Modelling the Libor transition: Implementing and extending the generalized forward market model

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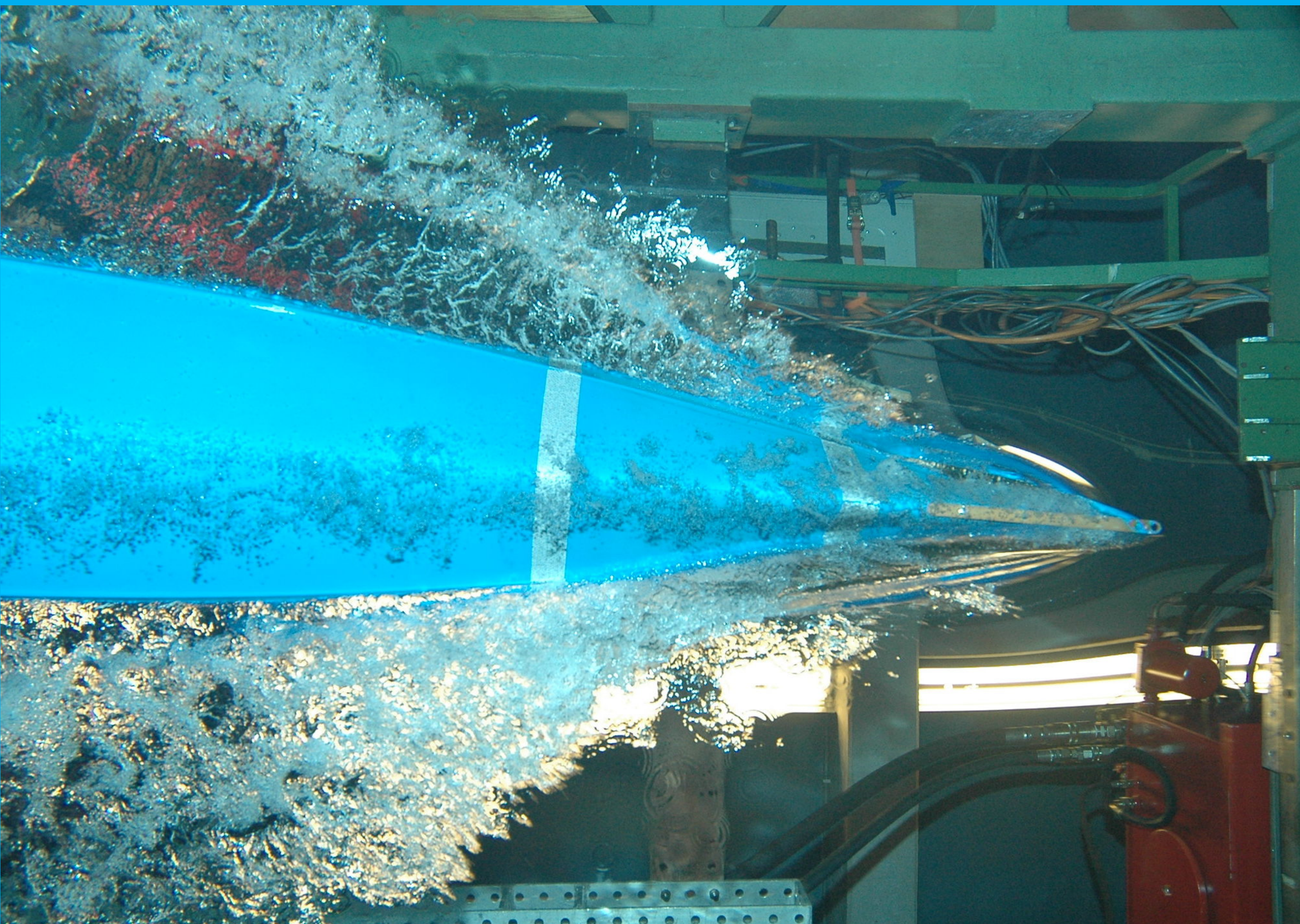

Modelling the Libor transition: Implementing and extending the generalized forward market model

by

Tim Hack

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Abstract

Interbank-offered-rates play a critical role in the hedging processes of banks, hedge funds or institutional investors. However, the financial stability board recommended to replace these rates by alternative risk-free-rates at the end of 2021. The new rates will be backward-looking rates and therefore, the payoff definitions of interest rate derivatives will change and the currently used Libor Market model to price exotic interest rate derivatives is no longer feasible. This thesis examines a new type of model, the forward market model, which is able to generate both the new backward-looking rates as the current forward-looking rates under the same stochastic process. Besides, contrary to the Libor Market Model, the dynamics under the risk-neutral measure can obtained. Consequently, the new forward market model should always be chosen over the Libor market model. Two issues regarding the forward market model are also considered in this thesis. First of all, the forward market model cannot deal with negative interest rate, this is solved by implementing a shifted version of the log-normal model. Second, a log-normal model is unable to reproduce the implied volatility smile which is present in the market. We solve this issue by combining the forward market model together with the SABR model. Under a few assumptions we derive the shifted SABR forward market model which hasn't been derived in the literature. The model is validated by pricing a new type of caplet that will be present in the post-Libor world, where the payoff won't be known until the payment date. We find that the implementation of this new shifted SABR-FMM can accurately price zero-coupon bonds and caplets in the market. Therefore, we conclude that this new type of model is a possible solution to price exotic interest rate derivatives in the post-Libor world.

Keywords: Market Models, Backward Rates, SABR, Interbank-offered-rates

Preface

This thesis was submitted for the degree of Master of Science in Applied Mathematics from the Delft University of Technology. I performed my thesis at the Pricing Model Validation team at Rabobank. I would like to thank my supervisors Natalia Borovykh and George Inkoom with their continuous support throughout the entire course of the thesis. Besides, I would also like to thank the entire PMV team. Even though I didn't meet them till seven months in my thesis due to COVID, I really enjoyed the days I was allowed to come to the office. My thanks also go to professor Kees Oosterlee who I first contacted to find a thesis project. He couldn't have set me up with a better project. He also always had some great input and feedback on my work. Finally, I would like to thank Antonis Papapantoleon and Fang Fang for being part of my thesis committee.

Tim Hack
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1 Introduction

The interest rate derivatives market is the biggest over-the-counter derivatives market in the world with an outstanding notional of approximately 495 billion dollar halfway 2020[1]. The most important number regarding interest rate derivatives is the interbank-offered-rate (IBOR) as the LIBOR, EURIBOR or TIBOR. From now on the term Libor will be used to address all interbank-offered-rates. The Libor constitute the costs of short-term interbank borrowing. Each day this rate is published as an average of a panel of banks which each give their own estimation on what the rates should be for different periods. For example, the three month Libor is what banks expect to pay on borrowing for a three month period from other banks. This means that Libors are not derived from actual transactions but are a projection of what banks think they should be. Almost everybody in the world has a relation to the Libor. It could be because of the variable interest rate on a mortgage or student loan but also in other matters.

The last couple of years the market of unsecured borrowing between banks hasn't been that active which gives banks a hard time in setting their Libor rates. Besides, during the financial crisis there were also widespread attempts to manipulate the Libor which weakened the believe in the benchmark, see [11]. In 2017 Bailey, [1], already questioned the future of LIBOR and this made the Financial Stability Board (FSB) review interest rate benchmarks after which they recommended to develop alternative rates. The new rates will replace the Libors at the end of 2021. These alternative nearly risk-free-rates will be different per country and will be overnight backward looking rates. There are a number of important differences between the new risk-free-rates and Libor rates. First of all, the new alternative RFR's are derived from actual market transactions. The second major difference is that the new risk-free-rates are backward-looking in time, whereas Libors are forward-looking rates.

These new rates will bring problems in modelling the prices of interest rate derivatives. The new RFR's are overnight rates and with the intention to use them as a substitution for Libors, they must be converted into term rates. This can be done in two ways:

1. Compounded setting-in-arrears: This means compounding the overnight rates over the application period. This method is backward looking and the backward rate is not known till the end of the period. For example, a three month forward rate at the moment is known at the start of the accrual period, in the post-Libor world we need to compound the overnight rate every day during this period to obtain the three month rate.
2. Market-implied prediction: This method is forward looking and the backward rate is known at the beginning of the period.

Most market participants are settling on using the backward-looking term version of the RFR's. This will bring problems in the pricing of certain interest rate derivatives. Take for instance the problem of a cap. A cap pays the buyer a certain amount of money (at time T_{i+1}) if the forward rate at time T_i is larger than a given strike. The cap contract is written at time t_0, which is today. Hence, at time T_i the Libor rate would be fixed and the payment would be known for time T_{i+1}. However, for the new backward looking rates the floating rate is not

[1] Source: BIS OTC Derivatives Statistics

known until time T_{i+1} since the rate has to be compounded every day over this period. Other products such as Libor-in-arrears or range accruals might even stop to exist when the Libors won't be reported anymore. This is due to the fact that payments are made immediately at the settlement date of the Libors. For more details see [32].

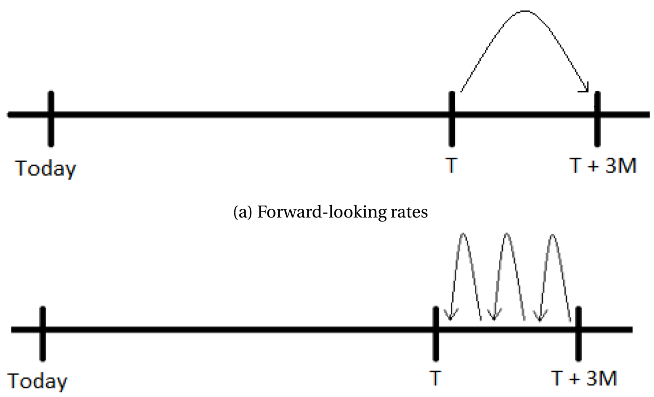
The current approach to price exotic interest rate derivatives in the market is to use the Libor market model. One major issue in the switch from Libors to backward-looking rates is the fact that the new rates keep evolving over their application period, since compounding takes place until the maturity time, see figure 1.1. In the Libor market model the simulation of the rates is only defined till the reset date, hence we cannot simulate the new type of rates. Since interest rate derivatives play a critical role in the hedging processes of banks, hedge funds or institutional investors it is of great importance to correctly price these derivatives in the new backward-looking rate environment. A possible solution is to fallback to instantaneous rate modeling. However, a market model is preferred over an instantaneous rate model. The authors in [25] presents the generalized forward market model as a solution to this problem.

The generalized forward market model gives the dynamics for both backward as forward looking rates under the same stochastic process. The concept of extended zero-coupon bonds is used to derive the dynamics of the new backward-looking rates. The forward market model should always be chosen over the Libor market model because of two reasons. First of all, in the new model we are still able to simulate the forward-looking rates, together with the new backward-looking rates. It will still be important to generate forward rates (Libors) even when the switch is made to the new rates at the end of 2021, because parties will still have contracts in their books which depend on forward rates and have an expiry of 20 or even 30 years. It is yet unknown how the true payoff of these derivatives will be calculated in the future, since at the moment there won't be any given Libor rates after 2021. The second reason, is due to the property that with the forward market model the dynamics under the risk-neutral measure can be obtained. This is possible for both the forward and backward rates. For the Libor market model this is impossible and an example of this application will be given using Eurodollar futures.

Since the transition away from Libors is relatively new, there are only a few studies that examine the problems that the new rates bring with them. The authors of [37] examined the pricing of caplets in a backward rate environment by extending the short rate models instead of extending the LMM. The paper only examines the pricing of caplets and doesn't incorporate the volatility smile or is able to work with negative interest rates. Besides [27] develop a model for pricing caps and swaptions but they also note that it could be that the model is not sufficient and should be extended, depending on what the market settles on in the future. There are also other problems concerning the Libor transition. For example, [18] looks at the new risk-free-rates and their alternatives identifying some issues. Other than that, the authors of [19] examine different types of derivatives and the changes in their payoff definitions.

The Libor market model has been extensively discussed in books and papers, besides the generalized forward market model is given to us by the authors of [25]. The forward market model does have some fallbacks which have not been examined extensively in the literature. Thus, apart from presenting established results from the literature about the Libor market model and generalized forward market model, we will also look into extensions of the forward market model which have not been discussed yet. First of all, the model cannot work with negative interest rates which are currently present in the market. To overcome this problem, we will extend the FMM to a shifted forward market model. Besides, the forward market model doesn't incorporate the volatility smile seen in the market. This was also an issue for the LMM and among one of the solutions the LMM was extended to a stochastic volatility model using for example the SABR model in [12]. We will also extend the FMM to a stochastic volatility model, specifically combine the model with the SABR model. There are no references where this has been performed yet, however [39] implemented the SABR model for solely caplets in a backward-looking rate environment. This does give a solution to price caplets in the post-Libor world, however with our extension to a SABR forward market model it is possible to price more exotic derivatives instead of only caplets. Finally, since vanilla derivatives only have payoffs at grid-points of the tenor structure the proposed FMM is sufficient to price these derivatives. Nevertheless, when more complex contracts should be priced, for example range accruals, simulation of off-grid rates are needed. In order to accomplish this, the FMM should be extended to a complete structure model. This topic is not considered in this thesis and for more information one could look at [26].

The thesis has been organised in the following way. We start with an introduction into interest rate derivative pricing. Then, we move on to explaining the Libor market model since the new model is an extension of this model. The fourth section of this MSc thesis will examine the new generalized forward market model. Chapter five gives some extensions of this model including the property that we can simulate rates under the risk-neutral measure. The sixth section extends the forward market model using the SABR model. Finally chapter seven ends with a discussion and proposes some future research topics for this MSc thesis.



(a) Forward-looking rates

(b) Backward-looking rates

Figure 1.1: (a) Forward-looking rates: the rate is fixed at the start of the application period. (b) Backward-looking rates: the rate keeps evolving over the accrual period and is unknown till the end of this period.

2

Background Information

In this chapter some basic definitions and assumptions are presented. These will be used throughout the rest of this thesis. We will cover pricing theory, the different measures that are used in the thesis, derive the forward rate and finally discuss some basic interest rate derivatives. A more thorough introduction into interest rates is presented by for example the authors of [4].

2.1. Definitions

First of all, an arbitrage free and complete economy is assumed. In the economy non-dividend paying securities are traded continuously from time 0 to time T. The prices of the securities are defined on the probability space $(\Omega, \mathscr{F}, P)$. Here the filtration is given by $\mathscr{F} = \{\mathscr{F}_t = \sigma\{W(s) : 0 \leq s \leq t\} : 0 \leq t \leq T\}$ and W is assumed to be an m-dimensional Brownian motion. The random variable V is called a contingent claim if $V(T)$ is $\mathscr{F}_T$ measurable. Here V(T) is the payoff amount at maturity time.

One of the most simple products is the zero-coupon bond. The zero-coupon bond is used as a building block for some theory and other derivatives.

Definition 2.1.1. (zero-coupon bond) A T-maturity zero-coupon bond, P(t,T), is a product which pays one unit of currency at maturity without any coupon payments. The value of a zero-coupon bond for $t \leq T$ is denoted as $P(t, T)$, where $P(T, T) = 1$.

Definition 2.1.2. (instantaneous forward rate) The interest over period $[T, T + \Delta]$ at time t, where $\Delta \downarrow 0$, is known as the instantaneous forward rate and given as

$$f(t, T) = -\frac{\partial \log(P(t, T))}{\partial T}.$$

Definition 2.1.3. (short rate) The short at time is t is defined as

$$r(t) = f(t, t).$$

Definition 2.1.4. (money-market account) The money-market-account, denoted by B(t), accrues money continuously over time at the risk-free rate. The change of the account is given by

$$dB(t) = r(t)B(t)dt, \qquad B(0) = 1,$$
$$B(t) = B(0)e^{\int_0^t r(u)du}.$$

When pricing interest rate derivatives the payoff function at maturity time is known. It is however important to obtain the value of the derivative for times $t \leq T$. In order to get these values the following pricing theorem will be used

Theorem 2.1.1. Let V(T) be the payoff of a contingent claim at time T. Assume V(T) is a $\mathscr{F}_T$-measurable random variable. The value of this contingent claim at any time $t < T$ under the measure Q^N, where N is the numéraire, is given by

$$V(t) = \mathbb{E}^{\mathbb{Q}^N}\left[\frac{N(t)}{N(T)} V(T) \mid \mathscr{F}_t\right]. \tag{2.1.1}$$

Proof. The proof of this theorem is presented in [36]. □

As an example the defined money-market-account could be used as numéraire resulting in the time t value of a derivative as

$$V(t) = \mathbb{E}^{\mathbb{Q}}\left[\frac{B(t)}{B(T)}V(T) \mid \mathscr{F}_t\right],$$

where the expectation under the risk-neutral measure is taken.

2.2. Measures

In this subsection the different measures which are used throughout the thesis are presented.

Definition 2.2.1. (risk-neutral measure) The risk-neutral measure is denoted as $\mathbb{Q}$ and has the money-market-account B(t) as numéraire. For all $t \leq T$ the following martingale property holds

$$\frac{V(t)}{B(t)} = \mathbb{E}^{\mathbb{Q}}\left[\frac{V(T)}{B(T)}|\mathscr{F}_t\right].$$

Here $\mathbb{E}^{\mathbb{Q}}$ is the conditional expectation with respect to the filtration $\mathscr{F}_t$.

Definition 2.2.2. (T-forward measure) The T-forward measure is denoted as $\mathbb{Q}^T$ and has a zero-coupon bond with maturity T as numéraire. For all $t \leq T$ the following martingale property holds

$$\frac{V(t)}{P(t,T)} = \mathbb{E}^{\mathbb{Q}^T}\left[\frac{V(T)}{P(T,T)}|\mathscr{F}_t\right].$$

Definition 2.2.3. (spot-Libor measure) The spot-Libor measure is denoted as $\mathbb{Q}^d$ and has the discrete bank-account as numéraire, defined in section 2.3. For all $t \leq T$ the following martingale property holds

$$\frac{V(t)}{B^d(t)} = \mathbb{E}^{\mathbb{Q}^d}\left[\frac{V(T)}{B^d(T)}|\mathscr{F}_t\right].$$

2.3. Forward rates

In this subsection the equation for the forward rates is derived. First of all, the following tenor structure is taken for $N \in \mathbb{N}$

$$0 = T_0 < T_1 < T_2 < \cdots < T_N, \qquad \tau_n = T_{n+1} - T_n.$$

This means that today, which is time zero, is taken as T_0. This could be different in other papers or books that are cited. Note that $n = 0, ..., N-1$. Also τ is the time between two tenor points. Unless stated otherwise in this thesis it is always considered to be equal to 0.25.

Forward rates denoted as $L(t, T_n, T_{n+1})$ consist of three time stamps. We have the current time (t) at which the forward rate is given for the settlement time (T_n) where the application of the rate starts and finally the maturity time (T_{n+1}) where the forward rate will end. This means that $L(t, T_n, T_{n+1})$ is the forward rate for the period $[T_n, T_{n+1}]$. Note that when t moves forward in time it will pass start / end points of forward rates. Therefore, if $t > T_n$, the set of forward rates shrinks since $L_n(t)$ no longer exists. An example is presented in figure 2.1. Note that rates which start at time zero are not forward rates but spot rates and hence are not given in the figure. In order to start with the LMM, we will first derive the simply compounded forward rate.

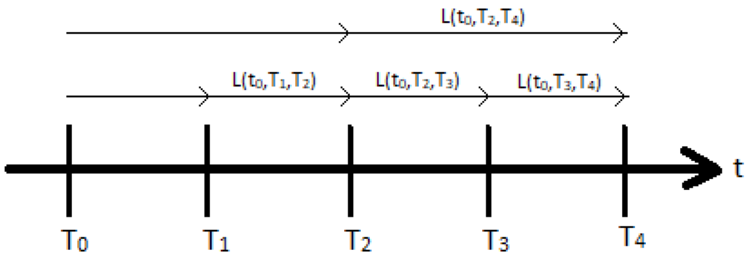


Figure 2.1: Different Libors

Assume two counterparties: Bank A and Bank B. Now bank B wants to borrow €1 from bank A at some time in the future T_1. Bank B will pay back one euro at time T_2 including interest K over the period $T_2 - T_1$. The value of this contract today will be equal to

$$\begin{aligned} V(t_0) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{-1}{B(T_1)} + \frac{1+K(T_2-T_1)}{B(T_2)}\right], \\ &= -P(t_0,T_1) + (1+(T_2-T_1)\cdot K)P(t_0,T_2). \end{aligned}$$

The fair rate for interbank lending is given by setting the value of this contract to zero hence

$$K = \frac{1}{(T_2-T_1)}\left(\frac{P(t_0,T_1)}{P(t_0,T_2)} - 1\right) = L(t,T_n,T_{n+1}) = L_n(t).$$

In other notation the simply compounded forward rate is given by:

$$\begin{aligned} L(t,T_n,T_{n+1}) &= \frac{1}{\tau_n}\frac{P(t,T_n)-P(t,T_{n+1})}{P(t,T_{n+1})}, \qquad \text{or:} \\ L_n(t)P(t,T_{n+1}) &= \frac{1}{\tau}(P(t,T_n)-P(t,T_{n+1})). \end{aligned} \tag{2.3.1}$$

This final notation will become useful in later derivations. It is a functional notation because the forward rate times the price of a zero-coupon bond is a tradable asset. Thus, we can apply the no-arbitrage pricing theorem. Libors aren't traded directly in the market so we cannot just apply this to the forward rate.

Besides the forward rate, two other equations will also be often used in this thesis. First of all, the bank account under simple interest rates. The bank account is worth 1 at time zero and at time T_n given by

$$B^d(T_n) = \prod_{i=0}^{n-1}(1+\tau L(T_i,T_i,T_{i+1})), \qquad B^d(0) = 1.$$

The zero-coupon bond, where $P(T,T) = 1$, can be expressed as

$$P(T_k,T_n) = \prod_{i=k}^{n-1}\frac{1}{1+\tau L(T_k,T_i,T_{i+1})}.$$

We are currently working with a tenor structure given by $[T_0, T_1, ..., T_N]$ however, time t is continuous. Consequently, t could be between two periods of the tenor structure. This gap must be closed somehow and in order to do so, the following index function $q(t)$ is defined as

$$T_{q(t)-1} \le t < T_{q(t)}. \tag{2.3.2}$$

Thus $q(t)$ rounds the continuous time upwards to the next tenor point. With this definition the bank account and zero-coupon bond can be defined for every time t instead of only tenor points.

$$B^d(t) = P(t,q(t))\prod_{i=0}^{q(t)-1}(1+\tau L(T_i,T_i,T_i+\tau)). \tag{2.3.3}$$

$$P(t,T_n) = P(t,q(t))\prod_{i=q(t)}^{n-1}\frac{1}{1+\tau L(t,T_i,T_i+\tau)}. \tag{2.3.4}$$

2.4. Interest rate derivatives

Interest rate derivatives are financial contracts that depend on the level of the interest rate. In this subsection, four simple interest rate derivatives are discussed which are used throughout this thesis.

2.4.1. Forward Rate Agreements

A forward rate agreement is a simple interest rate product which allows the buyer to lock-in an interest rate for a future time period.

Definition 2.4.1. (forward rate agreement) A forward rate agreement (FRA) for a future period $[T_i, T_{i+1}]$ is a contract where two parties agree to exchange a payment based on a fixed rate K and the floating Libor rate. The Libor is fixed at time T_i, but the payment is made at time T_{i+1}.

The payment of a FRA at time T_{i+1} is given by

$$V^{FRA}(T_{i+1}) = \tau(L(T_i, T_i, T_{i+1}) - K). \tag{2.4.1}$$

Using the definition of the T-forward measure, the value of the FRA at $t < T_{i+1}$ can be obtained by

$$V^{FRA}(t) = P(t, T_{i+1})\mathbb{E}^{T_{i+1}}\left[\tau(L(T_i, T_i, T_{i+1}) - K)|\mathscr{F}_t\right].$$

Since the forwards are martingales under this measure, we obtain

$$V^{FRA}(t) = P(t, T_{i+1})\tau(L(T_i, T_i, T_{i+1}) - K). \tag{2.4.2}$$

Figure 2.2 gives an example of the payoff structure of a forward rate agreement. The notional is chosen to be one unit.

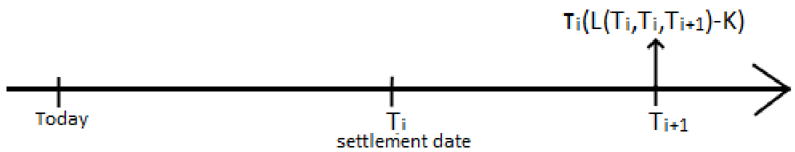


Figure 2.2: Cash flow forward rate agreement

2.4.2. Caps and Caplets

Caps are interest rate derivatives that provide insurance against increasing rates in the market. When the floating interest rate reaches a certain level called the cap rate, a payment is made to the cap holder. Similar products are floors which provide insurance against falling interest rates. Caps can be decomposed into caplets, these products are European options on the forward rates. Every caplet has a notional (N_i), strike price (K_i), a fixed date and a payment date. For the definition of caplets we use the one in [30].

Definition 2.4.2. Given two time points in the future, the reset date T_i and the payment date T_{i+1} with year-fraction $\tau_i = T_{i+1} - T_i$, the T_i-caplet with cap-rate K_i and notional N_i is a contract which pays at time T_{i+1}

$$Cpl_i(T_{i+1}) = \tau_i N_i \max(L_i(T_i) - K_i, 0). \tag{2.4.3}$$

The Libor is known at time T_i but the payment is not made till time T_{i+1}. A cap is the sum of m caplets where the notional and strike are kept constant. Hence caplets give insurance against increasing interest rates for one period while caps are multiple options consisting of different reset dates on the Libor. Figure 2.3 shows the cash flows of a cap holder where the notional is chosen to be one and $\tau_i = 1$. The value of a caplet can be obtained under the T_{i+1}-forward measure as

$$\begin{aligned} Cpl_i(t) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{N_i\tau_i}{B(T_{i+1})}\max(L_i(T_i) - K_i, 0)|\mathscr{F}_t\right], \\ &= N_i\tau_i P(t, T_{i+1})\mathbb{E}^{T_{i+1}}\left[\max(L_i(T_i) - K_i, 0)|\mathscr{F}_t\right]. \end{aligned} \tag{2.4.4}$$

Assuming the forward rates to be log-normally distributed, caplets can be priced using Black's (1976) formula, [2]. The price of a caplet at time t is given as

$$Cpl_i(t) = \tau P(t, T_{i+1}) N(L(0, T_i, T_{i+1})\Phi(d_1) - K\Phi(d_2)), \tag{2.4.5}$$
$$d_1 = \frac{\log\left(\frac{L(0,T_i,T_{i+1})}{K}\right) + \frac{1}{2}\sigma^2(T_i - t)}{\sigma\sqrt{(T_i - t)}},$$
$$d_2 = d_1 - \sigma\sqrt{(T_i - t)}.$$

Here $\Phi(\cdot)$ is a standard normal distribution function. The strike price determines the moneyness of a caplet. If $K < L$ the caplet is called in-the-money, if $K = L$ it is called at-the-money and finally if $K > L$ the caplet is out-the-money. Throughout this thesis, a notional of 10000 is used, which denotes the price of caplets in basis points.

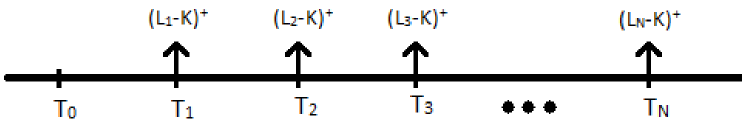


Figure 2.3: Cash flow of a cap

Black's formula depends on several parameters: the forward rate at time zero, the volatility σ, the strike price K and the maturity T. Usually the volatility parameter is extracted from the market, which is called the implied volatility parameter. The implied volatility of a caplet is the volatility parameter that can be used as input into Black's formula to obtain the market price of a caplet. Hence, instead of calculating the price of a caplet using Black's formula, the price of a caplet in the market is used as input to back-out the implied volatility from the market. In order to acquire the implied volatility the following equation has to be solved

$$\text{Black}(0, L, T, K, \sigma_{imp}) - Cpl(0) = 0. \tag{2.4.6}$$

To solve this kind of equation we can use the Newton-Rhapson method or Brent's method, see for more information [30].

2.4.3. Swaps

Swaps are a type of interest rate derivative where two sets of interest rates are exchanged.

Definition 2.4.3. (plain vanilla interest rate swap) An interest rate swap is a financial product where two parties swap a floating interest rate with a fixed interest rate. The exchange of the two rates is performed over multiple predetermined dates and the payoff is determined on a pre-specified amount. The fixed rate is also called the strike. The interest rate swap payer will receive the floating rate and pay the fixed rate. The payoff for an interest rate swap payer, where the payments are made at the future times $T_{n+1}, ..., T_m$ with fixed rate K is given as

$$V_{\text{payer}}(T_n, ..., T_m) = \sum_{k=n}^{m-1} \tau_k N(L(T_k, T_k, T_{k+1}) - K), \tag{2.4.7}$$

where N is the notional of the contract.

The cash flows for a vanilla interest rate swap can be found in figure 2.4. Here the cash flows on the top of the figure represent the floating payments, the bottom cash flows are the fixed payments. The value of a swap today can be derived using theorem 2.1.1. However, we won't use the swap that often in this thesis thus we refer to [30] for more information.

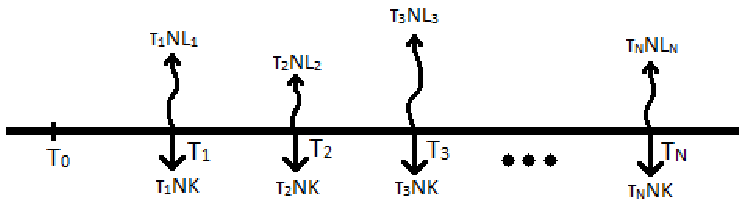


Figure 2.4: Cash flow of an interest rate swap

2.4.4. European swaptions

Swaptions are options on interest rate swaps. The definition of a swaption in [30] is used.

Definition 2.4.4. (European swaption) A European swaption gives the holder the right but not the obligation to enter into a swap contract at a future date for a predetermined strike price K. A payer (receiver) swaption is an option on a payer (receiver) swap. The swap can be entered at the expiry date of the swaption, T_n. The value of a payer swaption at time T_n is equal to

$$V_{\text{swpt}}(T_n) = \max(V_{\text{payer}}(T_n), 0) = N \cdot \max\left(\sum_{k=n}^{m-1} \tau_k P(T_n, T_{k+1})(L(T_n, T_k, T_{k+1}) - K), 0\right). \tag{2.4.8}$$

It is possible to determine today's value of the swaption and rewrite this into a more convenient notation. Swaptions are important liquid instruments for the calibration of market models. This thesis isn't focused on calibration. Therefore, we won't go into any further detail of swaptions. For more information we refer to [30].

3
Libor Market Model

Before moving to the forward market model, which gives the dynamics for the new risk-free-rates, we will look at the Libor market model (LMM) which is the currently used model in the market. Since the new generalized forward market model is an extension of the Libor market model it is important to understand its fundamentals, besides a good foundation can be made from the LMM where we can use similar derivations for the new model.

We start with a small introduction into the Libor market model after which the dynamics of the forward rates under different measures for the log-normal Libor market model are derived. After the dynamics are derived, we specify some parameters we need in order to start simulating the rates using two different methods. We end this chapter by comparing both methods to conclude which one we should use for the forward market model.

3.1. Introduction to the Libor Market Model

There are different ways to price interest rate derivatives. First there was Blacks formula [2], which could be used for simple plain vanilla interest rate derivatives. Then during the 1980's short rate models were introduced to price interest rate derivatives. These models simulated the instantaneous short rate under certain dynamics which differed by model. The short rate is the interest rate over an infinitesimally small period. Examples of short rate models are the Vasicek model, [38], and the Cox-Ingersoll-Ross (CIR) model, [7]. Since short rate models couldn't be calibrated to the market and only give the dynamics of a single point on the forward curve there was room for improvement.

This improvement came in the form of the Heath-Jarrow-Morton (HJM) model, [17]. The HJM model is a framework where instead of modelling just a single point on the forward curve, the entire dynamics of the forward rate curve are now obtained. The HJM model can be initialised by defining a certain instantaneous volatility, this leads to the forward rate which gives the short rate to price interest rate derivatives. Although the HJM-model can be calibrated to the market, it is not satisfactory that the instantaneous forward rates are not seen in the market or are not directly present in the payoff function of interest rate derivatives besides it is hard to calibrate the model to market data. This is the big advantage of the Libor Market Model (LMM).

The Libor market model, see [3] and [29] was introduced in the late 90's. The idea of a market model is to present the dynamics of observable processes directly, instead of using a hidden process which models the Libors. This happens for example in the HJM model. In the Libor market model the dynamics of forward rates are modelled as geometric Brownian motions. With these forward rates we can price interest rate derivatives since the payoffs can be written in terms of forward rates. Although forward rates cannot be traded in the market, they can be directly written in terms of zero coupon bonds and hence be observed in the market.

3.2. Dynamics of the forward rates

In this section we will derive the dynamics of the Libors. The Libor market model assumes the following dynamics for the forward rates

$$dL_i(t) = \mu_i(L_i(t), t)dt + \bar{\sigma}_i(t)dW_i(t) \tag{3.2.1}$$

Here W(t) is a Brownian motion defined on the probability space $(\Omega, \mathscr{F}_t, P)$, where $dW_i dW_j = \rho_{ij}dt$. The parameter $\mu_i(L_i(t), t)$ is the drift term which differs per forward rate. The parameter $\bar{\sigma}_i(t)$ is called the volatility coefficient. For the equation above the volatility coefficient is still very arbitrary with no structure imposed on this parameter, resulting in some flexibility.

