda)} \tag{56}$$

$$\geq e^{-\lambda n(x+\epsilon)}e^{n\varphi(\lambda)} \tag{57}$$

By letting $\epsilon \to 0$ going to zero, we get the resulting lower bound

$$\mathbb{P}\left(\frac{S_n}{n} \geq x\right) \geq e^{-\lambda n(x+\epsilon)}e^{n\varphi(\lambda)} \geq e^{-n(\lambda x - \varphi(\lambda))} \geq e^{-nI(x)} \tag{58}$$

It is said that S_n/n satisfies a large deviation principle with the rate function $I(x)$ and with the rate n. The previous logic yields that for large finite n, the following is the very sharp approximation for the probability of a large deviation of empirical mean bigger than x.

$$\mathbb{P}\left(\frac{S_n}{n} \geq x\right) \simeq e^{-nI(x)} \tag{59}$$

It turns out, that this approximation yields a much better expression for the tail of the distribution of the deviations of the empirical mean and it tells much more about the rate of the decay of large deviations of the empirical mean.

Note that the symbol $\simeq$ means the exponential equivalence. Two sequences α_n and β_n are called exponentially equivalent if and only if

$$\lim_{n\to\infty}\frac{1}{n}(\log(\frac{\alpha_n}{\beta_n}) = 0$$

Let us also establish another notation. Let $x > \mu$, by the following expression

$$\mathbb{P}\left(\frac{S_n}{n} \approx x\right) \simeq e^{-nI(x)}$$

we mean that the probability that S_n/n deviates from μ by an amount of order at least x is exponentially equivalent to $e^{-nI(x)}$, or, in other words, such deviation is described by the Large Deviation Principle with the rate function $I(x)$

Thus we formalize the first result of the Large Deviations Theory regarding a large deviation principle for empirical means of i.i.d. random variables, which came to be known as Cramer's Theorem.

Theorem 2.8 (Cramer's Theorem) *Let $\eta_1, \eta_2, ..., \eta_n$ be i.i.d. random variables with mean μ and with cumulant generating function φ. Let $S_n = \eta_1 + \eta_2 + ... + \eta_n$, then for every $x > \mu$ the following holds:*

$$\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{P}\left(\frac{S_n}{n} \geq x\right) = -\varphi^*(x) \tag{60}$$

$$\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{P}\left(\frac{S_n}{n} \leq x\right) = -\varphi^*(x) \tag{61}$$

where $\varphi^*(x)$ is a Fenchel-Legendre transform of cumulant generating function $\varphi = \ln\mathbb{E}e^{\lambda\eta_1}$

2.6.3 Basics of Abstract Large Deviation Theory

In this subsection, we give statements of abstract large deviations principle and two important theorems that allow one to derive LDPs of higher-order objects from already existing LDPs.

Let us give a general abstract Large Deviations Principle, which covers a bigger family of functionals of $\eta_1, \eta_2, ..., \eta_n$, and not just empirical mean.

Lemma 2.9 (Large Deviations Principle) *A family of probability measures* $(\mathbb{P}_n \ : \ n \in \mathbb{N})$ *satisfies a Large Deviation Principle (LDP) with rate* a_n *and with rate function* I*, if for any measurable set* A*,*

$$-\inf_{\theta\in int(A)} I(\theta) \leq \lim_{n\to\infty}\inf \frac{1}{a_n}\log\mathbb{P}_n(A) \leq \lim_{n\to\infty}\sup\frac{1}{a_n}\log\mathbb{P}_n(A) \leq -\inf_{\theta\in cl(A)} I(\theta) \tag{62}$$

where $cl(A)$ *and* $int(A)$*, are the closure and the interior of* A

However, it is not always possible to derive the rate function directly and thus directly prove LDP for any case. Let us next state the result from Large Deviations Theory that allows one to derive LDP for a random object from another known LDP via a "contraction".

Theorem 2.10 (Contraction Principle) *Let the families of probability measures* $(\mathbb{P}_n \ : \ n \in \mathbb{N})$ *and* $(\mathbb{P}'_n \ : \ n \in \mathbb{N})$ *be defined on* Ω *and* Ω' *correspondingly. Let* $(\mathbb{P}_n \ : \ n \in \mathbb{N})$ *satisfy some LDP on* Ω *with the rate function* I*.*
If there exists continuous function $f : \Omega \to \Omega'$ *such that*

$$\mathbb{P}'_n(A) = \mathbb{P}_n(f^{-1}(A)) \quad \forall A \in \Omega' \tag{63}$$

Then $(\mathbb{P}'_n \ : \ n \in \mathbb{N})$ *satisfies LDP on* Ω' *with the rate function* J

$$J(y) = \inf_{y:\ x=f^{-1}(y)} I(x), \qquad y \in \Omega' \tag{64}$$

Let us also state another important result, that will be quite useful along this study.

Theorem 2.11 (Varadhan's Lemma) *Let A family of probability measures* $(\mathbb{P}_n \ : \ n \in \mathbb{N})$ *satisfy the LDP on* Ω *with rate* n *and with the rate function* I*. Let* $F : \Omega \to \mathbb{R}$ *be a continuous function that is bounded from above. Then*

$$\lim_{n\to\infty}\frac{1}{n}\int_\Omega e^{nF(x)}\mathbb{P}_n(dx) = \sup_{x\in\Omega}[F(x) - I(x)] \tag{65}$$

The previously stated contraction principle is an extremely strong result that allows one to derive Large Deviation Principles for higher-level kinds of objects. For instance, it is possible to derive LDP for the following empirical measures $\mathcal{L}_n$.

$$\mathcal{L}_n = \frac{1}{n}\sum_{i=1}^{n}\delta_{X_i} \tag{66}$$

Note that there is a following have a connection between these empirical measures and previously defined empirical means

$$\frac{S_n}{n} = \frac{\sum_{i=1}^n X_i}{n} = \frac{\int x \sum_{i=1}^n \delta_{X_i}(dx)}{n} = \int x\frac{1}{n}\sum_{i=1}^{n}\delta_{X_i}(dx) = \int x\mathcal{L}_n(dx) \tag{67}$$

This might suggest, that the empirical measures should satisfy some large deviation principle since the underlying S_n satisfies LDP by the Cramer's Theorem. And this is actually the case, the $\mathcal{L}_n$ satisfies the so-called "Level 2" Large Deviation Principle. Let us state this quite surprising result

2.6.4 Sanov Theory

In this subsection, we state LDP results in increasing order of generalization, going from Cramer's Theorem up to Sanov's Theorem.

Let us remember the Cramer's Theorem. If i.i.d. random variables X_i with finite exponential moments are taking their values in $\mathbb{R}$, then empirical mean $\frac{1}{n}\sum X_i$ satisfies large deviation principle with the rate function $I(x) = \sup_\lambda(\lambda x - F(\lambda))$ with $F(\lambda) = \ln \mathbb{E}e^{\lambda X}$.

The natural question is, whether it is possible to derive rate functions when X_i's take their values from other spaces. For instance, it turns out that it is possible to derive large deviation principle in a case when X_i's are $\mathbb{R}^d$ valued.

Theorem 2.12 *Let X_i be $\mathbb{R}^d$-valued i.i.d random variables with finite exponential moments $\mathbb{E}\left(e^{\theta\|X\|}\right)$ for $\theta > 0$ and let $S_n = \sum_{i=1}^n X_i$. Then the empirical mean vector $\frac{S_n}{n}$ satisfies large deviation principle*

$$\mathbb{P}\left(\frac{S_n}{n} \approx X\right) \simeq e^{-nI(X)} \tag{68}$$

with the rate function

$$I(X) = \sup_{\lambda\in\mathbb{R}^d} \left(\langle\lambda, X\rangle - F(t)\right) \tag{69}$$

$$F(\lambda) = \ln \mathbb{E}\left(e^{\langle\lambda,X\rangle}\right) \tag{70}$$

where $\mathbb{E}(X_i) \neq X \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}^d$.

It is possible to further generalize the setting. We can let X_i's take values in some separable Banach Space B. Then one can derive the following large deviation principle

Theorem 2.13 *Let X_i be i.i.d random variables with values in a separable Banach space B. Let μ be a common distribution of X_i's. Assume that exponential moments $\mathbb{E}\left(e^{\theta\|X_i\|}\right)$ are finite for some $\theta > 0$. Let $S_n = \sum X_i$. Then the mean $\frac{S_n}{n}$ satisfies large deviation principle*

$$\mathbb{P}\left(\frac{S_n}{n} \approx X\right) \simeq e^{-nI(X)} \tag{71}$$

with the rate function

$$I(X) = \sup_{\lambda\in B^*} \left(\langle\lambda, X\rangle - F(\lambda)\right) \tag{72}$$

$$F(\lambda) = \ln \mathbb{E}\left(e^{\langle\lambda,X\rangle}\right) \tag{73}$$

where $\mathbb{E}(X_i) \neq X \in B$ and $\lambda \in B^*$. The B^* denotes dual space of B.

This general result has the following example. Let X_i be $\mathbb{R}$-valued i.i.d. random variables distributed with common distribution μ. Let us define the empirical measures

$$\mathcal{L}_n = \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$$

are elements of Banach space of finite signed measures $\mathcal{M}(\mathbb{R})$ on $\mathbb{R}$. Note that the dual space of $\mathcal{M}(\mathbb{R})$ is a space of all maps $\lambda_f : \mu \in \mathcal{M}(\mathbb{R}) \to \int f d\mu \in \mathbb{R}$ for functions $f \in \mathcal{C}_b(\mathbb{R})$ from the space of bounded continuous functions on $\mathbb{R}$. Then for ν deviating from the expectation of $\mathcal{L}_n$,

$$\mathbb{P}\left(\mathcal{L}_n \approx \nu\right) = e^{-nI(\nu)}$$

with the rate function

$$I(\nu) = \sup_{\lambda_f\in\mathcal{M}(\mathbb{R})^*} \left(\langle\lambda_f, \nu\rangle - F(\lambda_f)\right) = \tag{74}$$

$$= \sup_{f\in\mathcal{C}_b(\mathbb{R})} \left(\langle\lambda_f, \nu\rangle - F(f)\right) \tag{75}$$

where we note that optimizing $\langle\lambda_f, \nu\rangle - F(\lambda_f)$ over $\lambda_f \in \mathcal{M}(\mathbb{R})^*$, is the same as optimizing $\int f d\nu - F(f)$ over $f \in \mathcal{C}_b(\mathbb{R})$, where $F(f)$ is as follows

$$F(f) = \ln \mathbb{E}e^{\langle\lambda_f,\delta_X\rangle} = \ln \mathbb{E}e^{f(X)} = \ln \int e^{f(x)} d\mu(x) \tag{76}$$

Now substituting all in the rate function we get

$$I(\nu) = \sup_{f\in\mathcal{C}_b(\mathbb{R})} \left(\int f d\nu - \ln \int e^{f(x)} d\mu(x)\right) \tag{77}$$

This supremum gives Donsker-Varadhan's representation of relative entropy, i.e.

$$\sup_{f \in \mathcal{C}_b(\mathbb{R})} \left(\int f d\nu - \ln \int e^{f(x)} d\mu(x) \right) = H(\nu|\mu) = \begin{cases} \int \frac{d\nu}{d\mu} \ln \left(\frac{d\nu}{d\mu} \right) d\mu & \text{if } \nu \ll \mu \\ \infty & \text{if } \nu \not\ll \mu \end{cases} \tag{78}$$

(79)

where $H(\nu|\mu)$ is a relative entropy of ν from μ [5].

This means that the rate function for defined empirical measures $I(\nu) = H(\nu|\mu)$ is a relative entropy. This result is called Sanov's theorem

Theorem 2.14 (Sanov's Theorem) *Let X_i be i.i.d random variables with values in a separable Banach space B. Let μ be a common distribution of X_i's. Let us define the empirical measures*

$$\mathcal{L}_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{X_i}$$

Let $\nu \neq \mu$, then $\mathcal{L}_n$ satisfies large deviation principle

$$\mathbb{P}(\mathcal{L}_n \approx \nu) \simeq e^{-nH(\nu|\mu)} \tag{80}$$

with the rate function $H(\nu|\mu)$, which is an entropy from μ to ν:

$$H(\nu|\mu) = \begin{cases} \int \frac{d\nu}{d\mu} \ln \left(\frac{d\nu}{d\mu} \right) d\mu & \text{if } \nu \ll \mu \\ \infty & \text{if } \nu \not\ll \mu \end{cases} \tag{81}$$

where by $\nu \ll \mu$ we mean that ν doesn't assign the positive probabilities to events for which μ doesn't also do so

This is a quite surprising result, as it allows one to quantify the probabilities of the empirical measures to deviate from their asymptotic measures μ (to which they converge almost surely), and in turn to take the form of another measure ν.

One might notice that we have generalized a few functionals and tweaked the objects of interest. This justifies the "Level 2" name for such LDP.

3 Static Trapping Case

Here we discuss survival time assymptotics for Random Walk in a Static Random Environment. Firstly, we discuss the setup of the problem, in particular, we define a random static environment and survival functional. Then in the next subsection, we sketch a proof of the survival time asymptotics for the Independent Bernoulli hard traps model. At last, we state heuristics for the general i.i.d. environment of soft traps and state the general asymptotics divided into three classes depending on the underlying distribution of traps.

The trap environment η is drawn from some random distribution at a time zero and stays static in time. The random walk is left to traverse in this static random environment. This way, the generator of the process is exactly $L_1 f(\eta)$, where L_1 is the generator of the random walk, for the reasons that are going to be discussed in the next section .

In this section, we restrict the trapping configuration to be generated from i.i.d. distributions. This means, at a time $t = 0$ the lattice points "decide to be traps or normal points" independently of each other. There is no mixture between hard and soft traps. All traps are either defined to be a soft trap $\gamma < \infty$ for the whole configuration, or defined to be hard ones $\gamma = \infty$. Hard traps kill the Random Walk instantaneously upon touching. Soft traps do not kill the Random Walk in an instantaneous fashion, but they start an exponentially distributed killing timer. If the Random Walk leaves the trap before the "time" is up, it is free to travel further.

Let us define the survival time of the random walk in a random static environment functional.

$$Z_{\gamma,t} := \mathbb{P}(T \geq t) = \mathbb{E}_0^{X,\eta}\left(\exp\left[-\gamma \int_0^t \eta(X_s)ds\right]\right), \tag{82}$$

where $\mathbb{E}_0^{X,\eta} = \mathbb{E}_0^X \otimes \mathbb{E}^\eta = \mathbb{E}_0^X \mathbb{E}^\eta$. Note that the environment $\mathbb{E}^\eta$ expectation is taken ergodic translation-invariant measure ν_ρ.

3.1 Independent Bernoulli Hard Traps

In the special case when $\gamma = \infty$ (the hard traps) and when the random potential η is a collection of i.i.d. Bernoulli traps, the $Z_{\infty,t}$ can be thought as the expectation w.r.t. the random walk of the probability p that the site $x \in \mathbb{Z}^d$ is a trap to the power of the number of jumps realized by the random walk up to time t.

Then, given the Range of the random walk $R(X,t)$, which essentially counts a number of the unique sites to which the random walk has traversed, the $Z_{\infty,t}$ can be simplified as follows:

$$Z_{\infty,t} = \mathbb{E}_0^X\left(p^{R(X,t)}\right) = \mathbb{E}_0^X\left(e^{\ln p^{R(X,t)}}\right) = \mathbb{E}_0^X\left(e^{R(X,t)\ln p}\right) \tag{83}$$

From the results of the Donsker and Varadhan ([6]) we can choose the heuristics that the random walk chooses to stay within a spatial window of scale $1 \ll \alpha_t \ll \sqrt{t}$. In the (83) there are two competing forces: the exponential becoming large when the $R(X,t)$ is small and the costs of the random walk having a small range.

Now we have that the expectation is attained by the range of the random walk being inside a ball of the size α_t

$$e^{R(X,t)\ln p} \approx e^{c\alpha_t^d \ln p} = e^{c\alpha_t^d(-c)} = e^{-c_1\alpha_t^d} \tag{84}$$

Due to random walk scaling, for a certain c_2 we have

$$\mathbb{P}\left(\sup_{0\leq s\leq t} ||X(s)|| \leq \alpha_t\right) \approx \exp\left(-\frac{c_2 t}{\alpha_t^2}\right) \tag{85}$$

Thus we have the following asymptotics for the $Z_{\infty,t}$

$$Z_{\infty,t} \approx \exp\left\{-\inf_{1\ll\alpha_t\ll\sqrt{t}}\left(c_1\alpha_t^d + \frac{c_2 t}{\alpha_t^2}\right)\right\} = \exp\left\{-c_3 t^{\frac{d}{d+2}}\right\} \tag{86}$$

with the optimal scale $\alpha_t = t^{\frac{d}{d+2}}$, which equates the two terms in $c_1\alpha_t^d + \frac{c_2 t}{\alpha_t^2}$

This gives us the sub-exponential decay of survival probabilities in the case of Hard i.i.d. Bernoulli traps.

3.2 Independent General Soft Traps

This section mostly follows the material from section 3.3 of Wolfgang König book [7]. Here we state results for the case of general i.i.d. environment consisting of soft traps. We begin by deriving the rough lower and upper bounds on survival probability and then proceed by stating the general assymptotics divided into three classes, depending on the underlying environment distribution.

For the survival time in a static field of i.i.d. traps, one might notice that we can write the probability of the survival time in terms of the functional of local times of Random Walk over the expectation of the random walker.

