A Semi-Decentralized Tikhonov-Based Algorithm for Optimal Generalized Nash Equilibrium Selection

Benenati, Emilio; Ananduta, Wicak; Grammatico, Sergio

DOI
10.1109/CDC49753.2023.10383583

Publication date
2023

Document Version
Final published version

Published in
Proceedings of the 62nd IEEE Conference on Decision and Control (CDC 2023)

2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

A Semi-Decentralized Tikhonov-based Algorithm for Optimal Generalized Nash Equilibrium Selection

Emilio Benenati, Wicak Ananduta, and Sergio Grammatico

***Abstract*— To optimally select a generalized Nash equilibrium, in this paper, we consider a semi-decentralized algorithm based on a double-layer Tikhonov regularization algorithm. Technically, we extend the Tikhonov method for equilibrium selection to generalized games. Next, we couple such an algorithm with the preconditioned forward-backward splitting, which guarantees linear convergence to a solution of the inner layer problem and allows for a semi-decentralized implementation. We then establish a conceptual connection and draw a comparison between the considered algorithm and the hybrid steepest descent method, the other known distributed approach for solving the equilibrium selection problem.**

I. Introduction

Several multi-agent decision processes can be modelled as a game, that is, a set of inter-dependent optimization problems. In particular, if the agents are coupled not only through their respective objective functions, but also through a shared constraint set, then we label the setting as a *generalized* game. Application examples for generalized games include traffic routing [1], peer-to-peer energy markets [2] and cognitive radio networks [3]. A typical solution paradigm is the generalized Nash equilibrium (GNE), that is, an optimal situation for each agent given the decisions of the remaining agents, and especially the sub-class of variational GNEs (v-GNES), which has recently received widespread attention due to its stability properties [4].

Plenty efficient v-GNE seeking algorithms, e.g. [5]–[9], have been developed for games that satisfy a monotonicity condition. Crucially, monotone games admit in general an infinite number of v-GNEs (unless a much more restrictive *strong* monotonicity condition is imposed). An appealing method to deal with the non-uniqueness of the solution is to select a GNE that optimizes some desirable system-level objective, as proposed in [10]–[13]. The selection algorithms in [10], [11] use Tikhonov's regularization method and build on the literature of Variational Inequalities (VIs) to cast the selection problem as a VI-constrained VI, which is solved by finding a sequence of approximate solutions to regularized games. The algorithms in our previous works [12], [13], instead, rely on fixed point selection theory and use the hybrid steepest descent method (HSDM) [14], which pairs an appropriate nonexpansive operator with a gradient descent. While the latter is recently proposed as distributed algorithms for GNE selection, the former works for non-generalized games only.

The authors are with the Delft Center for Systems and Control (DCSC), TU Delft, the Netherlands. E-mail addresses: {e.benenati, w.ananduta, s.grammatico}@tudelft.nl.

This work was partially supported by the ERC under research project COSMOS (802348).

Our main contribution consists in devising a Tikhonov-based algorithm for equilibrium selection in generalized, monotone games. The proposed algorithm is semi-decentralized, in the sense that communication with a central coordinator is required, but its only duty is to broadcast signals aimed at optimizing the system-level objective function. Technically, we cast the GNE selection problem as a VI-constrained VI, which can be solved via a sequence of regularized sub-problems. Compared to [11], we propose to solve the resulting regularized sub-problems via the preconditioned forward-backward or pFB [7], which has linear convergence rate and allows one to distribute the computation burden among the agents. Compared with the Tikhonov-based GNE seeking algorithms in [5], [6] that compute the minimum-norm v-GNE, our proposed algorithm works for general convex selection functions. Secondly, we find a theoretical connection between the proposed Tikhonov method and the HSDM. Although neither method generalizes the other, the HSDM can be cast as a forward-backward step towards the solution of the Tikhonov regularized problem. Finally, in Section V, we compare the two methods numerically.

Notation: The Euclidean inner product and norm are denoted respectively by $\langle x, y\rangle$ and $\|\cdot\|$. For a symmetric matrix $\Psi \succ 0$, we denote the Ψ-induced norm by $\|\cdot\|_\Psi$ and define $\langle x, y\rangle_\Psi = \langle x, \Psi y\rangle$. Nonlinear set-valued operators are denoted in calligraphic letters, e.g. $\mathcal{T} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. For a matrix A, $\|A\|$ denotes its spectral norm. We denote the vector of all 1 (0) with dimension n by $\mathbf{1}_n$ ($\mathbf{0}_n$). The column stack operation is denoted by $\mathrm{col}(\cdot)$.

Operator theory: We denote by Id the identity operator. For a closed convex set C, $\mathrm{N}_C(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is its normal cone [15, Def. 6.38] and $\mathrm{proj}_C(x) = \mathrm{argmin}_{z\in C}\|x - z\|$. For an operator $\mathcal{T}\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, we denote its zero set by $\mathrm{zer}(\mathcal{T}) := \{x \in \mathrm{dom}(\mathcal{T}) \mid 0 \in \mathcal{T}(x)\}$, its fixed point set by $\mathrm{fix}(\mathcal{T}) := \{x \in \mathrm{dom}(\mathcal{T}) \mid x \in \mathcal{T}(x)\}$ and its resolvent by $\mathcal{J}_\mathcal{T} := (\mathrm{Id}+\mathcal{T})^{-1}$ [15, Def. 23.1]. An operator $\mathcal{T} : C \rightrightarrows \mathbb{R}^n$ is:

(μ-strongly) monotone: if, for any two pairs $(x, y), (x', y') \in \mathrm{gph}(\mathcal{T}) := \{(x, y) | x \in C, y \in \mathcal{T}(x)\}$, it holds $\langle y - y', x - x'\rangle \geq \mu\|x - x'\|^2$, with $\mu \geq 0$ (> 0);
Monotone-plus: if it is monotone and, for any two pairs $(x, y), (x', y') \in \mathrm{gph}(\mathcal{T})$ such that $\langle y - y', x - x'\rangle = 0$, it holds $(x, y') \in \mathrm{gph}(\mathcal{T})$ and $(x', y) \in \mathrm{gph}(\mathcal{T})$;
Lipschitz continuous: if there exists $L > 0$, such that, for all $x, x' \in C$, $\|\mathcal{T}(x) - \mathcal{T}(x')\| \leq L\|x - x'\|$;
Nonexpansive: if Lipschitz continuous with $L = 1$;
Contractive: if Lipschitz continuous with $L < 1$.

2023 62nd IEEE Conference on Decision and Control (CDC) | 979-8-3503-0124-3/23/$31.00 ©2023 IEEE | DOI: 10.1109/CDC49753.2023.10383583

Variational inequality (VI): For a closed convex set K and $\mathcal{T} : K \to K$, we denote by $\mathrm{VI}(K, \mathcal{T})$ the problem

$$\text{find } \boldsymbol{x}^* \in K \text{ s.t. } \inf_{\boldsymbol{x} \in K} \langle \mathcal{T}(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \geq 0,$$

and its solution set by

$$\mathrm{SOL}(K, \mathcal{T}) := \{\boldsymbol{x}^* \in K | \boldsymbol{x}^* \text{ solves } \mathrm{VI}(K, \mathcal{T})\}.$$

A point $\hat{\boldsymbol{x}}$ is an ε-approximate solution of $\mathrm{VI}(K, \mathcal{T})$ if

$$\varepsilon \geq \inf_{\boldsymbol{x} \in \mathrm{SOL}(K,\mathcal{T})} \|\boldsymbol{x} - \hat{\boldsymbol{x}}\|,$$

where $\varepsilon \geq 0$ denotes the approximation error.