We will use theorem 2.1.1 to derive the dynamics of the forward rates under different measures. From this theorem we can derive that the forward rate is a martingale under the appropriate T_{n+1}-measure. To show this we apply equation (2.1.1) under the T_{n+1}-measure. Together with its proper numéraire, $P(t, T_{n+1})$, we then obtain

$$\begin{aligned} \frac{L_n(t)P(t,T_{n+1})}{P(t,T_{n+1})} &= \mathbb{E}^{T_{n+1}}\left[\frac{L_n(S)P(S,T_{n+1})}{P(S,T_{n+1})}|\mathscr{F}_t\right], \\ L_n(t) &= \mathbb{E}^{T_{n+1}}\left[L_n(S)|\mathscr{F}_t\right]. \end{aligned} \tag{3.2.2}$$

Therefore, every forward rate is martingale under its own T_{n+1} measure. It is very important to remember that every forward measure only has one corresponding forward rate which is a martingale. Every other forward rate is not a martingale under that measure. Since we know that if the forward rate is a martingale, it must contain no drift term. Therefore, the dynamics under the T_{n+1} measure must be

$$dL_n(t) = \bar{\sigma}_n(t)dW_n^{T_{n+1}}(t).$$

It is also possible to obtain the dynamics of the forward rate under different measures. We know that $dL_n(t)$ is a martingale under the T_{n+1}-measure, however other forward rates contain a drift term under this measure. Although we will not use the dynamics of the Libors under an arbitrary T_M-forward measure for reasons later explained, the result of this derivation does help as a foundation for the dynamics under other measures. Hence, the derivation of the dynamics for forward rates under an arbitrary T_M-forward measure can be found in appendix A.

Another possibility would be to obtain the dynamics of the forward rates under the risk-neutral measure. This uses the continuous-time bank account as numéraire. It turns out that it is not possible in the Libor market model setting to obtain these dynamics. For a proof one can look at [4].

A different option is to use the previously defined discrete bank-account, equation (2.3.3), as a numéraire. This is also known as the spot-Libor measure, we will use this measure for simulating the forward rates. First we have to derive the dynamics of the forward rates under this measure.

3.2.1. Spot-Libor measure

Lemma 3.2.1. The process $L_n(t)$ under the spot-Libor measure Q^d is given by

$$dL_n(t) = \bar{\sigma}_n(t)\left(\sum_{k=q(t)}^{n}\frac{\tau_k\bar{\sigma}_k(t)}{1+\tau_k L_k(t)}dt + dW_n^d(t)\right), \tag{3.2.3}$$

where $W_k^d(t)$ is a Brownian motion under the measure.

Proof. To derive the dynamics for the spot-Libor measure we start from equation (2.1.1) and fill in the discrete bank-account as numéraire.

$$\frac{L_n(t)P(0,T_{n+1})}{B^d(0)} = \mathbb{E}^d\left[\frac{L_n(t)P(t,T_{n+1})}{B^d(t)}|\mathscr{F}_0\right],$$

which leads to

$$L_n(0) = \mathbb{E}^d\left[L_n(t)\frac{P(t,T_{n+1})}{B^d(t)}\frac{B^d(0)}{P(0,T_{n+1})}|\mathscr{F}_0\right].$$

Comparing this equation against (3.2.2) we can easily see the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^d}{d\mathbb{Q}^{T_{n+1}}} = \frac{B^d(t)}{B^d(0)} \frac{P(0,T_{n+1})}{P(t,T_{n+1})}. \tag{3.2.4}$$

Independent of using the simple interest rate bank account as numéraire, we can also look what happens if we use the shortest maturity bond (= $q(t)$ as maturity) as numéraire

$$L(0,T_n) = \mathbb{E}^{q(t)}\left[L(t,T_n)\frac{P(t,T_{n+1})}{P(t,q(t))}\frac{P(0,q(t))}{P(0,T_{n+1})}|\mathscr{F}_0\right].$$

Thus the Radon-Nikodym derivative is given by

$$\frac{d\mathbb{Q}^{q(t)}}{d\mathbb{Q}^{T_{n+1}}} = \frac{P(t,q(t))}{P(0,q(t))} \frac{P(0,T_{n+1})}{P(t,T_{n+1})}. \tag{3.2.5}$$

Comparing the Radon-Nikodym derivatives of equation (3.2.4) and (3.2.5) we see that only the first fraction is different. We also know that the formula's for the discrete bank-account and the zero-coupon bond are given in section 2.3. This means that it is possible to write out the fractions in both the Radon-Nikodym derivatives. Using the fact that $B(0) = 1$, we can easily see that the first fraction of equation (3.2.4) can be written as

$$\frac{B^d(t)}{B^d(0)} = P(t,q(t))\prod_{i=0}^{q(t)}(1+\tau L(T_i,T_i,T_i+\tau)).$$

Writing out the fraction of equation (3.2.5) we see that

$$\frac{P(t,q(t))}{P(0,q(t))} = P(t,q(t))\prod_{i=0}^{q(t)}(1+\tau L(T_i,T_i,T_i+\tau)).$$

Thus, both fractions are the same which means that both Radon-Nikodym derivatives are equal. Therefore, we can conclude that the dynamics of the forward rate under the spot-Libor measure is exactly the same as the dynamics under the $q(t)$-forward measure which has the shortest-maturity bond as numéraire. In appendix A we derived the formula for the dynamics of the forward rates under an arbitrary T_M-forward measure. All that remains is to use the q(t)-forward measure and fill this in to obtain the equation presented in the lemma. □

We have not discussed why the decision is made to simulate the forward rates under the spot-Libor measure. We have derived the dynamics under both an arbitrary T_M-forward measure in appendix A and the spot-Libor measure. Comparing both dynamics some differences are visible. First of all, it is not convenient to simulate the Libors under a T_M-forward measure. When we would like to simulate the Libors L_n for all different n, first the rates have to be simulated using a drift term running from $n+1$ till $M-1$ for the case where $M > n+1$. However, when we reach the point where $M < n+1$ the drift term changes sign and the sum now runs from $M+1$ till $n-1$. It is much simpler to stick with one type of drift term as for the spot-Libor measure. This argument is not valid for simulating under the terminal measure, which is the T_N-forward measure. In this case we will always have $M > n+1$. There is a second argument why the spot-Libor measure is preferred over a forward measure, which we will work out for the terminal measure. When simulating under the terminal measure the summation of the drift term will run from n+1 till N-1. Hence for simulating forward $L_n(t)$ the drift contains N - n - 1 terms. This means that for short forward rates (n being small) the drift contains many terms, while for longer forwards less and less terms are present in the drift term. This results in a more biased result for forwards with an application period near in the future compared to forwards with a maturity date far away in the future. In the case of the spot-Libor measure this difference in the number of summations in the drift term is much smaller. The drift summation contains n - q(t) + 1 terms. Therefore, it is obvious that over time the number of terms in the drift decreases. Hence the bias, due to the discretization of the drift, between the different Libors will be much closer to each other.

3.2.2. Log-normal Libor market model

In the previous section the volatility parameter was chosen very arbitrary. For Libor $L_n(t)$ we had as volatility parameter $\bar{\sigma}_n(t)$. This means that there is a lot of flexibility in choosing the structure for the volatility. We know that it is market practice to price exotic interest rate derivatives using the Libor market model. However, simple vanilla products as caps and caplets are actually priced with Black's formula. These products are extremely liquid hence it would be satisfactory that when pricing more exotic derivatives these products are priced consistently with the liquid cap/caplets. Since Black's formula assumes log-normally distributed forward rates, we will also work with the log-normal Libor market model. This means that we choose

$$\bar{\sigma}_i = \sigma_i(t)L_i(t),$$

where $\sigma_i(t)$ is called the instantaneous volatility. By going for the log-normal LMM, caps and caplets are priced consistent with Black's formula. It is of course possible to choose other functions for sigma bar which gives other type of models, for example we could make it a constant elasticity of variance model by specifying $\bar{\sigma}_i = \sigma_i(t)L_i(t)^\gamma$.

This log-normal specification of the volatility parameter gives us the final dynamics under the spot-Libor measure for the log-normal Libor market model

$$dL_n(t) = \sigma_n(t)L_n(t)\left(\sum_{k=q(t)}^{n} \rho_{nk}\frac{\tau_k\sigma_k(t)L_k(t)}{1+\tau_k L_k(t)}dt + dW_n^d(t)\right). \tag{3.2.6}$$

3.3. Parameters

Before moving on to the simulation of the forward rates some parameters have to be specified. Examining equation (3.2.6), it is clear that three different variables have to be stated. First of all, the Libors at time zero ($L_i(t_0)$) are necessary to start the simulation of a forward rate. These values can be extracted from the market using forward rate agreements. Second, the instantaneous volatility is needed ($\sigma_i(t)$), this volatility parameter can be different per forward rate and also possibly change over time. The instantaneous volatility can be derived by calibrating the model to the implied Black volatilities of caplets. This is possible because under the log-normal LMM caplets are priced consistently with Black's equation, they both assume log-normally distributed forward rates.

First a certain structure for the instantaneous volatilities have to be chosen. The authors of [4] give many different possibilities we could choose from. A form is chosen where the instantaneous volatilities will be constant over time but differ per Libor. This will give a structure as in table 3.1. A more complex structure for the instantaneous volatility could be chosen but since this is not the focus of this thesis we stick with a bit of simplicity.

Table 3.1: Instantaneous volatility structure

Instant. Volatility	t: $(T_0, T_1]$	$(T_1, T_2]$	$(T_2, T_3]$	$\cdots$	$(T_{N-1}, T_N]$
Fwd rate: $L_1(t)$	s_1	Dead	Dead	$\cdots$	Dead
$L_2(t)$	s_2	s_2	Dead	$\cdots$	Dead
$\vdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$L_N(t)$	s_N	s_N	s_N	$\cdots$	s_N

Forward rates exist till their starting date, this means that the corresponding instantaneous volatility no longer exists when the time is past this date. The parameter is considered 'dead' from this point.

Finally the instantaneous correlation needs to be specified. Many interest rate derivatives depend on multiple forward rates. All these rates have some correlation with each other. Since the value of caplets only depends on one forward rate they cannot be used to calibrate the instantaneous correlation between the forward rates. Instead, European swaptions are used for the calibration of the instantaneous correlations. The payoff formula, equation (2.4.8), depends on multiple forward rates. However, rather than performing all sorts of calibrations, we will assume the following instantaneous correlation structure

$$\begin{aligned} \rho_{ij} &= \prod_{k=i}^{j-1} \rho_{k,k+1}, \qquad && j > i \\ \rho_{ij} &= \rho_{ji}, && j < i. \end{aligned} \tag{3.3.1}$$

Here the correlation is assumed to be constant between two tenors. In the above equations the upper diagonal of the matrix has to be manually set.

3.4. Simulating forward rates

In this final paragraph of the chapter two different simulation techniques of the forward rates will be discussed. First, the classic Euler method will be examined after which the so-called log-Euler method is introduced. Using a numerical example it will also be shown that the log-Euler method is much faster compared to the Euler method achieving the same accuracy.

3.4.1. Euler method

There are multiple ways to simulate the dynamics of the different forward rates. One of the easiest approaches is by using the Euler method. Suppose that we would like to generate the forward dynamics under the spot-Libor measure. Recall that for the log-normal Libor market model the dynamics were given by the following equation

$$\frac{dL_n(t)}{L_n(t)} = \sigma_n(t) dW_t^d + \sigma_n(t) \sum_{k=q(t)}^{n} \frac{\tau L_k(t) \rho_{k,n} \sigma_k(t)}{1 + \tau L_k(t)} dt. \tag{3.4.1}$$

This equation can be discretized using very small time steps. Discretizing equation (3.4.1), very small increments in the forward rate over time t are can be obtained. This process can be applied to forward rates with different settlement dates T_n to obtain a set of different forward rates. Thus the dynamics are given as follows

$$L_n^{Euler}(t + \Delta t) = L_n(t) + L_n(t)\sigma_n(t)\mu(L_n(t), t)\Delta t + L_n(t)\sigma_n(t)(W_n^d(t + \Delta t) - W_n^d(t))$$

Taking a closer look at equation (3.4.1), six different parameters have to be defined in order to simulate the forward rates.

1. The change in Brownian motion. The distribution of dW is given by a normal distribution with mean zero and variance dt ($dW \sim N(0, dt)$). Hence, by generating standard normal variables it is easy to generate series of Brownian motions. Cholesky's decomposition is used in order to achieve the correlation between the Brownian motions.

2. The instantaneous volatility. A constant instantaneous volatility structure which depends on the forward rate is used. For forward rate $L(t, T_i, T_{i+1})$ the instantaneous volatility will be alive till time T_i after which it will be dead and the forward rate ceases to exist.

3. The Libors at time zero: $L_n(0)$. For now, own chosen values will be used.

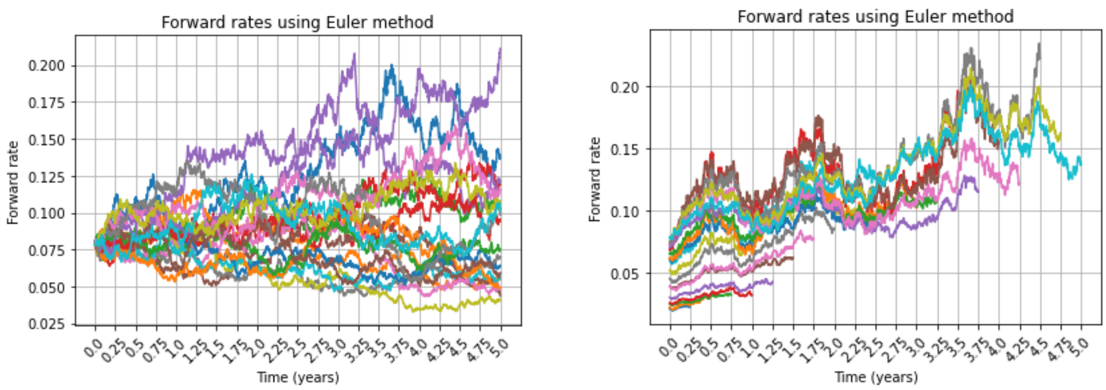
4. The instantaneous correlation. Here the previously defined structure is used.

5. The tenor structure. Forward rates will be simulated for every three months. Therefore, $\tau = 0.25$.

6. The discretization step. This depends on the number of steps we want to take till the end of the tenor structure. This is a trade-off between accuracy and computational time.

Finally, we start at time zero and work forward in time simulating at every time step the forward rates. However, when moving forward in time we eventually reach maturity dates of forward rates. These forward rates then cease to exist and at the end fewer and fewer forward rates are left.

Figure 3.1 shows some simulations of the forward rates. The left figure presents twenty different simulations of the Libor with an application five years till five years and three months from now. While the right figure shows 20 forward rates all with a different maturity time. Between two tenor points a total of 100 discretization steps were taken.



(a) Forward rates with same application period

(b) Forward rates with different settlement dates

Figure 3.1: Simulation of Libors

3.4.2. Predictor-corrector method

Simulating forward rates using Euler's method gives the required results. However, the simulation of forward rates with a maturity far away in the future will be time consuming. Simply due to the fact that more steps have to be taken. In this section a faster simulation technique is derived, where larger simulation steps can be taken. This automatically leads to less computations and hence a shorter computational time. In this thesis, all the interest rate derivatives that are considered only have settlement / payment dates at the pre-defined tenor points. Therefore, it is necessary to have a value for the Libors at tenor points in order to calculate their values. However, between these points there is no need for a value of the Libor. Consequently, the maximum step size we could take is the size between two tenors, which is τ. Thus when simulating forward rates it would be a good improvement to jump from T_0 to T_1 etc instead of taking very small steps in between two tenor points. It is actually possible to obtain an accurate approximation of the forward rates by solving the stochastic differential equation for the log-normal Libor market model

$$dL_i(t) = \mu_i(L_i(t), t)dt + \sigma_i(t)L_i(t)dW_i(t).$$

Applying Itô's lemma to the log dynamics

$$d\ln L_i(t) = \left(\mu_i(L_i(t), t) - \frac{1}{2}\sigma_i^2(t)\right)dt + \sigma_i(t)dW_i(t).$$

By integrating over $[T_0, t]$

$$\int_{T_0}^{t} d\ln L_i(u) = \ln L_i(t) - \ln L_i(T_0) = \int_{T_0}^{t} \mu_i(L_i(u), u)du - \int_{T_0}^{t} \frac{1}{2}\sigma_i^2(u)du + \int_{T_0}^{t} \sigma_i(t)dW_i(t).$$

Now by taking the exponential

$$L_i(t) = L_i(T_0)\exp\left(\int_{T_0}^{t} \mu_i(L_i(u), u)du - \int_{T_0}^{t} \frac{1}{2}\sigma_i^2(u)du + \int_{T_0}^{t} \sigma_i(t)dW_i(t)\right).$$

The next step is to discretize the tenor structure where this time large time steps are used. Consider T_k and T_{k+1} where $k \in [0, N-1]$. This will give the following discretization

$$\begin{aligned} L_i(T_{k+1}) &= L_i(T_k)\exp\left(\int_{T_k}^{T_{k+1}} \mu_i(L_i(u), u)du - \int_{T_k}^{T_{k+1}} \frac{1}{2}\sigma_i^2(u)du + \int_{T_k}^{T_{k+1}} \sigma_i(u)dW_i(u)\right), \\ &= L_i(T_k)\exp\left(X_{ik} + Y_{ik} + Z_{ik}\right). \end{aligned} \tag{3.4.2}$$

This equation allows us to jump from one arbitrary tenor point, T_k, to the next one T_{k+1}. Let's discuss each integral separately, starting with X_{ik}. Since the simulation of the forward rates is under the spot-Libor measure the appropriate drift equation can be filled in. This is done in equation (3.4.3).

$$
\begin{aligned}
X_{ik} &= \int_{T_k}^{T_{k+1}} \mu_i(L_i(u),u)du, \\
&= \int_{T_k}^{T_{k+1}} \sigma(u,T_i) \sum_{j=q(t)}^{i} \frac{\tau L(u,T_j)\rho_{j,i}\sigma(u,T_j)}{1+\tau L(u,T_j)} du.
\end{aligned} \tag{3.4.3}
$$

Taking a closer look at this equation, it is clear that the integral cannot be solved. This is due to the stochastic term ($L(u,T_j)$) present in the integral. In order to find a solution for this integral an approximation is made. The stochastic term is fixed to the known forward rate at time T_k. Thus we set $L(u,T_j) = L(T_k,T_j)$, where $T_k \leq u < T_{k+1}$. Therefore, the integral can be written as

$$
\begin{aligned}
X_{ik} &\approx \int_{T_k}^{T_{k+1}} \sigma_i(u) \sum_{j=q(t)}^{i} \frac{\tau L_j(T_k)\rho_{i,j}\sigma_j(u)}{1+\tau L_j(T_k)} du, \\
&= \sum_{j=q(t)}^{i} \nu_j(T_k) \int_{T_k}^{T_{k+1}} \sigma_i(u)\sigma_j(u)\rho_{i,j}(u)du.
\end{aligned} \tag{3.4.4}
$$

Here $\nu_j(T_k) = \frac{\tau L_j(T_k)}{1+\tau L_j(T_k)}$ and the integral in equation (3.4.4) is defined as follows

$$
C_{ij}(k) = \int_{T_k}^{T_{k+1}} \sigma_i(u)\sigma_j(u)\rho_{ij}(u)du.
$$

Using this integral it is immediately clear what the value of Y_{ik} in equation (3.4.2) is,

$$
Y_{ik} = -\frac{1}{2}C_{ii}(k). \tag{3.4.5}
$$

Where it is obvious that $\rho_{i,i} = 1$. Moving on to the final integral, Z_{ik}, which is a stochastic integral. Remember that for our model specifications the instantaneous volatility is assumed to be constant. This means that the value of Z_{ik} is equal to

$$
\begin{aligned}
Z_{ik} &= \int_{T_k}^{T_{k+1}} \sigma_i(u)dW_i(u), \\
&= \sigma_i \int_{T_k}^{T_{k+1}} dW_i(u) = \sigma_i(W_i(T_{k+1}) - W_i(T_k)).
\end{aligned} \tag{3.4.6}
$$

Hence, in order to simulate the forward rates from tenor point to tenor point the following equation can be used

$$
L_i(T_{k+1}) = L_i(T_k)\exp\left(\sum_{j=q(t)}^{i} \nu_j(T_K)C_{ij}(k) - \frac{1}{2}C_{ii}(K) + \sigma_i(W_i(T_{k+1}) - W_i(T_k))\right). \tag{3.4.7}
$$

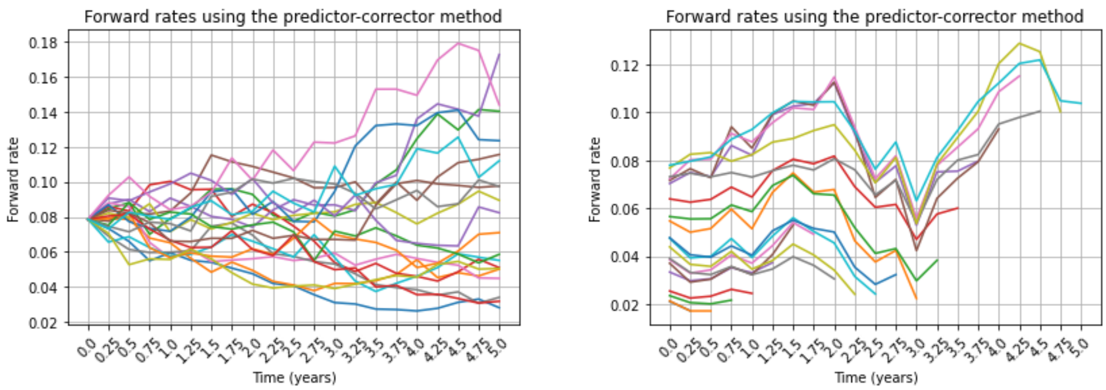
In equation (3.4.7) there is still an integral in the form of C_{ij}, however since the instantaneous volatilities and correlations are constant this integral can be calculated exact.

The above described method is known as the log-Euler method and was first introduced by the author of [23]. The log-Euler method can even be further improved by performing a drift correction also proposed in [23]. This is done by first calculating the forward rate value at the next tenor point ($L_i(T_{k+1})$) using the log-Euler method. This was done by approximating a stochastic term in a integral, where $L_i(u) = L_i(T_k)$, if $T_k \leq u < T_{k+1}$. Now with the new Libor value at T_{k+1} we can improve the approximation for the stochastic term by

$$
L_i(u) = \frac{L_i(T_{k+1}) + L_i(T_k)}{2}.
$$

This method is known as the predictor-corrector method. The author of [23] proved that this predictor-corrector approximation is better than the approximation presented in equation (3.4.7). This was only shown by using numerical examples.

Figure 3.2 shows two examples of simulated forward rates using the predictor corrector method. The left figure gives twenty simulations of $L(t,T_{20},T_{21})$. While the right figure gives a single simulation for twenty different forward rates. Comparing this to figure 3.1 it is obvious that over the same time period much less computational steps were taken. This results in a faster computational time.



(a) Simulation of $L(t, T_{20}, T_{21})$

(b) Forward rates with different settlement dates

Figure 3.2: Simulation of Libors using the predictor-corrector method

3.4.3. Comparing both simulation methods

Two different methods are proposed to simulate forward rates and both these methods can also be used in a later stadium to simulate the new risk-free-rates. The first method is the Euler's method where many small steps are taken to simulate the different Libors over time. A second method is the predictor-corrector method, where larger time steps are taken in order to decrease computational time. According to [23] the predictor-corrector method achieves the same accuracy as using the Euler method however in a much shorter time span. Two numerical tests are performed to check this statement instead of just assuming it is true.

Since the log-normal Libor market model prices caps and caplets in accordance with Black's formula, it is possible to compare simulated caplet prices from the implemented LMM with analytical caplet prices from Black's equation. By implementing a Monte Carlo simulation both the Euler method as the predictor-corrector method should result in approximately the same caplet price. The accuracy of both implementation methods can be checked, since we know the exact price from Black's equation. If the predictor-corrector method has the same accuracy as the Euler method, but is much faster this method should then be chosen for the new forward market model.

We can also compare both methods by valuating zero-coupon bonds. The exact current price of a zero-coupon bond is given by

$$P(0, T_n) = \prod_{i=0}^{n-1} \frac{1}{1 + \tau L(0, T_i, T_{i+1})}. \tag{3.4.8}$$

Hence, the price of this ZCB can be calculated analytically with the current forward rates (spot rates) calibrated from the market. However, it is also known that under the discrete bank account numéraire the zero-coupon bond price is given as follows

$$\frac{P(t, T_n)}{B(t)} = \mathbb{E}^d \left[\frac{P(S, T_n)}{B(S)} | \mathscr{F}_t \right].$$

Therefore,

$$\frac{P(0, T_n)}{B(0)} = \mathbb{E}^d \left[\frac{P(S, T_n)}{B(S)} | \mathscr{F}_0 \right].$$

Using the definition of a discrete bank account in equation (2.3.3) and taking $S = T_n$

$$\begin{aligned} P(0,T_n) &= \mathbb{E}^d\left[\frac{B(0)P(S,T_n)}{B(S)}\right], \\ &= \mathbb{E}^d\left[\frac{P(S,T_n)}{P(S,T_{q(S)})\prod_{i=0}^{q(S)-1}(1+\tau L(T_i,T_i,T_{i+1})}\right], \\ &= \mathbb{E}^d\left[\frac{P(T_n,T_n)}{P(T_n,T_{q(T_n)})\prod_{i=0}^{q(T_n)-1}(1+\tau L(T_i,T_i,T_{i+1})}\right], \\ &= \mathbb{E}^d\left[\frac{1}{P(T_n,T_{n+1})\prod_{i=0}^{n}(1+\tau L(T_i,T_i,T_{i+1})}\right], \\ &= \mathbb{E}^d\left[\frac{1}{\left(\prod_n^n \frac{1}{1+\tau L(T_n,T_n,T_{n+1})}\right)\prod_{i=0}^{n}(1+\tau L(T_i,T_i,T_{i+1})}\right] = \mathbb{E}^d\left[\prod_{i=0}^{n-1}\frac{1}{1+\tau L(T_i,T_i,T_{i+1})}\right]. \end{aligned} \tag{3.4.9}$$

The market gives the prices of zero-coupon bonds with maturity times T_n. Using equation (3.4.9) it is also possible to obtain the values of these zero-coupon bonds through a simulation of the forward rates. This simulation can be either with Euler's method or the predictor-corrector method.

Thus, there are two different ways to check the accuracy of both methods and inspect their computational time. For both methods a Monte Carlo simulation must be implemented in order to simulate the forward rates and obtain the caplet or zero-coupon bond value. The following steps are performed to derive the values

Validating a model

1. Set the instantaneous volatilities and correlations. Set the forward values at time zero.
2. Calculate the analytical price of the derivative.
   - 2a) Caplet: Using equation (2.4.5)
   - 2b) Zero-coupon-bond: Using equation (3.4.8)
3. Generate correlated Brownian motions using Cholesky's decomposition.
4. Generate the forward rates with different expiry dates.
   - 4a) Euler method: equation (3.4.1)
   - 4b) Predictor-corrector method: equation (3.4.7)
5. Obtain the simulated price of the derivative.
   - 5a) Caplet: equation (2.4.4)
   - 5b) Zero-coupon bond: equation (3.4.9)
6. Use the martingale property to obtain today's price of the derivative.
   - 6a) Caplet: For the caplets, the value at the payment date is known. The following martingale property can be used to obtain today's price

     $$\frac{Cap_j(0)}{B^d(0)} = \mathbb{E}^d\left[\frac{Cap_j(T_{j+1})}{B^d(T_{j+1})}\right]$$

   - 6b) Zero-coupon bond: No need, already obtained in step 5.
7. Perform step 3-6 a total of M times.
8. Take the average to obtain the simulated caplet price or ZCB price.

We will now present a bit more insight into the different steps. In step one we will work with our own chosen values. Later in this thesis the switch to actual market data is made. Step 2 is just the calculation of analytical prices using equations already discussed. Then in step three till five, both the Euler method as the predictor-corrector method are used to generate a path for different forward rates and obtain the payoff of a caplet at it's maturity time or the value of a zero-coupon bond. Since we have now only obtained the payoff of the caplet at the maturity time we have to perform step six to obtain today's simulated value. Step seven and eight are the Monte Carlo simulation.

For the Monte Carlo simulation antithetic sampling is used to reduce the number of paths needed to obtain accurate results. For a Monte Carlo simulation the value of a caplet at is given as

$$\tilde{V}_{caplet}(t_0) = \frac{1}{M}\sum_{j=1}^{M}(V_j(t_0)). \tag{3.4.10}$$

Here $V_j(t_0)$ is the value of a caplet from a single simulation. Note that we could also think of this equation in a vector form. Meaning that every input of the vector is the value of a caplet with a different settlement / payment date. The standard error of the simulation can be obtained through

$$s.e = \frac{\sqrt{\frac{1}{M-1}\sum_{j=1}^{M}(V_j(t_0) - \tilde{V}_{caplet}(t_0))^2}}{\sqrt{M}}. \tag{3.4.11}$$

The results of the validation for zero-coupon bonds can be found in table 3.2 and 3.3. Zero-coupon bonds with eight different maturities ranging from T_1 till T_8 were priced. A total of 50000 Monte Carlo paths were used and between two adjacent tenor points 64 discretization steps were taken, if Euler's method was applied. This number was chosen since a greater number of steps results in a longer computational time while less steps results in less accurate pricing. We decided that 64 discretization steps was a good trade-off between these two. Both tables show the analytical price extracted from the market and for obvious reasons these are the same in both tables. Note that for this simulation we used our own made up data. The simulated price column gives the zero-coupon bond prices obtained from the Monte Carlo simulation. The predictor-corrector method has the same accuracy compared to the Euler method. This can be concluded from approximately the same standard errors of both simulations. Besides both methods seem to be correctly implemented since the simulation prices are close to the true prices.

Next to the standard errors both tables also show the relative errors of the simulation. This is the difference in percentage of the simulated price from the analytical price. We could say with a 95% certainty that the simulated price is within ±1.96 standard errors of the true price. The computational time for Euler's method was 4850 seconds while the predictor-corrector method only took 168 seconds. This is a reduction of almost 29 times. We can thus definitely conclude that the predictor-corrector method should be chosen over Euler's method when implementing the forward market model. The reduction of 29 times isn't a constant number, increasing the number of zero-coupon bonds that are validated would make the gap between the computation times larger and vice versa.

Table 3.2: Pricing zero-coupon bonds using Euler's method

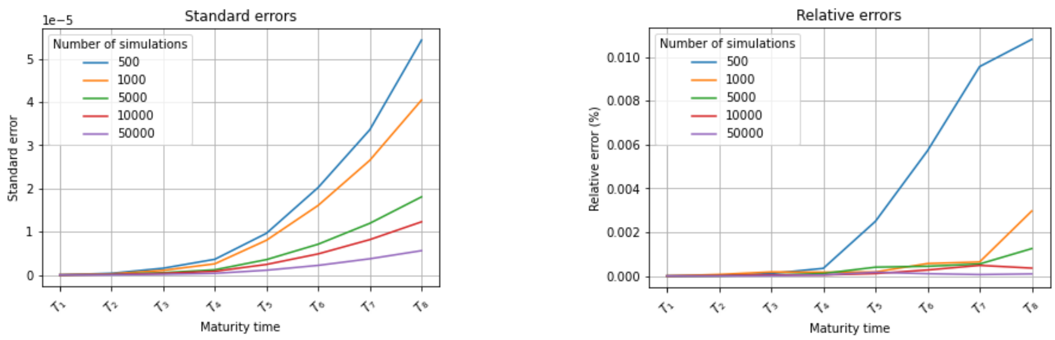
	Analytical value	Simulated value	Error (%)	Standard Error
$P(T_1)$	0.997	0.997	0	0
$P(T_2)$	0.992	0.992	1.02e-06	4.02e-08
$P(T_3)$	0.986	0.986	3.61e-06	1.59e-07
$P(T_4)$	0.979	0.979	1.34e-05	3.69e-07
$P(T_5)$	0.967	0.967	2.82e-05	1.10e-06
$P(T_6)$	0.953	0.953	7.00e-05	2.20e-06
$P(T_7)$	0.939	0.939	2.44e-04	3.72e-06
$P(T_8)$	0.923	0.923	3.78e-04	5.60e-06

Table 3.3: Pricing zero-coupon bonds using the predictor-corrector method

	Analytical value	Simulated value	Error (%)	Standard Error
$P(T_1)$	0.997	0.997	0	0
$P(T_2)$	0.992	0.992	1.03e-06	4.05e-08
$P(T_3)$	0.986	0.986	8.20e-07	1.61e-07
$P(T_4)$	0.979	0.979	2.41e-05	3.72e-07
$P(T_5)$	0.967	0.967	2.87e-05	1.09e-06
$P(T_6)$	0.953	0.953	3.85e-05	2.21e-06
$P(T_7)$	0.939	0.937	1.57e-04	3.75e-06
$P(T_8)$	0.923	0.923	3.47e-04	5.65e-06

We can also examine how the simulation performs for different number of Monte Carlo paths. These results are presented in figure 3.3 and 3.4, respectively for Euler's method and the predictor-corrector method. Both figures (a) show the decrease in standard error when the number of simulations is increased. From equation (3.4.11) it can be observed that increasing the number of Monte Carlo simulations ten times decreases the standard errors with a factor of $\sqrt{10}$. This results in a trade-off between accuracy and computational time which holds for both models. Again from both figures (a) it is clear that the standard error is approximately the same for both methods. While the predictor-corrector method had a much faster computational time.