We define the local times measure as $L_t(x)$ and cumulant generating function $H(t)$ as follows,

$$L_t(x) = \int_0^t \mathbb{1}(X_s = x)ds \tag{87}$$

$$H(t) = \ln \mathbb{E}^\eta \left(e^{-\eta(0)t}\right) \tag{88}$$

It is possible to obtain rough, but relatively simple upper and lower bounds on the survival probability

Lemma 3.1 *Given the local times measure as $L_t(x)$ and cumulant generating function $H(t)$ as defined above, the following rough bounds hold*

$$e^{H(\gamma t)-2dt} \leq Z_{\gamma,t} \leq e^{H(\gamma t)}, \quad t \in (0, \infty).$$

Proof: The lower bound can be obtained by restricting the expectation $Z_{\gamma,t} = \mathbb{E}_0^{\eta,X}\left[e^{-\gamma\int_0^t \eta(X_s)ds}\right]$ with respect to the random walk to the event $\bigcap_{s\in[0,t]}\{X_s = 0\}$ that the random walk does not leave the origin up to time t. The probability of such event is e^{-2dt} as the first jump time, $\tau = \inf\{t > 0 : X(t) \neq X(0)\}$, distributed exponentially with rate $2d$. Moveover this event, yields that $\int_0^t \eta(X(s))ds = t\eta(0)$. Thus,

$$Z_{\gamma,t} \geq \mathbb{E}^\eta \mathbb{E}_0^X \left[e^{-\gamma t\eta(0)}\mathbb{1}_{\{\tau>t\}}\right] = \mathbb{E}^\eta\left[e^{-\gamma t\eta(0)}\right]e^{-2dt} = e^{H(\gamma t)-2dt},$$

For the upper bound, we use Jensen's inequality in the exponential term in the Feynman-Kac representation to the probability measure on $[0, t]$ with Lebesgue density $1/t$:

$$\exp\left\{-\gamma\int_0^t \eta(X_s)\,ds\right\} \leq \int_0^t \frac{1}{t}\exp\{-\gamma t\eta(X_s)\}\,ds.$$

It is possible to interchange the expectation with respect to η with the time integral and the random walk expectation

$$\begin{aligned} Z_{t,\gamma} &\leq \mathbb{E}^\eta \mathbb{E}_0 \int_0^t \frac{1}{t}\left[\exp\{-\gamma t\eta(X_s)\}\right]ds = \int_0^t \frac{1}{t}\mathbb{E}^\eta\left[\mathbb{E}_0 e^{-\gamma t\eta(X_s)}\right]ds = \mathbb{E}^\eta e^{-\gamma t\eta(0)} \\ &= e^{H(\gamma t)} \end{aligned}$$

This yields an upper bound. The resulted bounds for the asymptotics hint that the survival probability asymptotics will be described by at least two terms. Furthermore, we will see that $e^{H(t)}$ will be replaced by a modified term.

□

Now, let us derive the sharp results. We will state the setup and the assumptions for the heuristical approach, as follows. The occupation times formula yields [7],

$$\int_0^t \eta(X_s)ds = \sum_{z\in\mathbb{Z}^d}\eta(z)L_t(z) \tag{89}$$

This in turn yields the following heuristical expansion of the survival time of a random walk,

$$Z_{\gamma,t} = \mathbb{E}_0^{X,\eta}\left(\exp\left[-\gamma\int_0^t \eta(X_s)ds\right]\right) = \mathbb{E}_0^{X,\eta}\left(\exp\left[-\gamma\sum_{z\in\mathbb{Z}^d}\eta(z)L_t(z)\right]\right) = \tag{90}$$

$$= \mathbb{E}_0^X\mathbb{E}^\eta\left(\prod_{z\in\mathbb{Z}^d}\exp\left[-\gamma\eta(z)L_t(z)\right]\right) = \mathbb{E}_0^X\left(\prod_{z\in\mathbb{Z}^d} e^{\ln\mathbb{E}^\eta\exp[-\gamma\eta(0)L_t(z)]}\right) = \tag{91}$$

$$= \mathbb{E}_0^X\left(\prod_{z\in\mathbb{Z}^d} e^{H(\gamma L_t(z))}\right) = \mathbb{E}_0^X\left(\exp\left[\sum_{x\in\mathbb{Z}^d} H(\gamma L_t(x))\right]\right) \tag{92}$$

The expression that we got can be called the total mass expansion,

$$Z_{\gamma,t} = \mathbb{E}_0^X\left(\exp\left[\sum_{x\in\mathbb{Z}^d} H(\gamma L_t(x))\right]\right) \tag{93}$$

Given this expansion, and the contraction principle from the Large Deviations theory, it is possible to analyze the behavior of the survival probability using large deviations of the occupation time measure of the random walk.

The survival time is the joint expectations over the path and over the potential. Thus, there is a contribution from both of these random objects, and it is possible to get an optimum in this functional, by a compromise joint strategy between two events, so that neither of them is too costly to plummet the survival time asymptotics.

It is possible to get the strategy that yields exponential costs from both of these opposing effects. Such a joint strategy of these random objects is a compromise between the two effects. Since each of them must contribute the exponential costs: the random trapping environment must yield low values in a suitable area, and the random path should not leave this area for time from zero to t. For making the latter not too costly, the area should be a centered ball. Hence, the main contribution to the survival time should come from a self-attractive behaviour of the random walk and a low input of the potential.

Talking from the perspective of intermittency, it turns out that the contribution of the sum over all $\mathbb{Z}^d$ of the solution is asymptotically optimally described by the sub-sum of a smaller spatial region, which can be centered at the origin with a radius $R\alpha(t)$ where $\alpha(t)$ will be our spatial scaling term [7].

Let us state the assumptions regarding the spatial scaling and the rescaling factors.

(A_1) : Let the rescaled occupation time measure be of the form

$$\hat{L}(y) = \frac{\alpha_t^d}{t}L(\lfloor\alpha_t y\rfloor) \qquad y\in[-R,R]^d \tag{94}$$

where the scale α_t is such that

$$\eta\left(\frac{\gamma t}{\alpha_t^d}\right)\alpha_t^d = \frac{t}{\alpha_t^2} \tag{95}$$

for some continuous $\eta(t) : \mathbb{R}^+ \to \mathbb{R}^+$.

(A_2) : One needs one more assumption on the generating function $H(t)$, namely that the following $\hat{H}(t) : \mathbb{R}^+ \to \mathbb{R}$ is not zero if $y \neq 1$:

$$\hat{H}(t) = \lim_{t\to\infty}\frac{H(yt) - yH(t)}{\eta(t)} \tag{96}$$

for some continuous $\eta(t) : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{t\to\infty}\eta(t)/t \in [0,\infty]$ from the definition of the $\alpha(t)$

This assumption comes from the fact that the rescaling from $\mathbb{Z}^d$ to the $\mathbb{R}^d$ case, requires that both rescaled $\hat{L}(y)$ and $\hat{H}(t)$ exist. One might notice, that it is not an overly restricting assumption. Let $\alpha(t) \to \infty$ as $t \to \infty$.

Now we can rewrite the $Z_{\gamma,t}$ expression in (93) as following

$$Z_{\gamma,t} = \mathbb{E}_0^X\left(\exp\left[\sum_{x\in\mathbb{Z}^d} H(\gamma L_t(x))\right]\right) = \tag{97}$$

$$= \mathbb{E}_0^X\left[\exp\left(\eta\left(\frac{\gamma t}{\alpha_t^d}\right)\sum_{x\in\mathbb{Z}^d}\frac{H\left(\frac{\gamma t}{\alpha_t^d}\tilde{L}_t\left(\frac{x}{\alpha_t}\right)\right)-\tilde{L}_t\left(\frac{x}{\alpha_t}\right)H\left(\frac{\gamma t}{\alpha_t^d}\right)}{\eta\left(\frac{\gamma t}{\alpha_t^d}\right)}\right)\right]e^{\alpha_t^d H\left(\frac{\gamma t}{\alpha_t^d}\right)} \geq \tag{98}$$

$$\geq e^{\alpha_t^d H\left(\frac{\gamma t}{\alpha_t^d}\right)}\mathbb{E}_0^X\left[\exp\left(\eta\left(\frac{\gamma t}{\alpha_t^d}\right)\sum_{x\in\mathbb{Z}^d}\hat{H}\left(\tilde{L}_t\left(\frac{x}{\alpha_t}\right)\right)\right)\right] \geq \tag{99}$$

$$\geq e^{\alpha_t^d H\left(\frac{\gamma t}{\alpha_t^d}\right)}\mathbb{E}_0^X\left[\exp\left(\eta\left(\frac{\gamma t}{\alpha_t^d}\right)\alpha_t^d\int_{\mathbb{R}^d}\hat{H}\left(\tilde{L}_t(y)\right)dy\right)\right] = \tag{100}$$

$$= e^{\alpha_t^d H\left(\frac{\gamma t}{\alpha_t^d}\right)}\mathbb{E}_0^X\left[\exp\left(\frac{t}{\alpha_t^2}\int_{\mathbb{R}^d}\hat{H}\left(\tilde{L}_t(y)\right)dy\right)\right] \tag{101}$$

which results in the term in (101) with an exponential rate of decay with rate $t\alpha(t)^{-2}$. This term is comparable with the large deviation of the probability of the Random Walk being restricted to a spatial window of scale α_t. Now the asymptotics of the survival probability can be derived using LDP for the random walk occupation time measure at scale α_t.

Lemma 3.2 ([7]) *The collection $(L_t)_{t\in(0,\infty)}$ satisfies a large-deviation principle with the rate $t\alpha(t)^{-2}$ and the rate function $g^2 \mapsto \|\nabla g\|_2^2$, that is,*

$$\mathbb{P}_0\left(L_t(\cdot)\approx g^2(\cdot) \text{ in } [-R,R]^d\right)\approx \exp\left\{-\frac{t}{\alpha(t)^2}\|\nabla g\|_2^2\right\},$$

for any L^2-normalised function $g\in H^1\left(\mathbb{R}^d\right)$ with support in $[-R,R]^d$.

To get the lower bound we insert the indicator of the event $\left\{L_t(\cdot)\approx g^2(\cdot) \text{ in } [-R,R]^d\right\}$ and optimize over g^2 and R. This yields that under the above assumptions

Theorem 3.3 ([7]) *Under assumptions* (A_1) *and* (A_2)*, the following lower bound on survival asymptotics holds*

$$Z_{\gamma,t}\geq \mathrm{e}^{\alpha(t)^d H\left(\frac{\gamma t}{\alpha(t)^d}\right)}\exp\left\{-\frac{t}{\alpha(t)^2}\left(\chi_\circ+o(1)\right)\right\},$$

where $\chi_\circ$ is given as

$$\chi_\circ=\inf\left\{\|\nabla g\|_2^2-\int_{\mathbb{R}^d}\widehat{H}\circ g^2 : g\in H^1\left(\mathbb{R}^d\right),\|g\|_2=1\right\}.$$

It is further possible to derive the following exact asymptotics. The proof is extremely technical and scattered across multiple papers, so we would like to state the main result without the proof.

Theorem 3.4 ([7]) *Under assumptions* (A_1) *and* (A_2) *we have the following cases for asymptotics,*

(C_1) : *If $\lim_{t\to\infty}\eta(t)/t=0$, then $\alpha(t)\to\infty$, and the asymptotics are of following form*

$$Z_{\gamma,t}=\mathrm{e}^{\alpha(t)^d H\left(\gamma t/\alpha(t)^d\right)}\exp\left\{-\frac{t}{\alpha(t)^2}\left(\chi_\circ+o(1)\right)\right\},$$

where $\chi_\circ$ is given as

$$\chi_\circ=\inf\left\{\|\nabla g\|_2^2-\int_{\mathbb{R}^d}\widehat{H}\circ g^2 : g\in H^1\left(\mathbb{R}^d\right),\|g\|_2=1\right\}.$$

(C_2) : *If $\lim_{t\to\infty}\eta(t)/t=1$, then $\alpha(t)\to 1$, and $\widehat{H}(y)=\rho y\log y$ for some $\rho\in(0,\infty)$, and the asymptotics are of following form*

$$Z_{\gamma,t}=\mathrm{e}^{H(\gamma t)}\exp\left\{-t\left(\chi_\rho+o(1)\right)\right\},$$

where χ_ρ *is given by*

$$\chi_\rho = \inf_{\varphi:\mathbb{Z}^d\to\mathbb{R}:\lim_{z\to\infty}\varphi(z)=-\infty}\left(\frac{\rho}{\mathrm{e}}\sum_{z\in\mathbb{Z}^d}\mathrm{e}^{\varphi(z)/\rho}-\lambda(\varphi)\right)$$

with $\lambda(\varphi)=\sup_{g\in\ell^2(\mathbb{Z}^d):\|g\|_2=1}\langle g,(L_1+\varphi)\,g\rangle$ *is defined as the top of the spectrum of* $L_1+\varphi$ *in* $\mathbb{Z}^d$*;*

(C_3) : *If* $\lim_{t\to\infty}\eta(t)/t=\infty$*, then* $\alpha(t)\to 0$ *and* $\rho=\infty$*, and the asymptotics are of following form*

$$Z_{\gamma,t}=\mathrm{e}^{H(\gamma t)}\exp\left\{-t\left(\chi_\infty+o(1)\right)\right\},$$

where $\chi_\infty = 2d$

One might notice that only when the environment of traps of such that the radius $\alpha(t)$ for the t-th moments of $\frac{1}{t}\int_0^t\eta(X_s)ds$ diverges do we get the subexponential decay of survival probability, in other cases when this island stays bounded away from zero and when this island shrinks we get the exponential decay.

4 Dynamic Traps and Environment Process

In this section, we establish the setup of our models of random walks in a dynamic random environment of traps. We formally define the random walk in a dynamic trapping environment process, which is referred to as the Environment Process. Moreover, here we derive the generator of the Environment Process, which will come out to be a sum of the generators of the Random Walk and the Trap process. At last, we define the functional of interest - the survival probability of the Random Walk in the dynamic environment.

Our choice of the Random Walk is of continuous time type, so it waits an exponential time before jumping to another lattice point different from its current position. In different models, we will define traps differently. In each model all traps are either "hard" - meaning the Random Walk is killed immediately upon touching the trap, or "soft" - meaning, if the Random Walk escapes the trap before the exponential killing time is up it lives, otherwise it is killed. In the dynamic setting, the initial η environment consists of the traps placed randomly as in the static setting, but the configuration of traps is changed over time according to some Markov Process.

In the Independent Spin Flip and Attractive Spin Flip models, each lattice point is flipped from being a normal site to being a trap with some rate. In the case of the Independent Spin Flip model each site is independent of the other, while in the Attractive Spin Flip model, the rates at which the sites are flipping are dependent on the configuration of traps. In other models, traps do not vanish or pop out of nothingness, they move along the lattice according to some law. In the Random Walking Traps Environment, the traps are independent Random Walkers with no limit on number of traps per site. In the Simple Symmetric Exclusion Process, the traps walk randomly to the nearest neighbors, waiting exponential time between the jumps, with the restriction of no more than one trap per site.

We call our state space E a space of all possible trap configurations and $\eta \in E$ a trap configuration. In this thesis, the following state spaces are considered.

- $E = \{0,1\}^{\mathbb{Z}^d}$ with at most one trap per site
- $E = \mathbb{N}^{\mathbb{Z}^d}$ with potentially multiple traps per site

As the distributions of traps change over time, let us define a Markov Process $\{\eta_t, t \geq 0\}$ as our trap process. The η_t takes values in E giving us the distribution of traps at a time t. We define $\eta_t(x)$ to be the value at the point $x \in \mathbb{Z}^d$ of the environment η_t at the time T. Thus we need to define a generator in a general form for the family of all of the different trapping processes that we study in this paper.

Definition 4.1 (Trap Process Generator) *The general form of generator of the trap process $\{\eta_t, t \geq 0\}$ is of the following form:*

$$L_0 f(\eta) = \lim_{t \to 0^+} \frac{\mathbb{E}^\eta f(\eta_t) - f(\eta)}{t} \tag{102}$$

As we work on a lattice space $\mathbb{Z}^d$, it is quite natural to assume some sort of the form of a translational invariance. The probability distributions environment should look the same if we translate the coordinate system, as there no special points in $\mathbb{Z}^d$ that we consider, nor any point is different from another.

Definition 4.2 (Translation operator) *Let η be a configuration of traps drawn from the space of all possible trap configurations E (which can be one of the state spaces discussed previously). Let τ_x be the translation operator, i.e. $\tau_x \eta$ is an environment seen from point x, with*

$$(\tau_x \eta)(y) = \eta(y + x)$$

Now let us focus on the generator of the Random Walk in the built setup. We assume translational symmetries of the Random Walk's jumping rates. Given the definition of the translational operator, the generator of the Random Walker of our choice, for all of the models, can be expressed as follows

Definition 4.3 (Shifted Random Walk Generator) *Let $X := (X_t)_{t \geq 0}$ be a continuous time random walk on $\mathbb{Z}^d$, its generator takes the following form*

$$(L_1 f)(\eta) = \sum_{x \in \mathbb{Z}^d} p(0, x)(f(\tau_x \eta) - f(\eta)) \tag{103}$$

Since we have formally defined the trapping process and the random walk, and their generators, it is plausible to start to think about how they should interact if we put both processes on the same space. We define this joined space as a product probability space, that has the following formal definition.