II. Equilibrium selection as a variational inequality

We consider the multi-agent decision process in which each of N agents aims at solving an optimization problem over the decision variables $x_i \in \mathcal{X}_i \subset \mathbb{R}^{n_i}$, where $i \in \{1, ..., N\} =: \mathcal{I}$. Let us denote $n := \sum_{i \in \mathcal{I}} n_i$, $\mathcal{I}_{-i} := \mathcal{I} \backslash \{i\}$, $\boldsymbol{x}_{-i} := \mathrm{col}((x_j)_{j \in \mathcal{I}_{-i}})$ and $\boldsymbol{x} := \mathrm{col}((x_j)_{j \in \mathcal{I}})$. Crucially, the decision problem associated with agent i is coupled to the decision variables of the remaining agents both through the objective function $J_i : \mathbb{R}^n \to \mathbb{R}$, and some constraints. We consider $m \in \mathbb{N}$ constraints of the form

$$\textstyle\sum_{i \in \mathcal{I}} A_i x_i \leq b, \tag{1}$$

where $A_i \in \mathbb{R}^{m \times n_i}$, for each $i \in \mathcal{I}$, and $b \in \mathbb{R}^m$. This problem is commonly referred to as a *generalized game* and we formalize it as follows:

$$\forall i \in \mathcal{I}\colon \begin{cases} \min\limits_{x_i \in \mathcal{X}_i} \quad J_i(x_i, \boldsymbol{x}_{-i}) & \text{(2a)} \\ \text{s.t.} \quad A_i x_i \leq b - \sum_{j \in \mathcal{I}_{-i}} A_j x_j. & \text{(2b)} \end{cases}$$

By defining $\boldsymbol{\mathcal{X}} := \Pi_{i \in \mathcal{I}} \mathcal{X}_i$, the collective feasible set of (2) is

$$\Gamma := \boldsymbol{\mathcal{X}} \cap \{\boldsymbol{x} \mid (1) \text{ holds true}\}.$$

We address the problem in (2) by examining generalized Nash equilibria, which are points from which no agent has an incentive to deviate unilaterally:

Definition 1. *A set of decision variables $\boldsymbol{x}^* \in \boldsymbol{\mathcal{X}}$ is a GNE for the game in* (2) *if, for each $i \in \mathcal{I}$,*

$$J_i(x_i^*, \boldsymbol{x}_{-i}^*) \leq J_i(x_i, \boldsymbol{x}_{-i}^*),$$

for any $x_i \in \mathcal{X}_i \cap \{y \in \mathbb{R}^{n_i} \mid A_i y \leq -\sum_{j \in \mathcal{I}_{-i}} A_j x_j^*\}$.

Existance of a GNE is guaranteed [16, Prop. 12.11] under the following, standard assumptions [8], [9]:

Assumption 1. *For each i, $J_i(\cdot, \boldsymbol{x}_{-i})$ is convex and continuously differentiable for any $\boldsymbol{x}_{-i}$.*

Assumption 2. *For all $i \in \mathcal{I}$, $\mathcal{X}_i$ is compact and convex; $\Gamma \neq \varnothing$ and it satisfies Slater's constraint qualification.*

Assumption 3. *The pseudogradient of the game in* (2)

$$F(\boldsymbol{x}) := \mathrm{col}((\nabla_{x_i} J_i(x_i, \boldsymbol{x}_{-i}))_{i \in \mathcal{I}})$$

is monotone and L_F-Lipschitz continuous.

For solving the problem in (2), we assume that the agents can exchange information over an undirected, connected communication network. We denote the set of neighbours of agent i in this network by $\mathcal{N}_i$. For simplicity, we consider the case where, for all $i \in \mathcal{I}$, J_i depends on x_i and the decision variables of (a subset of) $\mathcal{N}_i$, so that each agent is able to evaluate its cost function by communicating with their neighbours. Additionally, each agent maintains a local estimate of the dual variable λ_i for the shared constraints in (1) and an auxiliary variable ν_i to achieve consensus on dual variable estimates. We then define the following extended Karush-Kuhn-Tucker (KKT) operator [7], which includes both the optimality conditions for the problem in (2) and the consensus condition:

$$\begin{aligned} \mathcal{T}^{\mathrm{KKT}}(\boldsymbol{\omega}) :=& \mathcal{A}(\boldsymbol{\omega}) + \mathcal{B}(\boldsymbol{\omega}) + \mathcal{C}(\boldsymbol{\omega}), \\ \mathcal{A}(\boldsymbol{\omega}) :=& \mathrm{N}_{\boldsymbol{\mathcal{X}}}(\boldsymbol{x}) \times \mathrm{N}_{\mathbb{R}^{|\mathcal{I}|m}_{\geq 0}}(\boldsymbol{\lambda}) \times \{\mathbf{0}_{|\mathcal{I}|m}\}, \\ \mathcal{B} :=& \begin{bmatrix} F(\boldsymbol{x}) \\ -\bar{\mathcal{L}}\boldsymbol{\lambda} \\ 0 \end{bmatrix}, \ \mathcal{C}(\boldsymbol{\omega}) := \begin{bmatrix} \boldsymbol{A}^\top \boldsymbol{\lambda} \\ b - \boldsymbol{A}\boldsymbol{x} - \bar{\mathcal{L}}\boldsymbol{\nu} \\ \bar{\mathcal{L}}\boldsymbol{\lambda} \end{bmatrix}, \end{aligned} \tag{3}$$

where $\boldsymbol{\omega} = \mathrm{col}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu})$, $\bar{\mathcal{L}} = \mathcal{L} \otimes I_m$ and $\mathcal{L}$ is the Laplacian matrix of the communication graph. A subset of the GNEs of the game in (2) is characterized by the zero set of $\mathcal{T}^{\mathrm{KKT}}$ [7, Thm. 2]. These are called variational GNEs (v-GNEs) [4, Def. 3.10] and have been extensively studied with multiple efficient computation algorithms available. However, Assumptions 1–3 alone do not guarantee that $\mathrm{zer}(\mathcal{T}^{\mathrm{KKT}})$ is a singleton. Most of the algorithms in the literature compute an unspecified v-GNE among the possibly infinitely many. In contrast, our approach focuses on finding an optimally selected v-GNE according to the selection function ϕ, in the sense that we aim at solving

$$\min_{\boldsymbol{\omega}} \phi(\boldsymbol{\omega}) \quad \text{s.t. } \boldsymbol{\omega} \in \mathrm{zer}(\mathcal{T}^{\mathrm{KKT}}). \tag{4}$$

