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Dekking, F. M.

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On Hofstadter's G-Sequence

F. M. Dekking
CWI Amsterdam and Delft University of Technology
Faculty EEMCS
P.O. Box 5031
2600 GA Delft
The Netherlands
F.M.Dekking@math.tudelft.nl

Abstract

We characterize the entries of Hofstadter's G-sequence in terms of the lower and upper Wythoff sequences. This can be used to give a short and comprehensive proof of the equality of Hofstadter's G-sequence and the sequence of averages of the swapped Wythoff sequences. In the second part we give some results that hold when one replaces the golden mean by other quadratic algebraic numbers. In the third part we prove a close relationship between Hofstadter's G-sequence and a sequence studied by Avdivpahić and Zejnulahi.

1 Introduction

Hofstadter's G-sequence G is defined by $G(1) = 1, G(n) = n - G(G(n-1))$ for $n \geq 2$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$G(n)$	0	1	1	2	3	3	4	4	5	6	6	7	8	8	9	9	10	11	11

Table 1: $G(n) =$ A005206(n) for $n = 0, \ldots, 18$.

It was proved in 1988, independently in the two articles [8, 9] that there is a simple expression for Hofstadter's G-sequence as a slow Beatty sequence, given for $n \geq 0$ by

$$G(n) = \lfloor (n+1)\gamma \rfloor, \tag{1}$$

where $\gamma = (\sqrt{5}-1)/2$, the small golden mean.

The terminology 'slow Beatty sequence' comes from the paper [11] by Kimberling and Stolarsky. Actually, G is *not* a Beatty sequence: Beatty sequences are sequences $(\lfloor n\alpha \rfloor)$ with α a positive real number *larger* than 1. See, e.g., the papers [5, 15].

The paper by Kimberling and Stolarsky provides the following basic result.

Theorem 1. **[Kimberling and Stolarsky]** *Suppose that γ in $(0,1)$ is irrational, and let $s(n) = \lfloor (n+1)\gamma \rfloor$ for $n \geq 0$. Let A be the set $\{n \geq 0 : s(n+1) = s(n)\}$ and let $a(0) < a(1) < \cdots$ be the members of A in increasing order. Similarly, let b be the sequence of those n such that $s(n+1) = s(n)+1$. Then a is the Beatty sequence of $1/(1-\gamma)$, and b is the Beatty sequence of $1/\gamma$.*

When we apply this result to determine the value of $s(n)$ for a given n, then we need information on *two* entries, namely $s(n)$ and $s(n+1)$, and given this information we do not yet know for which m $s(n)$ will be equal to $a(m)$, respectively $b(m)$. The following theorem is more useful in this respect.

Theorem 2. *Suppose that γ in $(0,1)$ is irrational, and let $s(n+1) = \lfloor n\gamma \rfloor$ for $n \geq 0$. Define $L(n) = \left\lfloor \frac{1}{\gamma} n \right\rfloor$ and $U(n) = \left\lfloor \frac{1}{1-\gamma} n \right\rfloor$ for $n \geq 0$. Then*

$$s(L(n)) = n \quad \textit{and} \quad s(U(n)) = \frac{\gamma}{1-\gamma} n$$

for all $n \geq 0$.

Proof. By definition, the sequence L satisfies

$$\frac{1}{\gamma} n = L(n) + \varepsilon_n$$

for $n \geq 0$ and some ε_n with $0 \leq \varepsilon_n < 1$. This leads to

$$s(L(n)) = \big\lfloor (L(n)+1)\gamma \big\rfloor = \big\lfloor n + \gamma(1-\varepsilon_n) \big\rfloor = n,$$

since obviously $0 \leq \gamma(1-\varepsilon_n) < 1$.