Definition 4.4 (Product Space) *Let* $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ *and* $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ *be two probability spaces. The space* $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$ *is called their product space, if* $\mathbb{P}_1 \otimes \mathbb{P}_2(A_1 \times A_2) = \mathbb{P}_1(A_1)\mathbb{P}_2(A_2)$ *is a product measure, and* $\mathcal{F}_1 \otimes \mathcal{F}_2$ *defined to be minimal* σ*-algebra of subsets of* $\Omega_1 \times \Omega_2$ *that contains all sets of the form* $A_1 \times \Omega_2$ *and* $\Omega_1 \times A_2$*, where* $A_1 \in F_1$ *and* $A_2 \in F_2$*.*

We define the probability space of our combined processes as $(\mathbb{Z}^d \times E, \mathcal{F}^X \otimes \mathcal{F}^\eta, \mathbb{P}^X \otimes \mathbb{P}^\eta)$. Let us define the combined Markov Process and find its generator, which will be the main result of this section. We will refer to the combined Markov Process as an Environment process. Naturally speaking, the Environment Process is an environment as seen from the perspective of the Random walk. Formally it can be expressed quite elegantly as,

Definition 4.5 (Environment Process) *The Environment Process is a Markov Process* $\{\eta_t(X_t), t \geq\}$ *which is an environment from the perspective of Random Walk* X_t*, expressable in terms of translations in a following way*

$$\eta_t(X_t) = \tau_{X_t}\eta_t$$

The overall process Environment Process is quite sophisticated and its study is quite problematic, due to the trapping process and random walk living on the same product space, giving us a combined Environment Process with its own generator. Let us state the general non-closed form of the Environment Process generator

Definition 4.6 (Environment Process Generator) *The general form of the The Environment Process* $\{\eta_t(X_t), t \geq\}$ *denoted by* L *is of the following form:*

$$Lf(\eta) = \lim_{t \to 0^+} \frac{\mathbb{E}_0^{\eta,X} f(\eta_t) - f(\eta)}{t} \tag{104}$$

where $\mathbb{E}_0^{\eta,X}$ *is the expectation taken with respect to the product space* $(\mathbb{Z}^d \times E, \mathcal{F}^X \otimes \mathcal{F}^\eta, \mathbb{P}^X \otimes \mathbb{P}^\eta)$*, i.e.* $\mathbb{E}_0^{X,\eta} = \mathbb{E}_0^X \otimes \mathbb{E}^\eta = \mathbb{E}_0^X \mathbb{E}^\eta$*. Note that the environment* $\mathbb{E}^\eta$ *expectation is taken with respect to a reversible and ergodic translation-invariant measure* ν_ρ*, the definition of which will be discussed further in this section.*

Now given all of the prerequisites, let us observe that the generator of the environment process is just a sum of the Random Walk's and Trap process' generators. More generally speaking the generator of the coupling of the independent generators is a sum of these generators.

Theorem 4.1 *The generator* L *of the Environment process is of the following form:*

$$(Lf)(\eta) = (L_1 f + L_0 f)(\eta) \tag{105}$$

Proof:

$$Lf(\eta) = \lim_{t \to 0^+} \frac{E_0^{\eta,X} f(\eta) - f(\eta)}{t} = \lim_{t \to 0^+} \frac{1}{t}\left(\int f(\tau_{X_t}\eta_t) d(\mathbb{P}_0^X \otimes \mathbb{P}^\eta) - f(\eta)\right) \underset{theorem}{\overset{Fubini's}{=}} \tag{106}$$

$$= \lim_{t \to 0^+} \frac{1}{t}\left(\mathbb{E}^\eta \mathbb{E}_0^X f(\tau_{X_t}\eta_t) - f(\eta)\right) = \lim_{t \to 0^+} \frac{1}{t}\left(\sum_x p_t(0,x)\mathbb{E}^\eta f(\tau_x \eta_t) - f(\eta) \cdot 1\right) = \tag{107}$$

$$= \lim_{t \to 0^+} \frac{1}{t}\left(\sum_x p_t(0,x)\left[\mathbb{E}^\eta \tau_x f(\eta_t) - \tau_x f(\eta) + \tau_x f(\eta) - f(\eta)\right]\right) = \tag{108}$$

$$= \lim_{t \to 0^+}\left(\sum_x p_t(0,x)\left[\frac{\mathbb{E}^\eta \tau_x f(\eta_t) - \tau_x f(\eta)}{t}\right] + \sum_x \frac{p_t(0,x)}{t}\left[\tau_x f(\eta) - f(\eta)\right]\right) = \tag{109}$$

$$= \sum_x I(x=0) L_0(\tau_x f(\eta)) + \sum_x p(0,x)\left[\tau_x f(\eta) - f(\eta)\right] = \tag{110}$$

$$= L_0(\tau_0 f(\eta)) + L_1 f(\eta) = \tag{111}$$

$$= L_0 f(\eta) + L_1 f(\eta) \tag{112}$$

□

Now we will define invariant measures and the notion of ergodicity.

Let S_t be a Markov Semigroup, then for $\mu \in \mathcal{P}(E)$ we call the measure μS_t evolution of the measure μ after time t. In other words by μS_t we denote distribution of the process $\{\eta_t : t > 0\}$ at a time t when the process started from μ. Functionally this means,

$$\int f d\mu S_t = \int S_t f d\mu \qquad \forall f \in \mathcal{C}(E)$$

Definition 4.7 *A probability measure $\mu \in \mathcal{P}(E)$ is called invariant if*

$$\int S_t f d\mu = \int f d\mu \qquad \forall f \in \mathcal{C}(E)$$

Let us denote by $\mathcal{I}$ the set of all invariant measures on E.

We call a set $A \in E$ invariant if $S_t(\mathbb{1}_A) = \mathbb{1}_A$. Now we can define ergodicity of the measure μ

Definition 4.8 *A probability measure $\mu \in \mathcal{I}$ is called ergodic if all invariant sets have measre μ zero or one.*

Definition 4.9 *The process with semigroup S_t is called uniquely ergodic if $\mathcal{I} = \{\mu\}$ and for all $\nu \in \mathcal{P}(E)$ we have*

$$\lim_{t \to \infty} \nu S_t = \mu$$

Now we define the notion of the reversibility of the measure.

Definition 4.10 *A probability measure $\mu \in \mathcal{P}(E)$ is called reversible if*

$$\int (S_t f) g d\mu = \int f (S_t g) d\mu \qquad \forall f, g \in \mathcal{C}(E)$$

Note that a reversible measure is invariant.

More intuitively a reversible measure describes the following property. A measure $\mu \in \mathcal{P}(E)$ is reversible if and only if the process $\{\eta_t : 0 < t < T\}$ started from η_0 distributed according to μ has the same distribution as its time reversed analogue $\{\eta_{T-t} : 0 < t < T\}$.

We further assume that for each trapping process $\{\eta_t : t \geqslant 0\}$ that is going to be discussed in this paper there is a reversible and ergodic translation-invariant measure ν_ρ. Furthermore, we assume that the domain of L_0 is translation invariant. To formalize this,

- For any $f \in D(L_0)$, the translated $\tau_x f \in D(L_0)$ for all $x \in \mathbb{Z}^d$
- L_0 is self-adjoint on $L^2(\nu_\rho)$
- ν_ρ is ergodic for the process $\{\eta_t : t \geq 0\}$.
- $\tau_x \nu_\rho = \nu_\rho$, for all $x \in \mathbb{Z}^d$.

Now by using these assumptions, we can state an incredibly useful result

Theorem 4.2 ([8]) *The reversible and ergodic translation-invariant measure ν_ρ for the trapping process $\{\eta_t : t \geq 0\}$ generated by L_0 is again reversible and ergodic translation-invariant for the Environment Process $\{\eta_t(X_t) : t \geq 0\}$ generated by $L = L_1 + L_0$*

Now the object of the interest in the built setup is the probability that a Random Walker survives in a dynamic environment of traps for some $T \geq t$, defined to be

$$Z_{\gamma,t} = \mathbb{P}(T \geq t) = \mathbb{E}_0^{X,\eta}\left(\exp\left[-\gamma \int_0^t \eta_s(X_s) ds\right]\right) \tag{113}$$

where $\mathbb{E}_0^{\eta,X}$ is the expectation taken with respect to the product space $(\mathbb{Z}^d \times E, \mathcal{F}^X \otimes \mathcal{F}^\eta, \mathbb{P}^X \otimes \mathbb{P}^\eta)$, i.e. $\mathbb{E}_0^{X,\eta} = \mathbb{E}_0^X \otimes \mathbb{E}^\eta = \mathbb{E}_0^X \mathbb{E}^\eta$.

Further, we will study this functional under the previously given list of distinct trap processes.

5 Random Walk in an Independent Spin Flip Trap Dynamics

Our first example of the problem in a dynamic setting is the model of the random walk in the independent spin-flip trap process. Firstly we discuss the setup of the model. Then we proceed by proving a rough lower bound and sharp upper bound on the survival probability. Lastly, we prove that the decay is exponential under environment monotonicity assumptions and then provide an example of such an environment.

In this setting, the space of all trap configurations is $E = \{0,1\}^{\mathbb{Z}^d}$. Thus the environment (or the trap configuration) at any given time is a collection of spins 1's and 0's, where by 1 at a site $x \in \mathbb{Z}^d$ we denote a trap, and by 0 we denote a normal point. Then we all let traps independently flip at the same exponential rate c. The trap process begins from the configuration of i.i.d. Bernoulli distributed traps, which should also form an ergodic and reversible translation invariant measure ν_ρ. By [9] such measure exists.

Definition 5.1 (Reversible translational invariant and ergodic measure) *For the Independent Spin flip traps starting from i.i.d. Bernoulli distributed configuration the ergodic and reversible translation invariant measure is*

$$\nu_\rho = \bigotimes_{\mathbb{Z}^d} \textit{Bernoulli} \qquad \rho \in [0,1]$$

Let us quickly remind the reader about the generators of two Markov processes in this model. The generator of the Random walk is the same as was stated in the previous section, and the trapping process has the following generator

Definition 5.2 (Independent Spin Flip Process Generator) *Let trap process be independent-flips process $\eta(x,t) \in \{0,1\}$ with $x \in \mathbb{Z}^d$ defined on probability space $(E, \mathcal{F}^\eta, \mathbb{P}^\eta)$ with following generator L_0*

$$(L_0 f)(\eta) = \sum_x c(f(\eta^x) - f(\eta)) \tag{114}$$

where $\eta \in \{0,1\}^{\mathbb{Z}^d}$ is "frozen" configuration of traps on $\mathbb{Z}^d$ and η^x is a configuration with point $x \in \mathbb{Z}^d$ flipped, meaning following

$$\eta^x = \begin{cases} 1 - \eta(x), & \textit{if } y = x \\ \eta(y), & \textit{if } y \neq x \end{cases} \tag{115}$$

Note that the spins flip at rate c, with each lattice point flipping independently.

Now, let us define the overall Environment Process.

Definition 5.3 (Environment process) *Environment process is defined as the configuration of traps $\eta(t, X_t)$ at a time t as seen from the perspective of a random walker $(X_t)_{t\geq 0}$ $x \in \mathbb{Z}^d$ defined on probability space $(\mathbb{Z}^d \times E, \mathcal{F}^X \otimes \mathcal{F}^\eta, \mathbb{P}^X \otimes \mathbb{P}^\eta)$*

$$\eta(t, X_t) = \begin{cases} 1, & \textit{if a point } x \in \mathbb{Z}^d\textit{, at which random walker } X_t = x \textit{ is situated at a time } t\textit{, is a trap} \\ 0, & \textit{otherwise} \end{cases} \tag{116}$$

And the generator of the Environment Process takes the form of a sum of both generators of Random Walk (the L_1 generator) and ISF Trapping process (the L_0 generator) since both are independent Markov processes.

$$Lf(\eta) = L_0 f(\eta) + L_1 f(\eta)$$

In the following subsections, we are interested in the survival probability of the Random Walk, which we define in the following way

$$Z_{\gamma,t} = \mathbb{P}(T \geq t) = \mathbb{E}_0^{X,\eta}\left(\exp\left[-\gamma \int_0^t \eta_s(X_s) ds\right]\right) \tag{117}$$

In future subsections, we will observe that for the Independent Spin Flip process, the survival functional can be decomposed, which narrows down the focus to a single point of the environment, due to the independent nature of the traps. Further, it will become apparent that the study of the survival time asymptotics will rely on the observations of the so-called occupational time measure of the single point of origin. Throughout this thesis, we will resort to the indirect calculations of the survival probabilities via spectral methods and other sophisticated approaches.

Now we can proceed with quite simple calculations of the exponential lower bound on the decay of survival probabilities.

5.1 Lower Bound on Survival Probability

In this subsection, we will prove the rough lower bound on the survival functional and connect this lower bound to the special case of this model with infinite flip rates

Lemma 5.1 *The survival time of the Random Walk X_t among the independent spin-flip trap process can be exponentially bounded as follows*

$$\mathbb{P}(T \geq t) \geq e^{-\gamma\rho t}$$

where $\rho = \mathbb{E}^{\eta}\eta_s(\cdot)$ is a stationary distribution for $\eta_s(\cdot)$

Proof: The survival time functional can be bounded as follows

$$\mathbb{P}(T \geq t) = \mathbb{E}_0^{X,\eta}\left(\exp\left[-\gamma\int_0^t \eta(s,X_s)ds\right]\right) \geq \tag{118}$$

$$\geq \exp\left[-\gamma\int_0^t \mathbb{E}_0^{X,\eta}\eta(s,X_s)ds\right] = \tag{119}$$

$$= \exp\left[-\gamma\int_0^t \mathbb{E}^X\mathbb{E}_0^{\eta}\eta(s,X_s)ds\right] = \tag{120}$$

$$= \exp\left[-\gamma\int_0^t \sum_{x\in\mathbb{Z}^d}\mathbb{P}(X_s = x)(\mathbb{E}_0^{\eta}\eta(s,X_s))ds\right] = \tag{121}$$

$$= \exp\left[-\gamma\int_0^t \sum_{x\in\mathbb{Z}^d}\rho\mathbb{P}(X_s = x)ds\right] = \tag{122}$$

$$= e^{-\gamma\rho t} \tag{123}$$

□

Thus we have derived a very simple exponential lower bound, which, nevertheless, is exactly the limit when spin rates $c_x = \infty$ are infinite.

$$\lim_{c\to\infty}\mathbb{P}(T \geq t) = \lim_{c\to\infty}\mathbb{E}_0^{X,\eta}\left(\exp\left[-\gamma\int_0^t \eta(s,X_s)ds\right]\right) = \tag{124}$$

$$= \lim_{c\to\infty}\mathbb{E}^X\mathbb{E}_0^{\eta}\left(\exp\left[-\gamma\int_0^t \eta(s,X_s)ds\right]\right) = \tag{125}$$

$$= \lim_{c\to\infty}\mathbb{E}^X\left(\exp\left[-\gamma\int_0^t \rho ds\right]\right) = \tag{126}$$

$$= e^{-\gamma\rho t} \tag{127}$$

This happens since $c_x = \infty$ makes the random walk see the average environment, which essentially means that $\eta(s,\cdot) = \rho$ in distribution.

5.2 Upper Bound on Survival Probability

Here we derive an exponential upper bound for the survival probability in an agnostic way. We assume that the survival probabilities decay exponentially, i.e. there exists such an exponent $\tilde{\lambda}_{\max}$ (the definition of which will be apparent below) which dictates the decay rate. Then we bound this exponent using spectral techniques.

Theorem 5.2 *For the Random Walk in the Independent Spin Flip trap process, if the $\tilde{\lambda}_{\max}$ greatest eigenvalue of the operator $(L_0 + L_1 + V)$ exists then*

$$\mathbb{P}(T > t) \le e^{\tilde{\lambda}_{\max} t} \tag{128}$$

with $\tilde{\lambda}_{\max}$ being strictly negative. Note that $\mathbb{P}$ is taken with respect to ergodic and reversible measure.

Note that the existence of $\tilde{\lambda}_{\max} < 0$ is assumed here. We delegate proof of its existence under environment monotonicity assumptions to the next subsection. Then the final subsection is dedicated to expressing these assumptions in terms of non-degeneracy conditions on the rate function of occupational time measures of the system.

Proof: Given translation operator $\tau_{X_s}\eta(s,\cdot) = \eta(s, X_s)$ defined in the previous section and the fact that the environment doesn't have special points and translationally invariant, we can state that we have the following equality $\tau_{X_s}\eta(s,\cdot) = \eta(s,0)$ in distribution

$$\mathbb{P}(T \ge t) = \mathbb{E}_0^{\eta,X}\left(\exp\left[-\gamma\int_0^t \eta(s,X_s)ds\right]\right) = \int d\nu_\rho(\eta)\tilde{\mathbb{E}}_0^{\eta,X}\left(\exp\left[-\gamma\int_0^t \tau_{X_s}\eta(s,\cdot)ds\right]\right) = \tag{129}$$

$$= \int d\nu_\rho(\eta)\tilde{\mathbb{E}}^{\eta}\left(\exp\left[-\gamma\int_0^t \eta(s,0)ds\right]\right) = \int d\nu_\rho(\eta)\tilde{\mathbb{E}}^{\eta}\left(\exp\left[\int_0^t V(\eta_s)ds\right]\right) \tag{130}$$

where $V = -\gamma\eta(0)$ and $\tau_{X_s}\eta(s,\cdot)$ is an environment seen from the perspective of a random walk, with generator $\tilde{L}$ and ν_ρ is an ergodic and reversible measure of the environment process.

Let us observe that the inner expectation is the same as,

$$\mathbb{P}(T \ge t) = \int d\nu_\rho(\eta)\tilde{\mathbb{E}}^{\eta,X}\left(\exp\left[\int_0^t V(\eta_s)ds\right]\right) = \langle 1, e^{t(\tilde{L}+V)}1\rangle \tag{131}$$

where we used the fact that we proved in the "Feynman-Kac semigroup" section. We denote by $\tilde{L} = L_0 + L_1$ as a generator of the environment process (a sum of random walk's and trapping potential's generators) and $V = -\gamma\eta(0)$ is the so-called killing function. The previous inner product expression can be bounded by the greatest eigenvalue of $\tilde{L} + V$ in the following way

$$\langle 1, e^{t(\tilde{L}+V)}1\rangle \le \langle 1, 1\rangle e^{\tilde{\lambda}_{\max} t} \tag{132}$$

The $\tilde{\lambda}_{\max}$ greatest eigenvalue of the operator $(\tilde{L} + V)$ and can be found using the variational formula:

$$\tilde{\lambda}_{\max} = \sup_{g:\int g^2 d\nu_\rho = 1}\left(-\int V(\eta)g^2(\eta)\nu_\rho(d\eta) + (g, \tilde{L}g)\right) \tag{133}$$

Now we can find an upper bound the following way. Since by variational formula, we have the supremum that contains an inner product that involves $\tilde{L}$, which is a sum of inner products of its independent constituent generators. The random walk part can be thrown away giving us a strict bound:

$$\tilde{\lambda}_{\max} = \sup_{g:\int g^2 d\nu_\rho = 1}\left(-\int V(\eta)g^2(\eta)\nu_\rho(d\eta) + (g, \tilde{L}g)\right) = \tag{134}$$

$$= \sup_{g:\int g^2 d\nu_\rho = 1}\left(-\int V(\eta)g^2(\eta)\nu_\rho(d\eta) + (g, L_0 g) + (g, L_1 g)\right) \le \tag{135}$$

$$\le \sup_{g:\int g^2 d\nu_\rho = 1}\left(-\int V(\eta)g^2(\eta)\nu_\rho(d\eta) + (g, L_0 g)\right) = \lambda_{\max} \tag{136}$$

where we get the Random Walk out due to $-L_1$ being non-negative on L^2

Note that, due to only V and L_0 remaining, $\lambda_{\max}$ can be computed in the following way

$$\lambda_{\max} = \lim_{t\to\infty}\frac{1}{t}\ln \mathbb{E}^{\eta}\left(\exp\left[-\gamma\int_0^t \eta(s,0)ds\right]\right) = \lim_{t\to\infty}\frac{1}{t}\ln\langle 1, e^{t(L+V)}1\rangle \tag{137}$$

where the L and V are trap generator and killing function of a single point.

