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Process capability analysis considering asymmetric tolerance

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Abstract

The classical process capability indices are still the most prominently used by practitioners for asymmetrical tolerances even while not accurately reflecting on process capability. It appears that an adequate measure of capability for asymmetrical tolerances is yet to be discovered. This report formulates a list of five desirable PCI properties and explores some proposed indices developed for asymmetrical tolerances by comparing them to the properties. As none of the discussed indices satisfy all properties, four new proposals are made that improve upon existing indices. The new indices are related to process yield and centering, and compared to the existing indices. Further research is required to determine whether the new proposals are to be used in practice, but for now they serve as a source of inspiration in the development of PCIs for asymmetric tolerances.

Contents

1 Introduction

Process capability indices (PCIs) are a measure of the ability of a production process to produce products that meet the quality requirements of a customer. The requirements of the customer are in the form of a lower and upper bound together with a target value on a characteristic of the product. The characteristic of the produced products are measured and as processes naturally make errors from the desired value it is common for the measurement data to be normally distributed. Throughout this project we therefore do analysis on a normal process distribution. PCIs provide a way to relate the manufacturers process distribution to the customers specifications. PCIs relate the process with the specifications by providing a single-valued measure of process yield and targeting, and sometimes more criteria. In other words, a single score for capability. This is convenient in practice as practitioners don't have to worry about concepts like yield and squared error, which can be complicated for some. Furthermore, it makes communications between manufacturers and customers easier if they can agree upon a single reliable measure of process capability.

A process is said to have symmetric tolerance if the target value coincides with the midpoint of the specification limits. Most of the research work has focused on the development of PCIs for symmetric tolerance. As of today, there are 4 basic indices that provide a good reflection of the process capability for symmetric tolerance. They will be referred to as the classical PCIs. However, in practice it is common that the target value is not the midpoint of the specification limits. Or equivalently, that the specification limits are not on equal distance from the target value. In that case, we are said to be dealing with asymmetric tolerance. The classical PCIs no longer provide a good measure of process capability for asymmetric tolerances. For this reason, there is a need for PCIs that can measure process capability in the case of asymmetric tolerances.

There have been PCIs proposals that were developed to deal with asymmetric tolerance, however they still show shortcomings and it seems that an adequate PCI for asymmetric tolerances is yet to be discovered. Therefore, the aim of this project is to propose new ideas that improve upon existing PCIs. I have formulated a list of five important properties: (i) the PCI must be maximized at the target value, (ii) the PCI must take the direction of shift from the target value into account, (iii) the PCI must have a lower value on the further limit, (iv) the PCI must have non-negative values and (v) the PCI is a generalization of the classical PCIs. Together, the properties ensure that a PCI delivers index values that are consistent with process capability when considering asymmetric tolerance. I will explore some existing PCIs by comparing them to the properties. In order to contribute to development of PCIs for asymmetric tolerances, I will propose four ideas for a new PCI.

The remainder of this report is organized as follows. In section 2, we will take a look at the classical PCIs and how they reflect on yield and expected loss in the case of symmetric tolerances. In section 3, we will show what problems can occur when the classical PCIs are used in the case of asymmetric tolerances. Section 4 is devoted to the list of 5 properties that are desirable to have for a PCI dealing with asymmetric tolerance. Section 5 explores some existing PCIs that are developed to reflect on process capability in the case of asymmetric tolerances and compares them with the desirable properties from section 4. In section 6, some new PCI proposals are given that improve upon existing PCIs by satisfying more properties. Lastly, the report is brought to an end with a conclusion and discussion in section 7 and 8 respectively.

2 The classical process capability indices

2.1 Introduction

The capability of a process can be determined with respect to different criteria. Common criteria are yield and targeting. If it is of importance to the customer that the products are within the specification limits, he prioritizes maximizing the yield. If he wants to distinguish between values within the specification limits where it is important that the products are close to the target value, then he prioritizes maximizing targeting.

In this section, we will take a look at the most commonly used process capability indices. This includes C_p, C_{pk}, C_{pm} and C_{pmk}. We will also see how they are related to process yield and targeting in the case of symmetric tolerances. The relations will be stated directly and the derivations can be found in Wu et al. (2009). Before we state the relations, we will first define process yield and targeting.

2.1.1 Process yield

Process yield has been for some times the most common and standard criterion used in the manufacturing industries for judging process performance (Wu et al. (2009)). The process yield is the percentage of the process distribution that is within the specification limits. Formally defined as

$$\text{Yield} = \int_{LSL}^{USL} 1\, dF(x) = F(USL) - F(LSL)$$

where $F(x)$ is the process cumulative distribution function. Because we are assuming a normal process distribution, $X \sim N(\mu, \sigma^2)$, we can simplify the yield to

$$\begin{aligned}\text{Yield} &= P(X < USL) - P(X < LSL) \\ &= \Phi\left(\frac{USL-\mu}{\sigma}\right) - \Phi\left(\frac{LSL-\mu}{\sigma}\right) \\ &= \Phi\left(\frac{USL-\mu}{\sigma}\right) - 1 + \Phi\left(\frac{\mu-LSL}{\sigma}\right) \end{aligned} \tag{1}$$

where Φ is the cumulative distribution function of the standard normal distribution $N(0, 1)$.

2.1.2 Process targeting

The main disadvantage of yield is that it does not distinguish between values within the specification limits; they are all considered equally good. When we experience loss whenever values deviate from the target, we might also want to judge a process based on the distance of the values from the target. This criteria is called process targeting or process centering. The most common way measure process centering, is by looking at the expected squared loss, which can be defined by

$$\mathbb{E}\left[w(X-T)^2\right] = w\int_{-\infty}^{\infty}(x-T)^2\, dF(x) = w\left(\sigma^2 + (\mu-T)^2\right)$$

where w is a positive constant which can be set to scale the penalty.

A disadvantage of expected loss is the difficulty of setting a standard, as values reach from zero to infinity. In order to solve the problem of unbounded loss, Johnson (1992) defined the worth of the product $W(X) = W_T - w(X-T)^2$ where W_T is the worth of the product when X is exactly on target. He then proposed, for symmetric tolerance, the expected relative loss, defined by

$$L_e = \frac{\mathbb{E}[(X-T)^2]}{D^2}\left(\frac{A_0}{W_T}\right) \tag{2}$$

where A_0 is the loss at either specification limit and D is the distance from T to the specification limits. Note that this definition doesn't hold for asymmetric tolerance, as in that case loss is not equal on the specification limits. The expected relative loss provides a unitless measure of process targeting.

2.2 C_p

The first PCI is the precision index C_p, defined by

$$C_p = \frac{USL - LSL}{6\sigma} \tag{3}$$

We will refer to $USL - LSL$ as the width of the tolerance interval. Note that C_p is simply the ratio between the width of the tolerance interval and the process variability. C_p reflects on the percentage of tolerance interval used by the process, which can be computed by $(1/C_p) \times 100\%$. A higher C_p value means less of the tolerance interval is utilized and we say that the process is more precise. In practise, a value of 1 is the bare minimum and often a value of 1.33, 1.66 or even 2 is required. The process spread of the normal distribution relative to the specification interval is illustrated in figure 1.

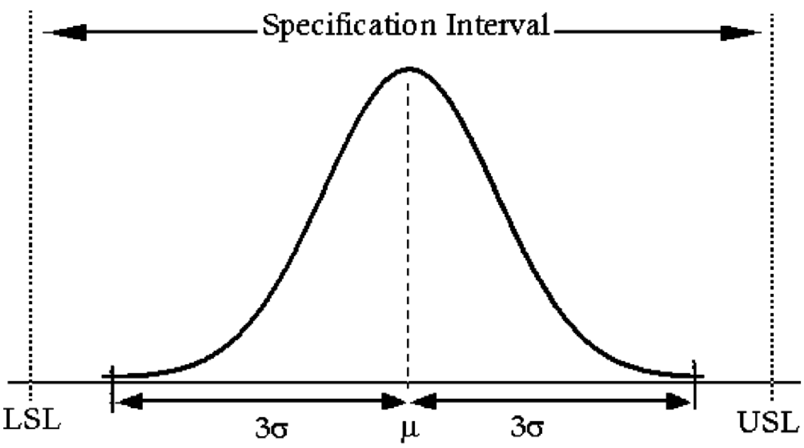


Figure 1: Process spread with specification interval. Note that this process is centered. From Wu et al. (2009).

2.2.1 C_p and process yield

C_p alone gives no indication of process yield, as it does not take the location of the distribution into consideration. However, if we assume the process to be centered ($\mu = M$), then C_p is actually directly related to the process yield. In that case, we can rewrite (1) into

$$\begin{aligned} \text{Yield} &= 2\Phi\left(\frac{USL - \mu}{\sigma}\right) - 1 \\ &= 2\Phi\left(\frac{USL - LSL}{2\sigma}\right) - 1 \\ &= 2\Phi\left(3C_p\right) - 1 \end{aligned}$$

For example, for a C_p value of 1 we have a process yield of 99.73%, or equivalently 0.27% non-conforming units, which is 2700 ppm (parts per million). At C_p of 1.33, we get 66 ppm non-conforming. At 1.66, 0.54 ppm. And with a C_p value of 2, we expect only 2 parts per billion to be non-conforming.

2.2.2 C_p and expected loss

C_p provides no information on process location and therefore cannot be related to loss.

2.3 C_{pk}

C_p does not take the location of the process into account, this can easily be seen by the fact that μ is not considered in the definition. The second index C_{pk} takes both the magnitude of the process variance and the process departure from the midpoint M into consideration. It is defined by

$$C_{pk} = \min\left\{\frac{USL-\mu}{3\sigma}, \frac{\mu - LSL}{3\sigma}\right\} = \frac{D - |\mu - M|}{3\sigma} \tag{4}$$

where $D = (USL - LSL)/2$ is half the width of the tolerance interval. C_{pk} considers the ratio between the distance from the mean to the closest specification limit and the process variability. Figure 1 might help to visualize the behavior of C_{pk}. If $C_{pk} < 0$, then the process mean lies outside of the specification limits. If $0 < C_{pk} < 1$, then the process mean is too close to at least one specification limit. If $C_{pk} \geq 1$, then the process distribution is well within the specification limits and we have a process yield of at least 99.73%. Again, a value of 1 is the minimum in practise, but a higher value can be required. For fixed σ, C_{pk} is maximized by $\mu = M$.

2.3.1 C_{pk} and process yield

As C_{pk} takes the distance from μ to the specification limits into consideration, we can say something about process yield. Actually, we can construct an interval for the process yield. Recall our formulation of yield (1) as

$$\text{Yield} = \Phi\left(\frac{USL-\mu}{\sigma}\right) - 1 + \Phi\left(\frac{\mu - LSL}{\sigma}\right)$$

Now by the definition of C_{pk} and the fact that Φ is an increasing function, we find Yield $\geq 2\Phi\left(3C_{pk}\right) - 1$. This can also be understood by seeing that C_{pk} gives an indication of the yield on the "bad" side of the process (the side which is closer to a specification limit). The yield on the other side is at least as good, therefore we arrive at a lower bound for the yield. We can derive an upper bound by seeing that for fixed C_{pk}, the yield is maximized if the process is completely within the specification limit on the "good" side. The yield would then be $\Phi\left(3C_{pk}\right)$. Note that the normal distribution is infinite, so we could never reach this upper bound. Now the relation between yield and C_{pk} can be summarized in the following inequality

$$2\Phi\left(3C_{pk}\right) - 1 \leq \text{Yield} < \Phi\left(3C_{pk}\right) \tag{5}$$

2.3.2 C_{pk} and expected loss

The expected relative squared loss based on C_{pk} can be expressed as

$$L_{C_{pk}} = \frac{A_0}{W_T}(1 - C_a)^2 + \frac{1}{9}\frac{A_0 C_a^2}{W_T C_{pk}^2} \tag{6}$$

where $C_a = 1 - \frac{|\mu - M|}{D}$ is a measure for centering. Wu et al. (2009) states that $L_{C_{pk}}$ is minimized for $C_a = \frac{9C_{pk}^2}{(1+9C_{pk})^2}$. And $L_{C_{pk}}$ increases if C_a decreases below this value. Note that we need C_a to express the expected relative loss, which is due to the fact that C_{pk} provides little information on the location of μ. This is shows that C_{pk} is not an adequate measure for loss.

2.4 C_{pm}

C_p and C_{pk} are yield based indices. They provide a good indication of capability when yield is the primary criteria. However, they do not take the target value into consideration. The next index, C_{pm} is developed to reflect on the degree of process targeting and is motivated by the idea of squared error loss. It is defined by

$$C_{pm} = \frac{USL - LSL}{6\tau} = \frac{D}{3\tau} \tag{7}$$

where $\tau = \sqrt{\sigma^2 + (\mu - T)^2}$ is a measure of the average deviation from the target value. It combines the variation relative to the process mean (σ^2) and deviation of the process mean from the target $(\mu - T)^2$. Note that $\tau^2 = \mathbb{E}[(X - T)^2]$. So τ^2 equals the expected loss as defined in section 2.1.2. For fixed σ, C_{pm} is maximized by $\mu = T$.

2.4.1 C_{pm} and process yield

The relation between the process yield and C_{pm} is a bit harder to derive than for C_p and C_{pk}. Wu et al. (2009) found that for $T = M$ (symmetric tolerance) and $1 - 1/(3C_{pm}) \leq C_a \leq 1$:

$$\text{Yield} = \Phi\left(\frac{2 - C_a}{\frac{1}{(3C_{pm})^2} - (1 - C_a)^2}\right) + \Phi\left(\frac{C_a}{\frac{1}{(3C_{pm})^2} - (1 - C_a)^2}\right) - 1 \tag{8}$$

where $C_a = 1 - \frac{|\mu - M|}{d}$ is measure of the distance from the mean to the midpoint M. Even though this relation is a bit cumbersome, it does prove that the process yield is explicitly related to C_{pm} and C_a for symmetric tolerances.

It must be emphasized that this one-to-one relation only holds when $T = M$. This can be seen by the fact that process yield is maximized by $\mu = M$ and C_{pm} by $\mu = T$. Thus when $T = M$, both are increasing when μ moves closer to M and both are maximized by $\mu = M$, and therefore a one-to-one relation can be established.

2.4.2 C_{pm} and expected loss

The expected relative loss based on C_{pm} can be expressed as

$$L_{C_{pm}} = \frac{1}{9}\frac{A_0}{W_T C_{pm}^2} \tag{9}$$

Wu et al. (2009) states that C_{pm} has the property of being the so-called a larger-the-better index; larger C_{pm} means less expected relative loss. Note that $L_{C_{pm}}$ is independent of C_a, as oppose to $L_{C_{pk}}$. This shows that C_{pm} is directly related to loss.

2.5 C_{pmk}

The last index, proposed by Choi and Owen (1990), is constructed by combining the yield-based index C_{pk} and the loss-based index C_{pm}. It is defined by

$$C_{pmk} = \min\left\{\frac{USL - \mu}{3\tau}, \frac{\mu - LSL}{3\tau}\right\} = \frac{D - |\mu - M|}{3\tau} \tag{10}$$

When the process mean μ deviates from the target value, the reduction of the value of C_{pmk} is larger than for the other indices. It can also be interesting to note that a requirement "$C_{pk} \geq C$" may not meet the requirement "$C_{pm} \geq C$" and vice versa. However, $C_{pmk} \geq C$ implies both $C_{pk} \geq C$ and $C_{pm} \geq C$, as $C_{pmk} \leq C_{pk}$ and $C_{pmk} \leq C_{pm}$. In general, for fixed μ, C_{pmk} is maximized for μ somewhere between M and T. In the case of symmetric tolerance, C_{pmk} is maximized by $\mu = M$.

2.5.1 C_{pmk} and process yield

In the case that $T = M$ (symmetric tolerance), using the relation found in section 2.4.1 and the relation $C_{pmk} = C_{pm} \cdot C_a$, we can derive

$$\text{Yield} = \Phi\left(\frac{2 - C_a}{\left(\frac{C_a}{3C_{pmk}}\right) - (1 - C_a)^2}\right) + \Phi\left(\frac{C_a}{\left(\frac{C_a}{3C_{pmk}}\right) - (1 - C_a)^2}\right) - 1 \tag{11}$$

Thus we see that the process yield is explicitly related to C_{pmk} and C_a. It must again be emphasized that this only holds for symmetric tolerances with the same reasoning as in section 2.4.1.

2.5.2 C_{pmk} and expected loss

The expected relative loss based on our last index C_{pmk} can be expressed as

$$L_{C_{pmk}} = \frac{1}{9}\frac{A_0 C_a^2}{W_T C_{pmk}^2} \tag{12}$$

Wu et al. (2009) states that for given C_{pmk}, $L_{C_{pmk}}$ increases as C_a increases and reaches it's maximum at $C_a = 1.0$. Wu concluded that C_{pmk} provided the best protection in terms of loss and is therefore preferred over C_{pk} and C_{pm} when measuring squared process loss.

2.6 Classical PCI superstructure

In this section, the focus has been on the differences between the classical PCIs. However, their similarities are easy to see and worth mentioning. First of all, one can notice that each index has a numerator that reflects on the process location and a denominator that reflects on the process variation. Secondly, C_p and C_{pm} only differ by the fact that σ is replaced by τ, and the same holds for C_{pk} and C_{pmk}. Notice that τ^2 is equal to the variation plus a term that accounts for the deviation from the target. So C_{pm} and C_{pmk} extend C_p and C_{pk} respectively by accounting for process targeting.