By definition, the sequence U satisfies for $n \geq 0$

$$\frac{1}{1-\gamma} n = U(n) + \varepsilon'_n,$$

for some ε'_n with $0 \leq \varepsilon'_n < 1$. This leads to

$$s(U(n)) = \Big\lfloor (U(n)+1)\gamma \Big\rfloor = \Big\lfloor \frac{\gamma}{1-\gamma} n + \gamma(1-\varepsilon'_n) \Big\rfloor = \frac{\gamma}{1-\gamma} n,$$

since obviously $0 \leq \gamma(1-\varepsilon'_n) < 1$. □

It is well-known (see [5]) that if α and β are two real numbers larger than 1, and moreover $1/\alpha + 1/\beta = 1$, then α and β form a *complementary Beatty pair*, which means that the two sets $\{\lfloor n\alpha \rfloor, n \geq 1\}$ and $\{\lfloor n\beta \rfloor, n \geq 1\}$ are disjoint, and that their union contains every positive integer. Note that for all γ in $(0,1)$ the sequences L and U form a complementary Beatty pair, since $\frac{1}{\gamma} > 1$, $\frac{1}{1-\gamma} > 1$ and $(\frac{1}{\gamma})^{-1} + (\frac{1}{1-\gamma})^{-1} = 1$.

2 Hofstadter and Wythoff

The most famous complementary Beatty pair is obtained by choosing $\alpha = \varphi$, and $\beta = \varphi^2$, where $\varphi := (1+\sqrt{5})/2$ is the golden mean. The Beatty sequences $L(n) = \lfloor n\varphi \rfloor$ and $U(n) = \lfloor n\varphi^2 \rfloor$ for $n \geq 1$ are known as the *lower Wythoff sequence* and the *upper Wythoff sequence*. The name has its origins in the paper [18].

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$L(n)$	1	3	4	6	8	9	11	12	14	16	17	19	21	22	24	25	27	29
$U(n)$	2	5	7	10	13	15	18	20	23	26	28	31	34	36	39	41	44	47

Table 2: $L(n) =$ A000201(n) and $U(n)=$A001950(n) for $n = 0, \dots, 18$.

We next turn our attention to sequence A002251, described as follows: start with the nonnegative integers; then swap $L(k)$ and $U(k)$ for all $k \geq 1$, where L and U are the lower and upper Wythoff sequences.

This means that this sequence, which we call W, is defined by

$$W(L(n)) = U(n) \quad \text{and} \quad W(U(n)) = L(n) \text{ for all } n \geq 1. \tag{2}$$

Regrettably, the sequence W was indexed starting with 0 in the *On-Line Encyclopedia of Integer Sequences* (OEIS). One of the unpleasant consequences of the useless index 0 is that sequence A073869 is not a clean Cesaró average of A002251. Another unpleasant consequence is that A073869 is basically a copy of A019444.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$W(n)$	0	2	1	5	7	3	10	4	13	15	6	18	20	8	23	9	26	28	11

Table 3: $W(n) =$ A002251(n) for $n = 0, \dots, 18$.

The sequence W has the remarkable property that the sum of the first $n+1$ terms is divisible by $n+1$. This leads to the sequence A073869, defined by A073869$(n) = \sum_{i=0}^{n} W(i)/(n+1)$.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$\overline{W}(n)$	0	1	1	2	3	3	4	4	5	6	6	7	8	8	9	9	10	11	11

Table 4: $\overline{W}(n) =$ A073869(n) for $n = 0, \dots, 18$.

The following theorem is a conjecture by Murthy in [14, A073869], but is proved in the long paper [17]. We give a new short proof.

Theorem 3. *The averaged Wythoff swap sequence* $\overline{W}$ *is equal to Hofstadter's G-sequence.*

Proof. The result holds for $n = 0, 1$. It suffices therefore to consider the sequence of differences. Subtracting $G(n-1) = \sum_{i=0}^{n-1} W(i)/n$ from $G(n) = \sum_{i=0}^{n} W(i)/(n+1)$, we see that we have to prove

$$(n+1)G(n) - nG(n-1) = W(n) \tag{3}$$

for all $n \geq 2$. But we know that there are only two possibilities for the recursion from $G(n-1)$ to $G(n)$. Therefore Equation (3) turns into the following two equations.

$$G(n) = G(n-1) \Rightarrow G(n) = W(n), \tag{4}$$

$$G(n) = G(n-1) + 1 \Rightarrow G(n) = W(n) - n. \tag{5}$$

It is not clear how to prove these equalities directly. However, we can exploit Theorem 1. According to this theorem with $s = G$, and $\gamma = (\sqrt{5}-1)/2$, and so $1/\gamma = \varphi$, $1/(1-\gamma) = \varphi^2$,

$$G(n) = G(n-1) \Leftrightarrow \exists M \text{ such that } n = U(M), \tag{6}$$

$$G(n) = G(n-1) + 1 \Leftrightarrow \exists M \text{ such that } n = L(M). \tag{7}$$