Now we specify the resulting L and V for independent spin flip dynamics. Without the loss of generality, we set flipping rates $c = 1$. Since the computations are done with respect to a single trap point and due to the Feynman-Kac formula, the L and V take the following form,

$$L = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad V = \begin{bmatrix} -\gamma & 0 \\ 0 & 0 \end{bmatrix} \qquad A := L + V = \begin{bmatrix} -\gamma - 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{138}$$

All is left to find the closed form of the exponential operator $e^{tA} = e^{t(L+V)}$ which will be quite computationally extensive, due to L and V not commutive between each other.

The eigenvalues of the matrix A are the following

$$\nu_1, \nu_2 = \frac{-\gamma - 2 \pm \sqrt{\gamma^2 + 4}}{2} =: \frac{-\gamma \pm \mu}{2} - 1 \tag{139}$$

The matrix Q of the eigenvectors of A can be computed to be the following:

$$Q = \begin{bmatrix} \frac{-\gamma+\mu}{2} & \frac{-\gamma-\mu}{2} \\ 1 & 1 \end{bmatrix} \qquad Q^{-1} = \frac{1}{\mu} \begin{bmatrix} 1 & \frac{\gamma+\mu}{2} \\ -1 & \frac{-\gamma+\mu}{2} \end{bmatrix} \tag{140}$$

Performing the diagonalization routine on the matrix A yields the following eigendecomposition

$$A = Q \Lambda Q^{-1} = \frac{1}{\mu} \begin{bmatrix} \frac{-\gamma+\mu}{2} & \frac{-\gamma-\mu}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{-\gamma+\mu}{2} - 1 & 0 \\ 0 & \frac{-\gamma-\mu}{2} - 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{\gamma+\mu}{2} \\ -1 & \frac{-\gamma+\mu}{2} \end{bmatrix} \tag{141}$$

Note that the powers of A are now easily computable

$$A^n = Q \Lambda^n Q^{-1} = \frac{1}{\mu^n} \begin{bmatrix} \frac{-\gamma+\mu}{2} & \frac{-\gamma-\mu}{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(\frac{-\gamma+\mu}{2} - 1\right)^n & 0 \\ 0 & \left(\frac{-\gamma-\mu}{2} - 1\right)^n \end{bmatrix} \begin{bmatrix} 1 & \frac{\gamma+\mu}{2} \\ -1 & \frac{-\gamma+\mu}{2} \end{bmatrix} \tag{142}$$

Let us finally compute the inner product involving the exponential operator e^{tA},

$$\langle 1, e^{tA} 1 \rangle = (1 \;\; 1) \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (1 \;\; 1) A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \tag{143}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{1}{\mu^n} ((\beta + 1)(\alpha + 1)(\beta - 1)^n - (\alpha - 1)(\beta - 1)(-\alpha - 1)^n) = \tag{144}$$

$$= (\beta + 1)(\alpha + 1) e^{\frac{\beta - 1}{\mu} t} - (\alpha - 1)(\beta - 1) e^{\frac{-\alpha - 1}{\mu} t} \tag{145}$$

where $\alpha = \frac{\gamma+\mu}{2}$, $\beta = \frac{\mu-\gamma}{2}$ and $\mu = \sqrt{\gamma^2 + 4}$. One can notice, that the powers of both exponents are negative, thus taking the logarithm and the limit with $t \to \infty$ will leave the larger exponent power surviving which is negative. Note that for general case when the flipping rates $c \neq 1$, one still gets the same result (by using substitution $\gamma := \gamma / c$ and taking factor c out of matrices in the previously derived expressions). Thus, we have the following

$$\lambda_{\max} = \lim_{t \to \infty} \frac{1}{t} \ln \langle 1, e^{t(L+V)} 1 \rangle < 0 \tag{146}$$

This concludes the sharp exponential upper bound for the decay of survival probabilities. This additionally suggests that the collection $\{\frac{1}{t} \int_0^t -\gamma \eta(s, 0) ds\}$ might satisfy Large Deviation Principle with the rate function $I(x) = \sup_{\lambda} \{\lambda x + \lambda_{\max}\}$ if $\tilde{\lambda}_{\max}$ exists, which will be discussed in the next subsection.

□

In the next section regarding the Simple Symmetric Exclusion traps, we state the Theorem 6.1 from the [8] that proves the sharp exponential decay of survival probability, assuming the Large Deviation Principle on the occupational time measures. In the following subsection regarding Attractive Spin Flip Dynamics, we state the model that is a generalization of the model in the current section, and we provide a proof of such Large Deviation Principle that makes Theorem 6.1 applicable to the model in this section.

5.3 Existence of the Exponent

In the previous section, we proved that if the exponent $\tilde{\lambda}_{\max}$ exists, then it must be negative. In the current section, we prove the existence of such exponent, which in combination with the previous result yields the exponential decay of survival probability of random walk in the independent spin flip environment.

Let $\psi(t,\eta) = \tilde{\mathbb{E}}^{\eta}(\exp[\int_0^t \eta_s(0)ds])$, where $\tilde{\mathbb{E}}^{\eta}$ is taken with respect to the path space measure of trap process starting from $\eta \in E$.

Theorem 5.3 *For the Random Walk in Independent Spin Flip trap process, the exponent*

$$\tilde{\lambda}_{\max} = \lim_{t\to\infty} \frac{1}{t} \ln \mathbb{P}(T > t) \tag{147}$$

exists and is finite, if $\psi(t,\eta)$ and $\tilde{\mathbb{E}}^{\eta}\psi(t,\eta_t)$ are positively correlated.

Note that by the reasons of the previous subsection, if such $\tilde{\lambda}_{\max}$ exists and is finite, then it is negative.

Proof: The proof relies on the fact that if

$$\ln \mathbb{P}(T > t) = \ln \mathbb{E}^{\eta}\mathbb{E}_0^X \left(\exp\left\{-\gamma \int_0^t \eta(s, X_s)ds\right\}\right) \tag{148}$$

is sub-additive in t then by Fekete's Lemma the exponent λ exists [10].

Let us prove the super-additivity of the survival time functional directly

$$\mathbb{E}_0^{\eta,X}\left(\exp\left\{-\gamma \int_0^{t+s} \eta(s,X_s)ds\right\}\right) = \int d\nu_\rho(\eta)\tilde{\mathbb{E}}_0^{\eta,X}\left(\exp\left\{-\gamma\int_0^{t+s} \tau_{X_s}\eta(s,\cdot)ds\right\}\right) = \tag{149}$$

$$= \int d\nu_\rho(\eta)\tilde{\mathbb{E}}^{\eta}\left(\exp\left\{-\gamma\int_0^{t+s}\eta_s(0)ds\right\}\right) \tag{150}$$

Now let us prove that $\tilde{\mathbb{E}}_0^{\eta,X}\left(\exp\left\{-\gamma\int_0^t \eta_s(0)ds\right\}\right)$ is sub-additive in t,

$$\tilde{\mathbb{E}}^{\eta}\left(\exp\left\{-\gamma\int_0^{t+s}\eta_s(0)ds\right\}\right) = \tilde{\mathbb{E}}^{\eta}\left(\tilde{\mathbb{E}}_0^{\eta,X}\left[\exp\left\{-\gamma\int_0^{t+s}\eta_s(0)ds\right\}\,\middle|\,\mathcal{F}_t\right]\right) = \tag{151}$$

$$= \tilde{\mathbb{E}}^{\eta}\left(\exp\left\{-\gamma\int_0^{t}\eta_s(0)ds\right\}\tilde{\mathbb{E}}^{\eta}\left[\exp\left\{-\gamma\int_t^{t+s}\eta_s(0)ds\right\}\,\middle|\,\mathcal{F}_t\right]\right) \tag{152}$$

Now we can revert the process inside the inner expectation to start from the sigma-algebra $\mathcal{F}_t$, meaning the process will start from η_t

$$\tilde{\mathbb{E}}^{\eta}\left(\exp\left\{-\gamma\int_0^{t+s}\eta_s(0)ds\right\}\right) = \tilde{\mathbb{E}}^{\eta}\left(\exp\left\{-\gamma\int_0^{t}\eta_s(0)ds\right\}\tilde{\mathbb{E}}^{\eta_t}\left[\exp\left\{-\gamma\int_0^{s}\eta_s(0)ds\right\}\right]\right) \tag{153}$$

This will yield,

$$\tilde{\mathbb{E}}^{\eta}\left(\exp\left\{-\gamma\int_0^{t+s}\eta_s(0)ds\right\}\right) = \psi(t,\eta)\tilde{\mathbb{E}}^{\eta}\left(\psi(t,\eta_t)\right) + \mathrm{Cov}\left(\psi(t,\eta), \tilde{\mathbb{E}}^{\eta}\left(\psi(t,\eta_t)\right)\right) \tag{154}$$

where $\psi(t,\eta) = \tilde{\mathbb{E}}^{\eta}(\exp[\int_0^t \eta_s(0)ds])$

Now, if we assume the positive correlations of $\psi(t,\eta_t)$ and $\tilde{\mathbb{E}}^{\eta}\left(\psi(t,\eta_t)\right)$, which might be proved for certain monotone environments, we get the following bound

$$\tilde{\mathbb{E}}_0^{\eta,X}\left(\exp\left\{-\gamma\int_0^{t+s}\eta_s(0)ds\right\}\right) = \psi(t,\eta)\tilde{\mathbb{E}}^{\eta}\left(\psi(t,\eta_t)\right) + \mathrm{Cov}\left(\psi(t,\eta), \tilde{\mathbb{E}}^{\eta}\left(\psi(t,\eta_t)\right)\right) \geq \tag{155}$$

$$\geq \psi(t,\eta)\tilde{\mathbb{E}}^{\eta}\left(\psi(t,\eta_t)\right) = f(\eta,t)g(\eta,s) \tag{156}$$

where f and g stand for the annealed versions of their F and G counterparts, Thus the problem reduces to the following,

$$\int d\nu_\rho(\eta)\tilde{\mathbb{E}}_0^{\eta,X}\left(\exp\left\{-\gamma\int_0^{t+s}\eta_s(0)ds\right\}\right) \geq \int d\nu_\rho(\eta) f(\eta,t)g(\eta,s) \geq \tag{157}$$

$$\geq \int d\nu_\rho(\eta) f(\eta,t)\int d\nu_\rho(\eta)g(\eta,s) \tag{158}$$

where the inequality part is due to f and g being simultaneously monotonous (if f is $\uparrow$ then g is as well, if f is $\downarrow$ then g is as well, since we assumed the positive correlations of f and g) by the FKG inequality.

The last result means that

$$\mathbb{E}_0^{\eta,X}\left(\exp\left\{-\gamma\int_0^{t+s}\eta(s,X_s)ds\right\}\right) \geq \mathbb{E}_0^{\eta,X}\left(\exp\left\{-\gamma\int_0^{t}\eta(s,X_s)ds\right\}\right)\mathbb{E}_0^{\eta,X}\left(\exp\left\{-\gamma\int_0^{s}\eta(s,X_s)ds\right\}\right) \tag{159}$$

which means that the survival time functional is time sub-additive. This concludes the proof. □

Thus given the proof from the previous subsection, we get that this $\lambda < 0$ is negative, and thus we get the sharp exponential decay of survival probabilities, although indirectly.

5.4 Attractive Spin Flip Dynamics

The interacting particle system we study in this section generalizes many Spin Flip systems studied in the literature. The Independent Spin Flip system is, in fact, an example of an Attractive Spin Flip system. The main goal of this section is to boost the assumptions of the previous subsection and to prove the applicability of the Theorem 6.1 from [8] to the case of Independent Spin Flip.

Firstly, we state large deviation estimates for the occupation time measure from the [11], particularly, strong large deviation estimates for the occupation times of ergodic systems. Secondly, using obtained large deviation estimates we prove that Theorem 6.1 regarding exponential decay of survival time from the [8] applicable to the case of attractive spin flip systems, and thus to the independent spin flip system.

Let us work in the same state space as usual $E = \{0,1\}^{\mathbb{Z}^d}$. As a spin-flip process, we again define a Markov process on a said state space E. We describe the evolution of the system through a family of the flip rates $\{c(x,\eta) : \eta \in E, x \in \mathbb{Z}^d\}$. Thus, let us define the generator of the trapping process as always

Definition 5.4 (Attractive Spin Flip Process Generator) *Let trap process be independent-flips process $\eta(x,t) \in \{0,1\}$ with $x \in \mathbb{Z}^d$ defined on probability space $(E,\mathcal{F},\mathbb{P}^\eta)$ with following generator L_0*

$$(L_c f)(\eta) = \sum_{i\in\mathbb{Z}^d} c(i,\eta)(f(\eta^i) - f(\eta)) \tag{160}$$

where η^x is a configuration of traps with point $x \in \mathbb{Z}^d$ flipped, $f(\eta)$ are local functions of η, and $c(i,\eta)$ form an attractive system

The system is said to be attractive if the rates $c(i,\eta)$ are obeying the following rule. For all η and ξ in E with $\eta \leqslant \xi$ (i.e., $\eta(i) \leqslant \xi(i), \forall i \in \mathbb{Z}^d$) the rates are as follows

$$\begin{cases} c(i,\eta) \leqslant c(i,\xi), & \text{if} \quad \eta(i) = \xi(i) = -1 \\ c(i,\eta) \geqslant c(i,\xi), & \text{if} \quad \eta(i) = \xi(i) = 1 \end{cases}$$

For this section we assume the translational invariance and finiteness of ranges of interaction of the flip-rates. To formalize this,

Assumption 5.4 *Flip rates $c(i,\eta)$ are translation invariant with finite range interactions. There exist a finite subset U_0 of $\mathbb{Z}^d$and a nonnegative function c_0 on E which is not identically zero, such that $c_0(\eta)$ depends on η only through the coordinates of η in U_0 and*

$$c(i,\eta) = c_0(\tau_i\eta) \quad \forall x \in \mathbb{Z}^d, \eta \in E \tag{161}$$

where τ_x is a shift operator on E defined as $(\tau_x\eta)(y) = \eta(x+y)$

The literature suggests that for the attractive spin systems the monotonicity assumptions of the Theorem 5.3 are satisfied [12], [13]. For measures μ_1, μ_2 on E if system is attractive then

$$\mu_1 \leq \mu_2 \implies \mu_1 S_t \leq \mu_2 S_t$$

where $\leq$ was in the weak sense, i.e. $\mu_1 \leq \mu_2$ whenever $\int f d\mu_1 \leq \int f d\mu_2$. Furthermore, by [12] we have that for the attractive system $\mu S_t \leq \mu$. Thus the assumptions made in the "Existence of Exponent" section regarding the independent spin flip problem can be proved for the attractive spin flip systems using above properties.

Now, as we have defined a setup for the model, we will proceed by defining the object of interest - occupation time measures, for which we will state large deviation estimate consequently.

Definition 5.5 *Define the occupation time measures at a time $t > 0$*

$$T_t = \frac{1}{t} \int_0^t \eta_s(0) ds \tag{162}$$

Given the previous setup we get the following quite useful result.

Theorem 5.5 *For $\forall \delta > 0$ there exists $\lambda_\delta > 0$:*

$$\sup_{\eta \in E} \mathbb{P}^0_\eta(T_t > \rho_+ + \delta \ or \ T_t < \rho_- - \delta) \leq e^{-\lambda_\delta t} \tag{163}$$

If the system is ergodic, then:

$$\sup_{\eta \in E} \mathbb{P}^0_\eta(|T_t - \rho| > \delta) \leq e^{-\lambda_\delta t} \tag{164}$$

Thus for the ergodic system, the ρ is a stationary measure for the T_t.

$$\lim_{t \to \infty} \mathbb{E}^\eta(T_t) = \rho \tag{165}$$

The previous Large Deviation estimate, as well as, the Large Deviations Principle from the [14] for the system with strictly positive rates gives a possibility of using the Theorem 6.1 from the [8] as its assumptions are satisfied by these results. Note that in the setup of the Independent Spin Flip, it is quite reasonable to assume space-time invariance as the flip rates do not interact with themselves, making the system obey all the previously stated assumptions. Furthermore, note that if the spin-flip rates are strictly positive the Large Deviation Estimates turn into the Large Deviation Principles. Furthermore, one can derive that the resulting rate function obeys the non-degeneracy and single zero conditions of the Theorem 6.1, which we proceed to prove.

