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Dyadic operators and the $T(1)$ theorem

In Chapter 11, we have mainly dealt with a situation, where a bounded linear operator on some $L^{p_0}(\mathbb{R}^d; X)$ space is given, and we have then explored its bounded extensions to other spaces including $L^p(\mathbb{R}^d; X)$ for $p \neq p_0$. We now turn to a somewhat different (and often more difficult) question of recognising such bounded operators to begin with.

Before addressing this question for the Calderón–Zygmund type operators of the kind studied in Chapter 11, we investigate a number of related objects in a simpler dyadic model. Besides serving as an introduction to some of the key techniques, it turns out that these dyadic operators can be, and will be, also used as building blocks of the proper singular integral operators towards the end of the chapter.

The dyadic operators will be of two essentially different types. The first class, which we vaguely refer to as "dyadic singular integrals" in Section 12.1, consist of a somewhat diverse family of relatives of the prototype dyadic shifts encountered in Chapter 5, where they we used to represent the prototype singular integral given by the Hilbert transform. It is thus only natural that a family of dyadic operators generalising this basic dyadic shift will serve as building block of the Calderón–Zygmund family of singular integrals generalising the Hilbert transform. Martingale techniques vaguely reminiscent of those in Section 5.1, but of somewhat higher complexity probably by necessity, will feature in the argument to put the UMD property of the underlying Banach space into action.

The second class of dyadic operators consists of so-called *paraproducts*, which we discuss in Section 12.2. These are new creatures of the non-convolution realm that we have entered and they will vanish (as we will see) as soon as we occasionally specialise our considerations to singular integral of the convolution form. However, for the representation the full class of Calderón–Zygmund operators they will turn out be quite essential.

The chapter will culminate in a lengthy treatment of the so-called $T(1)$ theorem, a general criterion for boundedness of singular integral operators. We will first discuss a version for abstract bilinear form in Section 12.3, and

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only then, in the final Section 12.4, turn to the task of checking the assumptions of the abstract result for singular integral operators with a Calderón–Zygmund kernel, of the kind that we met Chapter 11. However, in order to establish boundedness on $L^p(\mathbb{R}^d;X)$ from scratch, rather than extrapolating it from another $L^{p_0}(\mathbb{R}^d;X)$ space where it was already known (as in Chapter 11), somewhat stronger versions of the Calderón–Zygmund conditions will be needed, and the notion of R-boundedness from Chapter 8 will, once again, play a prominent role. While the results of this chapter will generically be established in arbitrary UMD spaces, it turns out that additional information about type and cotype, as studied in Chapter 7 can be traded against the precise kernel conditions, so that slightly larger classes of kernels are admissible under conditions of type and cotype of the underlying space.

12.1 Dyadic singular integral operators

In this section, we introduce and study a family of dyadic models of singular integrals, starting from the simplest case of Haar multipliers and proceeding to their more complicated relatives. All these operators will eventual come together as parts of a decomposition of general singular integral operators towards the end of the chapter.

Since our aim is not to assume, but to prove, the L^p-boundedness of the relevant operators, we will first define their action on appropriate spaces of test functions only.

Definition 12.1.1 (Classes of simple functions). *For a collection $\mathscr{C}$ of bounded Borel subsets of $\mathbb{R}^d$, let*

$$\begin{aligned}
S(\mathscr{C};X) &:= \operatorname{span}\Big\{\mathbf{1}_C\otimes x : C\in\mathscr{C}, x\in X\Big\},\\
S_0(\mathscr{C};X) &:= \Big\{f\in S(\mathscr{C};X) : \int_{\mathbb{R}^d} f(t)\,\mathrm{d}t = 0\Big\},\\
S_{\mathrm{loc}}(\mathscr{C};X) &:= \{f\in L^1_{\mathrm{loc}}(\mathbb{R}^d;X) : \mathbf{1}_C f\in S(\mathscr{C};X) \textit{ for all } C\in\mathscr{C}\},\\
S_\infty(\mathscr{C};X) &:= S_{\mathrm{loc}}(\mathscr{C};X)\cap L^\infty(\mathbb{R}^d;X).
\end{aligned}$$

It is easy to see that $S(\mathscr{C};X)\subseteq L^p(\mathbb{R}^d;X)$ for all $p\in[1,\infty]$, and that

$$S_0(\mathscr{C};X)\subseteq S(\mathscr{C};X)\subseteq S_\infty(\mathscr{C};X)\subseteq S_{\mathrm{loc}}(\mathscr{C};X).$$

Our primary case of interest will be when $\mathscr{C}=\mathscr{D}$ is a collection of dyadic cubes of $\mathbb{R}^d$ in the sense of Definition 11.1.6. In this case, $S(\mathscr{C};X)$ is dense in $L^p(\mathbb{R}^d;X)$ for all $p\in[1,\infty)$. In (12.2) below, we will add yet another space $S_{00}(\mathscr{D};X)\subseteq S_0(\mathscr{D};X)$ to this list, but its introduction requires some preliminaries.

12.1.a Haar multipliers

We begin with what is arguably the simplest class of operators deserving the name of "dyadic singular integrals". In essence, we have encountered these operators already, at least implicitly on the one-dimensional domain space $\mathbb{R}^1$, where we dealt with operators of the form

$$f \mapsto \sum_{I \in \mathscr{D}} \epsilon_I \langle f, h_I \rangle h_I$$

and showed their uniform boundedness on $L^p(\mathbb{R}; X)$ for arbitrary unimodular coefficients ϵ_I, assuming that $p \in (1, \infty)$ and X is a UMD space (see Theorem 4.2.13). We now wish to extend these consideration to the general Euclidean domain $\mathbb{R}^d$. This hardly presents any new challenges, and mainly serves as a warm-up for the subsequent considerations.

We first recall and extend the notation related to conditional expectations and martingale differences over the dyadic filtration of $\mathbb{R}^d$. For any cube

$$Q = a_Q + \ell(Q)[0,1)^d,$$

with sidelength $\ell(Q) > 0$ and "lower left" corner $a_Q \in \mathbb{R}^d$, we denote by

$$\mathrm{ch}(Q) := \Big\{ a_Q + \frac{1}{2}\ell(Q)([0,1)^d + \alpha) : \alpha \in \{0,1\}^d \Big\}$$

the collection of its 2^d "children" obtained by bisecting each of the intervals in the Cartesian product defining Q. In particular, for

$$Q \in \mathscr{D}_k := \{2^{-k}([0,1)^d + n) : n \in \mathbb{Z}^d\},$$

we have

$$\mathrm{ch}(Q) = \{Q' \in \mathscr{D}_{k+1} : Q' \subseteq Q\}.$$

For every cube Q, we define the conditional expectation and martingale difference projections (acting on $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$)

$$\mathbb{E}_Q f := \mathbf{1}_Q \fint_Q f \,\mathrm{d}x, \qquad \mathbb{D}_Q f := \sum_{Q' \in \mathrm{ch}(Q)} \mathbb{E}_{Q'} f - \mathbb{E}_Q f. \tag{12.1}$$

Then for every $k \in \mathbb{Z}$, we let

$$\mathbb{E}_k f := \mathbb{E}(f | \sigma(\mathscr{D}_k)) = \sum_{Q \in \mathscr{D}_k} \mathbb{E}_Q f,$$

$$\mathbb{D}_k f := \mathbb{E}_{k+1} f - \mathbb{E}_k f = \sum_{Q \in \mathscr{D}_k} \mathbb{D}_Q f.$$

We still want to express the martingale difference projections $\mathbb{D}_Q$ in terms of vector-valued extensions of rank-one operators on scalar-valued functions.

In dimension $d = 1$, the operators already have this form, as we recall from Lemma 4.2.11 and the preceding discussion:

$$\mathbb{D}_I f = \langle f, h_I \rangle h_I, \qquad h_I = |I|^{-1/2}(\mathbf{1}_{I_-} - \mathbf{1}_{I_+}),$$

where h_I is called the *Haar function* associated with the interval I.

In higher dimensions, there are various ways of constructing analogues of the Haar functions. For the present purposes, a standard tensor construction suffices. In $d = 1$, we denote

$$h_I^1 := h_I, \qquad h_I^0 := |I|^{-1/2}\mathbf{1}_I.$$

Lemma 12.1.2. *In general dimension $d \geqslant 1$, the (tensor-)Haar functions*

$$h_Q^\alpha(x) = h_{I_1\times\cdots\times I_d}^{(\alpha_1,\ldots,\alpha_d)}(x_1,\ldots,x_d) := \prod_{i=1}^d h_{I_i}^{\alpha_i}(x_i), \quad \alpha = (\alpha_1,\ldots,\alpha_d) \in \{0,1\}^d.$$

satisfy the following identity for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$:

$$\mathbb{D}_Q f = \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \langle f, h_Q^\alpha\rangle h_Q^\alpha =: \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \mathbb{D}_Q^\alpha f.$$

Proof. From the (obvious) orthogonality of one-dimensional Haar functions, it follows that

$$\langle h_Q^\alpha, h_Q^\beta\rangle = \prod_{i=1}^d \langle h_{I_i}^{\alpha_i}, h_{I_i}^{\beta_i}\rangle = \prod_{i=1}^d \delta_{\alpha_i,\beta_i} = \delta_{\alpha,\beta}.$$

Let H_Q be the space of scalar-valued functions supported on Q, constant on each dyadic child of Q, and of mean zero. Clearly $\dim H_Q = (2^d - 1)$ and $h_Q^\alpha \in H_Q$ for each $\alpha \in \{0,1\}^d \setminus \{0\}$. Since these h_Q^α are orthonormal and their number is equal to $\dim H_Q$, they must form an orthonormal basis of H_Q. On the other hand, one easily verifies that D_Q is the orthogonal projection of $L^2(\mathbb{R}^d)$ onto H_Q, so in particular $D_Q f = f$ for all $f \in H_Q$. Since the h_Q^α form an orthonormal basis, the claimed identity is true for all $f \in H_Q$. If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d; X)$ and $x^* \in X^*$, then $\langle D_Q f, x^*\rangle \in H_Q$ and thus

$$\langle \mathbb{D}_Q f, x^*\rangle = \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \langle\langle \mathbb{D}_Q f, x^*\rangle, h_Q^\alpha\rangle h_Q^\alpha = \Big\langle \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \langle \mathbb{D}_Q f, h_Q^\alpha\rangle h_Q^\alpha, x^*\Big\rangle,$$

where

$$\begin{aligned}\langle \mathbb{D}_Q f, h_Q^\alpha\rangle &= \sum_{Q'\in\mathrm{ch}(Q)} \langle \mathbb{E}_{Q'} f, h_Q^\alpha\rangle - \langle \mathbb{E}_Q f, h_Q\rangle \\ &= \sum_{Q'\in\mathrm{ch}(Q)} \langle f, \mathbb{E}_{Q'} h_Q^\alpha\rangle = \langle f, h_Q^\alpha\rangle.\end{aligned}$$

The claimed identity follows, since the functionals $x^* \in X^*$ separate the points $x \in X$ by the Hahn–Banach theorem. □

The functions h_Q^α, with $\alpha \in \{0,1\}^d \setminus \{0\}$, are referred to as *cancellative Haar functions*, as they all have vanishing mean. In contrast, $h_Q^0 = |Q|^{-1/2}\mathbf{1}_Q$ is the *non-cancellative Haar function* on Q. In the wavelet literature, the cancellative Haar functions are special cases of *mother wavelets*, while the non-cancellative Haar function is the *father wavelet*.

Lemma 12.1.3. *Let X be a Banach space and $p \in (1,\infty)$. Then the space of finite linear combinations of cancellative Haar functions with X-coefficients,*

$$S_{00}(\mathscr{D};X) := \operatorname{span}\Big\{h_Q^\alpha \otimes x : Q \in \mathscr{D}, \alpha \in \{0,1\}^d \setminus \{0\}, x \in X\Big\}, \qquad (12.2)$$

is dense in $L^p(\mathbb{R}^d;X)$.

Proof. The filtration generated by the dyadic cubes, $(\mathscr{F}_k)_{k\in\mathbb{Z}} := (\sigma(\mathscr{D}_k))_{k\in\mathbb{Z}}$ is σ-finite with respect to the Lebesgue measure on $\mathbb{R}^d$, and $\mathscr{F}_\infty := \sigma\big(\bigcup_{k\in\mathbb{Z}} \mathscr{F}_k\big)$ is the Borel σ-algebra of $\mathbb{R}^d$. Hence $E_k f \to f$ in $L^p(\mathbb{R}^d;X)$ as $k \to \infty$ for all $f \in L^p(\mathbb{R}^d;X)$ by the forward convergence of generated martingales (Theorem 3.3.2). On the other hand, $\mathscr{F}_{-\infty} := \bigcap_{k\in\mathbb{Z}} \mathscr{F}_k$ contains only sets of Lebesgue measure 0 (the empty set) or ∞ (the quadrants, and their unions), which means (by definition) that the Lebesgue measure is *purely infinite* on $\mathscr{F}_{-\infty}$. Thus $E_k f \to 0$ in $L^p(\mathbb{R}^d;X)$ as $k \to -\infty$ for all $f \in L^p(\mathbb{R}^d;X)$ by the backward convergence of martingales (Theorem 3.3.5).

Combining these observation about the limits at $\pm\infty$, it follows that functions of the form $\mathbb{E}_M f - \mathbb{E}_m f = \sum_{k=m}^{M-1} \mathbb{D}_k f$ are dense in $L^p(\mathbb{R}^d;X)$. Next, we make the following observations about each $\mathbb{D}_k$ appearing in this expansion. First, for any $P \in \mathscr{D}_m$, multiplication with $\mathbf{1}_P$ commutes with $\mathbb{D}_k$; second, $\mathbf{1}_P \mathbb{D}_k f$ is a finite linear combination of some $\mathbb{D}_Q f$, and finally, if $(P_i)_{i=1}^\infty$ is an enumeration of $\mathscr{D}_m$, then $\sum_{i=1}^N \mathbf{1}_{P_i} f \to f$ in $L^p(\mathbb{R}^d;X)$ as $N \to \infty$. Thus finite linear combinations of $\mathbb{D}_Q f$ are dense in $L^p(\mathbb{R}^d;X)$. Finally, Lemma 12.1.2 shows that $\mathbb{D}_Q f \in S_{00}(\mathscr{D};X)$, and completes the proof. □

Remark 12.1.4. One can check that

$$S_{00}(\mathscr{D};X) = \Big\{f \in S(\mathscr{D};X) : \int_D f = 0 \text{ for each quadrant } D \text{ of } \mathbb{R}^d\Big\}.$$

In particular, if $\mathscr{D}$ is a connected tree of dyadic cubes (i.e., every two cubes are contained in a common bigger dyadic cube), then $S_{00}(\mathscr{D};X) = S_0(\mathscr{D};X)$. Making this connectedness assumption would slightly simplify some considerations, but have the disadvantage of excluding the standard dyadic system (cf. Remark 11.1.9).

After these preparatory considerations, we are in a position to prove the first non-trivial estimates for operators of dyadic singular integral type. As one expects, the UMD property is used, but in this first estimate still in a relatively straightforward manner.

Proposition 12.1.5. *Let X be a UMD space, $p \in (1,\infty)$, and $f \in S_{00}(\mathscr{D};X)$. For any any $\alpha \in \{0,1\}^d \setminus \{0\}$ and coefficients $\lambda_Q \in \mathbb{K}$, we have the estimates*

$$\Big\| \sum_{Q\in\mathscr{D}} \lambda_Q \langle h_Q^\alpha, f\rangle h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d;X)} \leqslant \beta_{p,X} \sup_{Q\in\mathscr{D}} |\lambda_Q| \|f\|_{L^p(\mathbb{R}^d;X)},$$

$$\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle h_Q^\alpha, f\rangle h_Q^\alpha \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \leqslant \beta^+_{p,X} \|f\|_{L^p(\mathbb{R}^d;X)}.$$

Proof. Let us denote

$$\mathbb{D}_Q^\alpha f := \langle h_Q^\alpha, f\rangle h_Q^\alpha, \qquad \mathbb{D}_Q^{-\alpha} f := \mathbb{D}_Q f - \mathbb{D}_Q^\alpha f = \sum_{\gamma\in\{0,1\}^d\setminus\{0,\alpha\}} \langle h_Q^\gamma, f\rangle h_Q^\gamma.$$

Then $(\mathbb{D}_Q^\alpha f, \mathbb{D}_Q^{-\alpha} f)$ is a martingale difference sequence on Q, as each h_Q^γ with $\gamma \notin \{0,\alpha\}$ has average zero on the sets where h_Q^α is constant. Appropriately enumerated, $(\mathbb{D}_Q^\alpha f, \mathbb{D}_Q^{-\alpha} f)_{Q\in\mathscr{D}}$ also forms a martingale difference sequence. Estimating its martingale transform by a multiplying sequence of 0's and 1's, we obtain

$$\begin{aligned}\Big\| \sum_{Q\in\mathscr{D}} \lambda_Q \mathbb{D}_Q^\alpha f \Big\|_{L^p(\mathbb{R}^d;X)} &= \Big\| \sum_{Q\in\mathscr{D}} \Big(\lambda_Q \cdot \mathbb{D}_Q^\alpha f + 0 \cdot \mathbb{D}_Q^{-\alpha} f\Big) \Big\|_{L^p(\mathbb{R}^d;X)} \\ &\leqslant \beta_{p,X} \Big\| \sum_{Q\in\mathscr{D}} \Big(\mathbb{D}_Q^\alpha f + \mathbb{D}_Q^{-\alpha} f\Big) \Big\|_{L^p(\mathbb{R}^d;X)},\end{aligned}$$

For the other claim, we argue by the contraction principle and the randomised UMD inequality to see that

$$\begin{aligned}\Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \mathbb{D}_Q^\alpha f \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} &\leqslant \Big\| \sum_{Q\in\mathscr{D}} \Big(\varepsilon_Q \mathbb{D}_Q^\alpha f + \varepsilon'_Q \mathbb{D}_Q^{-\alpha} f\Big) \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\ &\leqslant \beta^+_{p,X} \Big\| \sum_{Q\in\mathscr{D}} \Big(\mathbb{D}_Q^\alpha f + \mathbb{D}_Q^{-\alpha} f\Big) \Big\|_{L^p(\mathbb{R}^d;X)},\end{aligned}$$

and in both cases we conclude by observing that

$$\sum_{Q\in\mathscr{D}} \Big(\mathbb{D}_Q^\alpha f + \mathbb{D}_Q^{-\alpha} f\Big) = \sum_{Q\in\mathscr{D}} \mathbb{D}_Q f = f.$$

□

For operator-valued coefficients $\lambda_Q \in \mathscr{L}(X,Y)$, the following variants of R-boundedness turn out to be relevant:

Definition 12.1.6. *For $p \in (1,\infty)$ and an operator family $\lambda = (\lambda_Q)_{Q\in\mathscr{C}} \subseteq \mathscr{L}(X,Y)$ indexed by a collection $\mathscr{C}$ of bounded Borel subsets of $\mathbb{R}^d$, we denote by $\mathscr{D}\mathscr{R}_p(\lambda)$ and $\mathscr{E}\mathscr{R}_p(\lambda)$ the smallest admissible constants such that the* *following estimates hold for all finitely non-zero families* $(x_Q)_{Q\in\mathscr{C}} \subseteq X$ *and* $(y_Q^*)_{Q\in\mathscr{C}} \subseteq Y^*$*:*

$$\begin{aligned}\sum_{Q\in\mathscr{C}} &|Q||\langle \lambda_Q x_Q, y_Q^*\rangle| \\ &\leqslant \mathscr{DR}_p(\lambda)\Big\| \sum_{Q\in\mathscr{C}} \varepsilon_Q x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \Big\| \sum_{Q\in\mathscr{C}} \varepsilon_Q y_Q^* \mathbf{1}_Q\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)},\end{aligned}$$

and

$$\Big\| \sum_{Q\in\mathscr{C}} \varepsilon_Q \lambda_Q x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \leqslant \mathscr{ER}_p(\lambda)\Big\| \sum_{Q\in\mathscr{C}} \varepsilon_Q x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}.$$

We refer to $\mathscr{DR}_p(\lambda)$ *as the* $\mathscr{DR}_p$*-bound of* λ*, and say that* λ *is* $\mathscr{DR}_p$*-bounded if* $\mathscr{DR}_p(\lambda) < \infty$*. The same convention applies to* $\mathscr{ER}_p$ *in place of* $\mathscr{DR}_p$*.*

Remark 12.1.7. The primary case of interest will be when $\mathscr{C} = \mathscr{D}$ is a system of dyadic cubes. In this case, it is useful to observe at once that the defining inequality of $\mathscr{ER}_p(\lambda)$ immediately extends to Haar functions h_Q^α in place of the indicators $\mathbf{1}_Q$:

$$\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \lambda_Q x_Q h_Q^\gamma\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \leqslant \mathscr{ER}_p(\lambda)\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q x_Q h_Q^\alpha\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}.$$

Proof. We have $h_Q^\alpha = \operatorname{sgn}(h_Q^\alpha)|Q|^{-1/2}\mathbf{1}_Q$ and hence, by the contraction principle,

$$\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q z_Q h_Q^\alpha\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} = \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q |Q|^{-1/2} z_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}$$

for both $(z_Q, Z) = \{(x_Q, X), (\lambda_Q x_Q, Y)\}$. Using this twice, with both α and γ, and in between the defining inequality of $\mathscr{ER}_p(\lambda)$ for $|Q|^{-1/2}x_Q$ in place x_Q, yields the claim. □

These notions are weaker than R-boundedness; we will shortly see that the converse fails in general.