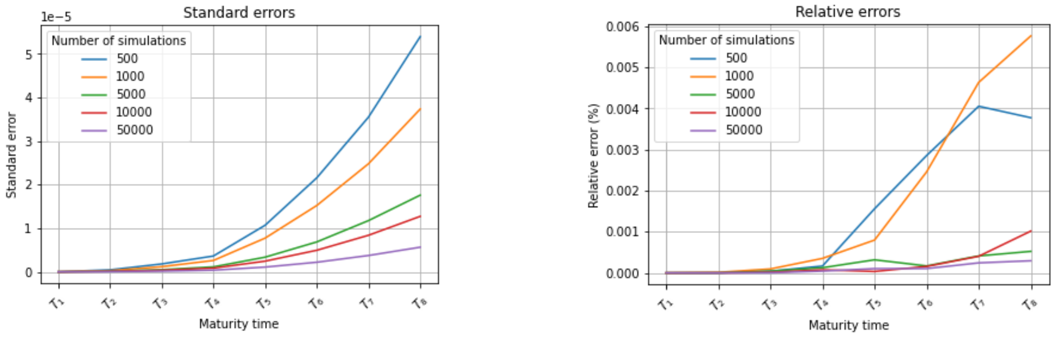
Both figures (b) show the relative error between the simulated and analytical price. These figures tell us less comparing them to the figures (a), since it is possible to have a large standard error but get lucky where the simulated price is extremely close the actual price. It is observable that when we increase the number of Monte Carlo simulations the pricing does get more accurate with less outliers far from the analytical price. To conclude, one method isn't superior compared to the other method. They both give accurate results and the standard errors are of the same magnitude. Even though the relative error for the predictor-corrector method is smaller this is seed dependent and could change for other simulations.



(a) Standard errors

(b) Relative errors

Figure 3.3: Pricing zero-coupon bonds using Euler's method



(a) Standard errors

(b) Relative errors

Figure 3.4: Pricing zero-coupon bonds using the predictor-corrector method

The validation of the model can also be examined by valuating different caplets. The results for these validations are presented in tables 3.4 and 3.5. Eight different caplets were priced where the settlement dates ranged from T_1 till T_8. Again, 50000 Monte Carlo paths were used and the distance between two tenor points was discretized using 64 steps. The column which shows the analytical prices was obtained using Black's formula. A notional of 10000 was used, which means that the prices of the caplets are presented in basis points. All caplets were priced at-the-money therefore, the different results can be compared to each other. Choosing one strike price for all the different caplets could mean that some are really far in-the-money or out-the-money this could result in extremely bad or fortunate results. Just as for the zero-coupon bond case, both Euler's method as the predictor-corrector method show comparable results. The standard errors are of the same order and all caplets are priced correctly. The computational time for Euler's method was 5347 seconds while the predictor-corrector method only took 166 seconds. This is a significant decrease in computational time, which only confirms the conclusion to prefer the predictor-corrector method over Euler's method. Finally, note that the relative error of table 3.4 and table 3.5 can be far of each other. This is however not a problem since we should look at the standard errors and these are very similar. It is further possible that sometimes the simulated price could be almost two standard errors from the analytical price. This happens in 5% of the simulations. Hence, it is seed dependent how close the simulated price will be to the analytical price. Increasing the number of Monte Carlo simulations will result in smaller relative errors, which means that the results of the two tables will be closer to each other.

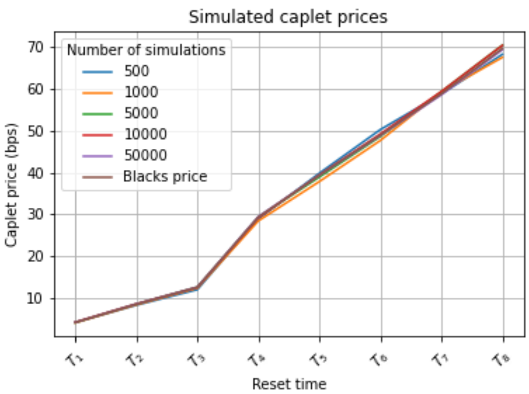
Table 3.4: Pricing caplets using Euler's method

	Analytical price	Simulated price	Error (%)	Standard Error
Cap(T_1, T_2)	4.09	4.09	0.04	0.0146
Cap(T_2, T_3)	8.43	8.45	0.20	0.0311
Cap(T_3, T_4)	12.48	12.49	0.10	0.04716
Cap(T_4, T_5)	29.29	29.24	0.15	0.1156
Cap(T_5, T_6)	39.36	39.25	0.26	0.1585
Cap(T_6, T_7)	48.93	48.76	0.35	0.2019
Cap(T_7, T_8)	58.82	58.67	0.26	0.2466
Cap(T_8, T_9)	69.67	69.52	0.21	0.2964

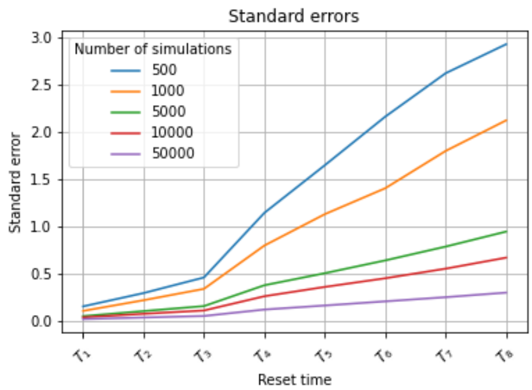
Table 3.5: Pricing caplets using the predictor-corrector method

	Analytical price	Simulated price	Error (%)	Standard Error
Cap(T_1, T_2)	4.09	4.09	0.03	0.0146
Cap(T_2, T_3)	8.43	8.43	0.04	0.0312
Cap(T_3, T_4)	12.48	12.52	0.34	0.0472
Cap(T_4, T_5)	29.29	29.36	0.24	0.1146
Cap(T_5, T_6)	39.36	39.36	0.0004	0.1591
Cap(T_6, T_7)	48.93	49.13	0.41	0.2025
Cap(T_7, T_8)	58.82	59.08	0.44	0.2470
Cap(T_8, T_9)	69.67	69.94	0.38	0.2967

The model performance can also be examined for different number of Monte Carlo paths. Figure 3.5 shows three different figures for Euler's method and figure 3.6 shows similar figures for the predictor-corrector method. The figures (a) show the simulated caplet prices for the different maturities together with a line for the analytical Black's caplet price. We see that for both methods the lines from 50000 simulations are close to the analytical line. Consider the third sub-figures, which show the relative errors comparing the analytical and simulated price. This shows that when the number of Monte Carlo paths is increased, the caplets are priced more and more accurately. From the second sub-figures, which contain the standard errors of the simulations per caplet we see that this decreases when there is an increase in the number of Monte Carlo paths. Hence, increasing the number of paths would result in even more accurate results. Finally, the results are similar for both methods which is as expected since the predictor-corrector method is just as accurate as Euler's method.



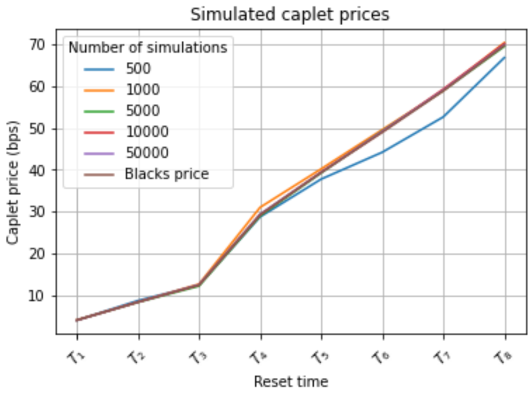
(a) Caplet prices



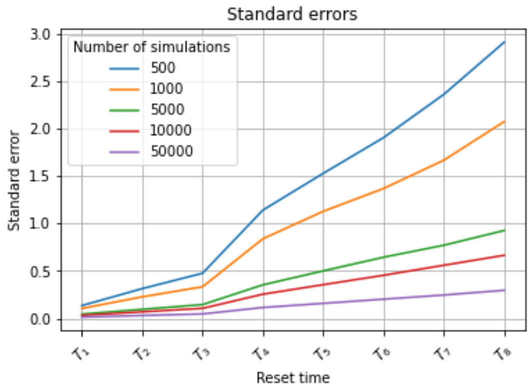
(b) Standard errors

(c) Relative errors price

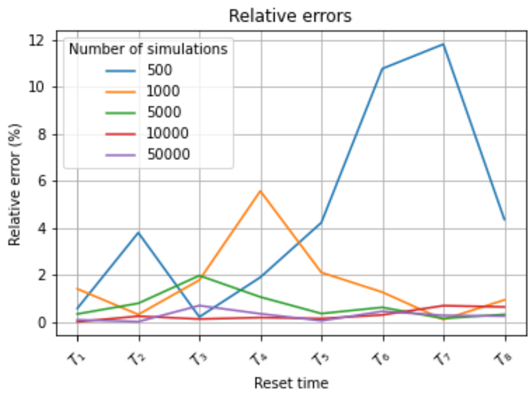
Figure 3.5: Pricing caplets using Euler's method



(a) Caplet prices



(b) Standard errors



(c) Relative errors price

Figure 3.6: Pricing caplets using the predictor-corrector method

Finally, we can also briefly compare the results for the validation of the caplets and the zero-coupon bonds. What is visible is that the accuracy for the caplets remains about the same for all valuations of the different caplets. This can be seen from the third sub-figures in figure 3.5 and figure 3.6. For the different maturities the relative error seems to increase and decrease randomly. Comparing this to the relative error figures for the zero-coupon bonds, it seems that the error increases for ZCB's with a larger maturity. Hence, the accuracy of pricing zero-coupon bonds decreases if the maturity increases. This is because for every single caplet only one Libor is taken into account in the payoff function. Contrary to zero-coupon bonds, looking at equation (3.4.9) we see that for ZCB's with a larger maturity more forward rates are needed to obtain the simulated value. This results in a larger bias for the zero-coupon bond value.

3.5. Summary

This chapter introduced the Libor market model which is currently used to price exotic interest rate derivatives. First, the dynamics of the forward rates under the spot-Libor measure, which uses the discrete bank-account as numéraire, were derived. In this thesis all simulations are performed under the spot-Libor measure since the bias of the drift term will be more evenly spread over the different rates compared to simulating under an arbitrary T_k-forward measure. For the forward rates a log-normal distribution was assumed. Caps and caplets are extremely liquid products in the market and are priced using Black's equation. By choosing a log-normal distribution, exotic interest rate derivatives are priced consistently with cap and caplet prices from the market. For the dynamics a constant structure for the instantaneous volatilties, which can differ per forward rate was assumed. Besides, we choosed an own structure for the instantaneous correlations even though they are usually calibrated to swaption prices.

The second part of the chapter introduced two simulation techniques to implement the Libor market model. The first one is the Euler method, which is based on a discretization of the time grid into very small time steps and simulates the rates over these steps. The second simulation method is the predictor-corrector method, which enabled us to simulate the forward rates from one tenor point to the next tenor point. The predictor-corrector method could be implemented since this thesis just considers simple interest rate derivatives with payoff functions only applicable at tenor points. Finally, the two simulation methods were compared to each other by examining their accuracy. This comparison was performed by pricing caplets and zero-coupon bonds and comparing these simulated prices to the analytical prices. The analytical caplet prices were obtained from Black's equation and the zero-coupon bond values could be directly obtained from the market. We saw that neither of the methods is superior compared to the other in terms of accuracy, they both priced the derivatives with the same accuracy. However, the predictor-corrector method had a much shorter computational time. Therefore, the predictor-corrector method will be preferred over the Euler method.

4

Generalized Forward Market Model

In this chapter we will present the generalized forward market model. This model extends on the Libor market model and gives the dynamics of the new risk-free-rates. The forward market model was first introduced by the authors of [25] and it can model both forward-looking as backward-looking rates under the same stochastic process. Besides the dynamics of both rates under the risk-neutral measure can be obtained which was impossible for the Libor market model.

We will start with a brief introduction about the forward market model. In the next paragraph we will define the concept of extended zero-coupon bonds after which we will move on to defining the new backward-looking rates. We will then derive the dynamics of the rates for the forward market model. When the dynamics are obtained we are able to do some simulations and validations which will be presented in the final paragraph.

4.1. Introduction to the forward market model

The forward market model can be seen as an extension of the Libor market model. The model is build up on the concept of extended zero-coupon bonds which will be defined in the next paragraph. The difference with the LMM is that for the Libor market model the rates were only defined till their starting point. For the forward market model, the rates are defined for every time, this includes the accrual period and the time after this period. This is exactly what we are looking for in a new type of model. Examining figure 4.1 we know that in the post-Libor world the backward-looking rates are unknown till the end of their accrual period. This means that they keep evolving till time T_2 in the figure. However in the Libor market model the evolution of the rates is only defined till the start of the period (T_1 in the figure). For the new forward market model the rates are also defined after that time point which is thus needed for the simulation of the new backward rates.

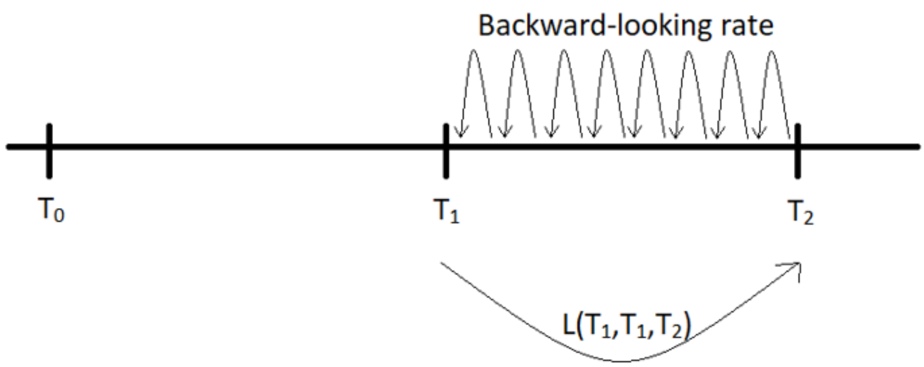


Figure 4.1: Extended Rates

Considering that banks have contracts which depend on forward rates with a maturity of 20 or even 30 years it is important that we are still able to generate forward rates. Since the FMM also gives the dynamics of forward rates this new model should always be chosen over the Libor market model.

To stress the importance of the new backward-looking rates and the forward market model once more. The Libors were widely used as a benchmark for other interest rates, for example on a mortgage or student loan. Besides many interest rate derivatives depend on the Libor. These derivatives are used by financial institutions to hedge against movements in the interest rates. When the Libor ceases to exist it will leave a giant hole in the financial world which can have an impact on everybody. It is thus of great importance to introduce a new benchmark for derivatives and other interest rates which comes in the form of the new backward-looking rates. With these new rates a new type of model must be introduced. Thus, the urgency for the FMM follows from the significance of the new backward rates.

4.2. Preliminaries

For the generalized forward market model we will use the tenor structure presented in chapter 2. However, the τ_n parameter which was the difference between two adjacent tenor points will be changed. From now on we assume $\tau_n = T_n - T_{n-1}$. Note that for the LMM the definition was $\tau_n = T_{n+1} - T_n$. Besides, the index function is chosen to be q(t) = $\min\{j : T_j \leq t\}$. This is the same index function as in [25] and different from the one we used for the LMM. This change is made because it can become really confusing when trying to change the derivations for the FMM using the index function for the LMM. We know that the value of a zero-coupon bond with maturity T at time t under the instantaneous risk-free-rate is given by

$$P(t,T) = \mathbb{E}\left[e^{-\int_t^T r(u)du} | \mathscr{F}(t)\right].$$

This definition for a zero-coupon bond is only defined for $t \leq T$ after which the contract expires. It is possible to extend this definition of a zero-coupon bond for any $t > T$ as follows:

$$P(t,T) = \mathbb{E}\left[e^{\int_T^t r(u)du} | \mathscr{F}(t)\right] = e^{\int_T^t r(u)du} = \frac{B(t)}{B(T)}. \tag{4.2.1}$$

The second equality sign comes from the fact that everything is $\mathscr{F}(t)$-measurable. This is the definition of an extended zero-coupon bond which has as maturity T but runs till time t. Note that $P(t,0) = B(t)$. This means that the bank-account is the same as an extended zero-coupon bond which expires immediately. The concept of extended zero-coupon bonds is not new. For example, it was already used by the authors of [9].

With the concept of extended zero-coupon bonds, the following self-financing strategy can be defined. If $t \leq T$ we buy the zero-coupon bond with maturity T and when $t > T$ we reinvest the proceeds in the risk free rate. Calling this strategy Z, then the value of the strategy over time is given as

$$Z_T(t) = \begin{cases} P(t,T) & \text{for } t \leq T \\ e^{\int_T^t r(u)du} = \frac{B(t)}{B(T)} & \text{for t > T} \end{cases}$$

Examining this strategy, if $t \leq T$ we are holding an ordinary zero-coupon bond. If t is larger than T we obtain the exact payoff as the extended zero-coupon bond in equation (4.2.1). Therefore, $Z_T(t) = P(t,T)$ for every T at all times t, due to the definition of the extended zero-coupon bond. We will refer to this strategy as the extended zero-coupon bond with maturity T.

The definition of the extended zero-coupon bond can be used as numéraire since it is a trade-able asset and it has the value of a self-financing strategy which is always positive. We call the measure, which has as numéraire the extended zero-coupon bond, the extended T-forward measure Q^T. Thus this is the equivalent martingale measure associated with the extended zero-coupon bond price $P(t,T)$. It is very important to remember that this measure is defined for every time t and not only for $t \leq T$ as for the classic forward measure. By the definition of the self-financing strategy the measure is a hybrid measure consisting of the classic T-forward measure up to T after which it switches to the risk-neutral measure of the continuous bank account. The fact that it is a hybrid measure also follows from the self-financing strategy Z. Therefore, because of the definition of extended zero-coupon bonds we will be able to extend the evolution of interest rates under all times t instead for only $t \leq T$.

4.3. Backward-looking and backward-looking forward rates

Before going over the generalized forward market model we must be clear about an important difference between the new rates we will see in the market. Here the difference between backward-looking and backward-looking forward rates is meant.

The backward-looking rate ($R(T_{j-1}, T_j)$) can be described as the rate we will actually see in the market at the end of an application period. This rate can also be referred to as the market risk-free term rate. Say for example that we took tenor points 3 months apart and we are at the end of an application period, so at time T_j. Now the backward-looking rate will be the rate which compounds all of the overnight rates over the period $[T_{j-1}, T_j]$. The overnight rates are known at the end of the period and given by the market. We can thus use the compounded setting-in-arrears method to obtain this backward-looking rate.

The backward-looking forward rate will be the backward-looking rate for a future period seen from an earlier time point. Thus, if we have not yet arrived at the end of an application period for the backward-looking rate, which means that $t < T_j$. We must give some projection of what the backward-looking rate will be over that period. This is also the rate that the forward market model will simulate for $t \in [T_0, T_j)$. We can simulate the projection of the backward-looking forward rate until the final time T_j. It makes sense that we would favor our backward-looking forward rate to be equal to the backward-looking rate at maturity time T_j. This will also come back in the definition of the backward-looking forward rate.

To make the difference between the two rates a bit more clear take a look at the timeline plot in figure 4.2. Here the backward-looking rate will only be known at time T_j where it is possible to compound all the given overnight risk-free-rates. On the contrary we will simulate the backward-looking forward rate over the timeline till we arrive at T_j where we want both the rates to be equal.

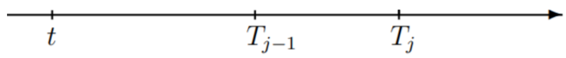


Figure 4.2: Timeline

4.3.1. Backward-looking rate

This section defines the backward-looking rate after which we can continue with the backward-looking forward rates. The backward-looking rate will be the daily compounded setting-in-arrears rate based on the overnight risk-free-rates. For every $j = 1, \dots, N$ the backward-looking rate will be defined as $R(T_{j-1}, T_j)$ with accrual period $[T_{j-1}, T_j]$. In order to do this, first r_i is defined to be the overnight rate on day i and δ_i to be the corresponding day-count fraction between T_{j-1} and T_j. As assumption n days are taken in the accrual period. The backward-looking rate for accrual period $[T_{j-1}, T_j]$ can then be approximated as

$$\begin{aligned} R(T_{j-1}, T_j) &= \frac{1}{\tau_j}\left(\prod_{i=1}^{n}(1 + r_i\delta_i) - 1\right) \\ &\approx \frac{1}{\tau_j}\left(e^{\int_{T_{j-1}}^{T_j} r(u)du} - 1\right) = \frac{1}{\tau_j}\left(\frac{B(T_j)}{B(T_{j-1})} - 1\right) = \frac{1}{\tau_j}\left(P_{j-1}(T_j) - 1\right). \end{aligned} \tag{4.3.1}$$

Here the limit of the day-count fraction is taken to zero, which gives the approximation. Finally, notice that the new defined extended zero-coupon bond is used in the final step. In order to obtain a value for the backward-looking rate we must wait till the end of the accrual period (T_j). Then all overnight rates are known, these can be compounded and the rate will be fixed.

Likewise, the forward rate can be examined over the same period. The forward is rate is known at the start of the accrual period (T_{j-1}) and fixed after this date. The following equation must hold in order to have no arbitrage:

$$L(T_{j-1}, T_j) = \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j)|\mathscr{F}_{T_{j-1}}\right]. \tag{4.3.2}$$

If the equation doesn't hold it means that the forward rate (over $[T_{j-1}, T_j]$) which is fixed at time T_{j-1} isn't equal to the expectation of what the backward-looking rate will be over the same accrual period. While the backward-looking rate should actually be a replacement for the forward rate.

4.3.2. Backward-looking in-arrears forward rates

This section will define the backward-looking forward rate. This rate will be denoted as $R(t, T_{j-1}, T_j)$, for convenience we often use the shorter notation $R_j(t)$. For example, $R_j(0)$ is today's projection of what the backward-looking rate for the future interval $[T_{j-1}, T_j)$ should be. This rate needs to be simulated since we have not arrived at the end of that period yet. The definition of the backward-looking forward rate is obtained from [25]:

Definition 4.3.1. (backward-looking forward rate) 'The backward-looking forward rate $R_j(t)$ at time t is the value of the fixed rate K_R in the swaplet paying: $\tau_j[R(T_{j-1}, T_j) - K_R]$ at time T_j such that the swaplet has zero value at time t.'

Notice that in the definition the backward-looking forward rate is chosen to be equal to the backward-looking rate at maturity time. This makes sense to do because the simulated rate will then be equal to what the true rate will be.

Since we have just given an expression for $R(T_{j-1}, T_j)$ in equation (4.3.1), it is possible to fill this in into the definition of the backward-looking forward rate. Therefore, we can solve it by setting the value of the swaplet equal to zero at time t. The payoff of the caplet at time T_j will be equal to $e^{\int_{T_{j-1}}^{T_j} r(u)du} - 1 - \tau_j K_R$. In order to set this value of the payoff to be equal to zero at time t, the expectation under the extended T_j forward measure is taken which gives the following expression

$$\mathbb{E}^{T_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)du} - 1 - \tau_j K_R|\mathscr{F}_t\right] = 0.$$

By linearity of the expectation operator and because in the definition $K_R = R_j(t)$, if the equation is zero we obtain

$$\begin{aligned}\tau_j K_R &= \mathbb{E}^{T_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)du} - 1|\mathscr{F}_t\right],\\ K_R &= \mathbb{E}^{T_j}\left[\frac{1}{\tau_j}e^{\int_{T_{j-1}}^{T_j} r(u)du} - 1|\mathscr{F}_t\right],\\ R_j(t) &= \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j)|\mathscr{F}_t\right].\end{aligned} \tag{4.3.3}$$

Comparing equation (4.3.2) with equation (4.3.3) it is clear that for each $j = 1, ..., N$ at $t = T_{j-1}$

$$L(T_{j-1}, T_j) = R_j(T_{j-1}). \tag{4.3.4}$$

Therefore, the backward-looking forward rate at the start of the accrual period is equal to the forward-looking rate over the same period.

It is also possible to derive an expression for the backward-looking forward rate under the risk-neutral measure, starting from equation (4.3.3)

$$\begin{aligned}R_j(t) = \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j)|\mathscr{F}_t\right] &= \mathbb{E}^{T_j}\left[\frac{1}{\tau_j}e^{\int_{T_{j-1}}^{T_j} r(u)du} - 1|\mathscr{F}_t\right],\\ 1 + \tau_j R_j(t) &= \mathbb{E}^{T_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)du}|\mathscr{F}_t\right].\end{aligned} \tag{4.3.5}$$

The Radon-Nikodym derivative is used to change measures:

$$\frac{dQ}{dQ^{T_j}} = \frac{B(T_j)}{B(t)}\frac{P(t, T_j)}{P(T_j, T_j)}.$$

By the definition of an expectation, the expectation in the right-hand side of equation (4.3.5) is the same as:

$$
\begin{aligned}
&\mathbb{E}^{T_j}\left[e^{\int_{T_{j-1}}^{T_j} r(u)du}|\mathscr{F}_t\right] = \int_\Omega e^{\int_{T_{j-1}}^{T_j} r(u)du} dQ^{T_j} = \int_\Omega e^{\int_{T_{j-1}}^{T_j} r(u)du} \frac{B(t)}{B(T_j)}\frac{P_j(T_j)}{P_j(t)} dQ,\\
&= \frac{1}{P_j(t)}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_j} r(u)du} e^{\int_{T_{j-1}}^{T_j} r(u)du}|\mathscr{F}_t\right] = \frac{1}{P_j(t)}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^{T_{j-1}} r(u)du}|\mathscr{F}_t\right], \qquad (4.3.6)\\
&= \frac{P_{j-1}(t)}{P_j(t)}.
\end{aligned}
$$

Where $\frac{1}{P_j(t)}$ is taken out of the expectation since it is $\mathscr{F}_t$-measurable. The last equality follows from the definition of extended zero-coupon bonds. Combining equation (4.3.5) together with (4.3.6), we can conclude that the formula for the backward-looking forward rate is equal to

$$
R_j(t) = \frac{1}{\tau_j}\left[\frac{P_{j-1}(t)}{P_j(t)} - 1\right]. \qquad (4.3.7)
$$

This formula is the same as the classic simply-compounded forward rate formula, however now the formula holds for every time t even after T_{j-1} where the classic forward rate stops. Thus, by the definition of the extended zero-coupon bond it is possible to give an expression for the new backward-looking rates which is defined for all time t.

From equation (4.3.7) we can see four different properties for the backward-looking forward rate:

1. Property 1: The backward-looking forward rate is a martingale under the extended T_j-forward measure. We can derive this by first rewriting equation (4.3.7) into the following form

$$
R_j(t)P_j(t) = \frac{1}{\tau}(P_{j-1}(t) - P_j(t))
$$

Examining the dynamics of the backward-looking forward rate under the extended T_j-forward measure we see that it is a martingale

$$
\begin{aligned}
\frac{R_j(t)P_j(t)}{P_j(t)} &= \mathbb{E}^{T_j}\left[\frac{R_j(S)P_j(S)}{P_j(S)}|\mathscr{F}_t\right],\\
R_j(t) &= \mathbb{E}^{T_j}\left[R_j(S)|\mathscr{F}_t\right].
\end{aligned}
$$

This is a similar derivation as for the forward-looking rates in the LMM. Just as for the LMM only one specific backward-looking forward rate is martingale under one specific extended T_j-forward measure. All other rates are martingales under other extended forward measures.

2. Property 2: The backward-looking forward rate is equal to the forward-looking spot rate at time T_{j-1}. Hence $R_j(T_{j-1}) = L(T_{j-1}, T_j)$. This follows easily from equation (4.3.7).

$$
R_j(T_{j-1}) = \frac{1}{\tau}\left(\frac{P_{j-1}(T_{j-1})}{P_j(T_{j-1})} - 1\right) = \frac{1}{\tau}\left(\frac{P_{j-1}(T_{j-1})}{P_j(T_{j-1})} - \frac{P_j(T_{j-1})}{P_j(T_{j-1})}\right) = \frac{1}{\tau}\left(\frac{P_{j-1}(T_{j-1}) - P_j(T_{j-1})}{P_j(T_{j-1})}\right) = L(T_{j-1}, T_j).
$$

3. Property 3: The backward-looking forward rate is equal to backward-looking rate at maturity time T_j. Hence $R_j(T_j) = R(T_{j-1}, T_j)$. First of all, this is true because of the definition of the backward-looking forward rate. The property can also be derived by filling in t = T_j in equation (4.3.7) and compare it with the equation for the backward-looking rate, equation (4.3.1).

$$
\begin{aligned}
R_j(T_j) &= \frac{1}{\tau}\left(\frac{P_{j-1}(T_j)}{P_j(T_j)} - 1\right) = \frac{1}{\tau}(P_{j-1}(T_j) - 1),\\
R(T_{j-1}, T_j) &= \frac{1}{\tau}(P_{j-1}(T_j) - 1).
\end{aligned}
$$

4. Property 4: The backward-looking forward rate is fixed after time T_j. Hence $R_j(t) = R(T_{j-1}, T_j), \quad t > T_j$. This is an important property since we don't want the backward-looking forward to evolve when

the accrual period is over. For the derivation we must assume that $t > T_j$, which means that everything up to time t is known. This includes time T_j and T_{j-1}. Writing out equation (4.3.7) for $t > T_j$, where the definition for extended zero-coupon bonds is used, we obtain

$$R_j(t) = \frac{1}{\tau_j}\left[\frac{P_{j-1}(t)}{P_j(t)} - 1\right] = \frac{1}{\tau_j}\left[\frac{P(t,T_{j-1})}{P(t,T_j)} - 1\right] = \frac{1}{\tau_j}\left[\frac{B(t)}{B(T_{j-1})}\frac{B(T_j)}{B(t)} - 1\right] = \frac{1}{\tau_j}\left[\frac{B(T_j)}{B(T_{j-1})} - 1\right].$$

Where in the last equation everything is known and stays fixed if $t > T_j$.

4.3.3. Forward-looking forward rate

In the introduction of this thesis it was already mentioned that one of the reasons the FMM should always be chosen over the LMM is because it is still possible to simulate the forward-looking forward rates under the same stochastic process. The classic forward rate (forward-looking forward rate) given by $L_j(t)$ which is a shorter notation for $L(t, T_{j-1}, T_j)$ is the rate seen from time t over some future period $[T_{j-1}, T_j]$. The definition given by [25] is used for this rate.

Definition 4.3.2. (forward-looking forward rate) 'The forward-looking forward rate is given by the value of the fixed rate K_L in the swaplet that pays $\tau_j[L(T_{j-1}, T_j) - K_L]$ at time T_j such that the swaplet has zero value at time t.' See figure 4.3

Here $L(T_{j-1}, T_j)$ is the actual forward rate which is only known at time T_{j-1}. Remember the same difference between backward-looking rates and backward-looking forward rates. Taking the expectation under the T_j-forward measure to obtain the value of the swaplet at time t:

$$0 = \mathbb{E}^{T_j}\left[\tau_j(L(T_{j-1}, T_j) - K_F)|\mathscr{F}_t\right].$$

From this equation it follows that $K_L = L_j(t) = \mathbb{E}^{T_j}\left[L(T_{j-1}, T_j)|\mathscr{F}_t\right]$. By using equation (4.3.2) we see that

$$L_j(t) = \mathbb{E}^{T_j}\left[R(T_{j-1}, T_j)|\mathscr{F}_t\right] = R_j(t).$$

Here a combination of equation (4.3.2) and equation (4.3.4) is used for the last equality. Thus, the backward-looking forward rate is the same as the forward-looking forward rate for $t \le T_{j-1}$. The forward-looking rate fixes at time T_{j-1} and equals $L(T_{j-1}, T_j)$ if $t > T_{j-1}$, while the backward-looking forward rate keeps evolving till time T_j. We can simulate the new backward-looking forward rates over the period $[0, T_{j-1}]$ using the new FMM. From the same simulation we immediately obtain the forward-looking rates since they have the exact same dynamics for $t \le T_{j-1}$.

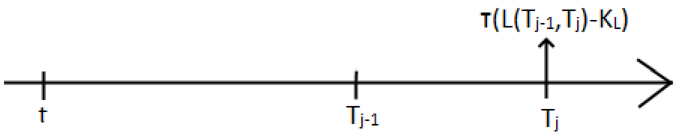


Figure 4.3: Forward-looking rate definition

4.4. The Generalized Forward Market Model

The generalized forward market assumes the same dynamics for the rates as for the LMM. The dynamics under the extended T_j-forward measure are introduced as

$$dR_j(t) = \mu_j(t)dt + \bar{\sigma}(t)dW_j(t), \tag{4.4.1}$$

where we are again free to choose the specific dynamics for $\bar{\sigma}$. It is known that each rate $R_j(t)$ is a martingale under its appropriate extended T_j-forward measure (Property 1). Therefore, under this measure the dynamics are driftless. Remember that each extended T_j-forward only has one backward-looking forward rate which behaves as a martingale. Just as for the LMM, the rates are assumed to be log-normally distributed. Thus, the log-normal forward market model is considered. The log-normal dynamics are chosen because for the same reasons as for the LMM. With the log-normal forward market model it is possible to price interest rate derivatives in the same way as the most liquid products (caps/caplets) in the market are priced. Consequently, the following specification for the volatility parameter is chosen

$$dR_j(t) = \sigma_j(t)R_j(t)\mathbb{1}_{\{t\leq T_j\}}dW_j(t). \tag{4.4.2}$$

Here the indicator function is used in order to guarantee the fourth property, which said that the rate is fixed after maturity time T_j.

