Constructing gradient-$\mathcal{S}_p$ quantum Markov semi-groups to obtain strong solidity results for von Neumann algebras

by

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Abstract

In this thesis we will for a quantum Markov semi-group $(\Phi_t)_{t\geq 0}$ on a finite von Neumann algebra $\mathcal{N}$ with a trace τ, investigate the property of the semi-group being gradient-$\mathcal{S}_p$ for some $p \in [1, \infty]$. This property was introduced in [12] (see also [9]) and has been studied in [9, 10, 12] for quantum Markov semi-groups on compact quantum groups and on q-Gaussian algebras. Beyond these classes the property gradient-$\mathcal{S}_p$ has not been studied; in particular for groups and their operator algebras no (non-trivial) examples were known before this thesis. The main aim of this thesis is therefore to construct interesting examples of quantum Markov semi-groups that possess the gradient-$\mathcal{S}_p$ property.

The reason why we are interested in constructing such semi-groups, is because they can be used to obtain properties like the Akemann-Ostrand property (AO^+) and strong solidity for the underlying von Neumann algebra. Over the last decade, these properties have become a topic of interest and have been studied for several von Neumann algebras, see [3, 8, 9, 10, 12, 23, 32, 33, 37, 41].

In this thesis we shall focus on group von Neumann algebras $(\mathcal{L}(\Gamma), \tau)$ for certain discrete groups Γ that possess the Haagerup property. Namely, for such groups there exists a proper, conditionally negative definite function ψ on Γ. We can then define an unbounded operator Δ_ψ on the GNS-Hilbert space $L^2(\mathcal{L}(\Gamma), \tau)$ as $\Delta_\psi(\lambda_{\mathbf{v}}) = \psi(\mathbf{v})\lambda_{\mathbf{v}}$ and consider the corresponding quantum Markov semi-group $(e^{-t\Delta_\psi})_{t\geq 0}$. For this semi-group we can investigate for what p it has the gradient-$\mathcal{S}_p$ property. In particular we will be considering group von Neumann algebras of Coxeter groups. Namely, a Coxeter group W possesses the Haagerup property by [4], and a proper conditionally negative function ψ on W is given by the minimal word length $\psi(\mathbf{w}) = |\mathbf{w}|$ w.r.t some set of generators. We will 'almost completely' characterize for what types of Coxeter systems the semi-group corresponding to the word length is gradient-$\mathcal{S}_p$. Moreover, in the cases that we get the gradient-$\mathcal{S}_2$ property, we obtain the Akemann-Ostand property (AO^+) and strong solidity for $\mathcal{L}(W)$.

Hereafter, we will also consider other quantum Markov semi-groups on $\mathcal{L}(W)$. We consider word lengths that arise by putting different weights on the generators, and consider the semi-groups associated to these proper, conditionally negative functions. From this we obtain (AO^+) and strong solidity for $\mathcal{L}(W)$ for some other cases.

Thereafter, we will generalize some of our results obtained for $\mathcal{L}(W)$ to the Hecke algebras $\mathcal{N}_q(W)$, which are q-deformations of $\mathcal{L}(W)$.

For the case of group von Neumann algebras $\mathcal{L}(\Gamma)$ for general groups, we shall examine for semi-groups induced by a proper, conditionally negative function ψ, how the gradient-$\mathcal{S}_p$ property of the semi-group $(\Phi_t)_{t\geq 0} := (e^{-t\Delta_\psi})_{t\geq 0}$ relates to the gradient-$\mathcal{S}_q$ property of the semi-group $(e^{-t\Delta_\psi^\alpha})_{t\geq 0}$ that is generated by the αth-root Δ_ψ^α of the generator.

Last, we will also show a method that allows us, for right-angled word hyperbolic Coxeter groups, to obtain (AO^+) and strong solidity for $\mathcal{L}(W)$ without building a gradient-$\mathcal{S}_p$ quantum Markov semi-group, but by using a slightly different method.

Contents

Preface

Throughout my study mathematics I got more and more interested in the field of analysis, and more specifically in the theory of operator algebras. When I had to choose a thesis project, it was not hard to decide that this should also be in the field of operator algebras. I contacted Martijn Caspers who suggested several possible projects to me. I choose this project on the subject of quantum Markov semi-groups as it seems very interesting.

I started working on my master thesis in November 2020, which was two months later than initially planned. This was due to a recovery from a car accident that I had in August, together with three friends of mine. This was a quite unexpected detour. Luckily, all of us have recovered very well, and were able to restart our study programs.

When I started working on the project, I began with reading into the literature. I very much liked the fact that the project involved different subjects. Namely, there was the operator theory about von Neumann algebras and quantum Markov semi-groups, and also there was the algebraic component of group theory. The project went very steady. The fact of a global pandemic was not really a problem, as I could just work from home, and have weekly meeting with Martijn online.

Now that I am almost finished, I want to thank Martijn very much for all his good help and feedback during the project. Also I want to thank the other members of the thesis committee. Last, I want to thank my family.

1. Introduction

In this thesis we will consider quantum Markov semi-groups and study a property of these semi-groups that was defined in [9, 12]. Quantum Markov semi-groups that possess that property have some interesting consequences, that were shown in the same paper. This property, called gradient-$\mathcal{S}_p$, was introduced to solve an open question in the theory of compact quantum groups, namely the strong solidity of free orthogonal quantum groups, see [9]. The property was studied further in [10] for more general compact quantum groups and [12] for q-Gaussian algebras. Beyond these classes the property gradient-$\mathcal{S}_p$ has not been studied; in particular for groups and their operator algebras no (non-trivial) examples were known before this thesis. The main aim of this thesis is therefore to construct interesting examples of quantum Markov semi-groups that possess the gradient-$\mathcal{S}_p$ property. Similar to [12] this will allow us to obtain interesting results for the underlying von Neumann algebra.

In this section we shall introduce the main topics of this thesis, without delving to deep in the theory. For a more advanced introduction on the topics we refer to the preliminaries. We shall moreover finish this section by giving an overview of the structure of this thesis.

In this section we start by giving some context on what quantum Markov semi-groups are. This we do by first considering classical Markov semi-groups which arise from random processes. After this we turn to quantum Markov semi-groups, which can be regarded as non-commutative analogues of the classical Markov semi-groups. We shall give the definition of these semi-groups, and show their connection to the classical case. We moreover state some properties of these semi-groups, and discuss their appearance in physics. Hereafter we shall also give multiple examples of quantum Markov semi-groups for the convenience of the reader. Thereafter we shall state what the gradient-$\mathcal{S}_p$ is about. We shall shortly discuss why we are interested in quantum Markov semi-groups that posses this property. Furthermore, we shall give some background on Coxeter groups. These groups play a significant role in this thesis as they can be used to build von Neumann algebras and quantum Markov semi-groups of which we can study the gradient-$\mathcal{S}_p$ property. Finally, we end by giving an outline of this thesis.

1.1. **Classical Markov semi-groups.** We give an overview of the classical Markov semi-groups, that originate from random processes. In a way they tell how a probability distribution behaves over time. We shall start by introducing discrete time, homogeneous Markov chains, and then turn to continuous time Markov chains. For more theory on classical Markov semi-groups we refer to [1, Chapter 1].

1.1.1. *Discrete time, homogeneous Markov Chains.* We let the state space $\mathcal{S} = \{s_1, ..., s_k\}$ be a finite set, whose elements we refer to as 'states'. A discrete time Markov chain on $\mathcal{S}$ then is a sequence $(X_n)_{n\in\mathbb{N}}$ of random variables, satisfying the Markov property. This is the property that

$$\mathbb{P}(X_{N+1} = x_{N+1}|X_0 = x_0, X_1 = x_1, ..., X_N = x_N) = \mathbb{P}(X_{N+1} = x_{N+1}|X_{N+1} = x_N) \tag{1}$$

for all $N \geq 0$, and all possible states $x_0, x_1, ..., x_N, x_{N+1} \in S$ for which the conditional expectations both exist. In words, the Markov property stands for the fact that the process is 'memoryless' in the sense that the process from time n only depends on the current value and not on the previous values. The Markov chain is moreover called homogeneous when $\mathbb{P}(X_{N+1} = a|X_N = b)$ is independent of the value of N.

A discrete time, homogeneous Markov chain $(X_n)_{n\geq 0}$ can be regarded as a random walk on the set of states. The random walk starts in some random state X_0, and moves from state s_i to state s_j in the next 'turn' according to a single value $p_{i,j} := \mathbb{P}(X_{N+1} = s_j|X_n = s_i)$ that does not depend on N. The $|\mathcal{S}| \times |\mathcal{S}|$ matrix P defined as $P_{i,j} = p_{j,i}$ consists of all these transition probabilities and then completely describes the behavior of the random walk. For example, if we have an initial distribution $\pi = (\pi_1, \pi_2, ..., \pi_{|S|})^T$ for the value of X_0, then the distribution for X_1 is given by $P\pi$. More generally it holds that $P^n\pi$ describes the probability distribution of X_n. The maps P^n thus describe how the initial distribution behaves over (discrete) time. They can be regarded as linear operators on $L^\infty(\mathcal{S}, \mu)$ where μ is the counting measure. The family of maps $(P^n)_{n\geq 0}$ together form a semi-group in the usual sense, that is $P^nP^m = P^{n+m}$. We moreover have that $\sum_{i\in\mathcal{S}}(P^n\pi)(i) = 1 = \sum_{i\in\mathcal{S}} \pi(i)$ as the probability of 'being somewhere' is always 1. More general we thus have that $\sum_{i\in\mathcal{S}}(P^nf)(i) = \sum_{i\in\mathcal{S}} f(i)$ for all $f \in L^\infty(\mathcal{S}, \mu)$. We also note that the maps P^n map the positive element $\pi \geq 0$ to a positive element $P^n\pi \geq 0$, as probabilities are always positive. More generally we thus have that P^n maps positive elements from $L^\infty(\mathcal{S}, \mu)$ to positive elements from $L^\infty(\mathcal{S}, \mu)$.

We summarize these results. The family of linear maps $(P^n)_{n\geq 0}$ on $L^\infty(\mathcal{S}, \mu)$ satisfies

(1) We have $P^0 = \mathrm{Id}_{L^\infty(S,\mu)}$.
(2) For $n, m \geq 0$ we have $P^n P^m = P^{n+m}$.
(3) For $n \geq 0$ and $f \in L^\infty(\mathcal{S}, \mu)$ we have $\int_{\mathcal{S}} P^n f d\mu = \int_{\mathcal{S}} f d\mu$.
(4) For $n \geq 0$ the map P^n is positive on $L^\infty(\mathcal{S}, \mu)$.

Now suppose that our random walk is also symmetric in the sense that moving from state s_j to state s_i has the same probability as moving from state s_i to state s_j. This is to say that $p_{i,j} = p_{j,i}$, and hence that the transition matrices P^n are self-adjoint. This means that we moreover have that

$$\int_S (P^n f) \cdot \overline{g} d\mu = \int_S f \cdot \overline{(P^n g)} d\mu \quad \text{for } f, g \in L^\infty(S, \mu) \text{ and } n \geq 0. \tag{2}$$

Moreover, for the function $1 \in L^\infty(\mathcal{S}, \mu)$ we then have that $(P1)_i = \sum_{j=1}^{|S|} P_{i,j} = \sum_{j=1}^{|S|} P_{j,i} = \sum_{j=1}^{|S|} \mathbb{P}(X_1 = s_j | X_0 = s_i) = 1$. This means that the maps P^n for $n \geq 1$ are moreover unital, that is $P^n 1 = 1$. The maps $(P^n)_{n\geq 0}$ corresponding to a discrete time, time homogeneous Markov chain that is moreover symmetric, have a connection to the semi-groups that we will consider. However, the semi-groups that we consider are actually continuous over time, that is we consider maps $(\Phi_t)_{t\geq 0}$ with $t \in \mathbb{R}_+$.

1.1.2. *Continuous time Markov chains.* We now turn to continuous time Markov chains, which are more closely related to quantum Markov semi-groups. A continuous time Markov chain $(X_t)_{t\in\mathbb{R}_+}$ on a set $\mathcal{S} = \{s_1, ..., s_k\}$ of states is a random process in which you change from a state to another random state like in the discrete case. However, in this case the moment when you move to the next state depends on an exponential random variable T. Equivalently, when in a state s_i at time $t = 0$, you move to another state s_j according to some exponential random variables $\{T_{i,j} : j \neq i\}$ with parameters $\{q_{i,j} > 0 : j \neq i\}$, where $q_{i,j}$ is the parameter of the exponential distribution $T_{i,j}$. Namely at time $T := \min_{j\neq i} T_{i,j}$ you move from state s_i to s_J, where $J = \min \arg_{j\neq i} T_{i,j}$.

A continuous time Markov chain $(X_t)_{t\geq 0}$ can entirely be described by a $|\mathcal{S}| \times |\mathcal{S}|$ matrix Q that is given by $Q_{i,j} = q_{j,i}$ whenever $j \neq i$ and otherwise by $Q_{i,i} = -\sum_{j\neq i} q_{i,j}$ (this is actually the adjoint of the transition rate matrix). We now denote the matrices $P_t := e^{-tQ} := \sum_{k\geq 0} \frac{(-tQ)^k}{k!}$. Now, if the initial distribution of X_0 is given by π, then it is true that the distribution of X_t is given by $P_t\pi$. The maps $(P_t)_{t\geq 0}$ thus describe how the probability distribution behaves over time. The matrices P_t can be regarded as linear maps on $L^\infty(\mathcal{S}, \mu)$ and satisfy similar properties as before, namely

(1) We have $P_0 = \mathrm{Id}_{L^\infty(S,\mu)}$.
(2) For $t, r \geq 0$ we have $P_t P_r = P_{t+r}$.
(3) For $t \geq 0$ and $f \in L^\infty(\mathcal{S}, \mu)$ we have $\int_{\mathcal{S}} P_t f d\mu = \int_{\mathcal{S}} f d\mu$.
(4) For $t \geq 0$ the map P_t is positive on $L^\infty(\mathcal{S}, \mu)$.

Also, when the Markov chain is symmetric in the sense that $q_{i,j} = q_{j,i}$ for all i, j then the matrix Q is self-adjoint. We then also have that P_t is self-adjoint for $t \geq 0$ so that $\int_{\mathcal{S}} (P_t f) \cdot \overline{g} d\mu = \int_{\mathcal{S}} f \cdot \overline{(P_t g)} d\mu$ holds for $g, f \in L^\infty(\mathcal{S}, \mu)$. Moreover, as Q is self-adjoint we have that $(Q1)_i = \sum_{j=1}^{|\mathcal{S}|} Q_{i,j} = \sum_{j=1}^{|\mathcal{S}|} Q_{j,i} = Q_{i,i} + \sum_{j\neq i} q_{i,j} = 0$. This shows that $Q1 = 0$ and hence that $P_t 1 = 1$ for $t \geq 0$ in this case.

Also, a property that we have for these Markov chains is that the map $t \mapsto P_t$ is continuous. Namely for $0 \leq t \leq r$ we have $\|P_t - P_r\| \leq \|P_t\| \cdot \|I - P_{r-t}\| \leq \|I - P_{r-t}\| \leq \sum_{k\geq 1} \frac{\|(t-r)Q\|^k}{k!} \leq e^{|t-r|\cdot\|Q\|} - 1$, which shows the continuity.

1.2. **Quantum Markov semi-groups.** We shall introduce here the definition of a quantum Markov semi-group on a von Neumann algebra. For a more precise introduction to this topic we refer to the preliminaries.

1.2.1. *Definition.* A von Neumann algebra $\mathcal{N}$ is a strongly closed $*$-subalgebra of the bounded operators $B(H)$ on some complex Hilbert space H, with $\mathrm{Id}_H \in \mathcal{N}$. We consider von Neumann algebras that are finite, which means that every isometry in $\mathcal{N}$ is a unitary. For such algebras there exists a normal faithful trace τ on $\mathcal{N}$. A quantum Markov semi-group on $\mathcal{N}$ is then defined a family $(\Phi_t)_{t\geq 0}$ of maps $\Phi_t : \mathcal{N} \to \mathcal{N}$ such that

(1) We have $\Phi_0 = Id_{\mathcal{N}}$.
(2) For $t, s \geq 0$ we have $\Phi_t \Phi_s = \Phi_{t+s}$.
(3) For $x \in \mathcal{N}$ he map $t \mapsto \Phi_t(x)$ from $[0, \infty) \to \mathcal{N}$ is continuous in the strong topology of $\mathcal{N}$.
(4) For $t \geq 0$ the map Φ_t is trace preserving, that is $\tau(\Phi_t(x)) = \tau(x)$ for $x \in \mathcal{N}$.
(5) For $t \geq 0$ the map Φ_t is unital completely positive.

(6) For $t \geq 0$ the map Φ_t is symmetric, that is $\tau(x\Phi_t(y)) = \tau(\Phi_t(x)y)$ for all $x, y \in \mathcal{N}$.

Before we examine this definition more closely, we show that the classical Markov semi-group $(P_t)_{t\geq 0}$ that we obtain from a continuous time (symmetric) Markov chain is in fact a quantum Markov semi-group. Indeed, the family $(P_t)_{t\geq 0}$ consists of linear maps on $L^\infty(\mathcal{S}, \mu)$ and this space is actually a finite von Neumann algebra. Moreover, a trace on $L^\infty(\mathcal{S}, \mu)$ is given by $\tau(f) = \int_S f d\lambda$. By what we have already shown, it follows directly that the maps $(P_t)_{t\geq 0}$ satisfy conditions (1),(2),(4) and (6). Property (3) also follows from the continuity that we had already shown. For condition (5) we note that we already know that the maps P_t are unital and positive. The notion of complete positivity is a condition that is even stronger than positivity. However, since the von Neumann algebra $L^\infty(\mathcal{S}, \mu)$ is abelian the notion of complete positivity coincides with positivity, see [35, Theorem 3.9.]. We thus obtain that the semi-group $(P_t)_{t\geq 0}$ is in fact a quantum Markov semi-group.

We elaborate some more on the definition of a quantum Markov semi-group. First of all, we note that the fact that Φ_t is trace preserving (condition (4)) will already follow from the fact that Φ_t is unital and symmetric. The reason we still included this condition is because there are also notions of quantum Markov semi-groups where the maps Φ_t are not assumed to be symmetric. Throughout this thesis we will only consider the symmetrical case, however.

We note that it follows from condition (4) and (5) and from the Kadison-Schwarts inequality that for $x \in \mathcal{N}$ we have

$$\tau(\Phi_t(x)^*\Phi_t(x)) \leq \tau(\Phi_t(x^*x)) = \tau(x^*x). \tag{3}$$

The maps Φ_t can therefore be extended to bounded operators on the GNS-Hilbert space $L^2(\mathcal{N}, \tau)$. The operators $(\Phi_t)_{t\geq 0}$ then form a C_0-semi-group on $L^2(\mathcal{N}, \tau)$. For such semi-groups there is an unbounded operator Δ on $L^2(\mathcal{N}, \tau)$ that we call its generator. This operator is defined by

$$\Delta(x) = -\lim_{t\downarrow 0} \frac{\Phi_t(x) - x}{t} \tag{4}$$

for those x where this limit exists. By the properties of Φ_t it moreover follows that Δ positive. Informally we shall sometimes write $e^{-t\Delta}$ to denote the operator Φ_t. Also, we shall simply write Φ_t for both the operator on $\mathcal{N}$ and that on $L^2(\mathcal{N}, \tau)$.

1.2.2. *Applications in physics.* The theory of quantum Markov semi-groups is a vast subject that originates in quantum mechanics. We give some context on how they appear in physics, for more on this theory see [28, 34].

In quantum mechanics, a physical system is described by a (complex) Hilbert space $\mathcal{H}$. The state of the system is given by a certain unit vector ψ_t. This vector describes all relevant physical properties of the system at time t and it changes over time according to the Schrodinger equation

$$i\hbar \frac{d}{dt}\psi_t = H(t)\psi_t. \tag{5}$$

Here $\hbar$ is a physical constant, and $H(t)$ is a closed densely defined operator called the Hamiltonian of the system. The operator corresponds to the total energy of the system. When this operator is time-independent the solution to the Schrödinger equation is given by $\psi_t = e^{-it\hbar H}\psi_0$. This shows the appearance of a semi-group. For physical reasons the maps of the semi-groups must be positive. Moreover, the need for the maps to be completely positive follows by combining multiple physical systems and considering them as a single system, see [29].

At a given moment, one can execute a measurement to observe certain physical quantities of the system, like for example the position, the momentum, or the spin of a particle. Such measurements are described by self-adjoint operators called observables. The spectrum of such operator consists of all possible outcomes that could be observed. Generally multiple outcomes are possible, in which case the system is said to be in superposition. What precise outcome is measured is probabilistic.

Let us for example measure the position of a particle on a line at time t. This measurement corresponds to some observable Q. The probability of observing a certain eigenvalue λ of Q is given by $|\langle P_\lambda\psi_t, \psi_t\rangle|^2$, where P_λ is the projection to the eigenspace of Q corresponding to the eigenvalue λ. After we preformed the measurement and found the particle's position to be λ_0, the state of the physical system immediately changes to the state $\psi_{t^+} = \frac{P_{\lambda_0}\psi_t}{\sqrt{\langle P_{\lambda_0}\psi_t, \psi_t\rangle}}$. At that moment the particles position is no longer in superposition, as we have just measured its value. Directly preforming the same measurement again will not change the outcome. However, over time the state of the physical system will again change according to the Schrödinger equation and one can obtain other outcomes for the measurements.

1.3. **Examples of quantum Markov semi-groups.** We have seen that the classical Markov semi-groups are examples of quantum Markov semi-groups. There are also many other ways to build quantum Markov semi-groups. A large variety of examples of quantum Markov semi-groups $(\Phi_t)_{t\geq 0}$ can be constructed on group von Neumann algebras using conditionally negative definite functions. This construction plays a central role in this thesis, and will be thoroughly discussed in section 5. At this point, however, we shall show some other ways to build quantum Markov semi-groups, with the goal to better illustrate the definition.

1.3.1. *Trivial quantum Markov semi-group.* For an arbitrary von Neumann algebra $\mathcal{N}$ with a trace τ we can build a trivial quantum Markov semi-group by setting $\Phi_t = Id_{\mathcal{N}}$ for $t \geq 0$. The generator of this semi-group is the operator $\Delta = 0$.

1.3.2. *Heat semi-group.* We let $\mathbb{T} \subseteq \mathbb{C}$ be the unit circle and consider the abelian von Neumann algebra $\mathcal{N} = L^\infty(\mathbb{T})$ of bounded functions on the torus. We let the trace τ be given by integration. We then have that the GNS-Hilbert space $L^2(\mathcal{N}, \tau)$ is just $L^2(\mathbb{T})$. An orthonormal basis for $L^2(\mathbb{T})$ is given by $\{e_k\}_{k\in\mathbb{Z}}$ where $e_k : \mathbb{T} \to \mathbb{C}$ is defined as $e_k(z) = z^k$. We can then define an unbounded operator Δ on $L^2(\mathbb{T})$ as $\Delta(e_k) = k^2 e_k$. This operator actually gives rise to a quantum Markov semi-group $(\Phi_t)_{t\geq 0}$ on $L^\infty(\mathbb{T})$, by setting $(\Phi_t)_{t\geq 0} := (e^{-t\Delta})_{t\geq 0}$.

1.3.3. *Semi-group on matrix algebra* $\mathrm{Mat}_2(\mathbb{C})$. We shall now give an example of a quantum Markov semi-group on a non-abelian von Neumann algebra. We denote $\mathcal{N} = \mathrm{Mat}_2(\mathbb{C})$ with the standard normalized matrix trace $\tau = \frac{1}{2}\,\mathrm{tr}$. Then for $t \geq 0$ we define the linear mapping $\Phi_t : \mathcal{N} \to \mathcal{N}$ as

$$\Phi_t(\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}) = \begin{pmatrix} a_{1,1} & e^{-t}a_{1,2} \\ e^{-t}a_{2,1} & a_{2,2} \end{pmatrix}. \tag{6}$$

These maps are clearly unital, and it can also be seen that these maps satisfy $\Phi_t\Phi_s = \Phi_{t+s}$ for $t, s \geq 0$. Moreover, for $A = (a_{i,j}), B = (b_{i,j}) \in \mathrm{Mat}_2(\mathbb{C})$ and $t \geq 0$ a computation shows that $\mathrm{tr}(A\Phi_t(B)) = \mathrm{tr}(\Phi_t(A)B)$. The fact that the maps Φ_t are also completely positive is a little more technical, but this follows from the fact that the matrix $M_t = \begin{pmatrix} 1 & e^{-t} \\ e^{-t} & 1 \end{pmatrix}$ for $t \geq 0$ is positive. Namely, the mappings Φ_t are actually Schur multipliers associated to the symbols M_t and these are completely positive (see [35, Theorem 3.7] for details).

1.3.4. *Example non-symmetric quantum Markov semi-group.* We give an example of a family of maps $(\Phi_t)_{t\geq 0}$ that form a non-symmetric quantum Markov semi-group, that is the maps lack the condition (6) in the definition of a quantum Markov semi-group. Let $\mathcal{N} = \mathrm{Mat}_n(\mathbb{C})$ with normalized matrix trace $\tau = \frac{1}{n}\,\mathrm{tr}$. For a self-adjoint matrix $b \in \mathrm{Mat}_n(\mathbb{C})$ we build maps $\Phi_t : \mathcal{N} \to \mathcal{N}$ by defining

$$\Phi_t(a) = e^{-itb}ae^{itb}. \tag{7}$$

That these maps are completely positive can be checked from the definition (see also [35, Theorem 4.1] and concluding remarks). Furthermore, they are clearly unital and preserve trace as $\mathrm{tr}(\Phi_t(a)) = \mathrm{tr}(e^{-itb}ae^{itb}) = \mathrm{tr}(ae^{itb}e^{-itb}) = \mathrm{tr}(a)$. These maps are generally not symmetric however. Thus the maps $(\Phi_t)_{t\geq 0}$ generally do not form a quantum Markov semi-group in the way we define it, and we will not study these.

1.4. **The gradient-$\mathcal{S}_p$ property and its importance.** For a quantum Markov semi-group and for $p \in [1, \infty]$, the gradient-$\mathcal{S}_p$ property was introduced in [12] (see also [9] for the case $p = 2$). This property will be the main study in this thesis. The gradient-$\mathcal{S}_p$ property of a semi-group $(e^{-t\Delta})_{t\geq 0}$ is involved with the question whether, for certain $a, b \in \mathcal{N}$, the map $\Psi^{a,b}$ given by

$$\Psi^{a,b}(x) = -\frac{1}{2}(\Delta(axb) + a\Delta(x)b - a\Delta(xb) - \Delta(ax)b) \tag{8}$$

is contained in the Schatten p-ideal $\mathcal{S}_p$, when considered as a map on $B(L^2(\mathcal{N}, \tau))$. We shall give the precise definition of this property only later in section 3. We now only say something about why we are interested in semi-groups that have this property. This is because, in the paper [12] it was shown that the existence of a semi-group with this property (under some additional conditions) can be used to obtain interesting results for the underlying von Neumann algebra, namely the Akemann-Ostrand property (AO^+), and strong solidity. These two properties have become a topic of interest and have been studied for several kinds of von Neumann algebras, see [3, 8, 9, 10, 12, 23, 32, 33, 37, 41]. The two properties are quite technical, and will only be discussed in section 4. At this moment however, it is important to highlight that for a von Neumann algebra $\mathcal{N}$ we are thus only interested in building a single semi-group that has the gradient-$\mathcal{S}_p$ property. Namely, this will already give us, for the von Neumann algebra $\mathcal{N}$, the results that we are interested in. We also note that the trivial semi-group is gradient-$\mathcal{S}_p$ for all $p \in [1, \infty]$, but we are not interested in this semi-group, as it does not satisfy some additional conditions that we need to find the results for the algebra $\mathcal{N}$. In particular, a condition is that the generator Δ needs to have compact resolvent.

1.5. **Coxeter groups.** Another topic that we introduce here are Coxeter groups. These groups play a significant role in this thesis as they can be used to construct von Neumann algebras together with quantum Markov semi-groups on these algebras. We shall give here an informal definition of these groups, and show how these groups can be interpreted geometrically.

A Coxeter group, is a group W that is generated by some finite set $S = \{s_1, ...s_n\}$. The generators are moreover assumed to satisfy certain Coxeter relations, which are relations of the form $(s_i s_j)^{m_{i,j}} = e$. Here e represents the unit of W, and the quantity $m_{i,j} \in \mathbb{N} \cup \{\infty\}$ is such that $m_{i,j} = 1$ when $i = j$ and $m_{i,j} \geq 2$ otherwise. We note that the restriction $m_{i,i} = 1$ implies that every generator has order 2. Furthermore, we note that with the property $m_{i,j} = \infty$ we mean that no relation of the form $(s_i s_j)^m$ for $m \geq 1$ exists. Another assumption that the Coxeter groups satisfy, is that the relations $(s_i s_j)^{m_{i,j}}$ for $s_i, s_j \in S$, are the only non-trivial relations that we have for elements in the group. That is, we can only apply algebraic manipulations that follow from these relations and the group axioms. A Coxeter group is thus completely determined by the number of generators and the values $m_{i,j}$.

Simple examples of Coxeter group are given by the Dihedral groups D_n, which represent the symmetries of regular n-sided polygons. For example the Dihedral group D_3, which is visualized in fig. 1, is a Coxeter group. This group is generated by the elements σ_1, σ_2 which are reflections in lines in the plane that have an angle of $\frac{\pi}{3}$. It can be seen that the element $\rho = \sigma_1\sigma_2$ is a rotation over an angle $\frac{2\pi}{3}$. The elements σ_1, σ_2 thus satisfy $(\sigma_1\sigma_2)^3 = e$. We thus obtain that D_3 is a group that is generated by some finite set $S = \{\sigma_1, \sigma_2\}$ of elements that satisfy the relations $\sigma_1\sigma_1 = \sigma_2\sigma_2 = (\sigma_1\sigma_2)^3 = e$. We moreover note that all other relations between element in D_3 follow from these relations. We thus have that D_3 is a Coxeter group. We note moreover that for a Coxeter group the generating set S is not unique. We could for example also have taken the set $S = \{\sigma_1, \sigma_1\sigma_2\sigma_1\}$.

Visualization of the dihedral group D_3

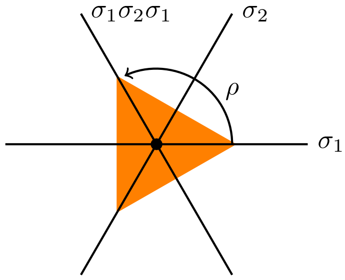


FIGURE 1. This is a visualization of the dihedral group D_3, which consists of the symmetries of an equilateral triangle. The group is generated by the two reflections σ_1 and σ_2. The element $\rho = \sigma_1\sigma_2$ is a rotation over an angle of $\frac{2\pi}{3}$. The set of all elements of D_3 is given by $\{e, \rho, \rho^2, \sigma_1, \rho\sigma_1, \rho^2\sigma_1\}$, where e is the identity element.

The groups D_n for $n \geq 1$ contain exactly $2n$ elements and are in particular finite. There are also Coxeter groups that are infinite. An example of an infinite Coxeter group is given by the infinite dihedral group D_∞. This is the group generated by the elements σ_1, σ_2 which are reflections in the complex plane in the lines $\Re(z) = 0$ and $\Re(z) = \frac{1}{2}$ respectively. These reflections satisfy no equation of the form $(\sigma_1\sigma_2)^n = e$ for any $n \geq 1$. This can be seen as the element $\rho = \sigma_1\sigma_2$ sends an integer k to $\rho(k) = \sigma_1(\sigma_2(k)) = \sigma_1(1-k) = k-1$. The group D_∞ then consists of the elements $D_\infty = \{\rho^k : k \in \mathbb{Z}\} \cup \{\rho^k\sigma_1 : k \in \mathbb{Z}\}$ and every relation these elements satisfy, follows from the relation

$\sigma_1^2 = \sigma_2^2 = e$ and the group axioms. This shows that D_∞ is also a Coxeter group.

We have up till now only considered Coxeter groups generated by two elements. In general a Coxeter group may have more generators and can have a more complicated structure. We note that a Coxeter group can also be infinite when all the values $m_{i,j}$ are finite. How to determine from the values $(m_{i,j})_{i,j}$ whether the Coxeter group is finite is not trivial, but this has been characterized, see [20, Theorem 1.3.3]. In this thesis we are actually only really interested in infinite Coxeter groups W, as they give rise to infinite dimensional von Neumann algebras $\mathcal{L}(W)$. On these von Neumann algebras we can build quantum Markov semi-groups using certain functions $\psi : W \to [0, \infty)$. Of particular interest will be the function ψ that is given by the minimal word length $|\mathbf{w}|$ of an element $\mathbf{w} \in W$. That is, $|\mathbf{w}|$ is the smallest integer $k \geq 0$ such that we can write $\mathbf{w} = w_1 ... w_k$ with $w_i \in S$.

1.6. **Structure and outline of thesis.** We summarize here what we do in each section of this thesis.

First, in section 2 we introduce notation, definitions and results that we will use throughout this text. This includes theory on von Neumann algebras, completely positive maps, quantum Markov semi-groups, bimodule structures, group von Neumann algebras, the Haagerup property, hyperbolic groups and Coxeter groups. Throughout the thesis we will at some points also introduce new definitions when needed.

In section 3 we shall introduce the gradient-$\mathcal{S}_p$ property defined in [12] that we will study in this thesis. In this section we will also do some calculations that will later be useful. Also we give here a condition that allows us to check more easily when a quantum Markov semi-group has the gradient-$\mathcal{S}_p$ property.

In section 4 we shall review some results from [12] that give some interesting implications for a von Neumann algebra $\mathcal{N}$, when a certain gradient-$\mathcal{S}_p$ quantum Markov semi-group exists. This shows why we are interested in semi-groups that possess the gradient-$\mathcal{S}_p$ property.

In section 5, we shall show a method to construct von Neumann algebras and quantum Markov semi-groups using certain discrete groups. In this thesis we shall mainly use the construction that we describe here to build these semi-groups. In this section also some useful notation is introduced that makes it easier to study the gradient-$\mathcal{S}_p$ property.

In section 6, we shall apply the construction from previous sections to Coxeter groups. We then obtain a certain quantum Markov semi-group of which we study the gradient-$\mathcal{S}_p$ property. Here we obtain results for what Coxeter systems our constructed semi-group is gradient-$\mathcal{S}_p$.

In section 7, we adapt the method from section 6 and consider slightly different semi-groups. We then obtain the results for what Coxeter groups our new semi-group is gradient-$\mathcal{S}_p$.

In section 8, we extend our results from section 6 and section 7 to semi-groups on other von Neumann algebras, namely Hecke-algebras.

In section 9, we consider multiple semi-groups, and try to relate the gradient-$\mathcal{S}_p$ property of certain semi-groups to the gradient-$\mathcal{S}_q$ property of another semi-group.

In section 10, we discuss a method that allows, for certain von Neumann algebras $\mathcal{N}$, to obtain the results we are interested in without building a quantum Markov semi-group that is gradient-$\mathcal{S}_p$, but with some slightly different approach.

Last, in section 11 we summarize all the results that we obtained in this thesis and restate the main theorems. Also we give some directions for future research.

2. Preliminaries

In this section we shall introduce some definitions, known results, and establish some notations that we shall use throughout this thesis. We assume that the reader is familiar with some basic theory on operator algebras (for example see chapter (1)-(5) from [30]), and state additionally needed theory here. Besides topics from operator theory we also introduce some group theoretical notions.

2.1. Von Neumann algebras, traces, GNS-representation, Schatten-p ideals.

Throughout this text we will always consider a finite a von Neumann algebras $\mathcal{N}$ equipped with a normal faithful finite trace τ. We state these definitions here, together with some related notions.

2.1.1. *von Neumann algebras.* A *von Neumann algebra* $\mathcal{N}$ on a Hilbert space H (always taken over $\mathbb{C}$) is a $*$-subalgebra of the space of bounded operators $B(H)$ that is closed in the strong operator topology and contains the identity Id_H. It holds true by [39, Corollary 1.13.3 and Theorem 1.16.7] that every von Neumann algebra $\mathcal{N}$ is the dual of a Banach space. This Banach space is unique up to isomorphism, and is called the *predual* of $\mathcal{N}$. A factor on H is a von Neumann algebra $\mathcal{N}$ for which $\mathcal{N}' \cap \mathcal{N} = \mathbb{C}\,\mathrm{Id}_H$ (here $\mathcal{N}' := \{a \in B(H) : ab = ba \text{ for } b \in \mathcal{N}\}$ denotes the *commutant* of $\mathcal{N}$). We say that two projections $p, q \in \mathcal{N}$ are *Murray-von Neumann equivalent* if there exists $v \in \mathcal{N}$ with $p = v^*v$ and $q = vv^*$, and this is written as $p \sim q$. A projection q is called finite if $q \sim p \leq q$ implies that $p = q$. If the unit $1 \in \mathcal{N}$ is a finite projection, then we also call $\mathcal{N}$ finite.

2.1.2. *Traces.* On a von Neumann algebra we consider a *trace*, which is a convex mapping $\tau : \mathcal{N}^+ \to [0, \infty]$ with $\tau(0) = 0$ such that $\tau(ab) = \tau(ba)$ for all $a, b \in \mathcal{N}^+$. A trace is called *finite* if $\tau(a) < \infty$ for all $a \in \mathcal{N}^+$. If τ is finite we can uniquely extend τ to a positive linear functional on $\mathcal{N}$, also denoted τ. This positive linear functional is then moreover *tracial*, that is $\tau(ab) = \tau(ba)$ for $a, b \in \mathcal{N}$. A trace is called *normal* if it preserves suprema of norm-bounded nets $\{x_i\}_{i \in I}$ of self adjoint elements. It is called *faithful* if $\tau(a) > 0$ whenever $a > 0$. It is true that every finite factor $\mathcal{N}$ possesses a unique normal finite faithful trace τ [43, Theorem V.2.6].

2.1.3. *GNS-representation.* For a von Neumann algebra $\mathcal{N}$ with a normal finite faithful trace τ we will denote $L^2(\mathcal{N}, \tau)$, or simply $L_2(\mathcal{N})$, for the GNS-Hilbert space. This is the Hilbert space completion of $\mathcal{N}$ with the inner product $\langle a, b \rangle_\tau = \tau(b^*a)$. We will furthermore denote $\|\cdot\|_{2,\tau}$ for the norm of $L^2(\mathcal{N}, \tau)$. Also we will denote Ω_τ for the cyclic vector in $L^2(\mathcal{N}, \tau)$ that implements the trace, which is given by $\Omega_\tau = 1 \in \mathcal{N}$.

2.1.4. *Schatten p-ideals.* Given a Hilbert space H and $p \in [1, \infty)$ we will write $\mathcal{S}_p$ for the Schatten p-class, which is an ideal in $B(H)$, consisting of all $a \in B(H)$ for which $\mathrm{Tr}(|a|^p) < \infty$. Here Tr is the trace on $B(H)$ defined by $\mathrm{Tr}(a) = \sum_{i \in I} \langle ae_i, e_i \rangle$, where $\{e_i\}_{i \in I}$ is an arbitrary orthonormal basis for H. This definition of Tr is in fact independent of the chosen orthonormal basis. Furthermore we will write $\mathcal{S}_\infty$ for the ideal of all compact operators in $B(H)$.

2.2. Tensor products, completely positive maps.

Throughout this thesis, we shall use different notions of tensor products. We give an overview of these different notions here. Also we introduce the definition of completely positive maps.