So we first have to prove that $n = U(M)$ implies $G(n) = W(n)$. This indeed holds, by an application of Theorem 2 and Equation (2):

$$G(n) = G(U(M) = L(M) = W(U(M)) = W(n).$$

Similarly, for the second case $n = L(M)$:

$$G(n) = G(L(M)) = M = U(M) - L(M) = W(L(M)) - L(M) = W(n) - n.$$

Here we applied $U(M) = L(M) + M$ for $M \geq 1$, a direct consequence of $\varphi^2 M = (\varphi+1)M$. □

In the comments of A073869 there is a scatterplot by Sloane—cf. Figure 1. The points have a nice symmetric distribution around the line $y = x$, since the points consists of all pairs $(L(n), U(n))$ and $(U(n), L(n))$ for $n = 1, 2, \ldots$. (Ignoring $(0, 0)$.) Apparently the points are almost lying on two lines. What are the equations of these lines? This is answered by the following proposition.

Proposition 4. *Let* W *be the Wythoff swap sequence, and* $\gamma = 1/\varphi$*. Then*

$$W(U(n)) = \lfloor \gamma U(n) \rfloor, W(L(n)) = \lfloor \varphi L(n) \rfloor + 1$$

for all $n \geq 1$.

Proof. Equation (4) and Equation (5) yield

$$W(n) = \begin{cases} G(n), & \text{if } G(n) = G(n-1); \\ G(n) + n, & \text{if } G(n) = G(n-1) + 1. \end{cases}$$

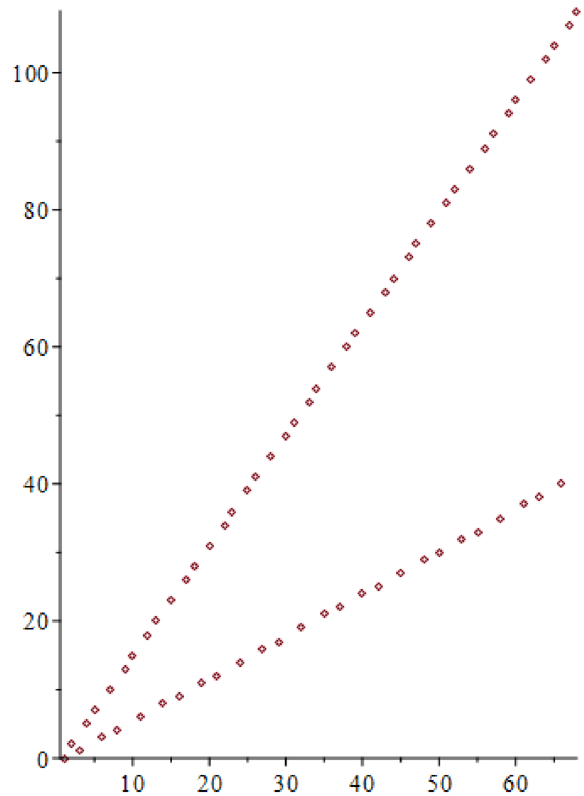

Figure 1: Scatterplot of the first 68 entries of W.

Since $G(n) = \lfloor (n+1)\gamma \rfloor$ by Equation (1), it follows from Equation (6) that

$$W(U(M)) = \lfloor U(M)\gamma \rfloor.$$

Since all $M = 1, 2, \ldots$ will occur, this gives the first half of the proposition.

For the second half of the proposition we perform the following computation under the assumption that $n = L(M)$:

$$G(n) + n = G(n-1) + n + 1 = \lfloor n\gamma \rfloor + n + 1 = \lfloor n(\gamma + 1) \rfloor + 1 = \lfloor n\varphi \rfloor + 1.$$

Now Equation (7) gives that $W(L(M)) = \lfloor \varphi L(M) \rfloor + 1$. □

Remark 5. Simple applications of Theorem 3 prove the conjectures in A090908 (terms $a(k)$ of A073869 for which $a(k) = a(k+1)$), and A090909 (terms $a(k)$ of A073869 for which $a(k-1), a(k)$ and $a(k+1)$ are distinct). It also proves the conjectured values of sequence A293688.