In equation (4.4.2), $\sigma_j(t)$ is again the instantaneous volatility. For the new backward-looking forward rates we must take a closer look at how the instantaneous volatility should behave over time. Especially for the accrual period, which is the period between the second-to-last and last tenor points where the compounding of overnight rates takes place. Consider figure 4.4 and assume that the difference between two tenor points is 3 months. Examining the period $[T_{j-1}, T_j]$ of the upper figure. Here we are at the start of the accrual period and have 90 days left with unknown overnight rates which need to be compounded in order to obtain the backward-looking rate. In the bottom picture only have a few days are left with unknown overnight rates and the majority is already known and fixed. These last few days won't have a great impact on what the realized backward-looking rate will be in the end. Since it is only a few days of compounding left compared to almost 90 days. Thus, the backward-looking rate is becoming less and less volatile as more daily rates are fixed. This leads to the conclusion that the volatility parameter of the backward-looking forward rate should decrease over its accrual period ($[T_{j-1}, T_j]$).

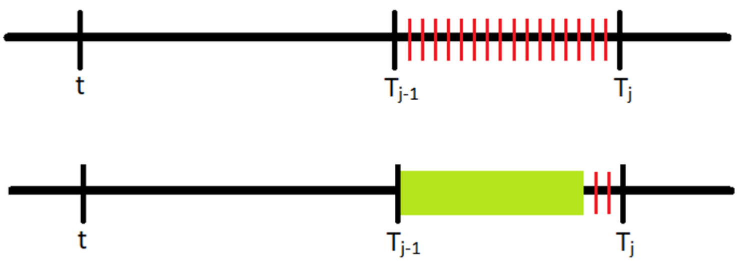


Figure 4.4: Decreasing volatility

This decrease in volatility will be incorporated into the dynamics as follows

$$dR_j(t) = \sigma_j(t)R_j(t)\gamma_j(t)dW_j(t). \tag{4.4.3}$$

Here $\gamma_j(t)$ is a differentiable and decreasing function in the period (T_{j-1}, T_j). We are free to choose our own γ function as long as it follows the two properties. In this thesis the following expression for γ is chosen: $\min\left(\frac{(T_j-t)^+}{(T_j-T_{j-1})}, 1\right)$. Three different periods for $\gamma_j(t)$ can be distinguished:

1. The period $[0, T_{j-1})$. Here $\gamma_j(t) = 1$, which means that the backward-looking forward rate has the same dynamics as the forward-looking forward rates for $t \leq T_{j-1}$.

2. The accrual period $[T_{j-1}, T_j)$. Here $\gamma_j(t)$ slowly decreases to zero. This means that the decrease in volatility in the dynamics are incorporated.

3. The period where $t \geq T_j$. Here $\gamma_j(t) = 0$, which means that the fourth property where the dynamics do not change after the maturity date still holds. Therefore, the indicator function is no longer needed.

The dynamics of each backward-looking forward rate under its appropriate extended T_j-forward measure is a martingale. However, just as for the LMM we would like to derive the dynamics under the spot-Libor measure due to a more evenly spread bias in the rates. The dynamics of the backward-looking rates under the risk-neutral and under the spot-Libor measure will now be derived. In order to obtain the dynamics under these measures the change of measure technique introduced by [4] is used. This gives the drift under the new measure given the dynamics from the measure we start from:

$$\text{Drift}(R_j; Q^N)(t) = \frac{dR_j(t)d\ln(N(t)/P(t,T_j))}{dt}. \tag{4.4.4}$$

Here the dynamics of the backward-looking forward rate ($dR_j(t)$) under the T_j-forward measure which has as numéraire $P(t,T_j)$ are known. We would like to obtain the dynamics under some arbitrary measure Q^N which has as numéraire N(t). For a derivation of this formula we refer to [4].

4.4.1. Dynamics under the risk-neutral measure

Lemma 4.4.1. The process $R_j(t)$ under the risk-neutral measure $\mathbb{Q}$ is given by

$$dR_j(t) = \sigma_j(t)\gamma_j(t)R_j(t)\left(\sum_{i=q(t)}^{j}\rho_{i,j}\frac{\tau_i\sigma_i(t)\gamma_i(t)R_i(t)}{1+\tau_iR_i(t)} + dW_j^{\mathbb{Q}}(t)\right). \tag{4.4.5}$$

where $W_j^{\mathbb{Q}}(t)$ is a Brownian motion under the measure.

Proof. To obtain the dynamics under the risk-neutral measure equation (4.4.4) is used. The switch is made from the extended T_j-forward measure to the risk-neutral measure, which means that in the equation: N(t) = B(t). Remember that for the LMM it was impossible to derive the dynamics for the forward rates under the risk-neutral measure. Filling this in gives

$$\text{Drift}(R_j;Q)(t) = \frac{dR_j(t)d\ln(B(t)/P(t,T_j)))}{dt}. \tag{4.4.6}$$

Writing out the natural-logarithmic term using the definitions of the bank-account and extended zero-coupon bond, the expression can be written as

$$\ln\frac{B(t)}{P(t,T_j)} = \ln\frac{P(t,0)}{P(t,T_j)} = \ln\prod_{i=1}^{j}\frac{P(t,T_{i-1})}{P(t,T_i)} = \ln\prod_{i=1}^{j}(1+\tau_iR_i(t)) = \sum_{i=1}^{j}\ln(1+\tau_iR_i(t)). \tag{4.4.7}$$

The second equality comes from the fact that it is a telescopic series. The third equality comes from equation (4.3.7). Filling this term back into equation (4.4.6) we obtain

$$\begin{aligned}
\text{Drift}(R_j;Q)(t) &= \frac{dR_j(t)d\sum_{i=1}^{j}\ln(1+\tau_iR_i(t))}{dt},\\
&= \sum_{i=1}^{j}\frac{dR_j(t)d\ln(1+\tau_iR_i(t))}{dt},\\
&= \sum_{i=1}^{j}\frac{\tau_i}{1+\tau_iR_i(t)}\frac{dR_j(t)dR_i(t)}{dt},\\
&= \bar{\sigma}_j(t)\gamma_j(t)\sum_{i=1}^{j}\rho_{i,j}\frac{\tau_i\bar{\sigma}_i(t)\gamma_i(t)}{1+\tau_iR_i(t)},\\
&= \sigma_j(t)R_j(t)\gamma_j(t)\sum_{i=1}^{j}\rho_{i,j}\frac{\tau_i\sigma_i(t)R_i(t)\gamma_i(t)}{1+\tau_iR_i(t)}.
\end{aligned}$$

Walking through this step by step: first, the term of equation (4.4.7) is filled in equation (4.4.6). Then the summation sign is taken to the front since the other terms do not depend on i. Next, the chain rule is applied for the derivative of the logarithmic term. Then the term dR_jdR_i is written out using the fact that under the extended T_j-forward measure it is martingale and is driftless. Finally, log-normal dynamics are chosen for $\bar{\sigma}$.

The drift for the backward-looking forward rates under the risk-neutral measure is now given. Therefore, the dynamics of the rates under the risk-neutral measure are given as

$$dR_j(t) = \sigma_j(t)\gamma_j(t)R_j(t)\left(\sum_{i=1}^{j}\rho_{i,j}\frac{\tau_i\sigma_i(t)\gamma_i(t)R_i(t)}{1+\tau_iR_i(t)} + dW_j^{Q}(t)\right).$$

Finally we take into account that $\gamma_i(t)$ is equal to zero if $t \geq T_i$ in the summation. Remember that $q(t) = \min\{j : T_j \geq t\}$. So for values smaller than q(t) it means that $t \geq T_i$ and the summation value is zero. □

4.4.2. Dynamics under the spot-Libor measure

Lemma 4.4.2. The process $R_j(t)$ under the spot-Libor measure Q^d is given by

$$dR_j(t) = \sigma_j(t)\gamma_j(t)R_j(t)\left(\sum_{i=q(t)+1}^{j} \rho_{i,j}\frac{\tau_i\sigma_i(t)\gamma_i(t)R_i(t)}{1+\tau_i R_i(t)} + dW_j^{Q^d}(t)\right). \tag{4.4.8}$$

where $W_j^{Q^d}(t)$ is a Brownian motion under the measure.

Proof. Starting from the equation of the discrete bank account, equation (2.3.3).

$$B_d(t) = P(t, q(t))\prod_{i=1}^{q(t)}(1+\tau L(T_i, T_{i-1}, T_i).$$

Using

$$L(t, T_{i-1})P(t, T_i) = \frac{1}{\tau}(P(t, T_{i-1}) - P(t, T_i)),$$
$$1+\tau L(t, T_{i-1}) = \frac{P(t, T_{i-1})}{P(t, T_i)}.$$

The discrete bank account equation can be written as

$$B_d(t) = P(t, q(t))\prod_{i=1}^{q(t)}\frac{P(t, T_{i-1})}{P(t, T_i)} = P(t, q(t))\prod_{i=1}^{q(t)}\frac{1}{P(T_{i-1}, T_i)},$$

To derive the dynamics under the spot-Libor measure we will choose $N(t) = B_d(t)$ and insert this into equation (4.4.4)

$$\text{Drift}(R_j; Q^d)(t) = \frac{dR_j(t)d\ln(B_d(t)/P(t, T_j)))}{dt} = \frac{dR_j(t)d\ln(P(t, q(t))/P(t, T_j)))}{dt}.$$

It is possible to work out the part in the logarithmic-term, again using a telescopic series

$$\ln\frac{P(t, q(t))}{P(t, T_j)} = \ln\prod_{i=q(t)+1}^{j}\frac{P(t, T_{i-1})}{P(t, T_i)} = \ln\prod_{i=q(t)+1}^{j}(1+\tau_i R_i(t)) = \sum_{i=q(t)+1}^{j}\ln(1+\tau_i R_i(t)).$$

Following the same reasoning as for the risk-neutral case, the drift term is equal to:

$$\text{Drift}(R_j; Q^d)(t) = \sigma_j(t)\gamma_j(t)R_j(t)\sum_{i=q(t)+1}^{j}\rho_{i,j}\frac{\tau_i\sigma_i(t)\gamma_i(t)R_i(t)}{1+\tau_i R_i(t)}.$$

This is the final drift form and proves the lemma. □

4.5. Validation of the model

In this final subsection of the chapter the implementation of the generalized forward market model is discussed. Before pricing exotic derivatives using the new FMM it is important to know if the model is implemented in a correct way. The implementation will be validated by pricing a new type of caplet, the backward-looking caplet and by pricing zero-coupon bonds. Just as for the LMM it is possible to validate the model using simple products as caplets since the analytical price of these products can be obtained. First a new type of caplet for the post-Libor world is introduced, after which the implementation of the FMM is presented and finally the validation of the generalized forward market model is discussed.

4.5.1. Caplets

In the post-Libor world, with the new backward-looking forward rates two type of caplets can be constructed for every accrual period $[T_{j-1}, T_j]$. The first one resembles the same kind of caplet as in today's Libor world. The payoff at time T_j with strike K of this caplet is equal to

$$\max(R(T_{j-1}, T_{j-1}, T_j) - K, 0). \tag{4.5.1}$$

Hence, this is a forward looking caplet, the payoff at T_j is equal to the backward-looking forward rate given at T_{j-1} over the interval $[T_{j-1}, T_j]$. Because the backward-looking forward rates are exactly equal to forward rates at T_{j-1} (property 2) this caplet can be modelled in the exact same way as a caplet using forward rates in the Libor market model. The same payoff is obtained and there is no need to validate this method anymore. Property two of the backward-looking forward rate assures that this forward-looking caplet will have exactly the same value as a caplet with reset date T_{j-1} in today's market.

The second type of caplet has a payoff at time T_j equal to

$$\max(R(T_j, T_{j-1}, T_j) - K, 0). \tag{4.5.2}$$

Hence, this is a backward-looking caplet, where the rate is not known at the reset date of the caplet but only at the payment date. To obtain the payoff of this caplet we thus look back over its accrual period. This type of caplet has more uncertainty in the payoff since the backward rate has more time to evolve. The rate for the forward-looking caplet is known at T_{j-1} while for the backward-looking caplet it can evolve till time T_j. This increase in uncertainty logically means that the price of the backward-looking caplet must be higher than the forward-looking one. This can also be derived by applying the tower property and using the martingale property of the backward-looking forward rates:

$$\begin{aligned}
\mathbb{E}^{T_j}\left[\max(R_j(T_j) - K, 0)|\mathscr{F}_t\right] &= \mathbb{E}^{T_j}\left[\mathbb{E}^{T_j}\left[\max(R_j(T_j) - K, 0)|\mathscr{F}_{T_{j-1}}\right]|\mathscr{F}_t\right] \\
&\geq \mathbb{E}^{T_j}\left[\max\left(\mathbb{E}^{T_j}\left[R_j(T_j)|\mathscr{F}_{T_{j-1}}\right] - K, 0\right)|\mathscr{F}_t\right] \\
&= \mathbb{E}^{T_j}\left[\max(R_j(T_{j-1}) - K, 0)|\mathscr{F}_t\right]
\end{aligned}$$

Here, in the first step the tower property is used.. The second step applies Jensen's inequality[1]. Finally, the martingale property of the backward-looking rates is used.

Just as in the Libor market model, log-normal dynamics for the backward-looking forward rates are assumed. Therefore, analytical caplet prices can be obtained using Black's formula. The volatility parameter used for Black's equation does need some modification for the backward-looking caplet. This is because the rates have slightly different dynamics compared to the rates currently present in the market. The new volatility parameter is given by

$$v_j^B(t) = \sigma_j^2 \int_t^{T_j} \left(\min\left(\frac{T_j - s}{T_j - T_{j-1}}, 1\right)\right)^2 ds.$$

Writing out this integral (appendix B), Black's volatility can be rewritten into

$$v_j^B(t) = \sigma_j^2 \left[(T_{j-1} - t)^+ + \frac{1}{3}\frac{(T_j - \max(T_{j-1}, t))^3}{(T_j - T_{j-1})^2}\right]. \tag{4.5.3}$$

For forward-looking caplets the volatility parameter doesn't change since the backward-looking rates coincide with forward-looking rates till their reset date. Therefore

$$v_j^F(t) = \sigma_j^2(T_{j-1} - t). \tag{4.5.4}$$

It again follows that the backward-looking caplet will be more expensive compared to a forward looking caplet. This results from the fact that for the backward-looking caplet the volatility will always be higher or equal since the $\frac{1}{3} \times$ (term) is added to v^B. Considering the case where all other risk factors are the same, a higher volatility results in a higher caplet price.

With these volatility parameters caplets can be priced using Black's formula

$$\begin{aligned}
Cpl_j^F(t) &= P_j(t) N \text{Black}(R_j(t), K, v_j^F(t)), \qquad t \leq T_{j-1}, \\
Cpl_j^B(t) &= P_j(t) N \text{Black}(R_j(t), K, v_j^B(t)), \qquad t \leq T_j.
\end{aligned} \tag{4.5.5}$$

Here

$$\text{Black}(R, K, v) = R\Phi\left(\frac{\ln\left(\frac{R}{K}\right) + \frac{1}{2}v}{\sqrt{v}}\right) - K\Phi\left(\frac{\ln\left(\frac{R}{K}\right) - \frac{1}{2}v}{\sqrt{v}}\right).$$

[1] $\phi(E[X]) \leq E[\phi(X)]$

4.5.2. Validation

The next step is implement and validate the log-normal forward market model. Instead of simulating forward rates, the new backward-looking forward rates will be simulated. However, the dynamics of these rates are exactly the same as the forward rates till the settlement date. Therefore, the forward rates are simulated under the same stochastic process. As for the LMM the rates will be simulated under the spot-Libor measure using the discrete bank-account as numéraire. As a reminder, the equation for the dynamics is presented below

$$dR_j(t) = \sigma_j(t)\gamma_j(t)R_j(t)\left(\sum_{i=q(t)+1}^{j} \rho_{i,j}\frac{\tau_i\sigma_i(t)\gamma_i(t)R_i(t)}{1+\tau_i R_i(t)}dt + dW_j^d(t)\right). \tag{4.5.6}$$

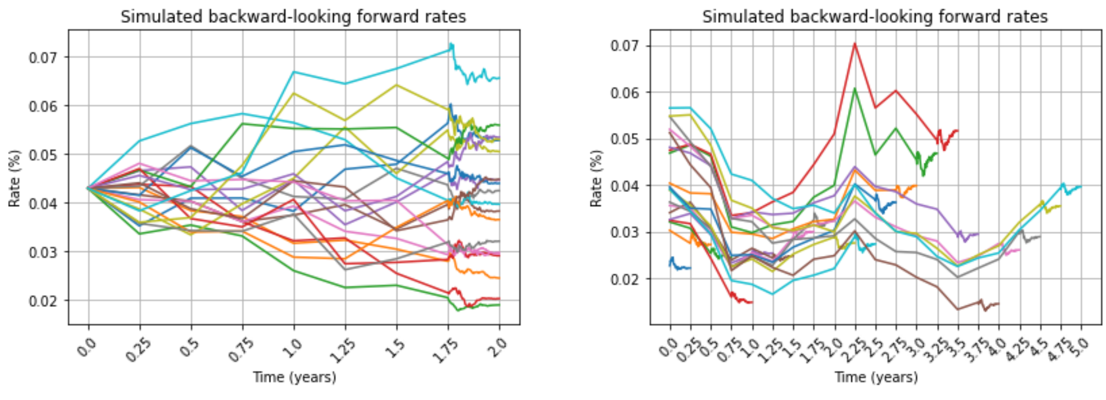
For the Libor market model two different simulation techniques were proposed. First of all, Euler's method which takes small increments over the entire time grid and second, the predictor-corrector method. This method had the same accuracy but was much faster. For the implementation of the log-normal FMM it would be favorable to simulate using the predictor-corrector method. However, observe the γ_j in equation (4.5.6) this accounts for a decreasing volatility of the backward rate during the accrual period, meaning between $[T_{j-1}, T_j]$. Therefore, the volatility is no longer constant over time in the accrual period of the backward-looking forward rates. The derivations in this thesis for the predictor-corrector method assumed a constant volatility parameter. If this is no longer the case, a stochastic integral has to be solved in order to obtain the correct expression for the predictor-corrector method. This won't be performed in this thesis and we will choose for a simpler solution. We will switch from the predictor-corrector method to the Euler method for the simulation of the accrual period of the backward rates. To summarize, for rate $R_j(t)$ the predictor-corrector method will be used from $[0, T_{j-1}]$ after which the switch to the Euler method will be made for the simulation of the final period $(T_{j-1}, T_j]$.

The simulation process is now similar as for the LMM, the steps can be found in Chapter 3. There are however two differences regarding the simulation of forward and backward rates that are good to address:

1. First of all, when the forward rates (L(t, T_i, T_{i+1})) were simulated the instantaneous volatility would be alive up to time T_i and it would be 'dead' for the period that the rate is applicable. For the backward rates, note that if the rate is over the period $[T_{i-1}, T_i]$ we still need a value for the instantaneous volatility at the end of the accrual period and hence σ is not 'dead' for the period that the rate is applicable.

2. Second, an important notational note should be made. For both the forward rates and backward rates we start counting the first rate as L_1 and R_1. However: $L_1 = L(t, T_1, T_2)$, which means that this is the rate for the period $[T_1, T_2]$, while $R_1 = R(t, T_0, T_1)$ which is the rate over the period $[T_0, T_1]$. This is important to keep in mind while implementing the simulation for the backward rates.

Figure 4.5 shows an example of the simulated dynamics of different backward-looking forward rates. Figure 4.5(a) shows 20 different rates all with an accrual period over $[T_7, T_8]$, while figure 4.5(b) shows twenty different rates with different accrual periods. The difference between the Euler method and the predictor correct method is clearly visible in both figures. Both figures also show the decreasing volatility during the accrual period. First large steps are taken after which very small increments are used for the accrual period. For all simulations, 64 discretization steps were used.



(a) (b)

Figure 4.5: Simulation of backward-looking forward rates

In order to guarantee that the simulation of the backward-looking forward rates is actually correct, the model should be validated. When the implemented model is validated, more exotic interest rate derivatives can be priced. First, the model is validated by comparing analytical zero-coupon bond prices to the simulated prices. By performing 50000 Monte Carlo simulations, 12 different zero-coupon bonds with a maturity ranging from T_1 till T_{12} are priced. For the simulation the accrual period is discretized into 64 steps.

The results for valuating zero-coupon bonds can be found in table 4.1. The standard errors and relative errors are extremely similar to the LMM implementation. From these results we can conclude that the model is implemented in a correct way. The first zero-coupon bond in the table with maturity T_1 has a relative and standard error of zero. Remember that the prices of the zero-coupon bonds are obtained using equation (3.4.9). From this equation we see that the first zero-coupon bond equals $P(0,T_1) = \mathbb{E}^d\left[\frac{1}{1+\tau L(T_0,T_0,T_1)}\right]$. Here $L(T_0,T_0,T_1)$ is the spot rate which is always given by the market. Hence, it is impossible to actually simulate the first zero-coupon bond value. Note that the results for table 4.1 are obtained using forward rates, since these are required in equation (3.4.9). Here the second property of the backward-looking forward rates is used. This property said that $L(T_i,T_i,T_{i+1}) = R(T_i,T_i,T_{i+1})$.

In this chapter there is not an overview of how the model performs for a different number of Monte Carlo simulations. This will be presented in chapter 5 where actual market data is used for the implementation instead of our own chosen values.

Table 4.1: Pricing zero-coupon bonds

	Analytical Price	Simulated price	Error (%)	Standard error
P(T_1)	0.997	0.997	0	0
P(T_2)	0.992	0.992	2.59e-05	4.98e-07
P(T_3)	0.985	0.985	7.18e-06	1.40e-06
P(T_4)	0.976	0.976	1.22e-04	2.09e-06
P(T_5)	0.967	0.967	3.16e-04	3.71e-06
P(T_6)	0.958	0.958	1.04e-04	7.20e-06
P(T_7)	0.948	0.948	1.62e-04	1.04e-05
P(T_8)	0.937	0.937	1.35e-04	1.24e-05
P(T_9)	0.925	0.925	6.24e-04	1.91e-05
P(T_{10})	0.913	0.913	1.27e-04	2.83e-05
P(T_{11})	0.897	0.897	7.11e-04	4.10e-05
P(T_{12})	0.881	0.881	6.92e-04	4.47e-05

The validation results by valuating backward-looking caplets can be found in table 4.2. Again a total of 50000 Monte Carlo simulations were used and the final discretization part was split up into 64 steps. Our own pre-chosen initial volatilities and rates at time zero were used. Finally, the choice was made to price ATM caplets, which means that the different backward-looking caplets can be compared to each other. A notional of 10000 is used to express the price in basis points. The results are similar to the validation of the LMM. The model is implemented in a correct way due to the simulated price being close to the analytical price. Also with the new backward-looking caplet it is now possible to have a caplet with an application period from $[T_0, T_1]$.

Remark. The validation was performed for multiple strike prices. A problem from the model, both LMM and FMM, is that for far OTM caplets the pricing is inaccurate. This is due to the fact that in the Monte Carlo simulation, payoff values on many paths are negative and are thus set to zero. This reduces the amount of effective Monte Carlo simulations.

Table 4.2: Pricing backward-looking caplets

	Analytical Price	Simulated price	Error (%)	Standard error
Cap(T_0, T_1)	3.72	3.75	0.90	0.0139
Cap(T_1, T_2)	17.02	17.04	0.13	0.0712
Cap(T_2, T_3)	28.01	28.00	0.04	0.1210
Cap(T_3, T_4)	27.22	27.41	0.68	0.1135
Cap(T_4, T_5)	42.10	42.37	0.64	0.1890
Cap(T_5, T_6)	60.08	60.03	0.07	0.2971
Cap(T_6, T_7)	62.48	62.59	0.18	0.3001
Cap(T_7, T_8)	54.56	54.56	0.006	0.2469
Cap(T_8, T_9)	98.83	98.50	0.33	0.5356
Cap(T_9, T_{10})	115.21	115.56	0.30	0.6578
Cap(T_{10}, T_{11})	156.71	156.73	0.01	0.9044
Cap(T_{11}, T_{12})	105.52	105.71	0.18	0.4874

4.6. Summary

This chapter introduced the generalized forward market model. In the post-Libor world the new rates keep evolving till the end of their accrual period. In the currently used Libor market model, the evolution of the rates is only defined till the start of their application period. The forward market model was able to simulate the new type of backward-looking rates. The generalized forward market model was built on the concept of extended zero-coupon bonds. First, the new backward-looking forward rate had to be defined, after which an expression was derived for this new rate. From this expression, four different properties for the backward-looking forward rates were evident. First of all, the backward-looking forward rate was a martingale under its appropriate extended forward measure. Second, the backward-looking forward rates were equal to the realized backward-looking rates at maturity, also the rates were fixed after maturity. Finally, the backward-looking forward rates were equal to the realized forward-looking rates at the start of the accrual period. This thesis considered the log-normal forward market model therefore, the simulated caplet prices will be consistent with the analytical prices of Black's equation.

The second part of this chapter derived the dynamics of the backward-looking forward rates under the spot-Libor measure and the risk-neutral measure. It was not possible to obtain the dynamics under the risk-neutral measure for the Libor market model. We saw that the volatility of the backward-looking forward rate should decrease during the accrual period. This was due to the fact that more and more overnight rates will be realized resulting in less uncertainty for the rate. This was incorporated in the dynamics with a volatility decreasing function γ. After the dynamics for the backward-looking forward rates were derived and some first simulation results were presented, it was important to actually validate the model implementation. The model could be validated by either pricing zero-coupon bonds or caplets. In the post-Libor world two types of caplets will be present in the market. The forward-looking caplet, where the payoff depends on the backward-looking rate at the start of the accrual period and hence, this is the exact same caplet we currently see in the market. The second type of caplet was the backward-looking caplet. This was a new type of caplet where the payoff depended on the backward-looking rate at maturity. Since the backward-looking caplet had more uncertainty due to the fact that the rates could evolve for a longer period, this new type of caplet would always be more expensive compared to forward-looking caplets. The implementation of the log-normal forward market model was validated in two ways, first by comparing simulated zero-coupon bond prices to the prices from market. Second, by comparing simulated backward-looking caplet prices to the analytical prices. To obtain an analytical price for the backward-looking caplets, the volatility parameter for Black's equation had to be adapted. We found that the implemented log-normal forward market model prices zero-coupon bonds and caplets with the same accuracy as the Libor market model and concluded that the model was implemented in a correct way.

5

Extensions of the forward market model

In this chapter the forward market model is extended in order to deal with negative interest rates which are present in the market. We will accomplish this by switching from a log-normal model to the shifted log-normal Libor market model. When this extension of the model is implemented, it is possible to make the switch from using our own data to actual market data. Using this market data, we can also examine how the model behaves when it prices zero-coupon bonds and caplets. Finally, the fact that the risk-neutral dynamics under the forward market model can be obtained is examined using Eurodollar futures as an example.

5.1. Shifted log-normal model

An important limitation of the log-normal Libor market model or log-normal forward market model is the fact that both models cannot deal with negative interest rates. Negative interest rates weren't a problem twenty years ago, actually if models produced negative rates it was seen as a pitfall of the model. However, after the financial crisis of 2008, people started to lose trust in the financial system. The lack of trust meant that people spent less and less money which had a negative impact on the economy. Central banks decided to step in and stimulate people to spend more money. By lowering interest rates they expected people, banks and companies to borrow money and invest this back into the economy. This would help the economy to grow. Eventually in 2014 negatives interest rates were reached. For more information on negative interest rates we refer to the authors of [30].

The forward market model prices caps/caplets consistent with Black's formula, this is possible for both forward-looking as backward-looking caplets. However inspecting the equations, (4.5.5), the following logarithmic term $\ln(\frac{L_i(t_0)}{K})$ is observed. As such, if the initial rates are negative, the fraction becomes negative and the logarithm of a negative number is not well defined. Note that if both the initial rate as the strike price are negative the fraction becomes positive, but remember that the caplet switches from a call to a put option. Hence, when dealing with negative rates our model isn't able to price caplets consistently with Black's equation. To solve this problem, some changes to the model have to be made.

Two classes of models to solve the problem are considered. The first one is the class of the local volatility models (LVM). For a local volatility model the volatility parameter is given as a function of time and the forward/backward rate. Local volatility models can deal with negative rates, examples are the Constant-Elasticity-of-Variance Model (CEV) [6] and the displaced diffusion model (DD) [35]. Both models must be calibrated to the market but are able to handle negative rates. Besides, both models partially solve the problem of capturing the implied volatility smile present in the market. However, the CEV and DD model are only able to capture an implied volatility skew. There are more advanced local volatility models which are also able to capture implied volatility smiles, which is presented by for example the authors of [5].

The second type of model which could possibly deal with negative interest rates are the stochastic volatility models (SVM). The main difference with LVM's is that this time the volatility is driven by its own Brownian motion. Using this type of model it is also possible to capture the stochastic behaviour of the volatility. Two commonly used stochastic volatility models are found in [21] and [20]. There are also other types of models such as jump-diffusion models, see for example [30], however this will not be discussed in this thesis.

Usually in practice a stochastic volatility model, like the SABR model, is used to incorporate the volatility smile or skew. Thus a stochastic volatility model allows to fit the model to the given smile or skew in the market. For now, the main interest is in solving the negative interest rate problem. The focus on this problem is important since if the model cannot deal with negative interest rates the given market data cannot be used.

Two often used models for dealing with negative rates are the displaced diffusion (DD) model and the shifted log-normal model. The DD model is a special type of shifted log-normal model. It is possible to rewrite the dynamics in a shifted log-normal form, shown by [28]. The DD model has a parameter which must be calibrated to the market while the shift for the shifted log-normal model doesn't need to be calibrated to the market. The shift is actually given in the market for every tenor point and can be used directly. Both models are not able to reproduce the implied volatility smile seen in the market. Since it is market practice to use a stochastic volatility model to fit the implied volatility curve we won't worry about this. The shifted log-normal model is chosen to deal with negative interest rates. The advantage is that no calibration is needed since the shift can be directly extracted from the market. Here simplicity is the driving force and if there is a need to reproduce the implied volatility smile/skew the model can be extended to a stochastic volatility model.

The basics of the shifted log-normal model are simple. The derivations are from a log-normal forward market model perspective thus using backward-looking forward rates. The same derivations can be applied on the Libor market model. First of all, the shifted process is defined as

$$\hat{R}_j(t) = R_j(t) + \theta_i,$$

where θ_i is the shift given by the market for every expiry and tenor point. Since the shift is a constant value and doesn't depend on time it is known that by Itô's lemma

$$d\hat{R}_j(t) = \hat{\sigma}_i \hat{R}_j(t) dW_j(t).$$

Thus the constant shift doesn't change the dynamics of the backward-looking forward rates, only the value at time zero. Because the dynamics don't change, it must mean that the shifted backward-looking forward rates are still log-normally distributed and we still work with a log-normal model. This results in the fact that the shifted rates will never reach negative values because of the property of a log-normal distribution. Before implementing the shifted log-normal model, the dynamics under the spot-Libor measure must be derived. Here we will use that the shift doesn't change the dynamics and follow the derivations given in [34].

Lemma 5.1.1. When $\hat{R}_j(t) = bR_j(t) + a$, the dynamics of the shifted backward-looking forward rates under the spot-Libor measure are given by

$$d\hat{R}_j(t) = bdR_j(t) = b\hat{R}_j(t)\gamma_j(t)\sigma_j(t)\left(\sum_{q(t)+1}^{j} \rho_{ij} \frac{\tau_i \sigma_i(t)\gamma_i(t)\hat{R}_i(t)}{1+\tau_i \frac{\hat{R}_i(t)-a}{b}} dt + dW^d(t)\right). \tag{5.1.1}$$

Proof. For a proof we refer to [34] since the derivations are similar to the proofs for the spot-Libor measure dynamics under the Libor market model or forward market model. □

Since in our case $b = 1$ and $a = \theta$ in equation (5.1.1), the dynamics of the backward-looking forward rates under the spot-Libor measure are given by

$$d\hat{R}_j(t) = \hat{R}_j(t)\gamma_j(t)\sigma_j(t)\left(\sum_{q(t)+1}^{j} \rho_{ij} \frac{\tau_i \sigma_i(t)\gamma_i(t)\hat{R}_i(t)}{1+\tau_i R_i(t)} dt + dW^d(t)\right). \tag{5.1.2}$$

With the new dynamics for the shifted log-normal forward market model we analyze how it can be used in valuating caplets. Consider a backward-looking caplet with strike price K and maturity date T_i. Today's price of this caplet is given by

$$\begin{aligned} Cpl_i(t_0) &= N\tau_i P(t_0, T_i) E^{T_i}\left[\max(R_i(T_i) - K, 0) | \mathscr{F}(t_0)\right], \\ &= N\tau_i P(t_0, T_i) E^{T_i}\left[\max(\hat{R}_i(T_i) - \theta_i - K, 0) | \mathscr{F}(t_0)\right], \\ &= N\tau_i P(t_0, T_i) E^{T_i}\left[\max(\hat{R}_i(T_i) - \hat{K}, 0) | \mathscr{F}(t_0)\right]. \end{aligned} \tag{5.1.3}$$

Here $\hat{K} = K + \theta_i$. Hence, the value of a caplet today doesn't change when applying a shift to the negative rates. The strike price has to be adapted with the shift which results in the same value as the original caplet. Now it is important to remember that $\hat{R}_i(t)$ still follows a log-normal distribution. Black's formula can be derived from the first payoff function in equation (5.1.3). Due to the log-normal property of $\hat{R}_i(t)$ the same derivation can be applied to the final payoff function in equation (5.1.3). This results in Black's formula for shifted rates, where the shifted positive rates and an adapted strike price are used as input. We have now arrived at our objective where the goal was to price a caplet in a negative interest rate environment. This is possible by shifting the rates and adapting the strike price. Black's formula for shifted rates is given by

$$Cpl_i(t_0) = N\tau P(t_0, T_i)\left[\hat{R}_i(t_0)N(d_1) - \hat{K}N(d_2)\right]. \tag{5.1.4}$$

Here $\hat{K} = K + \theta_i$ and $\hat{R}_i(t_0) = R_i(t_0) + \theta_i$. Besides

$$d_1 = \frac{\ln\left(\frac{\hat{R}_i(t_0)}{\hat{K}}\right) + \frac{1}{2}v}{\sqrt{v}}, \qquad d_2 = d_1 - \sqrt{v},$$

where v is the volatility parameter which is different for a forward-looking i.e. equation (4.5.4), or backward-looking caplet i.e. equation (4.5.3). One of the disadvantages of the shifted log-normal model is that it generates a flat implied volatility surface. This is now also clear since caplets would be priced using equation (5.1.4) however, the same equation would be used to back-out the implied volatility parameter resulting in a flat line. Black's formula for shifted rates is not derived since the shift on the rates doesn't change the dynamics hence the derivations are exactly the same as the original derivation of Black's formula.