2.2.1. *Different notions of tensor products.* For vector spaces V, W we shall write $V \otimes_{alg} W$ for the algebraic tensor product (see [7, Definition 3.1.1]).

When H_1, H_2 are Hilbert spaces we will write $H_1 \overline{\otimes} H_2$ to denote the Hilbert space completion of $H_1 \otimes_{\mathrm{alg}} H_2$ w.r.t. the inner product given by $\langle a \otimes b, c \otimes d \rangle := \langle a, c \rangle \langle b, d \rangle$.

When A, B are algebras we will endow $A \otimes_{alg} B$ with the multiplication given by $(a \otimes b)(c \otimes d) = ac \otimes bd$, which makes $A \otimes_{alg} B$ an algebra. When A, B are moreover C^*-algebras we can consider their universal representations $\pi_A : A \to B(H_A)$ and $\pi_B : B \to B(H_B)$. Then π_A and π_B are injective $*$-homomorphisms and by [30, Theorem 6.3.3] these give rise to an injective $*$-homomorphism $\pi : A \otimes_{alg} B \to B(H_A \overline{\otimes} H_B)$. This then gives us a C^*-norm $\|\cdot\|_{min}$ on $A \otimes_{alg} B$ given by $\|c\|_{min} := \|\pi(c)\|$. This norm is called the *spatial C^*-norm.* There may generally be multiple C^*-norms on $A \otimes_{alg} B$, however it is the case that $\|c\|_{min} \leq \|c\|$ for any C^*-norm $\|\cdot\|$ on $A \otimes_{alg} B$, see [30, Theorem 6.4.18]. Furthermore, a norm on $A \otimes_{alg} B$ can also be defined as

$$\|c\|_{max} = \max_{\rho \text{ is a } C^*\text{-norm}} \rho(c)$$

which is called the *maximal* C*-norm. We shall write $A \otimes_{min} B$ and $A \otimes_{max} B$ for the completions of $A \otimes_{alg} B$ w.r.t. the norms $\|\cdot\|_{min}$ and $\|\cdot\|_{max}$ respectively.

2.2.2. *Completely positive maps.* In order to introduce the theory of quantum Markov semi-groups, we must establish the definition of a completely positive map. A map $\varphi : A \to B$ between C^*-algebras is called *completely positive*, or simply *c.p*, if for all $n \geq 1$ the map $\varphi^{(n)} : A \otimes_{min} M_n(\mathbb{C}) \to B \otimes_{min} M_n(\mathbb{C})$ given by $\varphi^{(n)}(a \otimes e_{i,j}) = \varphi(a) \otimes e_{i,j}$ is a positive map. We will moreover call φ *unital completely positive*, or just *u.c.p.*, if A and B are moreover unital and $\varphi(1) = 1$. These maps have some interesting properties. As φ is in particular positive, we have that $\varphi(a^*) = \varphi(a)^*$ for $a \in A$. It also holds by the Kadison-Schwartz inequality [35, Proposition 3.3] that for a u.c.p map we have that $\varphi(a)^*\varphi(a) \leq \varphi(a^*a)$ for all $a \in A$. Furthermore, we have that $\|\varphi\| \leq 1$, [35, Proposition 3.2].

2.3. **Quantum Markov semi-group.** Let $\mathcal{N}$ be a finite von Neumann algebra with a normal faithful trace τ. A *quantum Markov semi-group* $(\Phi_t)_{t\geq 0}$ on the von Neumann algebra $(\mathcal{N}, \tau)$ is then defined as a family of unital completely positive maps $\Phi_t : \mathcal{N} \to \mathcal{N}$ with the properties that

- We have $\Phi_0 = Id_{\mathcal{N}}$.
- For $t, s \geq 0$ we have $\Phi_t \Phi_s = \Phi_{t+s}$.
- For $x \in \mathcal{N}$ the map $[0, \infty) \to \mathcal{N}$ given $t \mapsto \Phi_t(x)$ is continuous for the strong topology of $\mathcal{N}$.
- For $t \geq 0$ the map Φ_t is symmetric, that is $\tau(x\Phi_t(y)) = \tau(\Phi_t(x)y)$ for $x, y \in \mathcal{N}$.

We note here that the fact that Φ_t also preserves trace follows from the fact that it is unital and symmetric.

By the fact that Φ_t is u.c.p and preserves trace we have that

$$\|\Phi_t(x)\|_{2,\tau}^2 = \tau(\Phi_t(x)^*\Phi_t(x)) \leq \tau(\Phi_t(x^*x)) = \tau(x^*x) = \|x\|_{2,\tau}^2 \tag{9}$$

which shows that Φ_t extends to a contractive map on $L^2(\mathcal{N}, \tau)$. This operator we will also denote by Φ_t. By the fact that Φ_t is also symmetric we have that $\langle \Phi_t(x), y \rangle_\tau = \langle x, \Phi_t(y) \rangle_\tau$ for all $x, y \in \mathcal{N}$, which shows by density of $\mathcal{N}$ in $L^2(\mathcal{N}, \tau)$ that Φ_t is self-adjoint as an operator in $B(L^2(\mathcal{N}, \tau))$.

We show that the map $[0, \infty) \to B(L^2(\mathcal{N}, \tau))$ given by $t \mapsto \Phi_t$ is continuous for the strong operator topology. Namely, for $x \in L^2(\mathcal{N}, \tau)$ we have that

$$\|\Phi_t(x) - x\|_{2,\tau}^2 = \tau((\Phi_t(x) - x)^*(\Phi_t(x) - x)) \tag{10}$$
$$= \tau(\Phi_t(x^*)\Phi_t(x)) - \tau(\Phi_t(x^*)x) - \tau(x^*\Phi_t(x)) + \tau(x^*x) \tag{11}$$
$$= \tau(x^*\Phi_{2t}(x)) - \tau(x^*\Phi_t(x)) - \tau(x^*\Phi_t(x)) + \tau(x^*x). \tag{12}$$

Now as $t \to \Phi_t(x)$ is continuous for the strong topology of $\mathcal{N}$, it is continuous for the weak topology. This topology coincides on bounded sets in $\mathcal{N}$ with the σ-weak topology. Now since $\|\Phi_t(x)\| \leq \|x\|$ for $t \geq 0$ we obtain that $t \mapsto \Phi_t(x)$ is σ-weakly continuous. This then means that $\lim_{t\downarrow 0} \tau(x^*\Phi_t(x)) = \tau(x^*\Phi_0(x)) = \tau(x^*x)$. This thus means that $\|\Phi_t(x) - x\|_{2,\tau} \to 0$, which proves the claim.

The above conclusion actually says that the maps $(\Phi_t)_{t\geq 0}$ form a C_0*-semi-group* on $L^2(\mathcal{N}, \tau)$. For such semi-groups we can define a generator Δ of the semi-group as follows. We define Δ to be the unbounded operator on $L^2(\mathcal{N}, \tau)$ that is given for $x \in \mathrm{Dom}(\Delta)$ by

$$\Delta(x) = -\lim_{t\downarrow 0} \frac{\Phi_t(x) - x}{t} \tag{13}$$

where the limit is in the $\|\cdot\|_{2,\tau}$-norm. The domain $\mathrm{Dom}(\Delta)$ here is taken to be the set of all x in $L^2(\mathcal{N}, \tau)$ for which this limit exists. As $(\Phi_t)_{t\geq 0}$ defines a C_0-semi-group on $B(L^2(\mathcal{N}, \tau))$ we have by [42, Theorem 2.2.7] that Δ is closed and densely defined. It moreover holds by our properties of Φ_t that $\Delta(1) = 0$ and that Δ is symmetric as for $x, y \in \mathrm{Dom}(\Delta)$

$$\langle \Delta(x), y \rangle_\tau = -\lim_{t\downarrow 0} \frac{1}{t} \langle \Phi_t(x) - x, y \rangle_\tau \tag{14}$$
$$= -\lim_{t\downarrow 0} \frac{1}{t} \langle x, \Phi_t(y) - y \rangle_\tau \tag{15}$$
$$= \langle x, \Delta(y) \rangle_\tau \tag{16}$$

holds. It is even the case that Δ is positive as

$$\langle \Delta(x), x\rangle_\tau = -\lim_{t\downarrow 0} \frac{1}{t}\langle \Phi_t(x) - x, x\rangle_\tau \tag{17}$$

$$= \lim_{t\downarrow 0} \frac{1}{t}\left(\tau(x^*x) - \tau(x^*\Phi_t(x))\right) \tag{18}$$

$$= \lim_{t\downarrow 0} \frac{1}{t}\Big(\tau(x^*x) - \tau(\Phi_{\frac{t}{2}}(x^*)\Phi_{\frac{t}{2}}(x))\Big) \tag{19}$$

$$\geq \lim_{t\downarrow 0} \frac{1}{t}\Big(\tau(x^*x) - \tau(\Phi_{\frac{t}{2}}(x^*x))\Big) = 0. \tag{20}$$

In this text we shall sometimes write $(e^{-t\Delta})_{t\geq 0}$ to denote the semi-group $(\Phi_t)_{t\geq 0}$.

2.4. **Bimodule structures.** In order to understand the use of the gradient-$\mathcal{S}_p$ property we introduce the notion of bimodules, which is needed in section 4 and section 10. Let A be an algebra. An $A-A$ *bimodule* is an Hilbert space H together with two unital $*$-homomorphisms $\pi_l : A \to B(H)$ and $\pi_r : A^{op} \to B(H)$ whose images commute. Here A^{op} stands for the *opposite algebra.* This algebra just equals A as vector space and the element $a \in A$ is written as a^{op} to denote the element in A^{op}. The multiplication in A is defined in reversed order. This is to say that the multiplication $A^{op}\times A^{op} \to A^{op}$ is given by $a^{op}\cdot b^{op} = (b\cdot a)^{op}$. For a vector $\xi \in H$ and for $a, b \in \mathcal{A}$ we shall simply write $a\xi b$ to denote $\pi_l(a)\pi_r(b^{op})\xi = \pi_r(b^{op})\pi_l(a)\xi$. We note that this notation is not ambiguous as $(ab)\xi = \pi_l(ab)\xi = \pi_l(a)\pi_l(b)\xi = \pi_l(a)(b\xi) = a(b\xi)$ and similarly $\xi(ab) = \pi_r((ab)^{op})\xi = \pi_r(b^{op}a^{op})\xi = \pi_r(b^{op})\pi_r(a^{op})\xi = \pi_r(b^{op})(\xi a) = (\xi a)b$.

We note that when A is a C^*-algebra, then by [30, Theorem 6.3.7.] the $*$-homomorphisms π_l and π_r actually induce a unique $*$-homomorphism $\pi : A\otimes_{\max} A^{op} \to B(H)$ that satisfies $\pi(a\otimes b^{op}) = \pi_l(a)\pi_r(b^{op})$ for all $a, b \in A$.

When A is actually a von Neumann algebra, we will assume that the representations π_l, π_r are moreover normal. We note that when $(\mathcal{N}, \tau)$ is finite von Neumann algebra, then a $\mathcal{N}-\mathcal{N}$ bimodule is given by $L^2(\mathcal{N}, \tau)$. The $*$-homomorphisms $\pi_l : \mathcal{N} \to B(L^2(\mathcal{N}, \tau))$ and $\pi_r : \mathcal{N}^{op} \to B(L^2(\mathcal{N}, \tau))$ are then simply given by left and right multiplication, that is $\pi_l(a)b = ab$ and $\pi_r(a)b = ba$ for $a, b \in \mathcal{N}$. The mappings $\pi_l(a)$ and $\pi_r(a)$ extend uniquely to bounded mappings on $L^2(\mathcal{N}, \tau)$ as $\mathcal{N}$ is dense in $L^2(\mathcal{N}, \tau)$. We will call $L^2(\mathcal{N}, \tau)$ the trivial $\mathcal{N}-\mathcal{N}$-bimodule. Another $\mathcal{N}-\mathcal{N}$ bimodule is given by the Hilbert space tensor product $L^2(\mathcal{N}, \tau)\overline{\otimes}L^2(\mathcal{N}, \tau)$. The $*$-homomorphisms π_l, π_r are then given by $\pi_l(a)(b\otimes c) = ab\otimes c$ and $\pi_r(a)(b \otimes c) = b \otimes ca$ for $b \otimes c \in \mathcal{N} \otimes \mathcal{N}$. Again, these mappings extend to bounded maps on $L^2(\mathcal{N}, \tau)\overline{\otimes}L^2(\mathcal{N}, \tau)$. The tensor product $L^2(\mathcal{N}, \tau)\overline{\otimes}L^2(\mathcal{N}, \tau)$ together with these bimodule actions we will call the coarse bimodule.

Given an $\mathcal{N} - \mathcal{N}$ bimodule H, a *submodule* of H is a closed subspace of H that is invariant for the bimodule actions of $\mathcal{N}$. Moreover, given two $\mathcal{N} - \mathcal{N}$ bimodules H and K, we will say that K is *contained* in H when K is isomorphic to a submodule $\widetilde{H}$ of H. Here, with an *isomorphism* we mean an isomorphism between Hilbert spaces that preserves the bimodule actions. We will moreover say that K is *quasi-contained* in H if K is contained in the Hilbert space $\bigoplus_{i\in I} H_i$ for some index set I.

2.5. **Haagerup property for von Neumann algebras.** Let $\mathcal{N}$ be a finite von Neumann algebra with a normal faithful finite trace τ. We say that $(\mathcal{N}, \tau)$ has the *Haagerup* property whenever there exists a net of completely positive maps $\theta_i : \mathcal{N} \to \mathcal{N}$ for which $\tau \circ \theta_i \leq \tau$, and θ_i is L^2-compact, i.e. compact as an operator on $L^2(\mathcal{N}, \tau)$, and such that $\theta_i \to id_{L^2(\mathcal{N},\tau)}$ as $i \to \infty$ in the point-ultraweak topology of $B(L^2(\mathcal{N}, \tau))$. In [24, Prop. 2.4] was shown that if τ, τ' are two normal faithful finite traces on $\mathcal{N}$, and if $\mathcal{N}$ has the Haagerup property w.r.t. τ, then it also has the Haagerup property w.r.t. τ'. We thus do not have to specify the trace. In [25, Theorem. 1] was shown that, if $(\mathcal{N}, \tau)$ is a finite von Neumann algebra whose predual is separable, then if $\mathcal{N}$ has the Haagerup property we can find a quantum Markov semi-group $(\Phi_t)_{t\geq 0}$ on $\mathcal{N}$ for which Φ_t is a compact operator on $L^2(\mathcal{N}, \tau)$ for $t > 0$.

2.6. **Group von Neumann algebras.** A way of building von Neumann algebras with a trace τ is by using a topological group Γ. More specifically, we let Γ be a discrete group.

We shall denote $\mathbb{C}[\Gamma]$ for the *group ring* of Γ. This is the vector space over $\mathbb{C}$ with linear basis $\{g \in \Gamma\}$. Elements of $\mathbb{C}[\Gamma]$ can thus be written in the form $\sum_{g\in\Gamma} \alpha_g \cdot g$, where $\alpha_g \in \mathbb{C}$ is non-zero for only finitely many $g \in \Gamma$. The space $\mathbb{C}[\Gamma]$ is in fact a $*$-algebra by defining multiplication of the basis vectors in the natural way using the multiplication in Γ, and by defining the convolution as $g^* = g^{-1}$.

For $s \in \Gamma$ we can define operators $\lambda_s \in B(\ell_2(\Gamma))$ as $(\lambda_s f)(t) = f(s^{-1}t)$. These operators satisfy the equations $\lambda_s \lambda_r = \lambda_{sr}$ and $\lambda_s \delta_t = \delta_{st}$, where we denote $\delta_t = \chi_{\{t\}}$. Now this extends as $\lambda : \mathbb{C}[\Gamma] \to B(\ell_2(\Gamma))$ linearly to an injective $*$-homomorphism, called the *left regular representation*. Therefore we can, and generally will regard $\mathbb{C}[\Gamma]$ as a subset of $B(\ell_2(\Gamma))$, and denote its elements as $\sum_{g\in\Gamma} \alpha_g \lambda_g$. The representation also induces the norm $\|a\|_r := \|\lambda(a)\|$ on $\mathbb{C}[\Gamma]$. The norm closure of $\mathbb{C}[\Gamma]$ w.r.t. this norm is denoted $C^*_\lambda(\Gamma)$ and will also be regarded as a subalgebra of $B(\ell_2(\Gamma))$. The C^*-algebra $C^*_r(\Gamma)$ is called the *reduced C^*-algebra* of Γ. We can also construct the von Neumann Algebra $\mathcal{L}(\Gamma) := C^*_\lambda(\Gamma)''$, which is called the *group von Neumann algebra* of Γ. The canonical trace on $\mathcal{L}(\Gamma)$ is given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$. By [43, Proposition 7.9] we have that for a countable (discrete) group Γ that the group von Neumann algebra if finite. Also, when Γ is countable we have that the predual of $\mathcal{L}(\Gamma)$ is separable. Indeed, its predual is isomorphic to a quotient of the trace class operators $L^1(\ell_2(\Gamma))$, see [30, Theorem 4.2.9]. Now as the finite rank operators $F(\ell_2(\Gamma))$ are dense in $L^1(\ell_2(\Gamma))$, and as $F(\ell_2(\Gamma))$ is separable (since $\ell_2(\Gamma)$ is separable), we obtain that the predual of $L^1(\ell_2(\Gamma))$ is separable. Therefore, the predual of $\mathcal{L}(\Gamma)$ is separable. Hence, whenever $\mathcal{L}(\Gamma)$ has the Haagerup property, we can construct a quantum Markov semi-group.

2.7. **Haagerup property for groups.** To ensure that the group von Neumman algebra $\mathcal{L}(\Gamma)$ has the Haagerup property, we must have that Γ possesses the *Haagerup property for groups*. We will give a definition of this property for a discrete group Γ.

2.7.1. *Positive definite and conditionally negative definite.* A map $k : \Gamma \times \Gamma \to \mathbb{C}$, also called a *kernel*, is called *positive definite*, if for $n \geq 1$, $s_1, ..., s_n \in \Gamma$ and $c_1, ..., c_n \in \mathbb{C}$ we have $\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} k(s_i, s_j) \geq 0$. The map k is called *conditionally negative definite* if it takes values in $[0, \infty)$, if $k(s, s) = 0$ for $s \in \Gamma$ and if for $n \geq 1$, $s_1, ..., s_n \in \Gamma$ and $c_1, ..., c_n \in \mathbb{C}$ with $\sum_{i=1}^n c_i = 0$ we have $\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} k(s_i, s_j) \leq 0$. There are also other characterizations of positive definite and conditionally negative definite kernels that are useful. Namely, a kernel is positive definite if and only if we can find a Hilbert space H and a map $f : \Gamma \to H$ such that $k(r, s) = \langle f(r), f(s)\rangle$ for all $r, s \in \Gamma$, see [2, p. C.1.4]. Similarly, a kernel is conditionally negative definite if and only if we can find a Hilbert space H and a map $f : \Gamma \to H$ such that $k(r, s) = \|f(r) - f(s)\|_2^2$ for all $r, s \in \Gamma$, see [2, p. C.2.3]. A connection between positive definite and conditionally negative definite kernels is given by Schoenberg's theorem, [2, Theorem C.3.2.], that says that a kernel k satisfying $k(r, r) = 0$ and $k(r, s) = k(s, r)$ for all $r, s \in \Gamma$ is conditionally negative definite if and only if we have that for $t \geq 0$ that the kernel $(r, s) \mapsto e^{-tk(r,s)}$ is positive definite.

A function $\varphi : \Gamma \to \mathbb{C}$ is called *positive definite* if the kernel given by $(s, t) \mapsto \varphi(t^{-1}s)$ is positive definite. Likewise a function $\psi : \Gamma \to [0, \infty]$ is called *conditionally negative definite* if the kernel given by $(s, t) \mapsto \psi(t^{-1}s)$ is conditionally negative definite. We call ψ furthermore *proper* whenever $\{s \in \Gamma : \psi(s) < n\}$ is finite for all $n \geq 1$.

2.7.2. *Haagerup property.* For a discrete group Γ the *Haagerup property* if now defined as the property that there exists a sequence $(\varphi_n)_{n\geq 1}$ of positive definite functions on Γ such that $\varphi_n(e) = 1$, such that φ_n vanishes at infinity, and such that $\varphi_n \to 1$ point-wise. We note that if there exists a proper, conditionally negative definite function ψ on Γ, then we can define the functions $\varphi_t(s) := e^{-t\psi(s)}$ for $t > 0$ so that the sequence of functions $(\varphi_{\frac{1}{n}})_{n\geq 1}$ satisfy these conditions. Furthermore, if Γ has the Haagerup property, then a proper, conditionally negative definite function actually always exists, which follows from [7, Th. 12.2.4]. It holds true that the group von Neumann algebra $\mathcal{L}(\Gamma)$ possesses the Haagerup property if and only if the group Γ possesses the Haagerup property, see[7, Th. 12.2.9.]. We refer to [14] for additional theory on the Haagerup property for groups.

2.8. **Cayley graph, hyperbolic groups.** We shall give background to some group theoretic notions that we shall use.

2.8.1. *Cayley Graph.* Let Γ be a group that is generated by some finite set S. For such group we can define the *Cayley graph* of Γ. This is the simple graph $\mathsf{Cayley}_S(\Gamma)$ with vertex set Γ, in which two distinct vertices $u, w \in \Gamma$ share an edge if and only if $uw^{-1} \in S \cup S^{-1}$. This condition basically says that two vertices are connected if and only if they differ by just one element of S. Since S generates whole of Γ, we have that the graph $\mathsf{Cayley}_S(\Gamma)$ is actually connected. We can define a distance d on $\mathsf{Cayley}_S(\Gamma)$ by defining $d(u, v)$ to be the length of a shortest path from u to v. This makes $\mathsf{Cayley}_S(\Gamma)$ a metric space. We shall call a shortest path between u and v a *geodesic*. Such geodesic we can just denote by the set of vertices that the path traverses. Also, for $\delta > 0$ and a subset $U \subseteq \Gamma$ we shall write $B_\delta(U)$ to be the δ-neighborhood of U, that is we set $B_\delta(U) := \{g \in \Gamma : d(g, U) < \delta\}$.

2.8.2. *Hyperbolicity.* The Cayley graph of a finitely generated group can be used to define the notion of hyperbolicity. That is, let Γ be a group that is finitely generated by some set S. The group Γ is called *hyperbolic* if there exists $\delta > 0$ such that for all $u, v, w \in \Gamma$ and for all geodesics P_1, P_2,P_3 in $\mathsf{Cayley}_S(\Gamma)$ between the vertices u and v, between v and w and between w and u respectively, we have that $P_1 \subseteq B_\delta(P_2 \cup P_3)$ and $P_2 \subseteq B_\delta(P_1 \cup P_3)$ and $P_3 \subseteq B_\delta(P_1 \cup P_3)$. We note that the Cayley graph of Γ depends on the generating set S, but that the definition of hyperbolicity is in fact independent of the choice of the generating set, see [18, Theorem 12.5.3].

2.9. **Coxeter groups.** In this master thesis we will specifically consider group von Neumann algebras of Coxeter groups. These groups have the Haagerup property, and we can therefore consider a quantum Markov semi-group on its group von Neumann algebra of which we can study the gradient-$\mathcal{S}_p$ property.

A *Coxeter group* W is a group that is generated by some finite set $S = \{s_1, .., s_n\}$ that satisfy the relations $(s_i s_j)^{m_{i,j}} = e$ for some elements $(m_{i,j})_{1 \leq i,j \leq n}$ with $m_{i,i} = 1$ and $m_{i,j} \in \{2, 3, ..\} \cup \{\infty\}$ for $i \neq j$. Here $m_{i,j} = \infty$ means that the elements s_i and s_j satisfy no relation at all. Furthermore, the elements in W must be equal if and only if they can be shown to be equal by using these relations and the group axioms. We note here that in general, for a Coxeter group, the set S is not unique. Furthermore, we note that $(s_i s_j)^{m_{i,j}} = e$ implies that $(s_j s_i)^{m_{i,j}} = s_i s_i (s_j s_i)^{m_{i,j}} = s_i (s_i s_j)^{m_{i,j}} s_i = s_i^2 = e$. Therefore we can always assume that $m_{i,j} = m_{j,i}$. Furthermore, we note that $m_{i,j} = 2$ implies that $s_i s_j = (s_j s_i)^2 s_i s_j = s_j s_i$, i.e. the elements s_i and s_j commute. If all coefficients $m_{i,j}$ with $i \neq j$ are either 2 or ∞, then the Coxeter group is called *right-angled.*

Let S be a finite set and let $M = (m_{i,j})_{1 \leq i,j \leq n}$ be a symmetric matrix with diagonal elements equal to 1, and off-diagonal elements in the set $\{2, 3, ..\} \cup \{\infty\}$. We will write $W = \langle S|M \rangle$ for the Coxeter group generated by the finite set S subject to the relations $(s_i s_j)^{m_{i,j}} = e$, and such that all other relations follow from these and from the group axioms. We will moreover call $\langle S|M \rangle$ a *Coxeter system*, to empathize that we have specified the set S. Any element $\mathbf{w} \in W = \langle S|M \rangle$ can be written as $\mathbf{w} = w_1 w_2 .... w_k$ for some $k \in \mathbb{N}$ and $w_i \in S$. We shall therefore call elements $\mathbf{w} \in W$ *words.* We denote these words in bold to distinct them from the *letters* (i.e. generators in S). Note that a representation of $\mathbf{w}$ is not unique as we can for example add identities as e or $(s_i s_j)^{m_{i,j}}$ in the representation. We will define the *word length* of a word $\mathbf{w} \in W$ w.r.t. the generator set S as the minimal $k \geq 0$ for which we can write $\mathbf{w} = w_1 .... w_k$ with $w_i \in S$. We denote this as $|\mathbf{w}|_S$, or simply as $|\mathbf{w}|$ when the generator set S is understood. We will call a representation $\mathbf{w} = w_1 .... w_k$ with $k = |\mathbf{w}|$ *reduced.* Such reduced representation thus always exists, but is in general not unique. For example, if $m_{i,j}$ is odd then $s_j (s_i s_j)^{\frac{m_{i,j}-1}{2}} = s_i (s_j s_i)^{\frac{m_{i,j}-1}{2}}$ are two distinct reduced representations of the same word. Furthermore, we shall more generally call a representation $\mathbf{w} = \mathbf{w}_1 .... \mathbf{w}_k$ *reduced* if $|\mathbf{w}| = \sum_{i=1}^{k} |\mathbf{w}_i|$.

2.9.1. *Coxeter groups have Haagerup property.* All Coxeter groups satisfy the Haagerup property. Namely, on a Coxeter group $W = \langle S|M \rangle$ we can build a proper, conditionally negative function $\psi_S : W \to [0, \infty)$ by defining $\psi_S(\mathbf{w}) = |\mathbf{w}|$, i.e. ψ_S is the word length. This function is conditionally negative by [4], and moreover proper because $|\{\mathbf{w} \in W : \psi_S(\mathbf{w}) < n\}| \leq |S|^n$ for $n \geq 1$. This shows that the Coxeter group W possesses the Haagerup property.

2.9.2. *Word hyperbolic Coxeter groups.* For a Coxeter group it is custom to talk about *word hyperbolicity* instead of hyperbolicity. For a right-angled Coxeter group, the property that W is word hyperbolic is equivalent with the statement that W does not contain $\mathbb{Z}^2$ as a subgroup, see [18, Corollary 12.6.3.]

2.9.3. *Graph encoding information of Coxeter group.* To a Coxeter system $W = \langle S|M \rangle$ we can associate a certain graph that encodes the information of the Coxeter group. We shall denote $\mathsf{Graph}_S(W)$ for the complete simple graph with vertex set S. For distinct elements $s_i, s_j \in S$ we moreover label the edge $\{s_i, s_j\}$ with the quantity $M(s_i, s_j) = M(s_j, s_i)$. This graph is closely related to the Coxeter-Dynkin diagram of W, but in those diagrams the edges with label 2 have been omitted. For our purposes it is better to include those edges as well. We want to empathize that the graph $\mathsf{Graph}_S(W)$ generally depends on the set S of generators. For example, consider the Dihedral group $D_6 = \langle S|M \rangle$ where $S = \{s_1, s_2\}$ and $M(s_1, s_2) = 6$. We can choose an alternative generating set $\widetilde{S}$ as $\widetilde{S} = \{\sigma_1, \sigma_2, \rho\}$ where $\sigma_1 = s_1$, $\sigma_2 = s_2 s_1 s_2$ and $\rho = (s_1 s_2)^3$. These elements are of order 2, and generate the entire group. Furthermore they satisfy the relations $M(\sigma_1, \sigma_2) = 3$ and $M(\sigma_1, \rho) = M(\sigma_2, \rho) = 2$. Furthermore, all relations between the elements follow from these relations. Now it is clear that the labeled graph $\mathsf{Graph}_S(D_6)$ is not isomorphic to $\mathsf{Graph}_{\widetilde{S}}(D_6)$ as the graphs do not even have the same number of vertexes.

In the case of right-angled Coxeter groups, we have by [38] that for arbitrary Coxeter generator sets $S, \widetilde{S}$ the graphs $\mathsf{Graph}_S(W)$ and $\mathsf{Graph}_{\widetilde{S}}(W)$ are isomorphic (preserving labels).

2.9.4. *Geometric interpretation of Coxeter groups.* An intuitive, geometric way of looking at Coxeter groups is by considering them as reflection groups. Namely, for a Coxeter system $W = \langle S|M\rangle$ we can define the bilinear form $B : \mathbb{R}^{|S|} \times \mathbb{R}^{|S|} \to \mathbb{R}$ as $B(e_i, e_j) = -\cos(\frac{\pi}{m_{i,j}})$, and we can define the mappings σ_i on $\mathbb{R}^{|S|}$ as $\sigma_i(v) = v - 2B(v, e_i)e_i$. Then $\sigma_i(e_i) = -e_i$ and σ_i leaves the linear subspace $H_i := \{v \in \mathbb{R}^{|S|} : B(v, e_i) = 0\}$ invariant. Thus σ_i corresponds to some reflection in a linear subspace. Note that we have $e_i \perp H_i$ if and only if $m_{i,j} = 2$ for all j with $i \neq j$.

The group generated by the mappings $\{\sigma_i : i = 1, .., |S|\}$ is isomorphic to the Coxeter group W. It is clear that the relations $\sigma_i\sigma_i = e$ holds, since σ_i is a reflection. When $1 < m_{i,j} < \infty$ it can also be shown that the relationship $(\sigma_i\sigma_j)^{m_{i,j}} = I$ holds. Namely, it can be seen that $\sigma_i\sigma_j$ leaves the subspaces $H_i \cap H_j$ and $e_i\mathbb{R} + e_j\mathbb{R}$ invariant. On $H_i \cap H_j$ it acts as identity and a calculation shows that on $e_i\mathbb{R} + e_j\mathbb{R}$ it acts, w.r.t the basis (e_i, e_j) as the matrix

$$\sigma_i\sigma_j = \begin{pmatrix} 4B(e_i, e_j)^2 - 1 & 2B(e_i, e_j) \\ -2B(e_i, e_j) & -1 \end{pmatrix} \tag{21}$$

$$= \begin{pmatrix} 4\cos(\frac{\pi}{m_{i,j}})^2 - 1 & -2\cos(\frac{\pi}{m_{i,j}}) \\ 2\cos(\frac{\pi}{m_{i,j}}) & -1 \end{pmatrix}. \tag{22}$$

We can calculate its eigenvalues, which are $e^{\frac{2\pi i}{m_{i,j}}}$ and $e^{-\frac{2\pi i}{m_{i,j}}}$, which are distinct. We can thus diagonalize $\sigma_i\sigma_j$, and we see that we get $(\sigma_i\sigma_j)^{m_{i,j}} = I$.

For the case $m_{i,j} = \infty$ it can be seen that the vector $v = e_i + e_j$ is invariant under both σ_i and σ_j and that $\sigma_i\sigma_j(e_i) = \sigma_i(e_i + 2e_j) = \sigma_i(2v - e_i) = 2v + e_i$. Therefore $(\sigma_i\sigma_j)^m(e_i) = 2mv + e_j$ for $m \geq 1$ and thus no relation of the form $(\sigma_i\sigma_j)^m = I$ exists.

In the above, we have used here some notation as in [27, Section 5.3]. In [27, Section 5.4] it is also proven that the homomorphism from $W \to GL(V)$ given for generators as $s_i \mapsto \sigma_i$, is actually injective. Hence W is indeed isomorphic to the group generated by $\{\sigma_i : i = 1, .., |S|\}$.

3. The gradient-$\mathcal{S}_p$ property

In this section we shall introduce what the gradient-$\mathcal{S}_p$ property from [12] is precisely. In section 3.1 we give the precise definition of this property. There we also check whether the gradient-$\mathcal{S}_p$ property holds for some example. Thereafter, in section 3.2 we make some calculations that are useful throughout the rest of this thesis. Moreover, there we also give a sufficient condition for the gradient-$\mathcal{S}_p$ property, that is useful for showing that a semi-group has this property.

3.1. Definition of gradient-$\mathcal{S}_p$ property.

For $p \in [1, \infty]$ we state the definition of the gradient-$\mathcal{S}_p$ property from [12].

Definition 3.1. *Let $(\Phi_t)_{t\geq 0} = (e^{-t\Delta})_{t\geq 0}$ be a quantum Markov semi-group on a finite von Neumann algebra $(\mathcal{N}, \tau)$. Let $\mathcal{A}$ be a σ-weakly dense $*$-subalgebra of $\mathcal{N}$ for which $\mathcal{A}\Omega_\tau \subseteq \mathrm{Dom}(\Delta)$ and $\Delta(\mathcal{A}\Omega_\tau) \subseteq \mathcal{A}\Omega_\tau$ and so that furthermore, for $a \in \mathcal{A}$ the map $t \mapsto \Phi_t(a)$ is norm continuous. Fix $p \in [1, \infty]$. The quantum Markov semi-group $(\Phi_t)_{t\geq 0}$ is called gradient-$\mathcal{S}_p$ if for all $a, b \in \mathcal{A}$ the map $\Psi^{a,b} : \mathcal{N} \to \mathcal{N}$ given by*

$$\Psi^{a,b}(x) = -\frac{1}{2}(\Delta(axb) + a\Delta(x)b - \Delta(ax)b - a\Delta(xb)) \tag{23}$$

extends as $x\Omega_\tau \mapsto \Psi^{a,b}(x)\Omega_\tau$ to a bounded map on $L^2(\mathcal{N}, \tau)$ that is moreover in the Schatten p-class $\mathcal{S}_p$.

We recall that for a Hilbert space H, the Schatten p-class $\mathcal{S}_p$ for $p \in [1, \infty)$ is defined as the ideal in $B(H)$ of all elements $a \in B(H)$ that satisfy $\mathrm{Tr}(|a|^p) < \infty$. Here Tr denotes the trace on $B(H)$ that is given by $\mathrm{Tr}(a) = \sum_{i\in I}\langle ae_i, e_i\rangle$, where $\{e_i\}_{i\in I}$ is an (arbitrary) orthonormal basis for H. Furthermore, $\mathcal{S}_\infty$ is defined as the ideal of all compact operators.

We note that the gradient-$\mathcal{S}_p$ property generally depends on the choice of the algebra $\mathcal{A}$. Moreover, we note that an algebra $\mathcal{A}$ that satisfies all the stated conditions generally does not exist. However, in the cases that we consider there is an obvious candidate for the algebra $\mathcal{A}$, and we shall only consider the gradient-$\mathcal{S}_p$ property w.r.t this algebra. We shall furthermore sometimes write the map $\Psi^{a,b}$ as $\Psi^{a,b}_\Delta$ in order to clarify what semi-group $(e^{-t\Delta})_{t\geq 0}$ we consider.

3.1.1. *Example heat semi-group.*

We consider again the example of the heat semi-group from the introduction. This is the quantum Markov semi-group $(\Phi_t)_{t\geq 0} = (e^{-t\Delta})_{t\geq 0}$ on $L^\infty(\mathbb{T})$ with generator $\Delta(e_k) = k^2 e_k$. We are interested in whether this semi-group is gradient-$\mathcal{S}_p$ for some $p \in [1, \infty]$. We shall check whether this is the case w.r.t. the dense $*$-subalgebra $\mathcal{A} := \mathrm{Span}\{e_k : k \in \mathbb{Z}\}$ which satisfies all stated properties. We see that for $l, m \in \mathbb{Z}$ we have

$$\Psi^{e_l,e_m}(e_k) = -\frac{1}{2}(\Delta(e_{l+k+m}) + e_l\Delta(e_k)e_m - \Delta(e_{l+k})e_m - e_l\Delta(e_{k+m})) \tag{24}$$

$$= -\frac{1}{2}((l+k+m)^2 + k^2 - (l+k)^2 - (k+m)^2)e_{l+k+m} \tag{25}$$

$$= -lme_{l+k+m}. \tag{26}$$

This shows that the map Ψ^{e_l,e_m} is not compact on $L^2(\mathbb{T})$ when $l, m \neq 0$. Therefore, for $l, m \neq 0$ the map Ψ^{e_l,e_m} is not contained in the Schatten-p class $\mathcal{S}_p$. We thus find that the semi-groups $(\Phi_t)_{t\geq 0}$ is not gradient-$\mathcal{S}_p$ for any $p \in [1, \infty]$.

3.2. Examination of gradient-$\mathcal{S}_p$ property.

We shall have a closer look at the gradient-$\mathcal{S}_p$ property, by examining the map $\Psi^{a,b}$. We start by calculating the adjoint of the map $\Psi^{a,b}$, as an operator on $L^2(\mathcal{N})$. We then also calculate for $x \in \mathcal{N}$ the adjoint of the element $\Psi^{a,b}(x)$ of $\mathcal{N}$. These calculations are used at later points in this thesis. After these calculations we prove a useful lemma that allows us to more easily check whether a quantum Markov semi-group is gradient-$\mathcal{S}_p$ for some $p \in [1, \infty]$.

3.2.1. *Adjoint of $\Psi^{a,b}$ and adjoint of $\Psi^{a,b}(x)$.* Let $a, b \in \mathcal{A}$ and $x, y \in L^2(\mathcal{N})$. Since Δ is self-adjoint we have that

$$\langle \Psi^{a,b}(x), y\rangle = -\frac{1}{2}\left(\langle \Delta(axb), y\rangle + \langle a\Delta(x)b, y\rangle - \langle \Delta(ax)b, y\rangle - \langle a\Delta(xb), y\rangle\right) \tag{27}$$

$$= -\frac{1}{2}\left(\langle axb, \Delta(y)\rangle + \langle \Delta(x), a^*yb^*\rangle - \langle \Delta(ax), yb^*\rangle - \langle \Delta(xb), a^*y\rangle\right) \tag{28}$$

$$= -\frac{1}{2}\left(\langle x, a^*\Delta(y)b^*\rangle + \langle x, \Delta(a^*yb^*)\rangle - \langle ax, \Delta(yb^*)\rangle - \langle xb, \Delta(a^*y)\rangle\right) \tag{29}$$

$$= -\frac{1}{2}\left(\langle x, a^*\Delta(y)b^*\rangle + \langle x, \Delta(a^*yb^*)\rangle - \langle x, a^*\Delta(yb^*)\rangle - \langle x, \Delta(a^*y)b^*\rangle\right) \tag{30}$$

$$= \langle x, \Psi^{a^*,b^*}(y)\rangle \tag{31}$$

which shows that $(\Psi^{a,b})^* = \Psi^{a^*,b^*}$.