Lemma 5.6 *The rate function $I(x)$ of LDP of the occupation time measures $\{T_t\}$ is non-degenerate with the zero at the only point $x^* = \mathbb{E}^\eta(\eta_s(0))$*

Proof: Given that the underlying independent spin flip system is defined to be ergodic, we would have the following:

$$\mathbb{P}^0_\eta(|T_t - \rho| > \delta) \leq \sup_{\eta \in E} \mathbb{P}^0_\eta(|T_t - \rho| > \delta) \leq e^{-\lambda_\delta t} \tag{166}$$

This means that T_t satisfies some LDP with some rate function $I(x)$. Which implies that $T_t \to \rho$ as $t \to \infty$ $\mathbb{P}^0_\eta - a.s.$. This also implies that for any closed set F we have the following,

$$\frac{1}{t} \ln \mathbb{P}^0_\eta(T_t \in F) \leq \inf_{x \in F} I(x) \tag{167}$$

Given a positive $\varepsilon > 0$, the infimum of the rate function away from its stationary value ρ obeys the following inequality

$$\inf_{x \in [x^* + \varepsilon, x^* + 2\varepsilon]} I(x) > 0 \tag{168}$$

Thus we have that the rate function is positive for all inputs greater than ρ,

$$I(y) > 0 \quad \forall y > x^* \tag{169}$$

By the reverse logic

$$\inf_{x \in [x^* - \varepsilon, x^* - 2\varepsilon]} I(x) > 0 \tag{170}$$

Thus we have the reverse statement

$$I(y) > 0 \quad \forall y < x^* \tag{171}$$

Combining both results regarding the rate function we get

$$I(x) = 0 \quad \textit{iff} \quad x = x^* \tag{172}$$

□

This yields that in the case when the environment is independent spin flip traps the collection $\{T_t : t > 0\}$ of occupation measures satisfies the Large Deviation Principle with the non-degenerate function $I(x)$ which has the only zero at x^*, meaning that the conditions for the Theorem [8] are satisfied. Thus the theorem can be applied in the section regarding the Independent Spin Flip model, to derive the sharp exponential decay of survival probabilities.

6 Random Walk in Simple Symmetric Exclusion Trap Dynamics

In this section, we study the survival time of the Random Walk in a Simple Symmetric Exclusion environment of traps. We provide the statement and the proof of a quite useful theorem from [8], which allows one to derive exponential decay for survival time given certain requirements for the rate function for the occupational time measures of the process. At last, we apply this theorem for the case $d \geq 3$ by supporting it with occupational time measure LDP results from the literature.

The environment is defined to be $E = \{0,1\}^{\mathbb{Z}^d}$. The process can be thought as a collection of the particles situated on a lattice (1's being a particle, and 0's being free lattice points), each particle independently waiting exponential time before jumping to the closest neighboring lattice point (equiprobably in all directions since it is symmetric), if it is empty, otherwise waiting another exponentially distributed time. We define all traps be hard.

Let us remind the reader of the definition of the generator for the trap process.

Definition 6.1 (Simple Symmetric Exclusion Process Generator) *The generator of the exclusion process is given by:*

$$L_0 f(\eta) = \sum_{x\in\mathbb{Z}^d} \sum_{e\in\mathbb{Z}^d:|e|=1} \frac{1}{2d}\left(f(\eta^{x,x+e}) - f(\eta)\right), \tag{173}$$

where $\eta^{x,x+e}$ stands for exchanging occupations of points x and $x+e$

$$\eta^{x,x+e}(y) = \eta(y)(1-\delta_{y,x}-\delta_{y,x+e}) + \eta(x)\delta_{y,x+e} + \eta(x+e)\delta_{y,x} \tag{174}$$

For the proof of the existence of a reversible ergodic and translation invariant measure for the Simple Symmetric Exclusion trap process (and its form), we refer the reader to Chapter VIII, Section I of the book [15].

Definition 6.2 (Reversible and ergodic measure) *The Bernoulli product measure is reversible ergodic and translation invariant measure for the Simple Symmetric Exclusion dynamics*

$$\nu_\rho = \bigotimes_{\mathbb{Z}^d} \text{Bernoulli}(\rho) \qquad \rho \in [0,1]$$

Since we have the restriction of one trap particle per site, we can define the environment process in the same fashion as it was defined in the Independent Spin-Flip section.

Definition 6.3 (Environment process) *Environment process is defined as the configuration of traps $\eta(t, X_t)$ at a time t as seen from the perspective of a random walker $(X_t)_{t\geq 0}$ $x \in \mathbb{Z}^d$ defined on probability space $(\mathbb{Z}^d \times E, \mathcal{F}^X \otimes \mathcal{F}^\eta, \mathbb{P}^X \otimes \mathbb{P}^\eta)$*

$$\eta(t, X_t) = \begin{cases} 1, & \text{if a point } x \in \mathbb{Z}^d\text{, at which random walker } X_t = x \text{ is situated at a time } t\text{, is a trap} \\ 0, & \text{otherwise} \end{cases} \tag{175}$$

The generator of the Environment Process, as always, is a sum of both generators of Random Walk (the L_1 generator) and SSE Trapping process (the L_0 generator) since both are independent Markov processes.

$$Lf(\eta) = L_0 f(\eta) + L_1 f(\eta)$$

Now let us focus on the results from the literature. The large deviation principle for the random walk occupation time measure was proved by C.Landim [16] for $d \neq 2$, with the subsequent paper in 2004 by Chih-Chung Chang, C. Landim and Tzong-Yow Lee [17] where the corresponding large deviation principle for the case $d = 2$ is proved. Furthermore, the rate function of these LDP's for the case $d \geq 3$ satisfies the non-degeneracy conditions assumed by the next theorem from [8], which makes it instantly applicable.

Theorem 6.1 ([8]) *If $\{\eta_t : t \geq 0\}$ is such that the large deviation principle for the occupation time measure $\left\{\frac{1}{t}\int_0^t V(\eta_s)ds\right\}$ is satisfied with a non-degenerate rate function I with the only zero at the point $\rho = \mathbb{E}^\eta(V(\eta))$ then the following holds*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \mathbb{P}(T \geq t) < 0 \tag{176}$$

where $\mathbb{P}$ is taken in a sense of reversible and ergodic translation-invariant measure.

Proof:

$$\mathbb{P}(T \geq t) = \mathbb{P}\left(\int_0^t \eta(s, X_s)ds = 0\right) = \mathbb{P}\left(\exp\left\{-\int_0^t \eta(s, X_s)ds\right\} \geq 1\right) = \tag{177}$$

$$= \mathbb{P}\left(\exp\left\{-\int_0^t \tau_{X_s}\eta(s, \cdot)ds\right\} \geq 1\right) = \mathbb{P}\left(\exp\left\{-\int_0^t V(\eta)ds\right\} \geq 1\right) \leq \tag{178}$$

$$\leq \mathbb{E}\left(\exp\left\{-\int_0^t V(\eta)ds\right\}\right) \tag{179}$$

where $V(\eta) = \eta(0)$.

Now by the Feynman-Kac routine we have the following

$$\mathbb{E}\left(\exp\left\{-\int_0^t V(\eta)ds\right\}\right) \leq e^{-\lambda_{\max} t} \tag{180}$$

From the previous section, we know that $\lambda_{\max}$ can be found using Varadhan variational formula

$$\lambda_{\max} = \sup_{g: \int g^2 d\nu_\rho = 1} \left(-\int V(\eta)g^2(\eta)\nu_\rho(d\eta) + (g, L_0 g) + (g, L_1 g)\right) \leq \tag{181}$$

$$\leq \sup_{g: \int g^2 d\nu_\rho = 1} \left(-\int V(\eta)g^2(\eta)\nu_\rho(d\eta) + (g, L_0 g)\right) \leq \tag{182}$$

$$\leq \sup_{\mu \in \mathcal{P}(E), \mu \ll \nu_\rho} \left[-\int V(\eta)\mu(d\eta) + \left(\left(\frac{d\mu}{d\nu_\rho}\right)^{1/2}, L_0 \left(\frac{d\mu}{d\nu_\rho}\right)^{1/2}\right)\right] \textcircled{\leq} \tag{183}$$

$$\tag{184}$$

where in the last two steps we use the fact that the large deviation principle for the L_0 was proved in [16] and [17] with the appropriate non-degenerate rate function I_2 with the only zero at its stationary measure. Note that $\mathcal{P}(E)$ denotes the set of all probabilistic measures on E. Thus we have:

$$\textcircled{\leq} \sup_{\mu \in \mathcal{P}(E), \mu \ll \nu_\rho} \left[-\int V(\eta)\mu(d\eta) - I_2(\mu)\right] \tag{185}$$

By Varadhan's formula:

$$\lim_{t \to \infty} \frac{1}{t} \ln \mathbb{E}\left(\exp\left\{-\int_0^t V(\eta_s)ds\right\}\right) = \sup_{\mu \in \mathcal{P}(E), \mu \ll \nu_\rho} \left[-\int V(\eta)\mu(d\eta) - I_2(\mu)\right] \tag{186}$$

One of the last things to note is that by the contraction principle, the collection $\left\{\frac{1}{t}\int_0^t V(\eta_s)ds\right\}$ satisfies LDP with the following rate function:

$$I_1(x) = \inf_{\mu \in \mathcal{P}(E)} \left\{I_2(\mu) \,\middle|\, \int V(\eta)\mu(d\eta) = x\right\} \tag{187}$$

Now we have:

$$\lim_{t \to \infty} \frac{1}{t} \ln \mathbb{E}\left(\exp\left\{-\int_0^t V(\eta_s)ds\right\}\right) = \sup_{x \in \mathcal{R}^+} \left[-x - I_1(x)\right] \tag{188}$$

□

Thus we can conclude that for $d \geq 3$ there exists $\lambda > 0$, such that the following exponential upper bound of the survival probability is obtainable:

$$\mathbb{P}(T \geq t) \leq e^{-\lambda t} \tag{189}$$

The result in [17] proves the occupation time LDP with the rate $\sqrt{t}$ for the case $d = 1$ and with the rate $t/\log t$ for the case $d = 2$, so the corresponding survival probability, in principle, should also have the subexponential decay.

Conjecture 6.2 *The survival probability of the Random Walk among the Simple Symmetric Exclusion traps may have the following asymptotics*

$$\mathbb{P}(T \geq t) \simeq \begin{cases} \exp\left\{-c_1\sqrt{t} + o(t)\right\} & d = 1 \\ \exp\left\{-c_2\dfrac{t}{\log t} + o(t)\right\} & d = 2 \\ \exp\left\{-c_3 t + o(t)\right\} & d \geq 3 \end{cases} \tag{190}$$

for some positive constants c_1, c_2 *and* c_3

7 Random Walk among Independent Random Walking Traps

In this section, we study the survival time of the Random Walk among Independent Random Walking traps. Firstly, we prove the lower bounds on survival time in a setting with hard traps. Then we provide proof for the exact asymptotics in a setting with soft traps, which is a celebrated result by A. Drewitz et al.[10].

In this model, we define the random walk generator as previously and the trap process is defined to be a collection of independent random walkers. The state space is defined to be $E = \mathbb{N}^{\mathbb{Z}^d}$, where we count the number of traps at each site. We do not impose any restrictions on the number of traps per site, although we make the sites with a high number of traps update faster.

Definition 7.1 (Independent Random walking traps) *For $\eta \in E$ the trap process $\{\eta_t : t \geq 0\}$ starting from η is a collection of independent random walks,*

$$\left((\{X_i^x(t) : t \geq 0\})_{i=1}^{\eta(x)} \right)_{x \in \mathbb{Z}^d} \tag{191}$$

where $\eta(x)$ is a number of traps at site x and X_i^x means random walking trap starting from x. The number $\eta(x)$ of the random walks at site x is is defined to be Poisson distributed with mean ρ and random walking traps are defined to have jumping rate ν. This also forms the ergodic and the reversible translation-invariant measure of the process ν_ρ.

For the existence of reversible ergodic and translation invariant measure for the Independent Random Walking Traps Process we refer a reader to the [18].

Definition 7.2 (Reversible and ergodic measure) *The Poisson product measure is a reversible ergodic and translation invariant measure for the Independent Random Walks Dynamics*

$$\nu_\rho = \bigotimes_{\mathbb{Z}^d} Poisson(\rho) \qquad \rho \in (0, \infty)$$

As always we recall the definition of the generator of trapping potential, in this case the generator of the Random walking traps Process,

Definition 7.3 (Independent Random walking traps generator) *The generator of trapping potential is given by:*

$$Lf(\eta) = \sum_{x \in \mathbb{Z}^d} \sum_{e \in \mathbb{Z}^d : |e|=1} \frac{\eta(x)}{2d} \left(f(\eta^{x,x+e}) - f(\eta) \right), \tag{192}$$

where $\eta^{x,x+e}$ stands for removing trap at x and putting it at $x + e$

$$\eta^{x,x+e}(y) = \eta(y) - \delta_{y,x} + \delta_{y,x+e} \tag{193}$$

Definition 7.4 (Environment process) *The environment process $\eta(t, X(t))$ is defined to be the number of the walking traps at a site x where the random walker $X(t)$ is situated at a time t, with $\eta(t, x)$ to be of following form:*

$$\eta(t, x) = \sum_{y \in \mathbb{Z}^d, 1 \leq j \leq \eta(y)} I(X_j^y(t) = x) \tag{194}$$

Note that for each $t \geq 0$ the collection of $\eta(t, x)$ is a collection of i.i.d Poisson distributed random variables with mean ρ, so Markov process η_t is a stationary and reversible process [10]

For the purposes of this section, we need to define two survival functionals for the case of soft taps. Let us define the quenched survival probability (which depends on the random environment η):

$$Z_{t,\gamma}^\eta = \mathbb{E}_0^X \left(\exp \left\{ -\gamma \int_0^t \eta(s, X_s) ds \right\} \right) \tag{195}$$

The annealed survival probability (constantly referred to as just survival probability) is the same as in previous sections:

$$Z_{t,\gamma} = \mathbb{P}(T > t) = \mathbb{E}^{\eta} Z_{t,\gamma}^{\eta} = \mathbb{E}^{\eta} \mathbb{E}_0^X \left(\exp\left\{ -\gamma \int_0^t \eta(s, X_s) ds \right\} \right) \tag{196}$$

Note that $Z_{t,\gamma}$ due to Feynman-Kac formula also solves the following PDEs

$$\begin{cases} \frac{d}{dt} u(t,x) = \kappa L u(t,x) - \gamma \eta(t,x) u(t,x), \\ u(0,x) = 1 \end{cases}$$

where L is a shifted generator of a simple random walk on $\mathbb{Z}^d$.

In the next subsection, we will prove the lower bounds on the survival probability in a setting with hard traps. After that, we focus on studying the proof of exact asymptotics in the soft traps setting.

7.1 Exponential Lower Bound in a Hard Traps Setting

In the setting of the "hard" traps, instead of using Theorem 6.1 from the [8], as in the section regarding the simple symmetric exclusion process, it is possible to calculate things more explicitly using ranges of the random walks.

To have a reversible ergodic and translation invariant measure for the process we let the environment be Poisson distributed. Then such measure would be a product measure

$$\nu_\rho = \bigotimes_{\mathbb{Z}^d} Poisson(\rho) \qquad \rho \in (0, \infty)$$

Now it is possible to get a lower bound on survival time using the range between random walk and random walking traps and annealing it to the range of the difference between two simple random walks, which gives us the approximation via the range of one random walk with double speed.

Theorem 7.1 ([8]) *Let the random walker X_t travel in the random walking traps environment as defined above with the "hard traps". Then the survival probability is as follows*

$$\mathbb{P}(T \geq t) = \mathbb{E}^X \left(\exp[-\rho \mathbb{E}^Y (R(X - Y, t))] \right) \tag{197}$$

Proof:

$$\begin{aligned}
\mathbb{P}(T \geq t) &= \int \mathbb{P}(X_i^x(s) - X_s \neq 0 \forall i = 1, .., \eta(x), \forall x, \forall s \in [0, t)) d\nu_\rho = \\
&= \int d\mathbb{P}^X \int d\nu_\rho \prod_x \prod_{i=1}^{\eta(x)} \int d\mathbb{P}^{X_i} \mathbb{1}_{\{X_i(s) \neq X_s + x, \forall s \in [0,t)\}} \stackrel{ind}{=} \\
&\stackrel{ind}{=} \int d\mathbb{P}^X \int d\nu_\rho \prod_x \left(\int d\mathbb{P}^{X_i} \mathbb{1}_{\{X_i(s) \neq X_s + x, \forall s \in [0,t)\}} \right)^{\eta(x)} = \\
&= \int d\mathbb{P}^X \int d\nu_\rho \prod_x \left(\mathbb{P}^{X_i}(X_i(s) \neq X_s + x, \forall s \in [0, t)) \right)^{\eta(x)} = \\
&= \int d\mathbb{P}^X \prod_x \sum_{j=0}^{\infty} e^{-\rho} \frac{\rho^j}{j!} \mathbb{P}^{X_i}(X_i(s) \neq X_s + x, \forall s \in [0, t))^{\eta(x)} = \\
&= \int d\mathbb{P}^X \prod_x e^{-\rho} \sum_{j=0}^{\infty} \frac{\rho^j}{j!} \mathbb{P}^{X_i}(X_i(s) \neq X_s + x, \forall s \in [0, t))^j = \\
&= \int d\mathbb{P}^X \prod_x \exp(-\rho) \exp(\rho \mathbb{P}^{X_i}(X_i(s) \neq X_s + x, \forall s \in [0, t)) = \\
&= \int d\mathbb{P}^X \prod_x \exp\left[-\rho(1 - \mathbb{P}^{X_i}(X_i(s) \neq X_s + x, \forall s \in [0, t)) \right]
\end{aligned}$$

$$
\begin{aligned}
&= \int d\mathbb{P}^X \prod_x \exp\left[-\rho(\mathbb{P}^{X_i}(X_i(s) = X_s + x, \forall s \in [0,t))\right] \\
&= \int d\mathbb{P}^X \prod_x \exp\left[-\rho \int \sum_x \mathbb{1}(X_i(s) = X_s + x, \forall s \in [0,t)) d\mathbb{P}^{X_i}\right] \\
&= \int d\mathbb{P}^X \exp\left[-\rho \int R(X - X_i, s) d\mathbb{P}^{X_i}\right] \\
&= \mathbb{E}^X \exp\left[-\rho \mathbb{E}^Y R(X - Y, s)\right]
\end{aligned}
$$

□

Now that we have the above theorem, the following is the consequent

Theorem 7.2 ([8]) *The survival probability of the random walk among random walking "hard" traps has the following lower bound:*

$$\mathbb{P}(T \geq t) \geq \exp[-\rho \mathbb{E}^Z R(Z, 2t)] \tag{198}$$

Proof:

$$\mathbb{P}(T \geq t) = \mathbb{E}^X \exp\left[-\rho \mathbb{E}^Y R(X - Y, t)\right] \geq \exp\left[-\rho \mathbb{E}^X \mathbb{E}^Y R(X - Y, t)\right] = \exp\left[-\rho \mathbb{E}^Z R(Z, 2t)\right] \tag{199}$$

where we used Jensen's Inequality and the fact that $X - Y$ behaves like a random walk Z with doubled speed we have the last equality □

Now due to known expression for Expectation of a Range of the Random Walk in 2.5, we can conclude following subexponential lower bound for $d = 1, 2$ and exponential lower bound for $d \geq 3$:

$$\mathbb{P}(T \geq t) \geq \begin{cases} \sqrt{2t}, & \text{if } d = 1 \\ \frac{2\pi t}{\log 2t} + \mathcal{O}\left(t \frac{\log \log 2t}{\log^2 2t}\right), & \text{if } d = 2 \\ 2t\upsilon_d + \mathcal{O}((t^{2-d/2}), & \text{if } d \geq 3 \end{cases} \tag{200}$$

where υ_d is the probability of a random walk never coming back to the origin.