5.2. Implementing market data

All the correct adjustments are made to the forward market model in order to work with market data. The model should be able to price zero-coupon bonds, caplets but also other more exotic derivatives. In this subsection, the market data will be examined and some first simulation results of the shifted log-normal forward market model are presented.

The data that will be used consists of daily three months forward rates, starting at 09.30.2020, running all the way up to 12.31.2032. Besides, for every forward rate the corresponding instantaneous volatility is given. This means that no calibration has to be performed to obtain the instantaneous volatilities. Next, the discount factors for every date are also given, this is obtained from the zero-coupon bond prices present in the market. These discount factors can easily be converted back to zero-coupon bond prices. Finally, the market shift for negative rates is given, the shift is constant over time. The only unspecified variable is the instantaneous correlation between the different forward rates. This is usually obtained by a calibration to swaption prices. We will not perform this calibration in this thesis and choose our own instantaneous correlation matrix. An example of this matrix can be found in appendix C. A structure is chosen such that forward rates which are close to each other are more correlated compared to forward rates far away from each other. The instantaneous correlation matrix will not change over time.

Figure 5.1 shows the forward rates as seen in the market over the entire period. It is clear that currently the market shows negatives forward rates however, rates further away in the future are positive. The market prediction is that Libors will decrease slightly in the near future but increases and become positive for maturities further away in the future. Figure 5.2 shows the instantaneous volatility curve for the forward rates. The instantaneous volatilities show a clear humped shape, where the volatility increases in time after which it reaches some peak and starts to decrease again.

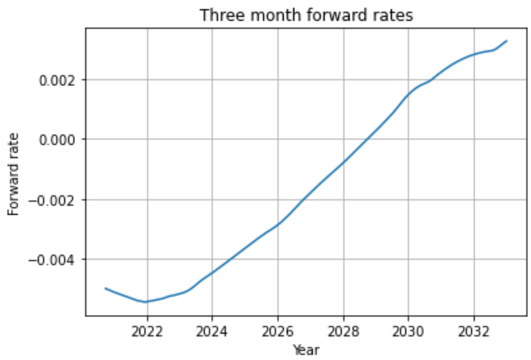


Figure 5.1: Forward rates from the market.

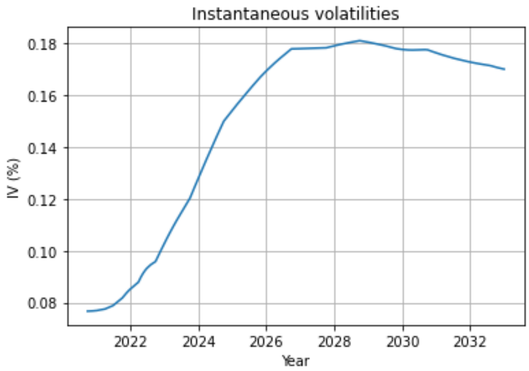


Figure 5.2: Instantaneous volatilities.

With the given data the assumption is made that today's date is the 30th of September 2020. However, we are free to choose our own starting date for the backward rates. This is because the market consists of all sorts of interest rate derivatives with different starting dates, thus different tenor points. To make this more clear, consider the following example and assume today is the 30th of September 2020. Suppose we would like to price a cap on the three months rate, where the first caplet has settlement date November, 2, 2020. So, $T_0 = 09.30.2020$ and the starting date $T_\alpha = 11.02.2020$. The difference between T_0 and T_α isn't three months, while from there we can continue the original tenor structure. Figure 5.3 gives a visual perspective. The difference between the first two tenors doesn't necessarily have to be τ apart depending on the derivative we would like to price.

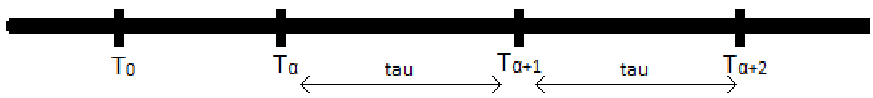


Figure 5.3: Timegrid

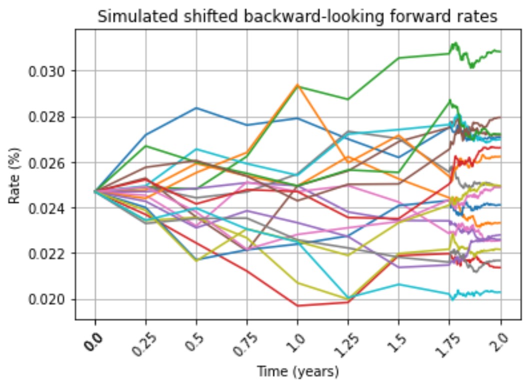
Because the difference between today's date and the starting date doesn't have to be τ apart, some changes to the implementation should be made. This is not interesting to consider and to keep the implementation simple the choice is made that the starting date of the rates and today's date are the same. Therefore a problem occurs for the first backward-looking forward rate which runs from $[T_0, T_1]$. The starting value for this rate is equal to the spot rate. However, the instantaneous volatility is not given by the market. This is because the instantaneous volatilities are usually calibrated to caplets and in the current forward rate world the $[T_0, T_1]$ caplet doesn't exist since the forward rate over this period is already known. As a solution, the instantaneous volatility is chosen to be zero and hence the first backward-looking forward rate doesn't change in the accrual period. Other solutions were to use historical volatility or to set the volatility equal to the instantaneous volatility of the next backward-looking rate.

5.2.1. First simulation results

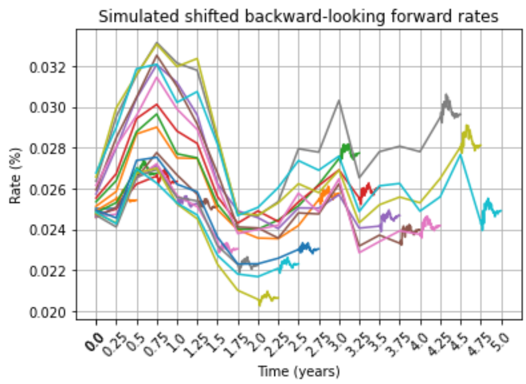
With the given market data it is possible to simulate the dynamics of the forward and backward rates over time. The implementation uses a combination of the predictor-corrector method together with Euler's method for the accrual period applied on equation (5.1.2). This is almost the exact same implementation as for the forward market model where we used our own chosen values, the only difference is the data that is used as input for the model. The starting values for the rates, remember that this is the same for forward as backward rates, will be negative. This problem is solved by shifting the starting rates then simulating them over time and at the end shifting the rates back. The rates at time zero and their corresponding instantaneous volatility can be found in appendix C.

The table in the appendix shows that today's date is the 30th of September 2020, where the choice for an instantaneous volatility of zero is taken for the first rate. Since $\tau = 0.25$ and today's date is taken to be the 30th of September we got the following structure: $T_0 = 09.30.2020$, $T_1 = 12.30.2020$, $T_2 = 03.30.2021$ etc. This results in a spot rate equal to $L(0, T_0, T_1) = -0.49754$ and we are able to simulate the first backward rate ($=R(t, T_0, T_1)$) with the chosen volatility parameter. The data between the different tenor points is left out since these numbers are not used in this thesis. Finally, notice that the shift from the market is chosen to be constant over time.

Figure 5.4 shows some first simulation results. The left figure presents 20 different simulations of $R(t, T_7, T_8)$, while the right figure shows simulations of different backward-looking forward rates. Both figures show the shifted simulated rates which won't reach negative values due to the log-normal property. To obtain the unshifted rates the market shift has to be subtracted from the shifted rates. The figures actually also show forward rates over time. For example, the left figure is also a simulation of $L(t, T_7, T_8)$ where the values at $T_7 = 1.75$ must be taken for this Libor. There is a clear difference between the use of the predictor-corrector method for the first part of the simulation after the switch to Euler's method is made for the accrual period. The accrual period was discretized into 64 steps.



(a)



(b)

Figure 5.4: Simulation of backward-looking forward rates

5.3. Validation

In this subsection, the shifted forward market model will be validated by pricing zero-coupon bonds. Besides, it is examined how the shifted log-normal forward market model behaves when pricing a simple interest rate derivative, namely the caplet. Both valuations will be obtained by performing a Monte Carlo simulation. Therefore, it is possible to see how the model performs by pricing zero-coupon bonds and comparing these to the given market prices for different zero-coupon bonds or by pricing backward or forward-looking caplets and compare these to the analytical prices from Black's formula.

The Monte Carlo steps are performed as given in paragraph 3.4.3. This time however backward-looking caplets will be priced instead of forward-looking caplets. Besides, the simulation uses a combination of Euler's method and the predictor-corrector method instead of using one of these methods. To obtain today's caplet prices the following martingale property is used

$$\frac{Cpl_j(0)}{B^d(0)} = E^d\left[\frac{Cap_j(T_{j+1})}{B^d(T_{j+1})}\right]$$

The formula for the discrete bank-account is given in equation (2.3.3). This equation uses the forward rates ($L(T_i, T_i, T_{i+1})$) to obtain an expression for the discrete bank-account. To obtain the value for the forward rate the second property of backward-looking forward rates will be used which says that $L(T_i, T_i, T_{i+1}) = R(T_i, T_i, T_{i+1})$. The first forward rate $L(0, T_0, T_1)$ is simply the spot-rate and therefore, the value of a zero-coupon bond with maturity T_1 cannot be simulated, it is already fixed.

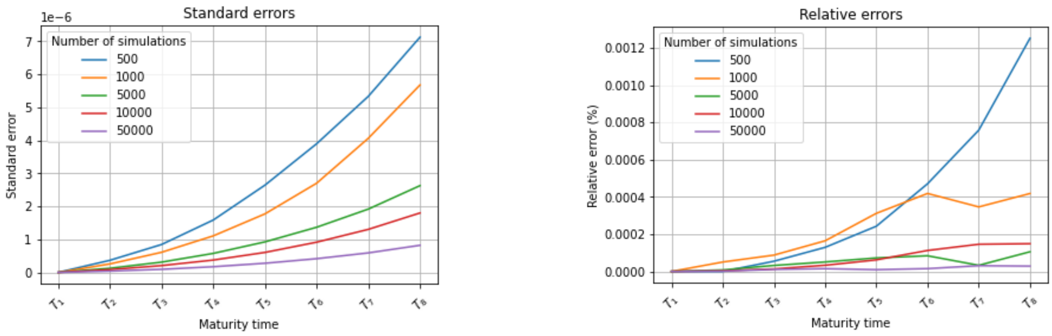
5.3.1. Zero-coupon bonds

First, the validation results for valuing zero-coupon bonds are presented. Table 5.1 shows the results for this validation. A total of 50000 Monte Carlo paths are used, where the rates were simulated using first the predictor-corrector method till their accrual period. In the accrual period the time was discretized into a grid of 64 steps. Besides, the simulation was based on the implementation of the shifted log-normal forward market model. At the end of the simulation it is possible to shift back to the original rates by subtracting the market shift. As can be seen from the table, we are dealing with negative interest rates since all the prices of the zero-coupon bonds are larger than one and increasing. This makes sense since figure 5.1 also shows that the market doesn't give any positive rates until 2028. We can definitely conclude that the shifted model is implemented in a correct way and that it is able to deal with market data. This is due to the fact that the zero-coupon bonds are priced extremely accurately.

Similar to the Libor market model case, we can inspect how the model behaves for different numbers of Monte Carlo paths. Figure 5.5 shows the standard errors and relative errors for pricing zero-coupon bonds with eight different maturities. Increasing the number of Monte Carlo paths to 50000 substantially increases the accuracy for pricing zero-coupon bonds. The standard errors and relative errors are much lower compared to the other simulations. There is no simulation with more than 50000 paths due to computational time, this is a trade-off since it would result in even more accurate results.

Table 5.1: Pricing zero-coupon bonds

	Analytical value	Simulated value	Error (%)	Standard Error
$P(T_1)$	1.0013	1.0013	0	0
$P(T_2)$	1.0025	1.0025	2.42e-06	3.86e-08
$P(T_3)$	1.0038	1.0038	7.93e-06	9.30e-08
$P(T_4)$	1.0052	1.0052	1.15e-05	1.72e-07
$P(T_5)$	1.0065	1.0065	9.39e-06	2.77e-07
$P(T_6)$	1.0079	1.0079	2.52e-05	4.16e-07
$P(T_7)$	1.0092	1.0092	4.68e-05	5.92e-07
$P(T_8)$	1.0106	1.0106	4.25e-05	8.22e-07



(a) Standard errors

(b) Relative errors

Figure 5.5: Pricing zero-coupon

5.3.2. Caplets

This section validates the model by pricing caplets. Equation (5.1.4) provides the analytical price of a caplet for the shifted log-normal model. Filling in the shifted rates at time zero together with the shifted strike price and the given instantaneous volatility results in the caplet price. This should be consistent with the simulation of caplet prices for both the unshifted as the shifted rates due to the same payoff function as in equation (5.1.3). Note that if the payoff is calculated using the shifted rates it should always be discounted using the unshifted rates.

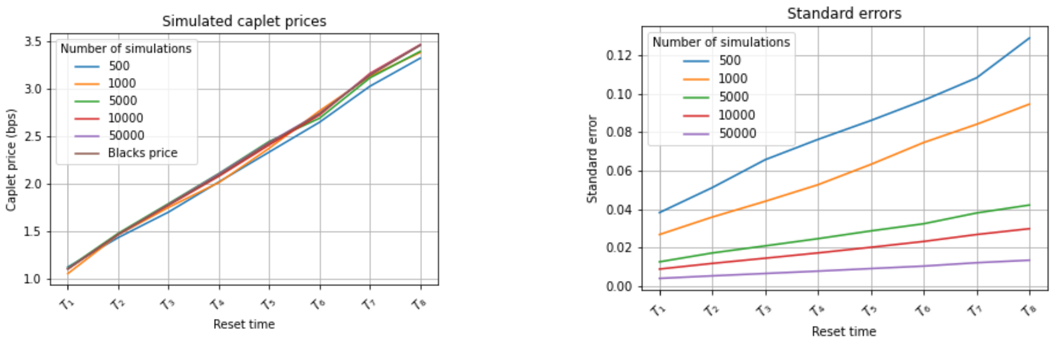
Table 5.2 present the results of a Monte Carlo simulation with 50000 paths for backward-looking caplets with different reset dates. For the discretization of the accrual period 64 steps were used. The table shows the results of pricing ATM caplets therefore, the results between different caplets can be compared. As always, a notional of 10000 is used to present the prices in basis points. The model seems to be implemented in a proper way and is able to price caplets with approximately the same accuracy as the Libor market model. The price of the first backward-looking caplet with accrual period $[T_0, T_1]$ has a value of zero. This is since an instantaneous volatility of zero is chosen for the first backward rate. Therefore, the rate doesn't change in the accrual period and the value at T_1 is exactly the same as the starting value at T_0. The table shows results for ATM caplets which means that the value of this first caplet is equal to zero. The floating rate will always be equal to the strike price.

It is also possible to examine how the accuracy of the pricing improves when the number of Monte Carlo paths increases. This is presented in figure 5.6. The figure consists of two sub-figures. Figure 5.6(a) shows how the different caplets are priced compared to Black's analytical price for a different number of Monte Carlo paths. Increasing the number of simulations definitely helps to improve the accuracy. This is also more visible in figure 5.6(b) where a decrease in the standard error for a higher number of simulations is shown.

Table 5.2: Pricing caplets

	Analytical price	Simulated price	Error (%)	Standard error
Cap(T_0, T_1)	0	0	0	0
Cap(T_1, T_2)	1.10	1.11	0.21	0.0039
Cap(T_2, T_3)	1.47	1.47	0.37	0.0053
Cap(T_3, T_4)	1.78	1.78	0.26	0.0065
Cap(T_4, T_5)	2.09	2.10	0.35	0.0077
Cap(T_5, T_6)	2.43	2.43	0.37	0.0091
Cap(T_6, T_7)	2.74	2.73	0.19	0.0104
Cap(T_7, T_8)	3.14	3.14	0.06	0.0121

Remark. There is still a possible small issue regarding the shifted log-normal model. This occurs when pricing far in-the-money caplets. When dealing with negative interest rates, this means that the strike prices would become very negative. It could happen that the shift given by the market does result in positive interest rates however the strike price could remain negative. This results in a negative value in the logarithmic term in Black's equation. In practice these products are almost never seen in the market because they are so far in-the-money.

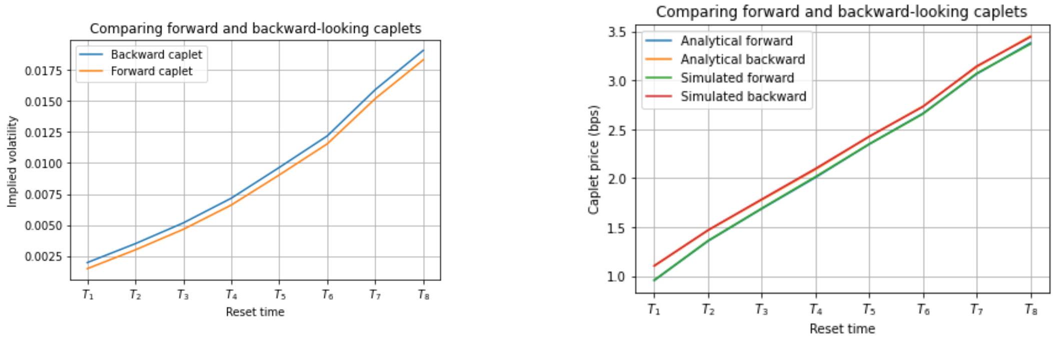


(a) Caplet prices

(b) Standard errors

Figure 5.6: Pricing backward-looking caplets

Finally, it is also interesting to examine the difference between forward-looking and backward-looking caplets. It was shown that the price of backward-looking caplets should be higher compared to forward-looking caplets, intuitively this also made sense. Figure 5.7 shows the pricing of eight different forward and backward-looking caplets. For both types of caplets we used the exact same parameters and 50000 simulations. In the left figure we see the prices of the different caplets given in their implied volatility. The backward-looking caplets always have a higher price. The right figure shows the prices of the caplets in basis points where the lines of the simulated prices overlap with the analytical prices. From these plots it is clear that backward-looking caplets will be more expensive compared to forward-looking caplets.



(a) Prices in implied volatility

(b) Prices in basis points

Figure 5.7: Pricing forward and backward-looking caplets

5.4. Interest rate futures

An important tool for pricing interest rate derivatives is the yield curve. The yield curve is constructed using the most liquid derivatives on the market, since it is assumed the price of these products are correctly given in the market. Examples of such products are interest rate futures. In this thesis we assume that the yield curve is given. As these products are extremely liquid, it is important that the pricing of interest rate futures is as accurate as possible. Therefore, we briefly will look into interest rate futures.

In this section one specific interest rate future is considered, the Eurodollar future (ED future). An ED futures contract is similar to a forward rate agreement however, there are some differences which we will come across. One important property of the new forward market model is the fact that it is possible to obtain risk-neutral dynamics for both backward-looking as forward-looking rates. This property is very useful when dealing with Eurodollar futures contracts.

5.4.1. Eurodollar futures

Eurodollar futures are very alike forward rate agreements (FRA) discussed in Chapter 2. Remember that with a forward rate agreement it is possible to lock in an interest rate over a specific period somewhere in the future. Forward rate agreements are contracts written on the Libor rate and traded in the over-the-counter market. The advantage of the OTC market is that we can set our own demands such as settlement time, the notional or the fixed rate of the contract. This is not possible for Eurodollar futures contracts where this is all fixed. This is an important property of these contracts since it assures liquidity.

Mostly banks and other large financial institutions trade with each other in the OTC market. However, the disadvantage is the exposure to counterparty risk. Therefore, entering a FRA in order to lock in an interest rate for some future time period will result in a risk that the counterparty will default, exposing one of the parties to changes in future interest rates again. For more information see [22].

An alternative for trading in the OTC market but entering a similar contract as a forward rate agreement is the Eurodollar futures contract. The Eurodollar futures contract which is traded on the Chicago Mercantile Exchange (CME) is an example of an interest rate future. ED futures are contracts on the three months spot Libor rate. The contracts are settled at maturity as

$$100(1 - L(T, T, T + \tau)), \tag{5.4.1}$$

where it is assumed that τ is equal to three months. There are no costs attached to entering an ED futures contract.

Although this is how the contract is settled at maturity, the cash flows of an ED futures contract are spread over the contract time. In order to reduce counterparty risk, daily margining is used. Therefore, when two parties enter an ED futures contract they both start with an initial margin on their account. Every day the change in rate is determined, if the underlying rate goes up a payment from the buyer's account to the seller's account is made and vice versa. When one parties margin account gets below some threshold, the party will have to refinance its account in order not to default.

The amount of money that has to be exchanged every day is determined by the change in the rate that day

$$N \cdot \left(1 - \frac{1}{4} F(t, T, T+\tau)\right).$$

Here $F(t, T, T+\tau)$ is known as the futures rate which, and this is important, is not exactly the same as the forward rate. N is the notional which is always equal to 1,000,000 and the $\frac{1}{4}$ comes from the fact that the contract is based on the three months rate. Hence, if the futures rate increases with 1bp it means that the buyer of the futures contract has to pay \$25 to the seller.

Examining more generally the marked-to-market concept, after both parties hold the contract for 1 day, the exchange in money has to be (per unit of notional)

$$\tau(1 - F(1d, T, T+\tau)) - \tau(1 - F(0, T, T+\tau)) = -\tau(F(1d, T, T+\tau) - F(0, T, T+\tau)).$$

The payoff on the second day is given as

$$\tau(1 - F(2d, T, T+\tau)) - \tau(1 - F(1d, T, T+\tau)) = -\tau(F(2d, T, T+\tau) - F(1d, T, T+\tau)).$$

In general this means

$$\tau(1 - F(t_{i+1}, T, T+\tau)) - \tau(1 - F(t_i, T, T+\tau)) = -\tau(F(t_{i+1}, T, T+\tau) - F(t_i, T, T+\tau)).$$

Extending this all the way up to maturity T, almost all terms will cancel out and the total amount of exchanged money over the period $[0, T]$ will be equal to:

$$-\tau(F(T, T, T+\tau) - \tau F(0, T, T+\tau)) = -\tau(L(T, T, T+\tau) - F(0, T, T+\tau)). \tag{5.4.2}$$

Here the last equality comes from the fact that the futures rate at maturity should equal the forward rate at maturity since otherwise there would be a delivery arbitrage opportunity. This is because the contract is defined in such a way that a payoff as in equation (5.4.1) must be assured at maturity time.

Figure 5.8 gives a visual representation of the daily margining for ED futures contracts. At time T_0 it is possible to enter the futures contract without making a payment, then daily margining takes place at days t_1, t_2 etc. At every day i a cash flow of C_i is interchanged. Here $C_i = N\tau(F(t_{i-1}) - F(t_i))$. Now at the maturity date of the contract, called T_n, the final payment is equal to $C_n = N\tau(L(T) - F(t_{n-1}))$.

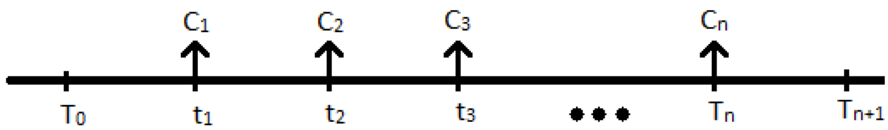


Figure 5.8: Timeline interest rate future

It is now also easier to see why for Eurodollar futures the counterparty risk is almost equal to zero. Every day the change in the futures rate is calculated to determine a cash flow between the seller and buyer of the contract. Since both parties started with an initial margin on their account the payment can either be added or subtracted from this account. When an account falls below a certain threshold, the party has to make an extra deposit to get back to a save margin. This reduces counterparty risk. If a party defaults, for example at time s ($0 \le t < s < T$) the exchange can just find another party who wants to enter an ED futures contract at time s. Remember that it is free to enter the contract. These two aspects combined make counterparty risk almost negligible since the probability that the exchange will default is extremely small. As long as the margin account of the defaulting party didn't get negative the exchange didn't lose any money. Thus, the maintenance margin must be chosen in such a way by the exchange that in most of the cases buyers/sells won't achieve negative margin accounts.

5.4.2. Futures rate

For the daily margining, the futures rate is used instead of the forward rate. These two rates aren't the same, only at maturity they are. The difference between the two rates is called the convexity adjustment. The difference between the two rates can be best explained using an example. The futures price quoted in the market at maturity must be (due to delivery arbitrage): $F(T) = 100(1 - L(T, T, T+\tau))$, however before maturity

$$F(t) \approx 100(1 - L(t, T, T+\tau)). \tag{5.4.3}$$

Consider that a person enters a futures contract at time t with maturity T. This contract is free to enter and over the contract period this person will pay/receive $F(t) - F(T) \approx 100(L(t, T, T+\tau) - L(T, T, T+\tau))$. At the same time this person could enter in a forward rate agreement with the same notional amount on the three-months Libor. At maturity, the payoff would be exactly $100(L(t, T, T+\tau) - L(T, T, T+\tau))$. Assume that the approximation in equation (5.4.3) is exact. If for example the forward rate goes up over the period $[t, T]$, this means that the counterparty of both the ED futures as the FRA has to pay money. For the FRA there is only a cashflow at the payment date, but for the futures contract payments are made every day. Since we assumed that the interest rates are rising the counterparty would of course want to pay this money as late as possible since the party can just put it into a money-market-account and turn it into more money. Hence, the FRA would have been a better choice for the counterparty. On the other hand, if forward rates would have dropped over the same period the counterparty would receive money. This time the ED futures would have been a better choice.

To summarize, interest rate futures resemble the payoff of forward rate agreements, but not exactly due to daily margining. This results into the fact that the rate for the daily margining cannot be the forward rate otherwise (dis)advantages are created. Thus, there is a difference between forward rates and the futures rate which is used in Eurodollar futures. This difference is called the convexity adjustment.

5.4.3. Convexity adjustment

The basics of Eurodollar futures contracts have now been discussed. However, we haven't considered yet why it is a favourable property to have the risk-neutral dynamics for these contracts. Following the authors of [34] we show why it is helpful to have the risk-neutral dynamics of the forward rates under the forward market model. The derivation starts with the following lemma

Lemma 5.4.1. Assuming continuously margining and the short rate r(t) to be positive and bounded. The futures rate over an arbitrary period $[T, S]$ at time t is a martingale under the risk-neutral measure. Besides the following equality holds

$$F(t, T, T+\tau) = \mathbb{E}^{\mathbb{Q}}\left[L(T, T, T+\tau)|\mathscr{F}_t\right]. \tag{5.4.4}$$

Proof. The proof starts with a cash flow from a futures contract. Over a small time period, the cash flow will be equal to

$$dF(t, T, T+\tau) = F(t+dt, T, T+\tau) - F(t, T, T+\tau).$$

Suppose that we have entered into a futures contract and then sell it at some arbitrary time point S where $t < S \leq T$. We want the value of this contract at time t and we have assumed continuously marked-to-market resettlements. Therefore, under the risk-neutral measure which has numéraire B(t)

$$V_{fut}(t) = B(t)\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^S \frac{dF(S, T, T+\tau)}{B(S)} + \frac{V_{fut}(S)}{B(S)} dS\right]. \tag{5.4.5}$$

Here we discount all cash flows proceeding from this strategy using the continuous-time bank account. Since the daily margining is continuously done, the integral enters in the equation. The second fraction are the proceeds from selling the contract. We also know that the value of a futures contract is always zero to enter thus: $V_{fut}(t) = V_{fut}(S) = 0$. This results into

$$\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^T \frac{dF(S, T, T+\tau)}{B(S)} dS\right] = 0.$$

If we look at this integral and remember that we have chosen the short rate to be almost surely be positive, we get

$$\mathbb{E}_t^{\mathbb{Q}}\left[dF(S, T, T+\tau)\right] = 0, \qquad t \leq S \leq T$$

The expectation of the change of the futures rate at arbitrary time t is equal to zero, it immediately follows that the futures rate is a $\mathbb{Q}$-martingale. This is because we expect that the futures rate under the risk-neutral measure won't change at any arbitrary time point. Thus, the best guess of what the futures rate will be in the future under the risk-neutral measure would be today's futures rate. Finally, since $F(T,T,T+\tau) = L(T,T,T+\tau)$ due to delivery arbitrage the second statement follows. This is due to the contract definition where we want to replicate the payoff $100(1-L(T,T,M))$. Hence, the futures rate at maturity should be the same as the forward rate. This concludes the proof. □

Note that in chapter 2 it was proven that the forward rate is also a martingale under its own forward measure. Combining this with the lemma above, this automatically results in the fact that the futures rate $F(t,T_n,T_{n+1})$ is a martingale under the T_{n+1}-forward measure.

From this lemma it is obvious why the risk-neutral measure is important for forward rates. The convexity adjustment can now be written as

$$F(t,T,M) - L(t,T,M) = E_t[L(T,T,M)] - E_t^T[L(T,T,M)], \tag{5.4.6}$$

In this thesis we don't go any deeper into the convexity adjustments. For more information about how to estimate the convexity adjustment we refer to [33] or [22].

Here we just consider the dynamics under the risk-neutral measure. Examining equation (5.4.6), in order to obtain the convexity adjustment, the dynamics of the forward rates under the risk-neutral measure are required. For the Libor market model it was impossible to obtain these dynamics. For the new forward market model it is possible to obtain the risk-neutral dynamics of the new backward-looking forward rates. The dynamics were given in equation (4.4.5). Combining this with the third property of the model, $R(T_i,T_i,T_{i+1}) = L(T_i,T_i,T_{i+1})$, we are able to obtain the forward rates under risk-neutral dynamics. Therefore, instead of approximating the futures rates and convexity adjustment at time $t = T_0$, the forward rates under the risk-neutral measure can be simulated which results in the futures rate and the convexity adjustment. Note that this is only possible for $t = T_0$ since in the expectation of equation (5.4.6) it is assumed that all information up to time t is known. This isn't the case if $t > T_0$.

Before moving on to deriving the futures rate we should first discuss how this is currently obtained. Every day Eurodollar futures are quoted in the market by traders as $100(1-F(t,T,M))$. This daily quote changes every day which enables us to calculate the cash flows for the daily margining. However, these quotes are just numbers and in order to obtain sensitivities or construct a yield curve the futures rate is approximated relating it to market data. One of the methods is to price futures using short rate models. Other methods can be used to approximate the convexity adjustment and combine this with forward rates.

Since with the given market data all information up to time T_0 is known, it is possible to give the futures rate $F(0,T,T+\tau)$ for different three months application periods. Comparing this to the Libors at time zero, we see that they are not exactly equal and thus convexity is present. This is a small practical application of the new forward market model. Note that the futures that are calculated using the implemented model are not the true futures rates since they are still model dependent. It might be interesting to see how it changes when a stochastic volatility model is implemented.

The forward rate is a martingale under the T-forward measure and using the property of the forward market model the convexity adjustment at time zero can be rewritten to

$$F(0,T,T+\tau) - L(0,T,T+\tau) = E_t[L(T,T,T+\tau)] - L(0,T,T+\tau) = E_t[R(T,T,T+\tau)] - L(0,T,T+\tau), \tag{5.4.7}$$

where we can simulate the dynamics for the backward-looking forward rates under the risk-neutral measure using equation (4.4.5).

Figure 5.9 shows the convexity present in the market for our model implementation. 300000 Monte Carlo paths were used to compute the futures rate for different application periods. Then the difference between the forward rate in the market and the futures rate is taken to derive the convexity adjustment. From the figure it is clear that for further away periods the convexity adjustment will be larger. It is also examined how the convexity adjustment will change if the instantaneous volatility would be different. The original calculations are done with the instantaneous volatilities given by the market, denoted as σ in the figure. We see that if the instantaneous volatility would increase this results in a larger difference between the forward and futures rate.

Note that the convexity adjustment doesn't necessarily have to be positive. This is just for the market conditions we use. From the figure it is clear that if the instantaneous volatility becomes smaller the convexity adjustment becomes smaller. This makes sense when examining the dynamics for simulating the future rates in equation (5.1.2). Every step the change of the futures rates depends on the instantaneous volatility (IV), both in the drift term as the diffusion term. If the IV goes to zero the change in rate per discretization step will go to zero. When $d\hat{R}$ in (5.1.2) approaches zero the range for the futures rate at maturity time will become smaller. For example choosing IV = 0 results in a flat line for the backward-looking forward rate, there is no change at all over time and the rate at maturity will be exactly equal to the starting value which is the forward rate used to calculate the convexity adjustment. Therefore, the convexity adjustment will be equal to zero. The same reasoning can be used to explain the fact that the convexity adjustment increases considering rates with an application period further in time. There is a longer time period for the futures rate to deviate from the starting rates.