Lemma 12.1.8. *For all Banach spaces* X *and* Y*, all operator families* $\lambda = (\lambda_Q)_{Q\in\mathscr{C}} \subseteq \mathscr{L}(X,Y)$ *and their adjoints* $\lambda^* := (\lambda_Q^*)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(Y^*, X^*)$*, and all* $p \in (1,\infty)$*, we have*

$$\begin{aligned}\sup_{Q\in\mathscr{C}} \|\lambda_Q\| \leqslant \mathscr{DR}_p(\lambda) &\leqslant \min\{\mathscr{ER}_p(\lambda), \mathscr{ER}_{p'}(\lambda^*)\}, \\ \mathscr{ER}_p(\lambda) &\leqslant \|x \mapsto \mathscr{R}_p(\{\lambda_Q : Q \ni x\})\|_{L^\infty(\mathbb{R}^d)} \leqslant \mathscr{R}_p(\lambda).\end{aligned}$$

Proof. The last two estimates are immediate. The first estimate follows by testing the defining condition of $\mathscr{DR}_p$ with only one non-zero pair (x_Q, y_Q^*) at a time. To see that $\mathscr{DR}_p(\lambda) \leqslant \mathscr{ER}_p(\lambda)$, for suitable scalars $|\eta_Q| = 1$, we have

$$\begin{aligned}\sum_{Q\in\mathscr{C}}|Q||\langle\lambda_Q x_Q, y_Q^*\rangle| &= \sum_{Q\in\mathscr{C}}\int \eta_Q\langle\lambda_Q x_Q\mathbf{1}_Q, y_Q^*\mathbf{1}_Q\rangle \\ &= \mathbb{E}\int\Big\langle \sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q\lambda_Q x_Q\mathbf{1}_Q, \sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\rangle \\ &\leqslant \Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q\lambda_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\Big\|\sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)} \\ &\leqslant \mathscr{E}\mathscr{R}_p(\lambda)\Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\Big\|\sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)},\end{aligned}$$

where we used Kahane's contraction principle and the definition of $\mathscr{E}\mathscr{R}_p(\lambda)$ to pull out the scalar η_Q and the operators λ_Q in the last step. Since $\langle\lambda_Q x_Q, y_Q^*\rangle = \langle x_Q, \lambda_Q^* y_Q^*\rangle$, and $\mathscr{D}\mathscr{R}_{p'}(\lambda)$ is defined by testing the expressions on the right over a more general choice of $x_Q^{**}\in X^{**}$ in place of $x_Q\in X$, it follows that

$$\mathscr{D}\mathscr{R}_p(\lambda)\leqslant\mathscr{D}\mathscr{R}_{p'}(\lambda^*)\leqslant\mathscr{E}\mathscr{R}_{p'}(\lambda^*)$$

by using what we already proved, but with λ^* in place of λ. □

Corollary 12.1.9. *If $\lambda=(\lambda_Q)_{Q\in\mathscr{C}}\subseteq\mathscr{L}(X)$ consists of scalar multiples of the identity, then*

$$\sup_{Q\in\mathscr{C}}|\lambda_Q| = \mathscr{D}\mathscr{R}_p(\lambda) = \mathscr{E}\mathscr{R}_p(\lambda) = \mathscr{R}_p(\lambda).$$

Proof. Lemma 12.1.9 shows that we have this chain with "$\leqslant$" in place of "$=$" throughout. On the other hand, Kahane's contraction principle guarantees that $\mathscr{R}_p(\lambda)=\sup_{Q\in\mathscr{C}}|\lambda_Q|$. Thus we have equality throughout. □

The following example of $\mathscr{D}\mathscr{R}_p$-bounded families will play a role in our investigation of criteria for boundedness of singular integral operators; the uniform boundedness of the quantities $|Q|^{-1}\langle T\mathbf{1}_Q,\mathbf{1}_Q\rangle$ is classically known as the *weak boundedness property* of the operator T.

Example 12.1.10. Suppose that $T\in\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))$, and define $\langle T(\mathbf{1}_Q),\mathbf{1}_Q\rangle\in\mathscr{L}(X,Y)$ by

$$\langle T\mathbf{1}_Q,\mathbf{1}_Q\rangle : x\mapsto\langle T(\mathbf{1}_Q x),\mathbf{1}_Q\rangle = \int_Q T(\mathbf{1}_Q x)\in Y.$$

For any collection $\mathscr{C}$ of bounded Borel subsets of $\mathbb{R}^d$, it follows that

$$\mathscr{D}\mathscr{R}_p\Big(\Big\{\frac{\langle T\mathbf{1}_Q,\mathbf{1}_Q\rangle}{|Q|}\Big\}_{Q\in\mathscr{C}}\Big)\leqslant\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}.$$

Proof. With suitable scalars $|\eta_Q|=1$, we have

$$\begin{aligned}\sum_{Q\in\mathscr{C}}|Q|\Big|\Big\langle\frac{\langle T\mathbf{1}_Q,\mathbf{1}_Q\rangle}{|Q|}x_Q,y_Q^*\Big\rangle\Big| &= \sum_{Q\in\mathscr{C}}\eta_Q\langle T(\mathbf{1}_Q x_Q),\mathbf{1}_Q y_Q^*\rangle\\ &= \mathbb{E}\Big\langle T\sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q x_Q\mathbf{1}_Q,\sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\rangle\\ &\leqslant \Big\|T\sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)}\Big\|\sum_{R\in\mathscr{C}}\varepsilon_R y_R^*\mathbf{1}_R\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y)},\end{aligned}$$

where

$$\begin{aligned}&\Big\|T\sum_{Q\in\mathscr{C}}\varepsilon_Q\eta_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)}\\ &\leqslant \|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\Big\|\sum_{Q\in\mathscr{C}}\varepsilon_Q x_Q\mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\end{aligned}$$

by the assumed boundedness of T and Kahane's contraction principle with the coefficients η_Q. □

While Example 12.1.10 will only play a role later, the weakening of R-boundedness has the following immediate application:

Theorem 12.1.11 (Haar multipliers). *Let X and Y be UMD spaces and $p\in(1,\infty)$. For $\alpha,\gamma\in\{0,1\}^d\setminus\{0\}$ and $\lambda=(\lambda_Q)_{Q\in\mathscr{D}}\subseteq\mathscr{L}(X,Y)$, consider the operator*

$$\mathfrak{H}_\lambda^{\alpha\gamma}: f\mapsto\sum_{Q\in\mathscr{D}}\lambda_Q\langle f,h_Q^\alpha\rangle h_Q^\gamma, \tag{12.3}$$

initially mapping $S_{00}(\mathscr{D};X)$ into $S_{00}(\mathscr{D};Y)$. Then $\mathfrak{H}_\lambda^{\alpha\gamma}$ extends to a bounded operator on $L^p(\mathbb{R}^d;X)$ if and only if $\mathscr{D}\mathscr{R}_p(\lambda)<\infty$, and in this case

$$\frac{\mathscr{D}\mathscr{R}_p(\lambda)}{\beta^-_{p,X}\beta^-_{p',Y^*}}\leqslant\|\mathfrak{H}_\lambda^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\leqslant\beta^+_{p,X}\beta^+_{p',Y^*}\mathscr{D}\mathscr{R}_p(\lambda). \tag{12.4}$$

Proof. Dualising $\mathfrak{H}_\lambda^{\alpha\gamma}f\in S_{00}(\mathscr{D};X)\subseteq L^p(\mathbb{R}^d;X)$ with $g\in S_{00}(\mathscr{D};Y^*)\subseteq L^{p'}(\mathbb{R}^d;Y^*)$, we arrive at

$$\begin{aligned}|\langle\mathfrak{H}_\lambda^{\alpha\gamma}f,g\rangle| &= \Big|\sum_{Q\in\mathscr{D}}\Big\langle\lambda_Q\langle f,h_Q^\alpha\rangle,\langle h_Q^\gamma,g\rangle\Big\rangle\Big|\\ &= \Big|\sum_{Q\in\mathscr{D}}|Q|\Big\langle\lambda_Q\frac{\langle f,h_Q^\alpha\rangle}{|Q|^{1/2}},\frac{\langle h_Q^\gamma,g\rangle}{|Q|^{1/2}}\Big\rangle\Big|\\ &\leqslant \mathscr{D}\mathscr{R}_p(\lambda)\Big\|\sum_{Q\in\mathscr{D}}\varepsilon_Q\langle f,h_Q^\alpha\rangle\frac{\mathbf{1}_Q}{|Q|^{1/2}}\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\times\\ &\qquad\times\Big\|\sum_{Q\in\mathscr{D}}\varepsilon_Q\langle g,h_Q^\alpha\rangle\frac{\mathbf{1}_Q}{|Q|^{1/2}}\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}.\end{aligned}\tag{12.5}$$

For a fixed $s \in \mathbb{R}^d$, the sequences $(\varepsilon_Q \mathbf{1}_Q(s)/|Q|^{1/2})_{Q\in\mathscr{D}}$ and $(\varepsilon_Q h_Q^\alpha)_{Q\in\mathscr{D}}$ have equal distribution; thus

$$\begin{aligned}\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle f, h_Q^\alpha\rangle \frac{\mathbf{1}_Q}{|Q|^{1/2}} \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} &= \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle f, h_Q^\alpha\rangle h_Q^\alpha \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\ &\leqslant \beta_{p,X}^+ \|f\|_{L^p(\mathbb{R}^d;X)}\end{aligned}$$

by Proposition 12.1.5 in the last step. Similarly, the last term in (12.5) is dominated by $\beta_{p',Y^*}^+ \|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}$. Hence

$$|\langle \mathfrak{H}_\lambda^{\alpha\gamma} f, g\rangle| \leqslant \mathscr{D}\mathscr{R}_p(\lambda)\beta_{p,X}^+\beta_{p',Y^*}^+ \|f\|_{L^p(\mathbb{R}^d;X)}\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)},$$

which proves the second estimate in (12.4).

Conversely, for finitely non-zero families $(x_Q)_{Q\in\mathscr{D}} \subseteq X$ and $(y_Q^*)_{Q\in\mathscr{D}} \subseteq Y^*$, we choose scalar $|\eta_Q| = 1$ such that $|\langle \lambda_Q x_Q, y_Q^*\rangle| = \eta_Q\langle \lambda_Q x_Q, y_Q^*\rangle$ and consider the functions

$$f := \sum_{Q\in\mathscr{D}} |Q|^{1/2}\eta_Q x_Q h_Q^\alpha \in S_{00}(\mathscr{D};X), \quad g := \sum_{Q\in\mathscr{D}} |Q|^{1/2} y_Q^* h_Q^\gamma \in S_{00}(\mathscr{D};Y^*).$$

Then

$$\begin{aligned}\mathfrak{H}_\lambda^{\alpha\gamma} f &= \sum_{Q\in\mathscr{D}} |Q|^{1/2}\eta_Q\lambda_Q x_Q h_Q^\gamma, \\ \langle \mathfrak{H}_\lambda^{\alpha\gamma} f, g\rangle &= \sum_{Q\in\mathscr{D}} |Q|\eta_Q\langle \lambda_Q x_Q, y_Q^*\rangle,\end{aligned}$$

and hence

$$\sum_{Q\in\mathscr{D}} |Q||\langle \lambda_Q x_Q, y_Q^*\rangle| \leqslant \|\mathfrak{H}_\lambda^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\|f\|_{L^p(\mathbb{R}^d;X)}\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)},$$

where

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^d;X)} &\leqslant \beta_{p,X}^-\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q|Q|^{1/2}\eta_Q x_Q h_Q^\alpha\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\ &= \beta_{p,X}^-\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)}\end{aligned}$$

by a similar equidistribution property as before. Similarly, we have

$$\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)} \leqslant \beta_{p',Y^*}^-\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q y_Q^* \mathbf{1}_Q\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)},$$

and combining the bounds, we have proved the first estimate in (12.4). □

Remark 12.1.12. Under stronger assumptions on the coefficients λ, one can improve the dependence on the UMD constants:

(1) If $X = Y$, $\alpha = \gamma$, and $\lambda \subseteq \mathbb{K} \cdot I_X$ is bounded, then $\mathfrak{H}_\lambda^{\alpha\alpha}$ extends to a bounded operator on $L^p(\mathbb{R}^d; X)$ of norm at most

$$\|\mathfrak{H}_\lambda^{\alpha\alpha}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leqslant \beta_{p,X} \|\lambda\|_\infty.$$

(2) If $\lambda \subseteq \mathscr{L}(X, Y)$ is R-bounded, then $\mathfrak{H}_\lambda^{\alpha\gamma}$ extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$ of norm at most

$$\|\mathfrak{H}_\lambda^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant \beta_{p,Y}^- \beta_{p,X}^+ \mathscr{E}\mathscr{R}_p(\lambda),$$

where a partial advantage over Theorem 12.1.11 comes from $\beta_{p,Y}^- \leqslant \beta_{p',Y^*}^+$.

Proof. (1): This is a restatement of the first estimate in Proposition 12.1.5.

(2): Since $(h_Q^\gamma)_{Q\in\mathscr{D}}$ is a martingale difference sequence, using the defining properties of various constants and the definition of $\mathscr{E}\mathscr{R}_p(\lambda)$ via Remark 12.1.7, we have

$$\begin{aligned}
&\Big\| \sum_{Q\in\mathscr{D}} \lambda_Q \langle f, h_Q^\alpha\rangle h_Q^\gamma \Big\|_{L^p(\mathbb{R}^d;Y)} \\
&\leqslant \beta_{p,Y}^- \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \lambda_Q \langle f, h_Q^\alpha\rangle h_Q^\gamma \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \\
&\leqslant \beta_{p,Y}^- \mathscr{E}\mathscr{R}_p(\lambda) \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle f, h_Q^\alpha\rangle h_Q^\alpha \Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\
&\leqslant \beta_{p,Y}^- \mathscr{E}\mathscr{R}_p(\lambda) \beta_{p,X}^+ \|f\|_{L^p(\mathbb{R}^d;X)},
\end{aligned}$$

where, in the last step, we used the second estimate in Proposition 12.1.5. □

Here is a nice class of examples of coefficients satisfying the dyadic R-boundedness condition:

Proposition 12.1.13. *Let Y be a UMD space and $p \in (1, \infty)$. Let $b \in L^\infty(\mathbb{R}^d; \mathscr{L}(X, Y))$, let $a = (a_Q)_{Q\in\mathscr{D}} \in \ell^\infty(\mathscr{D}; L^\infty(\mathbb{R}^d))$, and*

$$\lambda := (\lambda_Q)_{Q\in\mathscr{D}} := (\langle a_Q b\rangle_Q)_{Q\in\mathscr{D}}.$$

Then

$$\mathscr{E}\mathscr{R}_p((\langle a_Q b\rangle_Q)_{Q\in\mathscr{D}}) \leqslant \beta_{p,Y}^+ \|a\|_{\ell^\infty(L^\infty)} \|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))}$$

Thus, for $\alpha, \gamma \in \{0,1\}^d \backslash \{0\}$, the Haar multiplier $\mathfrak{H}_\lambda^{\alpha\gamma}$ extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$ of norm at most

$$\|\mathfrak{H}_\lambda^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant \beta_{p,Y}^- \beta_{p,Y}^+ \beta_{p,X}^+ \|a\|_{\ell^\infty(L^\infty)} \|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

Proof. The second claim is immediate from the first one in combination with Remark 12.1.12(2), so we concentrate on the first one. We may assume by scaling that $\|a_Q\|_{L^\infty(\mathbb{R}^d)} \leqslant 1$. Then

$$\begin{aligned}\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \langle a_Q b\rangle_Q x_Q \mathbf{1}_Q \Big\|_{L^p(\mathbb{R}^d;Y)} &= \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q \mathbb{E}_Q (a_Q b x_Q \mathbf{1}_Q) \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \beta^+_{p,Y} \Big\| b \sum_{Q\in\mathscr{D}} \varepsilon_Q x_Q \mathbf{1}_Q \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \beta^+_{p,Y} \|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))} \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q x_Q \mathbf{1}_Q \Big\|_{L^p(\mathbb{R}^d;Y)},\end{aligned}$$

where, in the first estimate, we applied Stein's inequality (Theorem 4.2.23) followed by Kahane's contraction principle with the scalar coefficients a_Q. □

The following result shows that result of Proposition 12.1.13 cannot be improved to usual R-boundedness; thus the notions $\mathscr{D}\mathscr{R}_p$ and $\mathscr{E}\mathscr{R}_p$ represent genuine relaxations:

Proposition 12.1.14. *For non-zero Banach spaces X and Y, the following are equivalent:*

(1) *X has type 2 and Y has cotype 2;*
(2) *for every $b \in L^\infty(0,1;\mathscr{L}(X,Y))$, the set $\{\langle b\rangle_Q : Q \in \mathscr{D}([0,1))\}$ is R-bounded;*
(3) *for every $b \in L^\infty(0,1;\mathscr{L}(X,Y))$, the function*

$$x \mapsto \mathscr{R}\Big(\big\{\langle b\rangle_Q : x \in Q \in \mathscr{D}([0,1))\big\}\Big)$$

is essentially bounded.

Proof. (1)⇒(2): For $b \in L^\infty(0,1;\mathscr{L}(X,Y))$, it is clear that the $\{\langle b\rangle_Q : Q \in \mathscr{D}([0,1))\}$ is uniformly bounded. Under the assumption (1), this implies R-boundedness by Proposition 8.6.1.

(2)⇒(3): This is clear.

(3)⇒(1): From the definition of R-boundedness, it is immediate that $\mathscr{R}(\mathscr{T}) = \sup(\{\mathscr{R}(\mathscr{F}) : \mathscr{F} \subseteq \mathscr{T} \text{ finite}\})$. So if some collection $\mathscr{T}$ is not R-bounded, it has finite subcollections $\mathscr{F}_n$ with $\mathscr{R}(\mathscr{F}_n) \to \infty$. Then the countable collection $\bigcup_{n=1}^\infty \mathscr{F}_n \subseteq \mathscr{T}$ also fails to be R-bounded.