Furthermore, for $a, b \in \mathcal{A}$ and $x \in \mathcal{N}$ we have

$$(\Psi^{a,b}(x))^* = -\frac{1}{2}(\Delta(axb) + a\Delta(x)b - a\Delta(xb) - \Delta(ax)b)^* \tag{32}$$

$$= -\frac{1}{2}(\Delta(b^*x^*a^*) + b^*\Delta(x^*)a^* - \Delta(b^*x^*)a^* - b\Delta(x^*a^*) \tag{33}$$

$$= \Psi^{b^*,a^*}(x^*). \tag{34}$$

3.2.2. *Condition to check gradient-$\mathcal{S}_p$ property.* We shall show in the following lemma that, to check the gradient-$\mathcal{S}_p$ property, it is sufficient to show that $\Psi^{u,w}$ is in $\mathcal{S}_p$ for all pairs $u, w \in \mathcal{A}_0$, where $\mathcal{A}_0 \subseteq \mathcal{A}$ is some self-adjoint subset that generates the entire algebra $\mathcal{A}$.

Lemma 3.2 (Condition that implies Gradient-$\mathcal{S}_p$ property)**.** *Let $(\mathcal{N}, \tau)$ be a finite von Neumann algebra and let $(\Phi_t)_{t\geq 0}$ be a quantum Markov semi-group on $\mathcal{N}$. Furthermore, let Δ be the generator of $(\Phi_t)_{t\geq 0}$ and let $\mathcal{A} \subseteq \mathcal{N}$ be an appropriate subalgebra of $\mathcal{N}$ to which we check the gradient-$\mathcal{S}_p$ property. Let $p \in [1, \infty]$. Then $(\Phi_t)_{t\geq 0}$ is gradient-$\mathcal{S}_p$ if and only if there is a self-adjoint subset $\mathcal{A}_0 \subseteq \mathcal{A}$ of elements that generates $\mathcal{A}$, such that for all pairs of generators $(s_i, s_j) \in \mathcal{A}_0^2$ we have that Ψ^{s_i,s_j} is in $\mathcal{S}_p$.*

Proof. The only if statement follows directly from the definition of gradient-$\mathcal{S}_p$. We will prove the other direction. For an element $a \in \mathcal{A}$ denote $|a|_{\mathcal{A}_0}$ for the minimal number k of elements $s_1, ..., s_k$ in $\mathcal{A}_0$ such that $a = s_1...s_k$. If such k does not exist, set $|a|_{\mathcal{A}_0} = \infty$. We note that, since $\mathcal{A}_0$ is self-adjoint, every element in $\mathcal{A}$ can be written as finite sum $a = \sum_{i=1}^{k} c_i a_i$ for some scalars $c_i \in \mathbb{C}$ and element $a_i \in \mathcal{A}$ with $|a_i|_{\mathcal{A}_0} < \infty$. Now if $u = \sum_{i=1}^{k_1} c_i u_i$ and $w = \sum_{i=1}^{k_2} d_i w_i$ for some integers $k_1, k_2 \in \mathbb{N}$, scalars $c_i, d_i \in \mathbb{C}$ and elements $u_i, w_i \in \mathcal{A}$ with $|u_i|_{\mathcal{A}_0}, |w_i|_{\mathcal{A}_0} < \infty$, then we have for $v \in \mathcal{A}$ that $\Psi^{u,w}(v) = \sum_{i=1}^{k_1}\sum_{j=1}^{k_2} c_i d_j \Psi^{u_i,w_j}(v)$. Thus, if all operators Ψ^{u_i,w_j} are in $\mathcal{S}_p$, then so is $\Psi^{u,w}$. Now, we prove for $u, w \in \mathcal{A}$ with $|u|_{\mathcal{A}_0}, |w|_{\mathcal{A}_0} < \infty$ that $\Psi^{u,w}$ is in $\mathcal{S}_p$. Namely, we prove by induction for $n \geq 1$ that $\Psi^{u,w}$ is in $\mathcal{S}_p$ for all $u, w \in \mathcal{A}$ with $|u|_{\mathcal{A}_0}, |w|_{\mathcal{A}_0} \leq n$.

Before we do the induction, note that for $u_1, u_2, v, w \in \mathcal{A}$ we have

$$\Psi^{u_1u_2,w}(v) = \Delta(u_1u_2vw) + u_1u_2\Delta(v)w - \Delta(u_1u_2v)w - u_1u_2\Delta(vw) \tag{35}$$

$$= (\Delta(u_1u_2vw) + u_1\Delta(u_2v)w - \Delta(u_1u_2v)w - u_1\Delta(u_2vw)) \tag{36}$$

$$+ u_1\left(\Delta(u_2vw) + u_2\Delta(v)w - \Delta(u_2v)w - u_2\Delta(vw)\right) \tag{37}$$

$$= \Psi^{u_1,w}(u_2v) + u_1\Psi^{u_2,w}(v) \tag{38}$$

and likewise for $u, v, w_1, w_2 \in \mathcal{A}$ we have

$$\Psi^{u,w_2w_1}(v) = \Psi^{u,w_1}(vw_2) + \Psi^{u,w_2}(v)w_1. \tag{39}$$

We now do the induction. First, by the assumption the statement holds for $n = 1$. Now, suppose that the statement holds for some $n \geq 1$. We show that the statement also holds for $n+1$. Namely, let $u, w \in \mathcal{A}$ with $|u|_{\mathcal{A}_0}, |w|_{\mathcal{A}_0} \leq n+1$. Then there are elements $u_1, u_2, w_1, w_2 \in \mathcal{A}$ with $|u_1|_{\mathcal{A}_0}, |u_2|_{\mathcal{A}_0}, |w_1|_{\mathcal{A}_0}, |w_2|_{\mathcal{A}_0} \leq n$ and $u = u_1u_2$ and $w = w_2w_1$. Now

$$\Psi^{u,w}(v) = \Psi^{u_1u_2,w}(v) \tag{40}$$

$$= \Psi^{u_1,w}(u_2v) + u_1\Psi^{u_2,w}(v) \tag{41}$$

$$= \Psi^{u_1,w_2w_1}(u_2v) + u_1\Psi^{u_2,w_2w_1}(v) \tag{42}$$

$$= \left(\Psi^{u_1,w_1}(u_2vw_2) + \Psi^{u_1,w_2}(u_2v)w_1\right) + u_1\left(\Psi^{u_2,w_1}(vw_2) + \Psi^{u_2,w_2}(v)w_1\right). \tag{43}$$

Now, by the induction hypothesis we have that $\Psi^{u_1,w_1}, \Psi^{u_1,w_2}, \Psi^{u_2,w_1}, \Psi^{u_2,w_2}$ are all in $\mathcal{S}_p$. Now since the $\mathcal{S}_p$ class forms an ideal in $B(L^2(\mathcal{N}, \tau))$ and since for $i = 1, 2$ the left and right multiplication $v \to u_i v$ respectively $v \to v w_i$ are bounded operators, we have that the four operators in eq. (43) are all in $\mathcal{S}_p$. Thus also their sum, $\Psi^{u,w}$, is in $\mathcal{S}_p$. This finishes the induction and thus shows that the associated semi-group is gradient-$\mathcal{S}_p$. $\square$

4. Implications of the gradient-$\mathcal{S}_p$ property for the von Neumann algebra

We shall in this section elaborate on why we are interested in quantum Markov semi-groups that possess the gradient-$\mathcal{S}_p$ property for certain $p \in [1, \infty]$. In short, this is because for a given von Neumann algebra $\mathcal{N}$, a quantum Markov semi-group $(\Phi_t)_{t \geq 0}$ on $\mathcal{N}$ that possesses this property can (under some additional conditions) be used to obtain interesting properties for the algebra $\mathcal{N}$. Therefore, for a von Neumann algebra $\mathcal{N}$, we are interested in the existence of a quantum Markov semi-group that has the gradient-$\mathcal{S}_p$ property for certain $p \in [1, \infty]$ (and satisfies some additional conditions).

In section 4.1 we shall show a direct consequence of the gradient-$\mathcal{S}_p$ property that was shown in [12, Section 3]. This consequence is the starting point for other results that follow from the gradient-$\mathcal{S}_p$ property. To understand these other results we give in section 4.2 some additional background. We finish with section 4.3, where we state the results from [12, Section 5] that follow from the gradient-$\mathcal{S}_p$ property for the von Neumann algebra.

4.1. Quasi-containment of gradient tensor products in coarse bimodule.

We shall give here a direct consequence of the gradient-$\mathcal{S}_p$ property. For this we first construct, using a quantum Markov semi-group, a certain bimodule called the gradient tensor product. We then give the proofs as in [12] that show the quasi-containment of this bimodule in the coarse bimodule, under assumption of the gradient-$\mathcal{S}_p$ property for $p = 2$. Also, some weaker results hold for other p. The quasi-containment that is obtained is the starting point for other results that follow from this.

4.1.1. *Construction of the gradient tensor product.* We let $(\mathcal{N}, \tau)$ be a finite von Neumann algebra, and we let $(\Phi_t)_{t \geq 0} = (e^{-t\Delta})_{t \geq 0}$ be a quantum Markov semi-group on $\mathcal{N}$. We will fix an appropriate dense $*$-subalgebra $\mathcal{A} \subseteq \mathcal{N}$ as in the definition of the gradient-$\mathcal{S}_p$ property (if such subalgebra exists). We shall moreover denote A for the C^*-completion of $\mathcal{A}$.

A *gradient form* $\Gamma : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is defined as

$$\Gamma(x, y) = \frac{1}{2}(\Delta(y)^* x + y^* \Delta(x) - \Delta(y^* x)). \tag{44}$$

This map can be regarded as an $\mathcal{A}$-valued inner product. Now for an $A - A$ bimodule $\mathcal{H}$, we can consider the (possibly degenerate) inner product on $\mathcal{A} \otimes_{alg} \mathcal{H}$ that is given by

$$\langle x \otimes \xi, y \otimes \eta \rangle = \langle \Gamma(x, y)\xi, \eta \rangle \qquad x, y \in \mathcal{A} \text{ and } \xi, \eta \in \mathcal{H}. \tag{45}$$

The Hilbert space obtained by quotienting out the degenerate part and taking the completion is then called the *gradient tensor product* and denoted as $\mathcal{H}_\nabla$. The element $x \otimes \xi$ in $\mathcal{H}_\nabla$ will be denoted as $x \otimes_\nabla \xi$. On $\mathcal{H}$ we can define an $\mathcal{A} - \mathcal{A}$ bimodule action as

$$a \cdot (x \otimes_\nabla \xi) = ax \otimes_\nabla \xi - a \otimes_\nabla x\xi \tag{46}$$

$$(x \otimes_\nabla \xi) \cdot a = x \otimes_\nabla \xi a. \tag{47}$$

By [17] this moreover extends to an $A - A$ bimodule. The fact that the right-action is well-defined is clear. We will show that the left-action is also well defined, by showing that the corresponding map $\pi_l : A \to B(\mathcal{H}_\nabla)$ is a unital $*$-homomorphism. First of all, since $\Delta(1) = 0$ we have that $\Gamma(1, 1) = 0$ and hence that $\|1 \otimes_\nabla \xi\|_2 = 0$ for $\xi \in \mathcal{H}$. This shows that $\pi_l(1)(x \otimes_\nabla \xi) = x \otimes_\nabla \xi$, so π_l is unital. For $a, b \in A$ and $x \in \mathcal{A}$ and $\xi \in \mathcal{H}$ we have

$$\pi_l(a)\pi_l(b)(x \otimes_\nabla \xi) = \pi_l(a)bx \otimes_\nabla \xi - \pi_l(a)b \otimes_\nabla x\xi \tag{48}$$

$$= (abx \otimes_\nabla \xi - a \otimes_\nabla bx\xi) - (ab \otimes_\nabla x\xi - a \otimes_\nabla bx\xi) \tag{49}$$

$$= abx \otimes_\nabla \xi - ab \otimes_\nabla x\xi \tag{50}$$

$$= \pi_l(ab)(x \otimes_\nabla \xi). \tag{51}$$

This shows that $\pi_l(a)\pi_l(b) = \pi_l(ab)$. Furthermore, for $a, x, y \in \mathcal{A}$ and $\xi, \eta \in \mathcal{H}$ we have

$$\langle \pi_l(a)(x \otimes_\nabla \xi), y \otimes_\nabla \eta\rangle = \langle ax \otimes_\nabla \xi - a \otimes_\nabla x\xi, z \otimes_\nabla \eta\rangle \tag{52}$$

$$= \langle \Gamma(ax, y)\xi - \Gamma(a, y)x\xi, \eta\rangle \tag{53}$$

$$= \frac{1}{2}\langle(\Delta(y)^*ax + y^*\Delta(ax) - \Delta(y^*ax))\xi, \eta\rangle \tag{54}$$

$$- \frac{1}{2}\langle((\Delta(y)^*a + y^*\Delta(a) - \Delta(y^*a))x\xi, \eta\rangle \tag{55}$$

$$= \frac{1}{2}\langle(y^*\Delta(ax) - \Delta(y^*ax) + \Delta(y^*a)x - y^*\Delta(a)y^*\Delta(a)x)\xi, \eta\rangle \tag{56}$$

$$= \langle\Psi^{y^*,x}(a)\xi, \eta\rangle. \tag{57}$$

From this it follows that also

$$\langle x \otimes_\nabla \xi, \pi_l(a)(y \otimes_\nabla \eta)\rangle = \overline{\langle\pi_l(a)(y \otimes_\nabla \eta), x \otimes_\nabla \xi\rangle} = \overline{\langle\Psi^{x^*,y}(a)\eta, \xi\rangle} = \langle\xi, \Psi^{x^*,y}(a)\eta\rangle. \tag{58}$$

Now by the calculations from section 3.2.1 we have that $(\Psi^{y^*,x}(a))^* = \Psi^{x^*,y}(a^*)$, from which it now follows that $\pi_l(a)^* = \pi_l(a^*)$. This shows that π_l is a $*$-homomorphism, and that the bimodule action is well-defined.

4.1.2. *Proving the quasi-containment in the coarse bimodule.* We now turn to prove the quasi-containment of the gradient tensor product in the coarse bimodule. For this we state the following lemma from [12, Lemma 2.2.] with proof. This lemma gives a condition for quasi-containment of bimodules that is useful.

Lemma 4.1. *Let $\mathcal{N}$ be a von Neumann algebra and $\mathcal{A}$ a σ-weakly dense $*$-subalgebra of $\mathcal{N}$ with norm closure A. Let H be an $A - A$ bimodule and let K be an $\mathcal{N} - \mathcal{N}$ bimodule. Suppose that there exists a dense subspace $D \subseteq H$ such that for every $\xi \in D$ there exists an $\eta \in K$such that for every $x, y \in \mathcal{A}$ we have*

$$\langle x\xi y, \xi\rangle = \langle x\eta y, \eta\rangle. \tag{59}$$

Then for every $\xi \in D$ the sub-bimodule $H_\xi := \overline{A\xi A}$ of H is contained in K as $A - A$ bimodules. Consequently H is quasi-contained in K.

Proof. We let H and K be as stated and assume the dense subspace D exists. Let $\xi \in D$. Then by assumption there exists $\eta \in K$ such that $\langle x\xi y, \xi\rangle = \langle x\eta y, \eta\rangle$ for all $x, y \in \mathcal{A}$. Now, let $a_1, b_1, a_2, b_2 \in \mathcal{A}$ and suppose that $a_1\xi b_1 = a_2\xi b_2$. Then we have for $c, d \in \mathcal{A}$ that

$$\langle c(a_1\eta b_1 - a_2\eta b_2)d, \eta\rangle = \langle ca_1\eta b_1 d - ca_2\eta b_2 d, \eta\rangle = \langle ca_1\xi b_1 d - ca_2\xi b_2 d, \xi\rangle = 0 \tag{60}$$

This means that $\langle a_1\eta b_1 - a_2\eta b_2, c^*\eta d^*\rangle = 0$ and hence that $a_1\eta b_1 - a_2\eta b_2 \perp A\eta A$. However, this means that $a_1\eta b_1 - a_2\eta b_2 = 0$. This calculation have showed us that we can define a map $U : A\xi A \to K$ by mapping $a\xi b \mapsto a\eta b$. We moreover note that $\|U(a\xi b)\|_2^2 = \langle a^*a\eta bb^*, \eta\rangle = \langle a^*a\xi bb^*, \xi\rangle = \|a\xi b\|_2^2$. This shows that U extends to an isometry on H_ξ. It is moreover clear that $U(ahb) = aU(h)b$ for every $h \in H_\xi$ and $a, b \in \mathcal{A}$, and this extends by continuity to all $a, b \in \mathcal{A}$.

Let Σ be the set of all families of unit vectors $(\xi_i)_i$ in H such that each is a sub $A - A$ bimodule of K and so that all spaces H_{ξ_i} are orthogonal with each other. We can then by Zorn's lemma take a maximal element $(\xi_i)_i$ in Σ. Let P_i be the orthogonal projection onto H_{ξ_i}. Suppose by contradiction that $P := \sum_i P_i \neq \mathrm{Id}_H$. We note that P commutes with the $A - A$ bimodule action as the bimodule actions keeps the subspaces H_{ξ_i} invariant.

Let $\xi \in D$ be such that $\xi_0 := (I - P)\xi \neq 0$ and fix $\eta \in K$ such that $\langle a\xi b, \xi\rangle = \langle a\eta b, \eta\rangle$ for all $a, b \in \mathcal{A}$. We now have for finitely many $a_i, b_i \in \mathcal{A}$ that

$$\|\sum_i a_i\xi_0 b_i\| = \|(I - P)\sum_i a_i\xi b_i\| \leq \|\sum_i a_i\xi b_i\| = \|\sum_i a_i\eta b_i\|. \tag{61}$$

We can thus define a contraction $v : \overline{A\eta A} \to \overline{A\xi_0 A}$ as $v(a\eta b) = a\xi_0 b = (I - P)a\xi b$ for $a, b \in \mathcal{A}$. As before such mapping is well-defined. We have moreover that v^*v commutes with the $A - A$ bimodule action as this holds for $(I - P)$. We now put $\eta' := (v^*v)^{\frac{1}{2}}\eta$ so that

$$\langle a\xi_0 b, \xi_0\rangle = \langle(I - P)a\xi b, (I - P)\xi\rangle = \langle a\eta' b, \eta'\rangle \tag{62}$$

for $a, b \in \mathcal{A}$. However, this means that $\xi_0 \in D$ and thus that H_{ξ_0} is a sub $A - A$ bimodule of K. It is moreover clear that ξ_0 is orthogonal to all other vectors ξ_i. Also, as $\xi_0 \neq 0$ we can scale the vector to obtain a unit vector. This contradicts the maximality. We thus have that $P = \mathrm{Id}_H$ which shows that we have an $A - A$ bimodule embedding

$$H = \bigoplus_i H_{\xi_i} \subseteq \bigoplus_i K \tag{63}$$

and we can extend the $A - A$ action to normal $\mathcal{N} - \mathcal{N}$ bimodule action using this embedding. □

We shall now turn to the following theorem that saids that the fact that the quantum Markov semi-group is gradient-$\mathcal{S}_2$ will give the quasi-containment of the $A-A$ bimodule $L^2(\mathcal{N})_\nabla$ in the coarse bimodule $L^2(\mathcal{N})\overline{\otimes}L^2(\mathcal{N})$, by using the previous lemma. This theorem was given in [12, Theorem 3.9.] where it was stated in some more generality. There, for $n \in \mathbb{N}$ the gradient-$\mathcal{S}_{2n}$ property was used to prove the quasi-containment of the n-fold $A-A$ bimodule $L^2(\mathcal{N})_{\nabla^{(n)}} := (...(L^2(\mathcal{N})_\nabla)_\nabla...)_\nabla$ in the coarse bimodule. We shall here only give the proof for $n = 1$ and this case is sufficient for the purposes of this thesis.

In what follows, we shall say that a vector $\xi_0 \in L_2(\mathcal{N}, \tau)_\nabla$ is *algebraic* if it is contained in the linear span of the elements $a \otimes_\nabla b$ for $a, b \in \mathcal{A}$. Furthermore for a Hilbert space H, we shall write $\overline{H}$ for its conjugation. We have that $\overline{H}$ as a set equals H. For an element $b \in H$ we will write $\overline{b}$ to denote the element in $\overline{H}$. The addition in $\overline{H}$ is given as in H and scalar-multiplication is given by $c \cdot \overline{b} = \overline{(\overline{c}b)}$ for $c \in \mathbb{C}$ and $b \in \overline{H}$. The inner product for $\overline{H}$ is given by $\langle \overline{a}, \overline{b} \rangle_{\overline{H}} = \overline{\langle a, b \rangle_H}$. It can be checked that $\overline{H}$ is indeed a Hilbert space.

Theorem 4.2. *Let $(\mathcal{N}, \tau)$ be a finite von Neumann algebra. Suppose the quantum Markov semi-group $(\Phi_t)_{t\geq 0}$ has the gradient-$\mathcal{S}_2$ property w.r.t. $\mathcal{A}$. Then for any algebraic $\xi_0 \in L_2(\mathcal{N})_\nabla$ we have that $\overline{A\xi_0 A}$ is contained in the coarse bimodule $L_2(\mathcal{N})\overline{\otimes}L_2(\mathcal{N})$ as an A bimodule.*

Proof. We fix $\alpha = a_0 \otimes_\nabla a_1 \Omega_\tau$ with $a_0, a_1 \in \mathcal{A}$. We define a functional $\rho : A \otimes_{\mathrm{alg}} A^{\mathrm{op}} \to \mathbb{C}$ as $\rho(x \otimes y^{\mathrm{op}}) = \langle x \cdot \alpha \cdot y, \alpha \rangle$. By [30, Theorem 6.3.7.] we can find a $*$-homomorphism $\pi : A \otimes_{\max} A^{\mathrm{op}} \to B(L^2(\mathcal{N})_\nabla)$ such that $\rho(x) = \langle \pi(x)\alpha, \alpha \rangle$. This shows that we have $\rho(x^*x) = \|\pi(x)\alpha\|^2_{2,\tau} \geq 0$ for $x \in A \otimes_{\mathrm{alg}} A^{\mathrm{op}}$. We shall now show that ρ is $\otimes_{\min}$ bounded. We define a map $\Theta : L^2(\mathcal{N}) \to L^2(\mathcal{N})$ as $\Theta(x) = a_1^* \Psi^{a_0^*, a_0^*}(x) a_1 \Omega_\tau$. This map is in $\mathcal{S}_2$ as $\Psi^{a_0^*, a_0^*}$ is in $\mathcal{S}_2$ by assumption. Moreover, by the calculations preceding this theorem, we have for $x, y \in A$ that

$$\rho(x \otimes y^{\mathrm{op}}) = \langle x \cdot \alpha \cdot y, \alpha \rangle \tag{64}$$

$$= \langle \Psi^{a_0^*, a_0^*}(x) a_1 y, a_1 \rangle \tag{65}$$

$$= \langle a_1^* \Psi^{a_0^*, a_0^*}(x) a_1, y^* \rangle = \langle \Theta(x), y^* \rangle. \tag{66}$$

For $\xi, \eta \in L^2(\mathcal{N}, \tau)$ we consider the rank 1 operator $\theta_{\xi,\eta}$ on $L^2(\mathcal{N}, \tau)$ given by $\theta_{\xi,\eta}(x) = \xi \langle x, \eta \rangle_\tau$. For such operator we have that

$$\langle x, \theta_{\xi,\eta}(y) \rangle = \langle x, \xi \rangle \overline{\langle y, \eta \rangle} \tag{67}$$

$$= \langle x \otimes \overline{y}, \xi \otimes \overline{\eta} \rangle. \tag{68}$$

Now, this means that for any finite rank operator θ we have that $\langle x, \theta(y) \rangle = \langle x \otimes \overline{y}, \zeta_\theta \rangle$ for some $\zeta_\theta \in L^2(\mathcal{N}) \otimes_{\mathrm{alg}} \overline{L^2(\mathcal{N})}$. Let θ be a finite rank operator on $L^2(\mathcal{N})$. Then we can write $\theta = \sum_{j=1}^n \theta_{\xi_j, \eta_j}$ for some $n \geq 1$ and vectors $\eta_1, ..., \eta_{|I|} \in L^2(\mathcal{N})$ and orthonormal vectors $\xi_1, ..., \xi_{|I|} \in L^2(\mathcal{N})$. We then have that $\zeta_\theta = \sum_{j=1}^n \xi_j \otimes \overline{\eta_j}$. Furthermore, let $(e_i)_{i\in I}$ be a orthonormal basis for $L^2(\mathcal{N})$, then we have

$$\|\theta\|_{\mathcal{S}_2}^2 = \sum_{i\in I} \langle \theta(e_i), \theta(e_i)\rangle \tag{69}$$

$$= \sum_{i\in I}\sum_{j=1}^n \langle \theta_{\xi_j,\eta_j}(e_i), \theta_{\xi_j,\eta_j}(e_i)\rangle \tag{70}$$

$$= \sum_{i\in I}\sum_{j=1}^n |\langle \eta_j, e_i\rangle|^2 \tag{71}$$

$$= \sum_{j=1}^n \|\eta_j\|^2 \tag{72}$$

$$= \sum_{j=1}^n \langle \xi_j \otimes \overline{\eta_j}, \xi_j \otimes \overline{\eta_j}\rangle \tag{73}$$

$$= \|\zeta_\theta\|_2^2. \tag{74}$$

Now as, by [30, Theorem 2.4.17.], the finite rank operators are dense in $\mathcal{S}_2$, we can define an isometry $U : \mathcal{S}_2(L^2(\mathcal{N})) \to L^2(\mathcal{N})\overline{\otimes}\overline{L^2(\mathcal{N})}$ by setting $U(\theta) = \zeta_\theta$ for finite rank operators and extending this to a bounded map. We can also build an isometry $J : L^2(\mathcal{N})\overline{\otimes}\overline{L^2(\mathcal{N})} \to L^2(\mathcal{N})\overline{\otimes}L^2(\mathcal{N}^{\mathrm{op}})$ as $J(x \otimes \overline{y}) = x \otimes (y^*)^{\mathrm{op}}$. Indeed, as $\langle y^{\mathrm{op}\,*}, w^{\mathrm{op}\,*}\rangle_\tau = \tau(w^{\mathrm{op}} y^{\mathrm{op}\,*}) = \tau((y^*w)^{\mathrm{op}}) = \tau(y^*w) = \langle w, y\rangle_\tau = \langle \overline{y}, \overline{w}\rangle_\tau$ we have that

$$\langle J(x \otimes \overline{y}), J(z \otimes \overline{w})\rangle = \langle x \otimes y^{\mathrm{op}\,*}, z \otimes w^{\mathrm{op}\,*}\rangle \tag{75}$$

$$= \langle x, z\rangle_\tau \langle y^{\mathrm{op}\,*}, w^{\mathrm{op}\,*}\rangle_\tau \tag{76}$$

$$= \langle x, z\rangle_\tau \langle \overline{y}, \overline{w}\rangle_\tau = \langle x \otimes \overline{y}, z \otimes \overline{w}\rangle \tag{77}$$

which shows that J is an isometry. We now find that

$$\rho(x \otimes y^{\mathrm{op}}) = \langle \Theta(x), y^*\rangle_\tau \tag{78}$$

$$= \langle \Theta, \theta_{y^*,x}\rangle_{\mathcal{S}_2} \tag{79}$$

$$= \langle \theta_{y^*,x}^*, \Theta^*\rangle_{\mathcal{S}_2} \tag{80}$$

$$= \langle JU(\theta_{x,y^*}), JU(\Theta^*)\rangle \tag{81}$$

$$= \langle J(x \otimes \overline{y^*}), JU(\Theta^*)\rangle \tag{82}$$

$$= \langle x \otimes y^{\mathrm{op}}, JU(\Theta^*)\rangle. \tag{83}$$

By [30, Theorem 6.4.19] we can consider $A \otimes_{\min} A^{\mathrm{op}}$ as a subspace of $B(L^2(\mathcal{N})\overline{\otimes}L^2(\mathcal{N}^{\mathrm{op}}))$. We then find

$$\rho(z) = \langle z(1 \otimes 1), JU(\Theta^*)\rangle| \leq \|z\|_{\min} \cdot \|1 \otimes 1\|_2 \cdot \|JU(\Theta^*)\|_2. \tag{84}$$

This shows that the map ρ is $\otimes_{\min}$-bounded. We show that its extension to $A \otimes_{\min} A^{\mathrm{op}}$ is moreover positive. Namely, let $w \in A \otimes_{\min} A^{\mathrm{op}}$ be positive. Then we can write $w = z^*z$ for some $z \in A \otimes_{\min} A^{\mathrm{op}}$. Now since $\mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}^{\mathrm{op}}$ is a self-adjoint subalgebra of $A \otimes_{\min} A^{\mathrm{op}}$ that is weakly dense, it follows from Kaplansky's density theorem, [43, Theorem II.4.8], that there exists a bounded net (z_i) in $\mathcal{A} \otimes_{\min} \mathcal{A}^{\mathrm{op}}$ converging strongly to z. Now, as the net is bounded and converges strongly to z we have that $z_i^*z_i \to z^*z$ weakly. Now since the weak and σ-weak topology coincide on bounded sets, we have that $z_i^*z_i \to z^*z = w$ σ-weak. Since ρ is normal this then means that $0 \leq \rho(z_i^*z_i) \to \rho(w)$, which shows that ρ is positive on $A \otimes_{\min} A^{\mathrm{op}}$.

Now, by the properties of ρ we have by [44, Chapter X] that there exists a $\zeta_\alpha \in L^2(\mathcal{N})\overline{\otimes}L^2(\mathcal{N})$ such that $\langle x \cdot \alpha \cdot y, \alpha\rangle = \langle x \cdot \zeta_\alpha \cdot y, \zeta_\alpha\rangle$. This thus holds for all $\alpha \in D := \mathrm{Span}\{a_0 \otimes_\nabla a_1\Omega_\tau : a_0, a_1 \in \mathcal{A}\}$. As the subspace D is dense in $L^2(\mathcal{N})_\nabla$ it now holds by lemma 4.1 that $L^2(\mathcal{N})$ is quasi-contained in the coarse bimodule $L^2(\mathcal{N})\overline{\otimes}L^2(\mathcal{N})$.

□

4.2. **Additionally needed background and definitions.** In theorem 4.2, a direct consequence of the gradient-$\mathcal{S}_2$ property was given. Other results subsequently follow from this. In order to understand these results, we state here some additional definitions that are needed.

4.2.1. *Completely bounded maps.* Let A, B be C*-algebras. Let $\varphi : A \to B$ be a bounded map. Then for $n \geq 1$ the map $\varphi^{(n)} : A \otimes_{\min} M_n(\mathbb{C}) \to B \otimes_{\min} M_n(\mathbb{C})$ given by $\varphi^{(n)}(a \otimes e_{i,j}) = \varphi(a) \otimes e_{i,j}$ is also bounded. We shall call φ *completely bounded* if the values $\|\varphi^{(n)}\|$ are moreover bounded, and we shall write $\|\varphi\|_{\mathrm{cb}} = \sup_{n \geq 1} \|\varphi^{(n)}\|$. We remark here the resemblance with the definition of completely positive maps.

4.2.2. *Herz-Schur-multipliers.* For a kernel $k : \Gamma \to \Gamma \to \mathbb{C}$ be a kernel. We define the *Schur-muliplier* $m_k : B(\ell^2(\Gamma)) \to B(\ell^2(\Gamma))$ to be the map that satisfies $\langle m_k(x)\delta_t, \delta_s\rangle = k(s,t)\langle x\delta_t, \delta_s\rangle$ for all $s, t \in \Gamma$ and $x \in B(\ell_2(\Gamma))$, whenever such map exists. Note that, in case the map exist, it is in fact unique. When k is a positive definite kernel then by [7, Theorem C.3] the Schur-multiplier exists and is a bounded u.c.p map. For a function φ on Γ we shall write m_φ for the Schur multiplier associated to the kernel $(s,t) \to \varphi(t^{-1}s)$, whenever this function m_φ exists. Note that this function exists in particularly when φ is positive definite. A function φ on Γ is called a *Herz-Schur-multiplier* whenever m_φ is completely bounded. We shall denote $B_2(\Gamma)$ for the Banach space of all Herz-Schur-multipliers equipped with the Herz-Schur norm $\|\varphi\|_{B_2} = \|m_\varphi\|_{\mathrm{cb}}$.

4.2.3. *Amenable groups.* Let Γ be a group. There are many equivalent definitions for amenability, see [7, Theorem 2.6.8]. One is that the group Γ is *amenable* if there exists a net (φ_i) of finitely supported positive definite function on Γ such that $\varphi_i \to 1$ point-wise. Comparing this to the definition of the Haagerup property, we see that all amenable groups posses the Haagerup property.

4.2.4. *Weak amenable groups.* Let Γ be a group. The group Γ is called *weakly amenable* if there exists a net (φ_i) of finitely supported function on Γ such that $\varphi_i \to 1$ point-wise and such that $\limsup \|\varphi_i\|_{B_2} < \infty$. We note that amenable groups are weak amenable. Namely, for an amenable group Γ there exists a net (φ_i) of finitely supported positive definite function on Γ such that $\varphi_i \to 1$ point-wise. We can moreover assume that $\varphi_i(e) = 1$ for all i. Now since φ_i is positive definite there exist Hilbert spaces H_i and functions $f_i : \Gamma \to H$ such that $\varphi_i(t^{-1}s) = \langle f_i(s), f_i(t)\rangle$ for $s, t \in \Gamma$. We get that the function f_i satisfies $\|f_i(s)\|_2^2 = \langle f_i(s), f_i(s)\rangle = \varphi_i(e) = 1$ for $s \in \Gamma$. Now, by [7, Theorem C.4] we have that the multiplier m_φ is completely bounded with $\|m_\varphi\|_{\mathrm{cb}} \leq 1$. This shows that $\limsup \|\varphi_i\|_{B_2} \leq 1$ holds, which shows weak amenability of Γ.

4.2.5. *Approximation properties.* A C*-algebra A satisfies the *completely bounded approximation property (CBAP)* if there exists a net of finite rank maps $\theta_i : A \to A$ such that for every $x \in A$ we have $\|\theta_i(x) - x\| \to 0$ and $\limsup_i \|\theta_i\|_{cb} < \infty$. A von Neumann algebra $\mathcal{N}$ satisfies the *W*-completely bounded approximation property (W*-CBAP)* if there exists a net of normal finite rank maps $\theta_i : \mathcal{N} \to \mathcal{N}$ such that for every $x \in \mathcal{N}$ we have $\theta_i(x) \to x$ weakly and $\limsup_i \|\theta_i\|_{cb} < \infty$. For a discrete group Γ we have by [7, Theorem 12.3.8.] that Γ is weakly amenable if and only if $C_r^*(\Gamma)$ has the CBAP if and only if $\mathcal{L}(\Gamma)$ has the W*-CBAP.

4.2.6. *Locally reflexive.* Let A be a unital C*-algebra. An *operator system* E is a closed self-adjoint subspace $E \subseteq A$ so that $1_A \in E$. Now let B be a C*-algebra (either unital or non-unital). We denote B^{**} for its double dual, which is unital. We now call B *locally reflexive* if for every finite dimensional operator system $E \subseteq B^{**}$ there exists a net (φ_i) of contractive c.p. maps $\varphi_i : E \to A$ so that $\varphi_i \to \mathrm{Id}_E$ in the point-ultraweak topology.

4.2.7. *Akemann-Ostrand property AO^+.* A finite von Neumann algebra $\mathcal{N}$ has the *Akemann-Ostrand property (AO^+)* if there exists a σ-weakly dense unital C*-subalgebra $A \subseteq \mathcal{N}$ such that:

(1) A is locally reflexive.
(2) There exists a u.c.p. map $\theta : A \otimes_{\min} A^{\mathrm{op}} \to B(L_2(\mathcal{N}))$ such that $\theta(a \otimes b^{\mathrm{op}}) - ab^{\mathrm{op}}$ is compact for all $a, b \in A$.

4.2.8. *Strong solidity.* A von Neumann algebra is called *diffuse* if it has no non-zero minimal projections. A von Neumann algebra $\mathcal{N} \subseteq B(H)$ is called *amenable* if there exists a completely positive map $\Phi : B(H) \to \mathcal{N}$ s.t. $\Phi(x) = x$ for $x \in \mathcal{N}$.

A von Neumann algebra $\mathcal{N}$ is now called *strongly solid* if for every diffuse amenable von Neumann subalgebra $\mathcal{M} \subseteq \mathcal{N}$ we have that the normalizer $\mathrm{Nor}_{\mathcal{N}}(\mathcal{M})$, defined by

$$\mathrm{Nor}_{\mathcal{N}}(\mathcal{M}) := \{u \in \mathcal{N} \mid u \text{ unitary and } u\mathcal{M}u^* = \mathcal{M}\},$$

generates an amenable von Neumann algebra, that is, $Nor_{\mathcal{N}}(\mathcal{M})''$ is amenable.

A sufficient condition for strong solidity is given by the following result.

Theorem 4.3. *[23, Theorem 4.2.1] Let $\mathcal{N}$ be a finite von Neumann algebra with separable predual. If $\mathcal{N}$ has condition AO^+ and satisfies the W*-CBAP, then it is strongly solid.*

4.2.9. *Cartan subalgebra.* Let $\mathcal{N}$ be a finite von Neumann algebra and $A \subseteq \mathcal{N}$ a subalgebra. We call the subalgebra $A \subseteq \mathcal{N}$ a *Cartan subalgebra* if

(1) A is maximal abelian, that is $A' \cap \mathcal{N} = A$.
(2) The group of normalizers generates $\mathcal{N}$, that is $\mathrm{Nor}_{\mathcal{N}}(A)'' = \mathcal{N}$.

4.3. **Final results that we are interested in.** The final results that we want to obtain using gradient-$\mathcal{S}_p$ quantum Markov semi-groups are the Akemann-Ostrand property and strong solidity for new kinds of von Neumann algebras. We give a historical overview of the study of these properties. Thereafter we give conditions under which we can obtain these properties using the gradient-$\mathcal{S}_2$ property.

4.3.1. *Historical overview of the study of* $(AO)^+$ *and strong solidity.* The properties (AO^+) and strong solidity for a von Neumann algebra were defined in the study of the existence of Cartan subalgebras. These Cartan are an important object of study in the theory of von Neumann algebras. We list results on this topic here.

It was proven by Voiculescu in [45] that the group von Neumann algebra $\mathcal{L}(\mathbb{F}_n)$ of the free group $\mathbb{F}_n$ does not have a Cartan subalgebra for $2 \leq n \leq \infty$. These von Neumann algebras formed the first examples satisfying this conditions. Later, in [40] it was proven by Shlyakhtenko that for $0 < \lambda < 1$, the type III_λ free Araki–Woods factor does no have a Cartan subalgebra. This result relied on the absence of Cartan subalgebras in $\mathcal{L}(\mathbb{F}_\infty)$.

Later in the work of Ozawa and Popa, in [32], strong solidity was defined and it was shown that $\mathcal{L}(\mathbb{F}_n)$ has this property. From this it also follows that $\mathcal{L}(\mathbb{F}_n)$ does not possess a Cartan subalgebra. The strong solidity has since then become a property of interest. In the continuation of their work, in [33], Ozawa and Popa proved the absence of a Cartan subalgebra and strong solidity for certain group von Neumann algebras.