7.2 The exact asymptotics on survival time in soft traps setting

In this subsection, we state asymptotic results for annealed and quenched survival probabilities and the existence of the Lyapunov exponent of the model. Then we prove the existence of the Lyapunov exponent and derive the exact asymptotics for annealed survival probability.

The exact asymptotics of survival probability for the case of soft traps were obtained by A. Drewitz et al. [10], and this subsection mainly follows this work. In the cases $d = 1, 2$ the survival probability decays subexponentially and the constants in the exponents are independent of κ and γ, and for $d \geq 3$ it decays exponentially with the Lyapunov exponent as a constant.

Theorem 7.3 ([10]) *Assume that $\gamma \in (0, \infty]$, $\kappa \geq 0$, $\rho > 0$, $\nu > 0$, then in a simple symmetric exclusion process the survival probability of the random walk has the following asymptotics*

$$\mathbb{P}(T \geq t) = \begin{cases} \exp\left\{-\nu\sqrt{\frac{8\rho t}{\pi}}(1 + \mathcal{O}(1))\right\} & d = 1 \\ \exp\left\{-\nu\pi\rho\frac{t}{\log t}(1 + \mathcal{O}(1))\right\} & d = 2 \\ \exp\left\{-\lambda_{d,\gamma,\kappa,\rho,\nu} t(1 + \mathcal{O}(1))\right\} & d \geq 3 \end{cases} \tag{201}$$

where $\lambda_{d,\gamma,\kappa,\rho,\nu}$ is a Lyapunov exponent

Surprisingly, the quenched survival probability always decays exponentially

Theorem 7.4 ([10]) *Assume that $\gamma \in (0, \infty]$, $\kappa \geq 0$, $\rho > 0$, $\nu > 0$. Then there exists deterministic quenched Lyapunov exponent $\hat{\lambda}_{d,\gamma,\kappa,\rho,\nu}$ such that:*

$$Z^{\eta}_{t,\gamma} \overset{\mathbb{P}-a.s.}{=} \exp\left\{-\hat{\lambda}_{d,\gamma,\kappa,\rho,\nu} t(1 + o(1))\right\} \qquad t \to \infty \tag{202}$$

The Annealed Lyapunov Exponent (can be called just Lyapunov exponent) is defined as follows:

Definition 7.5 (Lyapunov Exponent) *Assume that $\gamma \in (0, \infty]$, $\kappa \geq 0$, $\rho > 0$, $\nu > 0$. The Lyapunov exponent corresponding to the annealed survival probability $Z_{\gamma,t}$, is defined as:*

$$\lambda_{d,\gamma,\kappa,\rho,\nu} = -\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\eta} Z^{\eta}_{\gamma,t} \tag{203}$$

One might define a different rate from rate t (corresponding to $1/t$ term in the limit) in the Lyapunov exponent, for instance, the existence of the Lyapunov exponent with decay rate $t/\log t$ for exclusion process was proved in [17]. Essentially, the rate for the annealed Lyapunov exponent, tells how the tail of the annealed survival probability decays, if it was constructed appropriately

Proof: Here we prove the Theorem 7.3 (as in [10])

Let us prove that Lyapunov exponent $\lambda_{d,\gamma,\kappa,\rho,\nu}$ exists for SZRP in $d \geq 3$ and is finite

Define $\nu(t, y)$ for $y \in \mathbb{Z}^d$

$$\nu(t, y) = \mathbb{E}^{Y}_{y}\left(\exp\left\{-\gamma \int_0^t \delta_0(Y(s) - X(t-s))ds\right\}\right) \tag{204}$$

Now, annealed survival probability can be rewritten as follows:

$$Z_{\gamma,t} = \mathbb{E}^{\eta} u(0, t) = \mathbb{E}^{\eta}\mathbb{E}^{X}_{0}\left(\exp\left[-\gamma \int_0^t \eta(t-s, X(s))\right]\right) = \tag{205}$$

$$= \mathbb{E}^{X}_{0}\left(\exp\left[\nu \sum_{y\in\mathbb{Z}} (\nu_x(t, y) - 1)\right]\right) \tag{206}$$

Now, by the Feynman-Kac, this $\nu(t, y)$ exponential functional solves a certain system of PDE's (generator L in PAM is set to be the one of the random walk).

$$\begin{cases} \dfrac{\partial}{\partial t}\nu_X(t, y) = \rho L \nu_X(t, y) - \gamma \delta_{X(t)}(y)\nu_X(t, y) \\ \nu_X(0, y) = 1 \end{cases} \tag{207}$$

Which can be summed across $\mathbb{Z}^d$, by denoting $\Sigma_X(t) = \sum_{y\in\mathbb{Z}}(\nu_x(t, y) - 1)$, to form the following system of PDE's:

$$\begin{cases} \dfrac{\partial}{\partial t}\Sigma_X(t) = -\gamma \nu_X(t, X(t)) \\ \Sigma_X(0) = 0 \end{cases} \tag{208}$$

Integrating (208) from 0 to t gives us the following solution $\Sigma_X(t) = -\gamma \int_0^t \nu_X(s, X(s))ds$. Now the annealed survival probability can be rewritten as follows:

$$Z_{\gamma,t} = \mathbb{E}^{\eta} Z^{\eta}_{\gamma,t} = \mathbb{E}^{X}_{0}\left(\exp\left[-\nu\gamma\int_0^t \nu_X(s,X(s))ds\right]\right) \tag{209}$$

Now, let us assume the time-translational invariance under the following shift operator $\theta_{t_1}X(t) = (X(t_1+s) - X(t_1))_{s\geq 0}$. The following holds for $\nu_X(s,X(s))$ given $s > t_1$:

$$\nu_X(s,X(s)) = \mathbb{E}^{Y}_{X(s)}\left(\exp\left\{-\gamma\int_0^s \delta_0(Y(r) - X(s-r))dr\right\}\right) \geq \tag{210}$$

$$\geq \mathbb{E}^{Y}_{X(s)}\left(\exp\left\{-\gamma\int_0^{s-t_1} \delta_0(Y(r) - X(s-r))dr\right\}\right) = \tag{211}$$

$$= \nu_{\theta_{t_1}X}(s-t_1, \theta_{t_1}X(s)) \tag{212}$$

Let us note that for the annealed survival probability, given the $t_1, t_2 > 0$, the following holds:

$$\mathbb{E}^{\eta}Z^{\eta}_{\gamma,t_1+t_2} = \mathbb{E}^{X}_{0}\left(\exp\left[-\nu\gamma\int_0^{t_1}\nu_X(s,X(s))ds\right]\exp\left[-\nu\gamma\int_{t_1}^{t_1+t_2}\nu_X(s,X(s))ds\right]\right) \geq \tag{213}$$

$$\geq \mathbb{E}^{X}_{0}\left(\exp\left[-\nu\gamma\int_0^{t_1}\nu_X(s,X(s))ds\right]\exp\left[-\nu\gamma\int_0^{t_2}\nu_{\theta_{t_1}X}(s,\theta_{t_1}X(s))ds\right]\right) = \tag{214}$$

$$= \mathbb{E}^{\eta}(Z^{\eta}_{\gamma,t_1})\mathbb{E}^{\eta}(Z^{\eta}_{\gamma,t_2}) \tag{215}$$

This proves that $\log \mathbb{E}^{\eta} Z^{\eta}_{\gamma,t}$ is subadditive in t, and thus by the Fekete's Lemma the limit for $\lambda_{d,\gamma,\kappa,\rho,\nu}$ exists

Let us now prove the special case $\kappa = 0$, which would become useful for lower and upper bounds of asymptotics in $d = 1, 2$ and for general $\kappa > 0$.

When $\kappa = 0$ and $\gamma \in (0,\infty)$, the (209) becomes:

$$\mathbb{E}^{\eta}\left[Z^{\gamma}_{t,\eta}\right] = \exp\left\{-\nu\gamma\int_0^t v_0(s,0)ds\right\} \tag{216}$$

where $v_0(t,0)$ is the solution of the original system (207), with $X \equiv 0$. Given all of this, it is now sufficient to analyze the asymptotics of the $v_0(t,0)$. From the (204):

$$v_0(t,0) = \mathbb{E}^{Y}_{0}\left[\mathrm{e}^{-\gamma\int_0^t \delta_0(Y(s))\mathrm{d}s}\right] \tag{217}$$

Note that due to recurrence in $d = 1, 2$ the $v_0(t,0) \to 0^+$ and due to transience in $d \geq 3$ $v_0(t,0) \to C(d)$ for some constant dependent on dimension.
The Duhamel's principle dictates the following form for the $v_0(t,0)$:

$$v_0(t,0) = 1 - \gamma\int_0^t p_{\rho s}(0)v_0(t-s,0)ds \tag{218}$$

where $p_s(\cdot)$ is the transition probability kernel of a rate 1 simple symmetric random walk on $\mathbb{Z}^d$

The Laplace transforms of the $v_0(t,0)$ and the $p_t(0)$ are

$$\hat{v}_0(\lambda) = \int_0^{\infty} e^{-\lambda t}v_0(t,0)dt, \quad \hat{p}(\lambda) = \int_0^{\infty} e^{-\lambda t}p_t(0)dt \tag{219}$$

Then if we take Laplace transform of the (218) and solve for $\hat{v}_0(\lambda)$ we get the following:

$$\hat{v}_0(\lambda) = \frac{1}{\lambda}\cdot\frac{\rho}{\rho + \gamma\hat{p}(\lambda/\rho)} \tag{220}$$

The local central limit theorem for continuous time simple random walks (for $d = 1, 2$) tells us that $p_t(0) = \left(\frac{d}{2\pi t}\right)^{d/2}(1 + o(1))$ with $t \to \infty$, which gives us the following asymptotics as $\lambda \to 0^+$

$$\hat{p}(\lambda) = \begin{cases} \frac{1}{\sqrt{2\lambda}}(1+o(1)), & d=1, \\ \\ \frac{\ln\left(\frac{1}{\lambda}\right)}{\pi}(1+o(1)), & d=2, \\ \\ G_d(0)(1+o(1)), & d\geq 3, \end{cases} \tag{221}$$

where G_d is a green function associated with the random walk.

The previous asymptotics result for $\hat{p}(\lambda)$ is converted to the following asymptotics for $\hat{v}_0(\lambda)$ by (219) and (221):

$$\hat{v}_0(\lambda) = \begin{cases} \frac{\sqrt{2}\rho}{\gamma} \cdot \frac{1}{\sqrt{\lambda}}(1+o(1)), & d=1, \\ \\ \frac{\pi\rho}{\gamma} \cdot \frac{1}{\lambda \ln\left(\frac{1}{\lambda}\right)}(1+o(1)), & d=2, \\ \\ \frac{\rho}{\rho+\gamma G_d(0)} \cdot \frac{1}{\lambda}(1+o(1)), & d\geq 3. \end{cases} \tag{222}$$

Thus by Karamata's Tauberian theorem (given that $\hat{v}_0(\lambda)$ is monotonically decreasing) we get the following asymptotics for the original:

$$v_0(t,0) = \begin{cases} \frac{1}{\gamma}\sqrt{\frac{2\rho}{\pi}} \cdot \frac{1}{\sqrt{t}}(1+o(1)), & d=1, \\ \\ \frac{\pi\rho}{\gamma} \cdot \frac{1}{\ln t}(1+o(1)), & d=2, \\ \\ \frac{\rho}{\rho+\gamma G_d(0)}(1+o(1)), & d\geq 3, \end{cases} \tag{223}$$

which if integrated out in (216) gives the Theorem 7.3 for the case $\kappa = 0$ and $\gamma \in (0,\infty)$.

For the $\kappa = 0$ and $\gamma = \infty$, we have the following:

$$\mathbb{E}^\eta\left[Z^\gamma_{t,\eta}\right] = \mathbb{P}(\eta(s,0) = 0 \forall s \in [0,t]) = \exp\left\{-\nu \sum_{y\in\mathbb{Z}^d} \psi(t,y)\right\} \tag{224}$$

with $\psi(t,y) = \mathbb{P}^Y_y(\exists s \in [0,t] : Y(s) = 0)$.

Note that $\psi(t,y)$ solves the following system

$$\begin{cases} \frac{\partial}{\partial t}\psi(t,y) = \rho\Delta\psi(t,y), \quad y \neq 0, t \geq 0 \\ \psi(\cdot,0) \equiv 0 \\ \psi(0,\cdot) \equiv 1 \end{cases} \tag{225}$$

Thus if we sum it up, given the fact that $\sum_{x\in\mathbb{Z}^d} \Delta\psi(t,x) = 0$,the $\sum_{y\in\mathbb{Z}^d} \psi(t,y)$ solves the system:

$$\frac{d}{dt}\sum_{y\in\mathbb{Z}^d} \psi(t,y) = -\rho\Delta\psi(t,0) = \rho\left(1-\psi\left(t,e_1\right)\right) = \rho\phi\left(t,e_1\right) \tag{226}$$

where $e_1 = (1,0,\ldots,0), \phi\left(t,e_1\right) := 1-\psi\left(t,e_1\right)$

Thus, going back to (224):

$$\mathbb{E}^\eta\left[Z^\gamma_{t,\eta}\right] = \exp\left\{-\nu\rho\int_0^t \phi\left(s,e_1\right)ds\right\} \tag{227}$$

By going though calculations of generating functions and Tauberian theorems (Sect. 2.4 of [19]), the $\phi(t, e_1)$ (the probability that a rate 1 random walk that starts from e_1 doesn't hit 0 until time ρt) follows the asymptotics :

$$\phi(t, e_1) = \begin{cases} \sqrt{\frac{2}{\pi \rho t}}(1 + o(1)) & d = 1 \\ \\ \frac{\pi}{\ln t}(1 + o(1)) & d = 2 \\ \\ G_d(0)^{-1}(1 + o(1)) & d \geq 3 \end{cases} \tag{228}$$

Thus, by (227) we get the following asymptotics:

$$\ln \mathbb{E}^\eta \left[Z^\gamma_{t,\eta}\right] = \begin{cases} -\nu\sqrt{\frac{8\rho t}{\pi}}(1 + o(1)), & d = 1, \\ -\nu\pi\frac{\rho t}{\ln t}(1 + o(1)), & d = 2, \\ -\nu\frac{\rho t}{G_d(0)}(1 + o(1)), & d \geq 3, \end{cases} \tag{229}$$

which is a result for the Theorem 7.3 for the case $\kappa = 0$ and $\gamma = \infty$.

Now, we will give sharp lower and upper bounds for the survival probability in $d = 1, 2$ for the general $\kappa > 0$ and use the results that we got for $\kappa = 0$ to do so.

Firstly, let us get the lower bounds of the following form. For any $\gamma \in (0, \infty], \kappa \geq 0, \rho > 0$, and $v > 0$, we should aim for following bounds

$$\begin{aligned} \liminf_{t\to\infty} \frac{1}{\sqrt{t}} \log \mathbb{E}^\eta \left[Z^\gamma_{t,\eta}\right] &\geq -\nu\sqrt{\frac{8\rho}{\pi}}, \qquad d = 1, \\ \liminf_{t\to\infty} \frac{\ln t}{t} \log \mathbb{E}^\eta \left[Z^\gamma_{t,\eta}\right] &\geq -\nu\pi\rho, \qquad d = 2. \end{aligned} \tag{230}$$

We can restrict Random walk to a certain strategy to get these lower bounds. Let us constrict random walk to stay up to time t inside a ball B_{R_t} of radius R_t (which also will be a scaling function used in the argument) around the origin such that this ball remains clear of the traps until the time t. This leads us to the lower bound that is independent of γ and κ (due to ball being devoid of the traps). In the dimensions $d = 1, 2$ this bound is sharp since it is easier to create space-time regions clear of traps

Let E_t denote the event that $\eta(x) = 0$ for all $x \in B_{R_t}$. Let F_t denote the event that $X^x_j(s) \notin B_{R_t}$ for all $x \notin B_{R_t}, 1 \leq j \leq \eta(x)$, and $s \in [0, t]$. Lt G_t denote the event that X with $X(0) = 0$ does not leave B_{R_t} before time t. Immediately, it is evident that

$$\mathbb{E}^\eta \left[Z^\gamma_{t,\eta}\right] \geq \mathbb{P}(E_t \cap F_t \cap G_t) = \mathbb{P}(E_t)\,\mathbb{P}(F_t)\,\mathbb{P}(G_t) \tag{231}$$

Now let us estimate these three events, beginning with E_t:

$$\mathbb{P}(E_t) = \mathrm{e}^{-v(2R_t+1)^d} \tag{232}$$

For the $\mathbb{P}(G_t)$ we have to choose scale and use Donsker's invariance principle. Choose scale $1 \ll R_t \ll \sqrt{t}$ as $t \to \infty$. Now by Donsker's invariance principle there exists α such that:

$$\inf_{x\in B_{\sqrt{t}/2}} \mathbb{P}\left(X(s) \in B_{\sqrt{t}} \forall s \in [0, t], X(t) \in B_{\sqrt{t}/2} \mid X(0) = x\right) \geq \alpha. \tag{233}$$

Finally by partitioning $[0,t]$ into intervals of length R_t^2, using Markov property we get the following estimation for

$$\begin{aligned}\mathbb{P}(G_t) \geq & \mathbb{P}\left(X(s) \in B_{R_t} \forall s \in \left[(i-1)R_t^2, iR_t^2\right], \text{ and } X(iR_t^2) \in B_{R_t/2}, i = 1, 2, \cdots, \lceil t/R_t^2 \rceil\right) \geq \\ \geq & \alpha^{t/R_t^2} = \mathrm{e}^{t \ln \alpha / R_t^2}.\end{aligned} \tag{234}$$

For the final event estimation $\mathbb{P}(F_t)$, denote $\tilde{F}_t$ as the event that $X_j^x(s) \neq 0$ for all $x \in \mathbb{Z}^d$, $1 \leq j \leq \eta(x)$, and note that $\mathbb{P}(\tilde{F}_t)$ is exactly $\mathbb{E}^\eta\left[Z_{t,\eta}^\gamma\right]$ with $\kappa = 0$ and $\gamma = \infty$ for which we already proved the asymptotics. Next we will compare $\mathbb{P}(F_t)$ and $\mathbb{P}(\tilde{F}_t)$.