If (1) is not satisfied, then Proposition 8.6.1 says that the unit ball of $\bar{B}_{\mathscr{L}(X,Y)}$ of $\mathscr{L}(X,Y)$ is not R-bounded. By what we just observed, this means that we can find a sequence $\{u_k\}_{k=0}^\infty \subseteq \bar{B}_{\mathscr{L}(X,Y)}$ that fails to be R-bounded. Let $v_k := \frac{4}{3}u_k - \frac{1}{3}u_{k+1}$ and

$$b := \sum_{j=0}^\infty v_j \mathbf{1}_{[4^{-j-1},4^{-j})}.$$

Then $b \in L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))$ and $\|b\|_\infty = \sup_k \|v_k\| \leqslant \frac{5}{3}\sup_k \|u_k\| = \frac{5}{3}$. Moreover,

$$\langle b\rangle_{[0,4^{-k})} = 4^k \sum_{j=k}^{\infty} \frac{3}{4} 4^{-j} v_j = 4^k \frac{3}{4} \Big(\sum_{j=k}^{\infty} 4^{-j} \frac{4}{3} u_j - \sum_{j=k}^{\infty} 4^{-j} \frac{1}{3} u_{j+1} \Big) = u_k.$$

Then for each n, we have

$$\begin{aligned}
&\|x \mapsto \mathscr{R}(\{\langle b\rangle_I : x \in I \in \mathscr{D}([0,1))\})\|_{L^\infty(0,1)} \\
&\geqslant \mathscr{R}(\{\langle b\rangle_I : [0,4^{-n}) \in I \in \mathscr{D}([0,1))\}) \geqslant \mathscr{R}(\{\langle b\rangle_{[0,4^{-k})}\}_{k=0}^n) = \mathscr{R}(\{u_k\}_{k=0}^n),
\end{aligned}$$

and hence

$$\begin{aligned}
\infty = \mathscr{R}(\{u_k\}_{k=0}^\infty) &= \sup_{n\in\mathbb{N}} \mathscr{R}(\{u_k\}_{k=0}^n) \\
&\leqslant \|x \mapsto \mathscr{R}(\{\langle b\rangle_I : x \in I \in \mathscr{D}([0,1))\})\|_{L^\infty(0,1)}.
\end{aligned}$$

Thus (3) fails, and by contraposition this proves the claimed implication. □

Comparison of $\mathscr{D}\mathscr{R}_p$ and $\mathscr{E}\mathscr{R}_p$

In the rest of this section, we make a further comparison of the two relaxed notions of R-boundedness from Definition 12.1.6.. When Y is a UMD space—an assumption that we make a good part of the time—, these notions turn out to be equivalent. The universal bound $\mathscr{D}\mathscr{R}_p(\lambda) \leqslant \mathscr{E}\mathscr{R}_p(\lambda)$ was already observed in Lemma 12.1.8. The reverse estimate could be achieved essentially by concatenating a couple of results that we have treated earlier in these volumes, but it turns out that a slightly sharper quantitative bound can be achieved by also revisiting their proofs to establish the following proposition:

Proposition 12.1.15. *Let Y be a UMD space and $p \in (1,\infty)$. Let $\mathscr{E}_0 := \{\varnothing, \Omega\}$ be the trivial σ-algebra of a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ supporting a Rademacher sequence $(\varepsilon_n)_{n=1}^N$, and $(\mathscr{F}_n)_{n=1}^N$ be a σ-finite filtration of some measure space $(S, \mathscr{F}, \mu)$. Then, for all $f \in L^p(\Omega \times S; Y)$, we have*

$$\Big\| \sum_{n=1}^N \varepsilon_n \mathbb{E}(\varepsilon_n f | \mathscr{E}_0 \times \mathscr{F}_n) \Big\|_{L^p(\Omega\times S;Y)} \leqslant \beta^+_{p,Y} \|f\|_{L^p(\Omega\times S;Y)}.$$

Proof. Let $\mathscr{E}_n := \sigma(\varepsilon_1, \dots, \varepsilon_n)$ for $n = 1, \dots, N$. Then

$$\begin{aligned}
\mathbb{E}(\varepsilon_n f | \mathscr{E}_0 \times \mathscr{F}_n) &= \mathbb{E}(\mathbb{E}(\varepsilon_n f | \mathscr{E}_n \times \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_n) \\
&= \mathbb{E}(\varepsilon_n \mathbb{E}(f | \mathscr{E}_n \times \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_N),
\end{aligned}$$

where in the last step we note that for both $k \in \{n, N\}$, the conditional expectation of the function inside, given $\mathscr{E}_0 \times \mathscr{F}_k$, is obtained by simply integrating out the dependence on $\omega \in \Omega$. On the other hand, we have

$$\begin{aligned}
&\mathbb{E}(\varepsilon_n \mathbb{E}(f | \mathscr{E}_{n-1} \otimes \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_N) \\
&= \mathbb{E}(\mathbb{E}(\varepsilon_n \mathbb{E}(f | \mathscr{E}_{n-1} \times \mathscr{F}_n) | \mathscr{E}_{n-1} \times \mathscr{F}_N) | \mathscr{E}_0 \times \mathscr{F}_N) \\
&= \mathbb{E}(\mathbb{E}(\varepsilon_n | \mathscr{E}_{n-1} \times \mathscr{F}_N) \mathbb{E}(f | \mathscr{E}_{n-1} \times \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_N) \\
&= \mathbb{E}(0 \cdot \mathbb{E}(f | \mathscr{E}_{n-1} \times \mathscr{F}_n) | \mathscr{E}_0 \times \mathscr{F}_N) = 0.
\end{aligned}$$

Thus

$$\begin{aligned}\mathbb{E}(\varepsilon_n f|\mathscr{E}_0\times\mathscr{F}_n) &= \mathbb{E}(\varepsilon_n[\mathbb{E}(f|\mathscr{E}_n\times\mathscr{F}_n)-\mathbb{E}(f|\mathscr{E}_{n-1}\times\mathscr{F}_n)]|\mathscr{E}_0\times\mathscr{F}_N)\\ &= \mathbb{E}(\varepsilon_n[\mathbb{E}(f|\mathscr{G}_{2n})-\mathbb{E}(f|\mathscr{G}_{2n-1})]|\mathscr{E}_0\times\mathscr{F}_N)\\ &= \mathbb{E}(\varepsilon_n d_{2n}|\mathscr{E}_0\times\mathscr{F}_N),\end{aligned}$$

where

$$d_k := \begin{cases}\mathbb{E}(f|\mathscr{G}_k)-\mathbb{E}(f|\mathscr{G}_{k-1}), & k=2,\dots,2N,\\ \mathbb{E}(f|\mathscr{G}_1), & k=1,\end{cases}$$

are martingale differences relative to a filtration $(\mathscr{G}_k)_{k=1}^{2N}$ on $\Omega\times S$ defined by

$$\mathscr{G}_{2n} := \mathscr{E}_n\times\mathscr{F}_n, \qquad \mathscr{G}_{2n-1} := \mathscr{E}_{n-1}\times\mathscr{F}_n.$$

Then, noting that $\mathbb{E}(\ |\mathscr{E}_0\times\mathscr{F}_N)$ is constant in $\omega\in\Omega$, and denoting by $(\varepsilon'_k)_{k=1}^{2N}$ another Rademacher sequence on some $(\Omega',\mathscr{A}',\mathbb{P}')$, we have

$$\begin{aligned}&\Big\|\sum_{n=1}^N \varepsilon_n\mathbb{E}(\varepsilon_n f|\mathscr{E}_0\times\mathscr{F}_n)\Big\|_{L^p(\Omega\times S;Y)}\\ &= \Big\|\sum_{n=1}^N \varepsilon'_{2n}\mathbb{E}(\varepsilon_n d_{2n}|\mathscr{E}_0\times\mathscr{F}_N)\Big\|_{L^p(\Omega'\times\Omega\times S;Y)}\\ &= \Big\|\mathbb{E}(\sum_{n=1}^N \varepsilon'_{2n}\varepsilon_n d_{2n}|\mathscr{E}_0\times\mathscr{F}_N)\Big\|_{L^p(\Omega'\times\Omega\times S;Y)}\\ &\leqslant \Big\|\sum_{n=1}^N \varepsilon'_{2n}\varepsilon_n d_{2n}\Big\|_{L^p(\Omega'\times\Omega\times S;Y)} = \Big\|\sum_{n=1}^N \varepsilon'_{2n} d_{2n}\Big\|_{L^p(\Omega'\times\Omega\times S;Y)}\\ &\leqslant \Big\|\sum_{k=1}^{2N} \varepsilon'_k d_k\Big\|_{L^p(\Omega'\times\Omega\times S;Y)} \leqslant \beta^+_{p,Y}\Big\|\sum_{k=1}^{2N} d_k\Big\|_{L^p(\Omega\times S;Y)}\\ &= \beta^+_{p,Y}\|\mathbb{E}(f|\mathscr{G}_{2N})\|_{L^p(\Omega\times S;Y)} \leqslant \beta^+_{p,Y}\|f\|_{L^p(\Omega\times S;Y)},\end{aligned}$$

where, in the four estimates, we applied the contractivity of conditional expectation on L^p, Kahane's contraction principle with coefficients $\{0,1\}$, the definition of the UMD constant $\beta^+_{p,Y}$, and again the contractivity of conditional expectation on L^p. □

Remark 12.1.16. Proposition 12.1.15 is a simultaneous generalisation of Stein's inequality (Theorem 4.2.23),

$$\Big\|\sum_{n=1}^N \varepsilon_n\mathbb{E}(f_n|\mathscr{F}_n)\Big\|_{L^p(\Omega\times S;Y)} \leqslant \beta^+_{p,Y}\Big\|\sum_{n=1}^N \varepsilon_n f_n\Big\|_{L^p(\Omega\times S;Y)}, \tag{12.6}$$

for all $f_n\in L^p(S;Y)$, and the K-convexity inequality for UMD spaces (Proposition 4.3.10),

$$\Big\| \sum_{n=1}^{N} \varepsilon_n \mathbb{E}(\varepsilon_n f) \Big\|_{L^p(\Omega;Y)} \leqslant K_{p,Y} \|f\|_{L^p(\Omega;Y)}, \quad K_{p,Y} \leqslant \beta^+_{p,Y}, \tag{12.7}$$

for all $f \in L^p(\Omega;Y)$.

Namely, (12.6) is obtained from Proposition 12.1.15 by taking $f = \sum_{k=1}^{N} \varepsilon_k \otimes f_k$, in which case $\mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F}_n) = \mathbb{E}(f_n|\mathscr{F}_n)$, while (12.7) is the special case where $S = \{s\}$ is a singleton, or in other words f is independent of $s \in S$. Moreover, Proposition 12.1.15 shows that (12.7) holds equally well with real or complex Rademacher variable ε_n, provided only that we use the UMD constant $\beta^-_{p,Y}$ defined in terms of the same variables; in contrast, the proof of Proposition 4.3.10 was written for the real Rademacher variables r_n and made some explicit (although not essential) use of this choice.

Qualitatively, Proposition 12.1.15 could also be derived from the said two results, but with a quantitatively weaker conclusion; namely,

$$\begin{aligned}
&\Big\| \sum_{n=1}^{N} \varepsilon_n \mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F}_n) \Big\|_{L^p(\Omega\times S;Y)} \\
&= \Big\| \sum_{n=1}^{N} \varepsilon'_n \mathbb{E}(\mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F})|\mathscr{E}_0 \times \mathscr{F}_n) \Big\|_{L^p(\Omega'\times\Omega\times S;Y)} \\
&\leqslant \beta^+_{p,Y} \Big\| \sum_{n=1}^{N} \varepsilon'_n \mathbb{E}(\varepsilon_n f|\mathscr{E}_0 \times \mathscr{F}) \Big\|_{L^p(\Omega'\times\Omega\times S;Y)} \\
&= \beta^+_{p,Y} \Big\| \sum_{n=1}^{N} \varepsilon_n \mathbb{E}(\varepsilon_n f|\mathscr{E}_0) \Big\|_{L^p(S;L^p(\Omega;Y))} \leqslant \beta^+_{p,Y} K_{p,Y} \|f\|_{L^p(S;L^p(\Omega;Y))},
\end{aligned}$$

using the K-convexity inequality in $L^p(\Omega;Y)$, pointwise at each $s \in S$, in the last step.

Corollary 12.1.17. *If Y is a UMD space and $\lambda = (\lambda_Q)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(X,Y)$, then*

$$\mathscr{D}\mathscr{R}_p(\lambda) \leqslant \mathscr{E}\mathscr{R}_p(\lambda) \leqslant \beta^+_{p',Y^*} \mathscr{D}\mathscr{R}_p(\lambda).$$

Proof. We already proved the first inequality in Lemma 12.1.8. For the second inequality, we first note that, by Fubini's theorem,

$$\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q z_Q \mathbf{1}_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} = \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)} z_Q \mathbf{1}_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}, \tag{12.8}$$

where $n(Q) \in \mathbb{Z}$ is such that $Q \in \mathscr{D}_n$: This is because, pointwise at each $s \in \mathbb{R}^d$, there is exactly one dyadic $Q \ni s$ of each generation $n \in \mathbb{Z}$, and we can replace the sequence $(\varepsilon_Q)_{Q\ni s}$ by the equidistributed sequence $(\varepsilon_n)_{n\in\mathbb{Z}} = (\varepsilon_{n(Q)})_{Q\ni s}$. For $z_Q = \lambda_Q x_Q$ and $Z = Y$, we then dualise the right-hand side of (12.8) with $G \in L^{p'}(\Omega \times \mathbb{R}^d; Y^*)$:

$$\begin{aligned}\Big|\Big\langle \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)}\lambda_Q x_Q \mathbf{1}_Q, G\Big\rangle\Big| &= \Big|\sum_{Q\in\mathscr{D}_n} \Big\langle \lambda_Q x_Q, \langle \mathbb{E}(\varepsilon_{n(Q)}G)\rangle_Q\Big\rangle |Q|\Big| \\ &\leqslant \mathscr{D}\mathscr{R}_p(\lambda)\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)} x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \times \\ &\quad \times \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)} \langle \mathbb{E}(\varepsilon_{n(Q)}G)\rangle_Q \mathbf{1}_Q\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}.\end{aligned} \tag{12.9}$$

In the $L^{p'}(\Omega\times\mathbb{R}^d;Y^*)$ norm on the right, we write

$$\begin{aligned}\sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)}\langle \mathbb{E}(\varepsilon_{n(Q)}G)\rangle_Q \mathbf{1}_Q &= \sum_{n\in\mathbb{Z}} \varepsilon_n \sum_{Q\in\mathscr{D}_n} \mathbb{E}(\mathbb{E}(\varepsilon_n G)|\sigma(\mathscr{D}_n))\mathbf{1}_Q \\ &= \sum_{n\in\mathbb{Z}} \varepsilon_n \mathbb{E}(\mathbb{E}(\varepsilon_n G)|\sigma(\mathscr{D}_n)) = \sum_{n\in\mathbb{Z}} \varepsilon_n \mathbb{E}(\varepsilon_n G|\{\varnothing,\Omega\}\times\sigma(\mathscr{D}_n)).\end{aligned}$$

Thus, by a direct application of Proposition 12.1.15 in the UMD space Y^* in place of Y, it follows that

$$\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)}\langle \mathbb{E}(\varepsilon_{n(Q)}G)\rangle_Q \mathbf{1}_Q\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)} \leqslant \beta^+_{p',Y^*}\|G\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}.$$

Substituting back to (12.9), it follows by duality that

$$\begin{aligned}&\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)}\lambda_Q x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\ &\qquad \leqslant \beta^+_{p',Y^*}\mathscr{D}\mathscr{R}_p(\lambda)\Big\| \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)} x_Q \mathbf{1}_Q\Big\|_{L^p(\Omega\times\mathbb{R}^d;X)},\end{aligned}$$

and we can replace $n(Q)$ by Q on both sides according to (12.8) to obtain the claimed result. □

12.1.b Nested collections of unions of dyadic cubes

Before proceeding to more complicated dyadic singular integrals, we devote this intermediate section to elementary, although not entirely trivial, geometric–combinatorial considerations related to the dyadic cubes. Collecting the relevant auxiliary results here for easy reference will allow our subsequent analysis to flow with a nice tempo without annoying interruptions.

Definition 12.1.18 (Nestedness). *We say that two set E, F are* nested *if $E\cap F\in\{\varnothing, E, F\}$. A collection $\mathscr{E}$ of sets is called nested if any $E, F\in\mathscr{E}$ have this property.*

The fact that the collection $\mathscr{D}$ of dyadic cubes enjoys this property underlies many considerations that we have encountered in these volumes.

In the dyadic analysis of a singular integral operators that we undertake in this section, we will also need to deal with unions $Q_1 \cup Q_2$ of two dyadic cubes of the same size. A moment's thought confirms that two such sets will not be nested in general, yet quite frequently they still enjoy this property. Accordingly, a key to the related considerations will be the decomposition of collections of pairs of dyadic cubes into controllably many subcollections, where the nestedness of the unions $Q_1 \cup Q_2$ is valid.

Definition 12.1.19 (Strong nestedness). *Let $Q_1 \cup Q_2$ and $R_1 \cup R_2$ be two unions of some $Q_i, R_i \in \mathscr{D}$ with $\ell(Q_1) = \ell(Q_2)$ and $\ell(R_1) = \ell(R_2)$. We say that E and F are* strongly nested *if they are equal, or disjoint, or one of them, say $Q_1 \cup Q_2$, is contained not just in $R_1 \cup R_2$ but in a dyadic child of R_1 or R_2. A collection of such unions is called strongly nested if any two of its members have this property.*

Note that the dyadic cubes themselves, contained in this definition as a degenerate case with $Q_2 = Q_1$, clearly satisfy this strong nestedness. This notion is relevant for considerations dealing with Haar functions which, as we recall, are constant on the dyadic children of their supporting dyadic cubes; thus, if $Q_1 \cup Q_2$ and $R_1 \cup R_2$ are strongly nested, unequal but intersecting, then the smaller union is entirely contained in a set of constant value for any Haar function related to the larger union.

Our first (relatively simple) decomposition into strongly nested subcollections is the following:

Lemma 12.1.20. *Suppose that, for some $n \in \mathbb{N}$:*

(a) *$\mathscr{F} \subseteq \mathscr{D}$ is a finite subcollection;*
(b) *$\phi : \mathscr{F} \to \mathscr{D}$ is an injection with $\ell(\phi(Q)) = \ell(Q)$ for all $Q \in \mathscr{F}$;*
(c) *if $Q, R \in \mathscr{F}$ and $\ell(Q) < \ell(R)$, then $\ell(Q) < 2^{-n}\ell(R)$;*

and

$$\phi(Q) \subseteq Q^{(n)} \qquad \forall\, Q \in \mathscr{F}. \tag{12.10}$$

Then $\mathscr{F}$ can be partitioned into 3 subcollections $\mathscr{F}_i$ such that each collection $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i\}$ is strongly nested.

Proof. Step 1 – Let all assumptions of the lemma be in force until further notice. For each $Q \in \mathscr{F} \cup \phi(\mathscr{F})$, we define a *label* $r(Q) \in \{0, 1, 2\}$ such that $r(Q) \neq r(\phi(Q))$ for every $Q \in \mathscr{F}$ unless $\phi(Q) = Q$. This ensures that $Q \cup \phi(Q)$ and $R \cup \phi(R)$ are disjoint whenever $Q, R \in \mathscr{F}$ are two different cubes with $r(Q) = r(R)$ and $\ell(Q) = \ell(R)$.

Indeed, $Q \neq R$ implies $\phi(Q) \neq \phi(R)$. Since different dyadic cubes of equal size are disjoint, this implies that $Q \cap R = \varnothing = \phi(Q) \cap \phi(R)$. If $\phi(Q) = Q$ or $\phi(R) = R$, this already shows that $Q \cup \phi(Q)$ and $R \cup \phi(R)$ are disjoint. If $\phi(Q) \neq Q$ and $\phi(R) \neq R$, then $r(\phi(Q)) \neq r(Q) = r(R)$ implies $\phi(Q) \neq R$ and similarly $\phi(R) \neq Q$. By equal size again, this implies that $\phi(Q) \cap R = \varnothing = Q \cap \phi(R)$, giving the (strong) nestedness when $\ell(Q) = \ell(R)$.

To define such $r(R)$, let us denote $\phi^{\circ 0}(Q) = Q$, $\phi^{\circ k}(Q) = \phi(\phi^{\circ(k-1)}(Q))$ for $k \geqslant 1$. An *orbit* of ϕ is a set $\{\phi^{\circ k}(Q) : k = 0, \ldots, K\}$, where $Q \in \mathscr{F}$ and either $\phi^{\circ(K+1)}(Q) = Q$ (in this case the orbit is *cyclic*), or $Q \notin \phi(\mathscr{F})$ and $\phi^{\circ K}(Q) \notin \mathscr{F}$. For all $Q \in \mathscr{F} \cup \phi(\mathscr{F})$, we define $r(Q) \in \{0, 1, 2\}$ by alternating the values 0 and 1 on both non-cyclic orbits and cyclic orbits of even length, and in addition using the value 2 once on cyclic orbits of odd length. In this way, we ensure that $r(Q) \neq r(\phi(Q))$ for any $Q \in \mathscr{F}$ unless $Q = \phi(Q)$.