3 Generalizations

There is a lot of literature on generalizations of Hofstadter's recursion $G(n) = n - G(G(n-1))$. In most cases there is no simple description of the sequences that are generated by such recursions. An exception is the recursion $V(n) = V(n - V(n-1)) + V(n - V(n-4))$ analyzed by Balamohan et al. [4]. The sequence with initial values 1,1,1,1 generated by this recursion is sequence A063882. Allouche and Shallit [2] prove that the 'frequencies' of this sequence can be generated by an automaton. See the recent paper [7] for more results on this type of Hofstadter's recursions, known as Hofstadter Q-sequences. We consider the paper [6], that gives a direct generalization of Hofstadter's G-sequence.

Theorem 6. [Celaya and Ruskey] *Let $k \geq 1$, and let $\gamma = [0; k, k, k, \ldots]$. Assume $H(n) = 0$ for $n < k$, and for $n \geq k$, let*

$$H(n) = n - k + 1 - \Big(\sum_{i=1}^{k-1} H(n-i)\Big) - H(H(n-k).$$

Then $H(n) = \lfloor \gamma(n+1) \rfloor$ for $n \geq 1$.

As an example, take the case $k = 2$. In this case $\gamma = \sqrt{2} - 1$, the small silver mean. The recursion for what we call the Hofstadter Pell sequence is

$$H(n) = n - 1 - H(n-1) - H(H(n-2)).$$

Here Theorem 6 gives that

$$(H(n)) = \lfloor \gamma(n+1) \rfloor = 0, 0, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 5, 5, 6, 6, 7, 7, 7, 8, 8, 9, 9, 9, 10, 10, \ldots.$$

This is sequence A097508 in the OEIS.

Let $1/\gamma = 1 + \sqrt{2}$ and $1/(1-\gamma) = 1 + \frac{1}{2}\sqrt{2}$ form the Beatty pair given by Theorem 1. Let $L^{\mathrm{P}} = (\lfloor n(1+\sqrt{2}) \rfloor)$ and $U^{\mathrm{P}} = (\lfloor n(1 + \frac{1}{2}\sqrt{2}) \rfloor)$ be the associated Beatty sequences. One has $L^{\mathrm{P}} =$ A003151 and $U^{\mathrm{P}} =$ A003152.

According to Theorem 2, with R the slow Beatty sequence A049472 given by $R(n) = \lfloor \frac{1}{2}\sqrt{2} n \rfloor$, the following holds for the Hofstadter Pell sequence H:

$$H(L^{\mathrm{P}}(n)) = n \quad \text{and} \quad H(U^{\mathrm{P}}(n)) = R(n), \text{ for all } n \geq 1.$$

The sequence with L^{P} and U^{P} swapped is

A109250 $= 2, 1, 4, 3, 7, 9, 5, 12, 6, 14, 16, 8, 19, 10, 21, 11, 24, 26 \ldots.$

Apparently there is nothing comparable to the averaging phenomenon that occurred in the golden mean case.

Remark 7. See A078474, and in particular A286389 for two generalizations of Hofstadter's recursion, with conjectured expressions similar to Equation (1). The conjecture for A286389 was recently proved by Shallit [13].

For the recursion $a(n) = n - \lfloor \frac{1}{2} a(a(n-1)) \rfloor$ given in A138466, Cloitre proved that $(a(n))$ satisfies Equation (1) with $\gamma = \sqrt{3} - 1$. For generalizations of this, see A138467.

4 Greediness

There is a more natural way to define the Wythoff swap sequence W, which at first sight has nothing to do with Wythoff sequences. Venkatachala [17] considered the following greedy algorithm: $f(1) = 1$, and for $n \geq 2, f(n)$ is the least natural number such that

$$(a)\ f(n) \notin \{f(1), \ldots, f(n-1)\}; \quad (b)\ f(1) + f(2) + \cdots + f(n) \text{ is divisible by } n.$$

Surprisingly, it follows from Venkatachala's analysis that one has

$$W(n) = f(n+1) - 1$$

for all $n \geq 1$. The recent paper [3] studied a sequence z defined by a similar greedy algorithm: $z(1) = 1$, and for $n \geq 2, z(n)$ is the least natural number such that

$$(a)\ z(n) \notin \{z(1), \ldots, z(n-1)\}; \quad (b)\ z(1) + z(2) + \cdots + z(n) \equiv 1 \pmod{n+1}.$$

This entails that $(m(n))$, defined by $m(n) := (z(2) + \cdots + z(n))/(n+1)$ for $n \geq 1$, is a sequence of integers.