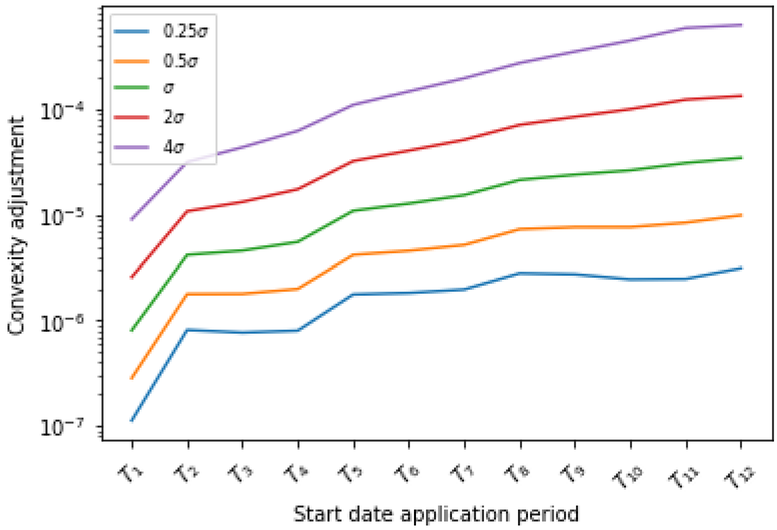


Figure 5.9: Convexity adjustment in the market. Here σ is the market volatility.

5.5. Summary

This chapter examined the problem of negative interest rates for the log-normal forward market model. Besides, we looked into a practical application of the model, where the dynamics under the risk-neutral measure could be used. In order to calibrate the forward market model, it should price caplets consistently with analytical caplet prices. However, with the current market conditions it was impossible to obtain analytical caplet prices. This was because negative interest rates were present in the market and therefore, Black's equation was undefined. There were multiple solutions to deal with negative interest rates, this thesis considered a simple solution in the form of a shifted log-normal model. The shifted forward market model shifted the original rates using a shift given by the market and therefore, no calibration was needed. It was assumed that the market shift was constant over time and equal for all different tenor points. The market shift didn't change the dynamics of the new backward-looking forward rates, which meant that the rates still followed a log-normal distribution. Therefore, it was still possible to obtain analytical caplet prices using the shifted Black's equation. The shifted log-normal forward market model was implemented and we found that backward-looking caplets, using actual market data, could be priced with the same accuracy as for the previous validations.

The second part of this chapter inspected a practical application of the forward market model. Eurodollar futures, which are similar to forward rate agreements but not traded on the OTC market were considered. The payoff function of an ED futures contract depended on the futures rate, which was slightly different from the forward rates due to daily margining for ED futures. The difference between the rates was called the convexity adjustment and with the dynamics of the rates under the risk-neutral measure it was actually possible to obtain the convexity adjustment. First, we derived that the futures rate was a martingale under the risk-neutral measure and that it could be obtained by taking the expectation of the forward rate under the risk-neutral measure. This meant that the convexity adjustment could be rewritten into terms of the forward rate under a forward measure and risk-neutral measure. It was only possible to obtain the convexity adjustment at T_0, since only all information up to that point was available. We found that with the market data we used, a convexity adjustment was present in the market. However, it should be noted that this was model dependent and that this convexity adjustment shouldn't be taken as the true convexity adjustment.

6

Stochastic Volatility Model

In this chapter we will examine the Stochastic Alpha Beta Rho stochastic volatility (SABR-SV) model in combination with the forward market model. A big issue with models that are consistent with Black's equation is that they are unable to reproduce the implied volatility smile. To price caplets consistently with Black's equation a constant volatility in the log-normal forward market model is assumed independent of the strike price of a caplet. However, the market shows an implied volatility smile or skew. Pricing caplets with a stochastic volatility model means that it is possible to back-out the implied volatility smile which is present in the market. In this chapter we will briefly dive into the problem of reproducing the implied volatility smile, then we cover some present literature about the SABR model and combining the Libor market model with the SABR model. The SABR-LMM is discussed since the new model builds on the theory of this model. The next subsections are dedicated to the new SABR forward market model, where we first derive the dynamics of this new model after which we focus on some implementation results.

6.1. The problem

One of the problems, next to the issue regarding negative rates, with the log-normal forward market model is that it assumes a constant implied volatility for different strike prices. To make this a bit more clear consider the new backward-looking caplet with reset date T_{i-1} and payment date T_i. For simplicity consider the notional to be equal to one unit. The payoff function of this caplet is given by

$$\tau(R(T_i, T_{i-1}, T_i) - K)^+.$$

From the equation the price today must be

$$Cpl_i(0) = P(0, T_i)\tau E^{T_i}\left[(R(T_i, T_{i-1}, T_i) - K)^+\right].$$

This equation can be solved which results in Black's formula with Black's volatility parameter given as

$$v_i^B(0) = \sigma_i^2(0)\int_0^{T_i}\left(\min\left(\frac{T_i - s}{T_i - T_{i-1}}, 1\right)\right)^2 ds.$$

From this formula it is immediately clear that the implied volatility has no dependence at all on the strike price K. Consequently, for a caplet with a fixed reset date and payment date the implied volatility will be constant across different strike prices if log-normal dynamics are assumed.

In the market however the implied volatility curve is not flat, there is actually a more smiled or skewed shape for the implied volatility curves. Log-normal market models are not able to reproduce these interest rate shaped implied volatility curves. Hence, if we would like to calibrate the instantaneous volatility to caplet market prices we will obtain inconsistent values for the implied volatility depending on the strike price which is considered. Common practice is to calibrate the volatility parameter to ATM prices. However, this could lead to a pricing error for more ITM / OTM caplets since they actually have a different volatility parameter.

An implied volatility smile exhibits higher volatilities for ITM and OTM strike prices while usually for ATM caplets the implied volatility is the lowest. This results in a smile shaped curve. The implied volatility skew also has high volatilties for ITM caplets, decreasing for strike prices which approach ATM values. The difference is that now it will keep on decreasing for ITM caplets. Intuitively the implied volatility smile makes sense, thinking about a caplet reaching far OTM or ITM strike prices would mean a more volatile market.

6.2. The stochastic alpha beta rho model

The Stochastic Alpha Beta Rho model (SABR) was first introduced by the authors of [14]. The concept of the SABR model is that the constant volatility parameter becomes a stochastic process. By pricing interest rate derivatives with a stochastic volatility model it is possible to reproduce the implied volatility smile seen in the market. There exists other stochastic volatility models, a few examples are given in [4]. In this thesis the SABR model is considered because of one important reason. Remember that log-normal dynamics were used for the Libor market model and forward market model so that both these models are consistent with Black's equation which prices the most liquid products in the market i.e., caps/caplets. For the SABR model it is still possible to obtain an approximation for the volatility parameter that can be plugged into Black's formula in order to price caplets. By choosing the SABR dynamics as the preferred stochastic volatility model, a model is chosen such that there is still an approximation for an analytical solution to price caplets. The dynamics of the SABR model are given as

$$\begin{aligned} dL(t) &= \sigma(t)C(L(t))dW(t), \\ d\sigma(t) &= \nu\sigma(t)dZ(t), \qquad\qquad \sigma(0) = \alpha. \end{aligned} \tag{6.2.1}$$

Here $0 \leq \beta \leq 1$, $\alpha, \nu > 0$. Note that if $\beta = 1$ and $\nu = 0$, the dynamics of the log-normal Libor market model are obtained. The dynamics given in equation (6.2.1) are assumed under the martingale measure for $L(t)$. For the SABR model the diffusion coefficient is chosen as $C(L) = L^{\beta}$, however it is much simpler to perform derivations using C(L). The correlation between the two Brownian motions is given as $E[dW(t)dZ(t)] = \eta dt$, while $E[dZ_i(t)dZ_j(t)] = r_{ij}dt$. There is no explicit solution for this model except for the case where $\beta = 0$, [13]. It is however possible to obtain an approximation for the model to price caplets or floorlets.

From the dynamics four parameters can be calibrated: β, $\sigma(0)$, r_{ij}, ν. Here β is usually chosen by traders instead of calibrated to the market. In this thesis the research isn't focused on calibration, this won't be discussed. In this thesis we will simply choose our own values. In section 6.5 different parameter values will be considered to see how this influences the solution.

For the dynamics and derivations of the SABR model in combination with the Libor market model we use [12]. To combine the SABR model with the Libor market model the following dynamics are assumed under the T_{n+1}-forward measure, under which $L_n(t)$ is a martingale

$$\begin{aligned} dL_n(t) &= C_n(t)dW_n(t) = \sigma_n(t)L_n(t)^{\beta}dW_n(t), \\ d\sigma_n(t) &= G_n(t)dt + D_n(t)dZ_n(t) = \nu_n(t)\sigma_n(t)dZ_n(t). \end{aligned} \tag{6.2.2}$$

Here $\nu_n(t)$ is called the volvol parameter, the volatility of the volatility can change over time and be different per rate. For the SABR-LMM model the drift function for the volatility parameter is chosen to be zero, $G_n(t) = 0$. Just as for the Libor market model it is more convenient to simulate the dynamics under the spot-Libor measure.

Lemma 6.2.1. The dynamics of the forward rates under the spot-Libor measure for the SABR-LMM are given by

$$\begin{aligned} dL_n(t) &= \sigma_n(t)L_n(t)^{\beta_n}\left(\sum_{k=q(t)}^{n}\frac{\rho_{kn}\tau_k\sigma_k(t)L_k(t)^{\beta}}{1+\tau_kL_k(t)}dt + dW_n(t)\right), \\ d\sigma_n(t) &= \nu_n(t)\sigma_n(t)\left(\sum_{k=q(t)}^{n}\frac{\eta_{kn}\tau_k\sigma_k(t)L_k(t)^{\beta}}{1+\tau_kL_k(t)}dt + dZ_n(t)\right). \end{aligned} \tag{6.2.3}$$

Proof. For a proof we refer to [12]. □

The derivation of these dynamics are not provided since in the next subsection a similar derivation is performed for the new SABR forward market model.

The implementation of the Libor market model and the forward market model have been validated by comparing analytical caplet prices with the simulated prices. This was possible since the assumption of log-normal dynamics and a constant volatility parameter were made. Therefore, the model priced caplets are consistent with Blacks76 formula. For the SABR-LMM model the volatility is stochastic. Consequently, the model doesn't price caplets consistent with Black's equation. For a model to be useful in practice it should be calibrated first. The solution to this problem is given by the authors of [14]. They derived an approximation for the implied volatility parameter which can be used as input for Black's formula. The value of a caplet is given by

$$\begin{aligned} Cpl_i(0) &= \tau P(0,T_i)N(L(0,T_i,T_{i+1})N(d_1) - KN(d_2)), \\ d_1 &= \frac{\log\left(\frac{L(0,T_i,T_{i+1})}{K}\right) + \frac{1}{2}\sigma^2 T_i}{\sigma\sqrt{T_i}}, \\ d_2 &= d_1 - \sigma\sqrt{T_i}. \end{aligned} \tag{6.2.4}$$

Here the volatility parameter $\sigma(L,K))$ depends on the strike K. The parameter is given by

$$\begin{aligned} \sigma(K,L) = & \frac{\alpha}{(LK)^{(1-\beta)/2}\left\{1+\frac{(1-\beta)^2}{24}\log^2 L/K + \frac{(1-\beta)^4}{1920}\log^4 L/K + \cdots\right\}} \cdot \left(\frac{z}{x(z)}\right) \\ & \left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(LK)^{1-\beta}} + \frac{1}{4}\frac{\rho\beta\nu\alpha}{(LK)^{(1-\beta)/2}} + \frac{2-3\rho^2}{24}\nu^2\right]t_{ex} + \cdots \right. \end{aligned} \tag{6.2.5}$$

Here

$$z = \frac{\nu}{\alpha}(LK)^{(1-\beta)/2}\log L/K,$$

and

$$x(z) = \log\left\{\frac{\sqrt{1-2\rho z + z^2} + z - \rho}{1-\rho}\right\}.$$

Since in this thesis we focus on pricing ATM caplets it is convenient to have a separate expression for the at-the-money volatility

$$\sigma_{ATM} = \sigma_B(L,K) = \frac{\alpha}{L^{(1-\beta)}}\left\{1 + \left[\frac{(1-\beta)^2}{24}\frac{\alpha^2}{L^{2-2\beta}} + \frac{1}{4}\frac{\rho\beta\alpha\nu}{L^{(1-\beta)}} + \frac{2-3\rho^2}{24}\nu^2\right]t_{ex} + \cdots \right. \tag{6.2.6}$$

For the derivations of all these equations we refer to [14].

The approximation for the volatility parameter is also known as Hagan's approximation. Although this explicit equation is a fast and an easy to implement approach to price caplets the formula's remain approximations. Therefore, this method is not arbitrage-free. Especially for low strikes the density function can become negative or not integrate to one. The density function can be implied from the caplet prices for different strike values. For more information on this arbitrage issue and a possible solution we refer to the authors of [15].

6.3. SABR forward market model

Now that a short introduction into the SABR model combined with the Libor market model has been discussed, the focus will be switched to a new type of model which is the SABR forward market model. This model hasn't been explored in the literature thus this section will be more detailed compared to the other models. First, the dynamics of the backward-looking rates with a stochastic volatility are derived, then some issues regarding the simulation are considered after which the model will be validated and some results are shown.

The SABR forward market model combines the generalized forward market model with the SABR stochastic volatility model. Contrary to the log-normal forward market model, this new model should be able to reproduce the implied volatility smile seen in the market. We start the derivations with the dynamics of the backward-looking rates. The backward-looking forward rate with accrual period $[T_{j-1}, T_j]$ must be a martingale under the extended T_j-forward measure, which is one of its properties. Therefore, the following dynamics for the backward-looking forward rates under the martingale measure are proposed

$$\begin{aligned} dR_j(t) &= \gamma_j(t)\sigma_j(t)R_j(t)^\beta dW_j(t), \qquad \gamma_j(t) = \min\left(\frac{(T_j-t)^+}{(T_j-T_{j-1})}, 1\right)^q \\ d\sigma_j(t) &= \nu_j(t)\sigma_j(t)dZ_j(t), \qquad \sigma_j(0) = \alpha_j. \end{aligned} \tag{6.3.1}$$

Here $\beta \in [0,1]$, α, $\nu > 0$. $W(t)$ and $Z(t)$ are both Q^{T_j} Brownian motions. The parameter β is fixed depending on the market we are considering. The Brownian motions are correlated as $E[dW_i(t)dZ_j(t)] = \eta_{ij}dt$ for all i, j. The $\gamma_j(t)$ function is again used to decrease the volatility in the accrual period since $R_j(t)$ becomes more deterministic reaching maturity time. Notice that a power parameter is added to the γ function which is called q. In the previous chapter we worked with $q = 1$, choosing a smaller q results in a slower decreasing volatility while choosing $q > 1$ results in a fast decreasing volatility in the accrual period.

Derivations with these dynamics can become very complicated and in this thesis we focus on the basics of the model, hence some simplifications and assumptions are made. We start with the most important assumption which must be understood well in order to proceed with the new model. Consider the correlations between the different Brownian motions, the correlation parameters between the Brownian motions of the backward rates are usually calibrated to swaptions prices. We do not know however how to calibrate the correlation between the Brownian motions of the volatility processes given by $E[dZ_i dZ_j] = r_{ij}dt$, these are unavailable in the market. Besides from practice it is known that more Brownian motions in the implementation results in more paths needed for accurate results during the simulation. Therefore, the assumption is made that the dynamics of the volatility is driven by a single Brownian motion. The dynamics is then given by

$$d\sigma(t) = \nu\sigma(t)dZ(t), \qquad \sigma(0) = 1.$$

Here the volatility of volatility parameter ν is also assumed to be constant over time hence $\nu(t) = \nu$. This choice is made for simplicity, in practice a piece-wise constant volvol is preferred. In the dynamics of equation (6.3.1) different starting values for the volatility process could be chosen depending on the rate, see the $\sigma_j(0) = \alpha_j$. A possibility was to choose α_j as the instantaneous volatility extracted from the market. Now that the volatility is driven by one process this is not possible anymore. In order to have enough free parameters to calibrate the model to caplets, the volatility process in the dynamics of the underlying rates is scaled by multiplying σ in $dR(t)$ with the constant α_j. Multiplying the volatility in the equation of the rates doesn't change anything about the dynamics. Putting this together results in the following dynamics

$$\begin{aligned} dR_j(t) &= \gamma_j(t)\alpha_j\sigma(t)R_j(t)^\beta dW_j(t), \qquad \gamma_j(t) = \min\left(\frac{(T_j-t)^+}{(T_j-T_{j-1})}, 1\right)^q \\ d\sigma(t) &= \nu\sigma(t)dZ(t), \qquad \sigma(0) = 1. \end{aligned} \tag{6.3.2}$$

The final assumption which is made is the fact that the Brownian motion of the single volatility process is uncorrelated with the Brownian motions of the backward-looking forward rates, $E[dWdZ] = 0$. This is a real simplification, the consequence is that the drift function of the volatility dynamics will be equal to zero.

6.3.1. Dynamics under different measures

The dynamics for the backward-looking rates are standard SABR dynamics when the accrual period is not reached yet, $t < T_{j-1}$. From these dynamics under the martingale measure the next steps for the SABR forward market model are considered. Here a distinction is made between two parts of the model. The calibration part which is about the dynamics under the martingale measure which leads to Hagan's approximation for caplet prices and the implementation part, where the dynamics under the spot-Libor measure are preferred.

(Calibration part) For the calibration part, the following parameter is defined

$$\Omega_j(t) = \alpha_j \sigma(t), \tag{6.3.3}$$

where α_j can differ per backward rate. Using this notation, equation (6.3.2) can be rewritten as

$$\begin{aligned} dR_j(t) &= \gamma_j(t)\Omega_j(t)R_j(t)^\beta dW_j(t), & \gamma_j(t) &= \min\left(\frac{(T_j - t)^+}{(T_j - T_{j-1})}, 1\right)^q \\ d\Omega_j(t) &= \alpha_j(t) d\sigma_j(t) = \alpha_j(t)\sigma(t)\nu dZ(t) = \Omega_j(t)\nu dZ(t), & \Omega_j(0) &= \alpha_j(t). \end{aligned} \tag{6.3.4}$$

This notation is equivalent to equation (6.3.2). Notice how similar this is to the original SABR dynamics. The advantage of writing the dynamics in this form is that the standard derivations presented in [14] and [39] to obtain Hagan's approximation for backward-looking and forward-looking caplets can be used. This would have been a problem with equation (6.3.2) where a single volatility process with $\sigma(0) = 1$ is used. Both dynamics are equivalent it is just rewritten in a way it is easier to work with.

Implementation part For the implementation we start working from equation (6.3.2). Just as for the LMM or log-normal FMM it is not preferred to simulate the rates under one specific (extended) forward measure due to the bias of the drift term being unevenly spread over the rates. The solution is again to derive the dynamics of the rates under the spot-Libor measure. To derive these dynamics equation (6.3.2) is slightly rewritten to

$$\begin{aligned} dR_j(t) &= C_j(t) dW_j(t), \\ d\sigma(t) &= D(t) dZ(t), \qquad \sigma(0) = 1. \end{aligned} \tag{6.3.5}$$

Here $C_j(t) = \gamma_j(t)\alpha_j(t)\sigma(t)R_j(t)^\beta$, $D_j(t) = \nu\sigma(t)$. This notation simplifies the derivation and in the end it is possible to fill in C_j and D_j.

Lemma 6.3.1. The SABR dynamics of the backward-looking forward rates under the spot-Libor measure, assuming no correlation between the Brownian motions of the underlying rate and the volatility process, are given by

$$\begin{aligned} dR_j(t) &= \alpha_j(t)\sigma_j(t)\gamma_j(t)R_j(t)^\beta \left(\sum_{i=q(t)+1}^{j} \rho_{i,j} \frac{\tau_i \alpha_i(t)\sigma_i(t)\gamma_i(t)R_i(t)^\beta}{1 + \tau_i R_i(t)} dt + dW_j^{Q^d}(t) \right), \\ d\sigma(t) &= \nu\sigma(t) dZ(t), \qquad \sigma(0) = 1. \end{aligned} \tag{6.3.6}$$

Proof. For the proof the change of measure technique given in equation (4.4.4) is used. This will result in the drift function for both the backward-looking rate and the volatility process under the new measure. The switch from the extended T_j-forward measure to the spot-Libor measure is made, hence $N(t) = B_d(t)$. Filling this into equation (4.4.4) results in

$$\text{Drift}(R_j; Q^d)(t) = \frac{dR_j(t) d\ln(B_d(t)/P(t, T_j)))}{dt} = \frac{dR_j(t) d\ln(P(t, q(t))/P(t, T_j)))}{dt}. \tag{6.3.7}$$

Working out the logarithmic part of the equation

$$\ln \frac{P(t, q(t))}{P(t, T_j)} = \ln \prod_{i=q(t)+1}^{j} \frac{P(t, T_{i-1})}{P(t, T_i)} = \ln \prod_{i=q(t)+1}^{j} (1 + \tau_i R_i(t)) = \sum_{i=q(t)+1}^{j} \ln(1 + \tau_i R_i(t)).$$

Filling this back into equation (6.3.7)

$$\begin{aligned} \text{Drift}(R_j; Q^d)(t) &= \frac{dR_j(t) d\sum_{i=q(t)+1}^{j} \ln(1 + \tau_i R_i(t))}{dt}, \\ &= \sum_{i=q(t)+1}^{j} \frac{dR_j(t) d\ln(1 + \tau_i R_i(t))}{dt}, \\ &= \sum_{i=q(t)+1}^{j} \frac{\tau_i}{1 + \tau_i R_i(t)} \frac{dR_j(t) dR_i(t)}{dt}, \\ &= C_j(t) \sum_{i=q(t)+1}^{j} \rho_{i,j} \frac{\tau_i C_j(t)}{1 + \tau_i R_i(t)}. \end{aligned}$$

This gives the drift of the backward-looking rates under the spot-Libor measure in terms of $C_j(t)$.

$$dR_j(t) = C_j(t)\left(\sum_{i=q(t)+1}^{j} \frac{\tau_i C_i(t)}{1+\tau_i R_i(t)} dt + dW_j^{Q^d}(t)\right). \tag{6.3.8}$$

Writing out $C_j(t)$ gives the dynamics for the backward-looking forward rates. Similarly the dynamics of the volatility under the spot-Libor measure can be obtained. For this derivation imagine that every $R_j(t)$ has its own volatility process which is called $\sigma_j(t)$. The assumption was made that the rates are driven by only one process as a consequence, $\sigma_1(t) = \sigma_2(t) = \cdots = \sigma_N(t)$. Starting again from equation (4.4.4)

$$\text{Drift}(\sigma_j; Q)(t) = \frac{d\sigma_j(t) d\ln(B_d(t)/P(t,T)))}{dt}. \tag{6.3.9}$$

Note that the part in the logarithm is exactly the same as in equation (6.3.7) hence this was already written out, resulting in

$$\begin{aligned}
\text{Drift}(\sigma_j; Q^d)(t) &= \frac{d\sigma_j(t) d\sum_{i=q(t)+1}^{j} \ln(1+\tau_i R_i(t))}{dt}, \\
&= \sum_{i=q(t)+1}^{j} \frac{d\sigma_j(t) d\ln(1+\tau_i R_i(t))}{dt}, \\
&= \sum_{i=q(t)+1}^{j} \frac{\tau_i}{1+\tau_i R_i(t)} \frac{d\sigma_j(t) dR_i(t)}{dt}, \\
&= D_j(t) \sum_{i=q(t)+1}^{j} \eta_{i,j} \frac{\tau_i C_j(t)}{1+\tau_i R_i(t)}, \\
&= 0.
\end{aligned} \tag{6.3.10}$$

Here the drift function is equal to zero since the correlation between the two Brownian motions W and Z was assumed to be equal to zero ($\eta_{ij} = 0$). Thus, the drift function is zero under the spot-Libor measure therefore we have the following dynamics for the volatility process

$$d\sigma_j(t) = D_j(t) dZ(t),$$

and because all the volatility processes are the same this can be written as

$$d\sigma(t) = D(t) dZ(t),$$

where writing out the $D(t)$ term leads to the lemma. □

The dynamics for the implementation are thus written differently compared to the dynamics for the calibration part. They both have advantages depending on what kind of properties of the model are considered. From equation (6.3.6), a few properties can be observed. First of all, before the rates reach the accrual period they follow standard SABR dynamics. Second, choosing $\beta = 1$ and $\nu \downarrow 0$ the original log-normal forward market dynamics are obtained. With the dynamics of the SABR forward market model it is possible to simulate the backward-looking forward rates together with a stochastic volatility parameter. There are however two more issues that need to be solved before moving on to the simulations.

6.3.2. Problem 1: Simulation

Since the switch from a log-normal type model to a stochastic volatility model is made, the log-normal property which assured positive rates is lost. Even though in Chapter 5 it was discussed that it isn't necessarily a bad property since the market does show negative interest rates, it is a problem for the SABR model. Consider equation (6.3.6) where $\beta < 1$, when the input for the model are positive rates it could happen that during the simulation a path becomes negative. When the rate is negative and we take a power which is smaller than one this value could be undefined, resulting in an error. Two possible solutions for this issue are proposed. Both solutions don't change the dynamics, the dynamics are just rewritten in such a way to deal with this issue. For the first solution, the dynamics of the backward-looking forward rates can be rewritten starting from equation (6.3.6)

$$dR_j(t) = \alpha_j(t)\sigma(t)\gamma_j(t)R_j(t)^\beta \left(\sum_{i=q(t)+1}^{j} \rho_{i,j} \frac{\tau_i \alpha_i(t)\sigma(t)\gamma_i(t)R_i(t)^\beta}{1+\tau_i R_i(t)} dt + dW_j^{Q^d}(t)\right).$$

For an arbitrary rate this can be written as

$$dR(t) = \hat{\mu} dt + \hat{\sigma} dW(t),$$

where $\hat{\mu}$ is the drift and $\hat{\sigma}$ is the diffusion. Both terms contain R^{β} which means that one R can be taken out of these terms and be put in the front. This results in the following dynamics

$$\frac{dR(t)}{R(t)} = \tilde{\mu} dt + \alpha\sigma(t)\gamma(t)R(t)^{(\beta-1)} dW(t).$$

Here $\tilde{\mu} = \frac{\hat{\mu}}{R(t)}$ and the diffusion term is written out. The first $R(t)^{\beta}$ term in the drift changes to $R(t)^{\beta-1}$. The result is that

$$dR(t) = \tilde{\mu} R(t) dt + \alpha\sigma(t)\gamma(t)R(t)R(t)^{(\beta-1)} dW(t).$$

Applying Itô's lemma, take $g(t,R) = \ln(R(t))$ hence $\frac{dg(t,R)}{dR(t)} = \frac{1}{R(t)}$, $\frac{d^2 g(t,R)}{dR(t)^2} = -\frac{1}{R(t)^2}$ and $\frac{dg(t,R)}{dt} = 0$. Therefore,

$$\begin{aligned} dg(t,R) &= \left(\tilde{\mu}\frac{1}{R(t)}R(t) - \frac{1}{2}\frac{1}{R(t)^2}(\alpha\sigma(t)\gamma(t)R(t)R(t)^{(\beta-1)})^2\right)dt + \frac{1}{R(t)}\alpha\sigma(t)\gamma(t)R(t)R(t)^{(\beta-1)}dW(t), \\ &= \left(\tilde{\mu} - \frac{1}{2}\alpha\sigma(t)^2\gamma(t)^2R(t)^{(2\beta-2)}\right)dt + \alpha\sigma(t)\gamma(t)R(t)^{(\beta-1)}dW(t). \end{aligned}$$

Using the fact that $dg(t,R(t)) = \ln(R(t))$, by integrating and filling in the drift term, the following expression is obtained

$$\begin{aligned} R_j(t+dt) = R_j(t)\exp\Bigg(\Bigg[\gamma_j(t)\alpha_j(t)\sigma(t)R_j(t)^{\beta-1}\sum_{i=q(t)+1}^{j}\rho_{i,j}\frac{\tau_i\gamma_i(t)\alpha_i\sigma(t)R_i(t)^{\beta}}{1+\tau_iR_i(t)} - \frac{1}{2}\sigma(t)^2\gamma(t)^2R^{2\beta-2}\Bigg]dt + \ldots \\ \alpha_j\sigma(t)\gamma(t)R^{(\beta-1)}dW_j(t)\Bigg). \end{aligned} \tag{6.3.11}$$

The dynamics of equation (6.3.6) are written in an equivalent form, but this time with an exponential term in the dynamics. Consequently, the model doesn't produce negative rates anymore. With these new dynamics in equation (6.3.11) a new problem arises due to the power $\beta - 1$ in the dynamics. Suppose $\beta = 0.5$, which isn't unusual in market practice. The problem is that the input values for the rates $R_j(t)$ are small, taking the power $\beta - 1 = -0.5$ of these rates results in a large number after which the exponential of this number is taken which results in an even larger number, this makes the interest rates explode over time. One easy solution is to simply choose $\beta = 0.9$, which won't result in exploding rates. However, some markets require a smaller β as input. For this, another solution is proposed starting from equation (6.3.2) again.

Instead of rewriting the dynamics of the backward-looking forward rates it is also possible to apply a reflecting scheme. The reflecting principle uses the original dynamics, where at the end of every simulation step the absolute function is taken

$$\begin{cases} R_j(t+\Delta) = \alpha_j\sigma(t)\gamma_j(t)R_j(t)^{\beta}\left((\sum_{i=q(t)+1}^{j}\rho_{i,j}\frac{\tau_i\alpha_i\sigma(t)\gamma_i(t)R_i(t)^{\beta}}{1+\tau_iR_i(t)})\Delta + dW_j^{Q^d}(t)\right), \\ R_j(t+\Delta) = |R_j(t+\Delta)|. \end{cases} \tag{6.3.12}$$

The result here is that when the simulation gives a negative value the reflection on the x-axis is taken, resulting in a positive rate. It makes sense that this method is more biased compared to rewriting the dynamics since paths where the rates reach small values could suddenly be projected onto larger rates. The expectation is that using mirroring requires more Monte Carlo paths to reach accurate results. The advantage is that when mirroring is used every β value can be used as an input.

6.3.3. Problem 2: Market data

The second problem to solve is the fact that negative interest rates in the market are present. Similar to the log-normal forward market model a shifted version of the SABR model will be implemented. The shift is given in the market and therefore the solution is simple. The shifted SABR model is already discussed in for example [24]. We start from the dynamics under the martingale measure, where a shift is applied to the backward-looking forward rates, i.e.,

$$\begin{aligned} dR_j(t) &= \alpha_j(t)\gamma_j(t)\sigma(t)(R_j(t)+\theta_j(t))^\beta dW_j(t), \\ d\sigma(t) &= \nu\sigma(t)dZ(t), \sigma(0) = \alpha. \end{aligned} \tag{6.3.13}$$

The assumption is made that the shift is constant for every rate and over time, hence $\theta_j(t) = \theta$. Again the dynamics under the spot-Libor measure are required. There are no changes made to the volatility process therefore, the dynamics remain the same and no derivations are necessary for the dynamics of $\sigma(t)$. For the backward-looking rates we can write

$$d(R_j(t)+\theta) = \gamma_j(t)\alpha_j(t)\sigma(t)(R_j(t)+\theta_j(t))^\beta dW_j(t),$$

since the shift doesn't change the dynamics because it is just a constant. Define $X_j(t) = R_j(t)+\theta$ which results in

$$dX_j(t) = \gamma_j(t)\alpha_j\sigma(t)X_j(t)^\beta dW_j(t) = C_j(t)X_j(t)^\beta dW_j(t).$$

This is a similar expression as in equation (6.3.5) and a similar derivation can be applied to obtain the dynamics under the spot-Libor measure.

Lemma 6.3.2. When $X(t) = R(t)+\theta$, the shifted SABR dynamics of the backward-looking forward rates under the spot-Libor measure, assuming no correlation between the two Brownian motions, are given by

$$\begin{aligned} dX_j(t) &= \alpha_j\sigma(t)\gamma_j(t)X_j(t)^\beta\left(\sum_{i=q(t)+1}^{j}\rho_{i,j}\frac{\tau_i\alpha_i\sigma(t)\gamma_i(t)X_i(t)^\beta}{1+\tau_iR_i(t)}dt + dW_j^{Q^d}(t)\right), \\ d\sigma(t) &= \nu\sigma(t)dZ_j(t), \qquad \sigma(0) = 1. \end{aligned} \tag{6.3.14}$$

Proof. The proof can be found in appendix D. We don't present it here since it is similar to the previous proof. □

With the full dynamics for the shifted SABR forward market model, simulation of the backward-looking forward rates should be possible. Besides zero-coupon bonds, caplets can now be priced using market data. There is one final remark regarding the simulation. For the Libor market model the simulation was performed using either the Euler discretization method or the predictor-corrector method, while for the log-normal forward market model the rates were simulated using a combination of both methods, where the Euler's discretization was performed for the accrual period. For the derivation of the predictor-corrector method the following integral had to be solved $\int_{T_k}^{T_{k=1}}\sigma_i(u)dW_i(u)$. For the SABR model the volatility parameter is a stochastic process over the entire simulation period. Consequently, a Brownian bridge must be used to solve this integral. This problem will not be considered in this thesis and therefore, the choice is made to solely use Euler's method for the simulation of the SABR-FMM. The fallback of this choice are the much longer computational times.