Assumption 4. *The selection function ϕ is convex, continuously differentiable, and coercive. Furthermore, its gradient is $L_{\nabla\phi}$-Lipschitz continuous.*

By [15, Prop. 23.39], (4) is a convex optimization problem under Assumptions 1, 3 and 4. However, $\mathrm{zer}(\mathcal{T}^{\mathrm{KKT}})$ can seldom be written in a closed form and, thus, (4) cannot be solved by standard optimization algorithms. To derive an algorithmic solution, we note that, by defining $\boldsymbol{\Omega} := \boldsymbol{\mathcal{X}} \times \mathbb{R}^{Nm}_{\geq 0} \times \mathbb{R}^{Nm}$ and by [17, Eq. (1.1.3)],

$$\boldsymbol{\omega}^* \in \mathrm{zer}(\mathcal{T}^{\mathrm{KKT}}) \Leftrightarrow \boldsymbol{\omega}^* \in \mathrm{SOL}(\boldsymbol{\Omega}, \mathcal{B} + \mathcal{C}). \tag{5}$$

Following the equivalence between convex optimization problems and VIs [17, Sec. 1.3.1], we then recast (4) as:

$$\mathrm{VI}(\mathrm{SOL}(\boldsymbol{\Omega}, \mathcal{B} + \mathcal{C}), \nabla\phi). \tag{6}$$

As discussed in [11], we can solve (6) by finding the ε_k-approximate solutions to a sequence of regularized subproblems, indexed by $k \in \mathbb{N}$:

$$\mathrm{VI}(\boldsymbol{\Omega}, \mathcal{B} + \mathcal{C} + \gamma_k \nabla\phi + \alpha(\mathrm{Id} - \boldsymbol{\omega})), \quad \boldsymbol{\omega} \in \boldsymbol{\Omega}. \tag{7}$$

The regularization weights α, $(\gamma_k)_{k \in \mathbb{N}}$ and tolerances $(\varepsilon_k)_{k \in \mathbb{N}}$ are chosen according to the following criteria:

Assumption 5. *The parameter α is positive; $(\gamma_k)_{k \in \mathbb{N}}$ and $(\varepsilon_k)_{k \in \mathbb{N}}$ are positive and non-negative sequences of real numbers, respectively, such that $\sum_{k \in \mathbb{N}} \gamma_k = \infty$, $\lim_{k \to \infty} \gamma_k = 0$ and $\varepsilon_k = 0$ for all $k > K$, where $K \in \mathbb{N}$.*

By denoting $\bar{\gamma} := \sup_{k\in\mathbb{N}} \gamma_k$, we observe the following properties of the operators that define (7):

Lemma 1. *Let Assumptions 3–5 hold true. For any $k \in \mathbb{N}$, $\alpha > 0$, and $\boldsymbol{\omega} \in \boldsymbol{\Omega}$:*

1) $\mathcal{B} + \mathcal{C} + \gamma_k \nabla\phi + \alpha(\mathrm{Id} - \boldsymbol{\omega})$ is α-strongly monotone,
2) $\mathcal{B} + \gamma_k \nabla\phi + \alpha(\mathrm{Id} - \boldsymbol{\omega})$ is L_G- Lipschitz continuous, where $L_G := \max(L_F, 2|\mathcal{N}_i|_{i\in\mathcal{I}}) + \bar{\gamma} L_{\nabla\phi} + \alpha$.

By applying [17, Cor. 2.2.5] and [17, Thm. 2.3.3a], we conclude that the regularized problem in (7) admits a unique solution. Proposition 1, which follows directly from [11, Thm. 2], formalizes a prototypical algorithmic solution to the problem in (6):

Proposition 1. *Let Assumptions 1–5 hold. Let $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}} \in \boldsymbol{\Omega}$ and, for every k, $\boldsymbol{\omega}^{(k+1)}$ be the ε_k-approximate solution of the VI in (7) with $\boldsymbol{\omega} = \boldsymbol{\omega}^{(k)}$. Then, the sequence $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ is bounded and each of its limit points is a solution of* (6).

III. Semi-decentralized equilibrium selection

In view of Proposition 1, next we derive an algorithm for generating a sequence $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ such that, for all k, $\boldsymbol{\omega}^{(k+1)}$ is a ε_k-approximate solution to (7) with $\boldsymbol{\omega} = \boldsymbol{\omega}^{(k)}$. The exact solution, denoted by $\boldsymbol{\omega}_k^*$, satisfies the monotone inclusion

$$0 \in (\mathcal{A} + \mathcal{B} + \mathcal{C} + \gamma_k \nabla\phi + \alpha(\mathrm{Id} - \boldsymbol{\omega}^{(k)}))(\boldsymbol{\omega}_k^*). \quad (8)$$

We then apply the preconditioned forward-backward (pFB) method, proposed in [7] in the context of decentralized GNE seeking, for solving (8). Let us define the following matrix:

$$\Psi = \mathrm{diag}(\boldsymbol{\rho}^{-1}, \boldsymbol{\tau}^{-1}, \boldsymbol{\sigma}^{-1}), \quad (9)$$

where $\boldsymbol{\rho} := \mathrm{diag}(\rho_i I_{n_i})_{i\in\mathcal{I}}$, $\boldsymbol{\tau} := \mathrm{diag}(\tau_i I_{n_i})_{i\in\mathcal{I}}$, $\boldsymbol{\sigma} := \mathrm{diag}(\sigma_i I_{n_i})_{i\in\mathcal{I}}$ collect the step sizes associated with the primal, dual and auxiliary variables, respectively, chosen according to the following design criterion:

Assumption 6. *Let*

$$r_i^{\mathrm{x}} := \max_{j=1,\dots,n_i} \textstyle\sum_{k=1}^m |[A_i]_{jk}|,$$
$$r_i^{\lambda} := \max_{j=1,\dots,n_i} \textstyle\sum_{k=1}^m |[A_i]_{jk}| + 2|\mathcal{N}_i|,$$
$$r_i^{\nu} := 2|\mathcal{N}_i|.$$

Furthermore, let $r = \max_{i\in\mathcal{I}}(r_i^{\mathrm{x}}, r_i^{\lambda}, r_i^{\nu})$ and $\delta > \max(\frac{L_G^2}{\alpha}, 2r)$, with L_G defined as in Lemma 1. For all $i \in \mathcal{I}$:

1) $(2\delta - r_i^{\mathrm{x}})^{-1} \le \rho_i \le (\delta + r_i^{\mathrm{x}})^{-1}$;
2) $(2\delta - r_i^{\lambda})^{-1} \le \tau_i \le (\delta + r_i^{\lambda})^{-1}$;
3) $(2\delta - r_i^{\nu})^{-1} \le \sigma_i \le (\delta + r_i^{\nu})^{-1}$.