We will finish this section by stating the superstructure containing all 4 basic indices, presented by Vännman (1997) as

$$C_p(u,v) = \frac{D - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \tag{13}$$

where $u, v \geq 0$. It is easy to see that $C_p(0,0) = C_p$, $C_p(1,0) = C_{pk}$, $C_p(0,1) = C_{pm}$ and $C_p(1,1) = C_{pmk}$.

The u, v parameters give rise to an infinite amount of indices, but we can disregard many possible values. Vännman (1997) noted that not much is gained by using non-integer values for u and v. In order to avoid negative index values, we take $u \in \{0,1\}$. For v we can take any non-negative integer depending on how strongly centering is valued. Thus besides values $u, v \in \{0,1\}$, it is only worth to consider larger values for v if centering is of higher priority.

The values 0 and 1 for u and v are considered standard values as they give back the classical PCIs. In the upcoming sections similar superstructures will be introduced for which the focus will be on u, v values $\{0,1\}$. Also, the same naming convention will be used. Meaning that m in the subscript is equivalent to taking $u = 0$ and $v = 1$ for example. These notations are used interchangeably.

3 Classical PCIs when considering asymmetric tolerance

3.1 Introduction

In practice, the cases of asymmetrical tolerance are not uncommon. In general, asymmetric tolerances reflect the customers view that deviations from the target are more tolerable in one direction than the other. It can also happen that a customer initially had symmetric specifications, but is willing to expand one of the specification limits if it occurs that the limit cannot be hold. An example of this is given in Boyles (1994). Another way of arriving at asymmetric limits is after transforming non-normal process data. The previously symmetric limits then also need to be transformed, after which they most likely become asymmetric.

In practice it is often seen that the classical PCIs are used even for asymmetric tolerances. In this section we take a look at how the classical PCIs can misrepresent the process capability when asymmetric tolerance are considered.

3.2 C_p with asymmetric tolerance

The simple index C_p is nothing more than a precision index. It does not take the target value T into account, in fact it says nothing about the location of the process distribution. If this is kept in mind, then C_p can still be used in the case asymmetric tolerance to get an indication of the precision of the process.

3.3 C_{pk} with asymmetric tolerance

C_{pk} is a yield-based index and only provides a lower bound for the process yield. Even though it takes the midpoint M into account, it fails to account for process centering, even for symmetric tolerances, as not much can be said about the location of the mean μ. This is best illustrated with an exaggerated example; figure 2 shows three processes with widely varying process means that all have $C_{pk} = 1$. This already shows that not much can be said about the location of the process mean. However, for symmetric tolerance we can consider process A and B equally capable, so there is no problem in having the same index value. But when the target is not specified at M, but for example $T = 2.5$ like in figure 2, then we clearly consider process B less capable than process A and the processes having the same index value is problematic.

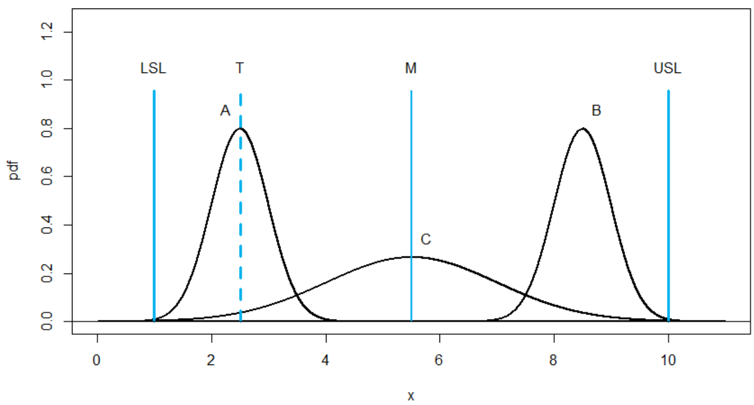


Figure 2: Process A and B have μ equal to 2.5 and 8.5 respectively, and both have $\sigma = 0.5$. Process C has $\mu = 5.5$ and $\sigma = 1.5$. All three processes have $C_{pk} = 1$.

Furthermore, the process yield and also C_{pk}, for fixed σ, are maximized by $\mu = M$, as noted in section 2.3. This is also illustrated in figure 3 where also see the symmetry of C_{pk} around M. While for asymmetric tolerance the customer has set the target value $T \neq M$ and we would like to maximize for $\mu = T$. In the case of asymmetric tolerances, maximizing yield and maximizing targeting around T are conflicting criteria. As C_{pk} only considers yield, it is not the best reflection of capability as specified by the customer.

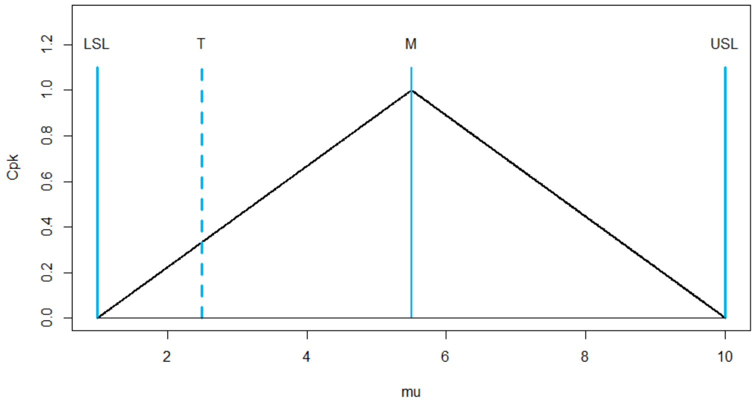


Figure 3: The value of C_{pk} for different μ ranging within the limits. σ is fixed and equal to 1.5.

3.4 C_{pm} with asymmetric tolerance

We've seen that C_p and C_{pk} don't take the target value into account. C_{pm} was developed to also reflect on process targeting by using the idea of squared error loss. C_{pm} has the property that it is maximized for $\mu = T$, which is desirable in the case of asymmetric tolerances. However, a shortcoming of C_{pm} is the fact that it does not account for the direction of the deviation from the target. This is due to the fact that the squared loss is symmetric around the target T. While for asymmetric tolerance it is clearly less desirable for the mean to deviate in the direction of the closer specification limit, as in that case yield decreases more rapidly. An example by Boyles (1994) is illustrated in figure 4. Processes A and B both have a C_{pm} index of 1.2, but the yield of process B is lower than that of process A.

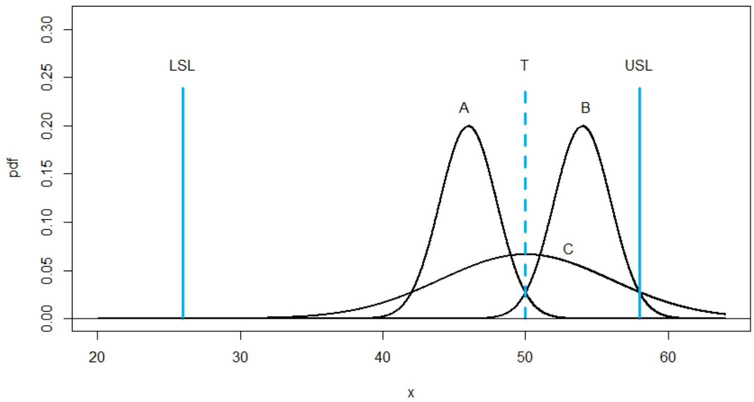


Figure 4: Processes A and B both have $C_{pm} = 1.2$. Process C has $C_{pm} = 1$, but expected percentage non-conforming of 6.7%.

Another problem is that for an on-target process, so where $\mu = T$, C_{pm} actually reduces to C_p. In section 2.2.1, it is already noticed that C_p can only evaluate the yield for centered processes. But for asymmetric tolerance, an on-target process is clearly not centered and thus nothing can be said about yield. This is also illustrated in figure 4 by process C, which has a C_{pm} value of 1, but expected percentage non-conforming of 6.7%. (Conventionally, an index value of 1 is associated with at most 0.27% non-conforming.)

Even though C_{pm} accounts for targeting, it considers losses on either side of T equally, creating a symmetric index around T. The value of C_{pm} for different μ is illustrated in figure 5 which shows the symmetry of C_{pm} around T.

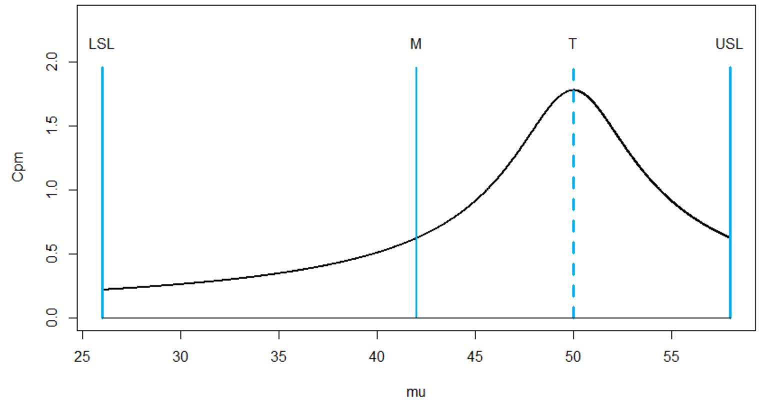


Figure 5: The value of C_{pm} for different μ ranging within the limits. σ is fixed and equal to 3.

3.5 C_{pmk} with asymmetric tolerance

The main problem is that C_{pmk} is maximized for μ somewhere between M and T (for fixed σ), as noted in section 2.5. This is also illustrated in figure 6. When considering asymmetric tolerance, the customer clearly values targeting and we would therefore like a PCI to be maximized by $\mu = T$.

In figure 6 we also see that C_{pmk} decreases more steeply towards the closer limit than to the further limit, which is desirable for asymmetric tolerances. However, when the process mean is exactly on either specification limit, then in both cases $C_{pmk} = 0$. We want a process on the further limit to have a lower index value than a process on the closer limit, because even though yield is equal, the process mean on the further limit is a lot further from the target value. This problem can also be seen from figure 6, where we see the value go to 0 for both limits.

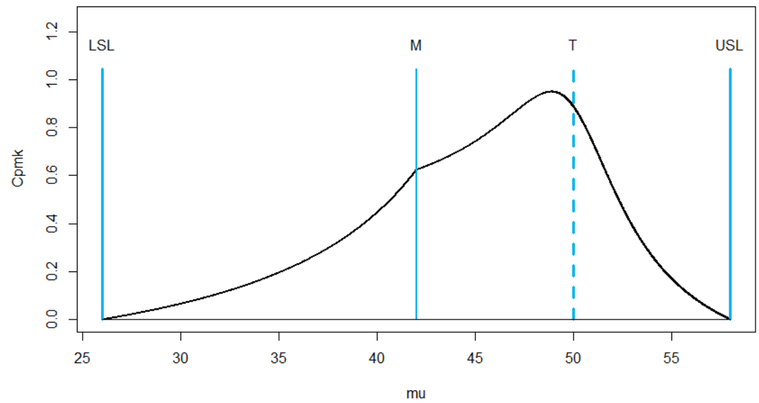


Figure 6: The value of C_{pmk} for different μ ranging within the limits. σ is fixed and equal to 3.

Other problems for the last index C_{pmk} in the case of asymmetric tolerance are less significant than the other PCIs and a bit harder to find. It cannot overstate the process yield due to the inequality $C_{pmk} \leq C_{pk}$ found in section 2.5. Process C from figure 4, which has a bad yield, has $C_{pmk} < 1$. Furthermore, it gives process B a lower index value than process A. However, this is due to the numerator of C_{pmk} and one could argue that this accounts for process yield. The component accounting for targeting, τ, is equal for process A and B. For asymmetric tolerance, we would like to distinguish between the directions of deviations from the target.

4 Desirable properties of a PCI when considering asymmetric tolerance

In this section, a list of desirable properties of a PCI when considering asymmetric tolerance is formulated. Some of which are found shortcomings in the previous section and others were found later when researching other PCIs. The list is (roughly) in order of importance.

(i) The PCI is maximized at $\mu = T$.

(ii) The PCI accounts for the direction of the deviation from the target: the index value must decrease more steeply when μ shifts from T towards the closer specification limit.

(iii) The PCI value must be lower on the further specification limit than on the closer specification limit.

(iv) The PCI value is always greater or equal to 0 within the specification limits.

(v) The PCI is a generalization of the classical PCIs. Meaning that when tolerances are symmetric, the PCI reduces to a classical PCI.

Properties (i) - (iii) make sure a PCI gives values that are consistent with capability. The target value is by definition the most optimal value, therefore we clearly want the index value to be maximized for T. If the process mean is to shift from the target value, then a shift towards the closer limit will result in a higher loss of yield than when it would shift towards the further limit. In other words, the capability decreases more quickly. For this reason, a PCI value must decrease more quickly when the process mean shifts towards the closer specification limit. If the process mean were to actually reach a specification limit, then process yield is 50% for both limits. However, the process mean is further from the target value on the further limit and thus less capable. Therefore, in order to keep the index value consistent, it must have a lower value on the further specification limit. We can see that properties (i)-(iii) together make sure that a PCI gives values that are consistent with process capability.

Property (iv) makes a PCI interpretable. PCIs are designed such that a process can be considered capable if the index value is at least 1. It is important to also agree upon when a process is incapable. The classical PCIs give a value of 0 when the process mean is on one of the specification limits, which is a good indication of the worst case scenario. If we would allow for negative values, then we lose this indication and it becomes difficult to interpret how incapable a process is. Besides the fact that negative values are impractical, they are also unconventional. An index value can be seen as a grade or rating, and it is unusual to give something a negative grade. Thus property (iv) makes a PCI interpretable and usable in practice.

The last property does not make the index better at indicating capability, but it is an important property if we want a PCI to be widely applicable. With property (v), a PCI can be used for both types of tolerances and is more likely to be used in practice.

5 Existing PCIs for asymmetric tolerances

In this section, we will look at the different approaches to deal with asymmetric tolerance and PCIs designed to reflect process capability for asymmetric tolerance. The indices are introduced and compared to the desirable properties from section 4. The indices are in chronological order by the date of the paper they are proposed in.

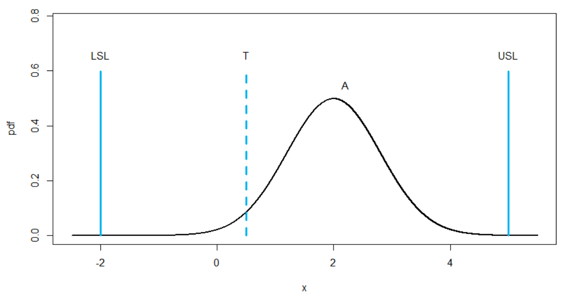
5.1 C_p^*

Some of the first approaches to dealing with asymmetric tolerance, were adjusting the asymmetric specification limits to symmetric ones and then use the classical PCIs on the new specification limits. We can denote the specification limits for asymmetric tolerance by $(T - D_l, T + D_u)$, where $D_l = T - LSL$ and $D_u = USL - T$.

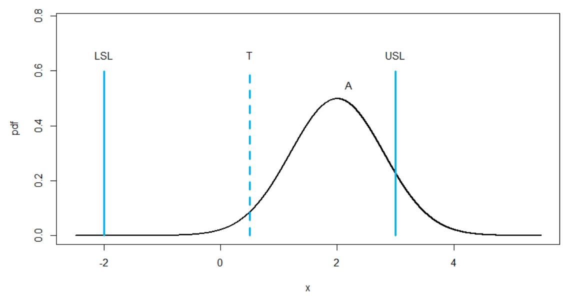
One way to adjust the specification limits, proposed by Kane (1986), is by replacing $(T - D_l, T + D_u)$ by $(T - d^*, T + d^*)$, where $d^* = \min\{D_l, D_u\}$. We can rewrite the superstructure 13 from section 2.6 as

$$C_p^*(u,v) = \frac{d^* - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \tag{14}$$

from which we can extract a new class of indices C_p^*, C_{pk}^*, C_{pm}^* and C_{pmk}^* in the same way as 2.6. With this approach, we can understate the process capability, as we are restricting to a proper subset of the actual specification limits. In figure 7 we can see the problem of understating process capability; process A (with $\mu = 2$ and $\sigma = 0.8$) is off target, but has a good yield and a C_{pk} value of 1.25. However, when adjusting the specification limits, process A has a C_{pk}^* value of 0.42.



(a) Process A with LSL = -2, USL = 5 and T = 0.5.



(b) Process A with adjusted specification limits using d^*. USL is 3 in this case.

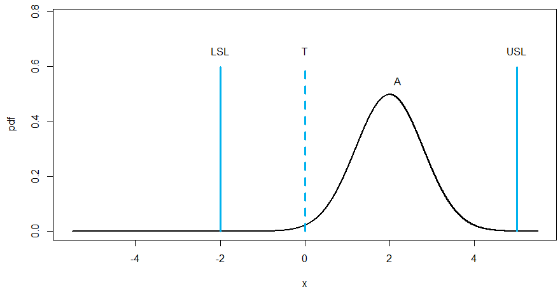
Figure 7: Illustrative example of C_p^*.