In [41], Sinclair proved strong solidity for group von Neumann algebras $\mathcal{L}(\Gamma)$ for certain discrete subgroups Γ of $SO(n,1)$ and $SU(n,1)$, where $SO(n,1)$ and $SU(n,1)$ are respectively the indefinite special orthogonal group and the generalized special unitary group.

In [22], Houdayer and Ricard proved the absence of Cartan subalgebras for free Araki-Woods factors, which generalized the result from Shlyakhtenko.

In [36], Popa and Vaes proved strong solidity results for $\mathcal{L}(\Gamma)$ for hyperbolic groups (see also [15, 16] from Chifian, Sinclair and Udrea for related results).

In [37], Popa and Vaes proved strong solidity results for certain von Neumann algebras using the Akemann-Ostrand condition (AO^+), which was a condition earlier defined by Ozawa in [31]. Thereafter, in [23], Isono more generally proved that the (AO^+) condition implies strong solidity under some conditions (this is theorem 4.3).

In [3], Boutonnet, Houdayer and Vaes, proved strong solidity for the free Araki-Woods factors, which improves the result from Shlyakhtenko.

In [8], Caspers proved strong solidity results for certain right-angled word hyperbolic Hecke algebras.

In [9] Caspers introduced the gradient-$\mathcal{S}_2$ property to prove strong solidity of the free orthonormal quantum groups.

In [10], Caspers used the gradient-$\mathcal{S}_2$ property to prove strong solidity results for a larger class of quantum groups. Also, using the non-commutative Riesz-transform it was shown here that these quantum groups have the AO^+ property, from which the strong solidity also follows. Furthermore, in [12], Caspers, Isono and Wasilewski introduced the gradient-$\mathcal{S}_p$ property for general $p \in [1, \infty]$, and they used this property to prove strong solidity for q-Gaussian algebras.

In this thesis we study this gradient-$\mathcal{S}_p$ property for the case of quantum Markov semi-groups on group von Neumann algebras. In particular we look at Coxeter groups. In some case we obtain (AO^+) and strong solidity similar to [9, 10, 12]. In section 8 we moreover study these properties for Hecke-algebras, which has also been studied in [8]. Furthermore, we note that in section 10, we will, like [10], use techniques involving the non-commutative Riesz transforms, to obtain (AO^+) for certain von Neumann algebras.

4.3.2. *Consequences of gradient-$\mathcal{S}_p$ property under additional conditions.* The quasi-containment that was proven in [12], was used in the same paper to prove the (AO^+) property, under some additional conditions. This can then by theorem 4.3 be used to prove strong solidity in some cases. We state here the assumptions under which we obtain the (AO$^+$) property. For this we introduce some notation first.

We shall call the semi-group $(\Phi_t)_{t\geq 0}$ *gradient coarse* if the $A-A$ bimodule action on $L^2(\mathcal{N})_\nabla$ extends to a normal $\mathcal{N}-\mathcal{N}$ bimodule action and further $L^2(\mathcal{N})_\nabla$ is weakly contained in the coarse bimodule. By lemma theorem 4.2 this is the case when $(\Phi_t)_{t\geq 0}$ has the gradient-$\mathcal{S}_2$ property (w.r.t. $\mathcal{A}$).

The generator Δ of the semi-group we will call *filtered* if it has compact resolvent and for every eigenvalue λ of Δ we have that there exists a (necessarily finite dimensional subspace) $\mathcal{A}(\lambda) \subseteq \mathcal{A}$ such that $\mathcal{A}(\lambda)\Omega_\tau$ equals the eigenspace of Δ at eigenvalue λ. Moreover, we assume that for an increasing enumeration $(\lambda_n)_{n\geq 1}$ of the eigenvalues of Δ, we have that

$$\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}(\lambda_n) \qquad\qquad \mathcal{A}(\lambda_l)\mathcal{A}(\lambda_k) \subseteq \bigoplus_{n=0}^{l+k} \mathcal{A}(\lambda_n) \tag{85}$$

where $\bigoplus$ denotes the algebraic direct sum. Furthermore, we will say that Δ has *subexponential growth* if the eigenvalues satisfy

$$\lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} = 1. \tag{86}$$

We now state the theorem that was obtained.

Theorem 4.4. *[12, Theorem 5.13] Let $\mathcal{N}$ be a finite von Neumann algebra and let $(\Phi_t)_{t\geq 0} = (e^{-t\Delta})_{t\geq 0}$ be a quantum Markov semi-group that is gradient coarse and suppose that Δ has compact resolvent, and is filtered with subexponential growth. Assume furthermore that A as defined above is locally reflexive. Then $\mathcal{N}$ satisfies (AO$^+$).*

We note that the result also follows when $\mathcal{N}$ is finite dimensional.

Remark 4.5. *Suppose $\mathcal{N}$ is finite dimensional, then it follows from the definition that $\mathcal{N}$ possess the (AO$^+$) property. Indeed, every operator on a finite dimensional space is compact, so we only need to check local reflexivity. However, this follows from the fact that $\mathcal{N}^{**} \simeq \mathcal{N}$ when $\mathcal{N}$ is finite dimensional.*

5. Gradient-$\mathcal{S}_p$ property for semi-groups on group von Neumann algebras

In this section we construct quantum Markov semi-groups on group von Neumann algebras $\mathcal{L}(\Gamma)$ of discrete groups Γ that possess the Haagerup property. These semi-groups $(\Phi_t)_{t\geq 0}$ are built using a proper, conditionally negative definite function ψ on the group Γ. In the rest of this thesis we will mainly look at semi-groups that are built in this particular way. In section 5.1 we construct the quantum Markov semi-groups and calculate the generators of these semi-groups. In section 5.2 we investigate in what cases these semi-groups are gradient-$\mathcal{S}_p$. There we also do some calculations and introduce some notation that will be used throughout the next sections. In section 5.3, we moreover check that under some conditions, the generators of the semi-groups satisfy the properties from section 4.3.2, i.e. that it is filtered and has subexponential growth.

5.1. **Construction of quantum Markov semi-groups using conditionally negative function.** We let Γ be a discrete group that possesses the Haagerup property and show how a quantum Markov semi-group $(\Phi_t)_{t\geq 0}$ on the group von Neumann algebra $\mathcal{L}(\Gamma) \subseteq B(\ell_2(\Gamma))$ can be constructed. Since Γ has the Haagerup property there exists a proper, conditionally negative definite function ψ on Γ. Now for $t \geq 0$ we define functions $\varphi_t : \Gamma \to \mathbb{R}$ as $\varphi_t(\mathbf{g}) = e^{-t\psi(\mathbf{g})}$, which, by Schoenberg's theorem, are positive definite. These functions moreover satisfy $\varphi_t(e) = 1$ and vanish at infinity. We now define a multiplier $m_{\varphi_t} : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ as

$$m_{\varphi_t}(\sum_{\mathbf{g}\in\Gamma} \alpha_{\mathbf{g}} \cdot \lambda_{\mathbf{g}}) = \sum_{g\in\Gamma} \varphi_t(g)\alpha_{\mathbf{g}} \cdot \lambda_{\mathbf{g}}. \tag{87}$$

From [7, Theorem 2.5.11] we obtain that m_{φ_t} extends to a normal u.c.p. map on the group von Neumann algebra $\mathcal{L}(\Gamma)$. This extension will be denoted as Φ_t. It is clear that $\Phi_0 = \mathrm{Id}_{\mathcal{L}(\Gamma)}$ and that $\Phi_{t_1}\Phi_{t_2} = \Phi_{t_1+t_2}$ for $t_1, t_2 \geq 0$. We will show that also the other conditions of a quantum Markov semi-group hold. First we prove symmetricity. For this we recall that the canonical trace on $\mathcal{L}(\Gamma)$ is given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$ and we thus see that for $\mathbf{g}, \mathbf{r} \in \Gamma$ we have

$$\langle \lambda_{\mathbf{r}}, \Phi_t(\lambda_{\mathbf{g}})\rangle_\tau = \tau(\Phi_t(\lambda_{\mathbf{g}^{-1}})\lambda_{\mathbf{r}}) = \varphi_t(\mathbf{g}^{-1})\tau(\lambda_{\mathbf{g}^{-1}}\lambda_{\mathbf{r}}) = \varphi_t(\mathbf{g}^{-1})\langle \lambda_{\mathbf{r}}\delta_e, \lambda_{\mathbf{g}}\delta_e\rangle = \varphi_t(\mathbf{r}^{-1})\mathbb{1}(\mathbf{r} = \mathbf{g}). \tag{88}$$

Similarly we obtain $\langle \Phi_t(\lambda_{\mathbf{r}}), \lambda_{\mathbf{g}}\rangle_\tau = \tau(\lambda_{\mathbf{g}^{-1}}\Phi_t(\lambda_{\mathbf{r}})) = \varphi_t(\mathbf{r})\mathbb{1}(\mathbf{r} = \mathbf{g})$. Now, as ψ is conditionally negative definite there is a Hilbert space H and a function $b : \Gamma \to H$ such that $\psi(\mathbf{w}^{-1}\mathbf{u}) = \|b(\mathbf{u}) - b(\mathbf{w})\|^2$ holds for all $\mathbf{u}, \mathbf{w} \in \Gamma$, hence

$$\psi(\mathbf{r}) = \psi(e\mathbf{r}) = \|b(\mathbf{r}) - b(e^{-1})\|^2 = \|b(e) - b(\mathbf{r})\|^2 = \psi(\mathbf{r}^{-1}e) = \psi(\mathbf{r}^{-1}). \tag{89}$$

This shows us that $\langle \lambda_{\mathbf{r}}, \Phi_t(\lambda_{\mathbf{g}})\rangle_\tau = \varphi_t(\mathbf{r}^{-1})\mathbb{1}(\mathbf{r} = \mathbf{g}) = \varphi_t(\mathbf{r})\mathbb{1}(\mathbf{r} = \mathbf{g}) = \langle \Phi_t(\lambda_{\mathbf{r}}), \lambda_{\mathbf{g}}\rangle_\tau$. Now, by extending this linearly we obtain that $\langle x, \Phi_t(y)\rangle_\tau = \langle \Phi_t(x), y\rangle_\tau$ for $x, y \in \mathbb{C}[\Gamma]$. By density this also means that $\langle \Phi_t(x), y\rangle_\tau = \langle x, \Phi_t(y)\rangle_\tau$ for $x, y \in L^2(\mathcal{L}(\Gamma), \tau)$. This shows that in particular for $x, y \in \mathcal{L}(\Gamma)$ we have $\tau(\Phi_t(x)y) = \langle y, \Phi_t(x^*)\rangle_\tau = \langle \Phi_t(y), x^*\rangle_\tau = \tau(x\Phi_t(y))$, which proves the maps Φ_t are symmetric.

We now prove for $x \in \mathcal{L}(\Gamma)$ that the function $t \mapsto \Phi_t(x)$ is continuous for the strong topology of $\mathcal{L}(\Gamma)$. First note that for $\mathbf{g} \in \Gamma$ we have that the mapping $t \mapsto \Phi_t(\lambda_{\mathbf{g}})$ is norm-continuous as by definition $\Phi_t(\lambda_{\mathbf{g}}) = e^{-t\psi(\mathbf{g})}\lambda_{\mathbf{g}}$. This implies that for $x \in \mathbb{C}[\Gamma]$ the mapping $t \mapsto \Phi_t(x)$ is also norm-continuous. Now let $x \in L^2(\mathcal{L}(\Gamma), \tau)$ and $\epsilon > 0$. As $\{\lambda_{\mathbf{g}}\}_{\mathbf{g}\in\Gamma}$ is an orthogonal basis for $L^2(\mathcal{L}(\Gamma), \tau)$, we can choose $y \in \mathbb{C}[\Gamma]$ such that $\|x - y\|_2 \leq \epsilon$. Now there exists $\delta > 0$ such that $\|\Phi_h(y) - y\|_2 \leq \epsilon$ whenever $0 \leq h \leq \delta$. For such values h we find

$$\|\Phi_h(x) - x\|_2 \leq \|\Phi_h(x - y)\|_2 + \|\Phi_h(y) - y\|_2 + \|y - x\|_2 \tag{90}$$

$$\leq \|x - y\|_2 + \|\Phi_h(y) - y\|_2 + \|y - x\|_2 \tag{91}$$

$$\leq 3\epsilon \tag{92}$$

where we used that the maps Φ_t are contractive on $L^2(\mathcal{L}(\Gamma), \tau)$. This shows that $\lim_{h\downarrow 0} \Phi_h(x) = x$, where convergence is in $\|\cdot\|_2$. We now let $x \in \mathcal{L}(\Gamma)$, let $\xi \in L^2(\mathcal{L}(\Gamma), \tau)$ and let $\epsilon > 0$. Then there is $\xi' \in \mathcal{L}(\Gamma)$ such that $\|\xi - \xi'\|_2 \leq \epsilon$. Now

$$\|(\Phi_h(x) - x)\xi\|_2 \leq \|(\Phi_h(x) - x)\xi'\|_2 + \|(\Phi_h(x) - x)(\xi - \xi')\|_2 \tag{93}$$

$$\leq \|\Phi_h(x) - x\|_2 \cdot \|\xi'\| + \|\Phi_h(x) - x\| \cdot \|\xi - \xi'\|_2 \tag{94}$$

$$\leq \|\Phi_h(x) - x\|_2 \cdot \|\xi'\| + 2\|x\|\epsilon. \tag{95}$$

We then obtain $\limsup_{h\downarrow 0} \|(\Phi_h(x) - x)\xi\|_2 \leq 2\|x\|\epsilon$ and hence $\lim_{h\to 0} \|(\Phi_h(x) - x)\xi\|_2 = 0$ as ϵ was arbitrary. This shows that $\lim_{h\downarrow 0} \Phi_h(x) \to x$ in the strong operator topology of $B(L^2(\mathcal{L}(\Gamma), \tau))$. Now, as the representation $\pi : \mathcal{L}(\Gamma) \to B(L^2(\mathcal{L}(\Gamma), \tau))$ given by $\pi(x)(y) = xy$ is faithful, we obtain by [26, Corollary 7.1.16] that $\lim_{h\downarrow 0} \Phi_h(x) \to x$ in the strong operator topology of $\mathcal{L}(\Gamma)$. We moreover obtain for $t > 0$ and $\eta \in \ell_2(\Gamma)$ that

$$\|(\Phi_{t+h}(x) - \Phi_t(x))\eta\| \le \|\Phi_t\| \cdot \|(\Phi_h(x) - x)\eta\| \le \|(\Phi_h(x) - x)\eta\| \to 0 \qquad \text{as } h \downarrow 0 \tag{96}$$

$$\|(\Phi_{t-h}(x) - \Phi_t(x))\eta\| \le \|\Phi_{t-h}\| \cdot \|(x - \Phi_h(x))\eta\| \le \|(x - \Phi_h(x))\eta\| \to 0 \qquad \text{as } h \downarrow 0 \tag{97}$$

which shows that $t \to \Phi_t(x)$ is continuous for the strong topology of $\mathcal{L}(\Gamma)$.

It now follows from these properties that $(\Phi_t)_{t\ge 0}$ defines a quantum Markov semi-group on $\mathcal{L}(\Gamma)$. This quantum Markov semi-group we will call *the semi-group associated to* ψ. As a semi-group on $L^2(\mathcal{L}(\Gamma), \tau)$ we can write $(\Phi_t)_{t\ge 0} = (e^{-\Delta_\psi t})_{t\ge 0}$, where Δ_ψ is the unbounded operator on $L^2(\mathcal{L}(\Gamma), \tau)$ that generates the semi-group. We can calculate Δ_ψ as follows. For $\mathbf{g} \in \Gamma$

$$\Delta_\psi(\lambda_{\mathbf{g}}) = -\lim_{t\to 0} \frac{\Phi_t(\lambda_{\mathbf{g}}) - \Phi_0(\lambda_{\mathbf{g}})}{t} = -\lim_{t\to 0} \frac{e^{-t\psi(\mathbf{g})} - 1}{t} \lambda_{\mathbf{g}} = \psi(\mathbf{g})\lambda_{\mathbf{g}}. \tag{98}$$

Now as the vectors $\{\lambda_{\mathbf{g}}\}_{\mathbf{g}\in\Gamma}$ form an orthogonal basis for $L^2(\mathcal{L}(\Gamma), \tau)$ this shows how Δ_ψ is defined. We note that when Γ is infinite we have, because ψ is proper, that Δ_ψ is not a bounded operator on $L^2(\mathcal{L}(\Gamma), \tau)$.

5.2. **Gradient-$\mathcal{S}_p$ property for semi-groups on group von Neumann algebras.** For a group Γ with the Haagerup property, and a proper, conditionally negative function ψ on Γ we let $(\Phi_t)_{t\ge 0} := (e^{-t\Delta})_{t\ge 0}$ be the quantum Markov semi-group on $\mathcal{L}(\Gamma)$ associated to ψ. In order to study the gradient-$\mathcal{S}_p$ property, we first have to specify what 'nice' subalgebra $\mathcal{A} \subseteq \mathcal{L}(\Gamma)$ we use. For this we will always take the $*$-subalgebra $\mathcal{A} := \mathbb{C}[\Gamma]$. We note that by the definition of the group von Neumann algebra, this $*$-subalgebra is σ-weakly dense in $\mathcal{L}(\Gamma)$. Moreover, clearly $\mathbb{C}[\Gamma] \subseteq \mathrm{Dom}(\Delta_\psi)$ and $\Delta_\psi(\mathbb{C}[\Gamma]) \subseteq \mathbb{C}[\Gamma]$. Furthermore, for $x \in \mathbb{C}[\Gamma]$ we already showed that the mapping $t \mapsto \Phi_t(x)$ is norm-continuous. This shows that $\mathbb{C}[\Gamma]$ indeed satisfies the properties that we need in order to define the gradient-$\mathcal{S}_p$ property.

The gradient-$\mathcal{S}_p$ property for $p \in [1, \infty]$ is now defined as the property that for $a, b \in \mathbb{C}[\Gamma]$ we have that the mapping $\Psi^{a,b}_{\Delta_\psi} : \mathcal{L}(\Gamma) \to \mathcal{L}(\Gamma)$ given by

$$\Psi^{a,b}_{\Delta_\psi}(x) = -\frac{1}{2}(\Delta_\psi(axb) + a\Delta_\psi(x)b - \Delta_\psi(ax)b - a\Delta_\psi(xb)) \tag{99}$$

extends to a bounded mapping on $L^2(\mathcal{L}(\Gamma), \tau)$ that is moreover in $\mathcal{S}_p$. We shall generally just write $\Psi^{a,b}$ for $\Psi^{a,b}_{\Delta_\psi}$ when the semi-group is understood.

We will introduce some notation here that makes it easier to study the gradient-$\mathcal{S}_p$ property of the semi-group. For $\mathbf{u}, \mathbf{w} \in \Gamma$ we define a function $\gamma^\psi_{\mathbf{u},\mathbf{w}} : \Gamma \to \mathbb{R}$ as

$$\gamma^\psi_{\mathbf{u},\mathbf{w}}(\mathbf{v}) = \psi(\mathbf{uvw}) + \psi(\mathbf{v}) - \psi(\mathbf{uv}) - \psi(\mathbf{vw})$$

and we simply write $\gamma_{\mathbf{u},\mathbf{w}}$ for $\gamma^\psi_{\mathbf{u},\mathbf{w}}$ when the function ψ is understood. We will keep ψ fixed in the following. We have that the function $\gamma_{\mathbf{u},\mathbf{w}}$ is related to the operator $\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}}$ as

$$\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{v}}}(\lambda_{\mathbf{v}}) = -\frac{1}{2}(\Delta_\psi(\lambda_{\mathbf{uvw}}) + \lambda_{\mathbf{u}}\Delta_\psi(\lambda_{\mathbf{v}})\lambda_{\mathbf{w}} - \Delta_\psi(\lambda_{\mathbf{uv}})\lambda_{\mathbf{w}} - \lambda_{\mathbf{u}}\Delta_\psi(\lambda_{\mathbf{vw}})) \tag{100}$$

$$= -\frac{1}{2}\gamma_{\mathbf{u},\mathbf{w}}(\mathbf{v})\lambda_{\mathbf{uvw}}. \tag{101}$$

We note that by the calculation from section 3.2.1 we moreover have that $(\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}})^* = \Psi^{\lambda^*_{\mathbf{u}},\lambda^*_{\mathbf{w}}} = \Psi^{\lambda_{\mathbf{u}^{-1}},\lambda_{\mathbf{w}^{-1}}}$ and that we thus get

$$|\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}}|^2(\lambda_{\mathbf{v}}) = \Psi^{\lambda_{\mathbf{u}^{-1}},\lambda_{\mathbf{w}^{-1}}}\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}}(\lambda_{\mathbf{v}}) \tag{102}$$

$$= -\frac{1}{2}\Psi^{\lambda_{\mathbf{u}^{-1}},\lambda_{\mathbf{w}^{-1}}}(\gamma_{\mathbf{u},\mathbf{w}}(\mathbf{v})\lambda_{\mathbf{uvw}}) \tag{103}$$

$$= \frac{1}{4}\gamma_{\mathbf{u}^{-1},\mathbf{w}^{-1}}(\mathbf{uvw})\gamma_{\mathbf{u},\mathbf{w}}(\mathbf{v})\lambda_{\mathbf{v}} \tag{104}$$

$$= \frac{1}{4}|\gamma_{\mathbf{u},\mathbf{w}}(\mathbf{v})|^2\lambda_{\mathbf{v}}. \tag{105}$$

This then means that $|\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}}|^p(\lambda_{\mathbf{v}}) = 2^{-p}|\gamma_{\mathbf{u},\mathbf{w}}(\mathbf{v})|^p\lambda_{\mathbf{v}}$ and therefore, as $\{\lambda_{\mathbf{v}}\}_{\mathbf{v}\in\Gamma}$ forms an orthonormal basis, we have that

$$\|\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}}\|_{\mathcal{S}_p} = \left(\sum_{\mathbf{v}\in\Gamma}\langle|\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}}|^p(\lambda_{\mathbf{v}}),\lambda_{\mathbf{v}}\rangle\right)^{\frac{1}{p}} = \frac{1}{2}\|\gamma_{\mathbf{u},\mathbf{w}}\|_{\ell_p(\Gamma)}. \tag{106}$$

Now for $p\in[1,\infty)$, in order to check whether $\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}}$ is in $\mathcal{S}_p$ we thus need to check whether $\gamma_{\mathbf{u},\mathbf{w}}\in\ell_p(\Gamma)$. Moreover, for $p=\infty$, the condition that $\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}}\in\mathcal{S}_p$ means that $\Psi^{\lambda_{\mathbf{u}},\lambda_{\mathbf{w}}}$ is a compact operator, which is precisely the case when $\gamma_{\mathbf{u},\mathbf{w}}\in c_0(\Gamma)$, i.e. when $\gamma_{\mathbf{u},\mathbf{w}}$ vanishes at infinity.

The above calculations, together with lemma 3.2, give us a simple condition to check for $p\in[1,\infty]$ whether the semi-group $(\Phi_t)_{t\geq 0}$ is gradient-$\mathcal{S}_p$

Lemma 5.1. *Let Γ be a discrete group, and let $\Gamma_0\subseteq\Gamma$ be a subset that generates the entire group. Then for $p\in[1,\infty)$, for a semi-group $(\Phi_t)_{t\geq 0}$ associated to some proper, conditionally negative function ψ on Γ, we have that if $\gamma^{\psi}_{u,w}\in\ell_p(\Gamma)$ for all $u,w\in\Gamma_0\cup\Gamma_0^{-1}$, then the semi-group $(\Phi_t)_{t\geq 0}$ is gradient-$\mathcal{S}_p$. The same holds true for $p=\infty$ when $\ell_p(\Gamma)$ is replaced with $c_0(\Gamma)$.*

Proof. We denote $\mathcal{A}_0 := \{\lambda_g : g\in\Gamma_0\cup\Gamma_0^{-1}\}\subseteq\mathbb{C}[\Gamma]$, which is a self-adjoint subset that generates $\mathbb{C}[\Gamma]$. We fix $p\in[1,\infty)$. Now, if for all $u,w\in\Gamma_0\cup\Gamma_0^{-1}$ we have that $\gamma^{\psi}_{u,w}\in\ell_p(\Gamma)$, then, by the calculation of $\|\Psi^{\lambda_u,\lambda_v}\|_{\mathcal{S}_p}$, we have that also $\Psi^{\lambda_u,\lambda_w}\in\mathcal{S}_p$ for all $u,w\in\Gamma_0\cup\Gamma_0^{-1}$. We thus find that $\Psi^{a,b}\in\mathcal{S}_p$ for all $a,b\in\mathcal{A}_0$, which shows by lemma 3.2 that $(\Phi_t)_{t\geq 0}$ is gradient-$\mathcal{S}_p$. The proof is similar for $p=\infty$. ☐

5.3. **Checking additional conditions to obtain (AO$^+$) and strong solidity.** Let ψ be a proper, conditionally negative definite function on a group Γ satisfying $\psi(\mathbf{uw})\leq\psi(\mathbf{u})+\psi(\mathbf{w})$ for $\mathbf{u},\mathbf{w}\in\Gamma$ and $\psi(\Gamma)=\mathbb{Z}_{\geq 0}$. We show that the generator Δ_ψ that we constructed in section 5.1 satisfies the properties from section 4.3.2, i.e. that it is filtered and has subexponential growth. These properties are needed to obtain the final results we are interested in.

We first show that Δ_ψ is filtered with respect to $\mathbb{C}[\Gamma]$. First of all we have that $(I+\Delta_\psi)^{-1}(\lambda_{\mathbf{v}}) = \frac{\lambda_{\mathbf{v}}}{1+\psi(\mathbf{v})}$ for $\mathbf{v}\in\Gamma$, which shows that $(I+\Delta_\psi)^{-1}$ is a compact operator as ψ is proper. The operator Δ_ψ thus has compact resolvent. Furthermore, we have that the finite dimensional subspaces

$$\mathbb{C}[\Gamma](l) := \mathrm{Span}\{\lambda_{\mathbf{v}}\in\mathbb{C}[\Gamma] : \psi(\mathbf{v}) = l\}\qquad\text{for integers } l\geq 0 \tag{107}$$

of $\mathbb{C}[\Gamma]$ are such that $\mathbb{C}[\Gamma](l)\Omega_\tau$ equals the eigenspace of Δ_ψ at the eigenvalue l. For these spaces we have that

$$\mathbb{C}[\Gamma] = \bigoplus_{l\geq 0}\mathbb{C}[\Gamma](l)\qquad\qquad \mathbb{C}[\Gamma](l)\mathbb{C}[\Gamma](k)\subseteq\bigoplus_{j=0}^{l+k}\mathbb{C}[\Gamma](j)\ \text{for } l,k\geq 0 \tag{108}$$

where $\bigoplus$ denotes the algebraic direct sum. The first equality holds because ψ only takes positive integers values and the second equality holds because $\psi(\mathbf{uw})\leq\psi(\mathbf{u})+\psi(\mathbf{w})$ for $\mathbf{u},\mathbf{w}\in W$. Last, we note that the eigenvalues $\{l : l\in\mathbb{Z}_{\geq 0}\}$ of Δ have subexponential growth as $\lim\limits_{l\to\infty}\frac{l+1}{l} = 1$.

6. Gradient-$\mathcal{S}_p$ property for semi-groups associated to word length on Coxeter Groups

In this section we will consider semi-groups on group von Neumann algebras of Coxeter groups. Namely, a Coxeter group $W = \langle S|M\rangle$ has the Haagerup property by [4], which allows us to construct semi-groups like we did in section 5.1. The function ψ_S on W given by the word length with respect to the generating set S defines a proper, conditionally negative definite function by [4]. In this section we shall only consider the quantum Markov semi-group $(\Phi_t)_{t\geq 0}$ on $\mathcal{L}(W)$ associated to this function ψ_S. With the tools described in section 5.2, we then investigate for what type of Coxeter groups W, this semi-group is gradient-$\mathcal{S}_p$ for some $p \in [1,\infty]$. We will obtain two results, theorem 6.7 and theorem 6.8, that together give an almost complete characterization for what Coxeter systems this is the case. Last we obtain a result, proposition 6.9, that clarifies the 'almost complete' characterization some more, by giving a more direct condition on the Coxeter group.

We state here two theorems that will give us results when the semi-group is gradient-$\mathcal{S}_2$.

Theorem 6.1. *For a Coxeter group W we have that $\mathcal{L}(W)$ satisfies the W*-CBAP. Further, $C_r^*(W)$ satisfies the CBAP and is in particular locally reflexive.*

Proof. Since any Coxeter group W is weakly amenable by [19], we obtain by [7, Theorem 12.3.8.] that $\mathcal{L}(W)$ satisfies W*-CBAP and that $C_r^*(W)$ satisfies CBAP. Now, the CBAP implies by [7, Definition 12.4.1 and Theorem 12.4.4 and Corollary 9.4.1] that $C_r^*(W)$ is in particular locally reflexive. ☐

Theorem 6.2. *Let $W = \langle S|M\rangle$ be a Coxeter group. Suppose the semi-group $(\Psi_t)_{t\geq 0}$ associated to the word length ψ_S is gradient-$\mathcal{S}_2$, then $\mathcal{L}(W)$ has the AO$^+$ property and is strongly solid.*

Proof. As the semi-group is gradient-$\mathcal{S}_2$, we have by theorem 4.2 that it is gradient coarse. We note moreover that $\mathcal{L}(W)$ is either finite dimensional (when W is finite) or that the operator Δ_{ψ_S} is filtered and has subexponential growth by section 5.3 as ψ_S satisfies $\psi_S(\mathbf{uw}) \leq \psi_S(\mathbf{u}) + \psi_S(\mathbf{w})$ for $\mathbf{u},\mathbf{w} \in W$ and $\psi_S(W) = \mathbb{Z}_{\geq 0}$ (when W is infinite). In the first case we obtain that $\mathcal{L}(W)$ has AO$^+$ by remark 4.5 and in the latter case we obtain this by theorem 4.4 and theorem 6.1. It now follows from theorem 4.3 and theorem 6.1 that $\mathcal{L}(W)$ is moreover strongly solid. ☐

We now make some observations that will help determine for what Coxeter groups W the semi-group $(\Phi_t)_{t\geq 0}$ associated to ψ_S is gradient-$\mathcal{S}_p$. First, since ψ_S only takes integer values we have for $\mathbf{u},\mathbf{w} \in W$ that $\gamma_{\mathbf{u},\mathbf{w}}^{\psi_S}$ is in $\ell_p(W)$ for some p, or in $c_0(W)$, if and only if $\gamma_{\mathbf{u},\mathbf{w}}^{\psi_S}$ is finite rank. Now since elements in the set S are its own inverse, we have by lemma 5.1 that the semi-group is gradient-$\mathcal{S}_p$ for some $p \in [1,\infty]$ if and only if for all pairs of generators $u, w \in S$ we have that $\gamma_{u,w}^{\psi_S}$ is finite rank. Moreover, if this is the case then directly we have that $(\Phi_t)_{t\geq 0}$ is gradient-$\mathcal{S}_p$ for all $p \in [1,\infty]$. We state this as the following remark.

Remark 6.3. *For a Coxeter group W, the semi-group on $\mathcal{L}(W)$ associated to the word length ψ_S is gradient-$\mathcal{S}_p$ for some $p \in [1,\infty]$ (or equivalently all $p \in [1,\infty]$), if and only if $\gamma_{u,w}^{\psi_S}$ is finite rank for all $u, w \in S$.*

We will thus investigate for generators $u, w \in S$ when precisely $\gamma_{u,w}^{\psi_S}$ is finite rank.

6.1. Describing support of the function $\gamma_{u,w}^{\psi_S}$.

The following lemma gives, for certain conditionally negative functions ψ, a simple formula for $|\gamma_{u,w}^{\psi}|$. Note that this lemma applies in particular to the word length ψ_S.

Lemma 6.4. *Let $W = \langle S|M\rangle$ be a Coxeter group. Suppose ψ is a conditionally negative function on W satisfying $\psi(\mathbf{w}) = \psi(w_1)+...+\psi(w_k)$ whenever $\mathbf{w} = w_1...w_k$ is a reduced expression. Then for generators $u, w \in S$ and an element $\mathbf{v} \in W$ we have that $|\gamma_{u,w}^{\psi}(\mathbf{v})| = 2\psi(u)\mathbb{1}(u\mathbf{v} = \mathbf{v}w) = 2\psi(w)\mathbb{1}(u\mathbf{v} = \mathbf{v}w)$.*

Proof. We first note that, since we have $u^2 = w^2 = e$ as they are generators, we have that

$$\gamma_{u,w}^{\psi}(\mathbf{v}) = \gamma_{u,w}^{\psi}(u\mathbf{v}w) = -\gamma_{u,w}^{\psi}(u\mathbf{v}) = -\gamma_{u,w}^{\psi}(\mathbf{v}w).$$

When $\mathbf{v}$ is fixed, we can then let $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$ be such that $|\mathbf{z}| = \min\{|\mathbf{v}|, |\mathbf{v}w|, |\mathbf{v}w|, |u\mathbf{v}w|\}$. Then we have $|\gamma_{u,w}^{\psi}(\mathbf{z})| = |\gamma_{u,w}^{\psi}(\mathbf{v})|$. Furthermore, because $|z|$ is minimal we have $|u\mathbf{z}| = |\mathbf{z}w| = |\mathbf{z}| + 1$. Thus, if $\mathbf{z} = z_1....z_k$ is a reduced expression for $\mathbf{z}$ we have that $uz_1...z_k$ and $z_1....z_kw$ are reduced expressions for $u\mathbf{z}$ respectively $\mathbf{z}w$. Therefore, $\psi(u\mathbf{z}) = \psi(u) + \psi(\mathbf{z})$ and $\psi(\mathbf{z}w) = \psi(\mathbf{z}) + \psi(w)$. Hence

$$\gamma_{u,w}^{\psi}(\mathbf{z}) = \psi(u\mathbf{z}w) + \psi(\mathbf{z}) - \psi(u\mathbf{z}) - \psi(\mathbf{z}w) \tag{109}$$

$$= \psi(u\mathbf{z}w) - \psi(\mathbf{z}) - \psi(u) - \psi(w). \tag{110}$$

Now, since $|u\mathbf{z}| = |\mathbf{z}| + 1$ we either have that $|u\mathbf{z}w| = |\mathbf{z}| + 2$ or $|u\mathbf{z}w| = |\mathbf{z}|$. We shall consider these two separate cases, from which the result will follow.

In the first case we have that $uz_1....z_kw$ is reduced so that $\psi(u\mathbf{z}w) = \psi(u)+\psi(\mathbf{z})+\psi(w)$ and therefore $|\gamma^{\psi}_{u,w}(\mathbf{v})| = |\gamma^{\psi}_{u,w}(\mathbf{z})| = 0$. We note that in this case also $u\mathbf{v} \neq \mathbf{v}w$. Namely, $u\mathbf{v} = \mathbf{v}w$ would imply $u\mathbf{z} = \mathbf{z}w$ and hence $u\mathbf{z}w = \mathbf{z}$, which contradicts that $|u\mathbf{z}w| = |\mathbf{z}| + 2$.

In the second case we have that $uz_1....z_kw$ is not reduced. Therefore, by the exchange condition (see [18, Theorem 3.3.4.]) and the fact that $|u\mathbf{z}w| = |\mathbf{z}| < |\mathbf{z}w|$ we have that $uz_1....z_kw$ is equal to $z_1...z_{i-1}z_{i+1}..z_kw$ for some index $1 \le i \le k$, or that $uz_1....z_kw = z_1....z_k$. Now, if the former, we also have that $u\mathbf{z} = z_1...z_{i-1}z_{i+1}...z_k$ so that $|u\mathbf{z}| < |\mathbf{z}|$ which is a contradiction. In this case we must thus have that $u\mathbf{z}w = \mathbf{z}$ and hence $u\mathbf{z} = \mathbf{z}w$. This then implies that $\psi(u\mathbf{z}w) = \psi(\mathbf{z})$ and $\psi(u) = \psi(u\mathbf{z}) - \psi(\mathbf{z}) = \psi(\mathbf{z}w) - \psi(\mathbf{z}) = \psi(w)$. In this case we thus obtain that

$$\gamma^{\psi}_{u,w}(\mathbf{z}) = \psi(u\mathbf{z}w) - \psi(\mathbf{z}) - \psi(u) - \psi(w) = -2\psi(u) = -2\psi(w) \tag{111}$$

which shows that $|\gamma^{\psi}_{u,w}(\mathbf{v})| = |\gamma^{\psi}_{u,w}(\mathbf{z})| = 2\psi(u) = 2\psi(w)$ in this case.

The result now follows from these cases. Namely, either we have that $|\gamma^{\psi}_{u,w}(\mathbf{v})| = 0$ and that $\mathbf{v}$ does not satisfy $u\mathbf{v} = \mathbf{v}w$, or we have that $|\gamma^{\psi}_{u,w}(\mathbf{v})| = 2\psi(u) = 2\psi(w)$ and that $\mathbf{v}$ does satisfy $u\mathbf{v} = \mathbf{v}w$. This thus shows us that $|\gamma^{\psi}_{u,w}(\mathbf{v})| = 2\psi(u)\mathbb{1}(u\mathbf{v} = \mathbf{v}w) = 2\psi(w)\mathbb{1}(u\mathbf{v} = \mathbf{v}w)$.

□

We note that for the word length ψ_S we have $\psi_S(s) > 0$ for all generators $s \in S$. Now by lemma 6.4, in order to see when $\gamma^{\psi_S}_{u,w}$ is finite-rank, we have to know what kind of words $\mathbf{v} \in W$ have the property that $u\mathbf{v} = \mathbf{v}w$. For this we introduce some notation.

For distinct $i, j \in \{1, ..., |S|\}$ we will, whenever the label $m_{i,j}$ is finite, denote $k_{i,j} = \lfloor \frac{m_{i,j}}{2} \rfloor \ge 1$. Now if $m_{i,j}$ is even, then $m_{i,j} = 2k_{i,j}$ and we set $\mathbf{r}_{i,j} = s_i(s_js_i)^{k_{i,j}-1}$. If $m_{i,j}$ is odd, then $m_{i,j} = 2k_{i,j} + 1$ and we set $\mathbf{r}_{i,j} = (s_is_j)^{k_{i,j}}$. Furthermore we set

$$a_{i,j} = s_i \qquad b_{i,j} = \begin{cases} s_i & m_{i,j} \text{ even} \\ s_j & m_{i,j} \text{ odd} \end{cases} \qquad c_{i,j} = s_j \qquad d_{i,j} = \begin{cases} s_j & m_{i,j} \text{ even} \\ s_i & m_{i,j} \text{ odd} \end{cases}. \tag{112}$$

Then $a_{i,j}$ and $b_{i,j}$ are respectively the first and last letter of the word $\mathbf{r}_{i,j}$. Furthermore when $m_{i,j}$ is even we have $c_{i,j}\mathbf{r}_{i,j} = s_js_i(s_js_i)^{k_{i,j}-1} = (s_js_i)^{k_{i,j}} = (s_is_j)^{k_{i,j}} = \mathbf{r}_{i,j}s_j = \mathbf{r}_{i,j}d_{i,j}$ and when $m_{i,j}$ is odd we have $c_{i,j}\mathbf{r}_{i,j} = s_j(s_is_j)^{k_{i,j}} = s_i(s_js_i)^{k_{i,j}} = \mathbf{r}_{i,j}s_i = \mathbf{r}_{i,j}d_{i,j}$. Thus in either case $c_{i,j}\mathbf{r}_{i,j} = \mathbf{r}_{i,j}d_{i,j}$.