Let us also define the preconditioning matrix

$$\Phi = \Psi + \begin{bmatrix} \mathbf{0} & -\boldsymbol{A}^\top & \mathbf{0} \\ -\boldsymbol{A} & \mathbf{0} & -\bar{\mathcal{L}} \\ \mathbf{0} & -\bar{\mathcal{L}} & \mathbf{0} \end{bmatrix}. \quad (10)$$

Lemma 2. *Under Assumption 6, $\Phi \succcurlyeq \delta I$ and $\frac{\delta}{\|\Phi\|} \ge \frac{1}{2}$.*

The pFB operator for the inclusion in (8) reads as

$$\begin{aligned} \mathcal{T}_k^{\mathrm{pFB}} =& (\mathrm{Id} + \Phi^{-1}(\mathcal{A} + \mathcal{C}))^{-1} \\ & (\mathrm{Id} - \Phi^{-1}(\mathcal{B} + \gamma_k \nabla\phi + \alpha(\mathrm{Id} - \boldsymbol{\omega}^{(k)}))). \end{aligned} \quad (11)$$

The following result formalizes the convergence of the fixed-point iteration generated by $\mathcal{T}_k^{\mathrm{pFB}}$ to the solution of (7).

Algorithm 1 Tikhonov-pFB for optimal GNE selection

Initialization. Let α, $(\varepsilon_k)_{k\in\mathbb{N}}$ and $(\gamma_k)_{k\in\mathbb{N}}$ satisfy Assumption 5. Let $\boldsymbol{\omega}^{(0)} \in \Pi_{i\in\mathcal{I}}(\mathcal{X}_i) \times \mathbb{R}_{\ge 0}^{|\mathcal{I}|m} \times \mathbb{R}^{|\mathcal{I}|m}$. Let $\boldsymbol{\rho}, \boldsymbol{\sigma}, \boldsymbol{\tau}$ satisfy Assumption 6.

Outer iteration: for $k \in \mathbb{N}_0$

1) Each agent $i \in \mathcal{I}$ sets:

$$(x_i^{(k,0)}, \lambda_i^{(k,0)}, \nu_i^{(k,0)}) \leftarrow y_i^{(k,0)} \leftarrow \omega_i^{(k)}. \quad (14)$$

2) **Inner iteration: for** $t \in \mathbb{N}_0$

-For each agent $i \in \mathcal{I}$:

a) Receive $x_j^{(k,t)}$, $\lambda_j^{(k,t)}, \nu_j^{(k,t)}$ from agent $j \in \mathcal{N}_i$ and $\nabla\phi_{\omega_i}(\boldsymbol{y}^{(k,t)})$ from the coordinator.

b) Update:

$$\begin{aligned} x_i^{(k,t+1)} =& \mathrm{proj}_{\mathcal{X}_i}\Big[x_i^{(k,t)} - \rho_i\big(\nabla_{x_i} J_i(\boldsymbol{x}^{(k,t)}) + \\ & A_i^\top \lambda_i^{(k,t)} + \gamma_k \nabla_{x_i}\phi(\boldsymbol{y}^{(k,t)}) + \alpha(x_i^{(k,t)} - x_i^{(k,0)})\big)\Big], \end{aligned} \quad (15)$$

$$\begin{aligned} \nu_i^{(k,t+1)} =& \nu_i^{(k,t)} - \sigma_i \Big(\textstyle\sum_{j\in\mathcal{N}_i}(\lambda_i^{(k,t)} - \lambda_j^{(k,t)}) \\ & + \alpha(\nu_i^{(k,t)} - \nu_i^{(k,0)}) + \gamma_k \nabla_{\nu_i}\phi(\boldsymbol{y}^{(k,t)})\Big). \end{aligned} \quad (16)$$

c) Receive $\nu_j^{(k,t+1)}$ from agent $j \in \mathcal{N}_i$.

d) Update:

$$\begin{aligned} \lambda_i^{(k,t+1)} =& \mathrm{proj}_{\mathbb{R}^m_{\ge 0}}\Big[\lambda_i^{(k,t)} + \tau_i\Big(\alpha(\lambda_i^{(k,t)} - \lambda_i^{(k,0)}) - b_i \\ & + A_i(2x_i^{(k,t+1)} - x_i^{(k,t)}) + \textstyle\sum_{j\in\mathcal{N}_i}\Big(2\nu_i^{(k,t+1)} - 2\nu_j^{(k,t+1)} \\ & - \nu_i^{(k,t)} + \nu_j^{(k,t)} - \lambda_i^{(k,t)} + \lambda_j^{(k,t)} - \nabla_{\lambda_i}\phi(\boldsymbol{y}^{(k,t)})\Big)\Big)\Big]. \end{aligned} \quad (17)$$

-Coordinator:

a) Set $\boldsymbol{y}^{(k,t+1)} \leftarrow (\boldsymbol{x}^{(k,t+1)}, \boldsymbol{\lambda}^{(k,t+1)}, \boldsymbol{\nu}^{(k,t+1)})$.

b) Communicate $\nabla_{\omega_i}\phi(\boldsymbol{y}^{(k,t+1)})$ to each agent $i \in \mathcal{I}$.

c) If the following is satisfied,

$$\|\boldsymbol{y}^{(k,t+1)} - \boldsymbol{y}^{(k,t)}\|_\Phi \le (1-\beta)\varepsilon_k, \quad (18)$$

terminate inner iteration. Each agent then sets

$$\omega_i^{(k+1)} = y_i^{(k,t+1)}. \quad (19)$$

Lemma 3. *Let Assumptions 1–6 hold and $\boldsymbol{y}^0 \in \boldsymbol{\Omega}$. Then, for all $k \in \mathbb{N}$, the sequence $(\boldsymbol{y}^{(t)})_{t\in\mathbb{N}}$ generated by the fixed-point iteration*

$$\boldsymbol{y}^{(t+1)} = \mathcal{T}_k^{\mathrm{pFB}}(\boldsymbol{y}^{(t)}) \qquad \forall\, t \in \mathbb{N}, \quad (12)$$

where $\mathcal{T}_k^{pFB}$ is defined in (11)*, converges linearly in the Φ-induced norm to $\boldsymbol{\omega}_k^*$ in* (8) *and*

$$\|\boldsymbol{y}^{(t)} - \boldsymbol{\omega}_k^*\|_\Phi \le \|\boldsymbol{y}^{(t+1)} - \boldsymbol{y}^{(t)}\|_\Phi/(1-\beta), \quad (13)$$

with $\beta := (1 + \frac{L_G^2}{\delta^2} - \frac{2\alpha}{\|\Phi\|}) < 1$.

Remark 1. *Differently from the pFB operator in [7], the one in* (11) *has additional regularization terms and thus achieves the desired linear rate.*

The linear convergence of $\mathcal{T}_k^{\mathrm{pFB}}$ guarantees that, for $\varepsilon_k > 0$, the iterate $\boldsymbol{\omega}^{(k+1)}$ in Proposition 1 is found within a finite number of inner iterations, whose termination can be

based on a simple stopping criterion derived from (13). The proposed method, which results from the expansion of the pFB operator, is illustrated in Algorithm 1, where we use t as the inner iteration index.