6.4. Validation

The dynamics of the backward-looking forward rates have now been derived, hence the model can be implemented and validated by pricing either zero-coupon bonds or caplets. The rates will be simulated using only Euler's method where between two tenor points a discretization of 64 steps is taken. For the following results the expression in equation (6.3.11) will be used, since the reflecting method requires more simulated paths. The following parameter values are chosen

$$q = 1, \qquad \beta = 0.9, \qquad \nu = 0.1.$$

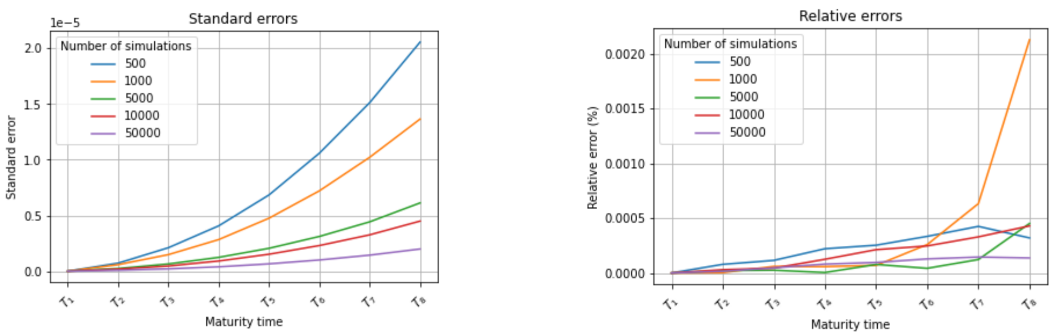
Finally, the different α_j parameters are chosen to be equal to the instantaneous volatilities given by the market. The first backward-looking rate runs from $[T_0, T_1]$ and since the market data doesn't offer a volatility parameter for this rate a value of zero is chosen. Consequently, the first backward-looking rate remains constant.

6.4.1. Zero-coupon bonds

Table 6.1 present the results where eight different zero-coupon bonds are priced by implementing the shifted SABR forward market model. A total of 50000 Monte Carlo paths were used. The maturities range from T_1 till T_8. What stands out in the table is that the first zero-coupon bond is priced exactly. This makes sense, since the first ZCB only depends on $L(0, T_0, T_1)$ which is the spot-rate and this is fixed and given by the market. It is apparent from this table that the model is implemented in a correct way. The zero-coupon bonds are priced with the same accuracy compared to the implementation of the other models in this thesis. The values of the zero-coupon bonds are larger than one due to the negative interest rates which are present in the market. Besides, figure 6.1 shows two sub figures where we can see that the standard error and relative error improve for a larger amount of Monte Carlo paths, which is consistent with the results previously presented in the thesis.

Table 6.1: Pricing zero-coupon bonds with the SABR-FMM

	Analytical Price	Simulated price	Error (%)	Standard error
P(T_1)	1.0013	1.0013	0	0
P(T_2)	1.0025	1.0025	3.55e-07	7.80e-08
P(T_3)	1.0038	1.0038	1.55e-05	2.04e-07
P(T_4)	1.0052	1.0052	1.43e-05	3.96e-07
P(T_5)	1.0065	1.0065	8.76e-06	6.61e-07
P(T_6)	1.0079	1.0079	1.54e-05	1.01e-06
P(T_7)	1.0092	1.0092	3.88e-05	1.44e-06
P(T_8)	1.0106	1.0106	8.36e-05	1.99e-06



(a) Standard errors

(b) Relative errors

Figure 6.1: Pricing zero-coupon with the SABR-FMM

6.4.2. Caplets

In the post-Libor world two types of caplets will be present in the market. First, the forward-looking caplet where the floating rate is $R(T_{j-1}, T_{j-1}, T_j)$. Since the dynamics of the backward-looking caplet follow the standard SABR dynamics till the start of the accrual period $[T_{j-1}, T_j]$, forward-looking caplets can be priced with the original Hagan's approximation for caplets. The second type of caplet is the backward-looking caplet, and for this caplet the floating rate is equal to $R(T_j, T_{j-1}, T_j)$. The backward-looking caplet has extra uncertainty in its price, besides the dynamics are different compared to forward rates. Therefore, Hagan's approximation must be adapted to obtain the correct caplet price. The forward-looking caplets value is given by

$$\begin{aligned} Cpl_j^F(0) &= P(0,\tau_1)\pi\left(T_{j-1}, K, R_j(0), \sigma_{IV}^F\right), \\ \sigma_{IV}^F &\approx \sigma_{hagan}\left(T_{j-1}, K, R_j(0), \alpha, \beta, \rho, \nu\right). \end{aligned} \tag{6.4.1}$$

Here the expression for the ATM volatility parameter is given in equation (6.2.6). The value of a backward-looking caplet is given as

$$\begin{aligned} Cpl_j^B(0) &= P(0,\tau_1)\pi\left(T_j, K, R_j(0), \sigma_{IV}^B\right), \\ \sigma_{IV}^B &\approx \sigma_{hagan}\left(T_j, K, R_j(0), \hat{\alpha}, \beta, \hat{\rho}, \hat{\nu}\right). \end{aligned} \tag{6.4.2}$$

Here a few differences can be seen. First of all, the parameters α, ρ and ν must be adapted for the backward-looking caplets. Second the settlement time of the backward-looking caplet is T_j contrary to T_{j-1} for the forward-looking caplet. The correct parameters to price backward-looking caplets are obtained from [39].

Theorem 6.4.1. The SABR parameters to obtain today's backward-looking caplet value are equal to

$$\hat{\rho} = \rho \frac{3\lambda^2 + 2qT_{j-1}^2 + T_j^2}{\sqrt{\gamma}(6q+4)}, \quad \hat{\nu}^2 = \nu^2 \gamma \frac{2q+1}{\lambda^3 T_j}, \quad \hat{\alpha}^2 = \frac{\alpha^2}{2q+1} \frac{\lambda}{T_j} e^{\frac{1}{2}HT_j}, \tag{6.4.3}$$

where

$$\lambda = 2qT_{j-1} + T_j, \quad H = \nu^2 \frac{\lambda^2 + 2qT_{j-1}^2 + T_j^2}{2T_j \lambda (q+1)} - \hat{\nu}^2$$

$$\gamma = \lambda \frac{2\lambda^3 + T_j^3 + \left(4q^2 - 2q\right) T_{j-1}^3 + 6qT_{j-1}^2 T_j}{(4q+3)(2q+1)} + 3q\rho^2 \left(T_j - T_{j-1}\right)^2 \frac{3\lambda^2 - T_j^2 + 5qT_{j-1}^2 + 4T_{j-1}T_j}{(4q+3)(3q+2)^2}.$$

Proof. The proof can be found in appendix E. □

Pricing caplets using Hagan's approximation in the original SABR model, equation (6.2.5), leads to arbitrage issues for low strike prices, see for example [15]. This problem will still exist for backward-looking caplets, since only a few constants are changed in the volatility parameter. However, we are dealing with negative interest rates and therefore apply a shift to the rates and the strikes in equation (6.4.2). Therefore, this won't be a problem for every value of β, due to large positive strikes coming from the shift. In the results section we will examine if arbitrage issues arise.

Tables 6.2 and 6.3 present the values of backward-looking caplets for the implemented shifted SABR forward market model. Table 6.2 shows the results where the backward rates were simulated using equation (6.3.11). Table 6.3 shows the results of the implementation where the reflecting principle is used. Both simulations used 50000 Monte Carlo paths and discretized the grid between two tenors into 64 steps, the prices are given in basis points. The new model prices caplets with essentially the same accuracy as the log-normal forward market model. Interestingly, the results of the implementation where the reflecting principle is used is very similar to the case where the dynamics were rewritten to an exponential form. The expectation was that mirroring the rates would produce more biased results. The fact that both methods produce the same results is likely due to parameter choice. With $\beta = 0.9$ and $\nu = 0.1$ we are close to log-normal dynamics, besides a large market shift is applied compared to the values of the original rates. This results into large positive starting values for the simulation. Lowering the β parameter will result in negative rates since the model moves away from log-normal dynamics to normal dynamics.

Table 6.2: Pricing caplets with the SABR-FMM

	Analytical Price	Simulated price	Error (%)	Standard error
Cap(T_1, T_2)	1.60	1.60	0.18	0.0058
Cap(T_2, T_3)	2.12	2.11	0.50	0.0078
Cap(T_3, T_4)	2.58	2.58	0.12	0.0097
Cap(T_4, T_5)	3.03	3.03	0.02	0.0116
Cap(T_5, T_6)	3.52	3.51	0.25	0.0136
Cap(T_6, T_7)	3.97	3.94	0.65	0.0156
Cap(T_7, T_8)	4.55	4.52	0.56	0.0182

Table 6.3: Pricing caplets with the SABR-FMM (Reflecting principle)

	Analytical Price	Simulated price	Error (%)	Standard error
Cap(T_1, T_2)	1.60	1.60	0.19	0.0058
Cap(T_2, T_3)	2.12	2.11	0.50	0.0078
Cap(T_3, T_4)	2.58	2.58	0.11	0.0097
Cap(T_4, T_5)	3.03	3.03	0.02	0.0116
Cap(T_5, T_6)	3.52	3.51	0.26	0.0136
Cap(T_6, T_7)	3.97	3.94	0.65	0.0156
Cap(T_7, T_8)	4.55	4.52	0.56	0.0182

6.4.3. Implied volatility

Most results for valuating caplets in the coming sections are presented in implied volatilities. This is a convenient method to examine how different parameters affect the SABR forward market model implementation. Providing caplet prices in implied volatilities instead of in basis points has as a consequence that different results using different model parameters can be compared. However, we are considering backward-looking caplets. For backward-looking caplets it is sometimes more convenient to work with the variance or a square of the volatility.

To obtain the implied volatility curve of the backward-looking caplets the following steps are taken. Consider a caplet with settlement date T_{j-1}. First, equation (6.4.2) is used to price this caplet for different strike prices. Here, all of the parameters including the reset and payment date are kept constant and only the strike price changes. The next step is to back-out the implied volatility parameter. This was already briefly discussed in Chapter 2, the implied volatility parameter is the σ value in Black's equation to match the given caplet price in the market. Here a few aspects have to be considered.

1. The market shows negative interest rates, therefore, Black's formula for shifted rates has to be used. The equation was given by
$$Cpl_i(t_0) = N\tau P(t_0, T_i)\left[\hat{R}_i(t_0)N(d_1) - \hat{K}N(d_2)\right].$$
Here $\hat{K} = K + \theta_i$ and $\hat{R}_i(t_0) = R_i(t_0) + \theta_i$. Besides
$$d_1 = \frac{\ln\left(\frac{\hat{R}_i(t_0)}{\hat{K}}\right) + \frac{1}{2}v}{\sqrt{v}}, \qquad d_2 = d_1 - \sqrt{v},$$
2. In the post-Libor world, the term v in the above equation can take two forms. For forward-looking caplets it is $v^F = \sigma_j^2(T_{j-1} - t)$ and for backward-looking caplets it becomes
$$v_j^B(t) = \sigma_j^2 \int_t^{T_j} \left(\min\left(\frac{T_j - s}{T_j - T_{j-1}}, 1\right)\right)^2 ds. \tag{6.4.4}$$

This gives the option to report the implied volatilities of the backward-looking caplets in two different ways. First of all, the backward-looking caplet can be priced after which the price is used in Black's equation and we back-out the v^B parameter. Here we stop and won't derive σ, the implied volatility isn't reported in σ but in v^B. This is an accurate method to compare the implied volatilities with each other. A second possibility is to further reproduce the σ parameter using the backed-out v^B from Black's equation. To obtain the implied volatility in terms of σ however, we can not use equation (6.4.4). The reason is that this equation depends on the volatility decreasing parameter γ. This parameter is model dependent, consequently backing-out σ from v^B results in different implied volatilities depending on the model parameter γ. The solution to this problem is to reproduce σ using the expression
$$v^B = \tilde{\sigma}_j(T_j - t). \tag{6.4.5}$$

Thus, first back-out the value of v^B from Black's equation then reproduce σ using equation (6.4.5). This equation is similar to the v^F parameter for forward-looking caplets, however since a backward-looking caplet is a stochastic quantity till the payment date the expression has the term T_j instead of T_{j-1}. Using this method makes the implied volatility parameter free of the time value. In this thesis, the implied volatility is given as in [34] which uses the first method where the implied volatility is denoted as v^B.

6.5. Results

In this section the impact of the model parameters on the implied volatility curve is examined. Four different parameters are considered, β, ν, α and q. We also briefly look into the arbitrage issues that arise using Hagan's approximation.

6.5.1. Parameter: β

The first parameter that will be examined is β. This parameter controls the skewness of the implied volatility curve, [14]. If ν in equation (6.3.2) is equal to zero the model simplifies to the constant elasticity of variance (CEV) model. For $\beta = 0$ we see normal dynamics corresponding to Bacheliers model, while for $\beta = 1$ the model has log-normal dynamics corresponding to Black's model. There isn't one single correct parameter value for β. Typically β is not calibrated to the market but chosen by traders. The value depends on which market is considered, e.g. the β can differ between the Euro and USD market. Figure 6.2 shows the impact of β on the implied volatility curve. First a caplet with an expiry of 4 years is priced for different strikes using Hagan's approximation for backward-looking caplets. Then the shifted version of Black's formula is used to reproduce the implied volatility. The implied volatility value isn't denoted in σ but in υ^B. With our chosen parameters we see an implied volatility skew instead of a smile. The implied volatilities and therefore the caplets prices are higher if β decreases in value, this is due to the so-called leverage effect. The volatility of the rates increases when the rates fall and vice versa. This effect is stronger for smaller β's, a larger volatility results in higher caplet prices. Figure 6.2(c) shows the different curves for a β ranging from 0.5 till 1. Since the differences are large between $\beta = 0.5$ and $\beta = 1$ it almost seems that there are some flat lines in the figure. That is why figures (a) and (b) are zoomed in for a smaller range of β.

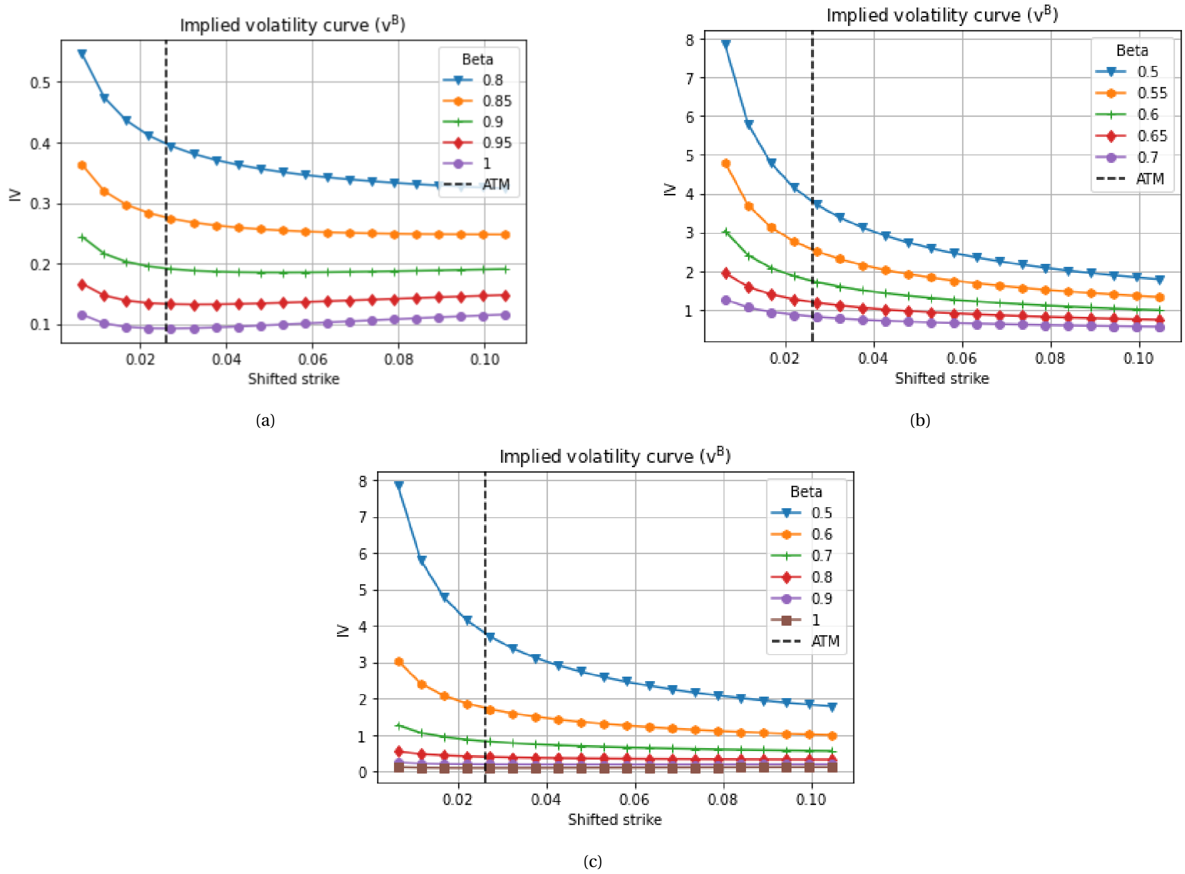


Figure 6.2: Implied volatility smile (T=4, $\nu = 0.1$)

It is also interesting to see the influence of β on the accuracy when caplets are priced. We mentioned that for lower values of β the reflection principle in equation (6.3.12) has to be used, which would result in more biased results. This effect is examined in two ways. First, the number of times reflection is applied for 10000 simulations is counted for different values of β keeping all other parameters constant. This is reported in table 6.4, where the number of times reflection had to be applied during the simulation of a backward-looking rate with $T_{j-1} = 1.75$ and $T_j = 2$ is counted. The more reflection is applied, the more the results should be biased. Note that one simulation consists of two paths, the normal one and the antithetic path and for this rate a total of 512 steps are taken from T_0 till $T_j = 2$. Therefore, it is possible that the reflection principle is applied much more than 10000 times. The table shows that for $\beta = 0.7$ and $\beta = 0.9$ the reflection principle is never used, which means that the backward-looking rate doesn't reach negative values. Therefore, for these values of β using the reflection principle will not be more biased compared to writing the dynamics as in equation (6.3.11). For $\beta = 0.5$, it seems that reflection is used, but significantly less as compared to smaller β values. To further examine how strong this effect is, the relative errors are considered.

The relative errors (%) are reported for eight different caplets and three different β values. Table 6.5 shows the results, we see that for a smaller value of β the caplets are priced less accurately, especially for $\beta = 0.2$. The table also shows that even though for $\beta = 0.5$ reflection was sometimes applied it only slightly increases the relative errors. Note that reporting the standard error doesn't contribute here since a lower β automatically results in higher caplet prices and therefore, the standard errors will be higher. Interestingly, caplets with an expiry further away in the future are priced much worse compared to caplets with a shorter expiry. This can have two reasons. First of all, caplets with a shorter expiry depend on backward rates which are alive for a shorter period. This means that the reflection principle will be applied fewer times compared to rates which are alive for a longer period. Second, caplets with a longer expiry depend on rates with a longer maturity. Simulating these rates with a longer maturity we need to obtain the drift term which depends on previous rates. If the rates in the drift term are all biased this will enlarge the bias in the longer maturity rate.

Table 6.4: Number of times reflection had to be applied for $R(t, 1.75, 2)$

	$\beta = 0.1$	$\beta = 0.3$	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$
Number of reflections	232175	218742	16088	0	0

Table 6.5: Relative error (%) for eight different caplets valuated using the reflecting principle

	Cap_0	Cap_1	Cap_2	Cap_3	Cap_4	Cap_5	Cap_6	Cap_7
$\beta = 0.2$	0	0.25	0.57	3.7	8.3	15	21	32
$\beta = 0.5$	0	0.21	0.51	0.10	0.05	0.33	0.79	0.69
$\beta = 0.9$	0	0.19	0.50	0.11	0.02	0.26	0.65	0.56

6.5.2. Parameter: ν

The next parameter is ν, which is the volatility of the volatility process, also known as the volvol. A higher volvol will result in a higher volatility process over time, while a lower volvol has as effect that the volatility process won't change that much with the extreme of $\nu = 0$ resulting in a constant volatility. Figure 6.3 shows the influence of the volvol parameter on the implied volatility curve. The results are for a caplet with an expiry of four years and $\beta = 0.9$. As can be seen from the figure, the parameter has an impact on the curvature of the implied volatility. Setting ν almost equal to zero results in the most flat line. This makes sense since with $\beta = 0.9$ the dynamics are close to log-normal dynamics with a constant volatility parameter and we know that this results in a flat implied volatility line. Increasing the parameter makes the curve more and more smiling.

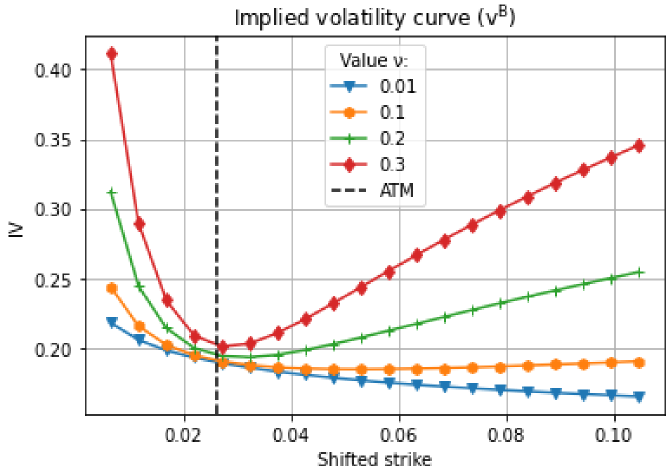


Figure 6.3: Implied volatility effect ν. T = 4, $\beta = 0.9$

6.5.3. Parameter: α

The third parameter we examine is α. Remember that for the SABR-FMM dynamics the choice was made to use only one volatility process, where $\sigma(0) = 1$. We multiplied the volatility in the dynamics of the backward rates with α. For the simulation the value of α was chosen to be equal to the instantaneous volatilities given by the market. We examine how different values of instantaneous volatilties would affect the implied volatility curve. This is performed by multiplying α to make it smaller or bigger. Figure 6.4 presents these results for a caplet with an expiry of four years, $\beta = 0.9$ and $\nu = 0.1$. A multiplier of one implies that the instantaneous volatilities given by the market are used. From the figure it is clear that α impacts the level of the implied volatility curve. A larger multiplier and thus bigger α results in a higher implied volatility value. This makes sense since a larger value for α results in larger values for $dR(t)$, see equation (6.3.14). Therefore, the caplets prices will be higher and thus also the implied volatilities.

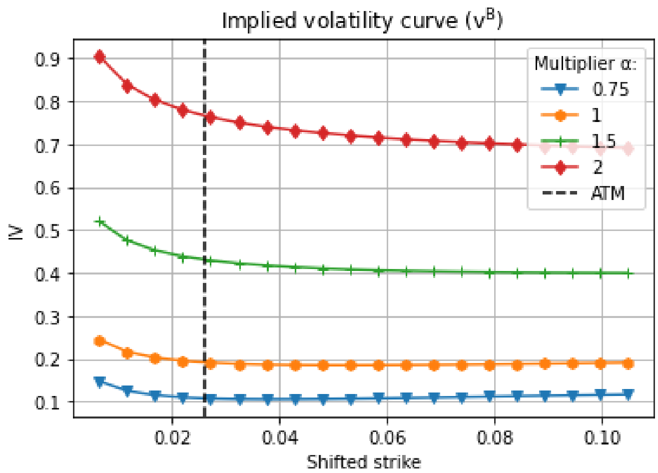


Figure 6.4: Implied volatility effect α. T = 4, $\beta = 0.9$, $\nu = 0.1$

6.5.4. Parameter: q

The final parameter we examine is a new one which is not present in the original SABR model. We look at q, which controls the volatility decreasing function and was given as

$$\gamma_j(t) = \min\left(\frac{(T_j - t)^+}{(T_j - T_{j-1})}, 1\right)^q.$$

Therefore, q controls how fast the volatility will decrease in the accrual period. Figure 6.5 shows the value of $\gamma(t)$ where $T_{j-1} = 0.25$ and $T_j = 0.5$. From this figure we see that for larger values of q the volatility will decrease faster compared to lower values. Figure 6.6 shows the impact of q on the implied volattility curve. What stands out is that the parameter has an effect on the level of the implied volatility curve, however much smaller compared to α. To understand this, consider the parameter α. We rewrote our dynamics with a single volatility process and the α parameter was used as a constant in the dynamics of the backward rates in front of every σ. The same goes for γ which only appears in the dynamics of the backward rates and always in front of σ, see for example equation (6.3.14). Therefore, intuitively it will have the same effect as α. However, γ only has an effect on the volatility during the accrual period and is much less strong compared to α. This is the reason that the level of the implied volatility changes much less compared to α.

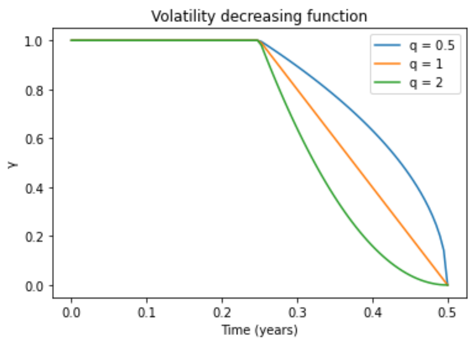


Figure 6.5: Function $\gamma(t)$ depending on q.

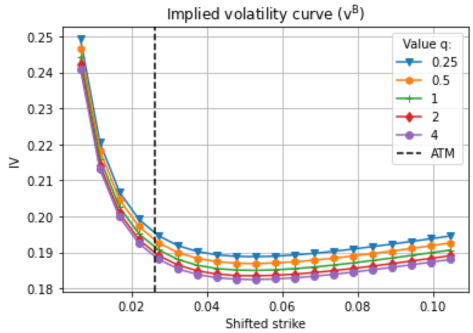


Figure 6.6: Implied volatility effect q. T = 4, $\beta = 0.9$, $\nu = 0.1$.

6.5.5. Arbitrage issue

The issue of Hagan's formula is that it is not arbitrage free, this problem is explained for example by the authors of [15]. This can be best shown through the example of a butterfly option.

Definition 6.5.1. (butterfly option) A butterfly option is an option strategy consisting of multiple call options. The value of this option at maturity T is given as

$$V_B(t,T,K,\Delta K) = C(t,T,K+\Delta K) - 2C(t,T,K) + C(t,T,K-\Delta K). \tag{6.5.1}$$

Here $\Delta K > 0$ is the call spread.

A butterfly option can also be defined as an option on the backward-looking rates using backward-looking caplets as call options. For simplicity, we will assume that the discount factor equals one for all t. Therefore, it can be shown for example in [30] that the relation between a butterfly option and the probability density function of the underlying is given by

$$f(K) = \frac{\partial^2 C}{\partial K^2}(K) \approx \frac{C(K+\Delta K) - 2C(K) + C(K-\Delta K)}{\Delta K^2} = \frac{\mathrm{V_B}(K,\Delta K)}{\Delta K^2}, \tag{6.5.2}$$

where $\Delta K \ll 1$.

Figure 6.7 shows two different plots which both show the value of a butterfly option. For both figures the call option was considered to be a backward-looking caplet with an expiry of four years. Besides

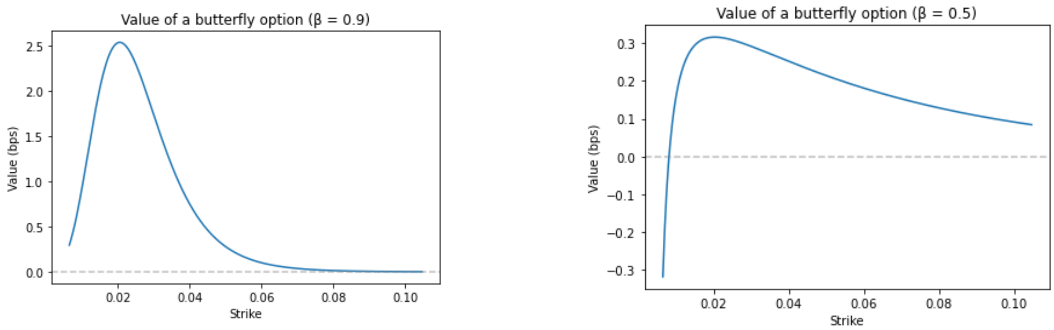
$$\nu = 0.1, \qquad q = 1, \qquad \Delta K = 0.005.$$

Figure 6.7(a) shows the option value for $\beta = 0.9$ while figure 6.7(b) shows the option value where $\beta = 0.5$. We see that the right figure shows negative values for the butterfly options which shouldn't be possible. Besides, from the relation in equation (6.5.2) this would imply negative PDF values at small strike prices, this should also be impossible. Therefore, we conclude that also in the post-Libor world there are still arbitrage issues with Hagan's formula for backward-looking caplets.

For $\beta = 0.9$, there aren't any arbitrage issues present. Whether or not negative density function values are implied, depends on the parameter choices that are made. Three choices have the biggest effect on the value of a butterfly option at small strikes:

1. The value of β: We see that for larger values of β there won't be any arbitrage issues present, considering the value of a butterfly option. For small values of β it is possible that the approximation is not arbitrage-free.
2. The shift by the market: The market shift has a positive effect on the arbitrage problem, meaning that without the market shift we are actually forced to use higher values of β to avoid any arbitrage issues.
3. The maturity of the caplet: Negative values for a butterfly option are more present for caplets with a longer maturity date. For caplets with a maturity closer in the future, smaller values of β can actually be used.

Therefore, when pricing the new backward-looking caplet in the post-Libor world with Hagan's approximation, we should be aware that there is a possibility that this is not arbitrage-free for small strikes. This depends on the parameter choices that are made, especially the value of β, the market shift and maturity of the caplet should be taken into consideration.



(a) $\beta = 0.9$

(b) $\beta = 0.5$

Figure 6.7: Value of a butterfly option

6.6. Summary

This chapter combined the forward market model with the Stochastic-Alpha-Beta-Rho (SABR) model. The combination with a stochastic volatility model was chosen since this model was able to reproduce the implied volatility curve present in the market. An issue with the log-normal forward market model was that it assumed a constant implied volatility line and therefore, it was unable to reproduce the implied volatility curve from the market. A stochastic volatility model was actually able to replicate this curve in the market. The choice for the SABR model was made since with this model it was still possible to obtain analytical caplet prices and therefore, the new SABR forward market model could be calibrated. A model should always be calibrated before using it in practice. We introduced the dynamics under the martingale measure for the SABR forward market model under a few assumptions. First of all, it was unknown how to calibrate the correlation parameters between the Brownian motions of the volatility processes. Therefore, we assumed that the volatility process is driven by a single Brownian motion. Besides, for simplicity it was assumed that the volatility of volatility parameter was constant over time and that the Brownian motions of the underlying rates were uncorrelated with the Brownian motion of the volatility process.

To simulate the backward-looking forward rates under the SABR forward market model, the dynamics under the spot-Libor measure were preferred. These dynamics were obtained by applying a change-of-measure technique. Most notable under the spot-Libor measure was the fact that the volatility process didn't had a drift term due to the assumptions we made. Since the model no longer had log-normal dynamics, the rates could reach negative values resulting in implementation issues. This problem could be solved in two ways. The first solution was to rewrite the dynamics using Itô's lemma, which was a working solution but only for large values of β. For smaller values of β the rates would explode. The second solution was to use the reflecting principle, where after every step the absolute value of the simulated rates were taken. Even though this solution worked for all β values, it would result in more biased results. Finally, we also needed to change from a SABR to a shifted SABR model, in order to deal with the negative interest rates from the market.

The next step was to validate the implementation of the shifted SABR forward market model. This was performed by comparing simulated zero-coupon bond and caplet prices to analytical prices. In the post-Libor world it was still possible to use Hagan's approximation for the new type of backward-looking caplets. However, the parameters for this approximation did need some slight adaptations. We found that the model was implemented in a correct way, since it priced zero-coupon bonds and caplets with the same accuracy as previously implemented models. Therefore, we could conclude that the shifted SABR forward market model would be a possible solution to price exotic interest rate derivatives in the post-Libor world.

The final part of this chapter examined the model behaviour for different parameter values. This was done by pricing a caplet for a variety of strikes using different parameter values and then backing-out the implied volatility curve. We found that the parameters β, ν and α had the same effect on the implied volatility curve as the classic SABR model. Besides, it was also possible to examine some new properties. First of all, we were able to control the volatility decreasing function with the parameter q. We saw that q had the same effect on the implied volatility curve as α, it shifts the level of the curve. Also, we examined the accuracy of pricing caplets for different values of β. We were forced to use the reflecting principle for the implementation of the SABR-FMM for small β values. We found that when $\beta < 0.5$, caplets were priced extremely inaccurately. Especially for caplets with a longer maturity time. Therefore, more Monte Carlo paths were required to obtain accurate pricing results.

Finally, we checked if pricing caplets with Hagan's approximation would be arbitrage-free in the post-Libor world. We found that for small values of β there could still be arbitrage issues, implying a negative density function. This problem heavily depended on the parameter choice of β and also on the market shift and maturity of the caplet which is considered. For a larger market shift or caplet with a shorter maturity time, a smaller value of β could be used without seeing any arbitrage issues.

7

Discussion and future research

In this final chapter we discuss what has been examined in this thesis and give some recommendations for future research.