We will now for generators $u, w \in S$ show for what kind of words $\mathbf{v} \in W$ with $|\mathbf{v}| \le |u\mathbf{v}|, |\mathbf{v}w|$ we have that $u\mathbf{v} = \mathbf{v}w$. In proposition 6.6 we then give a precise description of the support of $\gamma^{\psi_S}_{u,w}$.

Lemma 6.5. *For generators $u, w \in S$ and a word $\mathbf{v} \in W$ with $|\mathbf{v}| \le |u\mathbf{v}|, |\mathbf{v}w|$ we have $u\mathbf{v} = \mathbf{v}w$ if and only if $\mathbf{v}$ can be written in the reduced form $\mathbf{v} = \mathbf{r}_{i_1,j_1}.....\mathbf{r}_{i_k,i_k}$ so that $u = c_{i_1,j_1}$ and $w = d_{i_k,j_k}$ and so that for $l = 1, ..., k-1$ we have that $c_{i_{l+1},j_{l+1}} = d_{i_l,j_l}$ and $a_{i_{l+1},j_{l+1}} \notin \{s_{i_l}, s_{j_l}\}$ and $b_{i_l,j_l} \notin \{s_{i_{l+1}}, s_{j_{l+1}}\}$.*

Proof. First, suppose that $\mathbf{v}$ can be written in the given form $\mathbf{v} = \mathbf{r}_{i_1,j_1}.....\mathbf{r}_{i_k,i_k}$ with the given conditions on c_{i_l,j_l} and d_{i_l,j_l}. Then since we have $c_{i_l,j_l}\mathbf{r}_{i_l,j_l} = \mathbf{r}_{i_l,j_l}d_{i_l,j_l} = \mathbf{r}_{i_l,j_l}c_{i_{l+1},j_{l+1}}$ for $l = 1, ..., k-1$, and since $u = c_{i_1,j_1}$ and $w = d_{i_k,j_k}$ we have $u\mathbf{v} = \mathbf{v}w$, which shows the 'if' direction.

We now prove the opposite direction. First note that the statement holds for $\mathbf{v} = e$ as this can be written as the empty word. We now prove by induction to n that for $\mathbf{v} \in W$ with $|\mathbf{v}| \ge 1$ and $|\mathbf{v}| \le n$ and $|\mathbf{v}| \le |u\mathbf{v}|, |\mathbf{v}w|$ and $u\mathbf{v} = \mathbf{v}w$ for some $u, w \in S$, we can write $\mathbf{v}$ in the given form. Note first that the statement holds for $n = 0$, since then no such $\mathbf{v} \in W$ exists. Thus, assume that the statement holds for $n-1$, we prove the statement for n. Let $u, w \in S$ and $\mathbf{v} \in W$ be with $|\mathbf{v}| = n$ and $|u\mathbf{v}| = |\mathbf{v}w| = |\mathbf{v}| + 1$ and $u\mathbf{v} = \mathbf{v}w$. Let $(v_1, ..., v_n)$ be a reduced expression for $\mathbf{v}$. Then the expression $(u, v_1, ..., v_n)$ and $(v_1, ..., v_n, w)$ are reduced expressions for $u\mathbf{v} = \mathbf{v}w$. In particular we have $u \neq v_1$. Set $m := m_{u,v_1}$. Now, since $u\mathbf{v}$ and $\mathbf{v}w$ are equal and $u \neq v_1$, we can as in the proof of [18, theorem 3.4.2(ii)] find a reduced expression $(y_1, ..., y_{n+1})$ for $u\mathbf{v}$ with $n \ge m-1$ so that $(y_1, ..., y_m) = (u, v_1, u, v_1, ..., u)$ whenever m is odd, and $(y_1, ..., y_m) = (u, v_1, ..., u, v_1)$ whenever m is even. This is to say that if we let $i_0, j_0 \in \{1, ..., |S|\}$ be such that $v_1 = s_{i_0}$ and $u = s_{j_0}$, then we have that $\mathbf{r}_{i_0,j_0} = y_2...y_m$ and $c_{i_0,j_0} = s_{j_0} = u$. Note that by the proof of [18, theorem 3.4.2(ii)] we have in particular that $m < \infty$. Now moreover, since $y_1 = u$ we have that $(y_2, ..., y_{n+1}, w)$ is a expression for $\mathbf{v}w$, and this expression is reduced since $|\mathbf{v}w| = n + 1$.

Now suppose that $m = n + 1$, then $\mathbf{v} = \mathbf{r}_{i_0,j_0}$ and $i_0 \neq j_0$ since $u \neq v_1$. Now, we have $u = s_{j_0} = c_{i_0,j_0}$ and furthermore, since $\mathbf{r}_{i_0,j_0} d_{i_0,j_0} = c_{i_0,j_0}\mathbf{r}_{i_0,i_0} = u\mathbf{v} = \mathbf{v}w = \mathbf{r}_{i_0,j_0}w$, also $w = d_{i_0,j_0}$. Thus in this case we can write $\mathbf{v}$ in the given form.

Now suppose $m < n + 1$ and define $\mathbf{v}' = y_{m+1}...y_{n+1}$ and $u' = d_{i_0,j_0}$ and $w' = w$. Note that since $u = s_{j_0} = c_{i_0,j_0}$ and $u' = d_{i_0,j_0}$ we have

$$\mathbf{r}_{i_0,j_0}u'\mathbf{v}' = u\mathbf{r}_{i_0,j_0}\mathbf{v}' = u\mathbf{v} = \mathbf{v}w = \mathbf{r}_{i_0,j_0}\mathbf{v}'w'.$$

Therefore $u'\mathbf{v}' = \mathbf{v}'w'$. Moreover $|u'\mathbf{v}'| = |\mathbf{v}'w'| = |\mathbf{v}'|+1$ since $(y_{m+1}, ..., y_{n+1}, w)$ is a reduced expression for $\mathbf{v}'w$. Now, since also $|\mathbf{v}'| \geq 1$ and $|\mathbf{v}'| \leq n-1$ we have by the induction hypothesis that there is a reduced expression $\mathbf{v}' = \mathbf{r}_{i_1,j_1}....\mathbf{r}_{i_k,j_k}$ for some indices $i_l, j_l \in \{1, ..., |S|\}$ with $i_l \neq j_l$ so that $u' = c_{i_1,j_1}$ and $w' = d_{i_k,j_k}$ and so that for $l = 1, ..., k-1$ we have that $c_{i_{l+1},j_{l+1}} = d_{i_l,j_l}$ and $a_{i_{l+1},j_{l+1}} \notin \{s_{i_l}, s_{j_l}\}$ and $b_{i_l,j_l} \notin \{s_{i_{l+1}}, s_{j_{l+1}}\}$. Hence we can write $\mathbf{v} = \mathbf{r}_{i_0,j_0}\mathbf{v}' = \mathbf{r}_{i_0,j_0}.....\mathbf{r}_{i_k,j_k}$. We also have $u = s_{j_0} = c_{i_0,j_0}$ and $w = w' = d_{i_k,j_k}$ and $d_{i_0,j_0} = u' = c_{i_1,j_1}$. Furthermore, since $|\mathbf{v}| = n = (m-1) + (n-m+1) = |\mathbf{r}_{i_0,j_0}| + |\mathbf{v}'|$, and since the expression for $\mathbf{v}'$ is reduced we thus have that the expression for $\mathbf{v}$ is also reduced. Now suppose that $b_{i_0,j_0} \in \{s_{i_1}, s_{j_1}\}$. We note that $b_{i_0,j_0} \neq d_{i_0,j_0} = c_{i_1,j_1} \neq a_{i_1,j_1}$. Now as also $c_{i_1,j_1}, a_{i_1,j_1} \in \{s_{i_1}, s_{j_1}\}$ we obtain that $a_{i_1,j_1} = b_{i_0,j_0}$. However as $\mathbf{r}_{i_0,j_0}$ ends with b_{i_0,j_0} and as $\mathbf{r}_{i_1,j_1}$ starts with a_{i_0,j_0} we then obtain that $\mathbf{r}_{i_0,j_0}\mathbf{r}_{i_1,j_1}$ is not a reduced expression. This contradicts the fact that the expression for $\mathbf{v}$ is reduced. Likewise, if $a_{i_1,j_1} \in \{s_{i_0}, s_{j_0}\}$ we have because of the fact that $a_{i_1,j_1} \neq c_{i_1,j_1} = d_{i_0,j_0} \neq b_{i_0,j_0}$ and $d_{i_0,j_0}, b_{i_0,j_0} \in \{s_{i_0}, s_{j_0}\}$ that $a_{i_1,j_1} = b_{i_0,j_0}$. This then shows that $\mathbf{r}_{i_0,j_0}\mathbf{r}_{i_1,j_1}$ is not a reduced expression, which contradicts the fact that the expression for $\mathbf{v}$ is reduced. This proves the lemma. □

Proposition 6.6. *Let $u, w \in S$. Then we have $\mathbf{z} \in \mathrm{supp}(\gamma^{\psi_S}_{u,w})$ if and only if $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$, where $\mathbf{v}$ is a word as in lemma 6.5.*

Proof. It is clear that if $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$ where $\mathbf{v}$ is of the form of lemma 6.5, that we then have that $u\mathbf{z} = \mathbf{z}w$, and hence by lemma 6.4 that $\psi^{\psi_S}_{u,w}(\mathbf{z}) \neq 0$. For the other direction we suppose that $\mathbf{z} \in \mathrm{supp}(\gamma^{\psi_S}_{u,w})$. Then we have that $u\mathbf{z} = \mathbf{z}w$ holds by lemma 6.4. Now there is a $\mathbf{v} \in \{\mathbf{z}, u\mathbf{z}, \mathbf{z}w, u\mathbf{z}w\}$ such that $|\mathbf{v}| \leq |u\mathbf{v}|, |\mathbf{v}w|$. This word $\mathbf{v}$ moreover satisfies $u\mathbf{v} = \mathbf{v}w$ as we had $u\mathbf{z} = \mathbf{z}w$. Now, this means that $\mathbf{v}$ can be written in an expression as in lemma 6.5. Last, we note that $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$, which finishes the proof. □

6.2. **Parity paths in Coxeter diagram.** In proposition 6.6 we showed precisely for what kind of words $\mathbf{v} \in W$ we have $\mathbf{v} \in \mathrm{supp}(\gamma^{\psi_S}_{u,w})$. The question is now whether this support is finite for infinite. It follows from the proposition that the support is finite if and only if there exist only finitely many words $\mathbf{v} \in W$ that can be written in the form $\mathbf{v} = \mathbf{r}_{i_1,j_1}....\mathbf{r}_{i_k,j_k}$ with the condition from lemma 6.5. To answer the question on whether this is the case, we shall identify these expressions with certain walks in a graph.

We will let $\mathsf{Graph}_S(W) = (V, E)$ be the complete simple graph with vertex set $V = S$ and labels $m_{i,j}$ on the edges $\{s_i, s_j\} \in E$. We let $k \geq 1$ and $i_l, j_l \in \{1, .., |S|\}$ for $l = 1, ..., k$ and we let $P = (s_{j_1}, s_{i_1}, s_{j_2}, ..., s_{j_k}, s_{i_k})$ be a walk in $\mathsf{Graph}_S(W)$, which has even length. We will say that P is a *parity path* if the edges of P have finite labels, and if $i_l \neq j_l$ for all l and if for $l = 1, .., k-1$ we have $s_{j_{l+1}} = d_{i_l,j_l}$ and $i_{l+1} \notin \{i_l, j_l\}$. We will moreover call the parity path P, a *cyclic parity path* if the path $\overline{P} := (s_{j_1}, s_{i_1}, ..., s_{j_k}, s_{i_k}, s_{j_1}, s_{i_1})$ is a parity path.

The intuition for a parity path is that if you walk an edge with odd label, you have to stay there for one turn and then continue your walk over a different edge than you came from. Furthermore, when you walk an edge with an even label you have to return directly over the same edge, and then continue your walk using another edge. Note that in both cases you may still use same edges as before at a later point in your walk. A parity path is defined such that walking the path any number of times in a row, gives you a parity path.

We shall now show in the following two lemmas that the gradient-$\mathcal{S}_p$ property of the semi-group $(\Phi_t)_{t\geq 0}$ on $\mathcal{L}(W)$ associated to the word length ψ_S, is almost equivalent with the non-existence of parity paths in $\mathsf{Graph}_S(W)$.

Theorem 6.7. *Let $W = \langle S|M\rangle$ be a Coxeter system. Suppose there is a cyclic parity path*

$$P = (s_{j_1}, s_{i_1}, s_{j_2}, ..., s_{j_k}, s_{i_k})$$

in $\mathsf{Graph}_S(W)$ *in which the labels* $m_{i_l,j_l}, m_{i_l,i_{l+1}},\ m_{j_l,i_{l+1}}$ *are all unequal to* 2. *Then the semi-group* $(\Phi_t)_{t\geq 0}$ *associated to the word length* ψ_S *is not gradient-*$\mathcal{S}_p$ *for any* $p \in [1,\infty]$.

Proof. Suppose the assumptions hold. Then we have that there exists a parity path of the form $\overline{P} = (s_{j_1}, s_{i_1}, s_{j_2}, ..., s_{j_k}, s_{i_k}, s_{j_{k+1}}, s_{i_{k+1}})$ where $s_{i_1} = s_{i_{k+1}}$ and $s_{j_1} = s_{j_{k+1}}$. We will denote $\mathbf{v}_1 = \mathbf{r}_{i_1,j_1}...\mathbf{r}_{i_k,j_k}$. We note that by the definition of a parity path we have $d_{i_l,j_l} = s_{j_{l+1}} = c_{i_{l+1},j_{l+1}}$ for $l = 1,..,k-1$ and $d_{i_k,j_k} = s_{j_{k+1}} = s_{j_1} = c_{i_1,j_1}$. We now define $u = c_{i_1,j_1} = d_{i_k,j_k}$. Now we thus have $u\mathbf{v}_1 = \mathbf{v}_1 u$. This means by lemma 6.4 that $\gamma^{\psi_S}_{u,u}(\mathbf{v}_1) \neq 0$. We show that $\psi_S(\mathbf{v}_1) \geq k$. To see this, note that $a_{i_{l+1},j_{l+1}} = s_{i_{l+1}} \notin \{s_{i_l}, s_{j_l}\}$ by the definition of the parity path. Furthermore, since $b_{i_l,j_l} \neq d_{i_l,j_l} = c_{i_{l+1},j_{l+1}}$ and $b_{i_l,j_l} \neq a_{i_{l+1},j_{l+1}}$ (as $a_{i_{l+1},j_{l+1}} \notin \{s_{i_l}, s_{j_l}\} \ni b_{i_l,j_l}$) and $a_{i_{l+1},j_{l+1}} = s_{i_{l+1}} \neq s_{j_{l+1}} = c_{i_{l+1},j_{l+1}}$ we have that $b_{i_l,j_l} \notin \{a_{i_{l+1},j_{l+1}}, c_{i_{l+1},j_{l+1}}\} = \{s_{i_{l+1}}, s_{j_{l+1}}\}$. Now, since there are no labels m_{i_l,j_l} equal to 2 we have that the sub-words $\mathbf{r}_{i_l,j_l}$ contain both elements s_{i_l} and s_{j_l}. This means, since $a_{i_{l+1},j_{l+1}} \notin \{s_{i_l}, s_{j_l}\}$ and $b_{i_l,j_l} \notin \{s_{i_{l+1}}, s_{j_{l+1}}\}$, that the only sub-words of $\mathbf{v}_1$ of the form $(s_i, s_j, s_i, ..., s_i, s_j)$ or $(s_i, s_j, s_i, ..., s_j, s_i)$ are the sub-words of $\mathbf{r}_{i_l,j_l}$ for some $l = 1, ..., k$, and the words $(b_{i_l,j_l}, a_{i_{l+1},j_{l+1}})$ for $l = 1,..,k-1$. Now we have that $|\mathbf{r}_{i_l,j_l}| = m_{i_l,j_l} - 1$ and for $s_i = b_{i_l,j_l}$ and $s_j = a_{i_{l+1},j_{l+1}}$ we have $|s_i s_j| = 2 \leq \min\{m_{i_l,i_{l+1}}, m_{j_l,i_{l+1}}\} - 1 \leq m_{i,j} - 1$. Furthermore, there are no sub-words of $\mathbf{v}_1$ of the form (s_i, s_i). This means that the expression for $\mathbf{v}_1$ is M-reduced, and therefore, by [18, Theorem 3.4.2], that the expression is reduced. This means that $\psi_S(\mathbf{v}_1) \geq k$. Now, since we can create cyclic parity paths P_n by walking over P a n number of times, we can create $\mathbf{v}_n \in W$ with $\psi_S(\mathbf{v}_n) \geq nk$ and $\gamma^{\psi_S}_{u,u}(\mathbf{v}_n) \neq 0$. Therefore $\gamma^{\psi_S}_{u,u}$ is not finite rank, and hence the semi-group $(\Phi_t)_{t\geq 0}$ is not gradient-$\mathcal{S}_p$ for any $p \in [1,\infty]$. □

Theorem 6.8. *Let* $W = \langle S|M\rangle$ *be a Coxeter group. If there does not exist a cyclic parity path in* $\mathsf{Graph}_S(W)$ *then the semi-group* $(\Phi_t)_{t\geq 0}$ *associated to the word length* ψ_S *is gradient-*$\mathcal{S}_p$ *for all* $p \in [1,\infty]$.

Proof. Suppose that $(\Phi_t)_{t\geq 0}$ is not gradient-$\mathcal{S}_p$ for some $p \in [1,\infty]$. We will show that a cyclic parity path exists. Namely, since the semi-group is not gradient-$\mathcal{S}_p$, there exist by remark 6.3 generators $u, w \in S$ for which $\gamma^{\psi_S}_{u,w}$ is not finite rank. Set $m = \max\{m_{i,j} : 1 \leq i, j \leq |S|\} \setminus \{\infty\}$. We can thus let $\mathbf{z} \in \mathrm{supp}(\gamma^{\psi_S}_{u,w})$ be with $\psi_S(\mathbf{z}) > m|S|^2 + 2$. Then by proposition 6.6 there is a $\mathbf{v} \in \{\mathbf{z}, u\mathbf{z}, \mathbf{z}w, u\mathbf{z}w\}$ such that we can write $\mathbf{v}$ in reduced form $\mathbf{v} = \mathbf{r}_{i_1,j_1}....\mathbf{r}_{i_k,j_k}$ with the conditions as in lemma 6.5. Now define the path $P = (s_{j_1}, s_{i_1}, ..., s_{j_k}, s_{i_k})$. We show that this is a parity path. By the properties that we obtained from lemma 6.5, we have that $i_l \neq j_l$ and that $m_{i_l,j_l} < \infty$ for all l. Moreover $s_{j_{l+1}} = c_{i_{l+1},j_{l+1}} = d_{i_l,j_l}$ and $s_{i_l} = a_{i_l,j_l} \notin \{s_{i_{l+1}}, s_{j_{l+1}}\}$. This shows that P is a parity path. Note furthermore that since $\psi_S(\mathbf{v}) \geq \psi_S(\mathbf{z}) - 2 > m|S|^2$, we have that P has length $|P| = 2k \geq 2\frac{\psi_S(\mathbf{v})}{m} > 2|S|^2$. Therefore, there must exist indices $l < l'$ such that $(s_{j_l}, s_{i_l}) = (s_{j_{l'}}, s_{i_{l'}})$. The sub-path $(s_{j_l, s_{i_l}}, ..., s_{j_{l'-1}, j_{l'-1}})$ then is a cyclic parity path.

□

6.3. **Characterization of graphs that contain cyclic parity paths.** In the previous section, in theorem 6.7 and theorem 6.8 we have showed that the gradient-$\mathcal{S}_p$ property is almost equivalent to the non-existence of a cyclic parity path. We shall now characterize in proposition 6.9 precisely when a graph possesses a cyclic parity path. The content of this proposition is moreover visualized in fig. 2. Thereafter we state two corollaries that follow from this proposition and from theorem 6.7 and theorem 6.8. These corollaries give an 'almost' complete characterization of the types of Coxeter systems for which the semi-group associated to ψ_S is gradient-$\mathcal{S}_p$.

The following lemma show exactly when a cyclic parity path P in the graph $\mathsf{Graph}_S(W)$ exists.

Proposition 6.9. *Let us denote* $V = S$ *and* $E_0 = \{\{i,j\} : m_{i,j} \in 2\mathbb{N}\}$ *and* $E_1 = \{\{i,j\} : m_{i,j} \in 2\mathbb{N}+1\}$. *Then there does not exist a cyclic parity path* P *in* $\mathsf{Graph}_S(W)$ *if and only if* (V, E_1) *is a forest, and for every connected component* C *of* (V, E_1) *there is at most one edge* $\{t, r\} \in E_0$ *with* $t \in C$ *and* $r \notin C$, *and for every connected component* C *of* (V, E_1) *there is no edge* $\{t, t'\} \in E_0$ *with* $t, t' \in C$

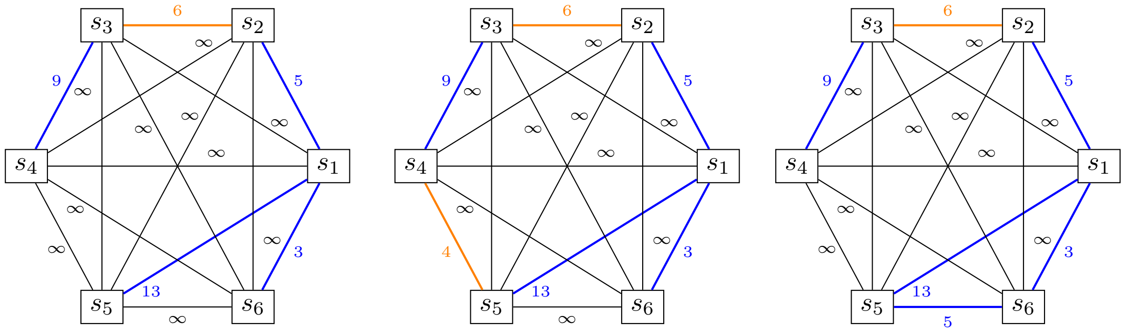
Proof. First suppose that (V, E_1) is not a forest. Then we can find a cycle $Q = (s_{j_1}, s_{j_2}, ..., s_{j_k}, s_{j_1})$ in (V, E_1). Now, since all edges are odd, this means that

$$P = (s_{j_1}, s_{j_2}, s_{j_2}, s_{j_3}, s_{j_3}, ..., s_{j_{k-1}}, s_{j_k})$$

is a cyclic parity path. Indeed, if we denote $j_{k+1} := j_1$ and $j_{k+2} := j_2$, then $j_l \neq j_{l+1}$ for $l = 1,..,k$ and we have $s_{j_{l+1}} = d_{j_{l+1},j_l}$ and $j_{l+2} \notin \{j_{l+1}, j_l\}$, which shows all conditions hold.

Now suppose that there is a connected component C of (V, E_1) for which there are two distinct edges $\{t_1, r_1\}, \{t_2, r_2\} \in E_0$ with $t_1, t_2 \in C$ and $r_1, r_2 \notin C$. If $t_1 = t_2$ then $r_1 \neq r_2$ and a cyclic parity

Graphs with and without a cyclic parity path



(A) Graph with no cyclic parity path

(B) Graph with a cyclic parity path

(C) Graph with a cyclic parity path

FIGURE 2. The graph $\mathsf{Graph}_S(W)$ is denoted for three different Coxeter systems $W = \langle S|M\rangle$ with $|S| = 6$. In each of the graphs the label $M(s_i, s_j)$ is shown on the edge $\{s_i, s_j\}$. We colored the edge orange when the label is even, we colored it blue when the label is odd, and we colored it black when the label is infinity. The relations we imposed on the generators are almost the same in the three cases. They only differ on the edges $\{s_4, s_5\}$ and $\{s_5, s_6\}$. The graph in (A) satisfies the assumptions of proposition 6.9 and hence does not contain a cyclic parity path. The graph in (B) does not satisfy the assumptions of the proposition as for the connected component $C = \{s_3, s_4\}$ of (V, E_1) there are two distinct edges $\{s_2, s_3\}$ and $\{s_4, s_5\}$ with even label and with (at least) one endpoint in C. Therefore the graph contains a cyclic parity path. One is given by $P = (s_3, s_2, s_3, s_4, s_4, s_5, s_4, s_3)$ (another cyclic parity path uses the node s_1) The graph in (C) does also not satisfy the assumptions of the proposition as it contains a cycle with odd labels. Here a cyclic parity path is given by $P = (s_1, s_5, s_5, s_6, s_6, s_1)$ (another cyclic parity path is obtained by walking in reverse order).

path is given by $P = (t_1, r_1, t_1, r_2)$. In the case that t_1 and t_2 are distinct there is a simple path $Q = (t_1, s_{i_1}, ..., s_{i_k}, t_2)$ in (V, E_1) from t_1 to t_2. The path

$$P = (t_1, s_{j_1}, s_{j_1}, s_{j_2}, s_{j_2}, ..., s_{j_k}, s_{j_k}, t_2, t_2, r_2, t_2, s_{j_k}, s_{j_k}, s_{j_{k-1}}, s_{j_{k-1}}, ..., s_{j_1}, s_{j_1}, t_1, t_1, r_1)$$

then is a cyclic parity path. Indeed, just as the previous case we have that the paths

$$P_1 := (t_1, s_{j_1}, s_{j_1}, s_{j_2}, s_{j_2}, ..., s_{j_k}, s_{j_k}, t_2)$$

and

$$P_2 := (t_2, s_{j_k}, s_{j_k}, s_{j_{k-1}}, s_{j_{k-1}}, ...s_{j_1}, s_{j_1}, t_1)$$

are parity paths, since they are obtained from a simple path in (V, E_1). We then only have to check that in the middle and at the start/end of the path P the conditions are satisfied. For the middle, we see that indeed $r_2 \notin \{s_{j_k}, t_2\}$ as the label of the edge between t_2 and r_2 is even. Furthermore, since P_1 is a parity path we have that $s_{j_k} \neq t_2$. Thus also $s_{j_k} \notin \{t_2, r_2\}$. Furthermore, if we let i, j be such that $t_2 = s_j$, $r_2 = s_i$, then since $m_{j_k,j}$ is odd, we have that $t_2 = d_{j,j_k}$ and since $m_{i,j}$ is even we have $t_2 = d_{i,j}$. This shows all conditions in the middle. The conditions at the start/end hold by symmetry. Thus P is indeed a cyclic parity path.

Now, suppose that there is a connected component C of (V, E_1) for which there exists an edge $\{t, t'\} \in E_0$ with $t, t' \in C$. Then we can, similar to what we just did, obtain a cyclic parity path by taking $t_1 = t$ and $t_2 = t'$ and $r_1 = t'$ and $r_2 = t$.

We now prove the other direction. Thus, suppose that (V, E_1) is a forest and that for every connected component C there is at most edge $\{t, r\} \in E_0$ with $t \in C$ and $r \in V$, and that for every connected component there is no edge $\{t, t'\} \in E_0$ with $t, t' \in C$. Suppose there exists a cyclic parity path $P = (s_{j_1}, s_{i_1}, ..., s_{j_k}, s_{i_k})$ in $(V, E_0 \cup E_1)$, we show that this gives a contradiction. Namely, first suppose that P only has odd edges. Then we have $s_{j_{l+1}} = d_{i_l, j_l} = s_{i_l}$ for $l = 1, .., k-1$ and $s_{j_1} = d_{i_k, j_k} = s_{i_k}$, and thus $P = (s_{i_k}, s_{i_1}, s_{i_1}, s_{i_2}, s_{i_2}, ..., s_{i_{k-1}}, s_{i_k})$. However, since also $i_{l+1} \notin \{i_l, j_l\} = \{i_l, i_{l-1}\}$, this means that $Q = (s_{i_1}, s_{i_2}, ..., s_{i_k}, s_{i_1})$ is a cycle in (V, E_1). But this is not possible since (V, E_1) is a forest, which gives the contradiction. We thus assume that there is an index l such that the label m_{i_l,j_l} is even. By choosing the starting point of P as j_l instead of j_1, we can assume that m_{i_1,j_1} is even. Now in that case we have $s_{j_2} = d_{i_1,j_1} = s_{j_1}$. We must moreover have $i_2 \notin \{i_1, j_1\}$ as P is a parity path. Now as the edges $\{i_1, j_1\}$ and $\{i_2, j_2\}$ are thus distinct, and share an endpoint, we obtain that m_{i_2,j_2} is odd. This means that $j_3 = d_{i_2,j_2} = i_2 \neq j_2$. Now the sub-path $(s_{j_2}, s_{i_2}, ..., s_{j_k}, s_{i_k}, s_{j_1}, s_{i_1})$ is also a parity path. Denote $j_{k+1} = j_1$ and $i_{k+1} = i_1$ and let $3 < k' \leq k+1$ be the smallest index such that $s_{j_{k'}} = s_{j_2}$. Note that such k' exists since $s_{j_{k+1}} = s_{j_1} = s_{j_2}$. Then the sub-path $P' := (s_{j_2}, s_{i_2}, ..., s_{j_{k'}}, s_{i_{k'}})$ is a parity path, and the labels m_{i_l,j_l} for $l = 2, ..., k'-1$ are odd since s_{j_2} is the only vertex in its connected component in (V, E_1) that is connected by an edge in E_0. Thus, just like previous case we have that $P' := (s_{i_{k'}}, s_{i_2}, s_{i_2}, s_{i_3}, ..., s_{i_{k'-1}}, s_{i_{k'}})$. Now this means that the path $Q = (s_{i_{k'}}, s_{i_2}, s_{i_3}, ..., s_{i_{k'}})$ contains a cycle, which is a contradiction with the fact that (V, E_1) is a forest. This proves the lemma. □

We now state two corollaries that directly follow from theorem 6.7, theorem 6.8 and proposition 6.9.

Corollary 6.10. *Let $W = \langle S|M \rangle$ be a Coxeter system and fix $p \in [1, \infty]$. Let us denote $E_0 = \{(i, j) : m_{i,j} \in 2\mathbb{N}\}$ and $E_1 = \{(i, j) : m_{i,j} \in 2\mathbb{N} + 1\}$. Then the semi-group $(\Phi_t)_{t\geq 0}$ on $\mathcal{L}(W)$ associated to the word length ψ_S is gradient-$\mathcal{S}_p$ if (S, E_1) is a forest, and if for every connected component C of (S, E_1) there is at most one edge $\{t, r\} \in E_0$ with $t \in C$ and $r \notin C$ and no edge $\{t, t'\} \in E_0$ with $t, t' \in C$.*

Corollary 6.11. *Let $W = \langle S|M \rangle$ be a Coxeter system satisfying $M(s_i, s_j) \neq 2$ for all $s_i, s_j \in S$. Fix $p \in [1, \infty]$. Let us denote $E_0 = \{(i, j) : M(s_i, s_j) \in 2\mathbb{N}\}$ and $E_1 = \{(i, j) : M(s_i, s_j) \in 2\mathbb{N} + 1\}$. Then the semi-group $(\Phi_t)_{t\geq 0}$ on $\mathcal{L}(W)$ associated to the word length ψ_S is gradient-$\mathcal{S}_p$ if and only if (S, E_1) is a forest, and for every connected component C of (S, E_1) there is at most one edge $\{t, r\} \in E_0$ with $t \in C$ and $r \notin C$ and no edge $\{t, t'\} \in E_0$ with $t, t' \in C$.*

In the cases that we have obtained the gradient-$\mathcal{S}_p$ property, we get by theorem 6.2 that $\mathcal{L}(W)$ has the (AO^+) property, and is strongly solid. We remark however the following result from [6, Example 5.1]

Proposition 6.12. *Let $W_i = \langle S_i|M_i \rangle$ be Coxeter systems for $i = 1, 2$ such that $\mathsf{Graph}_{S_i}(W)$ has no edges of even label, and such that the edges of odd label form a tree. Then if $\mathsf{Graph}_{S_1}(W_2)$ has the same set of labels as $\mathsf{Graph}_{S_2}(W_2)$ (counting multiplicities), then the Coxeter groups are equal, that is $W_1 = W_2$.*

Hence, it turns out that that the Coxeter groups are in some cases actually equal. In such case we have obtained the gradient-$\mathcal{S}_p$ property for multiple quantum Markov semi-groups.

7. Gradient-$\mathcal{S}_p$ semi-groups associated to weighted word lengths on Coxeter groups

In this section we will consider proper, conditionally negative definite functions on Coxeter groups that are different from the standard word length. We can then consider the quantum Markov semi-groups associated to these other functions, and study the gradient-$\mathcal{S}_p$ property of these semi-groups. We show that these other semi-groups may have the gradient-$\mathcal{S}_p$ properties in cases where the semi-group associated to the word length ψ_S fails to be gradient-$\mathcal{S}_p$. For $p \in [1,\infty]$ this gives us new examples of Coxeter groups W for which there exist a gradient-$\mathcal{S}_p$ quantum Markov semi-group on $\mathcal{L}(W)$. This is the main aim of this section. The structure of this section is as follows. We first prove, in section 7.1, for certain functions ψ that they are conditionally negative. Thereafter, we shall use these functions in section 7.2 to prove for certain Coxeter groups W that we can construct a gradient-$\mathcal{S}_p$ quantum Markov semi-group on $\mathcal{L}(W)$.

7.1. **Certain weighted word lengths define proper, conditionally negative functions.** For a Coxeter group W there are, besides the standard word length w.t.r. some generators, also other kinds of conditionally negative definite functions (see [5] for more on this). We shall show that for certain non-negative weights $\mathbf{x} = (x_1, ...x_{|S|})$ we can construct a conditionally negative functions $\psi_{\mathbf{x}}$ on W as the word length with respect to the weights $\mathbf{x}$ on the generators. Thereafter, we examine when these functions are moreover proper.

7.1.1. *Weighted word lengths that are conditionally negative.* We will denote $\widetilde{\mathsf{Graph}}_S(W)$ for the subgraph of $\mathsf{Graph}_S(W)$ that has vertex set S and edge set $E = \{(s_i, s_j) : 3 \leq M(s_i, s_j) < \infty\}$. We will furthermore denote $\mathcal{C}_i$ for the connected component in $\widetilde{\mathsf{Graph}}_S(W)$ that contains s_i. We have the following lemma.

Lemma 7.1. *Let $W = \langle S|M\rangle$ be a Coxeter group. Then if $\mathbf{x} \in [0,\infty)^{|S|}$ is such that $\mathbf{x}_i = \mathbf{x}_j$ whenever $\mathcal{C}_i = \mathcal{C}_j$, then the function $\psi_{\mathbf{x}} : W \to [0,\infty)$ given for a word $\mathbf{w} = w_1....w_k$ in reduced expression by $\psi_{\mathbf{x}}(\mathbf{w}) = \sum_{i=1}^{|S|} x_i|\{l : w_l = s_i\}|$ is well-defined, and is conditionally negative.*

Proof. Let $\mathbf{n} = (n_1, ..., n_{|S|}) \in \mathbb{N}^{|S|}$ be such that $\mathbf{n}_i = \mathbf{n}_j$ whenever $\mathcal{C}_i = \mathcal{C}_j$. We denote $S_{\mathbf{n}} = \{s_{i,k} : 1 \leq i \leq |S|, 1 \leq k \leq n_i\}$ for the set of letters. We then define $M_{\mathbf{n}} : S_{\mathbf{n}} \to \mathbb{N} \cup \{\infty\}$ as:

$$M_{\mathbf{n}}(s_{i,k}, s_{j,l}) = \begin{cases} M(s_i, s_j) & \mathcal{C}_i = \mathcal{C}_j \text{ and } k = l \\ 2 & \mathcal{C}_i = \mathcal{C}_j \text{ and } k \neq l \\ M(s_i, s_j) & \mathcal{C}_i \neq \mathcal{C}_j \end{cases}.$$

We set $\widetilde{W}_{\mathbf{n}} = \langle S_{\mathbf{n}}|M_{\mathbf{n}}\rangle$. We now define a homomorphism $\varphi_{\mathbf{n}} : W \to \widetilde{W}_{\mathbf{n}}$ given for generators by $\varphi_{\mathbf{n}}(s_i) = s_{i,1}s_{i,2}....s_{i,\mathbf{n}_i}$. We note that $\varphi_{\mathbf{n}}(s_i)^2 = s_{i,1}...s_{i,\mathbf{n}_i}s_{i,1}...s_{i,\mathbf{n}_i} = s_{i,1}^2...s_{i,\mathbf{n}_i}^2 = e$. Furthermore, when $\mathcal{C}_i = \mathcal{C}_j$ we have that $\mathbf{n}_i = \mathbf{n}_j$ and $(\varphi_{\mathbf{n}}(s_i)\varphi_{\mathbf{n}}(s_j))^m = (s_{i,1}....s_{i,\mathbf{n}_i}s_{j,1}....s_{j,\mathbf{n}_j})^m = (s_{i,1}s_{j,1})^m(s_{i,2}, s_{j,2})^m....(s_{i,\mathbf{n}_i}s_{j,\mathbf{n}_j})^m$. This means that in this case $(\varphi_{\mathbf{n}}(s_i)\varphi_{\mathbf{n}}(s_j))^{M(s_i,s_j)} = e$. If $\mathcal{C}_i \neq \mathcal{C}_j$ then either $M(s_i, s_j) = 2$ or $M(s_i, s_j) = \infty$. If $M(s_i, s_j) = 2$ then also $\varphi_{\mathbf{n}}(s_i)\varphi_{\mathbf{n}}(s_j) = s_{i,1}...s_{i,\mathbf{n}_i}s_{j,1}...s_{j,\mathbf{n}_j} = s_{j,1}...s_{j,\mathbf{n}_j}s_{i,1}...s_{i,\mathbf{n}_i} = \varphi_{\mathbf{n}}(s_i)\varphi_{\mathbf{n}}(s_j)$ holds. Therefore, we can extend $\varphi_{\mathbf{n}}$ to words $\mathbf{w} = w_1....w_k \in W$ by defining $\varphi_{\mathbf{n}}(\mathbf{w}) = \varphi_{\mathbf{n}}(w_1)...\varphi_{\mathbf{n}}(w_k)$. By what we just showed, this map is well-defined. Furthermore, from the definition it follows that this map is a homomorphism. Moreover, we note that if $\mathbf{w} = w_1...w_k \in W$ is a reduced expression, then $\varphi_{\mathbf{n}}(\mathbf{w}) = \varphi_{\mathbf{n}}(w_1)...\varphi_{\mathbf{n}}(w_k)$ is also a reduced expression. This means in particular that $\varphi_{\mathbf{n}}$ is injective. Furthermore, if we denote $\widetilde{\psi}_{\mathbf{n}}$ for the word length on $\widetilde{W}_{\mathbf{n}}$, then we have that for a word $\mathbf{w} = w_1....w_k \in W$ written in a reduced expression that

$$\widetilde{\psi}_{\mathbf{n}} \circ \varphi_{\mathbf{n}}(\mathbf{w}) = \sum_{l=1}^{k} \widetilde{\psi}_{\mathbf{n}}(\varphi_{\mathbf{n}}(w_l)) \tag{113}$$

$$= \sum_{i=1}^{|S|} \widetilde{\psi}_{\mathbf{n}}(\varphi_{\mathbf{n}}(s_i))|\{l : w_l = s_i\}| \tag{114}$$

$$= \sum_{i=1}^{|S|} \mathbf{n}_i|\{l : w_l = s_i\}|. \tag{115}$$

Now fix $\mathbf{x} \in [0,\infty)^{|S|}$ with $\mathbf{x}_i = \mathbf{x}_j$ whenever $\mathcal{C}_i = \mathcal{C}_j$. For $m \in \mathbb{N}$ define $\mathbf{n}_m \in \mathbb{N}^{|S|}$ by $(\mathbf{n}_m)_i = \lceil m x_i \rceil + 1 \in \mathbb{N}$. Now, for $\mathbf{w} \in W$ with reduced expression $\mathbf{w} = w_1 ... w_k$ we have

$$\left| \frac{1}{m}\widetilde{\psi}_{\mathbf{n}_m} \circ \varphi_{\mathbf{n}_m}(\mathbf{w}) - \sum_{i=1}^{|S|} x_i |\{l : w_l = s_i\}| \right| \leq \sum_{i=1}^{|S|} |\frac{(\mathbf{n}_m)_i}{m} - x_i| \cdot |\{l : w_l = s_i\}| \tag{116}$$

$$= \sum_{i=1}^{|S|} \frac{|\lceil m x_i \rceil + 1 - m x_i|}{m} |\{l : w_l = s_i\}| \tag{117}$$

$$\leq \sum_{i=1}^{|S|} \frac{2}{m} |\{l : w_l = s_i\}| \tag{118}$$

$$\leq \frac{2|\mathbf{w}|}{m}, \tag{119}$$

and hence $\frac{1}{m}\widetilde{\psi}_{\mathbf{n}_m} \circ \varphi_{\mathbf{n}_m}(\mathbf{w}) \to \sum_{i=1}^{|S|} x_i |\{l : w_l = s_i\}|$ as $m \to \infty$. This shows in particular that $\psi_{\mathbf{x}}$ is well defined. Now, since $\frac{1}{m}\widetilde{\psi}_{\mathbf{n}_m} \circ \varphi_{\mathbf{n}_m} \to \psi_{\mathbf{x}}$ point-wise, and since $\frac{1}{m}\widetilde{\psi}_{\mathbf{n}_m} \circ \varphi_{\mathbf{n}_m}$ is conditionally negative, we have by [2, Proposition C.2.4(ii)] that $\psi_{\mathbf{x}}$ is conditionally negative. □

We state two remarks to lemma 7.1.