Proposition 2. *Let $(\boldsymbol{\omega}^{(k)})_{k\in\mathbb{N}}$ be generated by Algorithm 1. Under Assumptions 1–6, for each k, $\boldsymbol{\omega}^{(k+1)}$ is an ε_k-approximate solution of* (7) *with $\boldsymbol{\omega} = \boldsymbol{\omega}^{(k)}$ and, if $\varepsilon_k > 0$, the condition in* (18) *is verified in a finite number of steps.*

Remark 2. *For Assumption 5 in Proposition 1 to be satisfied, the termination condition for the inner iterations needs to eventually ensure $\varepsilon_k = 0$. This is a stringent requirement, as the pFB algorithm only achieves an exact solution asymptotically. The authors of [11], alternatively, consider the less restrictive condition $\lim_{k\to\infty} \frac{\varepsilon_k}{\gamma_k} = 0$ in the case where the definition set of the VI-constrained VI is compact. Although $\boldsymbol{\Omega}$ does not satisfy this condition as the dual variables belong to an unbounded set, in practice the dual variables are bounded in view of [18, Prop. 3.3].*

IV. Theoretical connection and comparison with the Hybrid Steepest Descent Method

In our previous works [12], [13], we take a different algorithm design path, where we reformulate Problem (4) as a fixed point selection problem. This allows one to use the hybrid steepest descent method (HSDM) [14] to solve (4). Specifically, one has to find a quasi-shrinking mapping $\mathcal{T}$ [13, Def. 1] such that $\mathrm{fix}(\mathcal{T}) = \mathrm{zer}(\mathcal{T}^{\mathrm{KKT}})$. Then, the limit point of the sequence $(\boldsymbol{z}^{(k)})_{k\in\mathbb{N}}$, defined by

$$\boldsymbol{z}^{(k+1)} = \mathcal{T}(\boldsymbol{z}^{(k)}) - \gamma_k \nabla\phi(\mathcal{T}\boldsymbol{z}^{(k)}), \tag{20}$$

converges to the solution of (4) if $(\gamma_k)_{k\in\mathbb{N}}$ is square-summable but non-summable [14, Thm. 5]. The vanishing weight on $\nabla\phi$ is reminiscent of the Tikhonov regularization introduced in Section II. Indeed, the two methods are related, as shown next. Let us consider the exact solution to the VI in (7) with $\boldsymbol{\omega} = \boldsymbol{\omega}^{(k)}$. Then, from [17, Prop. 12.3.6],

$$\boldsymbol{\omega}^{(k+1)} = \mathcal{J}_{\frac{1}{\alpha}(\mathcal{A}+\mathcal{B}+\mathcal{C}+\gamma_k\nabla\phi)}(\boldsymbol{\omega}^{(k)}) =: \mathcal{T}_k^{\mathrm{Tik}}(\boldsymbol{\omega}^{(k)}). \tag{21}$$

The properties of the HSDM update in (20) depend on the choice of $\mathcal{T}$. Let us consider the particular case $\mathcal{T} = \mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}}$, which can be shown to be quasi-shrinking via [13, Lem. 1]. We rewrite (20) with this particular choice of $\mathcal{T}$ as:

$$\begin{aligned} \boldsymbol{v}^{(k+1)} &= \mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}}(\boldsymbol{z}^{(k)}), \\ \boldsymbol{z}^{(k+1)} &= \boldsymbol{v}^{(k+1)} - \gamma_k \nabla\phi(\boldsymbol{v}^{(k+1)}), \end{aligned} \tag{22}$$

where we introduced the auxiliary sequence $(\boldsymbol{v}^{(k)})_{k\in\mathbb{N}}$. By rearranging (22), we note that this sequence evolves as

$$\boldsymbol{v}^{(k+1)} = \mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}} \circ (\mathrm{Id} - \gamma_k\nabla\phi)(\boldsymbol{v}^{(k)}) =: \mathcal{T}_k^{\mathrm{HSDM}}(\boldsymbol{v}^{(k)}). \tag{23}$$

The operator $\mathcal{T}_k^{\mathrm{HSDM}}$ in (23) corresponds to the forward-backward (FB) splitting for the inclusion

$$0 \in \mathcal{A} + \mathcal{B} + \mathcal{C} + \gamma_k\nabla\phi.$$

From [15, Prop. 26.1iv] and [15, Prop. 23.38],

$$\mathrm{fix}(\mathcal{J}_{\frac{1}{\alpha}(\mathcal{A}+\mathcal{B}+\mathcal{C}+\gamma_k\nabla\phi)}) = \mathrm{fix}(\mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}} \circ (\mathrm{Id} - \gamma_k\nabla\phi)),$$

which implies

$$\mathrm{fix}(\mathcal{T}_k^{\mathrm{Tik}}) = \mathrm{fix}(\mathcal{T}_k^{\mathrm{HSDM}}). \tag{24}$$

	Tikhonov	Tikhonov (bounded set)	HSDM
param.	$\sum\gamma_k = \infty$, evt. $\varepsilon_k = 0$	$\sum\gamma_k = \infty$, $\lim_{k\to\infty}\frac{\varepsilon_k}{\gamma_k}=0$	$\sum\gamma_k = \infty$, $\sum\gamma_k^2 < \infty$
t (#inner iterations)	evt. ∞	$t \to \infty$	1
ϕ coercive	yes	no	no

TABLE I: Theoretical property differences of the Tikhonov method and HSDM for GNE selection.

Thus, we conclude that both the Tikhonov update in (21) and the HSDM step in (20) apply at each step k a single update of a fixed point iteration, and the two operators in (24) have the same fixed point set. This analogy is only theoretical, as in practice the operator $\mathcal{J}_{\mathcal{A}+\mathcal{B}+\mathcal{C}}$ cannot be implemented in a distributed fashion. Nevertheless, it outlines that both the Tikhonov method and the HSDM function by the same underlying principle of tracking the solutions to a sequence of regularized problems. Moreover, we note that (24) does not hold for a generic choice of $\mathcal{T}$, thus the HSDM includes algorithms that are not covered by the Tikhonov method. On the other hand, the operator $\mathcal{T}_k^{\mathrm{Tik}}$ cannot be rewritten in terms of an FB operator. Therefore, we should conclude that neither method is a generalization of the other. The key differences between the two frameworks are summarized in Table I. In addition, we remark that the Tikhonov framework can be paired with any (splitting) methods for strongly monotone games to obtain a decentralized algorithm. Meanwhile, the HSDM requires methods for monotone games that are quasi-shrinking, such as the forward-backward-forward (FBF) splitting [15, Sec. 26.6]. Thus, Tikhonov-based methods can benefit from a larger pool of available algorithms.