7.1. Discussion

The last couple of years the market of unsecured borrowing between banks hasn't been that active which gives banks a hard time in setting their Libor rates. Therefore, at the end of 2021, alternative rates will replace the Libors. These alternative nearly risk-free-rates will be different per country and will be overnight backward looking rates. These new rates must be converted into term rates for which a compounded setting-in-arrears method will be used. Compounding the rates gives problems for pricing interest rate derivatives. The current approach to price exotic interest rate derivatives in the market is to use the Libor market model. One major issue in the switch from Libors to the new backward-looking rates is the fact that the new rates keep evolving over their application period since compounding takes place until the maturity time. In the Libor market model the simulation of the rates is only defined till the reset date, hence we cannot simulate the new type of rates. Since interest rate derivatives play a critical role in the hedging process of banks, hedge funds or institutional investors it is of great importance to correctly price these derivatives in the new backward-looking rate environment. The solution to this problem is the generalized forward market model. The new model is built up on the existing Libor market model, therefore first the theory of the LMM was considered.

First, the dynamics of the forward rates under the Libor market model were derived. The dynamics of the forward rates were derived under the spot-Libor measure, since for this measure the bias of the drift term is spread more evenly over all rates. For this thesis the dynamics for the log-normal Libor market model were considered, and the analytical price of a caplet could be obtained by Black's76 formula. With the dynamics we were able to implement the Libor market model using two simulation methods. First of all, Euler's method which was based on a discretization of the entire time grid into small steps. The second possibility was to solve the stochastic differential equation for the forward rates by applying Itô's lemma and making the assumption of a constant drift function per step. This method was called the predictor-corrector method and with this method the step size could be increased to proceed from tenor to tenor point. Both methods were compared to each other by pricing zero-coupon bonds or caplets. The analytical price of a caplet was obtained through Black's equation and then compared to the simulated caplet price. The conclusion was that both methods price the derivatives with the same accuracy considering a Monte Carlo simulation with 50000 paths and discretizing the grid between two tenor points in 64 steps. The advantage was that the predictor-corrector method was about 30 times faster when eight different caplets or zero-coupon bonds were priced. With the theory and implementation of the Libor market model the step to the new generalized forward market model could be made.

The generalized forward market model built on the concept of extended zero-coupon bonds. An extended zero-coupon bond was similar to a normal zero-coupon bond however defined for all time t, thus also for $t > T$. Due to this concept, the dynamics of the underlying rates of the model were given for all times t, contrary to the Libor market model where the underlying rate was only defined till their settlement date. The forward market model was still able to present the dynamics of forward-looking rates and give the dynamics under the risk-neutral measure therefore, it should always be chosen over the Libor market model. Using the definition of the new backward-looking forward rate given in [25] an expression for this new rate was derived. From this expression four properties were apparent. The backward-looking forward rate was a martingale under the appropriate measure, the backward-looking forward rate was equal to the realized backward-looking rate at the maturity time, the backward-looking forward rate was equal to the forward-looking spot rate at the settlement date and finally the backward-looking forward rate was fixed after maturity. The next step was to introduce the dynamics of the backward-looking forward rates under the martingale measure. The dynamics of the rates contained a new parameter which decreases the volatility in the accrual period of the rate. This was due to the fact that the backward-looking forward rates became more certain during the accrual period since more and more overnight rates were realized. With the new dynamics under the martingale measure a change-of-measure technique was applied to obtain the dynamics under the spot-Libor measure. Besides the dynamics under the risk-neutral measure could be obtained, which was impossible for the Libor market model. With the dynamics, the model could be implemented and validated. Again, log-normal dynamics were considered for the model hence caplets were priced consistently with Black's equation. For the implementation a combination was chosen of the predictor-corrector method and Euler's method. The validation was again performed by pricing zero-coupon bonds and caplets. In the post-Libor world two kinds of caplets will exist. The forward-looking caplet where the payoff depends on the value of the floating rate at the start of the accrual period, which is the same kind of caplet that is currently in the market. Second, the backward-looking caplet where the payoff depends on the value of the backward-looking rate at maturity time. This was a new type of caplet where the payoff was not known till the payment date. To obtain the analytical caplet price of the backward-looking caplet, Black's formula was slightly adapted. It was shown that backward-looking caplets were more expensive than forward-looking caplets due to the extra uncertainty. The model implementation priced zero-coupon bonds and caplets in a correct way and with essentially the same accuracy as the Libor market model.

The first implementation and validation of the log-normal forward market model was performed with our own chosen parameters and data. The next step was to implement market data and perform some simulations using this data. We saw that currently, the market shows negative interest rates. Black's equation was not defined for negative values therefore, the switch was made to the shifted log-normal forward market model. The shifted forward market model dealt with negative interest rates by shifting the initial values to positive values. The shift was given by the market and was assumed to be fixed over time and constant per rate. The dynamics for the backward-looking forward rates were derived, including the shift in the dynamics. These new dynamics were consistent with the shifted Black's formula. Consequently, market data could be used to price caplets and zero-coupon bonds. We also looked at a practical example of the fact that the risk-neutral dynamics of the forward-looking and backward-looking rates could be derived. This was done by examining Eurodollar futures, which were a type of interest rate future. Eurodollar futures were based on the futures rate which was slightly different than the forward rate, the difference was called the convexity adjustment. The convexity adjustment came from the fact that daily margining was used for Eurodollar futures to reduce counterparty risk. Today's convexity adjustment could be obtained by simulation when the dynamics of the forward rates under the risk-neutral measure were known. Using the properties of the backward-looking forward rates this was possible and hence, the convexity adjustment could be calculated. We saw that there was a convexity adjustment present in the market which increases if the instantaneous volatilities of the market rises. The obtained convexity adjustment was still model dependent so we could not conclude that this was the true convexity adjustment in the market.

Finally, a model which has not been discussed in the literature was considered in this thesis. The shifted SABR forward market model. One of the disadvantages of the log-normal forward market model was that it was not able to reproduce the implied volatility smile in the market. To solve this issue we combined the forward market model with a stochastic volatility model. The SABR model was a type of stochastic volatility model which had a closed-form solution for valuating caplets. Therefore, the choice was made to combine the forward market model with the SABR model. Consequently, we would still be able to price the liquid caplets consistently with an approximation of the analytical caplet prices, which resulted in a possibility to calibrate the model. This was important since a model first has to be calibrated before using it in practice. The dynamics for the SABR-FMM were proposed where the assumption was made that the underlying rates depend on one volatility process. This assumption was made because we didn't know how to calibrate the correlation parameters for different volatility processes, also less Brownian motions usually leads to more accurate results. One of the parameters of the SABR model was β, where choices of this parameter resulted in negative or exploding rates. Consequently, the model implementation ran into problems. The solution was to either rewrite the model using Itô's lemma or to use the reflecting principle. Both of these methods were applied under the spot-Libor measure. The expectation was that the reflecting principle would be more biased. With the dynamics under the spot-Libor measure it was possible to simulate and validate the new model. However, since the market showed negative rates the shifted SABR model had to be implemented, where again a constant shift was chosen. For the implementation, Euler's method was chosen because for the predictor-corrector method a complicated integral had to be solved. The model valued zero-coupon bonds and caplets with the same accuracy as the previous models if $\beta > 0.5$. For smaller values more Monte Carlo paths were needed, since the reflecting principle was required for the simulation and this gets extremely biased for small β values. The analytical price of backward-looking caplets under the SABR assumptions could be obtained using an adapted form of Hagan's approximation for the volatility parameter. The implementation of the shifted SABR-FMM accurately priced zero-coupon bonds and caplets with the given market data. We now knew that the model was implemented in a correct way and that caplets could be used to calibrate model parameters. Therefore, we are able to conclude that this new model is a possible solution to price exotic interest rate derivatives in the post-Libor world. The final part of this thesis examined how the model parameters influenced the implied volatility smile. First β was examined, which affected the skewness of the implied volatility curve. Besides, we saw that when $\beta < 0.5$ more Monte Carlo paths were required to accurately price caplets. The volvol, ν, had an affect on the curvature of the implied volatility smile. The parameter α controlled the level of the implied volatility smile, the same held for the volatility decreasing function, controlled by q, although this effect was much smaller compared to α. We also showed that using Hagan's approximation in the post-Libor world is not arbitrage-free. Just as for the classic SABR formula, a negative density function could be implied for low strikes. We saw that this dependent on the parameter choices that were made. For small values of β the value of a butterfly option became negative, which implied a negative density function. Other parameter that should be taken into consideration are the market shift and the maturity of the caplet. For a larger market shift or a shorter maturity caplet, smaller values for β could be chosen.

7.2. Future work

The following topics can be examined to expand the research of this thesis:

- The generalized forward market model can be further extended to a complete structure model. This extension of the model is able to provide the evolution of all points on the yield curve. Currently, we can only describe the evolution of a spanning set of forward rates. The theory of this extension of the model is described by the authors of [26].
- This thesis didn't focus on the calibration of the different model parameters. It would be interesting to see how this should be performed in the post-Libor world. We will be dealing with new kinds of interest rate derivatives for which the calibration has to take place. For example, we saw that the volatility decreasing function has an impact on the implied volatility curve. However, we do not know yet how to choose this function and if it can be calibrated to the market.
- Moreover, several solutions can be implemented to solve the arbitrage issues for Hagan's formula. Possibilities are to follow the authors of [15]. They present a method for pricing options under the SABR model resulting in arbitrage-free prices. Second, the stochastic collocation method can be used, introduced in [10].
- Furthermore, the predictor-corrector method could be implemented for the accrual period in the log-normal forward market model or for the entire simulation of the SABR forward market model. In this thesis we went for the simple Euler's method when a stochastic integral had to be solved to obtain the dynamics for the predictor-corrector method. However, it is actually possible to solve the stochastic integral resulting in an expression to simulate the rates from one tenor point to the next tenor point. This would result in a much faster computational time. Remember, if we would like to price exotic interest rate derivatives for which small values of β were necessary, we were forced to use the reflecting principle. We saw that this had a negative impact on the accuracy when pricing interest rate derivatives and therefore, significantly more Monte Carlo paths were needed to obtain accurate prices. This meant that to obtain accurate prices the computational time would also be much larger. Therefore, implementing the predictor-corrector method for the simulation would help solving this problem.

- Finally, more variance reductions techniques, besides antithetic variables, can be applied to the Monte Carlo simulation. Implementing these techniques can result in smaller computational time or it can result in more accurate results. Examples of other variance reduction techniques are using control variates or importance sampling, for more information on variance reduction techniques we refer to [34].

Appendices

A

Dynamics of Libors under an arbitary T_M-forward measure

Lemma A.1. The process $L_n(t)$ under an arbitrary T_M-forward measure is given by

$$
\begin{aligned}
dL_n(t) &= \bar{\sigma}_n(t)\left(-\sum_{k=n+1}^{M-1}\frac{\tau\bar{\sigma}_k(t)}{1+\tau L_k(t)}dt + dW_n^{T_M}(t)\right), && \text{if } M > n+1,\\
dL_n(t) &= \bar{\sigma}_n(t)dW_n^{T_M}(t), && \text{if } M = n+1, \qquad \text{(A.1)}\\
dL_n(t) &= \bar{\sigma}_n(t)\left(\sum_{k=M+1}^{n-1}\frac{\tau\bar{\sigma}_k(t)}{1+\tau L_k(t)}dt + dW_n^{T_M}(t)\right), && \text{if } M < n+1.
\end{aligned}
$$

where $W_t^{T_M}(t)$ is a Brownian motion under the appropriate measure.

Proof. To derive the dynamics for an arbitrary T_M-forward measure we start with deriving the dynamics of $dL_n(t)$ under the T_n measure. We use the fact that $L_n(t)$ is a martingale under the T_{n+1} measure. Starting from the no-arbitrage pricing theorem

$$\frac{L_n(t)P(t,T_{n+1})}{P(t,T_n)} = \mathbb{E}^{T_n}\left[\frac{L_n(S)P(S,T_{n+1})}{P(S,T_n)}|\mathscr{F}_t\right]. \tag{A.2}$$

This is no longer a martingale which means that the stochastic differential equation contains a drift term. Moving forward from equation A.2

$$L_n(t) = \mathbb{E}^{T_n}\left[L_n(S)\frac{P(S,T_{n+1})}{P(S,T_n)}\frac{P(t,T_n)}{P(t,T_{n+1})}|\mathscr{F}_t\right].$$

From this equation we see that the Radon-Nikodym derivative ([31]) between both measures is

$$\frac{d\mathbb{Q}^{T_{n+1}}}{d\mathbb{Q}^{T_n}} = \frac{P(0,T_n)}{P(0,T_{n+1})}\frac{P(t,T_{n+1})}{P(t,T_n)}.$$

Filling in the formula for a zero-coupon bond, this can be rewritten to

$$\frac{1+\tau L(t,T_n)}{1+\tau L(0,T_n)}. \tag{A.3}$$

By applying Girsanovs theorem ([8]), which tells us that if there is some process under measure A (Here T_{n+1}) then for a certain function Y(t) a new Brownian motion under new measure B (Here T_n) can be defined, using: $dW^B(t) = -Y(t)dt + dW^A(t)$. Where the function Y(t) must satisfy the following equation:

$$\frac{d\mathbb{Q}^B}{d\mathbb{Q}^A} = exp\left(-\frac{1}{2}\int_0^t Y^2(s)ds + \int_0^t Y(s)dW^A(t)\right). \tag{A.4}$$

We won't derive the Y(t) function but we just assume we have the correct function as $\frac{\tau\bar{\sigma}_n(t)}{1+\tau L_n(t)}$. The fact that this is the correct equation can be checked by rewriting the Radon-Nikodym derivative in A.3 in such a way that it becomes the same as equation A.4.

Thus, by Girsanovs theorem the relation between the two Brownian motions under the measure T_{n+1} and T_n is given by:

$$dW_t^{T_n} = dW_t^{T_{n+1}} - Y(t)dt = dW_t^{T_{n+1}} - \frac{\tau\bar{\sigma}_n(t)}{1+\tau L_n(t)}dt. \tag{A.5}$$

This means that the dynamics of $dL_n(t)$ under the T_n-measure is given by

$$dL_n(t) = \bar{\sigma}_n(t)\left(Y(t)dt + dW_t^{T_n}\right) = \bar{\sigma}_n(t)\left(\frac{\tau\bar{\sigma}_n(t)}{1+\tau L_n(t)}dt + dW_t^{T_n}\right).$$

To obtain the dynamics of $dL_n(t)$ under an arbitary measure T_M. We just apply equation A.5 recursively. Let's consider the case where $M > n+1$

$$\begin{aligned}
dW_t^{T_{n+1}} &= dW_t^{T_{n+2}} - \frac{\tau\bar{\sigma}_{n+1}(t)}{1+\tau L_{n+1}(t)}dt, \\
&= dW_t^{T_{n+3}} - \sum_{j=n+1}^{n+2}\frac{\tau\bar{\sigma}_j(t)}{1+\tau L_j(t)}dt, \\
&\vdots \\
&= dW_t^{T_M} - \sum_{j=n+1}^{M-1}\frac{\tau\bar{\sigma}_j(t)}{1+\tau L_j(t)}dt
\end{aligned}$$

The same reasoning can be used to consider the case where $M < n+1$. This means that for every $L_n(t)$ for which we know the dynamics under the T_{n+1}-measure we can now switch to an arbitrary T_M measure where we have just derived the drift term. This gives the formula's given in the lemma and concludes the proof. □

B

Volatility parameter backward-looking caplet

Lemma B.1. The volatility parameter in Black's formula for a backwark-looking caplet is given by

$$v_j^B(t) = \sigma_j^2 \left[(T_{j-1} - t)^+ + \frac{1}{3} \frac{(T_j - \max(T_{j-1}, t))^3}{(T_j - T_{j-1})^2} \right]. \tag{B.1}$$

Proof. The derivations starts from the fact that the volatility parameter is given by

$$v_j^B(t) = \sigma_j^2 \int_t^{T_j} \left(min \left(\frac{T_j - s}{T_j - T_{j-1}}, 1 \right) \right)^2 ds.$$

To achieve this, a similar derivation as for the volatility parameter for the standard Black's equation can be performed. An example can be found in [4]. What remains is to solve the integral. Note that for the chosen $\gamma_j(s)$ function there is a clear difference depending on the value of t. Two cases are distinguished

1. t $\leq T_{j-1}$, which means that the period $[t, T_{j-1}]$ exists and that $\gamma_j(s)$ will be equal to one for this period.
2. t $> T_{j-1}$, which means that $\gamma_j(s)$ will be equal to zero for [0,t]. The interval $T_{j-1} - t$ will be negative and this part of the integral should be equal to zero. Finally, this means that you always have to take $\max(T_{j-1}, t)$.

$$\begin{aligned}
\int_t^{T_j} \left(\min \left(\frac{T_j - s}{T_j - T_{j-1}}, 1 \right) \right)^2 ds &= \int_t^{T_{j-1}} 1 ds + \int_{T_{j-1}}^{T_j} \left(\frac{T_j - s}{T_j - T_{j-1}} \right)^2 ds \\
&= \max(T_{j-1} - t, 0) + \int_{T_{j-1}}^{T_j} \left(\frac{T_j - s}{T_j - T_{j-1}} \right)^2 ds \\
&= \max(T_{j-1} - t, 0) + \frac{T_j^2}{(T_j - T_{j-1})^2} \int_{T_{j-1}}^{T_j} 1 ds - \frac{2T_j}{(T_j - T_{j-1})^2} \int_{T_{j-1}}^{T_j} s ds + \frac{1}{(T_j - T_{j-1})^2} \int_{T_{j-1}}^{T_j} s^2 ds \\
&= \max(T_{j-1} - t, 0) + \frac{T_j^2 (T_j - \max(T_{j-1}, t))}{(T_j - T_{j-1})^2} - \frac{T_j (T_j^2 - \max(T_{j-1}, t)^2)}{(T_j - T_{j-1})^2} + \frac{1}{3} \frac{T_j^3 - \max(T_{j-1}, t)^3}{(T_j - T_{j-1})^2} \\
&= \max(T_{j-1} - t, 0) + \frac{\frac{1}{3} T_j^3 - T_j^2 \max(T_{j-1}, t) + T_j \max(T_{j-1}, t)^2 - \frac{1}{3} \max(T_{j-1}, t)^3}{(T_j - T_{j-1})^2} \\
&= \max(T_{j-1} - t, 0) + \frac{1}{3} \frac{(T_j - \max(T_{j-1}, t))^3}{(T_j - T_{j-1})^2}.
\end{aligned}$$

Plugging this back into equation (B.1) proofs the lemma.

□

C

Market data

Example of an instantaneous correlation matrix for four forward rates.

$$\rho = \begin{bmatrix} 1 & 0.95 & 0.9025 & 0.8574 \\ 0.95 & 1 & 0.95 & 0.9025 \\ 0.9025 & 0.95 & 1 & 0.95 \\ 0.8574 & 0.9025 & 0.95 & 1 \end{bmatrix}$$

Table C.1: Market data

Date	3-Month forward rate (%)	Instantaneous volatility	Market shift (%)
30-sep-20	-0,49754	0	3
30-dec-20	-0,50905673	0,076814586	3
30-mrt-21	-0,51892859	0,077353785	3
30-jun-21	-0,52913103	0,078742794	3
30-sep-21	-0,53848412	0,081308423	3
30-dec-21	-0,54164065	0,085047	3
30-mrt-22	-0,53673077	0,087747754	3
30-jun-22	-0,53082298	0,093204097	3
30-sep-22	-0,52200786	0,095737477	3
30-dec-22	-0,51567517	0,10244403	3
30-mrt-23	-0,50590914	0,108666098	3
30-jun-23	-0,48696892	0,114434526	3
29-sep-23	-0,46520858	0,119722425	3
29-dec-23	-0,44689376	0,127493865	3
29-mrt-24	-0,42724409	0,135087089	3
28-jun-24	-0,40621279	0,142456255	3
30-sep-24	-0,38479798	0,149885772	3
30-dec-24	-0,36454192	0,15405519	3
31-mrt-25	-0,34421235	0,158091603	3
30-jun-25	-0,32381019	0,16199687	3
30-sep-25	-0,30546967	0,165909772	3
30-dec-25	-0,28697462	0,169311839	3
30-mrt-26	-0,26383721	0,172387743	3
30-jun-26	-0,23594007	0,175262174	3
30-sep-26	-0,20665378	0,177984908	3
30-dec-26	-0,18037661	0,178062055	3
30-mrt-27	-0,15467656	0,178126143	3
30-jun-27	-0,1295793	0,178215566	3
30-sep-27	-0,10530936	0,178314513	3
30-dec-27	-0,08085615	0,17914798	3
30-mrt-28	-0,05482302	0,179877763	3
30-jun-28	-0,02747725	0,180536255	3
29-sep-28	-0,00050097	0,181155109	3
29-dec-28	0,025735372	0,180583077	3
30-mrt-29	0,053439659	0,17992305	3
29-jun-29	0,082167605	0,179210598	3
28-sep-29	0,115170506	0,178328343	3
31-dec-29	0,147289738	0,177776828	3
29-mrt-30	0,169910328	0,17753846	3
28-jun-30	0,184583229	0,177603962	3
30-sep-30	0,200646435	0,177628363	3

D Shifted SABR dynamics

To switch from the martingale measure to the spot-Libor measure the drift under the new measure has to be derived. Using the change of measure technique, which has been performed multiple times during this thesis the drift can be acquired. Starting from

$$\text{Drift}(X_j;Q^d)(t) = \frac{dX_j(t)d\ln(B_d(t)/P(t,T_j)))}{dt} = \frac{dR_j(t)d\ln(P(t,q(t))/P(t,T_j)))}{dt}. \tag{D.1}$$

The logarithmic part has already been worked out in Chapter 6. Filling this in gives as result

$$\begin{aligned}
\text{Drift}(X_j;Q^d)(t) &= \frac{dX_j(t)d\sum_{i=q(t)+1}^{j}\ln(1+\tau_i R_i(t))}{dt}, \\
&= \sum_{i=q(t)+1}^{j} \frac{dX_j(t)d\ln(1+\tau_i R_i(t))}{dt}, \\
&= \sum_{i=q(t)+1}^{j} \frac{\tau_i}{1+\tau_i R_i(t)} \frac{dX_j(t)dR_i(t)}{dt}, \\
&= \alpha_j(t)\sigma(t)\gamma_j(t)X_j(t)^{\beta} \sum_{i=q(t)+1}^{j} \rho_{i,j} \frac{\tau_i\alpha_i(t)\sigma(t)\gamma_i(t)X_i(t)^{\beta}}{1+\tau_i R_i(t)}.
\end{aligned} \tag{D.2}$$

This is the drift parameter for the dynamics of the backward-looking forward rates under the spot-Libor measure for the shifted SABR forward market model. Filling this in provides the equation in the lemma.

E

Proof of Theorem 6.4.1

For the proof we will follow the author of [39]. In that paper a case of the results by the authors of [16] will be used to obtain the SABR parameters.

Theorem E.1. ([16]) Consider for $\epsilon > 0$ the following dynamics

$$\begin{aligned} \mathrm{d}R_\epsilon(t) &= \epsilon\phi(t)R_\epsilon(t)^\beta\sigma_\epsilon(t)\mathrm{d}B(t) \\ \mathrm{d}\sigma_\epsilon(t) &= \epsilon\nu\sigma_\epsilon(t)\mathrm{d}W(t), \quad \sigma_\epsilon(0) = 1. \end{aligned} \tag{E.1}$$

For a given expiry $T > 0$, the authors of [16] show that the implied volatility of European options under the dynamics given in equation (E.1) coincides with the implied volatility under the standard SABR model with the following parameters

$$\hat{\alpha} = \Delta \mathrm{e}^{\frac{1}{4}\epsilon^2\Delta^2 GT}, \quad \hat{\rho} = \frac{b}{\sqrt{c}}, \quad \hat{\nu} = \Delta\sqrt{c}, \tag{E.2}$$

where the constants are defined as

$$\begin{aligned} &\Delta^2 = \frac{v(T)}{T}, \quad b = \frac{2\rho\nu}{v^2(T)}\int_0^T (v(T)-v(s))\phi(s)\mathrm{d}s, \quad G = \frac{2\nu^2}{v^2(T)}\int_0^T v(T)-v(s)\mathrm{d}s - c, \\ &c = \frac{3\nu^2}{v^3(T)}\int_0^T (v(T)-v(s))^2\,\mathrm{d}s + \frac{9}{v^3(T)}\int_0^T w^2(s)\phi^2(s)\mathrm{d}s - 3b^2. \end{aligned} \tag{E.3}$$

Here v: u $\mapsto \int_0^u \phi^2(s)ds$, and w: u $\mapsto \rho\ \nu \int_0^u \phi(s)ds$.

This result is applied to the SABR forward market model given in equation (6.3.1). For the dynamics in the results we set $\epsilon = 1$, $\phi(t) = \alpha\gamma(t)$ and $T = T_j$. The parameters derived in this thesis apply for the case that none of the backward-looking rates are realized yet. Therefore, t starts before the accrual period and when time moves forward the accrual period can be reached. Consequently, the function $\phi(t)$ is given by

$$\phi(t) = \begin{cases} \alpha, & 0 \le t \le \tau_0, \\ \alpha\left(\frac{\tau_1 - t}{\tau_1-\tau_0}\right)^q, & \tau_0 \le t \le \tau_1 \end{cases}.$$

Here we have set $\tau_0 = T_{j-1}$ which is the start of the accrual period and $\tau_1 = T_j$ which denotes the maturity time. The next step is to obtain the integrals given in theorem E.1. We start with deriving $v(u)$

$$v(u) = \int_0^u \phi^2(s)ds.$$

First consider the case where $0 \le u \le \tau_0$

$$\begin{aligned} v(u) &= \int_0^u \phi^2(s)ds, \\ &= \int_0^u \alpha^2 ds = \alpha^2\int_0^u 1ds = \alpha^2 u. \end{aligned}$$

The case where $\tau_0 \le u \le \tau_1$ is given by

$$
\begin{aligned}
v(u) &= \int_0^u \phi^2(s)ds = \int_0^{\tau_0} \alpha^2 ds + \int_{\tau_0}^u \phi^2(s)ds,\\
&= \int_0^{\tau_0} \alpha^2 ds + \int_{\tau_0}^u \left(\alpha\left(\frac{\tau_1 - s}{\tau_1-\tau_0}\right)^q\right)^2 ds = \alpha^2\tau_0 + \int_{\tau_0}^u \left(\alpha\left(\frac{\tau_1 - s}{\tau_1-\tau_0}\right)^q\right)^2 ds,\\
&= \alpha^2\tau_0 + \frac{\alpha^2}{(\tau_1-\tau_0)^{2q}}\int_{\tau_0}^u (\tau_1 - s)^{2q} ds = \alpha^2\tau_0 + \frac{\alpha^2}{(\tau_1-\tau_0)^{2q}}\left[-\frac{(\tau_1-s)^{2q+1}}{2q+1}\right]_{\tau_0}^u,\\
&= \alpha^2\tau_0 + \frac{\alpha^2}{(\tau_1-\tau_0)^{2q}}\left(-\frac{(\tau_1-u)^{2q+1}}{2q+1} + \frac{(\tau_1-\tau_0)^{2q+1}}{2q+1}\right),\\
&= \alpha^2\tau_0 + \alpha^2\frac{(\tau_1-\tau_0)}{2q+1} - \alpha^2\frac{(\tau_1-u)^{2q+1}}{(\tau_1-\tau_0)^{2q}(2q+1)} = \frac{\alpha^2}{2q+1}\left(2q\tau_0 + \tau_1 - \frac{(\tau_1-u)^{2q+1}}{(\tau_1-\tau_0)^{2q}}\right).
\end{aligned}
$$

Therefore, the function v is given as

$$
v(u) = \begin{cases} \alpha^2 u, & 0 \le u \le \tau_0 \\ \frac{\alpha^2}{2q+1}\left(2q\tau_0 + \tau_1 - \frac{(\tau_1-s)^{2q+1}}{(\tau_1-\tau_0)^{2q}}\right), & \tau_0 \le u \le \tau_1 \end{cases}.
$$

Next the function w is derived, first consider the case $0 \le u \le \tau_0$

$$
w(u) = \rho\nu \int_0^u \phi(s)ds = \rho\nu\alpha\int_0^u 1ds = \rho\nu\alpha u.
$$

For the case that $\tau_0 \le u \le \tau_1$ w becomes

$$
\begin{aligned}
w(u) &= \rho\nu\int_0^u \phi(s)ds = \rho\nu\int_0^{\tau_0}\phi(s)ds + \rho\nu\int_{\tau_0}^u \phi(s)ds = \rho\nu\alpha\tau_0 + \rho\nu\int_{\tau_0}^u\phi(s)ds,\\
&= \rho\nu\alpha\tau_0 + \rho\nu\int_{\tau_0}^u \alpha\left(\frac{\tau_1-t}{\tau_1-\tau_0}\right)^q ds = \rho\nu\alpha\tau_0 + \rho\nu\frac{\alpha}{(\tau_1-\tau_0)^q}\int_{\tau_0}^u(\tau_1-s)^q ds,\\
&= \rho\nu\alpha\tau_0 + \rho\nu\frac{\alpha}{(\tau_1-\tau_0)^q}\left[-\frac{(\tau_1-s)^{q+1}}{(q+1)}\right]_{\tau_0}^u = \rho\nu\alpha\tau_0 + \rho\nu\frac{\alpha}{(\tau_1-\tau_0)^q}\left(-\frac{(\tau_1-u)^{q+1}}{(q+1)} + \frac{(\tau_1-\tau_0)^{q+1}}{(q+1)}\right),\\
&= \frac{\rho\nu\alpha}{q+1}\left(q\tau_0 + \tau_1 - \frac{(\tau_1-u)^{q+1}}{(\tau_1-\tau_0)^q}\right).
\end{aligned}
$$

Consequently, the function w is given by

$$
w(u) = \begin{cases} \rho\nu\alpha u, & 0 \le u \le \tau_0, \\ \frac{\rho\nu\alpha}{q+1}\left(q\tau_0 + \tau_1 - \frac{(\tau_1-u)^{q+1}}{(\tau_1-\tau_0)^q}\right), & \tau_0 \le u \le \tau_1 \end{cases}.
$$

The next step is to fill in the functions v and w into the constants, this results in the following expressions

$$
\begin{aligned}
\Delta^2 = \frac{v(T)}{T} = \frac{v(\tau_1)}{\tau_1} &= \frac{\alpha^2}{2q+1}\left(2q\tau_0 + \tau_1 - \frac{(\tau_1-\tau_1)^{2q+1}}{(\tau_1-\tau_0)^{2q}}\right)\frac{1}{\tau_1},\\
&= \frac{\alpha^2}{2q+1}\left(2q\tau_0+\tau_1\right)\frac{1}{\tau_1} = \frac{\alpha^2\lambda}{\tau_1(2q+1)}.
\end{aligned}
$$

The other three constants are given as

$$
\begin{aligned}
b &= \frac{\nu\rho}{\alpha}(2q+1)\frac{3\tau^2 + 2q\tau_0^2 + \tau_1^2}{2\tau^2(3q+2)}, \quad c = \frac{\nu^2}{\alpha^2}\frac{\gamma(2q+1)^2}{\tau^4}\\
G &= \frac{\nu^2}{\Delta^2}\frac{\tau^2 + 2q\tau_0^2 + \tau_1^2}{2\tau_1\tau(q+1)} - c,
\end{aligned}
$$

where the exact derivations are not presented. These are just extremely long integrals and calculations which do not contribute to the report.

Filling these constants in provides the SABR parameters

$$\hat{\alpha} = \Delta \mathrm{e}^{\frac{1}{4}\epsilon^2\Delta^2 GT} = \frac{\alpha^2\lambda}{\tau_1(2q+1)}\mathrm{e}^{\frac{1}{4}\Delta^2 G\tau_1},$$

$$\hat{\nu}^2 = \Delta^2 c = \frac{\alpha^2\lambda}{\tau_1(2q+1)}\frac{\nu^2}{\alpha^2}\frac{\gamma(2q+1)^2}{\lambda^4} = \nu^2\gamma\frac{(2q+1)}{\lambda^3\tau_1},$$

$$\hat{\rho} = \frac{b}{\sqrt{c}} = \frac{\frac{\nu\rho}{\alpha}(2q+1)\frac{3\lambda^2+2q\tau_0^2+\tau_1^2}{2^2(3q+2)}}{\sqrt{\frac{\nu^2}{\alpha^2}\frac{\gamma(2q+1)^2}{\lambda^4}}} = \frac{\nu\rho}{\alpha}(2q+1)\frac{3\lambda^2+2q\tau_0^2+\tau_1^2}{2\lambda^2(3q+2)}\frac{\alpha}{\nu}\frac{\lambda^2}{\sqrt{\gamma}(2q+1)} = \frac{\rho}{\sqrt{\gamma}}\frac{3\lambda^2+2q\tau_0^2+\tau_1^2}{2(3q+2)}.$$

Finally in theorem 6.4.1 the constant $H = \nu^2\frac{\lambda^2+2q\tau_0^2+\tau_1^2}{2\tau_1\lambda(q+1)} - \hat{\nu}^2$ was defined. Now consider

$$\Delta^2 G = \Delta^2\left(\frac{\nu^2}{\Delta^2}\frac{\lambda^2+2q\tau_0^2+\tau_1^2}{2\tau_1\lambda(q+1)} - \frac{\hat{\nu}^2}{\Delta^2}\right) = \nu^2\frac{\lambda^2+2q\tau_0^2+\tau_1^2}{2\tau_1\lambda(q+1)} - \hat{\nu}^2 = H$$

Substituting this into the above parameters gives the theorem.

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