Denote $\tau_{B_{R_t}}$ as a stopping time for X (here we mean random walking trap as X) to hit B_{R_t}, and τ_0 as stopping time for X (here we mean random walking trap as X) to hit 0. Now,

$$\ln \mathbb{P}(F_t) = -\nu \sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_x^X\left(\tau_{B_{R_t}} \leq t\right) \tag{235}$$

$$\ln \mathbb{P}(\tilde{F}_t) = -\nu \sum_{y \in \mathbb{Z}^d \setminus B_0} \mathbb{P}_x^X\left(\tau_{B_0} \leq t\right) \tag{236}$$

Now the following holds,

$$\sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_x^X\left(\tau_{B_{R_t}} \leq t\right) \geq \sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_x^X\left(\tau_0 \leq t\right) = \sum_{y \in \mathbb{Z}^d} \mathbb{P}_x^X\left(\tau_0 \leq t\right) - \sum_{y \in B_{R_t}} \mathbb{P}_x^X\left(\tau_0 \leq t\right). \tag{237}$$

Thus,

$$\ln \mathbb{P}(F_t) \leq \ln \mathbb{P}\left(\tilde{F}_t\right) + v \sum_{y \in B_{R_t}} \mathbb{P}_x^X\left(\tau_0 \leq t\right) \leq \ln \mathbb{P}\left(\tilde{F}_t\right) + v\left(2R_t + 1\right)^d. \tag{238}$$

For $\epsilon > 0$, we have

$$\begin{aligned}\sum_{y \in \mathbb{Z}^d} \mathbb{P}_x^X\left(\tau_0 \leq t + \epsilon t\right) \geq & \sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_x^X\left(\tau_{B_{R_t}} \leq t, \tau_0 \leq t + \epsilon t\right) \\ & \geq \inf_{z \in \partial B_{R_t}} \mathbb{P}_z^X\left(\tau_0 \leq \epsilon t\right) \sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_x^X\left(\tau_{B_{R_t}} \leq t\right),\end{aligned}$$

Dividing by the infimum term in r.h.s, we get

$$\sum_{y \in \mathbb{Z}^d \setminus B_{R_t}} \mathbb{P}_y^Y\left(\tau_{B_{R_t}} \leq t\right) \leq \frac{\sum_{y \in \mathbb{Z}^d} \mathbb{P}_y^Y\left(\tau_0 \leq t + \epsilon t\right)}{\inf_{z \in \partial B_{R_t}} \mathbb{P}_z^Y\left(\tau_0 \leq \epsilon t\right)}, \tag{239}$$

And thus by (235)

$$\ln \mathbb{P}(F_t) \geq \frac{\ln \mathbb{P}\left(\tilde{F}_{t+\epsilon t}\right)}{\inf_{z \in \partial B_{R_t}} \mathbb{P}_z^X\left(\tau_0 \leq \epsilon t\right)}. \tag{240}$$

Now, combining the results in (237) and (240), the fact that $\mathbb{P}\left(\tilde{F}_t\right)$ follows the desired asymptotics with $\kappa = 0$ and $\gamma = \infty$, and that $\epsilon > 0$ can be chosen to be arbitrarily small, this concludes in

$$\ln \mathbb{P}(F_t) = -v\sqrt{\frac{8\rho t}{\pi}}(1 + o(1)) = \ln \mathbb{P}(\tilde{F}_t).$$

For $R_t = \sqrt{t/\ln t}$ we have

$$\ln \mathbb{P}(E_t) = -v(2\sqrt{t/\ln t} + 1) \quad \text{and} \quad \ln \mathbb{P}(G_t) \geq \ln \alpha \ln t,$$

Substituting estimation of these probabilities in (231) results in desired lower bound (230) for $d = 1$

When $d = 2$, firstly choose $R_t = \ln t$. This results in $\inf_{z\in\partial B_{\ln t}} \mathbb{P}_z^X(\tau_0 \leq \epsilon t) \to 1$ as $t \to \infty$. By the same computations as for $d = 1$:

$$\ln \mathbb{P}(F_t) = -v\pi\rho\frac{t}{\ln t}(1 + o(1)) = \ln \mathbb{P}\left(\tilde{F}_t\right).$$

With the other probabilities having asymptotics

$$\ln \mathbb{P}(E_t) = -v(2\ln t + 1)^2 \quad \text{and} \quad \ln \mathbb{P}(G_t) \geq \frac{t\ln\alpha}{\ln^2 t}$$

Again, substituting estimation of these probabilities in (231) results in desired lower bound (230) for $d = 2$

For the upper bound let us state the Pascal principle:

Lemma 7.5 (Pascal Principle) *Let η be the random field generated by a collection of irreducible symmetric random walks $\left\{X_j^x\right\}_{y\in\mathbb{Z}^d, 1\leq j\leq\eta(x)}$ on $\mathbb{Z}^d$ with jump rate $\rho > 0$. Then for all piecewise constant $X : [0, t] \to \mathbb{Z}^d$ with a finite number of discontinuities, we have*

$$\mathbb{E}^\eta\left[\exp\left\{-\gamma\int_0^t \eta(s, X(s))\mathrm{d}s\right\}\right] \leq \mathbb{E}^\eta\left[\exp\left\{-\gamma\int_0^t \eta(s, 0)\mathrm{d}s\right\}\right]$$

The pascal principle tells that the survival probability is maximized when $X \equiv 0$ and does not move. Which means that combining it with proof of Theorem 7.3 for the case of $\kappa = 0$ gives us the desired upper bound. Which in turn proves the Theorem 7.3 in full generality. □

8 Interpolation between static and dynamic traps of 1D time-discrete model

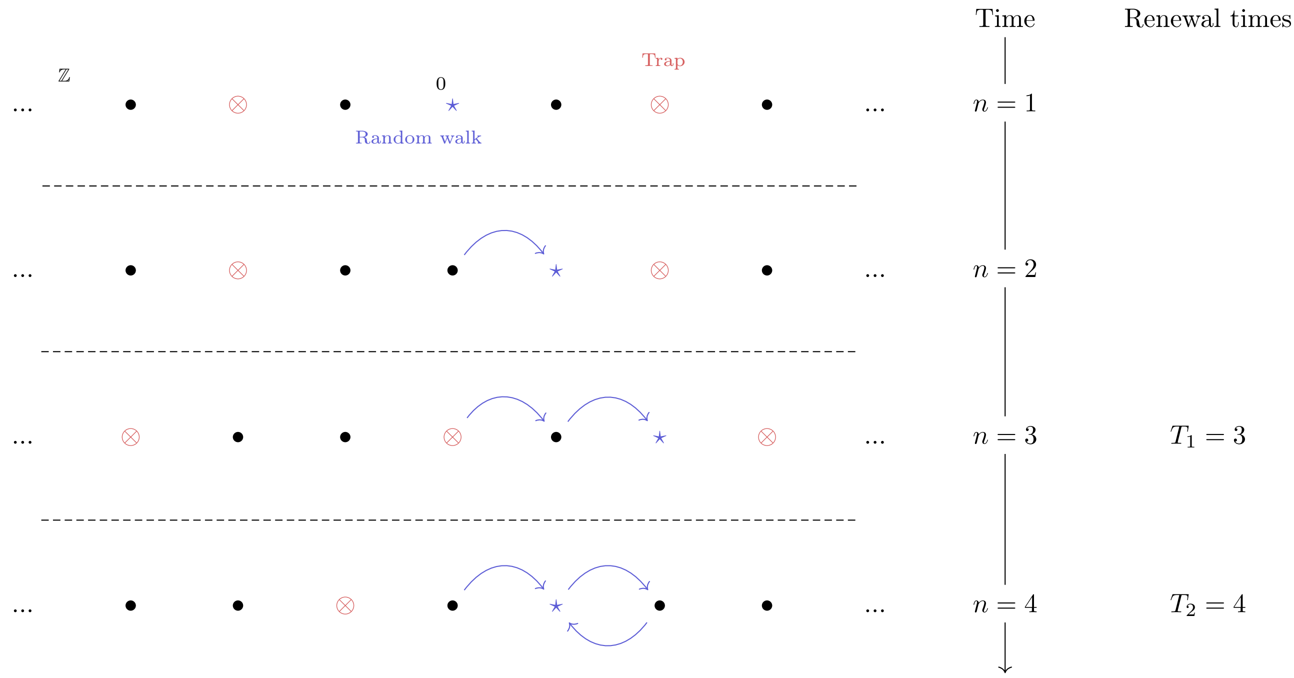

Figure 3: Random walk in an environment with random renewal times

In the previous sections, we have discussed settings with both static and dynamic environment of traps. The survival probability in a static setting happens to have a subexponential tail, compared to a strict exponential decay of survival probability in a dynamic setting. Intuitively, if we slow down the dynamic environment we should get some interpolation of survival probabilities from exponential decay to subexponential. In the literature, very little is written regarding this phenomenon, and almost nothing is known regarding the exact nature of such interpolation. In this section, we propose a simplified model which features interpolation between static and dynamic environment of traps. The model of choice is a discrete-time random walk in a discrete-time environment of hard traps on a lattice $\mathbb{Z}$, where the whole environment is updated at specific random "renewal times".

We define the random dynamic environment as follows. Let $E = \{0, 1\}^{\mathbb{Z}}$ be a space of all possible trap configurations on $\mathbb{Z}$.

Definition 8.1 (Traps) *Define the configuration of traps being i.i.d Bernoulli random variables with probability ρ of a site being a trap and $(1 - \rho)$ of a site being a normal point. The trap is defined to be hard: it kills the random walk if it touches it.*

Now we will define the way we want the environment to be dynamic. Although, all points of the environment should have individual "clocks" that tell the traps at which discrete timestep they should update, we will update all traps simultaneously at random steps. One should note that between these random environment update steps, which we will call renewal times, the environment is static. Let us formally define these renewal times and their difference more formally

Definition 8.2 (Renewal Times) *Let the sequence $T_1, T_2, ..., T_k, ...$ be a random sequence of renewal times (with $T_0 = 0$), which dictates at which iterations the random environment η is drawn again from E, independently of previous choices of environment.*

Assumption 8.1 (Renewal Times Differences) *Define the renewal times differences* $\tau_i = T_i - T_{i-1}$ *as i.i.d random variables. The tail of* τ_i *will turn out to be important.*

The main idea of this section is to gain some insight into the interpolation of survival probability asymptotics of the random walk in a built random environment between the slow updating regime (the random environment is such that the renewal times difference is exponential: $\mathbb{E}(\tau_i) = \infty$) and the fast regime (the renewal times difference is sub-exponential $\mathbb{E}(\tau_i) = a < \infty$) by controlling the distribution of τ_i.
Now let us define the Random Walk in this discrete setting.

Definition 8.3 (Random walk in discrete-time) *For* $n \in \mathbb{N}$ *and* $S_1, S_2, ..., S_n, ...$ *the* $\{-1, 1\}$*-valued sequence of i.i.d. random variables, define the random walk* (X_n) *as:*

$$X_n = \sum_{i=1}^{n} S_i \tag{241}$$

Let us also denote $R(n_1, n_2)$ as a range of random walk between discrete times n_1, n_2.

Definition 8.4 (Random Walk Range) *Range of the random walk* X_n *on* $\mathbb{Z}^d$ *between discrete times* n_1, n_2 *is defined to be*

$$R(n_1, n_2) = \sum_{x} \mathbb{1}_{\{X_s = x, \exists s \in \{n_1, n_2\}\}} \tag{242}$$

Now that the definition of the model is complete, we would like to prove the exponential decay of survival time after n steps and then present results regarding the proposed interpolation in future sections.

8.1 Fast environment: exponential bounds on decay of survival probability

In this subsection, we first proof the exponential decay of survival time after the random T_n update of the environment and then use this result with a combination of the Large Deviations Theory to prove that the survival probability after some deterministic n can be bounded exponentially from below and above.

Let S be a survival time of a random walk, then the probability that survival time is greater than n-th environment renewal time T_n is:

$$\mathbb{P}(S > T_n) \tag{243}$$

Let us proceed with the first lemma of this subsection

Lemma 8.2 *Let the difference of renewal times* τ_i *be i.i.d. distributed. Let* T_i *be random environment renewal times with* $T_k = \sum_{i=1}^{k} \tau_i$*. Let traps be i.i.d. Bernoulli random variables with parameter* ρ*. Then the following holds*

$$\mathbb{P}(S > T_n) = \left(\mathbb{E}^{\tau_1}\left[(1-\rho)^{R(0,\tau_1)}\right]\right)^n$$

where $R(0, \tau_1)$ *is the range of the random walk from time 0 to a first random renewal time* $T_1 = \tau_1$*.*

Proof: Let us first note that

$$\mathbb{P}(S > T_n) = \mathbb{E}^{T_i}\left(\mathbb{P}(S > T_n | T_i, 0 \le i \le n-1)\right) \tag{244}$$

where we condition inside term on the collection of known collection of T_i's up to T_n.
The survival time probability conditioned on a collection of T_i's essentially is the product across all T_i's collection of the probabilities that the environment, which is static between renewal times T_i's (thus $\eta(x, k) = \eta(x)$ for $T_i \le k \le T_{i+1}$), from the perspective of a random walker X_k is devoid of traps. This yields in the following expansion,

$$\mathbb{P}(S > T_n | T_i, 0 \le i \le n-1) = \mathbb{P}((\eta(X_k) = 0 \;\; \forall\, 0 \le k \le T_1) \cap \tag{245}$$

$$\cap(\eta(X_k) = 0 \;\; \forall\, T_1 \le k \le T_2) \; \cap ... \tag{246}$$

$$...\cap(\eta(X_k) = 0 \;\; \forall\, T_{n-1} \le k \le T_n)) = \tag{247}$$

$$= \prod_{k=1}^{n} (1-\rho)^{R(T_{k-1}, T_k)} \tag{248}$$

Thus the annealed survival probability of a random walk, with respect to environment renewal times, would take the following form

$$\mathbb{P}(S > T_n) = \mathbb{E}^{T_i}\left(\mathbb{P}(S > T_n | T_i)\right) = \tag{249}$$

$$= \mathbb{E}^{T_i}\left(\prod_{k=1}^{n}(1-\rho)^{R(T_{k-1},T_k)}\right) \overset{ind}{=} \tag{250}$$

$$\overset{ind}{=} \left(\mathbb{E}^{\tau_1}\left[(1-\rho)^{R(0,\tau_1)}\right]\right)^n \tag{251}$$

This concludes the proof □

Now, let us prove the exponential decay for the expression $\mathbb{P}(S > T_n)$ using the previous lemma.

Theorem 8.3 (Survival time after random update of the Environment) *Let τ_i be i.i.d. distributed by law π, with finite exponential moments $\mathbb{E}(e^{\lambda\tau}) < \infty$ for some $\lambda > 0$. Let T_i be random environment renewal times with $T_k = \sum_{i=1}^{k}\tau_i$. Then the following holds*

$$\mathbb{P}(S > T_n) \simeq e^{-\lambda_1 n}$$

where $\simeq$ is a logarithmic equivalence defined as in (19)

Proof: Proving the above statement can be done by proving that the following limit

$$\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{P}(S > T_n) < 0$$

is finite and negative.

We can do it, utilizing the previous Lemma 8.2

$$\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{P}(S > T_n) = \lim_{n\to\infty}\frac{n}{n}\ln\mathbb{E}^{\tau_1}\left[(1-\rho)^{R(0,\tau_1)}\right] = \ln\mathbb{E}^{\tau_1}\left[(1-\rho)^{R(0,\tau_1)}\right] =: -\lambda_1 < 0 \tag{252}$$

where we used the fact that $0 < \mathbb{E}^{T_i}(1-\rho)^{R(0,\tau_1)} < 1$ with a strong inequality (since τ_1 being at least 1 and $\rho < 1$).

Thus we have $\mathbb{P}(S > T_n) \simeq e^{-\lambda_1 n}$. □

Let us now prove one of the important results of this section in our studies.