V. Numerical simulations

We test the proposed algorithm on 100 game instances with 10 agents, where the pseudogradient is in the form

$$F(\boldsymbol{x}) = Q_F\boldsymbol{x} + \boldsymbol{c}_F.$$

The parameters $Q_F \succeq 0$ and $\boldsymbol{c}_F$ are randomly generated. We define the selection function

$$\phi(\boldsymbol{\omega}) = \|\boldsymbol{x}\|_{Q_\phi}^2 + \boldsymbol{c}_\phi^\top\boldsymbol{x} + \theta(\|\boldsymbol{\lambda}\|^2 + \|\boldsymbol{\nu}\|^2),$$

where $Q_\phi \succcurlyeq 0$ and $\boldsymbol{c}_\phi$ are as well randomly generated and $\theta = 10^{-3}$. For all i, we define the local constraint set $\mathcal{X}_i = \{\mathbb{R}^5 : \|x_i\|_\infty \le 1\}$. Furthermore, we let $A_i = I$, for all $i \in \mathcal{I}$, and $b = 2\cdot\mathbf{1}_5$.We then set $(\gamma_k)_{k\in\mathbb{N}}$ and $(\varepsilon_k)_{k\in\mathbb{N}}$ in Algorithm 1 to

$$\gamma_k = 10^{-3}k^{-\xi}; \quad \varepsilon_k = \begin{cases} 10^{-3}k^{-\xi\zeta} & \text{if } 10^{-3}k^{-\xi\zeta} \ge \underline{\varepsilon} \\ 0 & \text{if } 10^{-3}k^{-\xi\zeta} < \underline{\varepsilon} \end{cases}$$

where $\underline{\varepsilon}$ is set to the computer numerical precision. The parameter ξ controls the decay of the regularization weight γ_k, while ζ controls the decay rate of ε_k with respect to γ_k. We evaluate Algorithm 1 in terms of the residual computed for each outer iteration k and inner iteration t as

$$\mathfrak{R}(t) = \|\boldsymbol{y}^{(k,t)} - (\mathrm{Id} + \mathcal{A})^{-1} \circ (\mathrm{Id} - \mathcal{B} - \mathcal{C})(\boldsymbol{y}^{(k,t)})\|,$$

and in terms of the reduction of selection function ϕ with

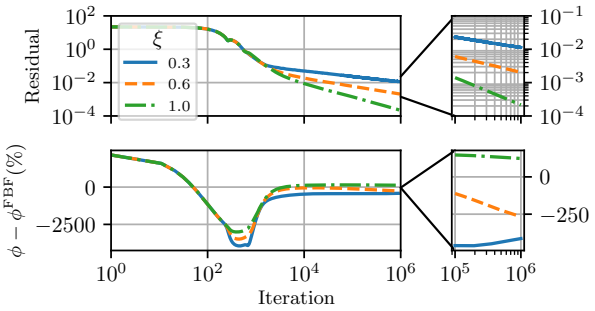

Fig. 1: Average performance of Algorithm 1 for $\zeta = 2$ and $\alpha = 1$.

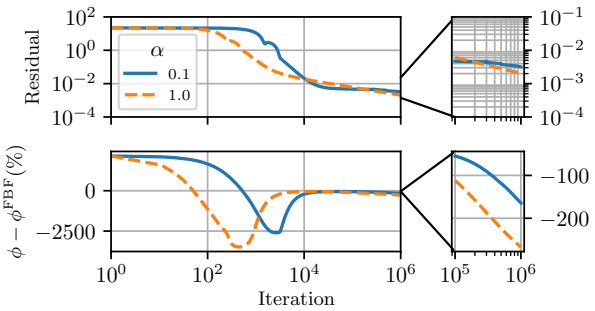

Fig. 2: Average performance of Algorithm 1 for $\xi = 0.6$ and $\zeta = 2$.

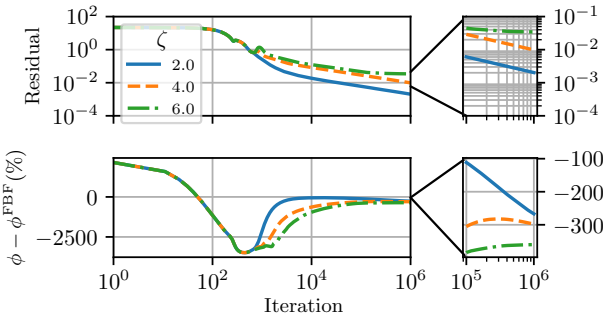

Fig. 3: Average performance of Algorithm 1 for $\alpha = 1$ and $\xi = 0.6$.

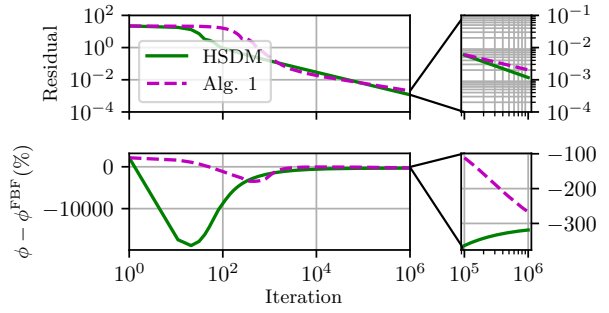

Fig. 4: Comparison between Algorithm 1 with $\xi = 0.6$, $\zeta = 2$, $\alpha = 1$ and the HSDM-FBF method [13, Alg. 1]. The x-axis indicates the cumulative number of inner iterations for Algorithm 1.

respect to the value obtained by the standard FBF algorithm (without optimal selection) [8, Alg. 2] in Figures 1–3. We observe that, for decreasing values of ξ, the algorithm achieves a lower selection function value and a larger residual (cf. Figure 1). This trade-off between convergence to a GNE (measured by the residual) and convergence to a ϕ-optimal point is expected, because a too slow decay of the regularization weight γ_k leads the algorithm to disregard the GNE seeking in order to compute the unconstrained optimal value of ϕ. Moreover, an increasing value of α improves the algorithm performance (cf. Figure 2). Finally, we observe that for increasing values of ζ, the algorithm reaches a higher residual and a lower selection function value (cf. Figure 3). In our experience, setting ζ too high might cause convergence failure as ε_k might become 0 before γ_k reaches a negligible value. In Figure 4 we compare Algorithm 1 with the HSDM paired with the FBF algorithm [13, Alg. 1]. The parameters for Algorithm 1 are chosen among the ones that performed reasonably well in both the performance metrics considered in Figures 1–3. We find the two algorithms to have similar convergence speed. One might find this surprising, as the Tikhonov method is double-layered; thus, one could expect slower convergence compared to the single-layer HSDM. This can be explained by noting that the slowdown caused by the double-layer iterations is compensated by the linear convergence of the pFB. In contrast, the HSDM uses the FBF, which only achieves sublinear convergence. Nevertheless, as observed in the first set of simulations, the Tikhonov method requires more careful parameter tuning than the HSDM.

VI. Conclusion

The generalized Nash equilibrium selection problem can be solved with a semi-decentralized algorithm based on the Tikhonov regularization method combined with the preconditioned forward backward algorithm, which achieves linear convergence for the regularized sub-problems. Interestingly, both the Tikhonov regularization method and a particular instance of the hybrid steepest descent method seek at each iteration an approximate solution to the same regularized problem, indicating a conceptual connection. Although theoretically less practical (as shown in Table I), the Tikhonov algorithm demonstrates comparable convergence performance to the state-of-the-art in our simulations.