These sequences have been analyzed by Shallit [12] using the computer software `Walnut`. Our Theorem 9 is an improvement of [12, Theorem 6]. In the proof of Theorem 9 we need the values of the Wythoff sequences at the Fibonacci numbers.

Lemma 8. *Let L and U be the Wythoff sequences. Then*

$$L(F_{2k}) = F_{2k+1} - 1; \tag{8}$$
$$U(F_{2k}) = F_{2k+2} - 1; \tag{9}$$
$$L(F_{2k-1}) = F_{2k}; \tag{10}$$
$$U(F_{2k-1}) = F_{2k+1}. \tag{11}$$

for all $k \geq 1$.

Proof. These equations can be derived from [3, Lemma 2.D]. Another, easy, proof is based on recalling that $L(m)$ gives the position of the m^{th} 0 in the infinite Fibonacci word $0100101\ldots$ generated by the morphism $\mu : 0 \mapsto 01, 1 \mapsto 0$ (see, e.g., [1, Corollary 9.1.6]). The infinite Fibonacci word is the limit of the words $\mu(0) = 01, \mu^2(0) = 010, \mu^3(0) = 01001, \mu^4(0) = 01001010, \ldots$.

Let $|w|$, $|w|_0$, and $|w|_1$ denote the length, the number of 0's and the number of 1's of a word w. Then it is easy to see that

$$|\mu^m(0)| = F_m, \quad |\mu^m(0)|_0 = F_{m-1}, \quad |\mu^m(0)|_1 = F_{m-2}, \tag{12}$$

for all $m \geq 1$, where μ^m is the m^{th} iterate of μ. Since $\mu^m(0)$ ends in 01 for odd m, Equation (12) with $m = 2k+1$ implies that Equation (8) holds. Similarly, since $\mu^m(0)$ ends in 10 for even m, one obtains Equation (9). That Equation (10) is correct follows from Equation (12) with $m = 2k$, since $\mu^m(0)$ ends with 0 for odd m. Similarly, since $\mu^m(0)$ ends with 1 for odd m, one obtains Equation (11) with $m = 2k+1$. □

Theorem 9. *Let $(z(n))$ and $(m(n))$ be the Avdivpahic and Zejnulahi sequences. Let W be the Wythoff swap sequence. Then for all $n \geq 1$ we have*

$$\begin{aligned}
&\text{(a) } z(n) = W(n) \quad \text{except if } n = F_{2k+1}-1 \text{ or } n = F_{2k+1},\\
&\text{In fact, } z(F_{2k+1}-1) = F_{2k}-1, z(F_{2k+1}) = F_{2k+2},\\
&\qquad W(F_{2k+1}-1) = F_{2k+2}-1, W(F_{2k+1}) = F_{2k}.\\
&\text{(b) } m(n) = \overline{W}(n) \quad \text{except if } n = F_{2k+1}-1.\\
&\text{In fact, } m(F_{2k+1}-1) = F_{2k}-1, \overline{W}(F_{2k+1}-1) = F_{2k}.
\end{aligned}$$

Proof. Part (a): We use the formula for $z(n)$ proved in the paper [3]:

$$z(n) = \begin{cases} F_{k-1} - 1, & \text{if } n = F_k - 1;\\ F_{k+1}, & \text{if } n = F_k;\\ L(k), & \text{if } n = U(k);\\ U(k), & \text{if } n = L(k). \end{cases}$$

So z is the swapping of L and U for indices $n \neq F_k - 1$ and $n \neq F_k$. We first handle the case of the Fibonacci numbers with an odd index. Here we have to prove that

$$W(F_{2k+1} - 1) = F_{2k+2} - 1 \tag{13}$$
$$W(F_{2k+1}) = F_{2k}. \tag{14}$$

We start with Equation (13). We have to show that there exists m such that either the pair of equations $L(m) = F_{2k+1} - 1$, and $U(m) = F_{2k+2} - 1$, or the pair of equations $U(m) = F_{2k+1} - 1$, and $L(m) = F_{2k+2} - 1$ holds. The first pair of these swapping equations, with the value $m = F_{2k}$, is equal to the equations (8), and (9), as we see from Lemma 8.