Theorem 8.4 *Let τ_i be i.i.d. distributed by law π with mean $\mathbb{E}(\tau) = a$, and with finite exponential moments $\mathbb{E}(e^{\lambda\tau}) < \infty$ for some $\lambda > 0$. Let T_i be random environment renewal times with $T_k = \sum_{i=1}^{k}\tau_i$. Then the following holds*

$$\lim_{n\to\infty}\frac{1}{n}\ln\mathbb{P}\left(S > T_n \middle| \left|\frac{T_n}{n} - a\right| \le \varepsilon\right) = \sup_{\nu\in G(a,\varepsilon)}\left(\int \ln\psi d\nu - H(\nu|\pi)\right) < 0 \tag{253}$$

where $\psi(\tau) = \mathbb{E}^{T_i}(1-\rho)^{R(0,\tau)}$, the set of measures $G(a,\varepsilon) = \{\mu \ : \ a - \epsilon \le \int x d\mu \le a + \epsilon\}$, and $H(\nu|\pi)$ is a relative entropy with the strict inequality

Proof: The appropriate setup is needed for us to proceed. Let us denote by $\psi(t)$ the following exponent of a range of random walk:

$$\psi(\tau) = \mathbb{E}^{T_i}(1-\rho)^{R(0,\tau)} \tag{254}$$

Now the survival probability after random time T_n conditioned on the collection T_i's becomes the following product

$$\mathbb{P}(S > T_n | T_i, 0 \le i \le n-1) = \prod_{i=1}^{n}\psi(\tau_i) \tag{255}$$

Define by Λ_n the empirical measures of the form:

$$\Lambda_n = \frac{1}{n}\sum_{i=1}^{n} \delta_{\tau_i} \tag{256}$$

By Sanov's theorem, since we have finite exponential moments $\mathbb{E}(e^{\lambda\tau})$, given that $\tau_i \sim \pi$, we have that the empirical measures Λ_n satisfies a Large Deviation Principle with the rate function I with $I(\mu) = H(\mu|\pi)$ where H is a relative entropy (also called Kullback–Leibler divergence) from π to μ.

$$\mathbb{P}(\Lambda_n \approx \mu) \simeq e^{-nI(\mu)} \tag{257}$$

Now let us derive the Large Deviation Principle for Λ_n for the conditioned probabilistic measures.

$$\mathbb{P}(\Lambda_n \approx \mu \mid \Lambda_n \in G) \simeq e^{-nI_G(\mu)} \tag{258}$$

where the rate function $I_G(\mu)$ is following:

$$I_G(\mu) = \begin{cases} I(\mu) - \inf_{\nu\in G} I(\nu) & \mu \in G \\ \infty & \text{otherwise} \end{cases} \tag{259}$$

An important note that if we have $\pi \in G$, then since $H(\mu|\pi)$ is nonnegative and relative entropy of π to itself is zero $H(\pi|\pi) = 0$, then the infimum zero in the definition I_G is attained, i.e.
$\inf_{\nu\in G} I(\nu) = I(\pi) = H(\pi|\pi) = 0$. Thus, if π is in G, we have the following:

$$I_G(\mu) = \begin{cases} I(\mu) & \mu \in G \\ \infty & \text{otherwise,} \end{cases} \tag{260}$$

It is straightforward to note the following connection between the empirical measures Λ_n from (256) and the empirical means T_n/n. Along the way, we define the resulting integral as $Q(\cdot)$ for the sake of simplicity of the notations in the future,

$$\frac{T_n}{n} = \frac{\sum_{i=1}^{n}\tau_i}{n} = \frac{\int x\sum_{i=1}^{n}\delta_{\tau_i}(dx)}{n} = \int x\frac{1}{n}\sum_{i=1}^{n}\delta_{\tau_i}(dx) = \int x\Lambda_n(dx) =: Q(\Lambda_n) \tag{261}$$

We proceed by looking at the quenched survival probability after said random time T_n,

$$\mathbb{P}(S > T_n) = \mathbb{E}^{T_i}\left(\mathbb{P}(S > T_n|T_i, 0 \le i \le n-1)\right) = \mathbb{E}^{T_i}\left(\prod_{i=1}^{n}\psi(\tau_i)\right) = \mathbb{E}^{T_i}\left(e^{\sum_{i=1}^{n}\ln\psi(\tau_i)}\right) \tag{262}$$

Measure theoretically speaking, the sum in the power of the exponent in the above expression can be thought of as an integral taken with respect to previously defined empirical measures. We also define the resulting integral as $F(\cdot)$:

$$\mathbb{P}(S > T_n) = \mathbb{E}^{T_i}\left(e^{\sum_{i=1}^{n}\ln\psi(\tau_i)}\right) = \mathbb{E}^{T_i}\left(e^{n\int\ln\psi(x)\frac{1}{n}\sum_{i=1}^{n}\delta_{\tau_i}(dx)}\right) = \mathbb{E}\left(e^{n\int\ln\psi(x)\Lambda_n(dx)}\right) =: \mathbb{E}\left(e^{nF(\Lambda_n)}\right) \tag{263}$$

Let us also recall that given the finite exponential moments $\mathbb{E}(e^{\lambda\tau})$ for some $\lambda > 0$ implies that the mean $\mathbb{E}(\tau) = a$ being finite, and also gives us the Large Deviations Principle for the empirical mean (by Cramer Theorem). By the weak of the large numbers the empirical mean T_n/n should be near $\mathbb{E}(\tau) = a$. With a little abuse of the notation let us derive the expression for the survival probability after a random T_n conditioned on the event $|T_n/n - a| \le \epsilon$ for some controllable ϵ.

$$\mathbb{P}\left(S > T_n \middle| \left|\frac{T_n}{n} - a\right| \le \epsilon\right) = \mathbb{E}\left(e^{nF(\Lambda_n)} \middle| \left|\frac{T_n}{n} - a\right| \le \epsilon\right) = \tag{264}$$

$$= \mathbb{E}\left(e^{nF(\Lambda_n)} \middle| a - \epsilon \le \frac{T_n}{n} \le a + \epsilon\right) = \tag{265}$$

$$= \mathbb{E}\left(e^{nF(\Lambda_n)} \middle| a - \epsilon \le Q(\Lambda_n) \le a + \epsilon\right) \tag{266}$$

$$= \mathbb{E}\left(e^{nF(\Lambda_n)} \middle| \Lambda_n \in G(a, \varepsilon)\right) \tag{267}$$

Note that the set $G(a,\varepsilon)$ can be viewed as a set of measures $G(a,\varepsilon) = \{\mu \ : \ a - \epsilon \leq \int x d\mu \leq a + \epsilon\}$. Furthermore, π is in this set $G(a,\varepsilon)$, as $\int x d\pi = \mathbb{E}(\tau) = a$ which obviously satisfies $a - \epsilon \leq a \leq a + \epsilon$.

Now, let us proceed by deriving the rate function for the survival probability after random renewal time conditioned on the event that the empirical mean is close to $c \neq a$.

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{P}\left(S > T_n \middle| \left|\frac{T_n}{n} - c\right| \leq \epsilon\right) = \lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left(e^{nF(\Lambda_n)} \ \middle| \ \Lambda_n \in G(a,\varepsilon)\right) \tag{268}$$

Note that the above functional is of Varadhan's Lemma type, by

$$\lim_{n\to\infty} \frac{1}{n} \ln \mathbb{E}\left(e^{nF(\Lambda_n)} \ \middle| \ \Lambda_n \in G(a,\varepsilon)\right) = \lim_{n\to\infty} \frac{1}{n} \ln \int_{G(a,\varepsilon)} e^{nF(\Lambda_n)} \mathbb{P}(d\Lambda_n) \tag{269}$$

Recall that we have derived that the family of the conditioned probability measures $\{\mathbb{P}(\cdot|\ \Lambda_n \in G), \ n \in \mathbb{N}\}$ satisfies the Large Deviation Principle with the rate function I_G. And since $\pi \in G(a,\varepsilon)$, we have the equality $I_{G(a,\varepsilon)} = I$, where I is the rate function for Λ_n. Thus by Varadhan's Lemma, we have:

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_{G(a,\varepsilon)} e^{nF(\Lambda_n)} \mathbb{P}(d\Lambda_n) = \sup_{\nu \in G(a,\varepsilon)} (F(\nu) - I_G(\nu)) \tag{270}$$

Finally, substituting everything:

$$\sup_{\nu \in G(a,\varepsilon)} (F(\nu) - I(\nu)) = \sup_{\nu \in G(a,\varepsilon)} \left(\int \ln \psi d\nu - H(\nu|\pi)\right) \tag{271}$$

Note that if the optimal $\nu = \pi$ we have that $I(\nu) = H(\nu|\pi) = H(\pi|\pi) = 0$ and $F(\nu) = F(\pi) < 0$, otherwise, if the optimal $\nu \neq \pi$ we have $F(\nu) \leq \ln(1-\rho) < 0$ thus $\sup_{\nu \in G(a,\varepsilon)}(F(\nu) - I(\nu)) < \ln(1-\rho) < 0$. Thus

$$\sup_{\nu \in G(a,\varepsilon)} (F(\nu) - I(\nu)) < 0 \tag{272}$$

□

Now given the previous results, it is possible to exponentially bound the survival probability from below and above.

Theorem 8.5 (Exponnetial lower and upper for survival probability) *Let τ_i be i.i.d. distributed by law π, with finite exponential moments $\mathbb{E}(e^{\lambda\tau}) < \infty$ for some $\lambda > 0$. Let T_i be random environment renewal times with $T_k = \sum_{i=1}^{k} \tau_i$. The survival probability after some deterministic time n can be bounded as follows*

$$e^{-\lambda_* n} \leq \mathbb{P}(S > n) \leq e^{-\lambda^* n}, \quad \lambda_*, \lambda^* > 0 \tag{273}$$

Proof: To obtain the exponential upper bound, we can decompose the survival probability as follows.

$$\mathbb{P}(S > (a+\varepsilon)n) = \mathbb{P}\left(S > (a+\varepsilon)n \middle\| \frac{T_n}{n} - a\right| \leq \varepsilon\right) \mathbb{P}\left(\left|\frac{T_n}{n} - a\right| \leq \varepsilon\right) + \tag{274}$$

$$+ \mathbb{P}\left(S > (a+\varepsilon)n \middle\| \frac{T_n}{n} - a\right| \geq \varepsilon\right) \mathbb{P}\left(\left|\frac{T_n}{n} - a\right| \geq \varepsilon\right) \leq \tag{275}$$

$$\leq \mathbb{P}\left(S > T_n \middle\| \frac{T_n}{n} - a\right| \leq \varepsilon\right) + \mathbb{P}\left(\left|\frac{T_n}{n} - a\right| \geq \varepsilon\right) \simeq \tag{276}$$

$$\simeq e^{n \sup_{\nu \in G(a,\varepsilon)}(F(\nu) - I(\nu))} + e^{n\varphi(\varepsilon)} \simeq \tag{277}$$

$$\simeq e^{-\lambda^* n} \tag{278}$$

where we used the fact that $\mathbb{P}(|T_n/n - a| \leq \varepsilon) \approx 1$, then we have used the Lemma (8.2) to get logarithmic equivalence of the first term and the fact that T_n/n satisfies LDP by the Cramer's theorem to get logarithmic equivalence of the second term by $e^{n\varphi(\varepsilon)}$ with $\varphi(\varepsilon) < 0$. Then in the last line, we used the fact that the largest exponent survives.

Now, for the lower bound, we need to look at the $\mathbb{P}(S > (a-\varepsilon)n)$.

$$\mathbb{P}(S > (a-\varepsilon)n) \geq \mathbb{P}\left(S > (a-\varepsilon)n \middle| \left|\frac{T_n}{n} - a\right| \geq \varepsilon\right) \geq \tag{279}$$

$$\geq \mathbb{P}\left(S > T_n \middle| \left|\frac{T_n}{n} - a\right| \geq \varepsilon\right) \tag{280}$$

$$\simeq e^{n \sup_{\nu \in G(a,\varepsilon)}(F(\nu) - I(\nu))} = \tag{281}$$

$$= e^{-\lambda_* n} \tag{282}$$

where we have used the Theorem 8.4

Combining both results yields,

$$e^{-\lambda_* n} \leq \mathbb{P}(S > n) \leq e^{-\lambda^* n}, \quad \lambda_*, \lambda^* > 0$$

which concludes the proof

□

8.2 Slow environment: subexponential lower bound for survival probability

In this section, we observe the interpolation between slow and fast regimes of the previously stated discrete-time model. It is quite natural to think that if we let the difference of the renewal times to be large enough, the model should start to exhibit the behavior of the static regime. Thus, we present the conditions for the exponential and sub-exponential decay of the survival probability of the Random Walk in this discrete model. These conditions happen to oppose each other, in a sense giving tunable instrument to switch between sub-exponential and exponential regimes.

Let us move on to stating the conditions for the sub-exponential lower bound decay of survival time. To some degree, the conditions on the next theorem are opposite to those of the fast regime model.

Theorem 8.6 *Let τ_i be i.i.d. distributed with infinite exponential moments $\mathbb{E}(e^{\lambda\tau}) = \infty$ for any $\lambda > 0$, with $\mathbb{P}(\tau > n)$ decaying with at most time rate of $n^{d/d+2}$. Let T_i be random environment renewal times with $T_k = \sum_{i=1}^{k} \tau_i$. The survival probability after some deterministic time n*

$$\mathbb{P}(S > n) \geq e^{-\lambda n^{d/d+2}}, \quad \lambda > 0 \tag{283}$$

Proof: We provide a rough, but simple sub-exponential lower bound to prove this theorem.

$$\mathbb{P}(S > n) \geq \mathbb{P}(S > n \ \cap T_1 > n) = \mathbb{P}(S_{static} > n \ \cap T_1 > n) = \mathbb{P}(S_{static} > n)\,\mathbb{P}(T_1 > n) = \tag{284}$$

$$= \mathbb{P}(S_{static} > n)\,\mathbb{P}(\tau_1 > n) = e^{-n^{\frac{d}{d+2}}} e^{-\alpha(n)} \tag{285}$$

Since we assume the distribution of τ, the $\alpha(n)$ is a parameter. We need τ to follow the sub-exponential type of distribution such that $\alpha(n) \leq n^{d/d+2}$, to get the sub-exponential decay of survival probability.

$$\mathbb{P}(S > n) \geq e^{-n^{\frac{d}{d+2}}} \tag{286}$$

□

This result combined with the Theorem 8.5 proves that the model interpolates between the static and dynamic regimes.

The interpolation logic is as follows. The Cramer's theorem for the empirical mean T_n/n is implied by the finiteness of the exponential moments $\mathbb{E}(e^{\lambda\tau}) < \infty$ for some $\lambda > 0$ making the survival probability decay exponentially (in a rough sense, not strictly) as in the previous section. And in the case $\mathbb{E}(e^{\lambda\tau}) = \infty$ for $\forall\lambda$ it implies sub-exponential decay of $\mathbb{P}(\tau > n)$ and thus by the previous result it implies the subexponential lower bound on the decay of the survival probability.

It is also worth mentioning that the interpolation that we get might not be abrupt, as we might not get the sudden jump from n to $n^{d/d+2}$ of the survival decays, from the $n^{d/d+2}$ side. If we weaken the conditions of the

Theorem 8.5 by letting $\mathbb{P}(\tau > n)$ decaying with the rate α faster than $n^{d/d+2}$ and slower than n then we might get the following result in a weak form

$$e^{c_1 n} < e^{\alpha(n)} \leq \mathbb{P}(S > n) \leq e^{c_2 n^{d/d+2}}$$

Essentially, this would result in an interpolation between the static and dynamic regimes of the model. If this result is true, it also gives rise to a new question - whether the interpolation is smooth and of the same rate as $\mathbb{P}(\tau > n)$, or does it abruptly jump from $n^{d/d+2}$ to n.

9 Conclusion

To sum up, a wide variety of rich and rigorous mathematical toolkits, such as theory of Markov processes, semigroups and generators theory, and the large deviations theory were studied in this thesis. Furthermore, it was shown how these theories were applied in the quite sophisticated field of Interacting Particle Systems through the lens of the problem regarding the Random Walk in a dynamic and static environment of traps.

A wide range of different models of random walk among dynamic random environments of traps were discussed and studied. The collection of the results regarding the decay of survival probability of the random walk in these environments gives a great overview of the present and past results in the literature regarding this topic. Moreover, it gives perspective on possible future research directions.

For some models, like Independent Spin Flip, it was proven that the rate of decay of survival probability of the random walk decays exponentially. We have also observed large deviation estimates for the occupation times of a much more general Attractive Spin Flip model.

Also, the recent strong results from the literature regarding the exact asymptotics for the decay of the survival probabilities of the random walk among random walking traps were presented in full generality.

In the last section, the interpolation between the static and dynamic regimes for the discrete-time model was shown, albeit not sharp and not in a closed form. We have proved that it suffices to let renewal times difference to be distributed sub-exponentially (at most with the $n^{d/(d+2)}$ rate of decay), to have the same sub-exponential lower bound on the decay of a survival time, same as in the static regime. Furthermore, it raises a question regarding much weaker conditions on the distribution of renewal times. Notably, it is possible to have deterministic renewal times, which would yield the same conclusions, and would connect our results to other papers in literature, such as "cooling" environment models. The last section suggests the following questions that should be researched in the future:

- As the sub-exponential distribution of the renewal time difference yields a sub-exponential lower bound on the rate of decay of survival probabilities, the question of exact sub-exponential asymptotics is open.
- Further generalizations, may include the limiting case of the model to get a continuous time model. This would give further insights into asymptotics in the continuous-time case.
- Is interpolation, caused by letting the difference of the renewal times to be distributed subexponentially, smooth or abrupt? Is survival probability $\mathbb{P}(S > n)$ decays the same as $\mathbb{P}(\tau > n)$ in the interpolating regime (i.e. is $\mathbb{P}(S > n) \simeq \mathbb{P}(\tau > n)$ if $\mathbb{E}(\tau) = \infty$?)

The overview, given by this thesis, shows that there are many still open problems regarding the Random walks in random environemnts subfield of Interacting Particle Systems. The lack of a general approach for deriving the survival time asymptotics in the literature, such that it would be applicable across different models with the different environment of traps, might suggest that the models might have quite different exact asymptotics.

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