5.2 C_p'

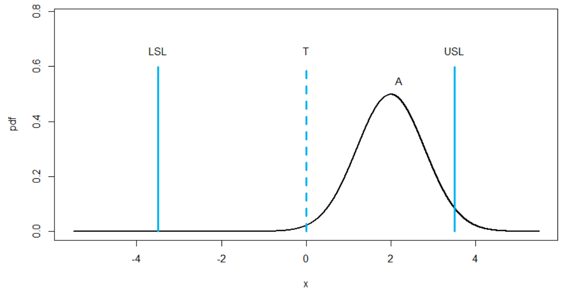
The second way to adjust the specification limits, proposed by Kane (1986), is by replacing $(T - D_l, T + D_u)$ by $(T - d', T + d')$, where $d' = (D_l + D_u)/2$. In this case, we can rewrite the superstructure 13 as

$$C_p'(u,v) = \frac{d' - u|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \tag{15}$$

from which we can extract a new class of indices C_p', C_{pk}', C_{pm}' and C_{pmk}' in the same way as 2.6. With this approach, we can either understate or overstate the process capability, depending on the location of μ relative to T. Figure 8 shows the problem of understating process yield with C_p'; process A has a C_{pk} value of 1.25 and a C_{pk}' value of 0.625 Figure 9 shows the problem of overstating process yield with C_p'; process A has a C_{pk} value of 0.708 and a C_{pk}' value of 1.33.

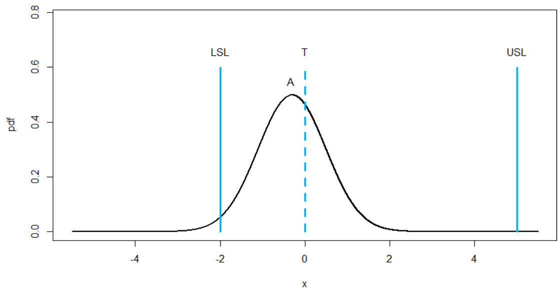


(a) Process A with LSL = -2, USL = 5 and T = 0,5.

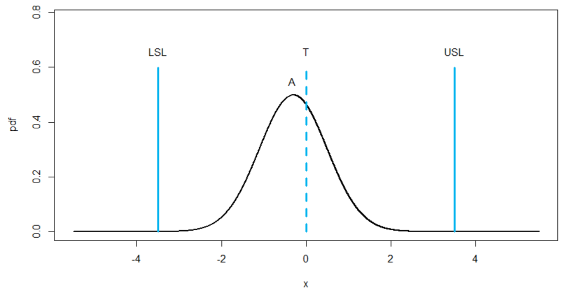


(b) Process A with adjusted specification limits using d'.

Figure 8: Illustrative example of understating process capability using C'_p.



(a) Process A with LSL = -2, USL = 5 and T = 0.



(b) Process A with adjusted specification limits using d'.

Figure 9: Illustrative example of overstating process capability using C'_p.

We see that the main problem for with C^*_p and C'_p is the fact that these indices fail to reflect on yield, which is understandable from the fact that we adjust the specification limits. Another problem is that deviations of μ from T are evaluated without considering direction. This is because C^*_p and C'_p are classical indices on the new specification limits and we've seen in the previous section that the classical PCIs do not take direction into account.

There is one advantage of using C^*_p and C'_p. Namely, they are maximized for $\mu = T$, or equivalently $\mu = M$, as $T = M$.

5.3 S_{pmk}

Boyles (1994) introduced a smooth function

$$S(x, y) = \frac{\Phi^{-1}\left(\frac{\Phi(x)+\Phi(y)}{2}\right)}{3} \tag{16}$$

and proposed a smooth generalization of C_{pmk} defined by

$$S_{pmk} = S\left(\frac{USL - \mu}{\tau}, \frac{\mu - LSL}{\tau}\right) \tag{17}$$

where $\tau = \sqrt{\sigma^2 + (\mu - T)^2}$. This index is proposed as an index for asymmetrical tolerances, because at this time C_{pmk} was considered a PCI for asymmetrical tolerances. Boyles states that C_{pmk} should be viewed as an approximation to S_{pmk}. In order to illustrate the value of S_{pmk} for different μ, an example process is used to plot some values in figure 10.

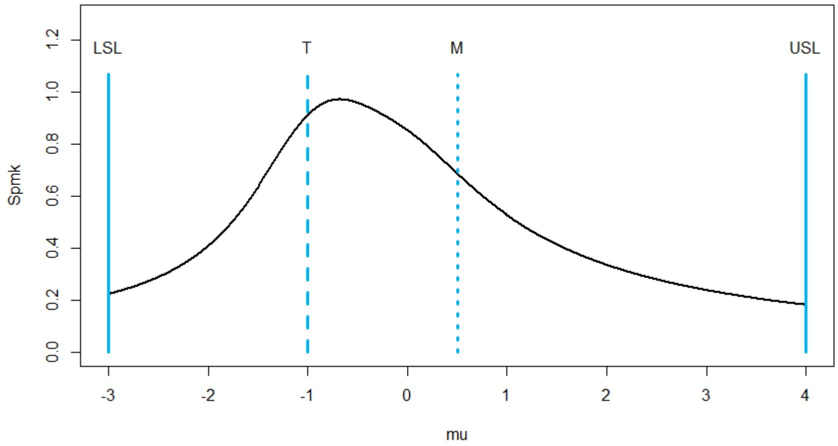


Figure 10: S_{pmk} values for different μ with $\sigma = 0.8$.

Properties of S_{pmk}:

- S_{pmk} is maximized for μ somewhere between T and M, thus property (i) does not hold.
- From figure 10, we see that S_{pmk} decreases more steeply when μ shifts towards the closer limit, thus property (ii) holds.
- In general if μ on one of the specification limits then either x or y is 0 and the other is $\frac{2d}{\tau}$. τ will be larger when μ is on the further limit. Thus we see that property (iii) holds in general. Also, in figure 10 we have $Spmk = 0.22$ on LSL and 0.18 on USL.
- Boyles states that $S_{pmk} \geq C_{pmk}$. We have that $C_{pmk} \geq 0$, as $D \geq |\mu - M|$. So we have that $S_{pmk} \geq 0$ and we conclude that property (iv) holds.
- S_{pmk} generalizes C_{pmk} in being a smooth version, however it does not reduce to C_{pmk} for symmetrical tolerances. Thus property (v) does not hold.

As concluded in section 3, C_{pmk} has shortcomings when considering asymmetric tolerance. S_{pmk} is nothing more than a smooth generalization of C_{pmk}. We actually see the same shortcomings, the most clear of which is the fact that S_{pmk} is not maximized by $\mu = T$, which we can also see in figure 10. S_{pmk} is therefore not a good representation of process capability when considering asymmetric tolerance.

5.4 $C^{\star}_{pm}$

We can define relative squared loss by

$$\text{Symmetric loss} = \frac{(X-T)^2}{D^2} \tag{18}$$

and when taking the expectation, we get expected relative loss defined as $\frac{\sigma^2+(\mu-T)^2}{D^2}$, which is a special case of the expected relative loss defined in section 2.1.2. Note that this is exactly what we find in the definition of C_{pm}. Recall that $C_{pm} = \frac{D}{3\sqrt{\sigma^2+(\mu-T)^2}}$. If we denote the expected relative loss by λ', then $C_{pm} = \frac{1}{3\sqrt{\lambda'}}$.

A problem with this definition of relative loss is that it does not take the direction of the deviation of μ from T into account. This is also a shortcoming of C_{pm} mentioned in section 3. This problem can be solved by using a piecewise loss function

$$\text{Asymmetric loss} = \begin{cases} \frac{A_l(X-T)^2}{D_l^2} & \text{if} \quad X \leq T \\ \frac{A_u(X-T)^2}{D_u^2} & \text{if} \quad X > T \end{cases} \tag{19}$$

where A_l and A_u represent monetary losses when X is at the respective specification limit. Notice that we are just taking relative loss on each side of T separately. Now by taking the expectation of this new loss function, Boyles (1994) derives a new form of expected relative loss, denoted by λ, as

$$\lambda = \sigma^2 \left(\frac{A_l h(\zeta)}{D_l^2} + \frac{A_u h(-\zeta)}{D_u^2} \right) \tag{20}$$

with

$$h(\zeta) = (1+\zeta^2)\Phi(\zeta) + \zeta\phi(\zeta) \tag{21}$$
$$h(-\zeta) = (1+\zeta^2)(1-\Phi(\zeta)) - \zeta\phi(\zeta) \tag{22}$$

where $\zeta = \frac{T-\mu}{\sigma}$. By taking $A_l = A_u = 1$, we define

$$C^\star_{pm} = \frac{1}{3\sqrt{\lambda}} \tag{23}$$

The definition of λ seems complicated, but it is only the result of the piecewise loss function. It is easy to see that in the case of symmetric tolerances, the asymmetric loss function (19) reduces to the symmetric loss function (18). Which results in $\lambda' = \lambda$ and in turn $C^\star_{pm}$ reduces to C_{pm}. So we see that $C^\star_{pm}$ is a generalization of C_{pm}.

Figure 11 illustrates the values of $C^\star_{pm}$ for different values of μ of a process with $\sigma = 0.8$. First of all, it noticeable that the values are relatively high with respect to the other indices we've discussed; between T and M we see that the index value is above 1, while for the other indices the values was always below 1. Secondly, it is clear to see that $C^\star_{pm}$ is not maximized for $\mu = T$. This is different from C_{pm}, which actually was maximized at T. This is due to the use of asymmetric loss, where loss "costs" more on the side of the closer specification limit.

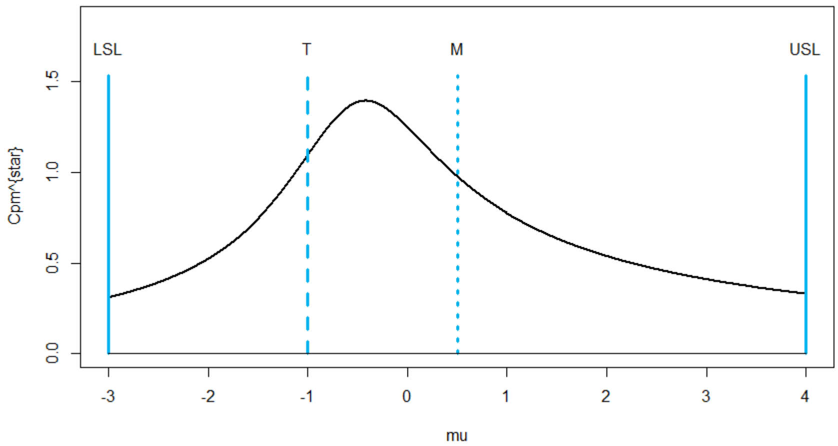


Figure 11: $C^\star_{pm}$ values for different μ.

Properties of $C^\star_{pm}$:

- From figure 11 it clear to see that $C^\star_{pm}$ is not maximized for $\mu = T$. Thus property (i) does not hold.
- By using an asymmetric loss function, $C^\star_{pm}$ takes direction of process shift into account. In figure 11 we also see that $C^\star_{pm}$ decreases more steeply towards the closer limit. Thus property (ii) holds.
- In the example from figure 11 we have $C^\star_{pm} = 0.31$ for μ on LSL and $C^\star_{pm} = 0.33$ for μ on USL. Thus property (iii) does not hold.
- As the expected loss λ is always greater or equal to 0, we have that $C^\star_{pm} \geq 0$. Thus property (iv) holds.

- As mentioned in this section, $C^{\star}_{pm}$ reduces to C_{pm} for symmetrical tolerances, so property (v) holds.

5.5 C^{+}_{pm}

Boyles (1994) stated that $C^{\star}_{pm}$ lacks calibration with process yield. He derives a new index of the same form as $C^{\star}_{pm}$, where he takes $A_l = A_u = A$ such that the index is equal to C_{pk} when $\mu = T$. He finds

$$A(r) = \frac{2}{1 + \min(r^2, r^{-2})}$$

where $r = D_l/D_u$. Substituting this A into λ in equation (20), we get a new index denoted by C^{+}_{pm}. Note that $A \geq 1$, with equality when $r = 1$, which is only the case for symmetric tolerance. We see that in the case of symmetric tolerance $C^{+}_{pm} = C^{\star}_{pm} = C_{pm}$.

Figure 12 illustrates the values of C^{+}_{pm} for different values of μ of a process with $\sigma = 0.8$. It looks exactly the same as figure 11, but only the index values are scaled down. This is due to the relation $C^{+}_{pm} = \frac{C^{\star}_{pm}}{\sqrt{A(r)}}$. We see that C^{+}_{pm} shares the properties of $C^{\star}_{pm}$, except the scale of the values. Boyles concludes that C^{+}_{pm} provides better protection with respect to process yield than $C^{\star}_{pm}$.

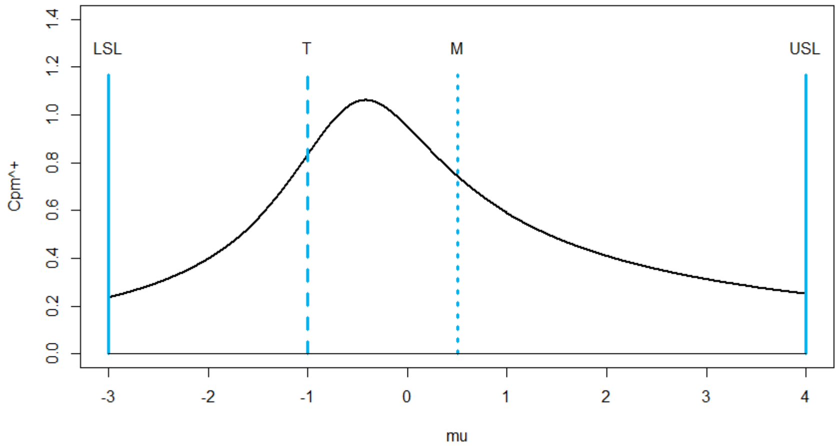


Figure 12: $C^{\star}_{pm}$ values for different μ.

Properties of C^{+}_{pm}:

- From figure 12 it clear to see that C^{+}_{pm} is not maximized for $\mu = T$. Thus property (i) does not hold.
- By using an asymmetric loss function, C^{+}_{pm} takes direction of process shift into account. In figure 12 we also see that C^{+}_{pm} decreases more steeply towards the closer limit. Thus property (ii) holds.
- In the example from figure 11 we have $C^{\star}_{pm} = 0.24$ for μ on LSL and $C^{\star}_{pm} = 0.25$ for μ on USL. Thus property (iii) does not hold.
- As the expected loss λ with the new $A(r)$ is still always greater or equal to 0, we have that $C^{\star}_{pm} \geq 0$. Thus property (iv) holds.
- As mentioned in this section, $C^{\star}_{pm} = C^{+}_{pm} = C_{pm}$ for symmetrical tolerances, so property (v) holds.

5.6 $C_{pa}(1, 1)$

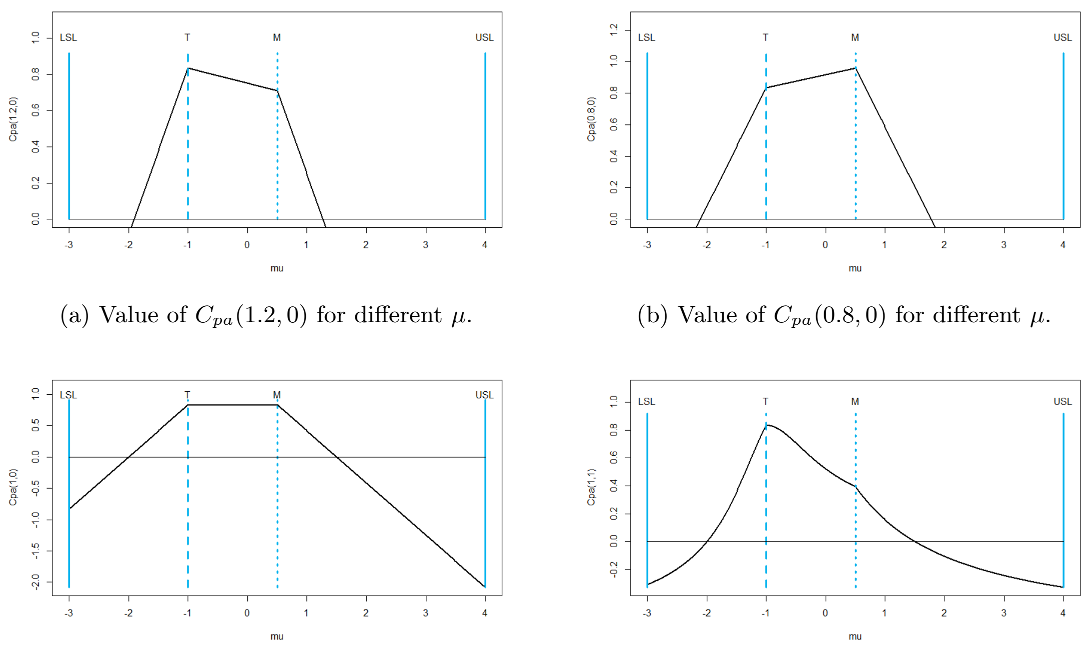
Vännman (1997) proposed a superstructure for asymmetrical tolerance defined by

$$C_{pa}(u, v) = \frac{d - |\mu - M| - u|\mu - T|}{3\sqrt{\sigma^2 + v(\mu - T)^2}} \tag{24}$$

When using standard values $u, v \in \{0, 1\}$, we get the following indices

$$\begin{aligned}
C_{pa}(0,0) &= \frac{d - |\mu - M|}{3\sigma} = C_{pk} \\
C_{pa}(1,0) &= \frac{d - |\mu - M| - |\mu - T|}{3\sqrt{\sigma^2}} \\
C_{pa}(0,1) &= \frac{d - |\mu - M|}{3\sqrt{\sigma^2 + (\mu - T)^2}} = C_{pmk} \\
C_{pa}(1,1) &= \frac{d - |\mu - M| - |\mu - T|}{3\sqrt{\sigma^2 + (\mu - T)^2}}
\end{aligned}$$

Note that this structure is similar to the superstructure of the classical PCIs (section 2.6). We can even see that for $u, v = 0$ we have $C_{pa}(0,0) = C_{pk}$. Also, for $u = 0$ we have $C_{pa}(0,1) = C_{pmk}$. Both of which we concluded not to be suitable for asymmetric tolerance. Moreover, if we want C_{pa} to be maximized at $\mu = T$, we need $|\mu - T|$ to weigh more than $|\mu - M|$, from which we conclude that we can only consider values of $u \geq 1$ in the case of asymmetric tolerances. This is illustrated in figure 13a and 13b. The variable v accounts for targeting, so in the case of asymmetric tolerances it would be reasonable to take $v > 0$. When $v = 0$ we also see that C_{pa} is not maximized for only $\mu = T$, which is illustrated in figure 13c. What value you use for v depends on how much you value targeting. Using $v = 1$, which is a reasonable standard value, figure 13d illustrates the value of C_{pa}.