Remark 7.2. *By lemma 7.1, in the case of a right-angled Coxeter group $W = \langle S|M \rangle$ we have that every weight $\mathbf{x} \in [0,\infty)^{|S|}$ defines a conditionally negative function.*

Remark 7.3. *For a general Coxeter group $W = \langle S|M \rangle$ and arbitrary non-negative weights $\mathbf{x} \in [0,\infty)^{|S|}$ the weighted word length is not well-defined. Indeed, if $s_i, s_j \in S$ are such that $M(s_i,s_j)$ is odd, then for $k_{i,j} := \lfloor \frac{M(s_i,s_j)}{2} \rfloor$ we have that $(s_i s_j)^{k_{i,j}} s_i$ and $s_j (s_i s_j)^{k_{i,j}}$ are two reduced expressions for the same word, but the values of $|\{l : w_l = s_i\}|$ and $|\{l : w_l = s_j\}|$ depend on the choice of the reduced expressions.*

We shall now turn to examine when a weighted word length is proper. In that case we can study the gradient-$\mathcal{S}_p$ property of the associated semi-group.

7.1.2. *Weighted word lengths that are proper.* Let us fix a Coxeter system $W = \langle S|M \rangle$. Let $\mathcal{I} \subseteq S$ be a subset of the generators such that for $i = 1, .., |S|$ either $\mathcal{C}_i \subseteq \mathcal{I}$ or $\mathcal{C}_i \cap \mathcal{I} = \emptyset$. We can define non-negative weights $\mathbf{x} \in [0,\infty)^{|S|}$ by $\mathbf{x}(i) = \chi_{\mathcal{I}}(i)$. Then by lemma 7.1 we have that $\psi_{\mathbf{x}}$ defines a conditionally negative function on W, and we shall denote this function by $\psi_{\mathcal{I}}$. We give the following characterization on when the function $\psi_{\mathcal{I}}$ is moreover proper.

Proposition 7.4. *The function $\psi_{\mathcal{I}}$ is proper if and only if the elements $S \setminus \mathcal{I}$ generate a finite subgroup.*

Proof. Indeed, if the generated group H is infinite, then $\psi_{\mathcal{I}}$ is not proper as $\psi_{\mathcal{I}}|_H = 0$. On the other hand, if the generated group H contains $N < \infty$ elements, then for a reduced expression $\mathbf{w} = w_1 .... w_k \in W$ we can not have that $w_l, w_{l+1}, ... w_{l+N} \in S \setminus \mathcal{I}$ for some $1 \leq l \leq k - N$ as the expressions $w_l, w_l w_{l+1}, w_l w_{l+1} w_{l+2}, ..$ would all be distinct elements in H. This thus implies that $\psi_{\mathcal{I}}(\mathbf{w}) > \frac{|\mathbf{w}|}{N+1} - 1$ which shows that $\psi_{\mathcal{I}}$ is proper in this case. □

7.2. **Gradient-$\mathcal{S}_p$ property with respect to weighted word length $\psi_{\mathcal{I}}$.** We let $W = \langle S|M \rangle$ be a Coxeter system. When a subset $\mathcal{I} \subseteq S$ is such that $\psi_{\mathcal{I}}$ defines a proper, conditionally negative definite function on W, we can study the gradient-$\mathcal{S}_p$ property of the associated semi-group. We shall only do this for right-angled Coxeter groups. For such group, by remark 7.2, we have that $\psi_{\mathcal{I}}$ defines a conditionally negative function for all $\mathcal{I} \subseteq S$. If the set $\mathcal{I}$ then is moreover such that the elements in $S \setminus \mathcal{I}$ pair-wise commute, then $\psi_{\mathcal{I}}$ is also proper.

Now, when the above property on the set $\mathcal{I}$ is satisfied, we can study the gradient-$\mathcal{S}_p$ property of the associated semi-group. For this we note that the functions $\psi_{\mathbf{x}}$ satisfies $\psi_{\mathbf{x}}(\mathbf{w}) = \psi_{\mathbf{x}}(w_1) + .. + \psi_{\mathbf{x}}(w_k)$ when $\mathbf{w} = w_1 ... w_k$ is a reduced expression. Therefore, by lemma 6.4 we have that $\gamma_{u,w}^{\psi_{\mathbf{x}}}(\mathbf{v}) \neq 0$ for $u, w \in S$ and $\mathbf{v} \in W$ if and only if $u\mathbf{v} = \mathbf{v}w$ and $\psi(u) > 0$.

Theorem 7.5. *Let $W = \langle S|M \rangle$ be a right-angled Coxeter group and let $\mathbf{x} \in [0,\infty)^{|S|}$ and $p \in [1,\infty]$. Furthermore, suppose the function $\psi_{\mathbf{x}}$ is proper. Then, the semi-group $(\Phi_t)_{t \geq 0}$ induced by $\psi_{\mathbf{x}}$ is gradient-$\mathcal{S}_p$ if and only if there do not exist generators $r, s, t \in S$ with $M(r,s) = M(r,t) = 2$ and $M(s,t) = \infty$ and $\psi_{\mathbf{x}}(r) > 0$.*

Proof. Suppose that $(\Phi_t)_{t\geq 0}$ is not gradient-$\mathcal{S}_p$ for some $p \in [1,\infty]$. We will show the generators with the given properties exits. Namely, there are generators u, w for which $\gamma^{\psi_{\mathbf{x}}}_{u,w}$ is not finite rank. We can thus let $\mathbf{v} \in W$ with $|\mathbf{v}| > |S|+1$ be such that $\gamma^{\psi_{\mathbf{x}}}_{u,w}(\mathbf{v}) \neq 0$. Then $u\mathbf{v} = \mathbf{v}w$ and $\psi_{\mathbf{x}}(u), \psi_{\mathbf{x}}(w) > 0$ by lemma 6.4. We note moreover that, by [18, Lemma 3.3.3] we have that $u = w$ because these elements are conjugate, and the Coxeter group is right-angled. We can now let $\mathbf{z} \in \{\mathbf{v}, u\mathbf{v}, \mathbf{v}w, u\mathbf{v}w\}$ be such that $|\mathbf{z}| \leq |u\mathbf{z}|, |\mathbf{z}w|$. Then the equality $u\mathbf{z} = \mathbf{z}w$ also holds. Therefore, we can write $\mathbf{z}$ in reduced form $\mathbf{z} = \mathbf{r}_{i_1,j_1}....\mathbf{r}_{i_k,j_k}$ with the conditions as in lemma 6.5. Now, as $M(s_{i_l}, s_{j_l}) < \infty$ we must have $M(s_{i_l}, s_{j_l}) = 2$ for $l = 1, ..., k$. Hence $\mathbf{z} = s_{i_1}s_{i_2}...s_{i_k}$. Furthermore $s_{j_{l+1}} = s_{j_l}$ for $l = 1, .., k-1$ since $M(s_{i_l}, s_{j_l})$ is even. We define $r = s_{j_1}$. Then $r = c_{i_1,j_1} = u$ so that $\psi_{\mathbf{x}}(r) > 0$. Furthermore, since $k = |\mathbf{z}| \geq |\mathbf{v}| - 1 > |S|$ there exist indices $l < l'$ such that $M(s_{i_l}, s_{i_{l'}}) = \infty$. We then set $s = s_{i_l}$ and $t = s_{i_{l'}}$. Then $M(s, r) = M(s_{i_l}, s_{j_l}) = 2$ and likewise $M(t, r) = 2$. This shows all stated properties hold for r, s, t.

For the other direction, suppose that there exist $r, s, t \in S$ with $M(r, s) = M(r, t) = 2$ and $M(s, t) = \infty$ and $\psi_{\mathbf{x}}(r) > 0$. Define the words $\mathbf{v}_n = (st)^n$. Then we have $|\mathbf{v}_n| = 2n$ and hence $\{\mathbf{v}_n\}_{n\geq 1}$ are all distinct. Moreover, we have $r\mathbf{v}_n = \mathbf{v}_n r$ and $\psi_{\mathbf{x}}(r) > 0$. This means by lemma 6.4 that $\gamma^{\psi_{\mathbf{x}}}_{r,r}(\mathbf{v}_n) = \psi_{\mathbf{x}}(r) > 0$ for $n \geq 1$. Thus the semi-group $(\Phi_t)_{t\geq 0}$ is not gradient-$\mathcal{S}_p$. □

As in section 6 it follows that when the semi-group is gradient-$\mathcal{S}_2$ we obtain that $\mathcal{L}(W)$ has the (AO$^+$) property and is strongly solid. Indeed, it is clear from section 5.3 that when W is infinite the operator $\Delta_{\psi_{\mathbf{x}}}$ is filtered w.r.t $\mathbb{C}[W]$ and has subexponential growth. The result then follows analogue to theorem 6.2. We now state a useful corollary that follows this fact and the previous lemma.

Corollary 7.6. *Let $W = \langle S|M\rangle$ be a right-angled Coxeter group and let $p \in [1,\infty]$. Furthermore set*

$$S_0 = \{r \in S : \exists s, t \in S : M(r, s) = M(r, t) = 2 \text{ and } M(s, t) = \infty\} \tag{120}$$

and $\mathcal{I} = S \setminus S_0$. Then, if the elements in S_0 pairwise commute, we have that the function $\psi_{\mathcal{I}}$ on W induces a gradient-$\mathcal{S}_p$ semi-group $(\Phi_t)_{t\geq 0}$. In particular $\mathcal{L}(W)$ has the AO$^+$ property and is strongly solid.

The set S_0 can also be described as the set of all the generators that are contained in multiple maximal cliques. Here with a *clique* we mean a set of generators that pair-wise commute.

Example of right-angled Coxeter group with gradient-$\mathcal{S}_p$ semi-group

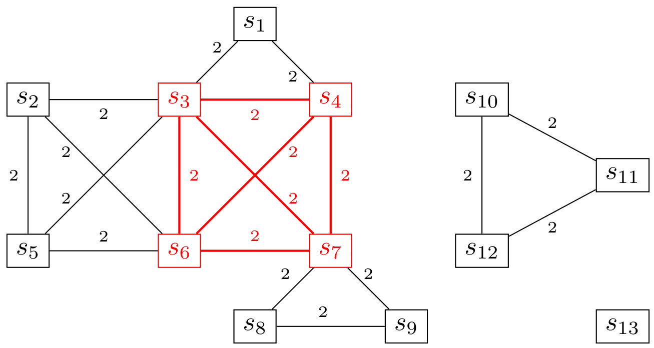


FIGURE 3. In the above, the graph $\mathsf{Graph}_S(W)$ is denoted for a certain right-angled Coxeter group. The edges with label ∞ have been omitted. The set S_0 has been denoted in red. As all elements in this set pairwise commute, we obtain that the function $\psi_{\mathcal{I}}$ with $\mathcal{I} = S \setminus S_0$ induces a gradient-$\mathcal{S}_p$ quantum Markov semi-group for all p.

We note that the (AO$^+$) and strong solidity results from corollary 7.6 were already known, as they follow from [21, Lemma 6.2.8] and from [36, Theorem 1.4]. The techniques we use here are different however.

We give a simple example of a right-angled Coxeter group $W = \langle S|M\rangle$ for which we can find a subset $\mathcal{I} \subseteq S$ so that $\psi_{\mathcal{I}}$ is proper, and so that the associated semi-group is gradient-$\mathcal{S}_p$ for $p \in [1,\infty]$. We furthermore note that in our example, the semi-group associated to the standard word length ψ_S is not gradient-$\mathcal{S}_p$ for any $p \in [1,\infty]$. Another example is shown in fig. 3

Example 7.7. *Let $S = \{s_1, s_2, s_3, s_4\}$ and define M as $M(s_i, s_j) = 1$ whenever $i = j$, as $M(s_i, s_j) = 2$ whenever $|i-j| = 1$, and as $M(s_i, s_j) = \infty$ whenever $|i-j| \geq 2$. We set $W = \langle S|M\rangle$ and denote $\mathcal{I} = \{s_1, s_4\}$. Then since the elements in $S\setminus\mathcal{I} = \{s_2, s_3\}$ commute, they generate a finite subgroup. This* *thus means that $\psi_{\mathcal{I}}$ is proper. Now, since there is only one generator in $\{s_2, s_3, s_4\}$ that commutes with s_1, and also only one generator in $\{s_1, s_2, s_3\}$ that commutes with s_4 we have that the generators r, s, t as in theorem 7.5 do not exist. This means that the semi-group $(\Phi_t)_{t \geq 0}$ associated to $\psi_{\mathcal{I}}$ is gradient-$\mathcal{S}_p$ for all $p \in [1, \infty]$. However, for this Coxeter group, the semi-group associated to the world length ψ_S is not gradient-$\mathcal{S}_p$ for any $p \in [1, \infty]$. Indeed, if we set $r = s_2$, $s = s_1$ and $t = s_3$ then theorem 7.5 shows that this is not the case.*

8. Gradient-$\mathcal{S}_p$ property for semi-groups on Hecke Algebras

Let $W = \langle S|M\rangle$ be a Coxeter system. In this section we will, instead of looking at a semi-group on the group von Neumann algebra $\mathcal{L}(W)$, consider semi-groups on the Hecke algebras $\mathcal{N}_q(W)$. These Hecke algebras are deformations of $\mathcal{L}(W)$, depending on the parameter q. In section 8.1 we shall give a definition of these algebras. In section 8.2 we shall then study the gradient-$\mathcal{S}_p$ property of semi-groups on these algebras. Last, in section 8.3 we shall, under some assumptions, consider semi-groups on certain Hecke algebras and prove the gradient-$\mathcal{S}_2$ property of these semi-groups.

8.1. **Definition of Hecke algebra.** We will give a definition of these Hecke algebras. Let us fix $q = (q_s)_{s\in S}$ with $q_s > 0$ for $s \in S$ and such that $q_s = q_t$ whenever $s, t \in S$ are conjugate in W. In this text we shall call such tuples *Hecke tuples*. Moreover, we will denote $p_s(q) = \frac{q_s-1}{\sqrt{q_s}}$. We can as in [18, Theorem 19.1.1] define for $s \in S$ the operators $T_s^{(q)} : \ell_2(W) \to \ell_2(W)$ given by

$$T_s^{(q)}(\delta_{\mathbf{w}}) = \begin{cases} \delta_{s\mathbf{w}} & |s\mathbf{w}| > |\mathbf{w}| \\ \delta_{s\mathbf{w}} + p_s(q)\delta_{\mathbf{w}} & |s\mathbf{w}| < |\mathbf{w}| \end{cases}.$$

For these operators we have

$$\langle T_s^{(q)}(\delta_{\mathbf{w}}), \delta_{\mathbf{z}}\rangle = \langle \delta_{s\mathbf{w}}, \delta_{\mathbf{z}}\rangle + \langle p_s(q)\delta_{\mathbf{w}}, \delta_{\mathbf{z}}\rangle \mathbb{1}(|s\mathbf{w}| < |\mathbf{w}|) \tag{121}$$

$$= \langle \delta_{\mathbf{w}}, \delta_{s\mathbf{z}}\rangle + \langle \delta_{\mathbf{w}}, p_s(q)\delta_{\mathbf{z}}\rangle \mathbb{1}(|s\mathbf{z}| < |\mathbf{z}|) \tag{122}$$

$$= \langle \delta_{\mathbf{w}}, T_s^{(q)}(\delta_{\mathbf{z}})\rangle \tag{123}$$

that is $(T_s^{(q)})^* = T_s^{(q)}$.

Now, for a word $\mathbf{w} \in W$ with reduced expression $\mathbf{w} = w_1....w_k$ we can set $T_{\mathbf{w}}^{(q)} = T_{w_1}^{(q)}.....T_{w_k}^{(q)}$, which is well-defined by [18, Theorem 19.1.1]. We note that we have $(T_{\mathbf{w}}^{(q)})^* = T_{\mathbf{w}^{-1}}^{(q)}$ and $T_{\mathbf{w}}^{(q)}(\delta_e) = \delta_{\mathbf{w}}$. Furthermore for $s \in S$ and $\mathbf{w} \in W$ they satisfy the equations

$$T_s^{(q)} T_{\mathbf{w}}^{(q)} = T_{s\mathbf{w}}^{(q)} + p_s(q) T_{\mathbf{w}}^{(q)} \mathbb{1}(|s\mathbf{w}| < |\mathbf{w}|) \tag{124}$$

$$T_{\mathbf{w}}^{(q)} T_s^{(q)} = T_{\mathbf{w}s}^{(q)} + p_s(q) T_{\mathbf{w}}^{(q)} \mathbb{1}(|\mathbf{w}s| < |\mathbf{w}|). \tag{125}$$

Note that the first equation holds by the proof of [18, Theorem 19.1.1], and the second equation follows by taking the adjoint on both sides.

We will denote $\mathbb{C}_q[W]$ for the $*$-algebra spanned by the linear basis $\{T_{\mathbf{w}}^{(q)} : \mathbf{w} \in W\}$. We furthermore denote $C_q^*(W) \subseteq B(\ell_2(W))$ for the C^*-algebra obtained by taking the closure of $\mathbb{C}_q[W]$. Finally, we define the Hecke von Neumann-algebra $\mathcal{N}_q(W)$ (or simply $\mathcal{N}_q$) as the strong closure of $C_q^*(W)$. We equip the von Neumann algebra with the faithful finite trace $\tau(x) = \langle x\delta_e, \delta_e\rangle$. We note here that when $q = (q_s)_{s\in S}$ is taken as $q_s = 1$ for $s \in S$, then $(\mathcal{N}_q, \tau)$ is simply the group von Neumann algebra $\mathcal{L}(W)$. The group von Neumann algebra is thus a special case of a Hecke algebra.

When the tuple $q = (q_s)_{s\in S}$ is fixed, we will simply write $T_{\mathbf{w}}$ instead of $T_{\mathbf{w}}^{(q)}$ and p_s instead of $p_s(q)$.

8.2. **Gradient-$\mathcal{S}_p$ property for semi-groups on Hecke-algebras.** Let $W = \langle S|M\rangle$ be a Coxeter system. Fix a Hecke tuple $q = (q_s)_{s\in S}$ with the stated properties, we will consider semi-groups on the von Neumann algebra $\mathcal{N}_q$. We let ψ be a proper conditionally negative definite function on W. We define an unbounded operator $\Delta_\psi^{(q)}$ on $L^2(\mathcal{N}_q, \tau)$ by putting $\Delta_\psi^{(q)}(T_{\mathbf{w}}) = \psi(\mathbf{w})T_{\mathbf{w}}$ for the orthogonal basis vectors $\{T_{\mathbf{w}} : w \in W\}$. While a proper, conditionally negative function always induces a quantum Markov semi-group on the algebra $\mathcal{L}(W)$, this is not the case on $\mathcal{N}_q(W)$. We can define for $t \geq 0$ a mapping $\Phi_t : \mathbb{C}_q[W] \to \mathbb{C}_q[W]$ as $\Phi_t(T_{\mathbf{w}}) = e^{-t\psi(T_{\mathbf{w}})}T_{\mathbf{w}}$, but this map may generally not extend to a u.c.p map on $\mathcal{N}_q(W)$. In this subsection we shall work under the assumption that ψ is such that Ψ_t actually extends to a u.c.p. map.

Assumption 8.1. *We have that ψ is a proper, conditionally negative function for which, for $t \geq 0$ the function $\Phi_t : \mathbb{C}_q[W] \to \mathbb{C}_q[W]$ given by $\Phi_t(T_{\mathbf{w}}) = e^{-t\psi(T_{\mathbf{w}})}T_{\mathbf{w}}$ extends to a u.c.p. map on $\mathcal{N}_q(W)$.*

If assumption 8.1 is satisfied then $(\Phi_t)_{t\geq 0}$ forms a quantum Markov semi-group on $\mathcal{N}_q$. Indeed, the maps $(\Phi_t)_{t\geq}$ then form a semi-group of u.c.p. maps. The fact that Φ_t is symmetric follows, as in section 5.1, from the fact that $\tau(\Phi_t(T_{\mathbf{r}})T_{\mathbf{g}}) = \tau(T_{\mathbf{g}}\Phi_t(T_{\mathbf{r}}))$ for all $\mathbf{r}, \mathbf{g} \in W$. Continuity of the map $t \mapsto \Phi_t(x)$ w.r.t the strong topology of $\mathcal{N}_q(W)$ follows also similar to section 5.1, which shows that $(\Phi_t)_{t\geq 0}$ is a quantum Markov semi-group.

Under assumption 8.1 we can study the gradient-$\mathcal{S}_p$ property of this semi-group. This we will do now. In order to do this, we first have to fix our appropriate $*$-subalgebra $\mathcal{A} \subseteq \mathcal{N}_q$. For this, we can in fact just take $\mathcal{A} = Span\{T_{\mathbf{w}} : \mathbf{w} \in W\}$, which is a σ-weakly dense $*$-sub-algebra of $\mathcal{N}_q$. Moreover $\mathcal{A} \subseteq D(\Delta)$ and $\Delta(\mathcal{A}) \subseteq \mathcal{A}$ and $t \mapsto \Phi_t(a)$ is norm-continuous for $a \in \mathcal{A}$. This means that we can check the gradient-$\mathcal{S}_p$ property with respect to the sub-algebra $\mathcal{A}$. Moreover, since the set $\mathcal{A}_0 := \{T_s : s \in S\}$ forms a self-adjoint set that generates the $*$-algebra $\mathcal{A}$, we have by lemma 3.2 that in order to check the gradient-$\mathcal{S}_p$ property for $(\Phi_t)_{t\geq 0}$ we only have to check that Ψ^{T_u,T_w} is in $\mathcal{S}_p$ for generators $u, w \in S$. To check when this is the case we shall make some calculations to obtain a simplified expression for Ψ^{T_u,T_w}.

8.2.1. *Simplified expression for the operator* Ψ^{T_u,T_w}. We make some calculations to obtain an explicit, simplified expression for the operator Ψ^{T_u,T_w} for $u, w \in S$. We will fix $u, w \in S$ and let $\mathbf{v} \in W$. We have by the multiplication rules that

$$T_u(T_{\mathbf{v}}T_w) = T_uT_{\mathbf{v}w} + T_uT_{\mathbf{v}}p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) \tag{126}$$
$$= T_{u\mathbf{v}w} + p_uT_{\mathbf{v}w}\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|) \tag{127}$$
$$+ (T_{u\mathbf{v}} + p_uT_{\mathbf{v}}\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|))p_w\mathbf{1}(|\mathbf{v}w| < |\mathbf{v}|). \tag{128}$$

We can now make the following calculations

$$\Delta_\psi(T_uT_{\mathbf{v}}T_w) = \psi(u\mathbf{v}w)T_{u\mathbf{v}w} + \psi(\mathbf{v}w)p_uT_{\mathbf{v}w}\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|) \tag{129}$$
$$+ \psi(u\mathbf{v})T_{u\mathbf{v}}p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) + \psi(\mathbf{v})p_uT_{\mathbf{v}}p_w\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|)\mathbf{1}(|\mathbf{v}w| < |\mathbf{v}|) \tag{130}$$
$$T_u\Delta_\psi(T_{\mathbf{v}})T_w = \psi(\mathbf{v})(T_{u\mathbf{v}w} + p_uT_{\mathbf{v}w}\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|)) \tag{131}$$
$$+ \psi(\mathbf{v})(T_{u\mathbf{v}} + p_uT_{\mathbf{v}}\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|))p_w\mathbf{1}(|\mathbf{v}w| < |\mathbf{v}|) \tag{132}$$
$$T_u\Delta_\psi(T_{\mathbf{v}}T_w) = \psi(\mathbf{v}w)T_uT_{\mathbf{v}w} + \psi(\mathbf{v})T_uT_{\mathbf{v}}p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) \tag{133}$$
$$= \psi(\mathbf{v}w)(T_{u\mathbf{v}w} + p_uT_{\mathbf{v}w}\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|)) \tag{134}$$
$$+ \psi(\mathbf{v})(T_{u\mathbf{v}} + p_uT_{\mathbf{v}}\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|))p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) \tag{135}$$
$$\Delta_\psi(T_uT_{\mathbf{v}})T_w = \psi(u\mathbf{v})T_{u\mathbf{v}}T_w + \psi(\mathbf{v})p_uT_{\mathbf{v}}T_w\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|) \tag{136}$$
$$= \psi(u\mathbf{v})(T_{u\mathbf{v}w} + T_{u\mathbf{v}}p_w\mathbb{1}(|u\mathbf{v}w| < |u\mathbf{v}|)) \tag{137}$$
$$+ \psi(\mathbf{v})p_u(T_{\mathbf{v}w} + T_{\mathbf{v}}p_w\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|))\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|). \tag{138}$$

Now by collecting all terms we get

$$-2\Psi^{T_u,T_w}_{\Delta_\psi}(T_{\mathbf{v}}) = \Delta_\psi(T_uT_{\mathbf{v}}T_w) + T_u\Delta_\psi(T_{\mathbf{v}})T_w - T_u\Delta_\psi(T_{\mathbf{v}}T_w) - \Delta_\psi(T_uT_{\mathbf{v}})T_w \tag{139}$$
$$= (\psi(u\mathbf{v}w) + \psi(\mathbf{v}) - \psi(\mathbf{v}w) - \psi(u\mathbf{v}))T_{u\mathbf{v}w} \tag{140}$$
$$+ [(\psi(u\mathbf{v}) + \psi(\mathbf{v}) - \psi(\mathbf{v}))\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) - \psi(u\mathbf{v})\mathbb{1}(|u\mathbf{v}w| < |u\mathbf{v}|)]T_{u\mathbf{v}}p_w \tag{141}$$
$$+ [(\psi(\mathbf{v}w) + \psi(\mathbf{v}) - \psi(\mathbf{v}w))\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|) - \psi(\mathbf{v})\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|)]p_uT_{\mathbf{v}w} \tag{142}$$
$$+ (\psi(\mathbf{v}) + \psi(\mathbf{v}) - \psi(\mathbf{v}) - \psi(\mathbf{v}))\mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|)\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|)p_uT_{\mathbf{v}}p_w \tag{143}$$
$$= \gamma^\psi_{u,w}(\mathbf{v})T_{u\mathbf{v}w} \tag{144}$$
$$+ \psi(u\mathbf{v})(\mathbb{1}(|\mathbf{v}w| < |\mathbf{v}|) - \mathbb{1}(|u\mathbf{v}w| < |u\mathbf{v}|))T_{u\mathbf{v}}p_w \tag{145}$$
$$+ \psi(\mathbf{v})(\mathbb{1}(|u\mathbf{v}w| < |\mathbf{v}w|) - \mathbb{1}(|u\mathbf{v}| < |\mathbf{v}|))p_uT_{\mathbf{v}w} \tag{146}$$
$$= \gamma^\psi_{u,w}(\mathbf{v})T_{u\mathbf{v}w} \tag{147}$$
$$+ \psi(u\mathbf{v})\left(\frac{|\mathbf{v}| - |\mathbf{v}w| + 1}{2} - \frac{|u\mathbf{v}| - |u\mathbf{v}w| + 1}{2}\right)T_{u\mathbf{v}}p_w \tag{148}$$
$$+ \psi(\mathbf{v})\left(\frac{|\mathbf{v}w| - |u\mathbf{v}w| + 1}{2} - \frac{|\mathbf{v}| - |u\mathbf{v}| + 1}{2}\right)p_uT_{\mathbf{v}w} \tag{149}$$
$$= \gamma^\psi_{u,w}(\mathbf{v})T_{u\mathbf{v}w} + \frac{1}{2}(|u\mathbf{v}w| + |\mathbf{v}| - |\mathbf{v}w| - |u\mathbf{v}|)(\psi(u\mathbf{v})T_{u\mathbf{v}}p_w - \psi(\mathbf{v})p_uT_{\mathbf{v}w}) \tag{150}$$
$$= \gamma^\psi_{u,w}(\mathbf{v})T_{u\mathbf{v}w} + \frac{1}{2}\gamma^{\psi_S}_{u,w}(\mathbf{v})(\psi(u\mathbf{v})T_{u\mathbf{v}}p_w - \psi(\mathbf{v})p_uT_{\mathbf{v}w}) \tag{151}$$

where ψ_S is the proper conditionally negative function given by the word length. Now, when $u\mathbf{v} \neq \mathbf{v}w$ we have by lemma 6.4 that $\gamma^{\psi_S}_{u,w}(\mathbf{v}) = 0$. In the other case that $u\mathbf{v} = \mathbf{v}w$ holds, we have $|\frac{1}{2}\gamma^{\psi_S}_{u,w}(\mathbf{v})| = \psi_S(u) = 1$. In this case the elements u and w are also conjugate and therefore $p_u = p_w$. Combining these facts we obtain the simplified formula for Ψ^{T_u,T_w} given by

$$\Psi^{T_u,T_w}_{\Delta_\psi}(T_{\mathbf{v}}) = -\frac{1}{2}\left(\gamma^{\psi}_{u,w}(\mathbf{v})T_{u\mathbf{v}w} + \frac{1}{2}\gamma^{\psi_S}_{u,w}(\mathbf{v})(\psi(u\mathbf{v}) - \psi(\mathbf{v}))T_{u\mathbf{v}}p_w\right). \tag{152}$$

8.2.2. *Bound on $\mathcal{S}_2$-norm of Ψ^{T_u,T_w}.* Using the simplified expression for Ψ^{T_u,T_w} that we have obtained, we shall now give an upper bound on the $\mathcal{S}_2$ norm of Ψ^{T_u,T_w}. When this upper bound is finite it can be used to show to show that the semi-group that we considered is gradient-$\mathcal{S}_2$. In the following, we shall work under the assumption that $\psi(\mathbf{uw}) \leq \psi(\mathbf{u}) + \psi(\mathbf{w})$ holds for $\mathbf{u},\mathbf{w} \in W$.

We let $u, w \in S$. Using the expression for Ψ^{T_u,T_w} that we found, and using the fact that $\{T_{\mathbf{v}}\}_{\mathbf{v}\in W}$ is an orthonormal basis for $L^2(\mathcal{N}_q(W),\tau)$ we obtain that for the $\mathcal{S}_2$-norm of Ψ^{T_u,T_w} we have the following bound

$$\|\Psi^{T_u,T_w}_{\Delta_\psi}\|^2_{\mathcal{S}_2} = \sum_{\mathbf{v}\in W}\langle \Psi^{T_u,T_w}_{\Delta_\psi}(T_{\mathbf{v}}), \Psi^{T_u,T_w}_{\Delta_\psi}(T_{\mathbf{v}})\rangle \tag{153}$$

$$= \frac{1}{4}\sum_{\mathbf{v}\in W}\left[|\gamma^{\psi}_{u,w}(\mathbf{v})|^2 + \frac{1}{4}|\gamma^{\psi_S}_{u,w}(\mathbf{v})|^2|\psi(u\mathbf{v}) - \psi(\mathbf{v})|^2|p_u|^2\right] \tag{154}$$

$$\leq \frac{1}{4}\left(\|\gamma^{\psi}_{u,w}\|^2_{\ell_2(W)} + \frac{1}{4}|\psi(u)|^2p_u^2\|\gamma^{\psi_S}_{u,w}\|^2_{\ell_2(W)}\right). \tag{155}$$

We are then thus interested in functions ψ for which this bound is finite for all $u, w \in S$.

8.3. **Building gradient-$\mathcal{S}_2$ quantum Markov semi-groups.** We will now find Coxeter systems $W = \langle S|M\rangle$ together with proper, conditionally negative functions ψ on W satisfying $\psi(\mathbf{uw}) \leq \psi(\mathbf{u})+\psi(\mathbf{w})$ for all $\mathbf{u},\mathbf{w} \in W$ for which the bound in eq. (155) is finite for all $u, w \in S$. In certain cases we know that the function ψ actually induces a quantum Markov semi-group, i.e. that assumption 8.1 is satisfied. In those cases we thus obtain a gradient-$\mathcal{S}_2$ quantum Markov semi-group. We shall give examples In section 8.3.1 we shall, for an arbitrary Hecke tuple $q = (q_s)_{s\in S}$ and for certain right-angled Coxeter group W construct a proper conditionally negative function ψ that induces a gradient-$\mathcal{S}_2$ quantum Markov semi-group on $\mathcal{N}_q(W)$. Thereafter, in section 8.3.2 we shall give other examples of Coxeter groups W and proper, conditionally negative functions ψ on W such that the bound from eq. (155) is finite for all $u, w \in S$. However, in these cases we do not know for what tuples $q = (q_s)_{s\in S}$ the function ψ induces a quantum Markov semi-group on $\mathcal{N}_q(W)$. For these cases we thus only obtain that if ψ induces a quantum Markov semi-group on $\mathcal{N}_q(W)$, then it is also gradient-$\mathcal{S}_2$.

8.3.1. *Gradient-$\mathcal{S}_2$ quantum Markov semi-groups.* We let $W = \langle S|M\rangle$ be a right-angled Coxeter group for which

$$S_0 = \{r \in S : \exists s,t \in S : M(r,s) = M(r,t) = 2 \text{ and } M(s,t) = \infty\} \tag{156}$$

generates a finite group, i.e. the elements in S_0 commute. This is similar to the condition in corollary 7.6. We now denote $\mathcal{I} = S \setminus S_0$ and we consider the conditionally negative definite function $\psi_{\mathcal{I}}$ on W. This function is proper as S_0 generates a finite group. We now fix a tuple $q = (q_s)_{s\in S}$. Because W is right-angled it follows by the results [8, Corollary 3.4] and [11, Proposition 2.30] that $\psi_{\mathcal{I}}$ satisfies assumption 8.1. This means that $\psi_{\mathcal{I}}$ actually induces a quantum Markov semi-group on $\mathcal{N}_q(W)$. We show that this semi-group is gradient-$\mathcal{S}_2$. Namely, it follows from corollary 7.6 that for all $u, w \in S$ we have that $\gamma^{\psi_{\mathcal{I}}}_{u,w} \in \ell_2(\Gamma)$. Now, if we have $u \in S_0$ then $\psi_{\mathcal{I}}(u) = 0$ and hence by eq. (155) we have $\|\Psi^{T_u,T_w}_{\Delta_{\psi_{\mathcal{I}}}}\|^2_{\mathcal{S}_2} \leq \frac{1}{4}\|\gamma^{\psi_{\mathcal{I}}}_{u,w}\|^2_{\ell_2(\Gamma)} < \infty$. In the other case that $u \in S \setminus S_0 = \mathcal{I}$ we have that $\psi_{\mathcal{I}}(u) = 1 = \psi_S(u)$ and therefore by lemma 6.4 we have

$$|\gamma^{\psi_{\mathcal{I}}}_{u,w}(\mathbf{v})| = 2\psi_{\mathcal{I}}(u)\mathbb{1}(u\mathbf{v} = \mathbf{v}w) = 2\psi_S(u)\mathbb{1}(u\mathbf{v} = \mathbf{v}w) = |\gamma^{\psi_S}_{u,w}(\mathbf{v})|. \tag{157}$$

This means that in this case $\gamma^{\psi_S}_{u,w} = \gamma^{\psi_{\mathcal{I}}}_{u,w} \in \ell_2(\Gamma)$ and therefore

$$\|\Psi^{T_u,T_w}_{\Delta_{\psi_{\mathcal{I}}}}\|^2_{\mathcal{S}_2} \leq \frac{1}{4}\left(\|\gamma^{\psi_{\mathcal{I}}}_{u,w}\|^2_{\ell_2(\Gamma)} + \frac{1}{4}|\psi_{\mathcal{I}}(u)|^2 \cdot \|\gamma^{\psi_S}_{u,w}\|^2_{\ell_2(\Gamma)}\right) < \infty. \tag{158}$$

We thus obtain that in either case we have that $\|\Psi^{T_u,T_w}_{\Delta_{\psi_{\mathcal{I}}}}\|_{\mathcal{S}_2} < \infty$. Now, by the observations made in the begin of section 8.2 we obtain that the semi-group $(\Phi_t)_{t\geq 0}$ on $\mathcal{N}_q(W)$ that is associated to this function $\psi_{\mathcal{I}}$ is gradient-$\mathcal{S}_2$. We note moreover that the operator $\Delta_{\psi_{\mathcal{I}}}$ is filtered and has subexponential growth w.r.t. $\mathbb{C}_q[W]$ when W is infinite. As Hecke-algebras are moreover locally reflexive by [13, Theorem 0.5], [7, Corollary 9.4.1] we obtain similar to theorem 6.2 and corollary 7.6 that the Hecke-algebra $\mathcal{N}_q(W)$ has the (AO^+) property in this case. As by [8] we have for right-angled Coxeter groups W that $\mathcal{N}_q(W)$ also has the W*CBAP, we then moreover obtain strongly solid for $\mathcal{N}_q(W)$ by theorem 4.3, whenever we obtained (AO^+).

8.3.2. *Other possible quantum Markov semi-groups.* Let us fix a Coxeter system $W = \langle S|M\rangle$ for which $\gamma^{\psi_S}_{u,w}$ is in $\ell_2(W)$ for $u, w \in S$. These Coxeter systems were 'almost completely' characterized in section 6. Let $q = (q_s)_{s\in S}$ be a Hecke tuple. We will now make the assumption that ψ_S satisfies assumption 8.1 for $\mathcal{N}_q(W)$. Under this assumption we have that ψ_S induces a quantum Markov semi-group $(\Phi_t)_{t\geq 0}$ on $\mathcal{N}_q(W)$. Now, since the quantity $\|\gamma^{\psi_S}_{u,w}\|_{\ell_2(W)}$ is finite for all $u, w \in S$, we have by the bound in eq. (155) that $\|\Psi^{T_u,T_w}_{\Delta_{\psi_S}}\|_{\mathcal{S}_2}$ is finite for all $u, w \in S$. By the observations at the start of section 8.2 this means that $(\Phi_t)_{t\geq 0}$ is then gradient-$\mathcal{S}_2$. We thus obtained that if the proper, conditionally negative function ψ_S on W induces a quantum Markov semi-group on $\mathcal{N}_q(W)$, then this semi-group is moreover gradient-$\mathcal{S}_2$. As in section 8.3.1 we obtain in such case that the Hecke-algebra $\mathcal{N}_q(W)$ has the (AO^+) property.