Step 2 – It remains to check the strong nestedness in the case of $Q, R \in \mathscr{F}$ with $\ell(Q) < \ell(R)$, hence $\ell(Q) < 2^{-n}\ell(R)$. If $Q \cup \phi(Q)$ intersect $R \cup \phi(R)$, then one of $P \in \{Q, \phi(Q)\}$ intersects one of $S \in \{R, \phi(R)\}$. Since $\ell(P) < 2^{-n}\ell(S)$ and the cubes are dyadic, this implies that $P^{(n)} \subsetneq S$. Since $\phi(Q) \subseteq Q^{(n)}$, we have $\phi(Q)^{(n)} = Q^{(n)}$, and hence $Q \cup \phi(Q) \subseteq Q^{(n)} \subsetneq S$, confirming strong nestedness in the case of $\ell(Q) < \ell(R)$. □

In the lack of (12.10), the situation is somewhat more complicated. Suitable substitute conditions are provided in the following:

Lemma 12.1.21. *Assume conditions* (a) *through* (c) *as well as:*

(d) $\phi(Q) \subseteq 3Q^{(n)}$ *for all* $Q \in \mathscr{F}$;
(e) $3Q \subseteq Q^{(n)}$ *for all* $Q \in \mathscr{F} \cup \phi(\mathscr{F})$.

Then $\mathscr{F}$ *can be partitioned into nine subcollections* $\mathscr{F}_i$ *such that each collection* $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i\}$ *is strongly nested.*

Proof. Step 1 – We define the label $r(Q) \in \{0, 1, 2\}$ exactly as in the proof of Lemma 12.1.20 to ensure that $r(Q) \neq r(\phi(Q))$ unless $Q = \phi(Q)$. This gives the nestedness of the sets $Q \cup \phi(Q)$ for cubes of a fixed sidelength, as before.

Step 2 – We claim that, for each $Q \in \mathscr{F} \cup \phi(\mathscr{F})$, there can be at most one $R \in \mathscr{F} \cup \phi(\mathscr{F})$ such that

$$Q \subsetneq R, \qquad 3Q^{(n)} \not\subseteq R_Q, \tag{12.11}$$

where R_Q is the unique dyadic child of R that contains $Q \subsetneq R$.

In fact, let R be as above, and $Q \subsetneq R \subsetneq S \in \mathscr{F} \cup \phi(\mathscr{F})$, thus $Q^{(n)} \subsetneq R$, $R^{(n)} \subsetneq S$ by (c). By (e) applied to the cube R, we then have $3Q^{(n)} \subseteq 3R \subseteq R^{(n)} \subseteq S_R$, so indeed S will not satisfy the condition (12.11) that R does, and this proves the uniqueness of R.

Step 3 – For each $P \in \mathscr{F}$, we define a second label $s(P) \in \{0, 1, 2\}$ in such a way that if $(r(P), s(P)) = (r(S), s(S))$, then (12.11) does not hold for either $R = S$ or $R = \phi(S)$. This will ensure strong nestedness for the subcollection with constant pairs of labels $(r(P), s(P))$.

Indeed, suppose that $P, S \in \mathscr{F}$ have $(r(P), s(P)) = (r(S), s(S))$ where $\ell(P) < \ell(S)$ and $P \cup \phi(P)$ intersects $S \cup \phi(S)$. Hence (at least) one of $Q \in \{P, \phi(P)\}$ intersects (at least) one of $R \in \{S, \phi(S)\}$ and thus $Q \subsetneq R$. By (d) and the failure of (12.11), we have $P \cup \phi(P) \subseteq (1 + 2^{n+1}Q) \subseteq R_Q$.

The required second label $s(P)$ is defined for each $P \in \mathscr{F}$ as follows. For all $P \in \mathscr{F}$ of maximal size, let $s(P) := 0$. Recursively, we proceed to the unlabelled cubes $P \in \mathscr{F}$ of maximal size. For these cubes, we first check whether (12.11) occurs with either $Q = P$ or $Q = \phi(P)$, and some $R \in \mathscr{F} \cup \phi(\mathscr{F})$. It could happen that $R \in \mathscr{F}$, or $R = \phi(S)$ with $S \in \mathscr{F}$, or both. We then require that $s(P)$ is chosen so that $(r(P), s(P)) \notin \{(r(R), s(R)), (r(S), s(S))\}$. If $S = R$, this is clearly one restriction on $S(P)$. But if $S \neq R = \phi(S)$, then $r(R) \neq r(S)$ by the alternating choice of r along the orbits, and we still get at most one restriction of the possible value of $s(P)$. Since different R and S may arise from the case $Q = P$ and $Q = \phi(P)$ we get altogether at most two restrictions on $s(P)$, and we can declare that $s(P)$ is the smallest remaining number in $\{0, 1, 2\}$. □

The next result relaxes the assumptions even further, at the cost of complicating the conclusions:

Lemma 12.1.22. *Assume conditions* (a) *through* (d). *Then $\mathscr{F}$ can be partitioned into 144 subcollections $\mathscr{F}_i$, and on each of them we have injections $\phi_{i,j} : \mathscr{F}_i \to \mathscr{D}$, $j = 0, 1, 2, 3$, where $\phi_{i,0}(Q) = Q$ and $\phi_{i,3}(Q) = \phi(Q)$ such that each collection*

$$\{\phi_{i,j}(Q) \cup \phi_{i,j+1}(Q) : Q \in \mathscr{F}_i\} \tag{12.12}$$

is strongly nested.

Proof. The idea is to combine the special cases treated in the two previous Lemmas 12.1.20 and 12.1.21, which had the additional assumptions (12.10) and (e), respectively; neither is assumed now.

For every $R \in \mathscr{D}$, consider the 2^{nd} cubes $Q \in \mathscr{D}$ with $Q^{(n)} = R$. Among them, there are $(2^n - 2)^d$ off-boundary cubes Q with $3Q \subseteq R$, while the number of boundary cubes is then

$$2^{nd} - (2^n - 2)^d = 2^{nd}[1 - (1 - 2^{1-n})^d] \leqslant 2^{nd} \cdot 2^{1-n} d \leqslant \frac{1}{2} 2^{nd}$$

if $n \geqslant \log_2(4d)$. When this is the case, we can define a permutation $\psi : \mathscr{D} \to \mathscr{D}$ with $\ell(\psi(Q)) = \ell(Q)$, $\psi(Q) \subseteq Q^{(n)}$ (as in (12.10)) such that $\psi(Q)$ is an off-boundary cube in $Q^{(n)}$ whenever Q is a boundary cube in $Q^{(n)}$.

Let us first divide $\mathscr{F}$ into four subcollection $\mathscr{F}_{u,v}$, where $u, v \in \{0, 1\}$, so that $Q \in \mathscr{F}_{u,v}$ is a boundary cube in $Q^{(n)}$ if and only if $u = 1$, whereas $\phi(Q)$ is a boundary cube in $\psi(Q)^{(n)}$ if and only if $v = 1$.

Case $\mathscr{F}_{0,0}$: By Lemma 12.1.21, we can divide $\mathscr{F}_{0,0}$ into nine subcollections $\mathscr{F}_i$ such that $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i\}$ is strongly nested. Letting $\phi_{i,1} = \phi_{i,2} = \phi_{i,3} = \phi$ in this case, we trivially have the strong nestedness of $\{\phi_{i,j}(Q) \cup \phi_{i,j+1}(Q) : Q \in \mathscr{F}_i\}$ for $j = 1, 2$ (since the collection is simply $\phi(\mathscr{F}_i) \subseteq \mathscr{D}$ in this case.

Case $\mathscr{F}_{0,1}$: On the collection $\mathscr{F}_{0,1}$, we consider the map $\psi \circ \phi$ and observe that it also satisfies (d); indeed, $\phi(Q) \subseteq 3Q^{(n)}$ lies inside one of the dyadic neighbours of $Q^{(n)}$, and ψ keeps it inside this same nth generation ancestor. Since $\phi(Q)$ is a boundary cube in $\phi(Q)^{(n)}$ for $Q \in \mathscr{F}_{0,1}$ by definition of this collection, $\psi(\phi(Q))$ is off-boundary in $\phi(Q)^{(n)} = \psi(\phi(Q))^{(n)}$ by definition of ψ, and hence $(\mathscr{F}_{0,1}, \psi \circ \phi)$ also satisfies (e) in place of $(\mathscr{F}, \phi)$. Then Lemma 12.1.21 shows that $\mathscr{F}_{0,1}$ can be divided into nine subcollections $\mathscr{F}'_a$ such that each $\{Q \cup \psi(\phi(Q)) : Q \in \mathscr{F}'_a\}$ is strongly nested. On the other hand, we can write

$$\{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}_{0,1}\} = \{R \cup \psi(R) : R \in \phi(\mathscr{F}_{0,1})\}.$$

Here $(\mathscr{F}_{0,1}, \psi)$ satisfies the assumptions of Lemma 12.1.20, and hence $\phi(\mathscr{F}_{0,1})$ can be divided into three subcollections $\mathscr{G}_b$ such that $\{R \cup \psi(R) : R \in \mathscr{G}_b)\}$ is strongly nested. This since ϕ is injective, this induces a decomposition of $\mathscr{F}_{0,1}$ into three subcollections where $\mathscr{F}''_b$ such that $\{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}''_b\}$ is strongly nested. Then, defining $\mathscr{F}_i = \mathscr{F}'_a \cap \mathscr{F}''_b$ for $i = (a, b)$, we find that both

$$\{Q \cup \psi(\phi(Q)) : Q \in \mathscr{F}_i\}, \qquad \{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}_i\}$$

are strongly nested, and there is in total $9 \cdot 3$ such collections $\mathscr{F}_i$ decomposing $\mathscr{F}_{0,1}$. So taking $\phi_{i,1} = \psi \circ \phi$ and $\phi_{i,2} = \phi_{i,3}$, we have the strong nestedness of the collections in (12.12), the case $j = 2$ for trivial reasons as in case $\mathscr{F}_{0,0}$.

Case $\mathscr{F}_{1,0}$: Similarly, on the collection $\mathscr{F}_{1,0}$, Lemma 12.1.20 applies to the mapping ψ to provide three subcollection $\mathscr{F}'_a$ such that $\{Q \cup \psi(Q) : Q \in \mathscr{F}'_a\}$ is strongly nested. And Lemma 12.1.21 applies to $(\psi(\mathscr{F}_{1,0}), \phi \circ \psi^{-1})$ to provide nine subcollections $\mathscr{F}''_b$ such that $\{\psi(Q) \cup \phi(Q) : Q \in \mathscr{F}''_b\}$ is strongly nested. So altogether we have $3 \cdot 9$ subcollection $\mathscr{F}_i = \mathscr{F}'_a \cap \mathscr{F}''_b$ such that

$$\{Q \cup \psi(Q) : Q \in \mathscr{F}_i\}, \qquad \{\psi(Q) \cup \phi(Q) : Q \in \mathscr{F}_i\}$$

are strongly nested. We can hence define $\phi_{i,1} = \psi$, $\phi_{i,2} = \phi_{i,3} = \phi$ to get the claimed conclusions.

Case $\mathscr{F}_{1,1}$: Finally, on the collection $\mathscr{F}_{1,1}$, Lemma 12.1.20 applies to both $(\mathscr{F}_{1,1} : \psi)$ and to $(\psi \circ \phi(\mathscr{F}_{1,1} : \psi^{-1})$ to provide three subcollections $\mathscr{F}'_a$ and three other $\mathscr{F}''_b$ such that $\{Q \cup \psi(Q) : Q \in \mathscr{F}'_a\}$ and $\{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}''_b\}$ are strongly nested. And we check that Lemma 12.1.21 applies to $(\psi(\mathscr{F}_{1,1}), \psi \circ \phi \circ \psi^{-1})$ to provide nine subcollections $\mathscr{F}'''_c$ such $\{\psi(Q) \cup \psi(\phi(Q)) : Q \in \mathscr{F}'''_c\}$ is strongly nested. Then with $\mathscr{F}_i = \mathscr{F}'_a \cap \mathscr{F}''_b \cap \mathscr{F}'''_c$ we obtain $3^2 \cdot 9$ subcollections such that $\{Q \cup \psi(Q) : Q \in \mathscr{F}_i\}$, $\{\psi(Q) \cup \psi(\phi(Q)) : Q \in \mathscr{F}_i\}$, and $\{\psi(\phi(Q)) \cup \phi(Q) : Q \in \mathscr{F}_i\}$ are strongly nested, and we can define $\phi_{i,1} = \psi$, $\phi_{i,2} = \psi \circ \phi$, $\phi_{i,3} = \phi$ in this case.

In total we have divided $\mathscr{F}$ into $9 + 2 \cdot 9 \cdot 3 + 9 \cdot 3^2 = 144$ subcollections $\mathscr{F}_i$ with required properties. □

Another variant of the conclusion with the same assumptions is as follows:

Lemma 12.1.23. *Assume conditions* (a) *through* (d)*. Then* $\mathscr{F}$ *can be partitioned into* 3^{3d+1} *subcollections* $\mathscr{F}_i$ *such that each collection*

$$\{Q^{[m(i)]} \cup \phi(Q)^{[m(i)]} : Q \in \mathscr{F}_i\} \subseteq \mathscr{D}^{m(i);3}$$

is strongly nested, where

(1) $\mathscr{D}^{m(i);3}$ *is one of the dilated dyadic systems from Proposition 11.3.11;*
(2) *for each* $P \in \mathscr{D}$*, we denote by* $P^{[m]}$ *the unique*

$$P^{[m]} \in \mathscr{D}^{m;3} \text{ with } P^{[m]} \supseteq P \text{ and } \ell(P^{[m]}) = 3\ell(P). \tag{12.13}$$

Proof. We have $Q \cup \phi(Q) \subseteq (1 + 2^{n+1})Q \subseteq 3Q^{(n)}$, where $Q^{(n)}$ is the nth generation dyadic ancestor of Q. Recall that the cubes $3R$, $R \in \mathscr{D}$, can be split into 3^d new dyadic-like systems $\mathscr{D}^{m;3}$ by Proposition 11.3.11. For each $Q \in \mathscr{F}$, let m_Q be the index such that $3Q^{(n)} \in \mathscr{D}^{m_Q;3}$, and let $Q' = Q^{[m_Q]}$, $Q'' = \phi(Q)^{[m_Q]}$ be as in (12.13). (Thus Q' is the three-fold expansions of one of the neighbours of Q; any of these contains Q, and exactly one of them belongs to the correct $\mathscr{D}^{m_Q;3}$; the same remark applies to Q'' and $\phi(Q)$ in place of Q' and Q.) Note that the same Q' can arise from 3^d different cubes Q, and the same Q'' from 3^d different $\phi(Q)$; however, by dividing $\mathscr{F}$ into 9^d subcollections $\mathscr{F}^a$, we ensure that Q is uniquely determined by Q', and $\phi(Q)$ by Q'', within each $\mathscr{F}^a$.

Let us then consider the collections $\mathscr{F}^{a,m} = \{Q' : Q \in \mathscr{F}^a, m_Q = m\} \subseteq \mathscr{D}^{m;3}$ for the 3^d different values of m. We can define $\Phi : \mathscr{F}^{a,m} \to \mathscr{D}^{m;3}$ by $\Phi(Q') = Q''$; this is well-defined since Q' uniquely determines Q, which determines $\phi(Q)$ and then Q''. The map Φ is also injective, since Q'' uniquely determines $\phi(Q)$, which (since ϕ is injective) determines Q and then Q'. Moreover, we have

$$\ell(\Phi(Q')) = \ell(Q'') = 3\ell(\phi(Q)) = 3\ell(Q) = \ell(Q').$$

Thus $\mathscr{F}^{a,m} \subseteq \mathscr{D}^{m;3}$ and Φ satisfy properties (a) and (b) in place of $\mathscr{F} \subseteq \mathscr{D}$ and ϕ, and the scale-separation property (c) is clearly inherited by Φ from ϕ. Moreover, the nth $\mathscr{D}^{m;3}$-ancestor of both $\Phi(Q') = Q''$ and Q' is clearly $3Q^{(n)}$ by construction, and hence Φ satisfies condition (12.10) of Lemma 12.1.20. The said lemma guarantees that $\mathscr{F}^{a,m}$ can be split into 3 subcollections $\mathscr{F}^{a,m}_j$, so that each

$$\{Q' \cup \Phi(Q') : Q' \in \mathscr{F}^{a,m}_j\} \subseteq \mathscr{D}^{m;3}$$

is strongly nested. Writing $i = (a, m, j)$, and defining

$$\mathscr{F}_i := \{Q \in \mathscr{F} : m_Q = m, Q^{[m]} \in \mathscr{F}^{a,m}_j\},$$

these are precisely the collections that we wanted to construct. Since a takes 9^d values, m takes 3^d values, and j takes 3 values, the number of these collections is $9^d \cdot 3^d \cdot 3 = 3^{3d+1}$, as claimed. □

Remark 12.1.24. In each of the Lemmas 12.1.20 through 12.1.23, we can drop assumption (c) at the cost of multiplying the required number of decomposing subcollections $\mathscr{F}_i$ by $n+1$.

Proof. For any $\mathscr{F} \subseteq \mathscr{D}$, consider the $n+1$ subcollection $\mathscr{F}^k := \{Q \in \mathscr{F} : \log_2 \ell(Q) \equiv k \mod (n+1)\}$. Each of these clearly satisfies (c). Moreover, any of the other properties (a) through (e) as well as (12.10), if valid for $\mathscr{F}$, is clearly inherited by each $\mathscr{F}^k$. Thus, if $\mathscr{F}$ satisfies the assumptions of any of these lemmas with the possible exception of (c), then each $\mathscr{F}^k$ satisfies all of the relevant assumptions, and the lemma in question provides a decomposition of $\mathscr{F}^k$ into some $\mathscr{F}_i^k$ with appropriate nestedness conditions. The required decomposition of the original $\mathscr{F}$ is then obtained simply as $\mathscr{F} = \bigcup_{k=0}^n \bigcup_i \mathscr{F}_i^k$, and clearly the number of collections in this decomposition is $n+1$ times as many as in the decompositions $\mathscr{F}^k = \bigcup_i \mathscr{F}_i^k$ given by the lemmas. □

12.1.c The elementary operators of Figiel

We will now study another family of dyadic singular integral operators with more complicated interactions between Haar functions at different locations. The first class of these operators combines the action of a Haar multiplier with a translation of the Haar functions. One might be tempted to refer to such operators as dyadic or Haar "shifts", but this name has been adopted for a somewhat different class of operators in the literature.

While the parameter n attached with these operators may appear like a technical detail at this point, it is essential for subsequent applications that one obtains a good dependence on n.

Theorem 12.1.25 (Figiel). *Let $\phi : \mathscr{D} \to \mathscr{D}$ be an injection with $\ell(\phi(Q)) = \ell(Q)$ and $\phi(Q) \subseteq 3Q^{(n)}$ for some $n \in \mathbb{N}$. Let X and Y be a UMD spaces, and let $p \in (1,\infty)$. Let $\lambda = (\lambda_Q)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(X,Y)$. Consider the mapping*

$$T_{\phi\lambda}^{\alpha\gamma} f = \sum_{Q\in\mathscr{D}} \lambda_Q \langle f, h_Q^\alpha \rangle h_{\phi(Q)}^\gamma, \tag{12.14}$$

initially from $S_{00}(\mathscr{D};X)$ to $S_{00}(\mathscr{D};Y)$. Let $A_d := 6\cdot(81)^d$.