(a) Value of $C_{pa}(1.2, 0)$ for different μ.

(b) Value of $C_{pa}(0.8, 0)$ for different μ.

(c) Value of $C_{pa}(1, 0)$ for different μ.

(d) Value of $C_{pa}(1, 1)$ for different μ.

Figure 13: Example process with $\sigma = 0.8$ and μ ranging between the specification limits.

In order to see what properties hold for C_{pa}, we consider $C_{pa}(1,1)$, as $u, v = 1$ can be considered standard values and are the only suitable values we've discussed when considering asymmetric tolerance. We concluded that C_{pa} is maximized by $\mu = T$ for $u \geq 1$, so property (i) holds. From figure 13d we can see that C_{pa} decreases more steeply when μ shifts towards the closer limit, so property (ii) holds. $C_{pa}(1,1) = -0.31$ for $\mu = LSL$ and $C_{pa}(1,1) = -0.33$ for $\mu = USL$ thus property (iii) holds. This is for $T = -1$, but when taking T anywhere between LSL and USL the property still holds. Property (iv) does not hold, as we can clearly see values below 0 in figure 13d. When tolerances are symmetric we can write $C_{pa}(u,v) = \frac{d - u'|\mu - M|}{3\sqrt{\sigma^2 + v(\mu - M)^2}}$, where $u' = 1 + u$. This is the classical superstructure for symmetrical tolerances, but with a modified u. This makes the index react stronger to decrease in yield and therefore the properties discussed in section 2 do not hold anymore. Thus property (v) does not hold.

5.7 C''_{pk}, C''_{pm} and C''_{pmk}

Chen and Pearn (2001) introduced a new superstructure

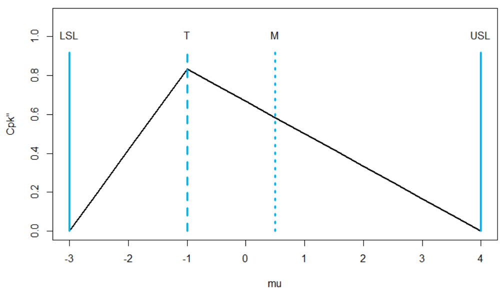
$$C''_p(u,v) = \frac{d^* - uF^*}{3\sqrt{\sigma^2 + vF^2}} \tag{25}$$

where

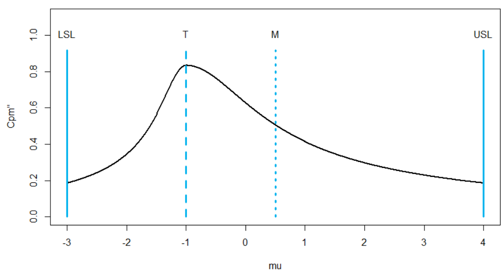
$$F = \max\left\{\frac{d(\mu - T)}{D_u}, \frac{d(T-\mu)}{D_l}\right\}$$
$$F^* = \max\left\{\frac{d^*(\mu - T)}{D_u}, \frac{d^*(T-\mu)}{D_l}\right\}$$

Recall that $d^* = \min\{D_l, D_u\}$. This superstructure yields a class of indices C''_p, C''_{pk}, C''_{pm} and C''_{pmk}. C''_{pk} was first proposed by Pearn and Chen (1998) as a generalization of C_{pk}. Notice that this new superstructure is C^*_p with $|\mu - M|$ replaced by F^* and $\mu - T$ replaced by F. This term accounts for taking direction of μ shift into account. It takes the distance of the shift relative to the distance to the limit, therefore shifts towards the closer limit weighs more than in the direction of the further limit.

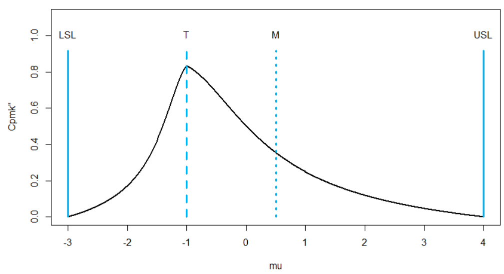
The value of three indices are illustrated in figure 14 for different μ. C''_p is left out, as it is equal to C^*_p from section 5.1. Note that this is not a process distribution, but a plot of index values for a range of values for μ given fixed σ.



(a) Value of $C''_{pk}(= C''_p(1,0))$ for different μ.



(b) Value of $C''_{pm}(= C''_p(0,1))$ for different μ.



(c) Value of $C''_{pmk}(= C''_p(1,1))$ for different μ.

Figure 14: Example process with $\sigma = 0.8$ and μ ranging between the specification limits.

Properties of C''_{pk}, C''_{pm} and C''_{pmk} indices (C''_p is not considered as it is equal to C^*_p from section 5.1):

- Note that $F, F^* \geq 0$, then the indices are maximized when F and F^* are minimized. Thus these indices obtain the maximum value at $\mu = T$ and property (i) holds.

- C''_p decreases more steeply when μ shifts towards a closer specification limit from T, which is caused by the terms F and F^* as they take deviations from T relative to the distance from T to the specification limit. This is also illustrated in figure 14. Thus property (ii) holds.
- If two processes with $\mu_A > T$ and $\mu_B < T$ satisfy $(\mu_A - T)/D_u = (T - \mu_B)/D_l$, then the processes have the same index value. So if $\mu_A = USL$ and $\mu_B = LSL$, then the index value is equal and thus property (iii) does not hold.
- If the process mean is on one the specification limits, then $F^* = d^*$ and $C''_{pk} = C''_{pmk} = 0$, and the value is greater than 0 within the limits. Also, $C_{pm} \geq 0$, as it only consists of positive terms. These relations are also illustrated in figure 14. Thus property (iv) holds.
- When tolerances are symmetric $(T = M)$, then $C''_p(u, v) = C_p(u, v)$, the classical PCI superstructure from section 2.6. As in that case $d^* = d = D_u = D_l$ and $\max\{\mu - M, M - \mu\} = |\mu - M|$. Thus property (v) holds.

5.8 C_p^{**}

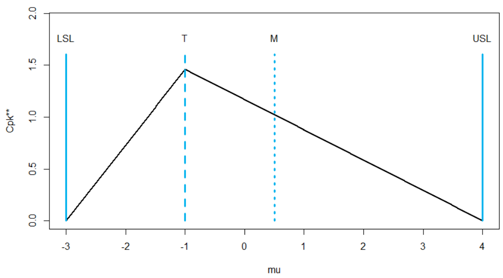
Grau (2005) comes up with two superstructures. The first of which is exactly C_p^*. The second one, denoted by C_p^{**} is quite similar and defined by

$$C_p^{**}(u, v) = \frac{d - uF}{3\sqrt{\sigma^2 + vF^2}} \tag{26}$$

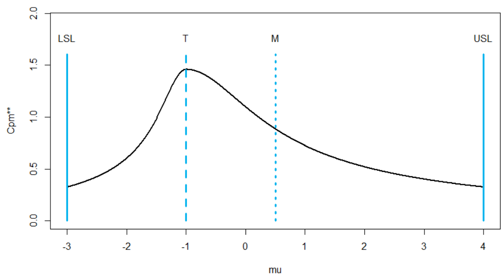
where

$$F = \max\left\{\frac{d(\mu - T)}{D_u}, \frac{d(T - \mu)}{D_l}\right\}$$

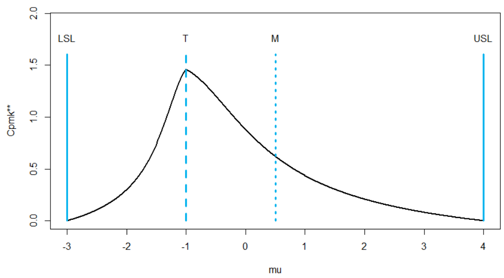
From the definition it is clear that C_p^{**} has the same structure as C''_p, the only difference being that d is not replaced by d^* in the numerator. Using d^* puts a restriction on the allowed variation to better be able to guarantee yield. In that sense, C_p^{**} is less strict and will produce higher index values. This is exactly what we see when we compare figure 15c, which shows values of C_p^{**} for different μ, with figure 14c. They have exactly the same structure but the index value scale is larger for C_p^{**}.



(a) Value of $C_{pk}^{**} (= C_p^{**}(1, 0))$ for different μ.



(b) Value of $C_{pm}^{**} (= C_p^{**}(0, 1))$ for different μ.



(c) Value of $C_{pmk}^{**} (= C_p^{**}(1, 1))$ for different μ.

Figure 15: Example process with $\sigma = 0.8$ and μ ranging between the specification limits.

From the definition, we can see that C_p^{**} is maximized for $\mu = T$, as $F = 0$ in that case. This can also be seen in figure 15. Thus property (i) holds. F accounts for direction of process shift as it takes the distance from T relative to the side of the shift. This is also visible in figure 15. Thus property (ii) holds. If μ is on either specification limit, then $F = d$ and $C_p^{**} = 0$. Thus $C_p^{**} \geq 0$ for μ within the specification limits and property (iv) holds. However, as $C_p^{**} = 0$ on both specification limits, property (iii) does not hold. When tolerances are symmetric $C_p^{**}(u,v) = C_p(u,v)$, so property (v) holds.

5.9 C_a''

Wu et al. (2010) proposed a generalization of $C_a = 1 - \frac{|\mu - M|}{D}$, an accuracy index for symmetrical tolerances. C_a is shortly mentioned in section 2.3.2, but not discussed in detail. The generalization C_a'' is defined by Wu as

$$C_a = 1 - \frac{F^*}{d^*} \tag{27}$$

where

$$F^* = \max\left\{\frac{d^*(\mu - T)}{D_u}, \frac{d^*(T-\mu)}{D_l}\right\}$$

A rather overcomplicated definition as d^* can be cancelled out. It is easy to see that what we have left is 1 substracted by the devation of μ from T relative to the distance of T to the specification limit. Thus C_a'' is a linear function going from 1 on T to 0 on the specification limit. The value of C_a'' for different μ is illustrated in figure 16. Note that we see the exact same structure as for C_{pk}'' in figure 14a, only now the maximal value is 1. This can be seen by the fact that we can rewrite $C_{pk}'' = \frac{d^*}{3\sigma} C_a^{**}$. In that sense, C_a'' does not provide a new way of calculating process capability.

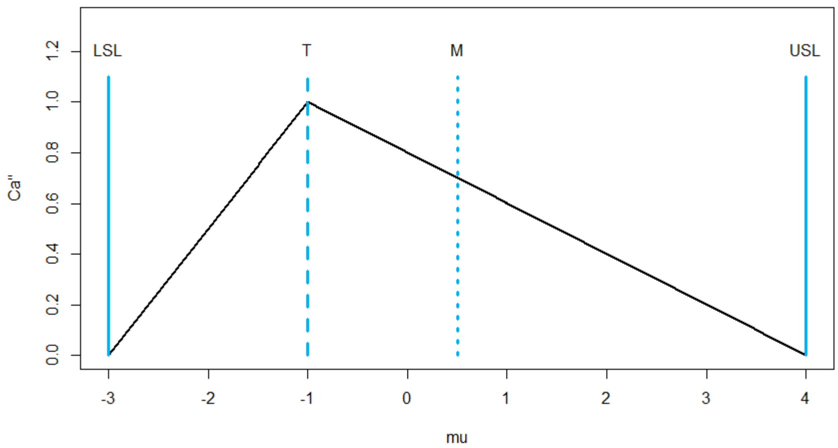


Figure 16: C_a'' values for different μ.

C_a'' is clearly maximized for $\mu = T$, so property (i) holds. From figure 16, we see that property (ii) holds. Property (iii) does not hold, as C_a'' is 0 on both specification limits. As $C_a'' \geq 0$, property (iv) holds. C_a'' reduces to C_a for symmetric tolerances and therefore property (v) holds, even though C_a is not mentioned as a classical PCI in section 2.

5.10 C_{pk}''', C_{pm}''' and C_{pmk}'''

Ganji and Gildeh (2016) noted a problem for $C_p''(u,v)$: if two processes A and B with the same variation σ where $\mu_A = USL$ and $\mu_B = LSL$, then they both have $C_{pk}'' = C_{pmk}'' = 0$ independent of the location of the target T. This can be seen by the fact that $F^* = d^*$ when μ is on one of the specification limits, which results in $C''_p(u, v) = 0$ when $u > 0$. It is also illustrated with an example in figure 14a and 14c where we see that the C''_p value is 0 on the limits. This is undesirable, because in the case of asymmetric tolerance we would like the process further from the target to have a lower index value.

Ganji solved this problem by proposing a new capability index $C'''_p(u, v)$, which improves $C''_p(u, v)$ by replacing F^* with A^* and F with A. It is defined as

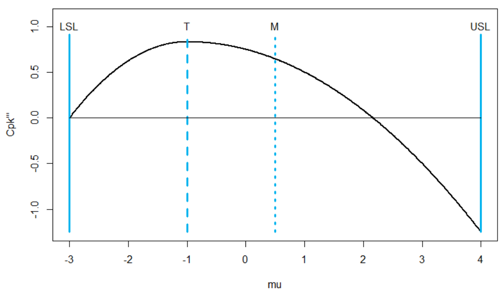
$$C'''_p(u, v) = \frac{d^* - uA^*}{3\sqrt{\sigma^2 + vA^2}} \tag{28}$$

where

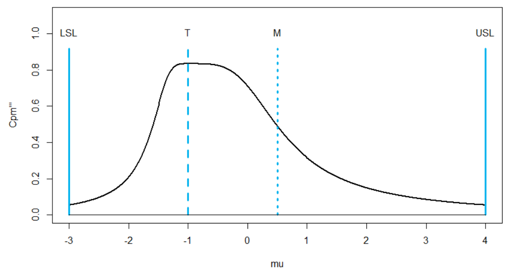
$$A^* = \begin{cases} \frac{(\mu - T)^2}{D_u} & \text{if} \quad \mu > T \\ \frac{(T-\mu)^2}{D_l} & \text{if} \quad \mu \leq T \end{cases}$$

$$A = \begin{cases} \frac{d(\mu - T)}{D_u} & \text{if} \quad \mu > T \\ \frac{d(T-\mu)}{D_l} & \text{if} \quad \mu \leq T \end{cases}$$

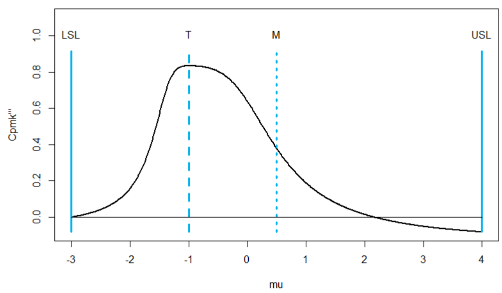
from which we can extract a new class of indices C'''_p, C'''_{pk}, C'''_{pm} and C'''_{pmk}. The value of three indices are illustrated in figure 17 for different μ. C'''_p is left out, as it is equal to C^*_p from section 5.1. We can see that the index value can be negative and is lower on the further specification limit.



(a) Value of $C'''_{pk}(= C'''_p(1, 0))$ for different μ.



(b) Value of $C'''_{pm}(= C'''_p(0, 1))$ for different μ.



(c) Value of $C'''_{pmk}(= C'''_p(1, 1))$ for different μ.

Figure 17: Example process with $\sigma = 0.8$ and μ ranging between the specification limits.