9. Semi-groups constructed using roots of generators

In this section we will consider the roots of generators of quantum Markov semi-groups. Namely, for a generator Δ of a quantum Markov semi-group, and for $\alpha \in (0,1)$ it holds true by [17, Section 10.4] that the root Δ^{α} also generates such semi-group. We can then study the gradient-$\mathcal{S}_p$ property of that semi-group. In particular, we want to relate the gradient-$\mathcal{S}_p$ property of the semi-group $(e^{-\Delta t})_{t\geq 0}$ to the gradient-$\mathcal{S}_q$ property of the semi-group $(e^{-\Delta^{\alpha} t})_{t\geq 0}$. In section 9.1 we will do this when $\Delta := \Delta_{\psi}$ is a generator of a semi-group on a group von Neumann algebra $\mathcal{L}(\Gamma)$, associated to some proper, conditionally negative function ψ. However, the result we obtain is not very strong as we are only able to do this under the assumption that the function $n \mapsto \#\psi^{-1}([n, n+1))$ has polynomial growth, and that the function ψ satisfies $\psi(\mathbf{uw}) \leq \psi(\mathbf{u}) + \psi(\mathbf{w})$ for all $\mathbf{u}, \mathbf{w} \in \Gamma$. In section 9.2 we show that some condition on the growth of $\#\psi^{-1}([n, n+1))$ is really necessary, as we give an example of a semi-group associated to some function ψ for which $(e^{-\Delta_{\psi}})_{t\geq 0}$ is gradient-$\mathcal{S}_p$ for all $p \in [1, \infty]$, but for $\alpha \in (0,1)$ the semi-group $(e^{-\Delta_{\psi}^{\alpha}})_{t\geq 0}$ is not gradient-$\mathcal{S}_p$ for any $p \in [1, \infty)$. This shows that we do need some condition on ψ in order to say something about the gradient-$\mathcal{S}_p$ property of $(e^{-t\Delta_{\psi}^{\alpha}})$.

9.1. **Roots of generators of semi-groups associated to a conditionally negative function.** Let us be given a discrete group Γ and a proper, conditionally negative definite function ψ on Γ. We can for $\alpha \in (0,1)$ consider the root of the positive operator Δ_{ψ} that generates the semi-group $(\Phi_t)_{t\geq 0}$ associated to ψ. By [17, Section 10.4] this operator Δ_{ψ}^{α} then also generates a semi-group on $\mathcal{L}(\Gamma)$, that we denote by $(\Phi_t^{\alpha})_{t\geq 0}$. We note that the generator Δ_{ψ}^{α} can be given explicitly by $\Delta_{\psi}^{\alpha}(\lambda_{\mathbf{v}}) = \psi^{\alpha}(\mathbf{v})\lambda_{\mathbf{v}}$. On $\mathbb{C}[\Gamma]$ the semi-group $(\Phi_t^{\alpha})_{t\geq 0}$ is then given by $\Phi_t^{\alpha}(\sum_{\mathbf{g}\in\Gamma} \alpha_{\mathbf{g}}\lambda_{\mathbf{g}}) = \sum_{\mathbf{g}\in\Gamma} e^{-t\psi^{\alpha}(\mathbf{g})}\alpha_{\mathbf{g}}\lambda_{\mathbf{g}}$. Define for $t \geq 0$ the function $\varphi_{t,\alpha} : \Gamma \to \mathbb{R}$ given by $\varphi_{t,\alpha}(x) = e^{-t\psi^{\alpha}(x)}$. Then Φ_t^{α} is given on $\mathbb{C}[\Gamma]$ by the function $m_{\varphi_{t,\alpha}}$ that multiplies point-wise with the function $\varphi_{t,\alpha}$. Now, since $(\Phi_t^{\alpha})_{t\geq 0}$ is a quantum Markov semi-group we have that the $m_{\varphi_{t,\alpha}}$ extends to a u.c.p. map on $\mathcal{L}(\Gamma)$. This means by [7, Theorem 2.5.11] that the function $\varphi_{t,\alpha}$ is positive definite. Now, since this holds for all $t \geq 0$, we have by Schoenberg's theorem [2, Theorem C.3.2] that the function ψ^{α} is conditionally negative. It is moreover clear that ψ^{α} is actually proper. Also we see that in fact we have that $\Delta_{\psi}^{\alpha} = \Delta_{\psi^{\alpha}}$, i.e. the semi-group generated by Δ_{ψ}^{α} is the semi-group associated to the proper, conditionally negative function ψ^{α}.

In the following lemma we will, for $\alpha \in (0,1)$ and for certain functions ψ relate the gradient-$\mathcal{S}_p$ property of the semi-group $(\Phi_t^{\Delta_{\psi^{\alpha}}})_{t\geq 0}$ to the gradient-$\mathcal{S}_q$ property of the semi-group $(\Phi_t^{\Delta_{\psi}})_{t\geq 0}$. We first introduce some notation. For functions $f, g : \mathbb{N} \to [0, \infty]$ we will write $f(n) \lesssim g(n)$ whenever there is a constant $c > 0$ such that $f(n) \leq c g(n)$ for all $n \geq 1$. A condition that we impose in the following lemma is that ψ is such that $n \mapsto \#\psi^{-1}([n, n+1))$ has polynomial growth. This means that we assume $\#\psi^{-1}([n, n+1)) \lesssim n^r$ for some $r \geq 0$. An example of a function ψ satisfying this condition is the word length $|\cdot|$ on the Coxeter group $W = \langle s_1, s_2 | M(s_1, s_2) = \infty\rangle$. Indeed, for that function we have $\#\psi^{-1}([n, n+1)) = \#\psi^{-1}(\{n\}) = 2$ is constant. Another condition we impose on ψ is that is satisfies $\psi(\mathbf{uw}) \leq \psi(\mathbf{u}) + \psi(\mathbf{w})$ for all $\mathbf{u}, \mathbf{w} \in \Gamma$. This condition is also satisfied by the word length.

Lemma 9.1. *Let Γ be a group with a proper, conditionally negative definite function ψ. Suppose the corresponding quantum Markov semi-group $(\Phi_t)_{t\geq 0}$ is gradient-$\mathcal{S}_q$ for some $q \in [1, \infty]$, and suppose we have for $n \in \mathbb{N}$ that $\#\psi^{-1}([n, n+1)) \lesssim n^r$ for some $r \geq 0$. Also suppose that ψ satisfies $\psi(\mathbf{uv}) \leq \psi(\mathbf{u}) + \psi(\mathbf{w})$ for all $\mathbf{u}, \mathbf{w} \in \Gamma$. Then for $\alpha \in (0,1)$ the semi-group $(\Phi_t^{\alpha})_{t\geq 0}$ associated to ψ^{α} is gradient-$\mathcal{S}_p$ for $p > \max\{\frac{r+1}{2-\alpha}, \frac{1}{\frac{1-\alpha}{r+1} + \frac{1}{q}}\}$.*

Proof. Fix $\mathbf{u}, \mathbf{w} \in \Gamma$. First note that since $(\Phi_t)_{t\geq 0}$ is gradient-$\mathcal{S}_q$ for some $q \in [1, \infty]$, we can find N_1 such that $\psi(\mathbf{v}) \geq N_1$ implies that $|\gamma_{\mathbf{u},\mathbf{w}}^{\psi}(\mathbf{v})| < 1$. We now set $N_2 = \psi(\mathbf{u}) + \psi(\mathbf{w})$ and $N = \max\{N_1, N_2\}$.

Now choose a $\mathbf{v} \in \Gamma$ with $y := \psi(\mathbf{v}) \geq 8N$. Also set $y_1 = \psi(\mathbf{uv}) - \psi(\mathbf{v})$ and $y_2 = \psi(\mathbf{vw}) - \psi(\mathbf{v})$, so that $\psi(\mathbf{uvw}) - \psi(\mathbf{v}) = y_1 + y_2 + \gamma_{\mathbf{u},\mathbf{w}}^{\psi}(\mathbf{v})$. We note that since ψ satisfies the triangle inequality we have $|y_1| \leq \psi(\mathbf{u})$ and $|y_2| \leq \psi(\mathbf{w})$. We define the function $g(x) = x^{\alpha}$. Then by using the mean value theorem multiple times we get

$$\begin{aligned}
|\gamma^{\psi^\alpha}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| &= |(\psi(\mathbf{uvw})^\alpha + \psi(\mathbf{v})^\alpha - \psi(\mathbf{uv})^\alpha - \psi(\mathbf{vw})^\alpha| \\
&= |(\psi(\mathbf{v}) + y_1 + y_2 + \gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v}))^\alpha - (\psi(\mathbf{v}) + y_1)^\alpha + \psi(\mathbf{v})^\alpha - (\psi(\mathbf{v}) + y_2)^\alpha| \\
&= |g(y + y_1 + y_2 + \gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})) - g(y + y_1) + g(y) - g(y + y_2)| \\
&\le |g(y + y_1 + y_2) - g(y + y_1) + g(y) - g(y + y_2)| \\
&+ |g(y + y_1 + y_2 + \gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})) - g(y + y_1 + y_2)| \\
&= |y_2 \cdot g'(c_1) - y_2 \cdot g'(c_2)| + |\gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| \cdot |g'(c_3)| \\
&= |y_2| \cdot |c_2 - c_1| \cdot |g''(c_4)| + |\gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| \cdot |g'(c_3)|
\end{aligned}$$

for some c_1 between $y + y_1 + y_2$ and $y + y_1$, some c_2 between y and $y + y_2$, some c_3 between $y + y_1 + y_2 + \psi^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})$ and $y + y_1 + y_2$ and for some c_4 between c_1 and c_2. It follows that we have the bounds

$$|c_1 - y| \le |y_1| + |y_1| \tag{159}$$

$$|c_2 - y| \le |y_2| \tag{160}$$

$$|c_3 - y| \le |y_1| + |y_2| + |\gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| \tag{161}$$

$$|c_4 - y| \le |y_1| + |y_2|. \tag{162}$$

Now since also $|y_1|, |y_2|, |\gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| \le N$ we have for $i = 1, 2, 3, 4$ that $|c_i - y| \le 4N \le \frac{y}{2}$ and thus $c_i \ge \frac{y}{2}$. Now since $|g'|$ and $|g''|$ are decreasing functions, we have that $|g'(c_3)| \le |g'(\frac{y}{2})|$ and $|g''(c_4)| \le |g''(\frac{y}{2})|$. Also we have that $|c_2 - c_1| \le |c_1 - y| + |c_2 - y| \le 3N$. We now define the following two constants

$$C_1 = N \cdot (3N)\alpha|\alpha - 1|2^{2-\alpha} \tag{163}$$

$$C_2 = \alpha 2^{1-\alpha}. \tag{164}$$

We then have that

$$|\gamma^{\psi^\alpha}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| \le |y_2| \cdot |c_2 - c_1| \cdot |g''(c_4)| + |\gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| \cdot |g'(c_3)| \tag{165}$$

$$\le N \cdot (3N)\alpha|g''(\frac{y}{2})| + |\gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| \cdot |g'(\frac{y}{2})| \tag{166}$$

$$\le N \cdot (3N)\alpha|\alpha - 1| \left(\frac{y}{2}\right)^{\alpha-2} + |\gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| \cdot \alpha \left(\frac{y}{2}\right)^{\alpha-1} \tag{167}$$

$$\le C_1 \psi(\mathbf{v})^{\alpha-2} + C_2|\gamma^{\psi}_{\mathbf{u},\mathbf{w}}(\mathbf{v})| \cdot \psi(\mathbf{v})^{\alpha-1}. \tag{168}$$

This inequality that we obtained thus holds for all $\mathbf{v} \in \Gamma$ with $\psi(\mathbf{v}) \ge 8N$. We shall now turn to show that the $\ell^p(\Gamma)$-norms of the right hand side are finite.

By assumption we have that $\#\psi^{-1}([n, n+1)) \lesssim n^r$ holds. Now for $p_1 > \frac{r+1}{2-\alpha}$ we have that $-1 > r + (\alpha - 2)p_1$ and hence

$$\|\psi^{\alpha-2}\chi_{\psi^{-1}([1,\infty))}\|_{\ell^{p_1}(\Gamma)} = \sum_{y \in \psi(\Gamma), y \ge 1} \#\psi^{-1}(\{y\}) \cdot y^{(\alpha-2)p_1} \tag{169}$$

$$\le \sum_{n \in \mathbb{N}} \#\psi^{-1}([n, n+1)) \cdot n^{(\alpha-2)p_1} \tag{170}$$

$$\lesssim \sum_{n \in \mathbb{N}} n^{r+(\alpha-2)p_1} < \infty. \tag{171}$$

Note also that since ψ is proper we have that $\psi(\mathbf{v}) < 1$ for only finitely many $\mathbf{v} \in \Gamma$. This shows that for $p_1 > \frac{r+1}{2-\alpha}$ we have that $\|\psi^{\alpha-2}\|_{\ell^{p_1}(\Gamma)}$ is finite. We now let $p_2 > \frac{1}{\frac{1-\alpha}{r+1} + \frac{1}{q}}$. We can set $q' = \frac{1}{\frac{1}{p_2} - \frac{1}{q}}$ so that $\frac{1}{p_2} = \frac{1}{q} + \frac{1}{q'}$ holds. Furthermore since $\frac{1}{p_2} - \frac{1}{q} < \left(\frac{1-\alpha}{r+1} + \frac{1}{q}\right) - \frac{1}{q} = \frac{1-\alpha}{r+1}$ we obtain that $q' > \frac{r+1}{1-\alpha}$. This means that we have the inequality $-1 > r + (\alpha - 1)q'$ and therefore

$$\|\psi^{\alpha-1}\chi_{\psi^{-1}([1,\infty))}\|_{\ell^{q'}(\Gamma)} = \sum_{y \in \psi(\Gamma), y \ge 1} \#\psi^{-1}(\{y\}) \cdot y^{(\alpha-1)q'} \tag{172}$$

$$\le \sum_{n \in \mathbb{N}} \#\psi^{-1}([n, n+1)) \cdot n^{(\alpha-1)q'} \tag{173}$$

$$\lesssim \sum_{n \in \mathbb{N}} n^{r+(\alpha-1)q'} < \infty. \tag{174}$$

Again, since $\psi(\mathbf{v}) < 1$ for only finitely many $\mathbf{v} \in \Gamma$, this shows that for $p_2 > \frac{1}{\frac{1-\alpha}{r+1}+\frac{1}{q}}$ we have that $\|\psi^{\alpha-1}\|_{\ell^{q'}(\Gamma)}$ is finite. Now since $\|\gamma^{\psi}_{\mathbf{u},\mathbf{w}}\|_{\ell^q(\Gamma)} = 2\|\Psi^{\mathbf{u},\mathbf{w}}_{\Delta_\psi}\|_{\mathcal{S}_q}$ is finite by assumption, we have by Hölder that $\|\gamma^{\psi}_{\mathbf{u},\mathbf{w}} \cdot \psi^{\alpha-1}\|_{\ell^{p_2}(\Gamma)}$ is finite. This shows us that for $p > \max\{\frac{r+1}{2-\alpha}, \frac{1}{\frac{1-\alpha}{r+1}+\frac{1}{q}}\}$ we have that

$$\|\gamma^{\psi^\alpha}_{\mathbf{u},\mathbf{w}}\chi_{\psi^{-1}([8N,\infty))}\|_{\ell^p(\Gamma)} \leq C_1\|\psi^{\alpha-2}\|_{\ell^p(\Gamma)} + C_2\|\gamma^{\psi}_{\mathbf{u},\mathbf{w}} \cdot \psi^{\alpha-1}\|_{\ell^{p_2}(\Gamma)} \tag{175}$$

and this quantity is finite. Since ψ is moreover proper we have that $\psi^{-1}([0,8N))$ is finite, and therefore we have that $\|\gamma^{\psi^\alpha}_{\mathbf{u},\mathbf{w}}\|_{\ell^p(\Gamma)}$ is finite. This shows that the semi-group generated by ψ^α is gradient-$\mathcal{S}_p$ for such p, and this proves the lemma. □

9.2. **Showing need of assumption of polynomial growth on ψ^{-1}.** We now show that for certain semi-groups $(\Phi_t)_{t\geq 0}$ on group von Neumann algebras on Coxeter groups, that, although the semi-groups may possess the gradient-$\mathcal{S}_p$ property for all $p \in [1,\infty]$, we have for $\alpha \in (0,1)$ that the semi-groups associated to the αth-power root of the generators generate semi-groups that are not gradient-$\mathcal{S}_p$ for any $p \in [1,\infty]$. This shows that in general we do need some condition of the growth of $n \mapsto \#\psi^{-1}([n,n+1))$.

Let $W = \langle S|M\rangle$ be a Coxeter system. We consider the semi-group $(\Phi_t)_{t\geq 0}$ associated to the word length ψ_S. In section 6 and section 7 we have classified for many Coxeter groups whether this semi-group is, or is not gradient-$\mathcal{S}_p$ for $p \in [1,\infty]$. For $\alpha \in (0,1)$ we let $(\Phi^\alpha_t)_{t\geq 0} := (e^{-t\Delta^\alpha_\psi})_{t\geq 0}$ be the semi-group generated by the αth-root of Δ_ψ. In the following lemma we show that $(\Psi^\alpha_t)_{t\geq 0}$ is not gradient-$\mathcal{S}_p$ for any $p \in [1,\infty]$ when there are three distinct elements $s_1, s_2, s_3 \in S$ such that $M(s_1,s_2) = M(s_2,s_3) = \infty$. However, by corollary 6.10 it is clear that there are Coxeter groups satisfying this condition for which the semi-group $(\Phi_t)_{t\geq 0} = (e_t^{-t\Delta_{\psi_S}})_{t\geq 0}$ is gradient-$\mathcal{S}_p$ for all $p \in [1,\infty)$. This thus shows that we generally need some assumption on the growth of the function $n \mapsto \psi^{-1}([n,n+1])$ in order to say something about the gradient-$\mathcal{S}_p$ property of the semi-group corresponding to the roots of the generator.

Lemma 9.2. *Let $W = \langle S|M\rangle$ be a Coxeter group. Suppose there are distinct elements $s_1, s_2, s_3 \in S$ s.t. $M(s_1,s_2) = M(s_2,s_3) = \infty$ then for $\alpha \in (0,1)$ the semi-group $(\Psi^\alpha_t)_{t\geq 0}$ associated to the function ψ^α_S is not gradient-$\mathcal{S}_p$ for any $p \in [1,\infty)$.*

Proof. Let $u = w = s_1$. For $n \geq 2$ let us denote

$$\mathcal{W}_{2n+1} = \{\mathbf{v} := s_2v_1s_2v_2s_2..s_2v_{n-1}s_2s_1s_2 \in W : v_i \in \{s_1,s_3\}\} \tag{176}$$

$$\mathcal{W}_{2n+2} = \{\mathbf{v} := s_2v_1s_2v_2s_2..s_2v_{n-1}s_2s_1s_3s_2 \in W : v_i \in \{s_1,s_3\}\}. \tag{177}$$

Let $n \geq 5$. We note that the expressions for the words $\mathbf{v} \in \mathcal{W}_n$ are reduced, and hence $|\mathbf{v}| = n$ for $\mathbf{v} \in \mathcal{W}_n$. Also we note that we have $|\mathcal{W}_n| \geq 2^{\frac{n}{2}-2}$.

For $x \geq 0$ we set

$$\gamma(x) = (x+2)^\alpha - 2(x+1)^\alpha + x^\alpha \tag{178}$$

so that we for $\mathbf{v} \in \mathcal{W}_n$ we have that

$$\gamma^{\psi^\alpha_S}_{u,w}(\mathbf{v}) = \psi_S(u\mathbf{v}w)^\alpha + \psi_S(\mathbf{v})^\alpha - \psi_S(u\mathbf{v})^\alpha - \psi_S(\mathbf{v}w)^\alpha = \gamma(n). \tag{179}$$

Now set $g(x) = x^\alpha$, then for $n \geq 1$ we have that

$$\gamma(n) = ((n+2)^\alpha - (n+1)^\alpha) - ((n+1)^\alpha - n^\alpha) = g'(c_2) - g'(c_1) \tag{180}$$

where $c_1 \in [n,n+1]$ and $c_2 \in [n+1,n+2]$ by the mean value theorem. Now, applying the mean value theorem again we obtain $c \in [c_1,c_2]$ such that

$$|\gamma(n)| = |g'(c_2) - g'(c_1)| = |(c_2-c_1)g''(c)| \geq \alpha|\alpha-1|(n+2)^{\alpha-2}(c_2-c_1).$$

Now in case that $c_2 - c_1 \leq \frac{1}{2}$ we obtain $c_3 \in [n+2,n+3]$ and $c' \in [c_2,c_3]$ such that

$$|\gamma(n+1)| = |g'(c_3) - g'(c_2)| = |(c_3-c_2)g''(c')| \geq \alpha|\alpha-1|(n+3)^{\alpha-2}(c_3-c_2).$$

Now since $c_2 - c_1 \leq \frac{1}{2}$ we then have that $c_3 - c_2 \geq c_3 - \frac{1}{2} - c_1 \geq 1 - \frac{1}{2} = \frac{1}{2}$. Hence either way we have for $n \geq 1$ that $\max\{|\gamma(n)|, |\gamma(n+1)|\} \geq \frac{1}{2}\alpha(1-\alpha)(n+3)^{\alpha-2}$.

We now have

$$\|\gamma_{u,w}^{\psi_S^\alpha}\|_{\ell^p(\Gamma)}^p = \sum_{\mathbf{v}\in\Gamma} |\gamma_{u,w}^{\psi_S^\alpha}(\mathbf{v})|^p \tag{181}$$

$$\geq \sum_{n\geq 5}\sum_{\mathbf{v}\in\mathcal{W}_n} |\gamma_{u,w}^{\psi_S^\alpha}(\mathbf{v})|^p \tag{182}$$

$$\geq \sum_{n\geq 5} |\gamma(n)|^p \cdot |\mathcal{W}_n| \tag{183}$$

$$\geq \sum_{n\geq 3} |\gamma(2n)|^p \cdot |\mathcal{W}_{2n}| + |\gamma(2n+1)|^p \cdot |\mathcal{W}_{2n+1}| \tag{184}$$

$$\geq \sum_{n\geq 3} \max\{|\gamma(2n)|^p, |\gamma(2n+1)|^p\} \cdot \min\{|\mathcal{W}_{2n}|, |\mathcal{W}_{2n+1}|\} \tag{185}$$

$$\geq \sum_{n\geq 3} \left(\frac{1}{2}\alpha(1-\alpha)(2n+3)^{\alpha-2}\right)^p \cdot 2^{n-2} = \infty. \tag{186}$$

This shows that $\|\Psi^{\mathbf{u},\mathbf{w}}_{\Delta_{\psi_S^\alpha}}\|_{\mathcal{S}_p} = \infty$ for $p \in [1,\infty)$, which shows that the semi-group is not gradient-$\mathcal{S}_p$ for $p \in [1,\infty)$. $\square$

10. Obtaining main results for $\mathcal{L}(W)$ for right-angled word hyperbolic Coxeter groups

Throughout this section we assume that $W = \langle S|M \rangle$ is a right-angled word-hyperbolic Coxeter group. For such Coxeter groups we will obtain in corollary 10.9 that $\mathcal{L}(W)$ has the (AO^+) property, and that it is strongly solid. These results were already known for general hyperbolic groups Γ, see [21, Lemma 6.2.8], [36, Theorem 1.4]. The method we present here however uses completely different tools. We note moreover that the method used in this section also differs somewhat from the proof methods done in previous sections. We give an outline of the proof we give in this section.

In section 10.1 we construct for the Coxeter group a certain bimodule $\mathcal{H}_W$. Thereafter in section 10.2 we give some conditions under which we obtain a certain quasi-containment of bimodules. In section 10.3 we examine these properties for the bimodule $\mathcal{H}_W$. In the final part, in section 10.4, we use the non-commutative Riesz-transform together with the results we already obtained to prove the existence of a u.c.p. map $\theta : C_r^*(W) \otimes_{\min} C_r^*(W)^{\mathrm{op}} \to B(\ell_2(W))$ as in the definition of the Akemann-Ostrand property. From this we then obtain that $\mathcal{L}(W)$ possesses the (AO^+) property and is strongly solid.

10.1. Construction of bimodule $\mathcal{H}_W$.

We let $W = \langle S|M \rangle$ be a word hyperbolic right-angled Coxeter group. We shall write $\mathsf{Cliq}_S(W)$ for all subsets of S of which the elements mutually commute. In the following, for every $\mathcal{I} \in \mathsf{Cliq}_S(W)$ we will define a $\mathbb{C}[W] - \mathbb{C}[W]$ bimodule $\mathcal{H}_{\mathcal{I}}$. This is done in a way similar to the construction in section 4.1. Thereafter, a single $\mathbb{C}[W] - \mathbb{C}[W]$ bimodule $\mathcal{H}_W$ is constructed as a tensor product of these bimodules.

For $\mathcal{I} \in \mathsf{Cliq}_S(W)$, define the unbounded mapping $\Delta_{\mathcal{I}} : L^2(\mathcal{L}(W)) \to L^2(\mathcal{L}(W))$ by

$$\Delta_{\mathcal{I}}(\lambda_{\mathbf{v}}) = \psi_{S\setminus\mathcal{I}}(\mathbf{v})\lambda_{\mathbf{v}}. \tag{187}$$

Note here that $\psi_{S\setminus\mathcal{I}}$ is the weighted word length defined in section 7. For $\mathcal{I} \in \mathsf{Cliq}(S)$ we moreover define mappings $\Gamma_{\mathcal{I}} : \mathbb{C}[W] \to \mathbb{C}[W]$ by

$$\Gamma_{\mathcal{I}}(a, b) = \frac{1}{2}(\Delta_{\mathcal{I}}(b)^* a + b^* \Delta_{\mathcal{I}}(a) - \Delta_{\mathcal{I}}(b^* a)). \tag{188}$$

Denote $c_{00}(W)$ for the set of finitely supported functions on W. For $\mathcal{I} \in \mathsf{Cliq}_S(W)$ we denote the (possible degenerate) inner product $\langle \cdot | \cdot \rangle_{\mathcal{I}}$ on $\mathcal{H}_{\mathcal{I},0} := \mathbb{C}[W] \otimes c_{00}(W)$ by

$$\langle a \otimes \xi, b \otimes \eta \rangle_{\mathcal{I}} = \langle \Gamma_{\mathcal{I}}(a, b)\xi, \eta \rangle. \tag{189}$$

We then define the Hilbert Space $\mathcal{H}_{\mathcal{I}}$ by quotienting out the degenerate part, and taking the completion. The element $a \otimes \xi \in \mathcal{H}_{\mathcal{I}}$ we will denote by $a \otimes_{\mathcal{I}} \xi$ to empathize what Hilbert space we use. Similar as in section 4.1, the space $\mathcal{H}_{\mathcal{I}}$ has a $\mathbb{C}[W] - \mathbb{C}[W]$ module structure given by

$$x \cdot (a \otimes_{\mathcal{I}} \xi) = xa \otimes_{\mathcal{I}} \xi - x \otimes_{\mathcal{I}} a\xi \tag{190}$$

$$(a \otimes_{\mathcal{I}} \xi) \cdot y = a \otimes_{\mathcal{I}} \xi y. \tag{191}$$

Now, if we have two $\mathbb{C}[W]$ bimodules H_1 and H_2, then we can construct a $\mathbb{C}[W]$ bimodule H as follows. We set $H = H_1 \overline{\otimes} H_2$ as Hilbert space, and denote the element $a \otimes b \in H$ as $a \otimes_W b$. The bimodule action on H is defined by

$$\lambda_g \cdot (a \otimes_W b) = \lambda_g a \otimes_W \lambda_g b \tag{192}$$

$$(a \otimes_W b) \cdot \lambda_g = a\lambda_g \otimes_W b\lambda_g \tag{193}$$

which makes H a $\mathbb{C}[W]$ bimodule. We can apply this construction to the bimodules $\mathcal{H}_{\mathcal{I}}$ for $\mathcal{I} \in \mathsf{Cliq}_S(W)$ to obtain a single bimodule $\mathcal{H}_W$ given by

$$\mathcal{H}_W = \bigotimes_{\mathcal{I} \in \mathsf{Cliq}_S(W)} \mathcal{H}_{\mathcal{I}}. \tag{194}$$

10.2. Conditions on coefficients that imply quasi-containment.

We shall now turn to showing results that will give quasi-containment of a certain submodule of $\mathcal{H}_W$ in the coarse bimodule. For this we first introduce some definitions and prove some lemmas here that will make it easier to prove this. In the next subsection we finish the argument.

Definition 10.1 (Coefficients). *Let Γ be a discrete group, H be a $\mathbb{C}[\Gamma]$ bimodule and let $\xi, \eta \in H$. If a map exists $T_{\xi,\eta} : \mathbb{C}[\Gamma] \to \mathbb{C}[\Gamma]$ such that $\tau(T_{\xi,\eta}(x)y) = \langle x\xi y, \eta \rangle$ for all $x, y \in \mathbb{C}[\Gamma]$, then this map is actually unique. Indeed, if $\widetilde{T_{\xi,\eta}}$ is another map with this property then $\tau((T_{\xi,\eta} - \widetilde{T_{\xi,\eta}})(x)y) = 0$ for all $x, y \in \mathbb{C}[\Gamma]$ so that $\widetilde{T_{\xi,\eta}} = T_{\xi,\eta}$. This unique map (if it exists) is called the coefficient of H at ξ, η. We* *shall simply write T_ξ for $T_{\xi,\xi}$. We will say that the coefficient $T_{\xi,\eta}$ is in $\mathcal{S}_p$ for some $p \in [1,\infty]$ if the map extends to a bounded operator $T_{\xi,\eta} : \ell^2(\Gamma) \to \ell^2(\Gamma)$ that is moreover in $\mathcal{S}_p$.*

We have the following proposition that follows from lemma 4.1 that gives us quasi-containment of H under some condition on the coefficients.

Proposition 10.2 (Quasi-containment)**.** *Let H be a bimodule over $\mathbb{C}[\Gamma]$. If there exists a dense subset $H_0 \subset H$ such that for any $\xi \in H_0$ the coefficient $T_{\xi,\xi}$ is in $\mathcal{S}_2$ then H is quasi-contained in the coarse bimodule $\ell^2(\Gamma) \otimes \ell^2(\Gamma)$.*

A subset $H_{00} \subseteq H$ (not necessarily being a subspace) is called *cyclic* if $H_0 := \mathrm{Span}\, \mathbb{C}[\Gamma] H_0 \mathbb{C}[\Gamma]$ is dense in H. The following lemma tells us that we can reduce to checking the property only for the coefficient for $\xi, \eta \in H_{00}$.

Lemma 10.3 (Reduction to cyclic subset)**.** *Suppose that $H_{00} \subseteq H$ is a cyclic subset whose coefficients $T_{\xi,\eta}$ for $\xi, \eta \in H_{00}$ are in $\mathcal{S}_2$. Then the coefficients $T_{\xi,\eta}$ for $\xi, \eta \in H_0 := \mathrm{span}\mathbb{C}[\Gamma] H_{00} \mathbb{C}[\Gamma]$ are in $\mathcal{S}_2$. Consequently, by proposition 10.2, H is quasi-contained in the coarse bimodule $\ell^2(\Gamma) \otimes \ell^2(\Gamma)$.*

Proof. Let $\xi' = \lambda_g \xi \lambda_h$ and $\eta' = \lambda_s \eta \lambda_t$ for some $\lambda_g, \lambda_h, \lambda_s, \lambda_t \in \Gamma$ and $\xi, \eta \in H_{00}$. We have that

$$\tau(T_{\xi',\eta'}(x)y) = \langle x\xi' y, \eta \rangle \tag{195}$$
$$= \langle x\lambda_g \xi \lambda_h y, \lambda_s \eta \lambda_t \rangle \tag{196}$$
$$= \langle \lambda_{s^{-1}} x \lambda_g \xi \lambda_h y \lambda_{t^{-1}}, \eta \rangle \tag{197}$$
$$= \tau(T_{\xi,\eta}(\lambda_{s^{-1}} x \lambda_g) \lambda_h y \lambda_{t^{-1}}) \tag{198}$$
$$= \tau(\lambda_{t^{-1}} T_{\xi,\eta}(\lambda_{s^{-1}} x \lambda_g) \lambda_h y) \tag{199}$$

this shows that $T_{\xi',\eta'}(x) = \lambda_{t^{-1}} T_{\xi,\eta}(\lambda_{s^{-1}} x \lambda_g) \lambda_h$. Therefore we have that $T_{\xi',\eta'}$ is in $\mathcal{S}_2$. It follows that the coefficients are in $\mathcal{S}_2$ for all elements in $\mathbb{C}[\Gamma] H_{00} \mathbb{C}[\Gamma]$, and hence also for all element in H_0. From this it follows by Proposition 10.2 that H is quasi-contained in the coarse bimodule $\ell_2(\Gamma) \otimes \ell_2(\Gamma)$. ☐

10.3. **Some coefficients for bimodule $\mathcal{H}_W$ are finite rank.** We shall now continue to work with the bimodule $\mathcal{H}_W$ that we constructed for a right-angled word-hyperbolic Coxeter group W. We shall show that certain $\xi, \eta \in \mathcal{H}_W$, the coefficient of $\mathcal{H}_W$ at ξ, η is finite rank. These coefficient are then $\mathcal{S}_p$ for $p \in [1, \infty]$.

10.3.1. *Coefficients for a subset.* Let us denote $\mathcal{H}_{00} \subseteq \mathcal{H}_W$ for the sets of all the vectors

$$\xi_{\mathbf{v}} := (\lambda_{\mathbf{v}} \otimes \delta_e) \otimes_W ... \otimes_W (\lambda_{\mathbf{v}} \otimes \delta_e) \tag{200}$$

with $\mathbf{v} \in W$. For $\xi_{\mathbf{u}}, \xi_{\mathbf{w}} \in \mathcal{H}_{00}$ we now inspect the coefficient $T_{\xi_{\mathbf{u}},\xi_{\mathbf{w}}}$. We have that

$$\tau(T_{\xi_{\mathbf{w}},\xi_{\mathbf{u}}}(\lambda_{\mathbf{v}})y) = \langle \lambda_{\mathbf{v}} \cdot \xi_{\mathbf{w}} \cdot y, \xi_{\mathbf{u}} \rangle \tag{201}$$
$$= \prod_{\mathcal{I} \in \mathsf{Cliq}_S(W)} \langle \lambda_{\mathbf{v}} \cdot (\lambda_{\mathbf{w}} \otimes_{\mathcal{I}} \delta_e) \cdot y, \lambda_{\mathbf{u}} \otimes_{\mathcal{I}} \delta_e \rangle_{\mathcal{I}} \tag{202}$$
$$= \prod_{\mathcal{I} \in \mathsf{Cliq}_S(W)} \langle \Psi^{\Delta_{\mathcal{I}}}_{\lambda_{\mathbf{u}^{-1}}, \lambda_{\mathbf{w}}}(\lambda_{\mathbf{v}}) \delta_e y, \delta_e \rangle \tag{203}$$
$$= \prod_{\mathcal{I} \in \mathsf{Cliq}_S(W)} \gamma^{\psi_{S \setminus \mathcal{I}}}_{\mathbf{u}^{-1},\mathbf{w}}(\mathbf{v}) \langle \lambda_{\mathbf{u}^{-1}\mathbf{v}\mathbf{w}} \delta_e y, \delta_e \rangle. \tag{204}$$

Now, we define the function

$$\widetilde{\gamma}_{\mathbf{u},\mathbf{w}}(\mathbf{v}) = \prod_{\mathcal{I} \in \mathsf{Cliq}_S(W)} \gamma^{\psi_{S \setminus \mathcal{I}}}_{\mathbf{u},\mathbf{w}}(\mathbf{v}). \tag{205}$$

Then, if $\widetilde{\gamma}_{\mathbf{u}^{-1},\mathbf{w}}(\mathbf{v}) = 0$ we have that $\tau(T_{\xi_{\mathbf{w}},\xi_{\mathbf{u}}}(\lambda_{\mathbf{v}})y) = 0$ for all $y \in \mathbb{C}[W]$. Hence we have $T_{\xi_{\mathbf{w}},\xi_{\mathbf{u}}}(\lambda_{\mathbf{v}}) = 0$ in this case. We thus have that $T_{\xi_{\mathbf{w}},\xi_{\mathbf{u}}}$ is finite rank whenever $\widetilde{\gamma}_{\mathbf{u}^{-1},\mathbf{w}}$ has finite support. In lemma 10.5 we shall show that the function $\widetilde{\gamma}_{\mathbf{u},\mathbf{w}}$ is actually finite rank for all $\mathbf{u}, \mathbf{w} \in W$ so that we obtain

Corollary 10.4. *Let W be a right-angled, word hyperbolic Coxeter group. Consider the subset $\mathcal{H}_{00} \subseteq \mathcal{H}_W$ defined above. For $\xi, \eta \in \mathcal{H}_{00}$ we have that the coefficient $T_{\xi,\eta}$ is finite rank.*

We shall now turn to prove lemma 10.5 from which this corollary follows.

10.3.2. *Proving the product from eq.* (205) *is finite rank.* In order to prove lemma 10.5 we shall introduce some notation here that we will use. A tuple $(w_1, ..., w_k)$ with $w_i \in S$ we will call *reduced* if the expression $w_1....w_k$ is reduced. Furthermore, we will call the tuple *semi-reduced* whenever $|w_1....w_k| + |\{l : w_l = e\}| = k$. We will say that a pair (i, j) with $i < j$ *collapses* for a tuple $(w_1, ..., w_k)$ whenever $w_i = w_j \neq e$ and the elements $\{w_l : i \leq l \leq j\}$ pair-wise commute. In that case we will call the tuple $(w_1, ..., w_{i-1}, e, w_{i+1}, ..., w_{j-1}, e, w_{j+1}, ..., w_k)$ the *tuple obtained from* $(w_1, .., w_k)$ *by collapsing on the pair* (i, j). We note that the word $w_1....w_k$ corresponding to $(w_1, .., w_k)$ is in fact the same as the word $w_1...w_{i-1}ew_{i+1}...w_{j-1}ew_{j+1}...w_k$ corresponding to the collapsed tuple. The notation that we introduced here is convenient because it keeps indices aligned correctly. We also note that a tuple $(w_1, ..., w_k)$ is semi-reduced if and only if we cannot collapse on any pair (i, j). Hence, for a general tuple we can obtain a semi-reduced tuple by subsequently collapsing on pairs $(i_1, j_1), ..., (i_q, j_q)$.

We will now prove the lemma.

Lemma 10.5. *For a right-angled word hyperbolic Coxeter group* W*, for* $\mathbf{u}, \mathbf{w} \in W$ *the function* $\widetilde{\gamma}_{\mathbf{u},\mathbf{w}} : W \to \mathbb{R}$ *given by*

$$\widetilde{\gamma}_{\mathbf{u},\mathbf{w}}(\mathbf{v}) = \prod_{\mathcal{I} \in \mathsf{Cliq}(S)} \gamma_{\mathbf{u},\mathbf{w}}^{\psi_{S \setminus \mathcal{I}}}(\mathbf{v})$$

has finite support.