Appendix

A. Proof of Lemma 1

Lemma 1.1 is immediate from the monotonicity of $\mathcal{B}$ [7, Lem. 5], $\mathcal{C}$ [15, Ex. 20.35] and $\nabla\phi$ (Assumption 4). From Gerschgorin's theorem, $\|\bar{\mathcal{L}}\| \leq 2\max(|\mathcal{N}_i|_{i\in\mathcal{I}})$. Thus, $\mathcal{B}$ is $\max(L_F, 2|\mathcal{N}_i|_{i\in\mathcal{I}})$-Lipschitz continuous from Assumption 3. Lemma 1.2 then follows immediately. ■

B. Proof of Proposition 1

Let us denote the solution set of (6) by $\mathcal{S}$. The proposition follows from [11, Thm. 2] if $\boldsymbol{\Omega}$ is closed and convex; $\mathcal{B}+\mathcal{C}$ is continuous and monotone; $\nabla\phi$ is continuous and monotone plus; $\mathcal{S}$ is bounded and not empty; and the set

$$\mathfrak{L}_1 := \{\boldsymbol{\omega}_2 \in \boldsymbol{\Omega} | \ \exists\ \boldsymbol{\omega}_1 \in \mathcal{S} \text{ s. t. } \nabla\phi(\boldsymbol{\omega}_2)^\top(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) > 0\}$$

is bounded. The conditions on $\boldsymbol{\Omega}$ and the continuity of $\mathcal{B}, \mathcal{C}, \nabla\phi$ follow from the assumptions. The monotonicity of $\mathcal{B}+\mathcal{C}$ is proven in Lemma 1. $\nabla\phi$ is monotone-plus from the convexity of ϕ and [15, Ex. 22.4i]. From [7, Thm. 2], $\operatorname{zer}(\mathcal{T}^{\mathrm{KKT}}) \neq \varnothing$. Let $\boldsymbol{\omega}_1 \in \operatorname{zer}(\mathcal{T}^{\mathrm{KKT}})$ and consider

$$\mathfrak{L}_2 := \{\boldsymbol{\omega}_2 \in \boldsymbol{\Omega} | \nabla\phi(\boldsymbol{\omega}_2)^\top(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \geq 0\}.$$

We apply the convexity inequality on ϕ to find

$$\phi(\boldsymbol{\omega}_1) - \phi(\boldsymbol{\omega}_2) \geq \nabla\phi(\boldsymbol{\omega}_2)^\top(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2), \quad \forall \boldsymbol{\omega}_2 \in \boldsymbol{\Omega}. \quad (25)$$

From the coercivity of ϕ, $\phi(\boldsymbol{\omega}_1) - \phi(\boldsymbol{\omega}_2) < 0$ for sufficiently large $\|\boldsymbol{\omega}_2\|$. Thus, $\mathfrak{L}_2$ is bounded. Therefore, from [17, Prop. 2.2.3], $\mathcal{S}$ is non-empty and compact. Consequently, as (25) holds for any $\boldsymbol{\omega}_1 \in \mathcal{S}$, we find for all $\boldsymbol{\omega}_2 \in \boldsymbol{\Omega}$,

$$\max_{\boldsymbol{\omega}_1 \in \mathcal{S}} \nabla\phi(\boldsymbol{\omega}_2)^\top(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) \leq \phi^* - \phi(\boldsymbol{\omega}_2),$$

$\phi^* := \max_{\boldsymbol{\omega}_1 \in \mathcal{S}} \phi(\boldsymbol{\omega}_1)$. By the coercivity of ϕ, $\phi^* - \phi(\boldsymbol{\omega}_2) < 0$ for sufficiently large $\|\boldsymbol{\omega}_2\|$. Therefore, $\mathfrak{L}_1$ is bounded. ■

C. Proof of Lemma 2 (sketch)

It follows from [7, Lemma 6] and Gerschgorin's theorem. ■

D. Proof of Lemma 3

The operator $(\mathrm{Id} + \Phi^{-1}(\mathcal{A}+\mathcal{C}))^{-1}$ is non-expansive in the Φ-induced norm, following [7, Lem. 7ii]. Denote $G = \mathcal{B} + \gamma_k\nabla\phi + \alpha(\mathrm{Id} - \boldsymbol{\omega}^{(k)})$. First, we observe

$$\|\Phi\| I \succcurlyeq \Phi \succcurlyeq \delta I \Rightarrow \|\Phi\|\|\boldsymbol{z}\|^2 \geq \|\boldsymbol{z}\|_\Phi^2 \geq \delta\|\boldsymbol{z}\|^2, \quad (26a)$$

$$\Phi \succcurlyeq \delta I \Rightarrow \Phi^{-1} \preccurlyeq \delta^{-1} I \Rightarrow \|\boldsymbol{z}\|_{\Phi^{-1}}^2 \leq \delta^{-1}\|\boldsymbol{z}\|^2, \quad (26b)$$

for any $\boldsymbol{z} \in \boldsymbol{\Omega}$. Furthermore, for any pair $\boldsymbol{z}, \boldsymbol{z}' \in \boldsymbol{\Omega}$,

$$\begin{aligned}
&\langle \boldsymbol{z} - \boldsymbol{z}', \Phi^{-1}(G\boldsymbol{z} - G\boldsymbol{z}')\rangle_\Phi \\
&\geq \langle \boldsymbol{z} - \boldsymbol{z}', \Phi^{-1}\gamma_k(\nabla\phi(\boldsymbol{z}) - \nabla\phi(\boldsymbol{z}')) + \alpha\Phi^{-1}(\boldsymbol{z} - \boldsymbol{z}')\rangle_\Phi \\
&\geq \alpha\|\boldsymbol{z} - \boldsymbol{z}'\|^2 \geq \tfrac{\alpha}{\|\Phi\|}\|\boldsymbol{z} - \boldsymbol{z}'\|_\Phi^2. \quad (27)
\end{aligned}$$

We use the fact that $\Phi^{-1}\mathcal{B}$ is cocoercive in the Φ-induced norm [7, Lem. 7i] (and, thus, monotone) in the first inequality, the monotonicity of $\nabla\phi$ in Euclidean norm in the second inequality, and (26a) in the third inequality. We then have that

$$\begin{aligned}
&\|\Phi^{-1}G(\boldsymbol{z}) - \Phi^{-1}G(\boldsymbol{z}')\|_\Phi^2 = \|G(\boldsymbol{z}) - G(\boldsymbol{z}')\|_{\Phi^{-1}}^2 \overset{(26b)}{\leq} \\
&\tfrac{1}{\delta}\|G(\boldsymbol{z}) - G(\boldsymbol{z}')\|^2 \leq \tfrac{L_G^2}{\delta}\|\boldsymbol{z} - \boldsymbol{z}'\|^2 \overset{(26a)}{\leq} \tfrac{L_G^2}{\delta^2}\|\boldsymbol{z} - \boldsymbol{z}'\|_\Phi^2, \quad (28)
\end{aligned}$$

where we use the Lipschitz continuity of G (Lemma 1). By expanding the square and from (27) and (28), we have that

$$\begin{aligned}
&\|(\mathrm{Id} - \Phi^{-1}G)(\boldsymbol{z}) - (\mathrm{Id} - \Phi^{-1}G)(\boldsymbol{z}')\|_\Phi^2 \\
&\leq (1 + \tfrac{L_G^2}{\delta^2} - \tfrac{2\alpha}{\|\Phi\|})\|\boldsymbol{z} - \boldsymbol{z}'\|_\Phi^2 = \beta\|\boldsymbol{z} - \boldsymbol{z}'\|_\Phi^2.
\end{aligned}$$