Properties of C'''_{pk}, C'''_{pm} and C'''_{pmk}:

- These indices obtain the maximum value at $\mu = T$, which is illustrated in figure 17. Thus property (i) holds.
- C'''_p decreases more steeply when μ shifts towards a closer specification limit from T, which is caused by the terms A and A^*, as they take deviation from T relative to the distance from T to the specification limit. Thus property (ii) holds.

- If the process mean is on the specification limits closest to T, then $C''_{pk} = 0$ and $C''_{pmk} = 0$. If μ is on the further specification limit, then $C'''_{pk}, C'''_{pmk} \leq 0$. This is because $A^* = D_u$ when $\mu = USL$ and $A^* = D_l$ when $\mu = LSL$. Recall that $d^* = \min\{D_u, D_l\}$, then we see that the numerator is 0 for the closer limit and has a negative value for the further limit. Thus property (iii) holds for C'''_{pk} and C'''_{pmk}. However, for C'''_{pm} we have $u = 0$, so A^* is not considered. We have that $A = d^2$ when μ is on either limit, thus C'''_{pm} has the same value when μ is on either limit. Thus property (iii) does not hold for C'''_{pm}.
- For the previous property we've seen that C'''_{pk} and C'''_{pmk} can get negative values, which is also illustrated in figure 17a and 17c. Thus property (iv) does not hold for C'''_{pk} and C'''_{pmk}. For C'''_{pm} we have $u = 0$, so it cannot take negative values. Thus property (iv) holds for C'''_{pm}.
- When tolerances are symmetric ($T = M$), then A and A^* do not reduce to $|\mu - M|$, so we do not get back the classical PCI superstructure from section 2.6. Thus property (v) does not hold.

$C'''_p(u, v)$ has another good property which is not included in section 4. Namely if the process mean coincides the target value and $C'''_p(u, v) = 1$ for all values of u and v, then we have at most 2700 ppm nonconforming. Equivalently, the yield is at least 99.73%. Furthermore, for $C'''_p(u, v) = 1.33$, this is only 66 ppm. And for $C'''_p(u, v) = 1.5$, this is 7 ppm. This is good property as these are the same bounds on yield as C_{pk} and are considered standard values.

Ganji succeeded in getting the index value lower on the further specification limit. However, this resulted in negative index values, as the index value is 0 on the closer limit. We see that the value can actually get high negative values, as seen for C'''_{pk} in figure 17c. The first problem is that this gives no standard for bad process capability, as it can go to minus infinity. Secondly, negative index values are difficult to interpret practice, as mentioned in section 4.

5.11 Y_p

The indices we've seen so far require an estimation of μ and σ in order to calculate the index. Estimation error can lead to a wrong indication of process capability. Wanga Ching-Hsin (2020) proposed a new PCI which which can be calculated directly from the process data and measures the capacity of processes for both symmetric and asymmetric tolerance. It is defined as

$$Y_p = \int_{LSL}^{USL} (1 - \delta)\, dF(x) \tag{29}$$

where

$$\delta = \begin{cases} \frac{X-T}{D_u} & X \geq T \\ \frac{T-X}{D_l} & X < T \end{cases} \tag{30}$$

In order to illustrate the distribution of Y_p for different μ, I took a sample of size 1000 for each μ and calculated Y_p, with given $\sigma = 0.8$. This would be a simple estimation of Y_p. Wang, Chen (2020) used a bootstrapping method to obtain a more precise estimate. However, for the sake of illustrating the index, this simple estimation will suffice. The plot of Y_p can be found in figure 18.

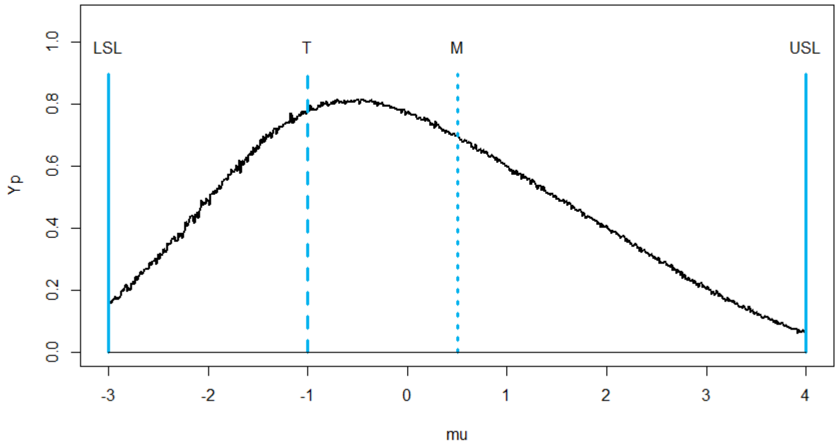


Figure 18: Y_p values for different μ.

Properties of Y_p:

- From figure 18 it is clear to see that Y_p is not maximized by $\mu = T$. Thus property (i) does not hold.
- By taking relative loss, Y_p does account for direction of μ shift and we see in figure 18 that the value decreases more quickly towards the closer limit, so property (ii) holds.
- As loss is higher when process data is on the further limit, Y_p takes a lower value on the further limit. This is also illustrated in figure 18. Thus property (iii) holds.
- $\delta \leq 1$ for μ within the specification limits, therefore Y_p will always take non-negative values and thus property (iv) holds.
- Y_p can be used for both symmetric and asymmetric tolerances, but it is not a generalization of the classical PCIs. So property (v) does not hold.

First of all, Y_p is not a generalization of the classical PCIs. Secondly, we see that it is not maximized for $\mu = T$. However, two good properties are Y_p being always greater or equal to 0 and the value on the closer limit being lower. Furthermore, the main advantage of Y_p is the fact that it can be used regardless of the process distribution.

5.12 Summary of PCIs for asymmetrical tolerance

We summarize this section by giving a table that shows the PCIs for asymmetrical tolerances and their desirable properties from section 4. C_p^* from section 5.1 and C_p' from section 5.2 are left out as they are approaches to asymmetrical tolerances rather than new indices. As we can see, none of the existing PCIs satisfy all the desirable properties.

PCI	(i)	(ii)	(iii)	(iv)	(v)
S_{pmk}		x	x	x	
$C_{pm}^{\star}$		x		x	x
C_{pm}^{+}		x		x	x
$C_{pa}(1,1)$	x	x	x		
C_{pk}''	x	x		x	x
C_{pm}''	x	x		x	x
C_{pmk}''	x	x		x	x
C_a^{**}	x	x		x	x
C_a''	x	x		x	x
C_{pk}'''	x	x	x		
C_{pm}'''	x	x		x	x
C_{pmk}'''	x	x	x		
Y_p		x	x	x	

Table 1: Overview of the existing PCIs and the properties from section 4 they satisfy.

6 New PCI proposals

As we have seen in the previous section, none of the discussed PCIs satisfy all of our desirable properties. In this section some new ideas are proposed that improve on the PCIs from the previous section by satisfying more desirable properties.

6.1 Proposal 1: satisfying property (iii) for $C_p''(u, v)$

$C_p''(u, v)$ already has 4 out of our 5 desirable properties. It does not have property (iii) because for two processes A and B with $\mu_A = USL$ and $\mu_B = LSL$, the index values are both 0.

Before we look at the cause of this, we note that it is convenient to define the relative departure of the process mean by

$$\beta = \begin{cases} \frac{\mu - T}{D_u} & \text{if} \quad \mu > T \\ \frac{T-\mu}{D_l} & \text{if} \quad \mu \leq T \end{cases} \tag{31}$$

which is 0 when $\mu = T$ and goes linearly to 1 when μ moves to either limit and thus more quickly towards the closer limit. This is illustrated in figure 19 by plotting $1 - \beta$, as $1 - \beta$ is more in line with index value plots because we want the relative departure β to be small. The term β is found in many PCIs for asymmetric tolerances. Namely, $F = d \cdot \beta$ in $C_p''(u, v)$ and C_a^{**} (section 5.7 and 5.8). $F^* = d^* \cdot \beta$ in $C_p''(u, v)$ and C_a'' (section 5.7 and 5.9). $A^* = |\mu - T| \cdot \beta$ in $C_p'''(u, v)$ (section 5.10).

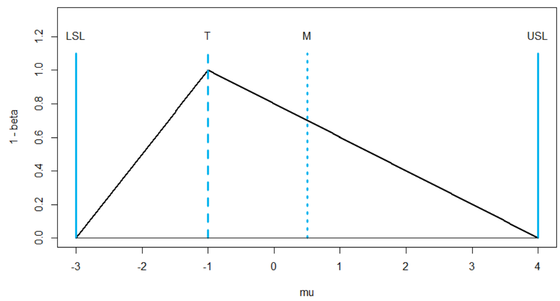


Figure 19: Value of $1 - \beta$ for different μ.

Using β, we can rewrite $C_p''(u, v)$ as

$$C_p''(u, v) = \frac{d^*(1 - u\beta)}{3\sqrt{\sigma^2 + vd^2\beta^2}} \tag{32}$$

From equation (32) we see that C_{pk}'' and C_{pmk}'' (which have $u = 1$) have the $1 - \beta$ term in the numerator. $1 - \beta = 0$ when the process mean is on either specification limit. This is the cause of C_{pk}'' and C_{pmk}'' having equal values, namely 0, on the specification limits and thus not satisfying property (iii).

We can adjust $C_p''(u, v)$ by taking F^+ instead of F^*. The new index, denoted by C_{p1}, is defined as

$$C_{p1}(u, v) = \frac{d^* - uF^+}{3\sqrt{\sigma^2 + vF^2}} \tag{33}$$

where

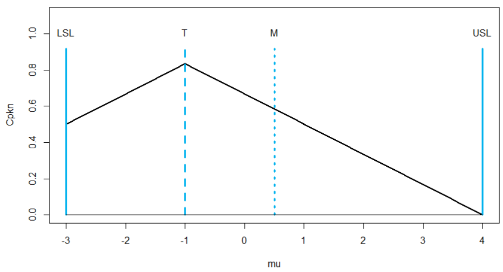
$$F = \max\left\{\frac{d(\mu - T)}{D_u}, \frac{d(T - \mu)}{D_l}\right\}$$
$$F^+ = \max\left\{\frac{d^*(\mu - T)}{d^+}, \frac{d^*(T - \mu)}{d^+}\right\}$$

with

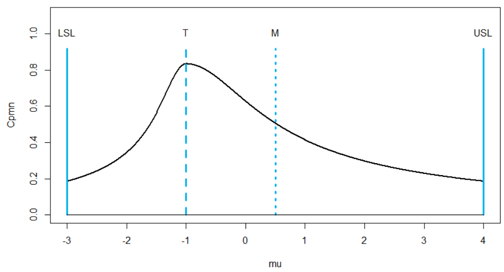
$$d^+ = \max\{D_u, D_l\}$$

We have that the numerator is still zero on further limit, but is equal to $d^*(1 - \frac{d^*}{d^+})$ on the closer limit. The idea is that when the target value is close to the midpoint, then the ratio between the specification widths ($\frac{d^*}{d^+}$) is close to one and the index value will be close to zero. But when the target value is close to a specification limit, then the ratio will be smaller and thus the index value larger. In this way, the index value on the closer limit depends on the location of the target relative to the specification limits.

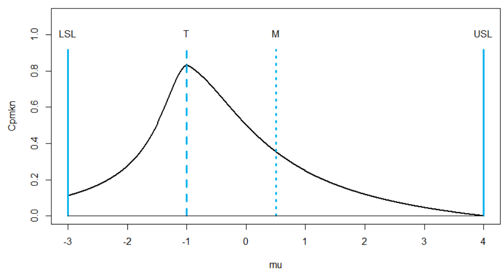
Figure 20 shows plots of the new indices for standard u, v values. One problem is that F^+ is symmetric around T. Therefore when $v = 0$, we have a symmetric index around T which does not satisfy property (ii). The symmetry is clear in figure 20a. For this reason, we should only consider $v > 0$. We also note that $C_{p1}(0,1) = C''_{pm}$, shown in figure 20b. More generally, $C_{p1}(0,v) = C''_p(0,v)$ for all v because the new term F^+ is not considered when $u = 0$. Thus only values $u, v > 0$ are of interested to us.



(a) Value of $C_{p1}(1,0)$ for different μ.



(b) Value of $C_{p1}(0,1)$ for different μ.



(c) Value of $C_{p1}(1,1)$ for different μ.

Figure 20: Plots of proposal 1

In order to look at the properties of new index we take $u, v = 1$, which can be considered standard values. $C_{p1}(1,1)$ is plotted in figure (20c).

- Note that $F, F^+ \geq 0$ and that they are 0 when $\mu = T$, then it is clear that $C_{p1}(1,1)$ is maximized by $\mu = T$. Thus property (i) holds.
- As noted earlier, the term F^+ is symmetric around T. However, F increases more quickly when μ moves to the closer limit. Thus with $v = 1 > 0$ the index decreases more steeply towards the closer limit and property (ii) holds.
- On the further limit $F^+ = d^*$ such that the index value $C_{p1}(1,1) = 0$. On the closer limit $F^+ = \frac{{d^*}^2}{d^+} \leq d^*$ such that the index value $C_{p1}(1,1) \geq 0$, with equality in the case of symmetric tolerances. Thus for asymmetric tolerances the index value on the further limit is lower than on the closer limit, and we see that property (iii) holds.
- From the above we also conclude that property (iv) holds.

- In section 5.7, we concluded that $C_p''(u,v)$ reduces to $C_p(u,v)$ (the classical PCI superstructure). In this new PCI, we've only substituted d^+ for D_u and D_l in F^* and d^+ also reduces to d for symmetrical tolerances. Thus $C_{p1}(u,v)$ also reduces to the classical PCI superstructure and property (v) holds.

As we can see, all our desirable properties hold for $C_{p1}(1,1)$. More generally, all properties hold for any $u \in (0,1]$ and $v \in (0,\infty)$.

6.1.1 Relation to process centering

Given a value c for the process capability index $C_p''(u,v)$, we can derive upper bounds for the relative departures $\lambda = \frac{\mu-T}{D_u}$ and $\lambda' = \frac{T-\mu}{D_l}$ by

$$\lambda_5 = \frac{d^*}{3c\sqrt{v}d + ud^*}$$

such that $\lambda, \lambda' \leq \lambda_5$ and

$$T - \lambda_5 D_l < \mu < T + \lambda_5 D_u$$

Details of the derivation can be found in the appendix. Given a value c for the new index $C_{p1}(u,v)$, we derive new upper bounds given by

$$\lambda_1 = \frac{d^*}{3c\sqrt{v}d + ud^*\frac{D_u}{d^+}}$$
$$\lambda_1' = \frac{d^*}{3c\sqrt{v}d + ud^*\frac{D_l}{d^+}}$$

such that $\lambda \leq \lambda_1$, $\lambda' \leq \lambda_1'$ and

$$T - \lambda_1' D_l < \mu < T + \lambda_1 D_u$$

Details of the derivation can be found in the appendix. Note that $\frac{D_u}{d^+} \leq 1$ and $\frac{D_l}{d^+} \leq 1$, such that $\lambda_1, \lambda_1' \geq \lambda_5$. Thus the upper bounds of the relative departure for the new PCI are higher then for C_p''. The upper and lower bounds are plotted in figure 21.

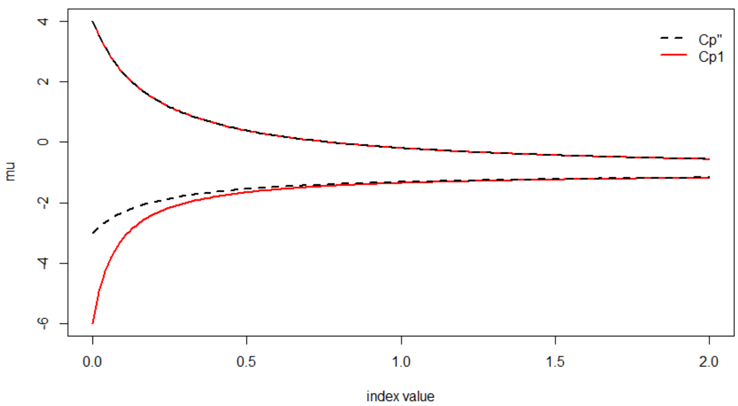


Figure 21: Upper and lower bound of μ for given index values of C_p'' and C_{p1} ranging from 0 to 2, with $(LSL, T, USL) = (-3, -1, 4)$.

6.1.2 Relation to process yield

Given a value c for the index $C_p''(u,v)$, we can derive an upper bound on the percentage of non-conforming products (NC) and Yield $= 1 - NC$. In the case that $D_u < D_l$, so T is closer to USL, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_u(1-\lambda_5)}{d^*}\right)\right)$$

and in the case that $D_l < D_u$, so T is closer to LSL, then

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_l(1-\lambda_5)}{d^*}\right)\right)$$

of which the derivations can be found in the appendix. In a similar way we can find these bounds when it is given that $C_{p1}(u,v) = c$. In the case that $D_u < D_l$, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_u(1-\lambda_1)}{d^*}\right)\right)$$

and in the case that $D_l < D_u$, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_l(1-\lambda_1')}{d^*}\right)\right)$$

Recall that $\lambda_1, \lambda_1' \geq \lambda_5$. This means that the upper bounds on NC for our new index are higher than for $C_p''(u,v)$. The upper bounds are plotted in figure 22.