(0) *If λ is R-bounded, or more generally if*

$$\min\{\mathscr{E}\mathscr{R}_p(\lambda), \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}})\} < \infty, \quad (\lambda_{\phi^{-1}})_Q := \begin{cases} \lambda_{\phi^{-1}(Q)}, & Q \in \phi(\mathscr{D}), \\ 0, & \text{else}, \end{cases}$$

then $T_{\phi\lambda}^{\alpha\gamma}$ extends boundedly from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$, with norm

$$\|T_{\phi\lambda}^{\alpha\gamma}\| := \|T_{\phi\lambda}^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant A_d(n+1)\beta_{p,Y}^- \beta_{p,X}^+ C(X,Y,p;\lambda),$$

where

$$\begin{aligned} C(X,Y,p;\lambda) &:= \min\{\beta_{p,X}^+ \cdot \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}), \quad \beta_{p,Y}^+ \cdot \mathscr{E}\mathscr{R}_p(\lambda)\} \\ &\leqslant \min\{\beta_{p,X}^+, \beta_{p,Y}^+\}\mathscr{R}_p(\lambda); \end{aligned}$$

(1) *If, in addition, Y has type $t \in [1,p]$ and X has cotype $q \in [p,\infty]$, or one of them has both, then we also have the estimate*

$$\|T_{\phi\lambda}^{\alpha\gamma}\| \leqslant A_d(n+1)^{1/t-1/q}\beta^-_{p,Y}\beta^+_{p,X}C(X,Y,p,q,t;\lambda)$$

where

$$\begin{aligned} C(X,Y,p,q,t;\lambda) := \min\Big\{&\tau_{t,X;p}\cdot\beta^+_{p,X}\cdot c_{q,X;p}\cdot\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}),\\ &\tau_{t,Y;p}\cdot\beta^+_{p,X}\cdot c_{q,X;p}\cdot\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}),\\ &\tau_{t,Y;p}\cdot\beta^+_{p,Y}\cdot c_{q,X;p}\cdot\mathscr{E}\mathscr{R}_p(\lambda),\\ &\tau_{t,Y;p}\cdot\beta^+_{p,Y}\cdot c_{q,Y;p}\cdot\mathscr{E}\mathscr{R}_p(\lambda)\Big\}\\ \leqslant C(X,Y,&p,q,t)\cdot\mathscr{R}_p(\lambda), \end{aligned}$$

and

$$\begin{aligned} C(X,Y,p,q,t) := \min\Big\{&\tau_{t,X;p}\beta^+_{p,X}c_{q,X;p},\ \tau_{t,Y;p}\beta^+_{p,X}c_{q,X;p},\\ &\tau_{t,Y;p}\beta^+_{p,Y}c_{q,X;p},\ \tau_{t,Y;p}\beta^+_{p,Y}c_{q,Y;p}\Big\}. \end{aligned} \tag{12.15}$$

(2) *If, in addition, $\lambda_Q \neq 0$ only when $\phi(Q) \subseteq Q^{(n)}$, then we have the alternative norm estimate*

$$\|T_{\phi\lambda}^{\alpha\gamma}\| \leqslant 3\cdot\beta_{p,Y}\beta^+_{p,X}\min\{c_{q,X;p},c_{q,Y;p}\}(n+1)^{1/q'}\mathscr{E}\mathscr{R}_p(\lambda).$$

(3) *For all $f \in L^p(\mathbb{R}^d;X)$ and $g \in L^{p'}(\mathbb{R}^d;Y^*)$, the extended operator has the absolutely convergent representation*

$$\langle T_{\phi\lambda}^{\alpha\gamma}f,g\rangle = \sum_{Q\in\mathscr{D}}\Big\langle\lambda_Q\langle f,h_Q^\alpha\rangle,\langle g,h^\gamma_{\phi(Q)}\rangle\Big\rangle.$$

When $\|f\|_{L^p(\mathbb{R}^d;X)} \leqslant$ and $\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)} \leqslant 1$, the corresponding absolute value series is dominated by the same upper bounds as those given for $\|T_{\phi\lambda}^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}$ above.

Remark 12.1.26. (1) In the prominent special case that $X = Y$, we have

$$\begin{aligned} C(X,X,p,q,t;\lambda) &= C(X,X,p,q,t)\cdot\min\{\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}),\mathscr{E}\mathscr{R}_p(\lambda)\},\\ C(X,X,p,q,t) &= \tau_{t,X;p}\cdot\beta^+_{p,X}\cdot c_{q,X;p}. \end{aligned}$$

(2) Case (0) of Theorem 12.1.25 is a special case of (1) using the trivial type and cotype exponents $t = 1$, $q = \infty$ with corresponding constants equal to one. The role of non-trivial type and cotype is to relax the dependence on the parameter n. The estimate obtained in case (2) is not strictly comparable to the other two bounds; its main advantage over the other two is achieving a quadratic bound in terms of the UMD constants, in contrast to the cubic bound in the other cases.

(3) Recalling the Haar multipliers $\mathfrak{H}_\lambda^{\alpha\gamma}$ from Theorem 12.1.11, one can check that, for any $\theta \in \{0,1\}^d \setminus \{0\}$,

$$T_{\phi\lambda}^{\alpha\gamma} = T_{\phi\mathbf{1}}^{\theta\gamma} \circ \mathfrak{H}_\lambda^{\alpha\theta} = \mathfrak{H}_{\lambda_{\phi^{-1}}}^{\theta\gamma} \circ T_{\phi\mathbf{1}}^{\alpha\theta}$$

where $\mathbf{1}$ is the constant sequence of all ones. Hence, for the qualitative conclusion of Theorem 12.1.25, it would suffice to consider just $X = Y$ and $\lambda = \mathbf{1}$, and then combine this special case with Theorem 12.1.11; however, the reader will quickly realise that this approach would produce a higher power of the UMD constants in the quantitative conclusion.

Before going into the proof, let us still formulate a corollary in the important special case when $\phi : \mathscr{D} \to \mathscr{D}$ is a bijection:

Corollary 12.1.27. *Let $\phi : \mathscr{D} \to \mathscr{D}$ be a bijection with $\ell(\phi(Q)) = \ell(Q)$ and $\phi(Q) \subseteq 3Q^{(n)}$ for some $n \in \mathbb{N}$. Let X and Y be a UMD spaces, and let $p \in (1,\infty)$. Suppose that Y has type $t \in [1,p]$ and X has cotype $q \in [p,\infty]$, or one of them has both. Let $\lambda = (\lambda_Q)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(X,Y)$ be R-bounded, consider the mapping $T_{\phi\lambda}^{\alpha\gamma}$ as in (12.14), and let*

$$\|T_{\phi\lambda}^{\alpha\gamma}\| := \|T_{\phi\lambda}^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}.$$

(1) *We have the norm estimate*

$$\|T_{\phi\lambda}^{\alpha\gamma}\| \leqslant 6 \cdot 3^{4d} \beta_{p,X}\beta_{p,Y}(n+1)^{1/t-1/q} \min\{C\mathscr{R}_p(\lambda), C^*\mathscr{R}_{p'}^*(\lambda)\}$$

where

$$C = C_{(12.15)}(X,Y,p,q,t), \qquad C^* := C_{(12.15)}(Y^*,X^*,p',t',q').$$

(2) *If, in addition, $\lambda_Q \neq 0$ only when $\phi(Q) \subseteq Q^{(n)}$, then we have the alternative norm estimate*

$$\|T_{\phi\lambda}^{\alpha\gamma}\| \leqslant 3 \cdot \beta_{p,X}\beta_{p,Y} \min\Big\{C(n+1)^{1/q'}\mathscr{R}_p(\lambda), C^*(n+1)^{1/t}\mathscr{R}_{p'}^*(\lambda)\Big\}.$$

where

$$C = \min\{c_{q,X;p}, c_{q,Y;p}\}, \qquad C^* = \min\{c_{t',Y^*;p'}, c_{t',X^*;p'}\}$$

Proof. The first versions of both bounds (i.e, using the first item of the respective minimums) above are simply those of Theorem 12.1.25, cases (1) and (2), where we estimated all UMD constants by $\beta_{p,Z}^{\pm} \leqslant \beta_{p,Z}$. The second versions of both bounds then follow by duality: When $\phi : \mathscr{D} \to \mathscr{D}$ is a bijection, one directly verifies that

$$(T_{\phi\lambda}^{\alpha\gamma})^* = T_{\phi^{-1},\lambda^*_{\phi^{-1}}}^{\gamma\alpha}$$

is an operator of the same form, acting from $\mathscr{Q}_{00}(\mathbb{R}^d;Y^*)$ to $\mathscr{Q}_{00}(\mathbb{R}^d;X^*)$ and eventually from $L^{p'}(\mathbb{R}^d;Y^*)$ to $L^{p'}(\mathbb{R}^d;X^*)$. If $Z \in \{X,Y\}$ has type t, then Z^*

has cotype t' with $c_{t',Z^*;p'} \leqslant \tau_{t,Z;p}$. (See Proposition 7.1.13; it is formulated for $p = t$, but the same short argument is easily modified to give the general statement.) If a UMD space Z has cotype q, then it has martingale type q (Proposition 4.3.13), hence Z^* has martingale cotype q' (Proposition 3.5.29), and thus cotype q' (as observed right before Proposition 4.3.13). Thus we can apply the case already handled, with (Y^*, X^*, p', t', q') in place of (X, Y, p, q, t), to get

$$\begin{aligned}\|T^{\alpha\gamma}_{\phi\lambda}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} &= \|T^{\gamma\alpha}_{\phi^{-1},\lambda_{\phi^{-1}}}\|_{\mathscr{L}(L^{p'}(\mathbb{R}^d;Y^*),L^{p'}(\mathbb{R}^d;X^*))}\\ &\leqslant 6\cdot 3^{4d}\beta_{p',Y^*}\beta_{p',X^*}(n+1)^{1/q'-1/t'}C(Y^*,X^*,p',t',q')\mathscr{R}_{p'}(\lambda^*).\end{aligned}$$

The claim then follows from $\beta_{p',Z^*} = \beta_{p,Z}$ and $1/q' - 1/t' = 1/t - 1/q$.

The second version of the second bound is obtained from the first version in the entirely similar way by duality. □

Proof of Theorem 12.1.25. Claim (0) is the special case $t = 1$, $q = \infty$ of (1), so we only need to prove the latter of the two. Let $\mathscr{F}$ be a finite collection of dyadic cubes. Then $\mathscr{F}$ and ϕ satisfy the assumptions of Lemma 12.1.23, except possibly the scale separation (c). By Remark 12.1.24, the lemma still applies to produce $3^{3d+1}(n+1)$ subcollections $\mathscr{F}_i \subseteq \mathscr{F}$ with the properties given in Lemma 12.1.23. Let us write $x_Q = \langle f, h^\alpha_Q\rangle$. Since the functions $(h^\gamma_Q)_{Q\in\mathscr{F}}$ form a martingale difference sequence, we have

$$\Big\|\sum_{Q\in\mathscr{F}}\lambda_Q x_Q h^\gamma_{\phi(Q)}\Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \beta^-_{p,Y}\Big\|\sum_{Q\in\mathscr{F}}\varepsilon_Q\lambda_Q x_Q h^\gamma_{\phi(Q)}\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)}.$$

From this point on, we have some flexibility as to when we want to "pull out" the coefficients λ_Q. For this reason, let us write $z_Q \in Z$ for a generic choice of either $z_Q = \lambda_Q x_Q \in Y$ or $z_Q = x_Q \in X$. We then continue with

$$\begin{aligned}\Big\|\sum_{Q\in\mathscr{F}}\varepsilon_Q z_Q h^\gamma_{\phi(Q)}\Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} &= \Big\|\sum_i\sum_{Q\in\mathscr{F}_i}\varepsilon'_i\varepsilon_Q z_Q h^0_{\phi(Q)}\Big\|_{L^p(\Omega'\times\Omega\times\mathbb{R}^d;Z)}\\ &\leqslant \tau_{t,Z;p}\Big(\sum_i\Big\|\sum_{Q\in\mathscr{F}_i}\varepsilon_Q x_Q h^0_{\phi(Q)}\Big\|^t_{L^p(\Omega\times\mathbb{R}^d;Z)}\Big)^{1/t},\end{aligned}$$

where, in the two steps above, we used the facts that

1. when multiplied by the random sign ε_Q, both the independent random sign ε'_i and the possible difference of the signs of $h^\alpha_{\phi(Q)}(t)$ and $h^0_{\phi(Q)}(t)$ are invisible to the norm; and
2. whenever Z has type $t \in [1, p]$, then so has $L^p(S; Z)$ (here: $S = \Omega\times\mathbb{R}^d$), and $\tau_{t,L^p(S;Z);p} \leqslant \tau_{t,Z;p}$ by Proposition 7.1.4.

For $Q \in \mathscr{F}_i$, let us denote by $E(Q) = Q^{[m(i)]}\cup\phi(Q)^{[m(i)]}$ the sets provided by Lemma 12.1.23 that form a strongly nested family, as guaranteed by the said lemma. In particular $E(Q) \supseteq Q\cup\phi(Q)$ and $|E(Q)| \leqslant 2\cdot 3^d|Q|$. (The inequality is due to the fact that the cubes $Q^{[m(i)]}$ and $\phi(Q)^{[m(i)]}$ are not necessarily different.) Hence

$$\mathbf{1}_{\phi(Q)} \leqslant \mathbf{1}_{\phi(Q)} \frac{2 \cdot 3^d}{|E(Q)|}|Q| = \mathbf{1}_{\phi(Q)} 2 \cdot 3^d \fint_{E(Q)} \mathbf{1}_Q \leqslant 2 \cdot 3^d \mathbb{E}_{E(Q)} \mathbf{1}_Q,$$

where the $\mathbb{E}_{E(Q)}$ are conditional expectations associated with a nested family, and hence with a filtration. This allows us to use Stein's inequality (Theorem 4.2.23) to the effect that

$$\begin{aligned} &\Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_{\phi(Q)} \Big\|_{L^p(\Omega \times \mathbb{R}^d; Z)} \\ &\leqslant 2 \cdot 3^d \Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q \mathbb{E}_{E(Q)} h^0_Q \Big\|_{L^p(\Omega \times \mathbb{R}^d; Z)} \\ &\leqslant 2 \cdot 3^d \cdot \beta^+_{p,Z} \Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|_{L^p(\Omega \times \mathbb{R}^d; Z)} \end{aligned} \tag{12.16}$$

Then

$$\begin{aligned} &\Big(\sum_i \Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|^t_{L^p(\Omega \times \mathbb{R}^d; Z)} \Big)^{1/t} \\ &\leqslant (3^{3d+1}(n+1))^{1/t - 1/q} \Big(\sum_i \Big\| \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|^q_{L^p(\Omega \times \mathbb{R}^d; Z)} \Big)^{1/q} \\ &\leqslant (3^{3d+1}(n+1))^{1/t - 1/q} c_{q,Z;p} \Big\| \sum_i \varepsilon'_i \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|_{L^p(\Omega' \times \Omega \times \mathbb{R}^d; Z)}, \end{aligned}$$

where, in the two steps above, we used

1. Hölder's inequality and counting of terms in the other factor; and
2. an application of the cotype q property of Z, recalling that this implies cotype q for $L^p(S; Z)$ (here: $S = \Omega \times \mathbb{R}^d$) with $c_{q, L^p(S;Z);p} \leqslant c_{q,Z;p}$ when $q \in [p, \infty]$ by Proposition 7.1.4.

By the invisibility of signs multiplying a random ε_Q, the last norm here is

$$\Big\| \sum_i \varepsilon'_i \sum_{Q \in \mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|_{L^p(\Omega' \times \Omega \times \mathbb{R}^d; Z)} = \Big\| \sum_{Q \in \mathscr{F}} \varepsilon_Q z_Q h^\alpha_Q \Big\|_{L^p(\Omega \times \mathbb{R}^d; Z)}.$$

If we did not already pull out the coefficients λ_Q, we do it at this point, after which we are left with

$$\Big\| \sum_{Q \in \mathscr{F}} \varepsilon_Q x_Q h^\alpha_Q \Big\|_{L^p(\Omega \times \mathbb{R}^d; X)} \leqslant \beta^+_{p,X} \|f\|_{L^p(\mathbb{R}^d; X)},$$

where the last step was a direct application of Proposition 12.1.5.

It remains to collect the various coefficients that we accumulated. In any case, the first estimate gave $\beta^-_{p,Y}$ and the last one $\beta^+_{p,X}$, but depending on where we pull out the coefficients λ_Q, we may use the constant of the space X or Y in place of the generic Z.

If we pull out the λ_Q before the application of Stein's inequality in (12.16), then λ_Q is the coefficient of $h^\gamma_{\phi(Q)}$, hence the coefficient of h^γ_R is $\lambda_{\phi^{-1}(R)}$, and thus an application of Remark 12.1.7 produces the factor $\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}})$. On the other hand, pulling out the λ_Q only after (12.16) leads to a "direct" application of Remark 12.1.7 and the factor $\mathscr{E}\mathscr{R}_p(\lambda)$.

Aside from the numerical factors $2 \cdot 3^d$ and $(3^{3d+1}(n+1))^{1/t-1/q}$, we get one of the following:

$$\begin{aligned}
&\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}) \times \tau_{t,X;p} \times \beta^+_{p,X} \times c_{q,X;p},\\
&\tau_{t,Y;p} \times \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}) \times \beta^+_{p,X} \times c_{q,X;p},\\
&\tau_{t,Y;p} \times \beta^+_{p,Y} \times \mathscr{E}\mathscr{R}_p(\lambda) \times c_{q,X;p},\\
&\tau_{t,Y;p} \times \beta^+_{p,Y} \times c_{q,Y;p} \times \mathscr{E}\mathscr{R}_p(\lambda),
\end{aligned}$$

where the order of the constants reflects the order of applying the related estimates: Before pulling out the coefficients λ_Q, we apply estimates on the Y side, and after that on the X side. Taking the minimum of the four terms, we arrive at the assertion of the theorem.

The alternative estimate (2)*:* In order to make efficient use of the additional assumption $\phi(Q) \subseteq Q^{(n)}$ when $\lambda_Q \neq 0$, we will need to modify the preceding considerations at various points.

Let $\mathscr{F}$ be a finite collection of dyadic cubes, and $\mathscr{F}^\lambda := \{Q \in \mathscr{F} : \lambda_Q \neq= 0\}$. Then $\mathscr{F}^\lambda$ and ϕ satisfy the assumptions of Lemma 12.1.20, except possibly the scale separation (c). By Remark 12.1.24, the lemma still applies to produce $3(n+1)$ subcollections $\mathscr{F}^\lambda_i \subseteq \mathscr{F}^\lambda$ with the properties given in Lemma 12.1.20. Let us write $x_Q = \langle f, h^\alpha_Q \rangle$. In the first step, we simply use the triangle inequality:

$$\Big\| \sum_{Q \in \mathscr{F}} \lambda_Q x_Q h^\gamma_{\phi(Q)} \Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \sum_i \Big\| \sum_{Q \in \mathscr{F}^\lambda_i} \lambda_Q x_Q h^\gamma_{\phi(Q)} \Big\|_{L^p(\mathbb{R}^d;Y)}.$$

The more interesting deviations from the previous case begin now.

Note that $h^\alpha_Q = |Q|^{-1/2}(\mathbf{1}_{Q^+_\alpha} - \mathbf{1}_{Q^-_\alpha})$ for suitable subsets $Q^\pm_\alpha \subseteq Q$ with $|Q^\pm_\alpha| = \frac{1}{2}|Q|$. If $Q \neq \phi(Q)$, we see that

$$\begin{aligned}
d^+_Q &:= \frac{1}{2}(h^\alpha_Q + h^\gamma_{\phi(Q)}) = \frac{1}{2}|Q|^{-1/2}(\mathbf{1}_{Q^+_\alpha \cup \phi(Q)^+_\gamma} - \mathbf{1}_{Q^-_\alpha \cup \phi(Q)^-_\gamma}),\\
d^-_Q &:= \frac{1}{2}(h^\alpha_Q - h^\gamma_{\phi(Q)}) = \frac{1}{2}|Q|^{-1/2}(\mathbf{1}_{Q^+_\alpha \cup \phi(Q)^-_\gamma} - \mathbf{1}_{Q^-_\alpha \cup \phi(Q)^+_\gamma})
\end{aligned}$$

form a martingale difference sequence (in either order) on $Q \cup \phi(Q)$, since either function has average zero on the sets where the other one is constant.

If $Q = \phi(Q)$ but $\alpha \neq \gamma$, then each of the sets $Q_\alpha^\pm \cap Q_\gamma^\pm$ has measure $\frac{1}{4}|Q|$, and once again

$$
\begin{aligned}
d_Q^+ &:= \frac{1}{2}(h_Q^\alpha + h_Q^\gamma) = |Q|^{-1/2}(\mathbf{1}_{Q_\alpha^+ \cap Q_\gamma^+} - \mathbf{1}_{Q_\alpha^- \cap Q_\gamma^-}),\\
d_Q^- &:= \frac{1}{2}(h_Q^\alpha - h_Q^\gamma) = |Q|^{-1/2}(\mathbf{1}_{Q_\alpha^+ \cap Q_\gamma^-} - \mathbf{1}_{Q_\alpha^- \cap Q_\gamma^+})
\end{aligned}
$$

form a martingale difference sequence (in either order) on $Q \cup \phi(Q) = Q$, since either function has average zero on the sets where the other one is constant.

Finally, if $Q = \phi(Q)$ and $\alpha = \gamma$, then the same definition gives $d_Q^+ = h_Q^\alpha$, $d_Q^- = 0$, which is also a (rather trivial) martingale difference sequence.