Proof. Let $\mathbf{u} = u_1....u_{n_1}, \mathbf{v} = v_1...v_{n_2}, \mathbf{w} = w_1....w_{n_3} \in W$ written in reduced expression. We will moreover assume that $|\mathbf{v}| > |\mathbf{u}| + |\mathbf{w}| + |S| + 2$. We will show that for such words we have $\widetilde{\gamma}_{\mathbf{u},\mathbf{w}}(\mathbf{v}) = 0$. This then shows that $\widetilde{\gamma}_{\mathbf{u},\mathbf{w}}$ has finite support.

Let $(u'_1, ..., u'_{n_1}, v'_1, ..., v'_{n_2})$ be the semi-reduced tuple obtained by subsequently collapsing the tuple $(u_1, .., u_{n_1}, v_1, .., v_{n_2})$ on pairs $(i'_1, j'_1), ..., (i'_{q_1}, j'_{q_1})$. Then we must have $i'_l \leq n_1$ and $j'_l > n_1$ since the expressions for $\mathbf{u}$ and $\mathbf{v}$ were reduced. Also $|\mathbf{uv}| = |\mathbf{u}| + |\mathbf{v}| - 2q_1$ and more generally for a weight $\mathbf{x} \in [0, \infty)^{|S|}$ we have

$$\psi_{\mathbf{x}}(\mathbf{uv}) = \psi_{\mathbf{x}}(\mathbf{u}) + \psi_{\mathbf{x}}(\mathbf{v}) - 2\sum_{l=1}^{q_1} \psi_{\mathbf{x}}(u_{i'_l}).$$

Likewise let $(v''_1, ..., v''_{n_2}, w''_1, ..., w''_{n_3})$ be the semi-reduced tuple obtained by subsequently collapsing the tuple $(v_1, .., v_{n_2}, w_1, .., w_{n_3})$ on pairs $(i''_1, j''_1), ..., (i''_{q_2}, j''_{q_2})$. Then we must have $i''_l \leq n_2$ and $j''_l > n_2$ since the expressions for $\mathbf{v}$ and $\mathbf{w}$ were reduced. Also $|\mathbf{vw}| = |\mathbf{v}| + |\mathbf{w}| - 2q_2$ and more generally for a weight $\mathbf{x} \in [0, \infty)^{|S|}$ we have

$$\psi_{\mathbf{x}}(\mathbf{vw}) = \psi_{\mathbf{x}}(\mathbf{v}) + \psi_{\mathbf{x}}(\mathbf{w}) - 2\sum_{l=1}^{q_2} \psi_{\mathbf{x}}(w_{j''_l - n_2}).$$

Let us denote $\mathcal{J} = \{v_j : j \in \{1, ..., n_2\} \setminus (\{j'_1 - n_1, ..., j'_{q_1} - n_1\} \cup \{i''_1, ..., i''_{q_2}\})\}$. Now since $n_2 = |\mathbf{v}| > |\mathbf{u}| + |\mathbf{w}| + |S| + 2 \geq q_1 + q_2 + |S| + 2$ we have that $|\mathcal{J}| \geq |S| + 2$. Hence, there are two elements $g_1, g_2 \in \mathcal{J}$ that do not mutually commute. Now, if $s_1, s_2 \in S$ commute with all elements in $\mathcal{J}$, then s_1, s_2 commute with both g_1 and g_2 so that by the word hyperbolicity of W we must have that also s_1 commutes with s_2. We now let $\mathcal{I}_0 \subseteq S$ be the set of all generators that commute with all elements in $\mathcal{J}$. Then by what we just mentioned we have that the elements in $\mathcal{I}_0$ pair-wise commute, i.e. $\mathcal{I}_0 \in \mathsf{Cliq}_S(W)$.

Now, for $i = 1, .., n_1$ let us set $\widetilde{u_i} = u'_i$ and for $i = 1, ..., n_3$ set $\widetilde{w_i} = w''_i$. Furthermore, for $i = 1, .., n_2$ set $\widetilde{v_i} = e$ whenever either $v'_i = v_i$ or $v''_i = e$ but not both, and set $\widetilde{v_i} = v_i$ otherwise. Let us also denote $\widetilde{\mathbf{u}} = \widetilde{u_1}..\widetilde{u_{n_2}}$, $\widetilde{\mathbf{v}} = \widetilde{v_1}...\widetilde{v_{n_2}}$ and $\widetilde{\mathbf{w}} = \widetilde{w_1}...\widetilde{w_{n_3}}$. We claim that then we have that $\widetilde{\mathbf{u}}\widetilde{\mathbf{v}}\widetilde{\mathbf{w}} = \mathbf{uvw}$. Namely, we have that $\mathbf{uvw} = \mathbf{u}v''_1...v''_{n_2}w''_1...w''_{n_3}$. Now we can collapse $(u_1, ..., u_{n_1}, v''_1, ..., v''_{n_2}, w''_1, ..., w''_{n_3})$ subsequently on the pairs (i'_l, j'_l) for $l = 1, .., q_1$ except when $v''_{j'_l - n_1} \neq v_{j'_l - n_1}$ for some $1 \leq l \leq q_1$, in which case $v_{j'_l - n_1} = e$. If this is the case then $j'_l - n_1 = i''_{k_l}$ for some $k_l \in \{1, ..., q_2\}$. In particular it follows that in this case $u_{i'_l} = v_{j'_l - n_1} = v_{i''_{k_l}} = w_{j''_{k_l} - n_2}$ and that this element commutes with all elements in $\mathcal{J}$. Therefore $u_{i'_l} \in \mathcal{I}_0$. We can then simply interchange the elements at index i'_l (which is $u_{i'_l}$) and the element at index j'_l (which is $v''_{j'_l - n_1} = e$). This manipulation does not change the word, and allows us to continue collapsing on the remaining pairs. Once we are done collapsing on all pairs we have obtained the tuple $(\widetilde{u}_1, ..., \widetilde{u_{n_1}}, \widetilde{v}_1, ..\widetilde{v_{n_2}}, \widetilde{w}_1, ..., \widetilde{w_{n_3}})$. This thus shows us that $\mathbf{uvw} = \widetilde{\mathbf{u}}\widetilde{\mathbf{v}}\widetilde{\mathbf{w}}$. It also shows us that $\widetilde{v_{j'_l - n_1}} \in \{e\} \cup \mathcal{I}_0$ for $l = 1, ..., q_2$. Note that also by definition $\widetilde{u_{i'_l}} = e$ for $l = 1, ..., q_1$ and $\widetilde{w_{j''_l - n_2}} = e$ for $l = 1, ..., q_2$. Therefore we also have that $\psi_{S \setminus \mathcal{I}_0}(\widetilde{u_{i'_l}}) = \psi_{S \setminus \mathcal{I}_0}(e) = 0$ for $l = 1, .., q_1$ and likewise

$\psi_{S\setminus\mathcal{I}_0}(\widetilde{w_{j_l''-n_2}}) = 0$ for $l = 1,..,q_2$. Furthermore $\psi_{S\setminus\mathcal{I}_0}(\widetilde{v_{j_l'-n_1}}) = 0$ for $l = 1,..,q_1$ and $\psi_{S\setminus\mathcal{I}_0}(\widetilde{v_{i_l''}}) = 0$ for $l = 1,..,q_2$.

Also, if we can collapse $(\widetilde{u_1},...\widetilde{u_{n_1}},\widetilde{v_1},...\widetilde{v_{n_2}},\widetilde{w_1},...,\widetilde{w_{n_3}})$ on some pair (i,j) then we must have $i \leq n_1$ and $j > n_1 + n_2$. Indeed otherwise either $(u_1',...,u_{n_1}',v_1',...,v_{n_2}')$ or $(v_1'',..,v_{n_2}'',w_1',...,w_{n_3}'')$ is not semi-reduced, which is a contradiction. Now let $(i_1,j_1),..,(i_q,j_q)$ be the pairs on which we can subsequently collapse $(\widetilde{u_1},...\widetilde{u_{n_1}},\widetilde{v_1},...\widetilde{v_{n_2}},\widetilde{w_1},...,\widetilde{w_{n_3}})$ to obtain a semi-reduced tuple. Then we thus must have $i_l \leq n_1$ and $j_l > n_1+n_2$. This thus implies that for $l = 1,..,q$ we have that $\widetilde{u_{i_l}} = \widetilde{w_{j_l}}$ commutes with the elements from $\mathcal{J}$. Therefore we have $\{\widetilde{u_{i_l}} : l = 1,..,q\} = \{\widetilde{w_{i_l}} : l = 1,..,q\} \subseteq \mathcal{I}_0$.

Now, we have that

$$\psi_{S\setminus\mathcal{I}_0}(\mathbf{uvw}) = \psi_{S\setminus\mathcal{I}_0}(\mathbf{u}) + \psi_{S\setminus\mathcal{I}_0}(\mathbf{v}) + \psi_{S\setminus\mathcal{I}_0}(\mathbf{w}) \tag{206}$$

$$- 2\left[\sum_{l=1}^{q_1}\psi_{S\setminus\mathcal{I}_0}(u_{i_l'}) + \sum_{l=1}^{q_2}\psi_{S\setminus\mathcal{I}_0}(w_{i_l''-n_2}) + \sum_{l=1}^{q}\psi_{S\setminus\mathcal{I}_0}(\widetilde{u_{i_l}})\right] \tag{207}$$

$$= \psi_{S\setminus\mathcal{I}_0}(\mathbf{uv}) + \psi_{S\setminus\mathcal{I}_0}(\mathbf{vw}) - \psi_{S\setminus\mathcal{I}_0}(\mathbf{v}) + 2\sum_{l=1}^{q}\psi_{S\setminus\mathcal{I}_0}(\widetilde{u_{i_l}}) \tag{208}$$

$$= \psi_{S\setminus\mathcal{I}_0}(\mathbf{uv}) + \psi_{S\setminus\mathcal{I}_0}(\mathbf{vw}) - \psi_{S\setminus\mathcal{I}_0}(\mathbf{v}). \tag{209}$$

This shows that $\gamma_{\mathbf{u},\mathbf{w}}^{\psi_{S\setminus\mathcal{I}_0}}(\mathbf{v}) = 0$. Therefore, as $\mathcal{I}_0 \in \mathsf{Cliq}_S(W)$ we obtain that $\widetilde{\gamma}_{\mathbf{u},\mathbf{w}}(\mathbf{v}) = 0$. Now as this holds for every $\mathbf{v} \in W$ with $|\mathbf{v}| > |\mathbf{u}| + |\mathbf{w}| + |S| + 2$, we obtain that $\widetilde{\gamma}_{\mathbf{u},\mathbf{w}}$ has finite support. ☐

10.4. **Proving results using non-commutative Riesz transform.** Let Γ be a discrete group. We will let $\Delta : \ell_2(\Gamma) \to \ell_2(\Gamma) \otimes \ell_2(\Gamma)$ be the co-multiplication which is the linear extension of the map given for $g \in \Gamma$ by

$$\Delta(\lambda_g) = \lambda_g \otimes \lambda_g. \tag{210}$$

We note that this is indeed an isometry as we have

$$\langle \Delta(\lambda_g), \Delta(\lambda_r)\rangle = \langle \lambda_g \otimes \lambda, \lambda_r \otimes \lambda_r\rangle = \langle \lambda_g, \lambda_r\rangle\langle \lambda_g, \lambda_r\rangle = \langle \lambda_g, \lambda_r\rangle. \tag{211}$$

Let Γ be a discrete group and H be a $\mathbb{C}[\Gamma]$ bimodule. We shall call a partial isometry $V : \ell_2(\Gamma) \to H$ *almost bimodular* if for all $x, y \in \mathbb{C}[\Gamma]$ the map $\ell_2(\Gamma) \to H$ given by $\xi \mapsto xV(\xi)y - V(x\xi y)$ is compact.

The arguments from lemma 10.6 and proposition 10.7 were shown to the author by Martijn Caspers and Mateusz Wasilewski. The author thanks them for allowing him to present their proofs here.

Lemma 10.6. *Let H_1 and H_2 be bimodules over $\mathbb{C}[\Gamma]$. Let $S_1 : \ell^2(\Gamma) \to H_1$ and $S_2 : \ell^2(\Gamma) \to H_2$ be partial isometries. We define a map $S_1 * S_2 : \ell_2(\Gamma) \to H_1 \otimes_\Gamma H_2$ as $(S_1 * S_2) := (S_1 \otimes S_2)\Delta$. Then if for $i = 1, 2$ the map S_i is almost bimodular, and if furthermore $\ker(S_i) = \mathrm{Span}\{\lambda_g : g \in I_i\}$ for some subset $I_i \subseteq \Gamma$ then $S_1 * S_2$ is an almost bimodular partial isometry with $\ker(S_1 * S_2) = \mathrm{Span}(\ker(S_1) \cup \ker(S_2))$.*

Proof. We first show that $S_1 * S_2$ is a partial isometry. Let $a, b \in \ell_2(\Gamma)$ be elements in $(\ker(S_1) \cup \ker(S_2))^\perp$. We can then write $a = \sum_{g\in\Gamma} \alpha_g\lambda_g \in \ell_2(\Gamma)$ and $b = \sum_{g\in\Gamma}\beta_g\lambda_g \in \ell_2(\Gamma)$ for some $\alpha_g, \beta_g \in \mathbb{C}$ for $g \in \Gamma$ that satisfy $\alpha_g = \beta_g = 0$ when $g \in I_1 \cup I_2$. We shall check that $\langle S_1 * S_2(a), S_1 * S_2(b)\rangle = \langle a, b\rangle$. We have

$$\langle (S_1 * S_2)(a), (S_1 * S_2)(b)\rangle = \langle \sum_{g\in\Gamma}\alpha_g S_1(\lambda_g) \otimes S_2(\lambda_g), \sum_{r\in\Gamma}\beta_r S_1(\lambda_r) \otimes S_2(\lambda_r)\rangle \tag{212}$$

$$= \sum_{g\in\Gamma}\sum_{r\in\Gamma}\alpha_g\overline{\beta_r}\langle S_1(\lambda_g) \otimes S_2(\lambda_g), S_1(\lambda_r) \otimes S_2(\lambda_r)\rangle \tag{213}$$

$$= \sum_{g\in\Gamma}\sum_{r\in\Gamma}\alpha_g\overline{\beta_r}\langle S_1(\lambda_g), S_1(\lambda_r)\rangle\langle S_2(\lambda_g), S_2(\lambda_r)\rangle \tag{214}$$

$$= \sum_{g\in\Gamma}\sum_{r\in\Gamma}\alpha_g\overline{\beta_r}\langle \lambda_g, \lambda_r\rangle\langle \lambda_g, \lambda_r\rangle \tag{215}$$

$$= \sum_{g\in\Gamma}\alpha_g\overline{\beta_g} = \langle a, b\rangle. \tag{216}$$

Now, let $a \in \ker(S_1) \cup \ker(S_2)$. Then we can write $a = \sum_{g\in\Gamma}\alpha_g\lambda_g$ where the complex numbers α_g satisfy $\alpha_g = 0$ when $g \notin I_1 \cup I_2$. We moreover note that when $g \in I_1 \cup I_2$ then $S_1(\lambda_g) = 0$ or $S_2(\lambda_g) = 0$ so that $S_1(\lambda_g) \otimes S_2(\lambda_g) = 0$. This shows that $(S_1 * S_2)(a) = \sum_{g\in I_1\cup I_2}\alpha_g S_1(\lambda_g) \otimes S_2(\lambda_g) = 0$. This shows that $S_1 * S_2$ is a partial isometry with $\ker(S_1 * S_2) = \mathrm{Span}(\ker(S_1) \cup \ker(S_2))$. We now show that it also is almost bimodular. For this we first note that for $u, w \in \Gamma$ and for $i = 1, 2$ we have that $a \mapsto \lambda_u S_i(a)\lambda_w$

is a partial isometry with kernel $\mathrm{Span}\{\lambda_g : g \in I_i\}$. Furthermore the map $S_i^{u,w} : \ell_2(\Gamma) \to H_i$ given by $S_i^{u,w}(a) = S_i(\lambda_u a \lambda_w)$ is a partial isometry with $\ker S_i^{u,w} = \mathrm{Span}\{\lambda_g : g \in \lambda_u^{-1} I_i \lambda_w^{-1}\}$. We compute that

$$\begin{aligned}
\lambda_u(S_1 * S_2)(\lambda_v)\lambda_w &= \lambda_u(S_1(\lambda_v) \otimes S_2(\lambda_v))\lambda_w && (217)\\
&= \lambda_u(S_1(\lambda_v) \otimes S_2(\lambda_v))\lambda_w && (218)\\
&= (\lambda_u S_1(\lambda_v)\lambda_w \otimes \lambda_u S_2(\lambda_v)\lambda_w) && (219)\\
&= (\lambda_u S_1 \lambda_w * \lambda_u S_2 \lambda_w)(\lambda_v) && (220)
\end{aligned}$$

so that we obtain that $\lambda_u(S_1 * S_2)\lambda_w = (\lambda_u S_1 \lambda_w) * (\lambda_u S_2 \lambda_w)$. Furthermore, we calculate

$$\begin{aligned}
(S_1 * S_2)(\lambda_u \lambda_v \lambda_w) &= S_1(\lambda_{uvw}) \otimes S_2(\lambda_{uvw}) && (221)\\
&= S_1^{u,w}(\lambda_v) \otimes S_2^{u,w}(\lambda_v) && (222)\\
&= (S_1^{u,w} * S_2^{u,w})(\lambda_v) && (223)
\end{aligned}$$

and hence obtain that $(S_1 * S_2)(\lambda_u a \lambda_w) = (S_1^{u,w} * S_2^{u,w})(a)$ for $a \in \ell_2(\Gamma)$. This then gives us that for $a \in \ell_2(\Gamma)$ we have

$$\begin{aligned}
\lambda_u(S_1 * S_2)(a)\lambda_w - (S_1 * S_2)(\lambda_u a \lambda_w) &= (\lambda_u S_1 \lambda_w * \lambda_u S_2 \lambda_w)(a) - (S_1^{u,w} * S_2^{u,w})(a) && (224)\\
&= ((\lambda_u S_1 \lambda_w - S_1^{u,w}) * \lambda_u S_2 \lambda_w)(a) && (225)\\
&\quad - (S_1^{u,w} * (S_2^{u,w} - \lambda_u S_2 \lambda_w))(a). && (226)
\end{aligned}$$

Now since both S_1 and S_2 are almost bimodular we have that $\lambda_u S_1 \lambda_w - S_1^{u,w}$ and $S_2^{u,w} - \lambda_u S_2 \lambda_w$ are compact. We shall now show that when $K : \ell_2(\Gamma) \to H_1$ is compact and $T : \ell_2(\Gamma) \to H_2$ is bounded then $K * T$ and $T * K$ are compact. This will then show by eq. (224) that $S_1 * S_2$ is almost bimodular. We can find a sequence $\{F_n\}_{n \geq 1}$ of finite subsets of Γ that increases to Γ. For a finite set F, we denote P_F for the orthogonal projection on the subspace of $\ell_2(\Gamma)$ of functions with support in F. This projection is finite rank, as F is finite. Now, we note now that P_F is such that $\Delta \circ P_F = (P_F \otimes \mathrm{Id}_{\ell^2(\Gamma)}) \circ \Delta = (\mathrm{Id}_{\ell^2(\Gamma)} \circ P_F) \circ \Delta$. We now show that $K * T$ can be approximated in norm by finite rank operators. Namely, we have

$$\|(K * T)P_F - K * T\| = \|(KP_F - K) * T\| \leq \|KP_F - K\| \cdot \|T\|. \tag{227}$$

Now by compactness of K we have that $\lim_{n\to\infty} \|KP_{F_n} - K\| = 0$. Therefore also $\lim_{n\to\infty} \|(K*T)P_{F_n} - K * T\|$. Now, since the operators $(K * T)P_{F_n}$ for $n \geq 1$ are finite rank, we have that $K * T$ is compact. The argument that $T * K$ is compact is similar. Now, by what we stated before this gives that $S_1 * S_2$ is almost bimodular, which finishes the proof.

□

For a set $\mathcal{J} = \{v_1, ..., v_n\} \in \mathsf{Cliq}(S)$ we will denote the word $\lambda_{\mathcal{J}} := v_1....v_n$. Note that since the elements in $\mathcal{J}$ commute the order of the elements v_i in the expression does not matter, so that $\lambda_{\mathcal{J}}$ is well-defined. We shall moreover call a word of which all letters commute a *clique word*. It can now be seen that for $\mathcal{I} \in \mathsf{Cliq}(S)$ we have that $\ker(\Delta_{\mathcal{I}}) = \mathrm{Span}\{\lambda_{\mathcal{J}} : \mathcal{J} \subseteq \mathcal{I}\}$ and that this is a finite dimensional subspace of $\ell_2(\Gamma)$.

For $\mathcal{I} \in \mathsf{Cliq}(S)$ we now introduce the Riesz-transform $R_{\mathcal{I}}$ associated to $\Delta_{\mathcal{I}}$. This is linear map $R_{\mathcal{I}} : \ell_2(W) \to \mathcal{H}_{\mathcal{I}}$ defined on $\ker(\Delta_{\mathcal{I}})^\perp$ by

$$R_{\mathcal{I}}(\lambda_{\mathbf{v}}) = \frac{\lambda_{\mathbf{v}} \otimes \delta_e}{\sqrt{\psi_{S\setminus\mathcal{I}}(\mathbf{v})}} \tag{228}$$

and on $\ker(\Delta_{\mathcal{I}})$ as 0. We prove that the Riesz-transform is a partial isometry that is almost bimodular.

Proposition 10.7. *For $\mathcal{I} \in \mathsf{Cliq}(S)$ the Riesz-transform $R_{\mathcal{I}}$ is an almost bimodular partial isometry with* $\ker(R_{\mathcal{I}}) = \mathrm{Span}\{\lambda_{\mathcal{J}} : \mathcal{J} \subseteq \mathcal{I}\}$.

Proof. We first show that it is a partial isometry. Namely, for $\lambda_{\mathbf{u}}, \lambda_{\mathbf{w}}$ that are in $\ker(\Delta_{\mathcal{I}})^{\perp}$ we have

$$\langle R_{\mathcal{I}}(\lambda_{\mathbf{u}}), R_{\mathcal{I}}(\lambda_{\mathbf{w}})\rangle = \frac{\langle \lambda_{\mathbf{u}} \otimes \delta_e, \lambda_{\mathbf{w}} \otimes \delta_e\rangle_{\mathcal{I}}}{\sqrt{\psi_{S\setminus\mathcal{I}}(\mathbf{u})\psi_{S\setminus\mathcal{I}}(\mathbf{w})}} \tag{229}$$

$$= \frac{\langle \Gamma_{\mathcal{I}}(\lambda_{\mathbf{u}}, \lambda_{\mathbf{w}})\delta_e, \delta_e\rangle}{\sqrt{\psi_{S\setminus\mathcal{I}}(\mathbf{u})\psi_{S\setminus\mathcal{I}}(\mathbf{w})}} \tag{230}$$

$$= \frac{\langle (\psi_{S\setminus\mathcal{I}}(\mathbf{w}^{-1}) + \psi_{S\setminus\mathcal{I}}(\mathbf{u}) - \psi_{S\setminus\mathcal{I}}(\mathbf{w}^{-1}\mathbf{u}))\lambda_{\mathbf{w}^{-1}\mathbf{u}}\delta_e, \delta_e\rangle}{2\sqrt{\psi_{S\setminus\mathcal{I}}(\mathbf{u})\psi_{S\setminus\mathcal{I}}(\mathbf{w})}} \tag{231}$$

$$= \mathbb{1}(\mathbf{u} = \mathbf{w})\frac{\psi_{S\setminus\mathcal{I}}(\mathbf{u}^{-1}) + \psi_{S\setminus\mathcal{I}}(\mathbf{u}) - \psi_{S\setminus\mathcal{I}}(e)}{2\sqrt{\psi_{S\setminus\mathcal{I}}(\mathbf{u})\psi_{S\setminus\mathcal{I}}(\mathbf{u})}} \tag{232}$$

$$= \mathbb{1}(\mathbf{u} = \mathbf{w}) = \langle \lambda_{\mathbf{u}}, \lambda_{\mathbf{w}}\rangle. \tag{233}$$

Therefore we have for $a, b \in \ker(\Delta_{\mathcal{I}})^{\perp}$ that $\langle R_{\mathcal{I}}(a), R_{\mathcal{I}}(b)\rangle = \langle a, b\rangle$. It now follows that $R_{\mathcal{I}}$ is a partial isometry with $\ker(R_{\mathcal{I}}) = \ker(\Delta_{\mathcal{I}}) = \mathrm{Span}\{\lambda_{\mathcal{J}} : \mathcal{J} \subseteq \mathcal{I}\}$.

In order to proof that $R_{\mathcal{I}}$ is almost bimodular we use the result [12, Theorem 5.12]. To use this result we need $\Delta_{\mathcal{I}}$ to be filtered and have subexponential growth. This is true when W is infinite, which follows from section 5.3. Moreover, when W is finite we have that $\ell_2(W)$ is finite dimensional and therefore $R_{\mathcal{I}}$ is almost bimodular in this case as well. In either case we thus obtain that $R_{\mathcal{I}}$ is almost bimodular.

□

Proposition 10.8. *Let $W = \langle S|M\rangle$ be a right-angled word-hyperbolic Coxeter group. There exists a u.c.p. map $\theta : C_r^*(W) \otimes_{\min} C_r^*(W)^{\mathrm{op}} \to B(\ell_2(W))$ such that for all $a, b \in C_r^*(W)$ we have that $\theta(a \otimes b^{\mathrm{op}}) - ab^{\mathrm{op}}$ is compact.*

Proof. By proposition 10.7 we have that for $\mathcal{I} \in \mathsf{Cliq}(S)$ the maps $R_{\mathcal{I}}$ are almost bimodular partial isometries, whose kernel is given by $\ker(R_{\mathcal{I}}) = \ker(\Delta_{\mathcal{I}}) = \mathrm{Span}\{\lambda_{\mathcal{J}} : \mathcal{J} \subseteq \mathcal{I}\}$. Therefore, by apply lemma 10.6 multiple times we have that the mapping $R : \ell_2(W) \to \mathcal{H}_W$ defined by

$$R = *_{\mathcal{I}\in\mathsf{Cliq}(S)} R_{\mathcal{I}} \tag{234}$$

is a partial isometry that is almost bimodular. Moreover, by explicit examination we see that the finite dimensional kernel of (234) is given by the linear span of $\{\lambda_{\mathcal{I}}, \mathcal{I} \in \mathsf{Cliq}(S)\}$. Furthermore, we have

$$R(\lambda_{\mathbf{v}}) := \otimes_{\mathcal{I}\in\mathsf{Cliq}(S)} R_{\mathcal{I}}(\lambda_{\mathbf{v}}) = \left(\prod_{\mathcal{I}\in\mathsf{Cliq}(S)} \psi_{S\setminus\mathcal{I}}(\mathbf{v})^{-\frac{1}{2}}\right)(\lambda_{\mathbf{v}} \otimes \delta_e) \otimes_W \ldots \otimes_W (\lambda_{\mathbf{v}} \otimes \delta_e) \in \mathcal{H}_{00} \tag{235}$$

for every $\mathbf{v} \in W$ not being a clique word. Let $K \subseteq \ell_2(W)$ be the closed linear span of $\{\lambda_{\mathcal{I}}, \mathcal{I} \in \mathsf{Cliq}(S)\}$ which is a finite dimensional subspace of $\ell_2(W)$. Let $p_K : \ell_2(W) \to K$ be the orthogonal projection onto K. Then p_K is finite rank and hence $1 - p_K$ is almost bimodular. Since p_K is the projection onto the kernel of R we have that $R^*R = 1 - p_K$ is Fredholm.

Let $L \subseteq \mathcal{H}$ be the smallest $\mathbb{C}[W]$-$\mathbb{C}[W]$ bimodule containing the range of R. Note that we have $\mathrm{Ran}(R) \subseteq L \subseteq \overline{\mathcal{H}_0}^{\|\cdot\|_2}$ where $\mathcal{H}_0 := \mathbb{C}[W]\mathcal{H}_{00}\mathbb{C}[W]$. Now, for every vectors ξ and η of the form (235) we have that the coefficients $T_{\xi,\eta}$ is Hilbert-Schmidt, see corollary 10.4 and lemma 10.3. Therefore $T_{\xi,\eta}$ is Hilbert-Schmidt for every $\xi, \eta \in L$ by lemma 10.3. It follows from proposition 10.2 that L is quasi contained in the coarse bimodule. We have now obtained that $R : \ell_2(W) \to L$ satisfies the assumptions of [12, Proposition 5.2] and the same proposition concludes the theorem. □

Corollary 10.9. *Let W be a right-angled word-hyperbolic Coxeter group. Then $\mathcal{L}(W)$ has the AO$^+$ property and is strongly solid.*

Proof. The C*-algebra $C_r^*(W) \subseteq \mathcal{L}(W)$ is a σ-weakly dense subalgebra. Now by theorem 6.1 we have that $C_r^*(W)$ is locally reflexive. Now the property from proposition 10.8 then gives us that $\mathcal{L}(W)$ has the AO$^+$ property, by its definition. As $\mathcal{L}(W)$ has W*-CBAP by theorem 6.1, and as it has a separable predual, we then obtain from theorem 4.3 that $\mathcal{L}(W)$ is strongly solid. □

11. Discussion and conclusions

In this thesis we have for quantum Markov semi-groups studied the gradient-$\mathcal{S}_p$ property from [9, 12]. We did this with the goal of obtaining new examples of von Neumann algebras that have the Akemann-Ostrand property (AO$^+$) and are strongly solid. Specifically we have studied the gradient-$\mathcal{S}_p$ property for certain quantum Markov semi-groups on the group von Neumann algebras $\mathcal{L}(W)$ for given Coxeter group W. We had moreover extended our study to Hecke algebras $\mathcal{N}_q(W)$. In certain cases we were able to obtain a gradient-$\mathcal{S}_p$ quantum Markov semi-group, which then gave us (AO$^+$) and strong solidity for the von Neumann algebra. Also, for certain Coxeter groups W we were able to obtain these properties for $\mathcal{L}(W)$ using a slightly different method. We will in this section restate the precise results that we obtained. This we will do in section 11.1, where we state and discuss our results for each section separately. Thereafter, in section 11.2 we shall moreover discuss possible directions for future research.

11.1. Summarizing results.

We shall summarize our results for each section.

In section 3 we stated the definition of the gradient-$\mathcal{S}_p$ property and we proved lemma 3.2 that shows that in order to prove the gradient-$\mathcal{S}_p$ property it is enough to check the defining condition only for pairs of elements in a self-adjoint generating set $\mathcal{A}_0$.

In section 4 we discussed the results that we want to obtain using gradient-$\mathcal{S}_p$ quantum Markov semi-groups. We moreover restated conditions from [12] that our quantum Markov semi-group must satisfy to obtain these results.

In section 5 we considered semi-groups on group von Neumann algebras that are built using a proper, conditionally negative function ψ. For these semi-groups we gave a specific condition on the function ψ that ensures that the semi-group is gradient-$\mathcal{S}_p$.

In section 6 we specifically considered group von Neumann algebras of Coxeter groups. We looked at the quantum Markov semi-groups associated to the standard word length and studied when this semi-group is gradient-$\mathcal{S}_p$. We obtained the following result.

Theorem 11.1. *Let $W = \langle S|M\rangle$ be a Coxeter system. We denote $E_0 = \{(i,j) : M(s_i,s_j) \in 2\mathbb{N}\}$ and $E_1 = \{(i,j) : M(s_i,s_j) \in 2\mathbb{N}+1\}$. Suppose W is such that (S,E_1) is a forest, and that for every connected component C of (S,E_1) there is at most one edge $\{t,r\} \in E_0$ with $t \in C$ and $r \notin C$, and no edge $\{t,t'\} \in E_0$ with $t,t' \in C$. Then the quantum Markov semi-group on $\mathcal{L}(W)$ associated to standard word length is gradient-$\mathcal{S}_p$ for all $p \in [1,\infty]$, and $\mathcal{L}(W)$ has the Akemann-Ostrand property (AO$^+$) and is strongly solid.*

We also obtained results on when the semi-group is not gradient-$\mathcal{S}_p$. These results combined give an almost complete characterization of the kind of Coxeter groups for which this semi-group is gradient $\mathcal{S}_p$ for some $p \in [1,\infty]$ (or equivalently for all $p \in [1,\infty]$). If we only consider Coxeter groups $W = \langle S|M\rangle$ for which no two generators $s,t \in S$ commute, then our characterization is actually complete. That is, we have obtained the following result.

Theorem 11.2. *Let $W = \langle S|M\rangle$ be a Coxeter system such that $M(s,t) \neq 2$ for all $s,t \in S$. We denote $E_0 = \{(i,j) : M(s_i,s_j) \in 2\mathbb{N}\}$ and $E_1 = \{(i,j) : M(s_i,s_j) \in 2\mathbb{N}+1\}$ and fix $p \in [1,\infty]$. We have that the semi-group on $\mathcal{L}(W)$ associated to the word length ψ_S is gradient-$\mathcal{S}_p$ if and only if (S,E_1) is a forest, and for every connected component C of (S,E_1) there is at most one edge $\{t,r\} \in E_0$ with $t \in C$ and $r \notin C$, and no edge $\{t,t'\} \in E_0$ with $t,t' \in C$.*

We emphasize that this does not give a classification of Coxeter groups for which $\mathcal{L}(W)$ is strongly solid. Indeed, from the gradient-$\mathcal{S}_2$ property we obtain strong solidity, but the fact that the semi-group is not gradient-$\mathcal{S}_2$ does not mean that $\mathcal{L}(W)$ is not strongly solid.

In section 7 we considered semi-groups on $\mathcal{L}(W)$ that are associated to other conditionally negative functions ψ, namely weighted word lengths. Here, we were able to construct gradient-$\mathcal{S}_p$ quantum Markov semi-groups on $\mathcal{L}(W)$ for certain right-angled Coxeter groups W. More specifically we assumed that the right-angled Coxeter group W is such that the elements in

$$S_0 := \{r \in S : \exists s,t \in S : M(r,s) = M(r,t) = 2 \text{ and } M(s,t) = \infty\} \tag{236}$$

mutually commute. The semi-group that we constructed was then actually the semi-group associated to the proper, conditionally negative function $\psi_{S\setminus S_0}$. We then showed that this semi-group is $\mathcal{S}_p$ for all $p \in [1,\infty]$.

In section 8 we have looked at the Hecke algebras $\mathcal{N}_q(W)$, and we tried to generalize our results from section 6 and section 7 to these algebras. We obtained in section 8.3.1 that the construction from section 7 for $p = 2$ actually applies to general Hecke algebras $\mathcal{N}_q(W)$ as well. We state the result here.

Theorem 11.3. *Let W be a right-angled Coxeter group for which the elements in*

$$S_0 := \{r \in S : \exists s, t \in S : M(r,s) = M(r,t) = 2 \text{ and } M(s,t) = \infty\} \tag{237}$$

all mutually commute. We then denote $\mathcal{I} = S \backslash S_0$, and we let $q = (q_s)_{s \in S}$ be an arbitrary Hecke tuple. We obtain that the function $\psi_{\mathcal{I}}$ induces a gradient-$\mathcal{S}_2$ quantum Markov semi-group on $\mathcal{N}_q(W)$. Furthermore we find that $\mathcal{N}_q(W)$ has the Akemann-Ostrand property (AO^+) and is strongly solid.

In section 8.3.2 we look at the Coxeter groups W for which the word length ψ_S induces a gradient-$\mathcal{S}_2$ quantum Markov semi-group on $\mathcal{L}(W)$. We do not know in those cases whether the word length ψ_S also induces a quantum Markov semi-group on the Hecke algebras $\mathcal{N}_q(W)$. However, we obtain for those cases that, for a Hecke tuple $q = (q_s)_{s \in S}$, if ψ_S induces a quantum Markov semi-group on $\mathcal{N}_q(W)$, then this semi-group is moreover gradient-$\mathcal{S}_2$. That is, we have

Theorem 11.4. *Let W be a Coxeter group, and let $q = (q_s)_{s \in S}$ be a Hecke tuple. Suppose the word length ψ_S induces a gradient-$\mathcal{S}_2$ quantum Markov semi-group on $\mathcal{L}(W)$. Furthermore, suppose that the word length ψ_S induces a quantum Markov semi-group on $\mathcal{N}_q(W)$. Then this semi-group on $\mathcal{N}_q(W)$ is also gradient-$\mathcal{S}_2$ and we obtain that the Hecke-algebra $\mathcal{N}_q(W)$ has the Akemann-Ostrand property (AO^+).*

In section 9 we looked at group von Neumann algebras for generally groups that have the Haagerup property. We aimed to relate the gradient-$\mathcal{S}_p$ of a semi-group $(e^{-\Delta_\psi})_{t \geq 0}$ associated to a proper, conditionally negative function ψ to the gradient-$\mathcal{S}_q$ property of a semi-group that is generated by a root Δ_ψ^α of the positive generator Δ_ψ. The conditions on the function ψ that we had to impose in order to get such results turned out to be quite restrictive. This result may therefore not be very useful.

In section 10, for a right-angled word-hyperbolic Coxeter group W, we constructed certain bimodules similar to [12]. These we used to construct a single bimodule $\mathcal{H}_W$. For this bimodule we showed that certain coefficients are finite rank, which then gave us the quasi-containment of certain bimodules. Using the non-commutative Riesz-transform we obtained our results for $\mathcal{L}(W)$, that is, we obtained

Theorem 11.5. *Let W be a right-angled word hyperbolic Coxeter group. Then $\mathcal{L}(W)$ has the Akemann-Ostrand property (AO^+) and is strongly solid.*

We note that the result from theorem 11.5 was already known in [36] for general hyperbolic groups. However, the proof that we gave uses a completely different method.

11.2. **Directions for future research.** We finish this thesis by stating questions that remain open, and discussing directions for future research. There are the following two main classification problems.

11.2.1. *Classification of Coxeter groups for which $\mathcal{L}(W)$ is strongly solid.* We have obtained results that say that $\mathcal{L}(W)$ is strongly solid for certain kinds of Coxeter groups. Ideally we would want to obtain a full classification for what kind of Coxeter groups this holds. To obtain new types of Coxeter groups for which $\mathcal{L}(W)$ is strongly solid, one could try to prove that the semi-group associated to the standard word length is gradient-$\mathcal{S}_p$ for some Coxeter groups for which this is not yet known. However, this can only give relatively few new examples. Perhaps one should adapt the semi-group that we are using in order to get more new results, or maybe one should use an entirely different method instead. Furthermore, in order to obtain a classification one should also study for what Coxeter groups $\mathcal{L}(W)$ is not strongly solid.

11.2.2. *Classification what Hecke algebras $\mathcal{N}_q(W)$ are strongly solid.* We have tried to extend our results for group von Neumann algebras to Hecke algebras. However in section 8.3.2 we do not know for what Hecke tuples $q = (q_s)_{s \in S}$ the function ψ_S will actually induce a quantum Markov semi-group on $\mathcal{N}_q(W)$. If we could, as in the right-angled case obtain that this is the case, then we obtained new examples of Hecke algebras that are strongly solid. Furthermore, if more results on the classifications of Coxeter groups W for which $\mathcal{L}(W)$ is strongly solid are obtained, one may also try to extend these to Hecke algebras as well.

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