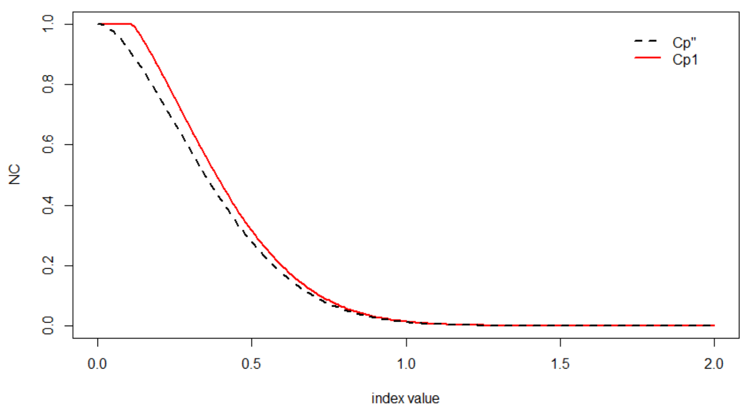


Figure 22: Upper bound of NC for given index values of C_p'' and C_{p1} ranging from 0 to 2, with $(LSL, T, USL) = (-3, -1, 4)$.

6.2 Proposal 2: satisfying property (iv) for C_{pk}''' and C_{pmk}'''

For C_{pk}''' and C_{pmk}''' we concluded that they lack properties (iv) and (v). By the increasing complexity of the indices, property (v) becomes harder to satisfy. In order to improve C_{pk}''' and C_{pmk}''' we will satisfy property (iv).

In order to see how we can improve $C_p'''(u,v)$ we look at the definition (section 5.10) and we can see that $A^* = D_u$ for $\mu = USL$ and $A^* = D_l$ for $\mu = LSL$. As $d^* = \min\{D_u, D_l\}$, we will get negative values in the numerator when the process mean is close to the further specification limit which results in negative index values. The index value will go to 0 when the process mean moves towards the closer limit. In this way, property (iii) is satisfied at the cost of property (iv).

Using the same idea as proposal 1, we scale A^* by multiplying with d^*/d^+. In this way, we will have that the numerator is zero on the further limit and equal to $\frac{d^*}{d^+}$ on the closer limit, such that the index value will always be greater or equal to 0.

Without further ado, we define $C_{p2}(u, v)$ as

$$C_{p2}(u, v) = \frac{d^* - uA^+}{3\sqrt{\sigma^2 + vF^2}} \tag{34}$$

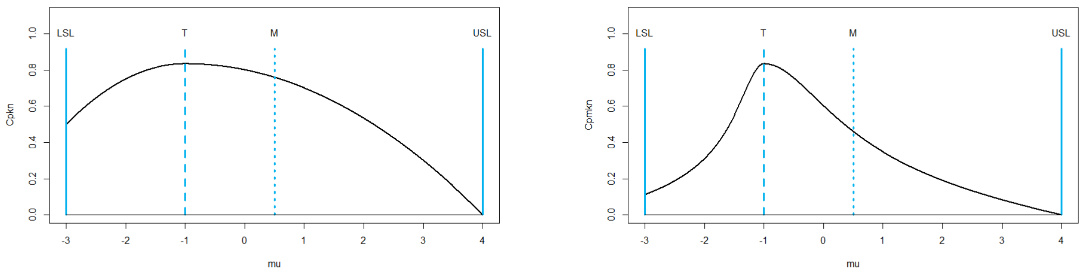
where

$$A^+ = \begin{cases} \frac{d^*}{d^+} \frac{(\mu - T)^2}{D_u} & \text{if} \quad \mu > T \\ \frac{d^*}{d^+} \frac{(T - \mu)^2}{D_l} & \text{if} \quad \mu \leq T \end{cases}$$

$$F = \begin{cases} \frac{d(\mu - T)}{D_u} & \text{if} \quad \mu > T \\ \frac{d(T - \mu)}{D_l} & \text{if} \quad \mu \leq T \end{cases}$$

with

$$d^+ = \max\{D_u, D_l\}$$



(a) Value of $C_{p2}(1, 0)$ for different μ.

(b) Value of $= C_{p2}(1, 1)$ for different μ.

Figure 23: Plots of proposal 2

Note that $C_{p2}(0, v) = C_p'''(0, v)$ as we've only adjusted the term in the numerator. To illustrate the behavior of the new index, figure 23 shows the values of $C_{p2}(1, 0)$ and $C_{p2}(1, 1)$. In order to look at the properties of the new index, we consider $C_{p2}(1, 0)$ and $C_{p2}(1, 1)$.

- A^+ is still 0 for $\mu = T$ and therefore both $C_{p2}(1, 0)$ and $C_{p2}(1, 1)$ are maximized for $\mu = T$, such that property (i) holds.
- If we recall that property (ii) holds for $C_p'''(u, v)$ and note that A^+ is A^* scaled, then it is true that (ii) also holds for $C_{p2}(1, 0)$ and $C_{p2}(1, 1)$.
- From the definition, we can see that $A^+ = d^*$ for μ on the further limit and $A^+ = \frac{d^*}{d^+} \cdot d^*$ for μ on the closer limit. In this way, we will have an index value of 0 when the process mean is on the further limit and a strictly positive value on the closer limit (as $\frac{d^*}{d^+} < 1$ for asymmetric tolerances). Thus the index value is lower on the further limit and property (iii) holds.
- From the previous point we can conclude that $C_{p1}(u, v) \geq 0$ within the limits, thus property (iv) holds.
- As mentioned earlier, it is hard to satisfy property (v) due to the increasing complexity of this index. The quadratic term in A^+ makes it such that $C_{p2}(1, 0)$ and $C_{p2}(1, 1)$ do not reduce to classical PCIs for symmetrical tolerances.

6.2.1 Relation to process centering

For a given value c for $C_p'''(u,v)$, Ganji and Gildeh (2016) derived upper bounds for λ and λ' given by

$$\lambda_6 = \frac{\sqrt{9vc^2d^2 + 4uD_ud^*} - 3\sqrt{v}dc}{2uD_u}$$
$$\lambda_6' = \frac{\sqrt{9vc^2d^2 + 4uD_ld^*} - 3\sqrt{v}dc}{2uD_l}$$

where λ_6 is an upper bound for λ and λ_6' is an upper bound for λ'. From which it follows that

$$T - \lambda_6' D_l < \mu < T + \lambda_6 D_u$$

Given a value c for the new index $C_{p2}(u,v)$, we derive new upper bounds given by

$$\lambda_2 = \frac{\sqrt{9vd^2c^2 + 4ud^*\frac{d^*}{d^+}D_u} - 3\sqrt{v}dc}{2u\frac{d^*}{d^+}D_u}$$
$$\lambda_2' = \frac{\sqrt{9vd^2c^2 + 4ud^*\frac{d^*}{d^+}D_l} - 3\sqrt{v}dc}{2u\frac{d^*}{d^+}D_l}$$

such that $\lambda \leq \lambda_2$, $\lambda' \leq \lambda_2'$ and

$$T - \lambda_2' D_l < \mu < T + \lambda_2 D_u$$

Details can be found in the appendix. Note that $\frac{d^*}{d^+} \leq 1$, such that $\lambda_2 \geq \lambda_6$ and $\lambda_2' \geq \lambda_6'$. Thus the bounds for the new PCI are actually more loose. The upper and lower bounds are plotted in figure 24.

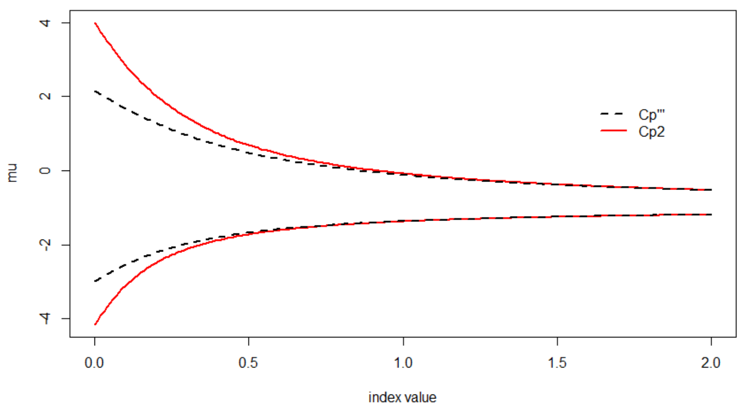


Figure 24: Upper and lower bound of μ for given index values of C_p'' and C_{p2} ranging from 0 to 2, with $(LSL, T, USL) = (-3, -1, 4)$.

6.2.2 Relation to process yield

Instead of calculating the process yield, we can calculate the percentage of non-conforming products (NC), where we have that $yield = 1 - NC$. Ganji and Gildeh (2016) derived lower bounds for NC given that $C_p'''(u,v) = c$. In the case that $D_u < D_l$, so T is closer to USL, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_u(1-\lambda_6)}{d^*}\right)\right)$$

and in the case that $D_l < D_u$, so T is closer to LSL, then

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_l(1-\lambda_6')}{d^*}\right)\right)$$

of which the derivations can be found in their appendix. In a similar way we can find these bounds when it is given that $C_{pn3}(u,v) = c$. In the case that $D_u < D_l$, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_u(1-\lambda_2)}{d^*}\right)\right)$$

and in the case that $D_l < D_u$, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_l(1-\lambda_2')}{d^*}\right)\right)$$

Recall that $\lambda_2, \lambda_2' \geq \lambda_5$. This means that the upper bounds on NC for our new index are higher and therefore less precise than for $C_p''''(u,v)$. The upper bounds are plotted in figure 25.

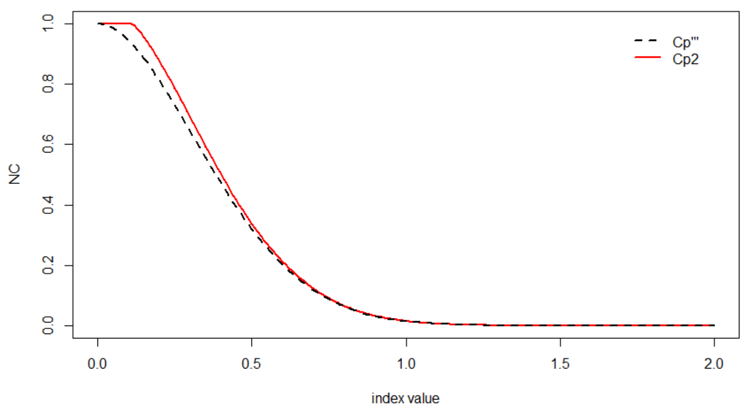


Figure 25: Upper bound of NC for given index values of C_p'''' and C_{p2} ranging from 0 to 2, with $(LSL, T, USL) = (-3, -1, 4)$.

6.3 Proposal 3: satisfying property (iv) for $C_p''''(u,v)$

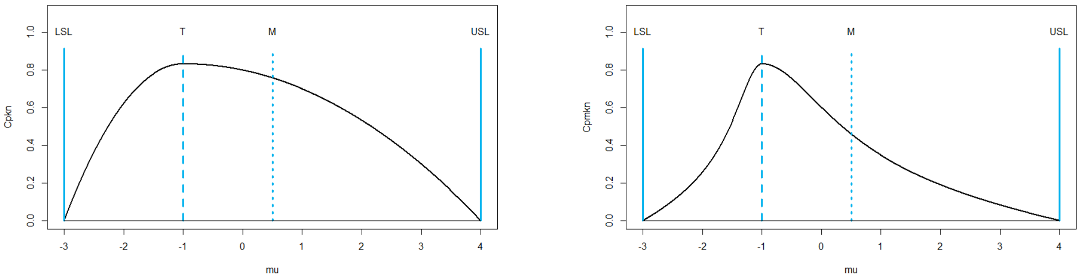
Another way to satisfy property (iv) for $C_p''''(u,v)$ is by taking A' for A^*. We define A' as

$$A' = \begin{cases} \frac{d^*(\mu-T)^2}{D_u^2} & \text{if} \quad \mu > T \\ \frac{d^*(T-\mu)^2}{D_l^2} & \text{if} \quad \mu \leq T \end{cases} \tag{35}$$

This is a similar term as F^* from $C_p'''(u,v)$. Recall that $F^* = d^* \cdot \beta$. Now note that $A' = d^* \cdot \beta^2$. We can define the new index $C_{p3}(u,v)$ by

$$C_{pn3}(u,v) = \frac{d^* - uA'}{3\sqrt{\sigma^2 + vF^2}} \left(= \frac{d^*(1-u\beta^2)}{3\sqrt{\sigma^2 + vF^2}} \right) \tag{36}$$

Figure 26 shows plots of the new index for μ ranging within the specification limits.



(a) Value of $C_{p3}(1,0)$ for different μ.

(b) Value of $C_{p3}(1,1)$ for different μ.

Figure 26: Plots of proposal 3

Note that $C_{pn3}(0,1) = C_p''(u,v)$ as $u = 0$ such that it doesn't take the new term into account. We will look which properties hold for $C_{p3}(1,0)$ and $C_{p3}(1,1)$.

- A' and F are both 0 for $\mu = T$, such that both $C_{p3}(1,0)$ and $C_{p3}(1,1)$ are maximized for $\mu = T$, thus satisfying property (i).
- The term A' increases more quickly towards the closer limit, as in that case we're dividing by the shorter distance. Also, F increases more quickly for the same reason. Property (ii) holds for both $C_{p3}(1,0)$ and $C_{p3}(1,1)$.
- As noted before, β ranges from 0 to 1 and thus also β^2. This way, we avoid negative index values. However, $A' = d^*$ for μ on either specification limit, causing the index value to be 0 on both limits. This holds for $C_{p3}(1,0)$ and $C_{p3}(1,1)$, thus they both do not satisfy property.
- From the previous point it can be concluded that the index value is minimized by 0 such that property (iv) holds for both.
- A' reduces to $\frac{(\mu-M)^2}{D}$ for symmetric tolerances, which is unequal to the term $|\mu - M|$ we have for the classical PCIs. Thus C_{p3} does not satisfy property (v).

We can see that we succeeded at satisfying property (iv) at the cost of property (iii).

6.3.1 Relation to process centering

Given a value c for our new process capability index $C_{pn3}(u,v)$, we can derive upper bounds for λ and λ' by

$$\lambda_3 = \frac{\sqrt{9vc^2d^2 + 4ud^{*2}} - 3\sqrt{v}dc}{2ud^*}$$

such that

$$T - \lambda_3 D_l < \mu < T + \lambda_3 D_u$$

If we compare λ_6 and λ_6' to λ_3, then we see that D_u and D_l are replaced by $d^* = \min\{D_u, D_l\} \leq D_u, D_l$ in the numerator and the denominator. However in the numerator d^* is inside the square root. Therefore we actually have that $\lambda_3 \geq \lambda_6, \lambda_6'$. Thus the bounds on μ by $C_p'''(u,v)$ are actually tighter than the bounds by the new index. The upper and lower bounds are plotted in figure 27.

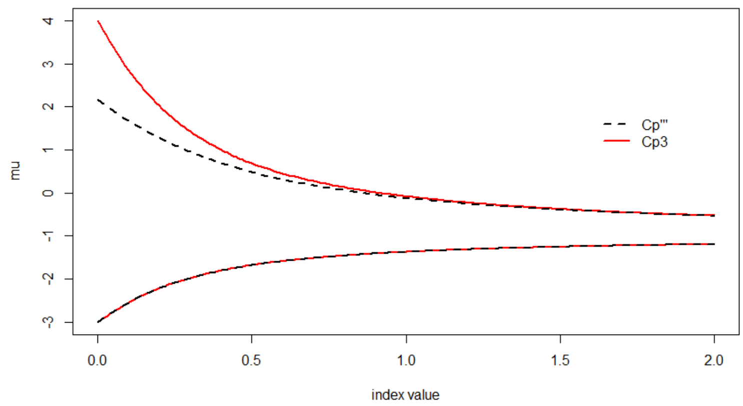


Figure 27: Upper and lower bound of μ for given index values of C_p''' and C_{p3} ranging from 0 to 2, with $(LSL, T, USL) = (-3, -1, 4)$.

6.3.2 Relation to process yield

Let it be given that $C_{pn3}(u, v) = c$. In the case that $D_u < D_l$, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_u(1-\lambda_3)}{d^*}\right)\right)$$

and in the case that $D_l < D_u$, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_l(1-\lambda_3)}{d^*}\right)\right)$$

Recall that $\lambda_3 \geq \lambda_6, \lambda_6'$. This actually means that our upper bounds for our new index are higher and therefore less precise than for $C_p'''(u, v)$. The upper bounds are plotted in figure 28 using an example set of specifications. We can see that the upper bounds are equal in this case. This is because we have $D_l < D_u$ and $\lambda_2' = \lambda_3$.