From Assm. 6 and Lemma 2, $\frac{L_G^2}{\delta^2} < \frac{L_G^2\alpha}{L_G^2\delta} \leq \frac{2\alpha}{\|\Phi\|}$, thus $\beta < 1$ and $(\mathrm{Id} - \Phi^{-1}G)$ is contractive. From [15, Prop. 26.1.iv] and [17, Eq. 1.1.3], $\operatorname{fix}(\mathcal{T}_k^{\mathrm{pFB}}) = \operatorname{zer}(\mathcal{A}+\mathcal{C}+G) = \mathrm{SOL}(\boldsymbol{\Omega}, \mathcal{C}+G)$ and the thesis follows from [15, Thm. 1.50]. ■

E. Proof of Proposition 2

By expanding the operators that define $\mathcal{T}_k^{\mathrm{pFB}}$, (15)–(17) are equivalent to $\boldsymbol{y}^{(k,t+1)} = \mathcal{T}_k^{\mathrm{pFB}}(\boldsymbol{y}^{(k,t)})$, for all $k \in \mathbb{N}$. From (18), (19), and Lemma 3,

$$\|\boldsymbol{\omega}^{(k+1)} - \boldsymbol{\omega}_k^*\|_\Phi \leq \|\boldsymbol{y}^{(k,t+1)} - \boldsymbol{y}^{(k,t)}\|_\Phi/(1-\beta) \leq \varepsilon_k.$$

Next, by triangle inequality, we have that

$$\begin{aligned}
\|\boldsymbol{y}^{(k,t+1)} - \boldsymbol{y}^{(k,t)}\|_\Phi &\leq \|\boldsymbol{y}^{(k,t+1)} - \boldsymbol{\omega}_k^*\|_\Phi + \|\boldsymbol{\omega}_k^* - \boldsymbol{y}^{(k,t)}\|_\Phi \\
&\overset{\{1\}}{\leq} (1+\beta)\|\boldsymbol{\omega}_k^* - \boldsymbol{y}^{(k,t)}\|_\Phi \overset{\{2\}}{\leq} \beta^t(\beta+1)\|\boldsymbol{\omega}_k^* - \boldsymbol{y}^{(k,0)}\|_\Phi,
\end{aligned}$$

where $\{1\}$ follows from the contractivity of $\mathcal{T}_k^{\mathrm{pFB}}$ (see the proof of Lemma 3) and $\{2\}$ from [15, Thm. 1.50iii]. Thus, if $\varepsilon_k > 0$, (18) holds for $t \geq \log_\beta\left(\frac{\varepsilon_k}{(\beta+1)\|\boldsymbol{\omega}_k^* - \boldsymbol{y}^{(k,0)}\|_\Phi}\right)$. ■

REFERENCES

[1] B. G. Bakhshayesh and H. Kebriaei, "Decentralized equilibrium seeking of joint routing and destination planning of electric vehicles: A constrained aggregative game approach," *IEEE Transactions on Intelligent Transportation Systems*, pp. 1–10, 2021.

[2] W. Ananduta and S. Grammatico, "Equilibrium seeking and optimal selection algorithms in peer-to-peer energy markets," *Games*, vol. 13, no. 5, p. 66, 2022.

[3] J. Wang, M. Peng, S. Jin, and C. Zhao, "A Generalized Nash Equilibrium Approach for Robust Cognitive Radio Networks via Generalized Variational Inequalities," *IEEE Transactions on Wireless Communications*, vol. 13, no. 7, pp. 3701–3714, 2014.

[4] F. Facchinei and C. Kanzow, "Generalized Nash equilibrium problems," *Annals of Operations Research*, vol. 175, no. 1, pp. 177–211, 2010.

[5] H. Yin, U. V. Shanbhag, and P. G. Mehta, "Nash equilibrium problems with scaled congestion costs and shared constraints," *IEEE Transactions on Automatic Control*, vol. 56, no. 7, pp. 1702–1708, 2011.

[6] A. Kannan and U. V. Shanbhag, "Distributed computation of equilibria in monotone Nash games via iterative regularization techniques," *SIAM Journal on Optimization*, vol. 22, no. 4, pp. 1177–1205, 2012.

[7] P. Yi and L. Pavel, "An operator splitting approach for distributed generalized Nash equilibria computation," *Automatica*, vol. 102, pp. 111–121, 2019.

[8] B. Franci, M. Staudigl, and S. Grammatico, "Distributed forward-backward (half) forward algorithms for generalized Nash equilibrium seeking," in *European Control Conference*, pp. 1274–1279, 2020.

[9] G. Belgioioso, A. Nedić, and S. Grammatico, "Distributed generalized Nash equilibrium seeking in aggregative games on time-varying networks," *IEEE Transactions on Automatic Control*, vol. 66, no. 5, pp. 2061–2075, 2020.

[10] G. Scutari, F. Facchinei, J.-S. Pang, and L. Lampariello, "Equilibrium selection in power control games on the interference channel," in *Proceedings IEEE INFOCOM*, pp. 675–683, 2012.

[11] F. Facchinei, J.-S. Pang, G. Scutari, and L. Lampariello, "VI-constrained hemivariational inequalities: distributed algorithms and power control in ad-hoc networks," *Mathematical Programming*, vol. 145, 2014.

[12] E. Benenati, W. Ananduta, and S. Grammatico, "On the optimal selection of generalized Nash equilibria in linearly coupled aggregative games," in *Conference on Decision and Control*, pp. 6389–6394, IEEE, 2022.

[13] E. Benenati, W. Ananduta, and S. Grammatico, "Optimal selection and tracking of generalized Nash equilibria in monotone games," *IEEE Transactions on Automatic Control*, 2023. in press.

[14] I. Yamada and N. Ogura, "Hybrid steepest descent method for variational inequality problem over the fixed point set of certain quasi-nonexpansive mappings," *Numerical Functional Analysis and Optimization*, vol. 25, no. 7-8, pp. 619–655, 2005.

[15] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*. CMS Books in Mathematics, Springer international publishing, 2017.

[16] D. P. Palomar and Y. C. Eldar, *Convex Optimization in Signal Processing and Communications*. Cambridge university press, 2009.

[17] F. Facchinei and J.-S. Pang, *Finite-dimensional variational inequalities and complementarity problems*. Springer, 2003.

[18] A. Auslender and M. Teboulle, "Lagrangian duality and related multiplier methods for variational inequality problems," *SIAM Journal on Optimization*, vol. 10, no. 4, pp. 1097–1115, 2000.