Next, we prove Equation (14). Here Lemma 8 gives that Equation (11) and Equation (10) solve the swapping equations.

We still have to handle the case with Fibonacci numbers with an even index. There we have to prove that

$$W(F_{2k} - 1) = z(F_{2k} - 1) = F_{2k-1} - 1 \tag{15}$$
$$W(F_{2k}) = z(F_{2k}) = F_{2k+1}. \tag{16}$$

We start with Equation (16). Here Lemma 8 gives that Equation (10) and Equation (11) solve the swapping equations.

Next, we prove Equation (15). Here Lemma 8 gives that Equation (8) and Equation (9) solve the swapping equations, both with k shifted by 1.

Part (b): The case $n = 1$: for $k = 1$, $F_3 - 1 = 1$, and $m(1) = 1 = \overline{W}(1) - 1$.
For $n \geq 2$ we have

$$(n+1)m(n) = z(2) + \cdots + z(n), \; (n+1)\overline{W}(n) = W(1) + \cdots + W(n).$$

We see that for $n = 2,\ 3m(2) = z(2) = 3$, and $3\overline{W}(n) = W(1) + W(2) = 2 + 1 = 3$. So also

$$(n+1)m(n) = 3 + z(3) + \cdots + z(n),\ (n+1)\overline{W}(n) = 3 + W(3) + \cdots + W(n).$$

Note furthermore that $z(F_{2k+1} - 1) + z(F_{2k+1}) = F_{2k} - 1 + F_{2k+2}$, and $\overline{W}(F_{2k+1} - 1) + \overline{W}(F_{2k+1}) = F_{2k+2} - 1 + F_{2k}$. Since these two sums are equal, the difference of 1 created at $n = F_{2k+1} - 1$ is 'repaired' at $n = F_{2k+1}$. This proves the first part of Part (b).

For the second part we have to see that $m(F_{2k+1}-1) = F_{2k} - 1$, or equivalently (see the proof of the first part of Part (b)), that $\overline{W}(F_{2k+1} - 1) = F_{2k}$.

In general we have (see Theorem 3)

$$\overline{W}(n) = G(n) = \lfloor (n+1)\gamma \rfloor = \lfloor (n+1)/\varphi \rfloor = \lfloor (n+1)\varphi \rfloor - (n+1) = L(n+1) - (n+1)$$

for all $n \geq 1$. So, using Equation (10), we obtain

$$\overline{W}(F_{2k+1} - 1) = L(F_{2k+1}) - F_{2k+1} = F_{2k+2} - F_{2k+1} = F_{2k}, \tag{17}$$

which ends the proof of Part (b). □

Let $(a(n), b(n))$ defined by the recurrences $a(n) = n - b(a(n-1)), b(n) = n - a(b(n-1))$ be the "married" functions of Hofstadter given in his book [10, p. 137]. Here $(a(n))$ is A005378 and $(b(n))$ is A005379.

Theorem 10. [**Stoll**][16] *Let $\gamma = (\sqrt{5} - 1)/2$ be the small golden mean. Then*

$$\begin{aligned} &\text{(a) } a(n) = \lfloor (n+1)\gamma \rfloor \quad \text{except if } n = F_{2k}-1 : a(F_{2k}-1) = \lfloor F_{2k}\gamma \rfloor + 1; \\ &\text{(b) } b(n) = \lfloor (n+1)\gamma \rfloor \quad \text{except if } n = F_{2k+1}-1 : b(F_{2k+1}-1) = \lfloor F_{2k+1}\gamma \rfloor - 1 \end{aligned}$$

for all $n \geq 1$.

It follows by combining Theorem 3, Theorem 9, Stoll's Theorem 10, and Equation (17) that

$$(b(n)) = (m(n)).$$

Then Stoll's theorem also gives an expression for $(a(n))$. See Shallit's paper [12] for proofs using the computer software Walnut.

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