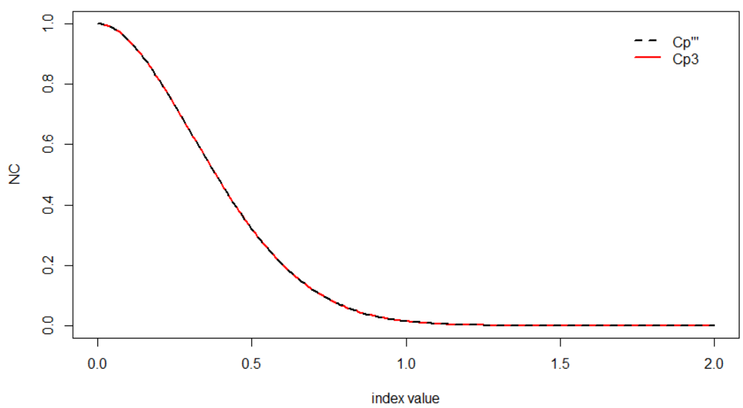


Figure 28: Upper bound of NC for given index values of C_p''' and C_{p3} ranging from 0 to 2, with (LSL,T,USL) = (-3,-1,4).

6.4 Proposal 4: satisfying property (iii) for $C_p''(u, v)$

Instead of only getting a positive index value on the closer limit, as we did for $C_{p1}(u, v)$ to satisfy property (iii), we can also make sure we never get a 0 index value. We can do this by slightly expanding the specification limits. We will extend the limits by γ defined as

$$\gamma = |M - T| \tag{37}$$

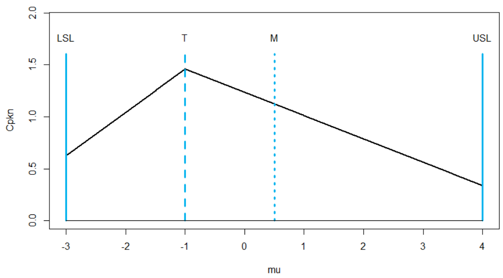
and now the idea is to define $C_p''(u, v)$ using the new limits $LSL' = LSL - \gamma$ and $USL' = USL + \gamma$. $C_p''(u, v)$ will be 0 on the new limits, but will have a strictly positive value on the real limits LSL and USL. We obtain a new index $C_{p4}(u, v)$ defined by

$$C_{p4}(u, v) = \frac{d'' - uF''}{3\sqrt{\sigma^2 + vF'^2}} \tag{38}$$

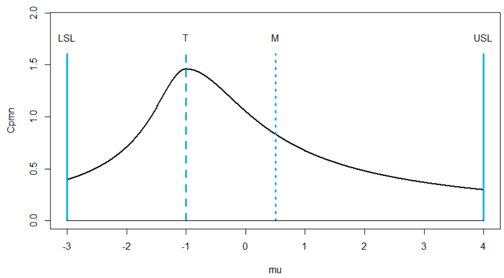
where

$$F' = \max\left\{\frac{d'(\mu - T)}{D_u'}, \frac{d'(T - \mu)}{D_l'}\right\}$$
$$F'' = \max\left\{\frac{d''(\mu - T)}{D_u'}, \frac{d''(T - \mu)}{D_l'}\right\}$$

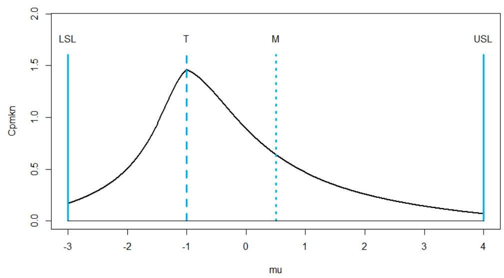
with $D_u' = D_u + \gamma$ and $D_l' = D_l + \gamma$. And $d' = d + \gamma$ and $d'' = d^* + \gamma$. So the index has the same properties as $C_p''(u, v)$ but is now 0 on LSL' and USL'. We chose γ this way, as it reduced to 0 for symmetric tolerance. This way, property (v) still holds. The values of the new index are plotted in figures 29 and 30.



(a) Value of $C_{p4}(1, 0)$.

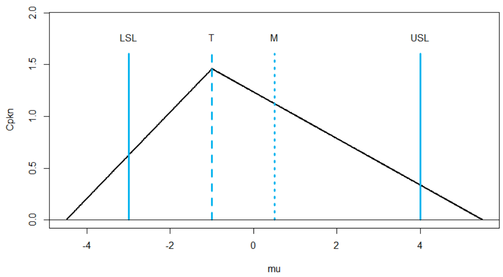


(b) Value of $C_{p4}(0, 1)$.

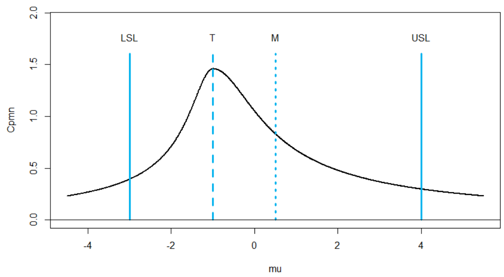


(c) Value of $C_{p4}(1, 1)$.

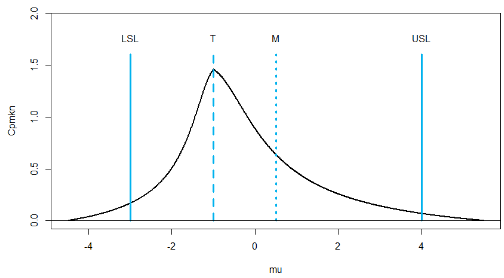
Figure 29: Plots of proposal 4 for various μ and fixed $\sigma = 0.8$



(a) Value of $C_{p4}(1, 0)$.



(b) Value of $C_{p4}(0, 1)$.



(c) Value of $C_{p4}(1, 1)$.

Figure 30: Plots of proposal 4 for various μ and fixed $\sigma = 0.8$: expanded view

We can show that the new index $C_{p4}(u, v)$ still satisfies the desirable properties from $C_p''(u, v)$. First note that the new index is just $C_p''(u, v)$ on extended limits, from which it easily follows that properties (i), (ii) and (iv) still hold. It was also noted that $\gamma = 0$ for symmetrical tolerances, which means that in that case $C_{p4}(u, v) = C_p''(u, v)$. Now because property (v) holds for $C_p''(u, v)$, it also holds for $C_{p4}(u, v)$.

We wanted to satisfy property (iii) and we can show that it holds. Recall the property of $C_p''(u, v)$ that two processes with $\mu_A > T$ and $\mu_B < T$ satisfy $(\mu_A - T)/D_u' = (T - \mu_B)/D_l'$, then the processes have the same index value. For the new index if $\mu_A = USL$ and $\mu_B = LSL$, then $(\mu_A - T)/D_u' \neq (T - \mu_B)/D_l'$. Now the process on the closer specification limit is also relatively closer to T. As the index value increases towards T, we have that the index value of the process on the closer limit is higher than on the further limit. Thus satisfying property (iii).

As a final note, we have that all properties hold for any $u \in (0, 1]$ and $v \in [0, \infty)$.

6.4.1 Relation to process centering

Given a value c for our new index $C_{pn4}(u, v)$, we can derive upper bounds for λ and λ' by

$$\lambda_4 = \frac{d^* + \gamma}{3c\sqrt{v}d + ud^* + (3c\sqrt{v} + u)\gamma}$$

such that

$$T - \lambda_4 D_l < \mu < T + \lambda_4 D_u$$

We actually have that $\lambda_4 \geq \lambda_5$. Thus the bounds for our new index $C_{p4}(u, v)$ are higher than the bounds of $C_p''(u, v)$. The upper and lower bounds are plotted in figure 31.

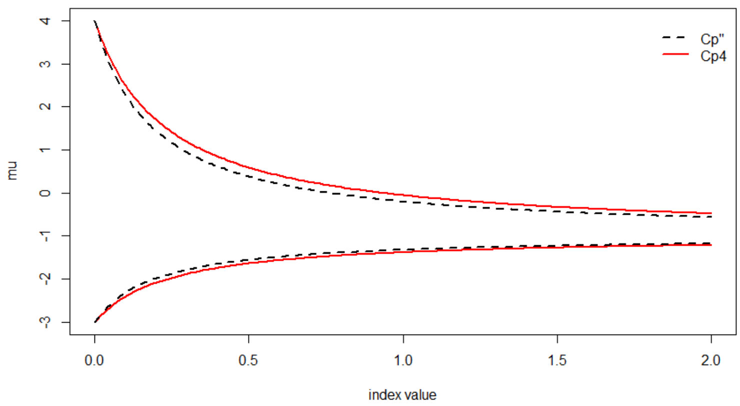


Figure 31: Upper and lower bound of μ for given index values of C_p'' and C_{p4} ranging from 0 to 2, with $(LSL, T, USL) = (-3, -1, 4)$.

6.4.2 Relation to process yield

Let it be given that $C_{p4}(u, v) = c$. In the case that $D_u < D_l$, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_u(1-\lambda_4)}{d^*}\right)\right)$$

and in the case that $D_l < D_u$, we have

$$NC \leq 2\left(1 - \Phi\left(\frac{3cD_l(1-\lambda_4)}{d^*}\right)\right)$$

Details can be found in the appendix. Recall that $\lambda_4 \geq \lambda_5$. This means that our upper bounds for our new index are higher than for $C_p''(u, v)$. The upper bounds are plotted in figure 32.

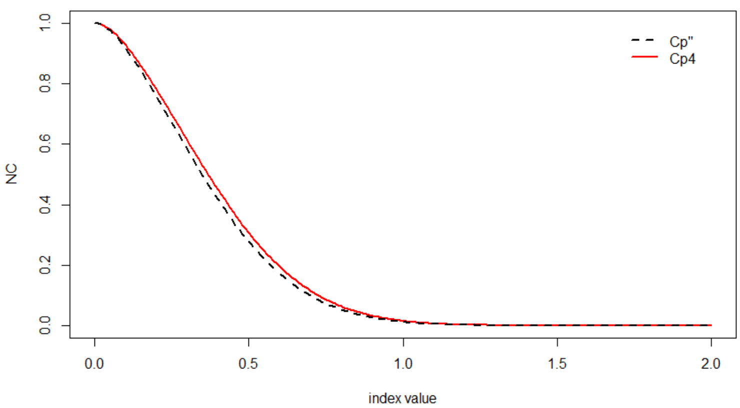


Figure 32: Upper bound of NC for given index values of C_p'' and C_{p4} ranging from 0 to 2, with $(LSL, T, USL) = (-3, -1, 4)$.

6.5 Estimation

In order to calculate the value of the PCIs, μ and σ are required. However, in practice we are dealing with measurement data and the exact μ and σ are unknown in general and therefore we can never calculate the exact

PCI value. In order to get an estimate of the index value, we replace the true mean and variance by the sample mean $\bar{x}$ and sample variance s^2 respectively.

Based on a sample of n measurements $x_1, x_2, ...x_n$ from a process X, we have

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$$
$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Now taking $\hat{\mu} = \bar{X}$ and $\hat{\sigma}^2 = S^2$ we can estimate our first index $C_{p1}(u, v)$ by

$$\hat{C}_{p1}(u, v) = \frac{d^* - u\hat{F}^+}{3\sqrt{\hat{\sigma}^2 + v\hat{F}^2}} \tag{39}$$

where

$$\hat{F} = \max\left\{\frac{d(\hat{\mu} - T)}{D_u}, \frac{d(T - \hat{\mu})}{D_l}\right\}$$
$$\hat{F}^+ = \max\left\{\frac{d^*(\hat{\mu} - T)}{d^+}, \frac{d^*(T - \hat{\mu})}{d^+}\right\}$$

We can obtain estimates for our other indices in the same way.

On the assumption that X has a $N(\mu, \sigma)$ distribution, then $\bar{X}$ has a $N\left(\mu, \frac{\sigma^2}{n}\right)$ distribution and S^2 has a $\frac{\sigma^2}{n-1}\chi^2$ distribution, and $\bar{X}$ and S^2 are mutually independent (Kotz and Johnson (2002)). It is possible to derive a confidence interval for our estimator, but the statistical distribution of $\hat{C}_{p1}(u, v)$ will be complicated. Due to the complexity, deriving the confidence intervals has not done, as this is not of great interest for this report. However, an example can be found in Zhang et al. (1990) where they developed a confidence interval for the classical PCI C_{pk}.

6.6 Summary of new PCIs

We summarize this section with a table showing the new PCIs and the properties they satisfy.

PCI	(i)	(ii)	(iii)	(iv)	(v)
$C_{p1}(1,1)$	x	x	x	x	x
$C_{p2}(1,0)$	x	x	x	x	
$C_{p2}(1,1)$	x	x	x	x	
$C_{p3}(1,0)$	x	x		x	
$C_{p3}(1,1)$	x	x		x	
$C_{p4}(u,v)$	x	x	x	x	x

Table 2: Overview of the new PCIs and the properties from section 4 they satisfy.

6.7 Comparison

In order to compare the values of our new proposed PCIs with the existing PCIs C_p'' and C_p''', we use the same setting as is in their respected papers; Chen and Pearn (2001) and Ganji and Gildeh (2016). This setting is often found in papers on process capability indices for asymmetric tolerance. The oldest paper found with this setting is Boyles (1994). It is however unclear what the specifications are based on, but are most likely chosen manually for illustrating purpose.

Cp'', Cp1 and Cp4							
A comparison based on (LSL, USL) = (26, 58) and T = 50.							
Mu	Cp''(1,0)	Cp4(1,0)	Cp''(0,1)	Cp4(0,1)	Cp''(1,1)	Cp1(1,1)	Cp4(1,1)
26	0.000	0.500	0.164	0.293	0.000	0.000	0.073
27	0.042	0.562	0.171	0.306	0.007	0.026	0.086
28	0.083	0.625	0.179	0.319	0.015	0.053	0.100
29	0.125	0.688	0.187	0.334	0.023	0.082	0.115
30	0.167	0.750	0.196	0.350	0.033	0.112	0.131
31	0.208	0.812	0.206	0.368	0.043	0.144	0.149
32	0.250	0.875	0.217	0.388	0.054	0.178	0.170
33	0.292	0.938	0.229	0.409	0.067	0.214	0.192
34	0.333	1.000	0.243	0.434	0.081	0.251	0.217
35	0.375	1.062	0.258	0.461	0.097	0.290	0.245
36	0.417	1.125	0.275	0.492	0.114	0.331	0.277
37	0.458	1.188	0.294	0.528	0.135	0.374	0.313
38	0.500	1.250	0.316	0.568	0.158	0.418	0.355
39	0.542	1.312	0.342	0.615	0.185	0.464	0.404
40	0.583	1.375	0.371	0.670	0.217	0.512	0.461
41	0.625	1.438	0.406	0.735	0.254	0.561	0.528
42	0.667	1.500	0.447	0.812	0.298	0.611	0.609
43	0.708	1.562	0.496	0.906	0.351	0.662	0.708
44	0.750	1.625	0.555	1.020	0.416	0.713	0.828
45	0.792	1.688	0.625	1.159	0.495	0.764	0.978
46	0.833	1.750	0.707	1.329	0.589	0.814	1.163
47	0.875	1.812	0.800	1.529	0.700	0.863	1.385
48	0.917	1.875	0.894	1.743	0.820	0.911	1.634
49	0.958	1.938	0.970	1.925	0.930	0.957	1.865
50	1.000	2.000	1.000	2.000	1.000	1.000	2.000
51	0.875	1.875	0.800	1.743	0.700	0.946	1.634
52	0.750	1.750	0.555	1.329	0.416	0.871	1.163
53	0.625	1.625	0.406	1.020	0.254	0.785	0.828
54	0.500	1.500	0.316	0.812	0.158	0.697	0.609
55	0.375	1.375	0.258	0.670	0.097	0.612	0.461
56	0.250	1.250	0.217	0.568	0.054	0.534	0.355
57	0.125	1.125	0.187	0.492	0.023	0.465	0.277
58	0.000	1.000	0.164	0.434	0.000	0.404	0.217

Table 3: Values of C_p'' and new proposals C_{p1} and C_{p4} for various μ and fixed $\sigma = 8/3$.

Table 3 shows values for the existing index $C_p''(u,v)$ with standard u,v values together with new proposals $C_{p1}(1,1)$ and $C_{p4}(u,v)$ with standard u,v values, which are supposed to improve upon C_p''. Recall that $C_p''(u,v)$ did not satisfy property (iii), which is also clearly visible in this example as the values on the limits are equal for all three instances. We can also see that this is different for $C_{p1}(1,1)$ and $C_{p4}(u,v)$, as they have a lower index value on the further limit than on the closer limit. Lastly, we can see that $C_{p4}(u,v)$ does not have index values of 0 on the boundaries, as intended. However, another effect of the expanded limits for $C_{p4}(u,v)$ is also clearly visible, namely we see significantly higher index values. Most notably is $C_{p4}(1,0) = 1$ on the upper specification limit. This is clearly an overestimation of process capability as the process yield is only 50% on the limit.