The conclusion of Lemma 12.1.20, that each $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i^\lambda\}$ is strongly nested, guarantees that the whole collection $\{d_Q^+, d_Q^-\}_{Q \in \mathscr{F}_i^\lambda}$ can be organised into a martingale difference sequence. Hence

$$
\begin{aligned}
&\Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \lambda_Q x_Q h_{\phi(Q)}^\gamma \Big\|_{L^p(\mathbb{R}^d;Y)}\\
&= \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \lambda_Q x_Q (d_Q^+ - d_Q^-) \Big\|_{L^p(\mathbb{R}^d;Y)}\\
&\leqslant \beta_{p,Y} \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q (d_Q^+ + d_Q^-) \Big\|_{L^p(\Omega \times \mathbb{R}^d;Y)}\\
&= \beta_{p,Y} \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Y)},
\end{aligned}
\tag{12.17}
$$

where we used the definition of UMD with signs $\pm\varepsilon_Q$ multiplying the martingale differences $d_Q^\pm$, followed by taking an average over the ε_Q. (It might appear at first glance that we could have used just the one-sided UMD$^-$ property to arrive at the same conclusion with the smaller constant $\beta_{p,Y}^-$, but this is not the case: an application of the one-sided UMD$^-$ property would give us independent random signs, say $\varepsilon_Q^\pm$, in front of each $d_Q^\pm$, and this is not what we want.)

For $z_Q \in \{x_Q, \lambda_Q x_Q\}$ and $Z \in \{X, Y\}$ we then have

$$
\begin{aligned}
&\sum_i \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q z_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Z)}\\
&\quad\leqslant (3(n+1))^{1/q'} \Big(\sum_i \Big\| \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q z_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Z)}^q \Big)^{1/q}\\
&\quad\leqslant (3(n+1))^{1/q'} c_{q,Z;p} \Big\| \sum_i \varepsilon_i' \sum_{Q \in \mathscr{F}_i^\lambda} \varepsilon_Q z_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Z)}\\
&\quad= (3(n+1))^{1/q'} c_{q,Z;p} \Big\| \sum_{Q \in \mathscr{F}^\lambda} \varepsilon_Q z_Q h_Q^\alpha \Big\|_{L^p(\Omega \times \mathbb{R}^d;Z)},
\end{aligned}
\tag{12.18}
$$

using in the last step the fact that the $\mathscr{F}^\lambda = \bigcup_i \mathscr{F}_i^\lambda$ is a disjoint partition, so the independent random signs ε_Q with $Q \in \mathscr{F}^\lambda$ do not "see" the multiplying signs ε'_i. Hence, pulling out the λ_Q either at the beginning or at the end of (12.18) (but in any case only after having replaced the translated $h^\gamma_{\phi(Q)}$ by h^α_Q in (12.17), which in contrast to what happened in the previous case of the proof), we obtain

$$\begin{aligned}\Big\| \sum_{Q\in\mathscr{F}} \lambda_Q x_Q h^\gamma_{\phi(Q)}\Big\|_{L^p(\mathbb{R}^d;Y)} &\leqslant \beta_{p,Y} \sum_i \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q h^\alpha_Q\Big\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \beta_{p,Y} \mathscr{E}\mathscr{R}_p(\lambda)(3(n+1))^{1/q'} \min\{c_{q,X;p}, c_{q,Y;p}\} \Big\| \sum_{Q\in\mathscr{F}^\lambda} \varepsilon_Q x_Q h^\alpha_Q\Big\|_{L^p(\mathbb{R}^d;X)}.\end{aligned}$$

Finally, recalling that $x_Q = \langle f, h^\alpha_Q\rangle$ and using the contraction principle to replace $\mathscr{F}^\lambda \subseteq \mathscr{F}$ by the finite set $\mathscr{F} = \{Q \in \mathscr{D} : \langle f, h^\alpha_Q\rangle \neq 0\}$, we obtain from Proposition 12.1.5 that

$$\Big\| \sum_{Q\in\mathscr{F}^\lambda} \varepsilon_Q x_Q h^\alpha_Q\Big\|_{L^p(\mathbb{R}^d;X)} \leqslant \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q x_Q h^\alpha_Q\Big\|_{L^p(\mathbb{R}^d;X)} \leqslant \beta^+_{p,X} \|f\|_{L^p(\mathbb{R}^d;X)},$$

which concludes the estimate.

The representation (3)*:* Let first $\mathscr{F} \subseteq \mathscr{D}$ be finite. For suitable $\eta_Q \in \mathbb{K}$ with $|\eta_Q| = 1$, we have

$$\begin{aligned}\sum_{Q\in\mathscr{F}} \Big|\Big\langle \lambda_Q \langle f, h^\alpha_Q\rangle, \langle g, h^\gamma_{\phi(Q)}\rangle\Big\rangle\Big| &= \Big\langle \sum_{Q\in\mathscr{F}} \eta_Q \lambda_Q \langle f, h^\alpha_Q\rangle h^\gamma_{\phi(Q)}, g\Big\rangle \\ &= \langle T^{\alpha\gamma}_{\eta\lambda,\phi} P_{\mathscr{F}} f, g\rangle,\end{aligned}$$

where $(\eta\lambda)(Q) := \eta_Q \lambda_Q$, and

$$P_{\mathscr{F}} f := \sum_{\substack{Q\in\mathscr{F} \\ \theta\in\{0,1\}^d\setminus\{0\}}} \langle f, h^\theta_Q\rangle h^\theta_Q \in S_{00}(\mathscr{D}; X)$$

is a Haar projection of f; the action of $T^{\alpha\gamma}_{\eta\lambda,\phi}$ is thus well-defined via the initial definition on this space. From the previous part of the theorem that we already proved, we have

$$\begin{aligned}&\sum_{Q\in\mathscr{F}} \Big|\Big\langle \lambda_Q \langle f, h^\alpha_Q\rangle, \langle g, h^\gamma_{\phi(Q)}\rangle\Big\rangle\Big| \\ &\qquad \leqslant \|T^{\alpha\gamma}_{\eta\lambda,\phi}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \|P_{\mathscr{F}} f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^p(\mathbb{R}^d;Y^*)}.\end{aligned}$$

We now apply this estimate with the increasing sequence of finite sets

$$\mathscr{F}_N := \{Q \in \mathscr{D} : 2^{-N} < \ell(Q) \leqslant 2^N, \operatorname{dist}_\infty(Q, 0) \leqslant 2^N\},$$

whose union is $\bigcup_{N=1}^{\infty} \mathscr{F}_N = \mathscr{D}$. The corresponding projection can be expressed as

$$P_{\mathscr{F}_N} f = \mathbf{1}_{F_N}(\mathbb{E}_N - \mathbb{E}_{-N})f, \quad F_N := \bigcup_{\substack{Q\in\mathscr{D}_{-N} \\ \mathrm{dist}_\infty(Q,0)\leqslant 2^N}} Q,$$

and this is seen to satisfy $\|P_{\mathscr{F}_N} f\|_{L^p(\mathbb{R}^d;X)} \leqslant 2\|f\|_{L^p(\mathbb{R}^d;X)}$ and $P_{\mathscr{F}_N} \to f$ in $L^p(\mathbb{R}^d;X)$ as $N\to\infty$. Thus

$$\begin{aligned}
&\sum_{Q\in\mathscr{D}} \Big|\Big\langle \lambda_Q\langle f, h_Q^\alpha\rangle, \langle g, h_{\phi(Q)}^\gamma\rangle\Big\rangle\Big| \\
&\quad = \lim_{N\to\infty} \sum_{Q\in\mathscr{F}_N} \Big|\Big\langle \lambda_Q\langle f, h_Q^\alpha\rangle, \langle g, h_{\phi(Q)}^\gamma\rangle\Big\rangle\Big| \\
&\quad \leqslant \|T_{\eta\lambda,\phi}^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \lim_{N\to\infty} \|P_{\mathscr{F}} f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^{p}(\mathbb{R}^d;Y^*)} \\
&\quad = \|T_{\eta\lambda,\phi}^{\alpha\gamma}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \|f\|_{L^p(\mathbb{R}^d;X)} \|g\|_{L^{p}(\mathbb{R}^d;Y^*)},
\end{aligned}$$

where $T_{\eta\lambda,\phi}^{\alpha\gamma}$ has the same norm estimate as $T_{\lambda,\phi}^{\alpha\gamma}$, since

$$\mathscr{E}\mathscr{R}_p(\eta\lambda) = \mathscr{E}\mathscr{R}_p(\lambda), \qquad \mathscr{E}\mathscr{R}_p((\eta\lambda)_{\phi^{-1}}) = \mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}})$$

by the contraction principle.

Thus we have shown the claimed absolute convergence, and hence the bilinear form

$$\mathfrak{t}_{\lambda\phi}^{\alpha\gamma}(f,g) := \sum_{Q\in\mathscr{D}} \langle \lambda_Q\langle f, h_Q^\alpha\rangle, \langle g, h_Q^\gamma\rangle\rangle$$

is well-defined and bounded from $L^p(\mathbb{R}^d;X)\times L^p(\mathbb{R}^d;Y)$ to $\mathbb{K}$. So is the bilinear form $\langle T_{\lambda\phi}^{\alpha\gamma} f, g\rangle$, where $T_{\lambda\phi}^{\alpha\gamma}$ denotes the bounded extension of the operator initially defined on $S_{00}(\mathscr{D};X)$. Moreover, these bilinear forms clearly coincide when $f\in S_{00}(\mathscr{D};X)$ and $g\in S_{00}(\mathscr{D};Y^*)$. By density, they must coincide for all f and g, and the proof is complete. □

The second class of operators that we deal with in this section have the additional twist of "tearing apart" the supports of Haar functions. The relevance of this feature will be justified in the appearance of this type of operators in the proof of the $T(1)$ theorem further below.

Theorem 12.1.28 (Figiel). *Let $\phi:\mathscr{D}\to\mathscr{D}$ be an injection with $\ell(\phi(Q)) = \ell(Q)$ and $\phi(Q)\subseteq 3Q^{(n)}$ for some $n\in\mathbb{N}$. Let X and Y be a UMD spaces and $p\in(1,\infty)$. Let $\lambda = (\lambda_Q)_{Q\in\mathscr{D}} \subseteq \mathscr{L}(X,Y)$, and consider the mapping*

$$U_{\phi\lambda}^\gamma : f \mapsto \sum_{Q\in\mathscr{D}} \lambda_Q\langle f, h_Q^\gamma\rangle(h_{\phi(Q)}^0 - h_Q^0), \tag{12.19}$$

initially from $S_{00}(\mathscr{D};X)$ to $S_0(\mathscr{D};Y)$. Let $B_d := 5200\cdot(81)^d$.

(0) *If $\lambda \subseteq \mathscr{L}(X,Y)$ is R-bounded, or more generally if $\mathscr{E}\mathscr{R}_p(\lambda) < \infty$, then $U^\gamma_{\phi\lambda}$ extends boundedly from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$ with norm*

$$\begin{aligned}\|U^\gamma_\phi\| &:= \|U^\gamma_\phi\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ &\leqslant B_d\cdot(n+1)\cdot\beta^-_{p,Y}\cdot\beta^+_{p,X}\cdot\min\{\beta^+_{p,X}\mathscr{R}_p(\lambda),\ \beta^+_{p,Y}\mathscr{E}\mathscr{R}_p(\lambda)\}.\end{aligned}$$

(1) *If, in addition, X or Y has cotype $q\in[p,\infty]$, then we also have*

$$\|U^\gamma_\phi\| \leqslant B_d(n+1)^{1-1/q}\beta^-_{p,Y}\beta^+_{p,X}\begin{cases}C(X,Y,p,q)\cdot\mathscr{R}_p(\lambda),\\ \beta^+_{p,Y}\cdot\min\{c_{q,X;p},\ c_{q,Y;p}\}\cdot\mathscr{E}\mathscr{R}_p(\lambda),\end{cases}$$

where

$$\begin{aligned}C(X,Y,p,q) &:= \min\left\{\beta^+_{p,X}c_{q,X;p},\quad \beta^+_{p,Y}c_{q,X;p},\quad \beta^+_{p,Y}c_{q,Y;p}\right\} \\ &= C_{(12.15)}(X,Y,p,q,1).\end{aligned} \tag{12.20}$$

(2) *If, in addition, we have $\lambda_Q\neq 0$ only when $\phi(Q)\subseteq Q^{(n)}$, then we have the alternative norm estimate*

$$\|U^\gamma_\phi\| \leqslant 6\cdot(n+1)^{1-1/q}\cdot\beta_{p,Y}\cdot\beta^+_{p,X}\cdot\min\{c_{q,X;p},c_{q,Y;p}\}\cdot\mathscr{E}\mathscr{R}_p(\lambda).$$

(3) *For all $f\in L^p(\mathbb{R}^d;X)$ and $g\in L^{p'}(\mathbb{R}^d;Y^*)$, the extended operator has the absolutely convergent representation*

$$\langle U^\gamma_{\phi\lambda}f,g\rangle = \sum_{Q\in\mathscr{D}}\Big\langle \lambda_Q\langle f,h^\gamma_Q\rangle, \langle g, h^0_{\phi(Q)}-h^0_Q\rangle\Big\rangle.$$

When $\|f\|_{L^p(\mathbb{R}^d;X)}\leqslant 1$ and $\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}\leqslant 1$, the corresponding absolute value series is dominated by the same upper bounds as those given for $\|U^\gamma_{\phi\lambda}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}$ above.

Remark 12.1.29. We have observations analogous to Remark 12.1.26:

(1) When $X=Y$, we have $C(X,X,p,q)=\beta^+_{p,X}c_{q,X;p}$.
(2) Case (0) of Theorem 12.1.28 is a special case of (1) using the cotype exponent $q=\infty$ with corresponding constant equal to one. The role of finite cotype is to relax the dependence on the parameter n. As in Theorem 12.1.25(2), the main point of the alternative bound (2) to improve the cubic dependence on the UMD constants to a quadratic one; in contrast to the situation in Theorem 12.1.25(2), when $X=Y$, the present alternative bound (2) is a strict improvement of (1), in view of the fact that $\beta_{p,X}\leqslant\beta^-_{p,X}\beta^+_{p,X}$ (Proposition 4.2.3).
(3) Recalling the Haar multipliers $\mathfrak{H}^{\alpha\gamma}_\lambda$ from Theorem 12.1.11, one can check that, for any $\theta\in\{0,1\}^d\setminus\{0\}$,

$$U^\gamma_{\phi\lambda} = U^\theta_{\phi\mathbf{1}}\circ\mathfrak{H}^{\gamma\theta}_\lambda,$$

where $\mathbf{1}$ is the constant sequence of all ones. Hence, for the qualitative conclusion of Theorem 12.1.28, it would suffice to consider just $X = Y$ and $\lambda = \mathbf{1}$, and then combine this special case with Theorem 12.1.11; however, the reader will quickly realise that this approach would produce a higher power of the UMD constants in the quantitative conclusion.

(4) In contrast to Theorem 12.1.25, our proof of Theorem 12.1.28 does not allow replacing the assumptions on λ by $\mathscr{E}\mathscr{R}_p(\lambda_{\phi^{-1}}) < \infty$. The related issue of when in the argument, and under what assumptions, we may pull out the coefficients λ_Q, is shortly discussed inside the proof.

Proof of Theorem 12.1.28. Claim (0) is the special case $q = \infty$ of (1), so it suffices to consider the latter of these two claims. Let $\mathscr{F} \subseteq \mathscr{D}$ be finite. An additional challenge compared to the proof of Theorem 12.1.25 is that, unlike the Haar functions $h^{\alpha}_{\phi(Q)}$, the functions $h^0_{\phi(Q)} - h^0_Q$ do not necessarily form a martingale difference sequence, preventing a straightforward introduction of the random signs in the initial step. Instead, a decomposition of $\mathscr{F}$ is necessary from the beginning.

Let us denote by $\mathscr{F}^k = \{Q \in \mathscr{F} : \log_2 \ell(Q) \equiv k \mod (n+1)\}$ the scale-separated subcollections of $\mathscr{F}$ as in Remark 12.1.24. Then $\mathscr{F}^k$ and ϕ satisfy the assumptions of both Lemmas 12.1.22 and 12.1.23. Let us denote the decomposing subcollections of $\mathscr{F}_k$ provided by Lemma 12.1.22 by $\mathscr{A}^k_a$ and those provided by Lemma 12.1.23 by $\mathscr{B}^k_b$, let $\mathscr{F}^k_i = \mathscr{A}^k_a \cap \mathscr{B}^k_b$ for $i = (a, b)$, and let $\mathscr{F}_i$ consists of an enumeration of all these $\mathscr{F}^k_i$. The total number of these $\mathscr{F}_i$ is then $144 \cdot 3^{3d+1} \cdot (n+1)$, and they satisfy the conclusions of both Lemmas 12.1.22 and 12.1.23.

We first make use of Lemma 12.1.22. For $Q \in \mathscr{F}_i$, we have

$$h^0_{\phi(Q)} - h^0_Q = h^0_{\phi_{i,3}(Q)} - h^0_{\phi_{i,0}(Q)} = \sum_{j=0}^{2} (h^0_{\phi_{i,j+1}(Q)} - h^0_{\phi_{i,j}(Q)}),$$

where each collection $\{\phi_{i,j}(Q) \cup \phi_{i,j+1}(Q) : Q \in \mathscr{F}_i\}$ is strongly nested. But this implies that each

$$(h^0_{\phi_{i,j+1}(Q)} - h^0_{\phi_{i,j}(Q)})_{Q \in \mathscr{F}_i}$$

is (or can be enumerated as) a martingale difference sequence. Note that here it is important that a smaller union $\phi_{i,j+1}(Q) \cup \phi_{i,j}(Q)$ is not just contained in a larger $\phi_{i,j+1}(R) \cup \phi_{i,j}(R)$, but entirely in (a dyadic child of) one of $\phi_{i,j+1}(R)$ or $\phi_{i,j}(R)$, where the function $h^0_{\phi_{i,j+1}(R)} - h^0_{\phi_{i,j}(R)}$ is constant.

Using this martingale difference property, we can then proceed as in the proof of Theorem 12.1.25. Let us abbreviate $x_Q := \langle f, h^{\gamma}_Q \rangle \in X$ and $y_Q := \lambda_Q x_Q \in Y$.

$$\begin{aligned}
&\Big\| \sum_{Q\in\mathscr{F}} y_Q (h^0_{\phi(Q)} - h^0_Q) \Big\|_{L^p(\mathbb{R}^d;Y)} \\
&\leqslant \sum_{j=0}^{2} \sum_i \Big\| \sum_{Q\in\mathscr{F}_i} y_Q (h^0_{\phi_{i,j+1}(Q)} - h^0_{\phi_{i,j}(Q)}) \Big\|_{L^p(\mathbb{R}^d;Y)} \\
&\leqslant \sum_{j=0}^{2} \sum_i \beta^-_{p,Y} \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q y_Q (h^0_{\phi_{i,j+1}(Q)} - h^0_{\phi_{i,j}(Q)}) \Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\
&\leqslant \beta^-_{p,Y} \sum_{j=0}^{3} \alpha_j \sum_i \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q y_Q h^0_{\phi_{i,j}(Q)} \Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)}, \qquad \begin{cases} \alpha_0 = \alpha_3 = 1, \\ \alpha_1 = \alpha_2 = 2, \end{cases}
\end{aligned}$$

where the first and the last steps were simply triangle inequalities.

As in the proof of Theorem 12.1.25, we have some flexibility on when to pull out the coefficients λ_Q, and we again proceed with a generic choice of $z_Q \in Z$ for either $y_Q \in Y$ or $x_Q \in X$. The norm to be estimated has exactly the same form as what we estimated (12.16) in the proof of Theorem 12.1.25 (using Lemma 12.1.23 in this step), and we can there read the bound

$$\begin{aligned}
&\Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_{\phi_{i,j}(Q)} \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} \\
&\qquad \leqslant 2\cdot 3^d \cdot \beta^+_{p,Z} \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}.
\end{aligned} \tag{12.21}$$

By Hölder's inequality, we have

$$\begin{aligned}
&\sum_i \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} \\
&\leqslant \big(144\cdot 3^{3d+1}\cdot(n+1)\big)^{1/q'} \Big(\sum_i \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|^q_{L^p(\Omega\times\mathbb{R}^d;Z)} \Big)^{1/q}.
\end{aligned}$$

Invoking cotype q of Z, and recalling that this implies cotype q of $L^p(S;Z)$ (here: $S = \Omega\times\mathbb{R}^d$) with constant $c_{q,L^p(S;Z);p} \leqslant c_{q,Z;p}$ when $q\in[p,\infty]$ by Proposition 7.1.4, we continue with

$$\begin{aligned}
&\Big(\sum_i \Big\| \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|^q_{L^p(\Omega\times\mathbb{R}^d;Z)} \Big)^{1/q} \\
&\leqslant c_{q,L^p(Z);p} \Big\| \sum_i \varepsilon'_i \sum_{Q\in\mathscr{F}_i} \varepsilon_Q z_Q h^0_Q \Big\|_{L^p(\Omega'\times\Omega\times\mathbb{R}^d;Z)} \\
&= c_{q,L^p(Z);p} \Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q z_Q h^\gamma_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}.
\end{aligned}$$

It is no later than here that we should to pull out the coefficients λ_Q, after which we are left with the final step, based on Proposition 12.1.5, that

$$\Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q x_Q h_Q^\gamma \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \leqslant \beta_{p,X}^+ \|f\|_{L^p(\mathbb{R}^d;X)}.$$

Under the assumption of R-boundedness of λ, depending on the moment of pulling out the coefficients λ_Q, the constants that we accumulate in the various steps with the option of estimating in $Z \in \{X, Y\}$ produce, aside from the numerical factors $2\cdot 3^d$ and $\big(144\cdot 3^{3d+1}\cdot(n+1)\big)^{1/q'}$, one of the products

$$\begin{aligned}
&\mathscr{R}_p(\lambda)\cdot\beta_{p,X}^+\cdot c_{q,X;p},\\
&\beta_{p,Y}^+\cdot\mathscr{R}_p(\lambda)\cdot c_{q,X;p},\\
&\beta_{p,Y}^+\cdot c_{q,Y;p}\cdot\mathscr{R}_p(\lambda).
\end{aligned}$$

In the latter two versions, i.e., pulling out the λ_Q only after making the step (12.21) with $Z = Y$, we might as well replace $\mathscr{R}_p(\lambda)$ by $\mathscr{E}\mathscr{R}_p(\lambda)$, thus leading to the possible upper bounds

$$\begin{aligned}
&\beta_{p,Y}^+\cdot\mathscr{E}\mathscr{R}_p(\lambda)\cdot c_{q,X;p},\\
&\beta_{p,Y}^+\cdot c_{q,Y;p}\cdot\mathscr{E}\mathscr{R}_p(\lambda).
\end{aligned}$$

(On the other hand, if we wanted to pull out the λ_Q before step (12.21), and thus apply (12.21) with $Z = X$, the coefficient λ_Q would be multiplying a Haar function $h^0_{\phi_{i,j}(Q)}$; this would lead to a constant of the type $\mathscr{E}\mathscr{R}_p(\lambda_{\phi_{i,j}^{-1}})$, where $\phi_{i,j}$ need not be the original ϕ from the assumptions of the theorem, but one of the auxiliary mappings produced by Lemma 12.1.22. This would lead to an unreasonably technical formulation of probably little practical value, which is why we have not included the resulting alternative upper bound in the statement of the theorem.)

Altogether, choosing the best of the possible alternative estimates, we arrive at

$$\begin{aligned}
&\Big\| \sum_{Q\in\mathscr{F}} x_Q(h^0_{\phi(Q)} - h^0_Q)\Big\|_{L^p(\mathbb{R}^d;X)} \|f\|^{-1}_{L^p(\mathbb{R}^d;X)}\\
&\leqslant \beta_{p,X}^- \sum_{j=0}^{3} \alpha_j (2\cdot 3^d)\big(144\cdot 3^{3d+1}\cdot(n+1)\big)^{1/q'}\beta_{p,X}^+\times\\
&\qquad\times\begin{cases} C(X,Y,p,q)\mathscr{R}_p(\lambda),\\ \beta_{p,Y}^+\min\{c_{q,X;p}, c_{q,Y;p}\}\mathscr{E}\mathscr{R}_p(\lambda),\end{cases}
\end{aligned}$$

where $C(X,Y,p,q)$ is as in the statement of the Theorem, and $\sum_{j=0}^3 \alpha_j = 1+2+2+1 = 6$.

The alternative estimate (2)*:* As in the previous proof of Theorem 12.1.25(2), we construct some auxiliary martingale differences. The initial considerations are identical:

Let again $\mathscr{F}$ be a finite collection of dyadic cubes, and $\mathscr{F}^\lambda := \{Q \in \mathscr{F} : \lambda_Q \neq 0\}$. Then $\mathscr{F}^\lambda$ and ϕ satisfy the assumptions of Lemma 12.1.20, except possibly the scale separation (c). By Remark 12.1.24, the lemma still applies to produce $3(n+1)$ subcollections $\mathscr{F}_i^\lambda \subseteq \mathscr{F}^\lambda$ with the properties given in Lemma 12.1.20. Let us write $x_Q = \langle f, h_Q^\alpha \rangle$. In the first step, we simply use the triangle inequality:

$$\Big\| \sum_{Q\in\mathscr{F}} \lambda_Q x_Q (h^0_{\phi(Q)} - h^0_Q) \Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \sum_i \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \lambda_Q x_Q (h^0_{\phi(Q)} - h^0_Q) \Big\|_{L^p(\mathbb{R}^d;Y)}.$$

The slight symmetry break between h_Q^α and $h^0_{\phi(Q)} - h^0_Q$ is also reflected in the construction of the auxiliary martingale differences. As in the proof of Theorem 12.1.25(2), we denote $Q_\alpha^\pm := Q \cap \{\operatorname{sgn}(h_Q^\alpha) = \pm 1\}$. If $\phi(Q) \neq Q$, we choose

$$\begin{aligned} d_Q^1 &:= \frac{1}{3}|Q|^{-1/2}(\mathbf{1}_{\phi(Q)\cup Q_\alpha^+} - 3\cdot \mathbf{1}_{Q_\alpha^-}), \\ d_Q^2 &:= \frac{1}{3}|Q|^{-1/2}(-\mathbf{1}_{\phi(Q)} + 2\cdot \mathbf{1}_{Q_\alpha^+}), \end{aligned}$$

where d_Q^2 has average zero on the sets where d_Q^1 is constant; note that, unlike in the proof of Theorem 12.1.25(2), the order matters now. Moreover, we can recover the original functions by

$$\begin{aligned} d_Q^1 + d_Q^2 &= \frac{1}{3}|Q|^{-1/2}\big((1-1)\mathbf{1}_{\phi(Q)} + (1+2)\mathbf{1}_{Q_\alpha^+} - 3\cdot \mathbf{1}_{Q_\alpha^-}\big) = h_Q^\alpha, \\ d_Q^1 - 2d_Q^2 &= \frac{1}{3}|Q|^{-1/2}\big((1+2)\mathbf{1}_{\phi(Q)}) + (1-4)\mathbf{1}_{Q_\alpha^+} - 3\cdot \mathbf{1}_{Q_\alpha^-}\big) = h^0_{\phi(Q)} - h^0_Q. \end{aligned}$$

If $\phi(Q) = Q$, then $h^0_{\phi(Q)} - h^0_Q = 0$, and we can simply set $d_Q^1 := h_Q^\alpha$ and $d_Q^2 = 0$, and the original functions are recovered by

$$h_Q^\alpha = d_Q^1 = d_Q^1 + d_Q^2, \qquad h^0_{\phi(Q)} - h^0_Q = 0 = 0 \cdot d_Q^1 - 2d_Q^2.$$

The conclusion of Lemma 12.1.20, that $\{Q \cup \phi(Q) : Q \in \mathscr{F}_i^\lambda\}$ is strongly nested, ensures that the full collection $\{d_Q^1, d_Q^2\}_{Q\in\mathscr{F}_i^\lambda}$, appropriately enumerated, is a martingale difference sequence. Hence

$$\begin{aligned} &\Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \lambda_Q x_Q (h^0_{\phi(Q)} - h^0_Q) \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &= \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \lambda_Q x_Q \big((1-\delta_{Q,\phi(Q)}) d_Q^+ - 2\cdot d_Q^-\big) \Big\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant 2\beta_{p,Y} \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q (d_Q^+ + d_Q^-) \Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\ &= 2\beta_{p,Y} \Big\| \sum_{Q\in\mathscr{F}_i^\lambda} \varepsilon_Q \lambda_Q x_Q h_Q^\alpha \Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \end{aligned} \tag{12.22}$$

as an application of the definition of UMD via martingale transforms with a multiplying sequences of numbers $\{0, 1, -2\} \times \varepsilon_Q$, and averaging over independent random ε_Q.

Except for the factor 2, the right side of (12.22) coincides with the right side of (12.17) from the proof of Theorem 12.1.25(2). Hence the rest of the estimate can be concluded by repeating the said proof *verbatim*.

The representation (3)*:* This is proved in the same way as the corresponding part of Theorem 12.1.25. □

12.2 Paraproducts

The notion of paraproducts arises from a number of considerations. Here we choose a point of departure that also motivates their name: they are objects that arise from a decomposition of the ordinary pointwise product of functions. While paraproducts certainly look more complicated than the regular product, it turns out that in certain respects they actually behave better. Another motivation is the key role that these objects play in the $T(1)$ theorem in Section 12.3. Some further connections will be discussed in the Notes.

Proposition 12.2.1. *Let $b \in L^1_{\mathrm{loc}}(\mathbb{R}^d; \mathscr{L}(X,Y))$, where X and Y are Banach spaces, and let $f \in S_{00}(\mathscr{D}; X)$. Then*

$$bf = \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \mathfrak{H}_b^{\alpha\gamma} f + \Pi_b f + \Pi_b^* f, \tag{12.23}$$

where $\mathfrak{H}_b^{\alpha\gamma}$ are Haar multipliers of the form

$$\mathfrak{H}_b^{\alpha\gamma} f := \sum_{Q\in\mathscr{D}} \langle \operatorname{sgn}(h_Q^\alpha h_Q^\gamma) b\rangle_Q \langle f, h_Q^\alpha\rangle h_Q^\gamma,$$

and the remaining terms are the paraproducts

$$\Pi_b f := \sum_{\substack{Q\in\mathscr{D}\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle b, h_Q^\alpha\rangle \langle f\rangle_Q h_Q^\alpha,$$

$$\Pi_b^* f := \sum_{\substack{Q\in\mathscr{D}\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle b, h_Q^\alpha\rangle \langle f, h_Q^\alpha\rangle \frac{\mathbf{1}_Q}{|Q|},$$

where the series of $\Pi_b^* f$ is finitely non-zero, and the non-zero terms in $\Pi_b f$ are attached to cubes contained in finitely many maximal ones, and the series converges (at least) conditionally along any decreasing order of the dyadic cubes contained in these maximal ones.

The notation Π_b^* is motivated by the easily verified duality relation

$$\langle \Pi_b^* f, g\rangle = \langle f, \Pi_{b^*} g\rangle, \qquad f \in S_{00}(\mathscr{D}; X), \quad g \in S_{00}(\mathscr{D}; Y),$$

where $b^* \in L^\infty(\mathbb{R}^d; \mathscr{L}(Y^*, X^*))$ is the pointwise adjoint of b.

Remark 12.2.2. The diagonal $\alpha = \gamma$ of the sum in (12.23) is

$$\sum_{\alpha \in \{0,1\}^d \setminus \{0\}} \mathfrak{H}_b^{\alpha\alpha} f = \sum_{\substack{Q \in \mathscr{D} \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \langle b\rangle_Q \langle f, h_Q^\alpha\rangle h_Q^\alpha$$

This has formally the same structure as $\Pi_b f$, but with the roles of b and f reversed, and hence (12.23) could be also written in the form

$$bf = \sum_{\substack{\alpha,\gamma \in \{0,1\}^d \setminus \{0\} \\ \alpha \neq \gamma}} \mathfrak{H}_b^{\alpha\gamma} f + \Pi_f b + \Pi_b f + \Pi_b^* f,$$

where the summation is empty in dimension $d = 1$ (since there is only one possible value of $\alpha \in \{0,1\} \setminus \{0\}$). It is also evident that $\Pi_b^* f$ is symmetric in b and f, and hence a more symmetric notation could also be preferred. However, we shall not pursue this point of view any further, since the roles played by the two functions b and f will be quite different in our main applications, so that such symmetries would be only misleading.

Proof of Proposition 12.2.1. It suffices to prove this for $f = x \otimes h_R^\theta$. Then

$$bf = (b - \langle b\rangle_R) f + \langle b\rangle_R f = \sum_{\substack{Q \subseteq R \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \langle b, h_Q^\alpha\rangle x \otimes h_Q^\alpha h_R^\theta + \langle b\rangle_R x \otimes h_R^\theta,$$

where the series converges (at least) conditionally along any decreasing order of the dyadic cubes $Q \subseteq R$, by the Martingale Converge Theorem 3.3.2, since this is a martingale difference expansion of the function $\mathbf{1}_R(b - \langle b\rangle_R)x \in L^1(\mathbb{R}^d; Y)$.

We observe that

$$h_Q^\alpha h_R^\theta = h_Q^\alpha \langle h_R^\theta\rangle_Q \qquad \forall Q \subsetneq R,$$

whereas

$$h_R^\theta h_R^\theta = \frac{\mathbf{1}_R}{|R|}, \qquad h_R^\alpha h_R^\theta \frac{h_R^{\alpha+\theta}}{|R|^{1/2}}, \quad \forall \alpha \neq \theta,$$

where we use modulo 2 addition in $\{0,1\}^d$. Hence

$$\sum_{\substack{Q \subsetneq R \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \langle b, h_Q^\alpha\rangle x \otimes h_Q^\alpha h_R^\theta = \sum_{\substack{Q \in \mathscr{D} \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \langle b, h_Q^\alpha\rangle \langle f\rangle_Q \otimes h_Q^\alpha = \Pi_b f,$$

observing that $\langle f\rangle_Q = \langle h_R^\theta\rangle_Q x = 0$ unless $Q \subsetneq R$. Moreover,

$$
\begin{aligned}
&\sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \langle b, h_R^\alpha\rangle x \otimes h_R^\alpha h_R^\theta + \langle b\rangle_R x \otimes h_R^\theta \\
&= \Big(\langle b, h_R^\theta\rangle x \otimes \frac{\mathbf{1}_R}{|R|} + \sum_{\alpha\in\{0,1\}^d\setminus\{0,\theta\}} \langle b, h_R^\alpha\rangle x \otimes \frac{h_R^{\alpha+\theta}}{|R|^{1/2}}\Big) + \frac{\langle b, h_R^0\rangle}{|R|^{1/2}} x \otimes h_R^\theta \\
&= \langle b, h_R^\theta\rangle x \otimes \frac{\mathbf{1}_R}{|R|} + \sum_{\alpha\in\{0,1\}^d\setminus\{\theta\}} \langle b, h_R^\alpha\rangle x \otimes \frac{h_R^{\alpha+\theta}}{|R|^{1/2}}.
\end{aligned}
$$

Using the orthogonality of the Haar functions, we see that

$$
\langle b, h_R^\theta\rangle x \otimes \frac{\mathbf{1}_R}{|R|} = \sum_{\substack{Q\in\mathscr{D}\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle b, h_Q^\alpha\rangle \langle f, h_Q^\alpha\rangle \otimes \frac{\mathbf{1}_Q}{|Q|} = \Pi_b^* f.
$$

Finally, with the change of variable $\gamma := \alpha + \theta$

$$
\begin{aligned}
&\sum_{\alpha\in\{0,1\}^d\setminus\{\theta\}} \langle b, h_R^\alpha\rangle x \otimes \frac{h_R^{\alpha+\theta}}{|R|^{1/2}} = \sum_{\gamma\in\{0,1\}^d\setminus\{0\}} \langle b, h_R^{\gamma+\theta}\rangle x \otimes \frac{h_R^\gamma}{|R|^{1/2}} \\
&= \sum_{\gamma\in\{0,1\}^d\setminus\{0\}} \langle b \operatorname{sgn}(h_R^\gamma h_R^\theta)\rangle_R x \otimes h_R^\gamma = \sum_{\gamma\in\{0,1\}^d\setminus\{0\}} \langle a_R^{\theta\gamma} b\rangle_R x \otimes h_R^\gamma \\
&= \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \sum_{Q\in\mathscr{D}} \langle a_Q^{\alpha\gamma} b\rangle_Q \langle f, h_Q^\alpha\rangle \otimes h_Q^\gamma = \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \mathfrak{H}_b^{\alpha\gamma} f,
\end{aligned}
$$

again by the orthogonality of the Haar functions in the penultimate step. □

Proposition 12.2.3. *Let X and Y be UMD spaces and $p \in (1,\infty)$. Let $b \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$. Then $\Lambda_b := \Pi_b + \Pi_b^*$, initially defined on $S_{00}(\mathscr{D}; X)$, extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$ of norm*

$$
\|\Lambda_b\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant \Big(1 + (2^d-1)^2 \beta_{p,Y}^- \beta_{p,Y}^+ \beta_{p,X}^+\Big) \|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))},
$$

and we have the identity

$$
bf = \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \mathfrak{H}_b^{\alpha\gamma} f + \Lambda_b f \qquad \forall\, f \in L^p(\mathbb{R}^d; X).
$$

We will obtain a far better estimate in Theorem 12.2.25, but it seems worthwhile recording this relatively simple bound as an illustration of the techniques that we have developed thus far.

Proof of Proposition 12.2.3. It is clear that pointwise multiplication by $b \in L^\infty(\mathbb{R}^d; \mathscr{L}(X,Y))$ defines a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$, for any Banach spaces X, Y and all $p \in [1, \infty]$. Moreover, the Haar multiplier $\mathfrak{H}_b^{\alpha\gamma}$ featuring in Proposition 12.2.3 have exactly the form considered in Proposition 12.1.13, and hence

$$\|\mathfrak{H}_b^{\alpha\gamma} f\|_{L^p(\mathbb{R}^d;Y)} \leqslant \beta_{p,Y}^{-} \beta_{p,Y}^{+} \beta_{p,X}^{+} \|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))} \|f\|_{L^p(\mathbb{R}^d;X)}.$$

By triangle inequality, it then follows from (12.23) that

$$\begin{aligned}\|\Lambda_b f\|_{L^p(\mathbb{R}^d;Y)} &\leqslant \|bf\|_{L^p(\mathbb{R}^d;Y)} + \sum_{\alpha,\gamma \in \{0,1\}^d \setminus \{0\}} \|\mathfrak{H}_b^{\alpha\gamma} f\|_{L^p(\mathbb{R}^d;Y)} \\ &\leqslant \|b\|_{L^\infty(\mathbb{R}^d;\mathscr{L}(X,Y))} \Big(1 + (2^d - 1)^2 \beta_{p,Y}^{-} \beta_{p,Y}^{+} \beta_{p,X}^{+}\Big) \|f\|_{L^p(\mathbb{R}^d;X)}\end{aligned}$$

for all $f \in S_{00}(\mathscr{D}; X)$, and hence Λ_b extends to a bounded operator from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y))$ with the asserted norm estimate. Since the claimed identity holds (by Proposition 12.2.1) for all $f \in S_{00}(\mathscr{D}; X)$, and each term is continuous with respect to the $L^p(\mathbb{R}^d; X)$ norm of f (as we just showed), it is immediate that this identity extends to all $f \in L^p(\mathbb{R}^d; X)$. □

As we shall see later, the operator Λ_b is not only as good as, but actually *better* than the pointwise product $f \mapsto bf$, in the sense that it remains a bounded operator for a broader class of functions b than just the bounded ones. As the reader will have guessed from the introduced notation, we will also be interested in the mapping properties of the individual paraproducts Π_b and Π_b^*.

While the paraproduct Π_b arouse from our analysis of the pointwise product with a multiplier b, in other considerations we will encounter similar series

$$\Pi f = \sum_{\substack{Q \in \mathscr{D} \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \pi_Q^\alpha \langle f \rangle_Q \otimes h_Q^\alpha$$

with some coefficient π_Q^α replacing the Haar coefficients $\langle b, h_Q^\alpha \rangle$ of a function b above. Formally, we always have $\pi_Q^\alpha = \langle b, h_Q^\alpha \rangle$ by choosing

$$\text{“} \quad b := \Pi(1) = \sum_{\substack{Q \in \mathscr{D} \\ \alpha \in \{0,1\}^d \setminus \{0\}}} \pi_Q^\alpha \otimes h_Q^\alpha \quad \text{”},$$

but giving a precise meaning for this series requires non-trivial considerations in general, and it is hence useful not to insist in the a priori existence of function b generating the coefficients in this way.

12.2.a Necessary conditions for boundedness

As we already saw in the analysis of the pointwise product bf, and we will see again in the $T(1)$ theorem below, paraproducts frequently appear in pairs of the form $\Pi_1 + \Pi_2^*$, where Π_1 is a paraproduct as in the previous section, and Π_2^* is the formal adjoint of another paraproduct. In other words, we are concerned with the operator

$$\Lambda f := \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big(\pi_Q^{1,\alpha}\langle f\rangle_Q h_Q^\alpha + \pi_Q^{2,\alpha}\langle f, h_Q^\alpha\rangle \frac{\mathbf{1}_Q}{|Q|}\Big). \tag{12.24}$$

Of course this covers both Π_1 and Π_2^* as special cases, by simply setting some of the coefficients $\pi_Q^{i,\alpha}$ equal to zero.