Cp''', Cp2 and Cp3 A comparison based on (LSL, USL) = (26, 58) and T = 50.						
Mu	Cp'''(1,0)	Cp2(1,0)	Cp3(1,0)	Cp'''(1,1)	Cp2(1,1)	Cp3(1,1)
26	−2.000	0.000	0.000	−0.329	0.000	0.000
27	−1.755	0.082	0.051	−0.301	0.075	0.051
28	−1.521	0.160	0.102	−0.272	0.148	0.102
29	−1.297	0.234	0.154	−0.243	0.219	0.154
30	−1.083	0.306	0.206	−0.212	0.287	0.206
31	−0.880	0.373	0.259	−0.181	0.353	0.259
32	−0.688	0.438	0.312	−0.149	0.416	0.312
33	−0.505	0.498	0.365	−0.116	0.476	0.365
34	−0.333	0.556	0.418	−0.081	0.533	0.418
35	−0.172	0.609	0.471	−0.044	0.588	0.471
36	−0.021	0.660	0.524	−0.006	0.639	0.524
37	0.120	0.707	0.576	0.035	0.688	0.576
38	0.250	0.750	0.627	0.079	0.733	0.627
39	0.370	0.790	0.677	0.126	0.775	0.677
40	0.479	0.826	0.725	0.178	0.813	0.725
41	0.578	0.859	0.771	0.235	0.848	0.771
42	0.667	0.889	0.814	0.298	0.880	0.814
43	0.745	0.915	0.854	0.370	0.908	0.854
44	0.812	0.938	0.891	0.451	0.932	0.891
45	0.870	0.957	0.923	0.543	0.953	0.923
46	0.917	0.972	0.950	0.648	0.970	0.950
47	0.953	0.984	0.971	0.762	0.983	0.971
48	0.979	0.993	0.987	0.876	0.992	0.987
49	0.995	0.998	0.997	0.965	0.998	0.997
50	1.000	1.000	1.000	1.000	1.000	1.000
51	0.984	0.995	0.971	0.787	0.993	0.971
52	0.938	0.979	0.891	0.520	0.973	0.891
53	0.859	0.953	0.771	0.349	0.941	0.771
54	0.750	0.917	0.627	0.237	0.895	0.627
55	0.609	0.870	0.471	0.157	0.839	0.471
56	0.438	0.812	0.312	0.095	0.772	0.312
57	0.234	0.745	0.154	0.044	0.696	0.154
58	0.000	0.667	0.000	0.000	0.611	0.000

Table 4: Values of C_p'''' and new proposals C_{p2} and C_{p3} for various μ and fixed $\sigma = 8/3$.

Table 4 shows values for the existing index $C_p''''(u, v)$ together with new proposals $C_{p2}(u, v)$ and $C_{p3}(u, v)$ for $u = 1$ and $v \in \{0, 1\}$, as for $u = 0$ we have $C_p''''(0, v) = C_{p2}(0, v) = C_{p3}(0, v)$. Recall that $C_p''''(1, 0)$ and $C_p''''(1, 1)$ did not satisfy (iv), which is clearly visible in this table especially for $C_p''''(1, 0)$ having negative values up to -2. $C_{p2}(u, v)$ and $C_{p3}(u, v)$ were designed to satisfy this property and in this example we can see that the values are greater or equal to 0. Notable however are the values of C_{p2} on the upper specification limit. An index value of ≥ 0.6 on a specification limit might be a case of overstating process capability. Furthermore, this example also illustrates C_{p3} having 0 values on both limits and therefore no longer satisfying (iii).

7 Conclusion

The purpose of this report was to explore existing PCIs developed to handle asymmetric tolerances and propose new ideas that improve upon the existing PCIs. In order to evaluate the existing PCIs, a list of five desirable properties was formulated: (i) the PCI must be maximized at the target value, (ii) the PCI must take the direction of shift from the target value into account, (iii) the PCI must have a lower value on the further limit, (iv) the PCI must have non-negative values and (v) the PCI is a generalization of the classical PCIs. None of the existing PCIs that were discussed satisfied all five properties.

Four proposals for new a PCI were made that are based on existing PCIs and adjusted to satisfy more of the properties. Two proposals, C_{p2} and C_{p3}, improve upon C_p''' by satisfying an additional property, but not all five. The other two proposals, C_{p1} and C_{p4} improve upon C_p'' and satisfy all five properties. Therefore, C_{p1} and C_{p4} are the most promising proposals.

The relationship of the new proposals with process yield and centering has also been investigated. Based on a given index value, upper bounds for process yield and centering were derived. However, these upper bounds were larger than for the existing PCIs C_p'' and C_p''', meaning that the existing PCIs actually have a closer relationship with yield and centering. Thus, even though the new proposals satisfy more of the properties, they give less information about process capability.

In order to compare the new proposals with the existing PCIs C_p'' and C_p''', a commonly seen asymmetrical specification setting was used and the index values were evaluated for various process means and a fixed standard deviation. The example showed that the values for C_{p4} were significantly higher than for the other indices and it was concluded that C_{p4} overstated process capability. Also, C_{p2} overstated process capability on the closer specification limit. C_{p1} and C_{p3} showed no direct problems.

Out of the four proposals, only C_{p1} should be considered as it is superior to the other proposals. However, it is arguable if C_{p1} is an improvement on the existing PCI C_p'' due to the higher upper bounds on process yield and centering. C_{p1}, as well as the other proposals, are not advised to be used in practice as they require further investigation and validation. However, the proposals, and the list of desirable properties, can be considered as an inspiration in the development of PCIs for asymmetrical tolerances.

8 Discussion

The greatest difficulty has been to satisfy property (iii), together with property (iv). It makes perfect sense for the index value to be zero on the further specification limit, as this is the worst case scenario (within the limits), but the challenge is to come up with a way to value processes located on the closer specification limit. The index value of proposals 1 and 2 on the closer specification limit is dependent on the ratio between the lengths of the two specification widths D_u and D_l. In this way, if the target value is relatively close to the midpoint, then the index value on the closer limit will be close to zero. But if the target value is relatively close to one of the specification limits, then we will see higher index values at the closer limit. I think that this is an intuitive approach to valuing processes located on the closer limit, but it requires further research to investigate if this does not overstate process capability.

One might have noticed that even though proposal 3 was constructed as an improvement on C_p''', in definition and behavior it looks more like C_p''. However, C_{p3} does not satisfy property (v), but C_p'' does. In that sense, I think that C_p'' is superior to C_{p3}. I believe that proposal 3 has also been considered by authors of C_p'' and C_p''' and was not proposed for this reason. As there is no advantage of using C_{p3} instead of C_p'', I don't believe proposal 3 should be investigated further.

Proposal also 4 requires some evaluation. The idea was to step away from the usual 0 values on the specification limits. While this was achieved, we saw that the index values were significantly higher than for the existing indices. This makes sense as we expanded the specification limits without correcting it in any way, therefore simply overstating process capability. C_{p4} can therefore be improved by finding a suitable correction such that it no longer overstates capability. Even though my proposal is not a viable PCI, I think the idea of having positive values on both specification limits deserves further research.

My proposals, like many other existing PCIs, are presented by a superstructure that is dependent of parameters u and v. In this project, I have only focused on $u, v \in \{0, 1\}$. The literature also seems to pay little attention to different u, v values. While it's not bad practice to use the standard values, I believe the u, v have the potential to be chosen in such a way that the PCI more accurately reflects the customers view on process yield and centering. I think that guidelines for picking u and v based on customer criteria are useful for practitioners and a good angle to further research existing PCIs.

All the upper bounds on non-conforming products and centering for the new proposals are higher than for the PCIs they are improving on. What it means that the bounds are higher, is that given a certain index value, the guarantee on yield and/or centering is lower. Or in other words, it guarantees less capability. For some time I thought this was not much of a problem as my main focus was to construct a consistent PCI by satisfying all, or most, of the desirable properties. However, a general difficulty with PCIs is trying to reflect on multiple criteria in one value. For this reason it is important that they have a close relation with the criteria, meaning the upper bounds should be low. Therefore, if at all possible, the proposals can be improved by redefining them in a way such that their upper bounds for yield and centering are lower.

This report, as well as many papers on PCIs, focuses on the theory and statistical properties behind the true PCI value. I say true PCI value, as in practice you are working with an estimate. I have only shortly introduced how to estimate the new PCIs, but there is a lot more to be said about the distribution of the estimate and what effect this has on practical use. Further research is definitely necessary in order for my proposals, and even some of the newer PCIs that were discussed, to have any practical use. In order to get an idea on how to use a PCI correctly; Cheng (1994) developed a procedure for determining whether a process is actually capable given an estimated value of C_p and C_{pm}.

A Upper bounds for process centering

The most common way to measure process centering in the case of asymmetric tolerance is by taking the departure ratio $\lambda = \frac{\mu - T}{D_u}$ when $\mu > T$ and $\lambda' = \frac{T-\mu}{D_l}$ when $\mu \leq T$. Given a PCI value c, we can derive upper bounds λ_i and λ'_i for the departure ratio's λ and λ' respectively. Then

$$\begin{aligned} \frac{\mu - T}{D_u} &\leq \lambda_i \implies \mu \leq T + \lambda_i D_u \\ \frac{T-\mu}{D_l} &\leq \lambda'_i \implies \mu \geq T - \lambda'_i D_l \end{aligned}$$

from which we obtain an interval for μ as

$$T - \lambda'_i D_l \leq \mu \leq T + \lambda_i D_u \tag{40}$$

Note that smaller λ_i, λ'_i imply a tighter interval for μ.

A.1 $C''_p(u, v)$

Let $C''_p(u, v) = c$ be given, then

$$c = \frac{d^* - uF^*}{3\sqrt{\sigma^2 + vF^2}} \leq \frac{d^* - uF^*}{3\sqrt{v}F}$$

if $\mu > T$, then

$$\begin{aligned} c &\leq \frac{d^* - ud^*\lambda}{3\sqrt{v}d\lambda} \\ 3c\sqrt{v}d\lambda &\leq d^* - ud^*\lambda \\ \lambda &\leq \frac{d^*}{3c\sqrt{v}d + ud^*} = \lambda_5 \end{aligned}$$

if $\mu \leq T$, then in the same way

$$\lambda' \leq \frac{d^*}{3c\sqrt{v}d + ud^*} = \lambda'_5$$

Note that $\lambda_5 = \lambda'_5$.

A.2 $C_{p1}(u, v)$

Let $C_{p1}(u, v) = c$ be given, then

$$c = \frac{d^* - uF^+}{3\sqrt{\sigma^2 + vF^2}} \leq \frac{d^* - uF^+}{3\sqrt{v}F}$$

If $\mu > T$, then

$$\begin{aligned} c &\leq \frac{d^* - u\frac{d^*}{d^+}D_u\lambda}{3\sqrt{v}d\lambda} \\ \lambda &\leq \frac{d^*}{3c\sqrt{v}d + u\frac{d^*}{d^+}D_u} = \lambda_1 \end{aligned}$$

If $\mu \leq T$, then in the same way

$$\lambda' \leq \frac{d^*}{3c\sqrt{v}d + u\frac{d^*}{d^+}D_l} = \lambda'_1$$

A.3 $C_{p2}(u, v)$

Let $C_{p2}(u, v) = c$ be given, then

$$c = \frac{d^* - uA^+}{3\sqrt{\sigma^2 + vF^2}} \leq \frac{d^* - uA^+}{3\sqrt{v}F}$$

If $\mu > T$, then

$$c \leq \frac{d^* - u\frac{d^*}{d^+}D_u\lambda^2}{3\sqrt{v}d\lambda}$$
$$u\frac{d^*}{d^+}D_u\lambda^2 + 3c\sqrt{v}d\lambda - d^* \leq 0$$

now λ is between the roots of the above equation. The larger root is therefore the upperbound for λ such that

$$\lambda \leq \frac{-3\sqrt{v}dc + \sqrt{9vd^2c^2 + 4ud^*\frac{d^*}{d^+}D_u}}{2u\frac{d^*}{d^+}D_u} = \lambda_2$$

If $\mu \leq T$, then in the same way

$$\lambda' \leq \frac{-3\sqrt{v}dc + \sqrt{9vd^2c^2 + 4ud^*\frac{d^*}{d^+}D_l}}{2u\frac{d^*}{d^+}D_l} = \lambda_2'$$

A.4 $C_{p3}(u, v)$

Let $C_{p3}(u, v) = c$ be given, then

$$c = \frac{d^* - uA'}{3\sqrt{\sigma^2 + vF^2}} \leq \frac{d^* - uA'}{3\sqrt{v}F}$$

If $\mu > T$, then

$$c \leq \frac{d^* - ud^*\lambda^2}{3\sqrt{v}d\lambda}$$
$$ud^*\lambda^2 + 3c\sqrt{v}d\lambda - d^* \leq 0$$

now λ is between the roots of the above equation. The larger root is therefore the upperbound for λ such that

$$\lambda \leq \frac{-3\sqrt{v}dc + \sqrt{9vd^2c^2 + 4ud^{*2}}}{2ud^*} = \lambda_3$$

If $\mu \leq T$, then in the same way

$$\lambda' \leq \frac{-3\sqrt{v}dc + \sqrt{9vd^2c^2 + 4ud^{*2}}}{2ud^*} = \lambda_3'$$

A.5 $C_{p4}(u, v)$

Let $C_{p4}(u, v) = c$ be given, then

$$c = \frac{d'' - uF''}{3\sqrt{\sigma^2 + vF'^2}} \leq \frac{d'' - uF''}{3\sqrt{v}F'}$$

if $\mu > T$, then

$$\begin{aligned}
c &\leq \frac{d'' - ud''\lambda}{3\sqrt{v}d'\lambda} \\
3c\sqrt{v}d'\lambda &\leq d'' - ud''\lambda \\
\lambda &\leq \frac{d''}{3c\sqrt{v}d' + ud''} = \frac{d^* + \gamma}{3c\sqrt{v}d + ud^* + (3c\sqrt{v} + u)\gamma} = \lambda_4
\end{aligned}$$

if $\mu \leq T$, then in the same way

$$\lambda' \leq \frac{d^* + \gamma}{3c\sqrt{v}d + ud^* + (3c\sqrt{v} + u)\gamma} = \lambda_4'$$

Note that $\lambda_4 = \lambda_4'$.

B Upper bounds for non conforming products

The number of non-conforming products (NC) is given by

$$NC = P(X < LSL) + P(X > USL)$$

If the target is closer to the upper limit, $D_u < D_l$, then, given a distance k between the mean and the target value ($|\mu - T| = k$), we have more non-conforming products when $\mu > T$. Thus

$$\begin{aligned} NC \leq 2P(X > USL) &= 2(1 - P(X < USL)) \\ &= 2\left(1 - \Phi\left(\frac{USL - \mu}{\sigma}\right)\right) \\ &= 2\left(1 - \Phi\left(\frac{(1-\lambda)D_u}{\sigma}\right)\right) \end{aligned}$$

Where the last equality follows from the fact that $\mu = T + \lambda D_u \implies USL - \mu = (1-\lambda)D_u$.

If the target is closer to the lower limit, $D_l < D_u$, then, given a distance k between the mean and the target value ($|\mu - T| = k$), we have more non-conforming products when $\mu \leq T$. Thus

$$\begin{aligned} NC \leq 2P(X < LSL) &= 2P(X < LSL) \\ &= 2\Phi\left(\frac{LSL - \mu}{\sigma}\right) \\ &= 2\left(1 - \Phi\left(\frac{\mu - LSL}{\sigma}\right)\right) \\ &= 2\left(1 - \Phi\left(\frac{(1+\lambda')D_u}{\sigma}\right)\right) \end{aligned}$$

Where the last equality follows from the fact that $\mu = T - \lambda' D_l \implies \mu - LSL = (1+\lambda')D_l$.

We want an upper bound for NC. Now NC is maximized when λ and σ is maximized. For λ we can use the obtained upper bounds from the previous section. An upper bound for σ can be found given an index value.

B.1 $C''_p(u, v)$

Let $C''_p(u, v) = c$ be given, then

$$\begin{aligned} c &= \frac{d^* - uF^*}{3\sqrt{\sigma^2 + vF^2}} \implies \\ \sigma^2 &= \left(\frac{d^* - uF^*}{3c}\right)^2 - vF^2 \implies \\ \sigma^2 &\leq \left(\frac{d^*}{3c}\right)^2 \implies \\ \sigma &\leq \frac{d^*}{3c} \end{aligned}$$

Now for $D_u < D_l$ we have that

$$NC \leq 2\left(1 - \Phi\left(3c\frac{(1-\lambda_5)D_u}{d^*}\right)\right)$$

and for $D_l < D_u$ we have

$$NC \leq 2\left(1 - \Phi\left(3c\frac{(1+\lambda_5)D_l}{d^*}\right)\right)$$

B.2 $C_{p4}(u, v)$

Let $C_{p4}(u, v) = c$ be given, then

$$
\begin{aligned}
c &= \frac{d'' - uF''}{3\sqrt{\sigma^2 + vF'^2}} \implies \\
\sigma^2 &= \left(\frac{d^* - uF^*}{3c}\right)^2 - vF^2 \implies \\
\sigma^2 &\leq \left(\frac{d''}{3c}\right)^2 \implies \\
\sigma &\leq \frac{d''}{3c} = \frac{d^* + \gamma}{3c}
\end{aligned}
$$

Now for $D_u < D_l$ we have that

$$NC \leq 2\left(1 - \Phi\left(3c\frac{(1-\lambda_4)D_u}{d^* + \gamma}\right)\right)$$

and for $D_l < D_u$ we have

$$NC \leq 2\left(1 - \Phi\left(3c\frac{(1+\lambda_4')D_l}{d^* + \gamma}\right)\right)$$

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