Compared to the operator Λ_b featuring in Proposition 12.2.3, we now allow possibly different coefficients $\pi_Q^{1,\alpha}$ and $\pi_Q^{2,\alpha}$ in the first and second term above, as this will be relevant in the $T(1)$ theorem. Via the duality relations

$$\langle \Lambda f, g\rangle = \langle f, \Lambda^* g\rangle = \mathfrak{L}(f,g), \qquad f\in S_{00}(\mathscr{D};X),\ g\in S_{00}(\mathscr{D};Y^*),$$

we define the formal adjoint

$$\Lambda^* g := \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big((\pi_Q^{1,\alpha})^*\langle g, h_Q^\alpha\rangle \frac{\mathbf{1}_Q}{|Q|} + (\pi_Q^{2,\alpha})^*\langle g\rangle_Q h_Q^\alpha\Big) \tag{12.25}$$

which has exactly the same form as Λ, only with different coefficients, and the associated bilinear form

$$\mathfrak{L}(f,g) := \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big(\Big\langle \pi_Q^{1,\alpha}\langle f\rangle_Q, \langle h_Q^\alpha, g\rangle\Big\rangle + \Big\langle \pi_Q^{2,\alpha}\langle f, h_Q^\alpha\rangle, \langle g\rangle_Q\Big\rangle\Big). \tag{12.26}$$

Lemma 12.2.4. *The series* (12.26) *is finitely non-zero whenever*

$$(f,g)\in \big(S_{00}(\mathscr{D};X)\times S(\mathscr{D};Y^*)\big)\cup\big(S(\mathscr{D};X)\times S_{00}(\mathscr{D};Y^*)\big).$$

In particular, we have

$$\mathfrak{L}(x\otimes \mathbf{1}_R, y^*\otimes h_R^\beta) = \langle \pi_R^{1,\beta}x, y^*\rangle, \quad \mathfrak{L}(x\otimes h_R^\beta, y^*\otimes \mathbf{1}_R) = \langle \pi_R^{2,\beta}x, y^*\rangle$$

for all $x\in X$, $y^*\in Y^*$, $R\in\mathscr{D}$ and $\beta\in\{0,1\}^d\setminus\{0\}$.

Proof. By symmetry, it is enough to consider $(f,g)\in S_{00}(\mathscr{D};X)\times S(\mathscr{D};Y^*)$. We may further assume that

$$f = x\otimes h_P^\beta, \qquad g = y^*\otimes \mathbf{1}_R$$

for some $x\in X, y^*\in Y^*$, $P,R\in\mathscr{D}$ and $\beta\in\{0,1\}^d\setminus\{0\}$, since general f and g are finite linear combinations of such functions.

For such f and g, the (Q,α) term in (12.26), is given by

$$\langle \pi^{1,\alpha}x, y^*\rangle\langle h_P^\beta\rangle_Q\langle h_Q^\alpha, \mathbf{1}_R\rangle + \langle \pi^{2,\alpha}x, y^*\rangle\langle h_P^\beta, h_Q^\alpha\rangle\langle \mathbf{1}_R\rangle_Q,$$

where $\langle h_P^\beta\rangle_Q \neq 0$ only if $Q \subsetneq P$, while $\langle h_Q^\alpha, \mathbf{1}_R\rangle \neq 0$ only if $R \subsetneq Q$; finally, $\langle h_P^\beta, h_Q^\alpha\rangle \neq \delta_{P,Q}\delta_{\alpha,\beta}$. Thus

$$\mathfrak{L}(x \otimes \mathbf{1}_P, y^* \otimes h_R^\beta) := \sum_{\substack{Q\in\mathscr{D} \\ R\subsetneq Q\subsetneq P \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \langle \pi_Q^{1,\alpha}x, y^*\rangle\langle h_P^\beta\rangle_Q\langle h_Q^\alpha, \mathbf{1}_R\rangle + \langle \pi_P^{2,\beta}x, y^*\rangle\langle \mathbf{1}_R\rangle_P,$$

which is clearly a finite sum. When $P = R$, the sum above is void, and we get

$$\mathfrak{L}(x \otimes \mathbf{1}_R, y^* \otimes h_R^\beta) = \langle \pi_R^{2,\beta}x, y^*\rangle.$$

The other case follows by symmetry. □

Although our main concern is L^p boundedness, we formulate the following lemma slightly more generally, since the additional generality comes essentially for free.

Lemma 12.2.5. *Let $p, q \in [1,\infty)$. A necessary condition for $\mathfrak{L}$ to satisfy*

$$|\mathfrak{L}(f,g)| \leqslant C\|f\|_{L^p(\mathbb{R}^d;X)}\|g\|_{L^{q'}(\mathbb{R}^d;Y^*)},$$

uniformly for all (f,g) of the form $(x\otimes\mathbf{1}_Q, y^*\otimes h_Q^\alpha)$ and $(x\otimes h_Q^\alpha, y^*\otimes\mathbf{1}_Q)$, is that

$$\|\pi_Q^{i,\alpha}\|_{\mathscr{L}(X,Y)} \leqslant C|Q|^\gamma, \qquad \gamma := \frac{1}{p} - \frac{1}{q} + \frac{1}{2} < \frac{3}{2}. \tag{12.27}$$

On the other hand, assuming the coefficient bound (12.27), the defining series (12.26) of $\mathfrak{L}(f,g)$ converges absolutely for all

$$(f,g) \in S(\mathscr{D};X)\times S(\mathscr{D};Y^*)$$

Proof. We have

$$\begin{aligned} |\langle \pi_Q^{1,\alpha}x, y^*\rangle| &= |\mathfrak{L}(x\otimes\mathbf{1}_Q, y^*\otimes h_Q^\alpha)| \\ &\leqslant C\|x\otimes\mathbf{1}_Q\|_{L^p(\mathbb{R}^d;X)}\|y^*\otimes h_Q^\alpha\|_{L^{q'}(\mathbb{R}^d;Y^*)} \\ &= C\|x\|_X|Q|^{1/p}\|y^*\|_{Y^*}|Q|^{1/q'-1/2} \\ &= C\|x\|_X\|y^*\|_{Y^*}|Q|^{1/p-1/q+1/2} \end{aligned}$$

and taking the supremum over $\|y^*\|_{Y^*} \leqslant 1$ and $\|x\|_X \leqslant 1$ proves the estimate for $i = 1$. The case $i = 2$ is entirely symmetric. Finally, note that $1/p, 1/q \in (0,1]$ so that $1/p - 1/q < 1$.

To prove the convergence, it is enough to consider $f = x\otimes\mathbf{1}_P$, $g = y^*\otimes\mathbf{1}_R$, and moreover, by symmetry, just the first half of $\mathscr{L}(f,g)$ with coefficients $\pi_Q^{1,\alpha}$. Now

$$|\langle \pi_Q^{1,\alpha}\langle f\rangle_Q, \langle g, h_Q^\alpha\rangle\rangle| = |\langle \pi_Q^{1,\alpha}x, y^*\rangle||\langle \mathbf{1}_P\rangle_Q|\langle \mathbf{1}_R, h_Q^\alpha\rangle|,$$

where

$$|\langle \pi_Q^{1,\alpha} x, y^* \rangle| \leqslant C|Q|^\gamma \|x\|_X \|y^*\|_{Y^*}, \quad \langle \mathbf{1}_P \rangle_Q \leqslant \frac{|P|}{|Q|}, \quad |\langle \mathbf{1}_R, h_Q^\alpha \rangle| \leqslant \frac{|R|}{|Q|^{1/2}},$$

and moreover the last pairing is non-zero only if $Q \supsetneq R$. Hence the absolute convergence of the series follows from the convergence of

$$\sum_{\substack{Q\in\mathscr{D} \\ Q\supsetneq R}} |Q|^{\gamma-3/2} = |R|^{\gamma-3/2} \sum_{k=1}^\infty 2^{kd(\gamma-3/2)} < \infty,$$

since this is as a geometric series with $\gamma - 3/2 < 0$. □

Lemma 12.2.6. *Suppose that the series defining Λf converges (even just conditionally) in $L^p(\mathbb{R}^d; Y)$ for some $f = \mathbf{1}_R \otimes x$, where $R \in \mathscr{D}$ and $x \in X$. Then*

$$(\mathbf{1}_R - E_R)\Lambda(\mathbf{1}_R \otimes x) = \sum_{\substack{Q\subseteq R \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \pi_Q^{1,\alpha} x \otimes h_Q^\alpha$$

Proof. We have

$$\mathbf{1}_R\Big(\pi_Q^{1,\alpha}\langle \mathbf{1}_R\otimes x\rangle_Q h_Q^\alpha + \pi_Q^{2,\alpha}\langle \mathbf{1}_R\otimes x, h_Q^\alpha\rangle \frac{\mathbf{1}_Q}{|Q|}\Big) = \begin{cases} \pi_Q^{1,\alpha} x\otimes h_Q^\alpha + 0, & Q\subseteq R, \\ y_{Q,R}^\alpha \otimes \mathbf{1}_R, & Q \not\subseteq R, \end{cases}$$

for some $y_{Q,R}^\alpha \in Y$, which is not difficult to find explicitly, but it is irrelevant for the present purposes. The assumed convergence in $L^p(\mathbb{R}^d; Y)$, and the boundedness of the conditional expectation E_R and the pointwise multiplier $\mathbf{1}_R$ on $L^p(\mathbb{R}^d; Y)$ guarantee that we can move $(\mathbf{1}_R - E_R)$ inside the defining series. Since $E_R(y_{Q,R}^\alpha \otimes \mathbf{1}_R) = y_{Q,R}^\alpha \otimes \mathbf{1}_R$, we have

$$(\mathbf{1}_R - E_R)\Lambda(\mathbf{1}_R \otimes x) = \sum_{\substack{Q\subseteq R \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \pi_Q^{1,\alpha} x \otimes h_Q^\alpha,$$

as claimed. □

Lemma 12.2.7. *Let Y be a Banach space, and $p \in [1,\infty)$. Let $y_Q^\alpha \in Y$ for all $Q \in \mathscr{D}$, $\alpha \in \{0,1\}^d \setminus \{0\}$. For each $R \in \mathscr{D}$ and $n \in \mathbb{N}$, consider the sum*

$$B_R^n := \sum_{\substack{Q\subseteq R \\ \ell(Q)>2^{-n}\ell(R) \\ \alpha\in\{0,1\}^d\setminus\{0\}}} y_Q^\alpha \otimes h_Q^\alpha$$

Suppose that, for every $R \in \mathscr{D}$, we have one of the following

(1) $B_R := \lim_{n\to\infty} B_R^n$ *exists in $L^p(\mathbb{R}^d; Y)$, or*

(2) *Y has the Radon–Nikodým property, and* $\sup_{n\in\mathbb{N}} \|B_R^n\|_{L^p(\mathbb{R}^d;Y)} < \infty$.

Then there exists a function $b \in L^p_{\mathrm{loc}}(\mathbb{R}^d;Y)$ *such that*

$$\mathbf{1}_R(b - \langle b\rangle_R) = B_R, \quad \langle b, h_R^\alpha\rangle = y_R^\alpha, \quad \forall R \in \mathscr{D},\ \alpha \in \{0,1\}^d \setminus \{0\}.$$

If, moreover, the supremum below is finite, then $b \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;Y)$ *and*

$$\sup_{\substack{Q\in\mathscr{D}\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \frac{\|y_Q^\alpha\|_Y}{|Q|^{1/2}} \leqslant \|b\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Y)} = \sup_{R\in\mathscr{D}} \inf_{c\in Y} \frac{\|B_R - c\|_{L^p(\mathbb{R}^d;Y)}}{|R|^{1/p}} \tag{12.28}$$

Proof. It is immediate to verify that $(B_R^n)_{n=0}^\infty$ is a martingale in $L^p(\mathbb{R}^d;Y)$. By the Martingale Convergence Theorem 3.3.16, it follows that (2) implies (1). Hence it suffices to prove the lemma under assumption (1).

We construct the function b via the correspondence established in Lemma 11.2.11. It is enough to construct $b|_S$ separately for each quadrant $S \subseteq \mathbb{R}^d$. So we fix a quadrant $S \subseteq \mathbb{R}^d$, and let

$$\Delta(s,t) := \sum_{\substack{Q\in\mathscr{D}(S)\\ \alpha\in\{0,1\}^d\setminus\{0\}}} (h_Q^\alpha(s) - h_Q^\alpha(t))y_Q^\alpha,$$

where we need to justify the convergence of this series in some sense. We will prove that it converges in $L^p_{\mathrm{loc}}(S\times S;Y)$. To this end, note that any bounded subset of $S\times S$ is contained in $R\times R$ for some $R\in\mathscr{D}(S)$. For $s,t\in R$, only $Q\in\mathscr{D}(S)$ with $Q\cap R\neq\varnothing$ can contribute to the series; moreover, if $Q \supsetneq R$, then h_Q^α is constant on R, and hence $h_Q^\alpha(s) - h_Q^\alpha(t) = 0$ for $s,t\in R$. Thus

$$\begin{aligned}(\mathbf{1}_{R\times R}\Delta)(s,t) &= \mathbf{1}_{R\times R}(s,t) \sum_{\substack{Q\in\mathscr{D}(R)\\ \alpha\in\{0,1\}^d\setminus\{0\}}} (h_Q^\alpha(s) - h_Q^\alpha(t))y_Q^\alpha \\ &= \mathbf{1}_{R\times R}(s,t)(B_R(s) - B_R(t)),\end{aligned} \tag{12.29}$$

where the (conditional) convergence in $L^p(R\times R, \mathrm{d}s\,\mathrm{d}t;Y)$ follows by Fubini's theorem from the assumed (conditional) convergence of each B_R in $L^p(R;Y)$.

Now that the convergence of the defining series of $\Delta(s,t)$ has been justified, it is immediate from the defining formula that $\Delta(s,t) + \Delta(t,u) = \Delta(s,u)$ for $s,t,u\in S$. By Lemma 11.2.11, we have the existence of $b : S\to Y$ such that $\Delta(s,t) = b(s) - b(t)$. substituting this into (12.29), we obtain

$$b(s) - b(t) = B_R(s) - B_R(t), \qquad \text{for } s,t\in R,$$

and hence $b(\cdot) = B_R(\cdot) + (b(t) - B_R(t)) \in L^p(R;Y) \subseteq L^1(R;Y)$. Taking the average over $t\in R$, it follows that

$$b(s) - \langle b\rangle_R = B_R(s) - \langle B_R\rangle_R = B_R(s), \qquad \text{for } s\in R,$$

observing that $\langle h_Q^\alpha\rangle_R = 0$ for all $Q \subseteq R$ that appear in the series of B_R. Then it also follows that

$$\langle b, h_R^\alpha\rangle = \langle \mathbf{1}_R(b - \langle b\rangle_R), h_R^\alpha\rangle = \langle B_R, h_R^\alpha\rangle = y_R^\alpha.$$

This also implies, for any $c \in Y$, that

$$\frac{\|y_Q^\alpha\|_Y}{|Q|^{1/2}} = \Big\|\Big\langle B_Q - c, \frac{h_Q^\alpha}{|Q|^{1/2}}\Big\rangle\Big\|_Y \leqslant \int_Q \|B_Q - c\|_Y \frac{1}{|Q|} \leqslant \Big(\fint_Q \|B_Q - c\|_Y^p\Big)^{1/p},$$

and (12.28) follows from the identity $B_Q = \mathbf{1}_Q(b - \langle b\rangle_Q)$, which implies that

$$\begin{aligned}\|b\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Y)} &:= \sup_{Q\in\mathscr{D}} \inf_{c\in Y} \|\mathbf{1}_Q(b - c)\|_{L^p(Q;Y)} \\ &= \sup_{Q\in\mathscr{D}} \inf_{c'\in Y} \|\mathbf{1}_Q(B_Q - c')\|_{L^p(Q;Y)}\end{aligned}$$

by a simple change of variable. □

Proposition 12.2.8. *Let X and Y be Banach spaces and $p \in (1,\infty)$. Let $\pi_Q^{i,\alpha} \in \mathscr{L}(X,Y)$, and let Λ be defined by the formal series in (12.24).*

(1) *If, for some $x \in Xm$ the series (12.24) defining Λf converges (even just conditionally) in $L^p(\mathbb{R}^d;Y)$ whenever $f = \mathbf{1}_R \otimes x$ for $R \in \mathscr{D}$, and these satisfy the testing condition*

$$\|\mathbf{1}_R\Lambda(\mathbf{1}_R \otimes x)\|_{L^p(\mathbb{R}^d;Y)} \leqslant \mathscr{T}_\Lambda^x |R|^{1/p},$$

then $\|\pi_Q^{1,\alpha}x\|_Y \leqslant \mathscr{T}_\Lambda^x |Q|^{1/2}$ and there is $b_1^x \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;Y)$ of norm $\|b_1^x\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Y)} \leqslant \mathscr{T}_\Lambda^x$ such that $\pi_Q^{1,\alpha}x = \langle b_1^x, h_Q^\alpha\rangle$.

(2) *If, in addition to (1), we have $X = Y$ and $\pi_Q^{1,\alpha} \in \mathbb{K}$, then $b_1^x = b_1 \otimes x$ for some $b_1 \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$ that is independent of x.*

(3) *If, for some $y^* \in Y^*$, the series (12.25) defining $\Lambda^* g$ converges (even just conditionally) in $L^{p'}(\mathbb{R}^d;Y^*)$ whenever $g = \mathbf{1}_R \otimes y^*$ for $R \in \mathscr{D}$, and these satisfy the testing condition*

$$\|\mathbf{1}_R\Lambda^*(\mathbf{1}_R \otimes y^*)\|_{L^{p'}(\mathbb{R}^d;X^*)} \leqslant \mathscr{T}_{\Lambda^*}^{y^*} |R|^{1/p'},$$

then $\|(\pi_Q^{2,\alpha})^* y^*\|_{X^*} \leqslant \mathscr{T}_{\Lambda^*}^{y^*} |Q|^{1/2}$ and there is $b_2^{y^*} \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;X^*)$ with $\|b_2^{y^*}\|_{\mathrm{BMO}^{p'}_{\mathscr{D}}(\mathbb{R}^d;X^*)} \leqslant \mathscr{T}_{\Lambda^*}^{y^*}$ and $(\pi_Q^{2,\alpha})^* y^* = \langle b_2^{y^*}, h_Q^\alpha\rangle$.

(4) *If, in addition to (3), we have $X = Y$ and $\pi_Q^{2,\alpha} \in \mathbb{K}$, then $b_2^{y^*} = b_2 \otimes y^*$ for some $b_2 \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$.*

Proof. (1): Let us fix an $x \in X$. By assumption and Lemma 12.2.6, the series

$$B_R^x := \sum_{\substack{Q\subseteq R \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \pi_Q^{1,\alpha} x \otimes h_Q^\alpha = (\mathbf{1}_R - E_R)\Lambda(\mathbf{1}_R \otimes x)$$

converges (conditionally) in $L^p(\mathbb{R}^d;Y)$. Since $E_R\Lambda(\mathbf{1}_R\otimes x)$ is constant on R, we have the uniform estimate

$$\inf_{c\in Y}\|B_R^x-c\|_{L^p(\mathbb{R}^d;Y)}\leqslant\|\mathbf{1}_R\Lambda(\mathbf{1}_R\otimes x)\|_{L^p(\mathbb{R}^d;Y)}\leqslant\mathscr{T}_\Lambda^x|R|^{1/p}.$$

By Lemma 12.2.7, there is a function $b_1^x\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;Y)$ such that

$$\|b_1^x\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Y)}\leqslant\mathscr{T}_\Lambda^x,\quad\langle b_1^x,h_Q^\alpha\rangle=\pi_Q^\alpha x$$

for all $Q\in\mathscr{D}$ and $\alpha\in\{0,1\}^d\setminus\{0\}$,and $\|\pi_Q^\alpha x\|_Y\leqslant\mathscr{T}_\Lambda|Q|^{1/2}\|x\|_X$, from which the claimed bound for $\|\pi_Q^\alpha\|_{\mathscr{L}(X,Y)}$ is immediate.

(2): Under the assumptions of this case, an inspection of the previous argument shows that all auxiliary functions in the construction of b_1^x have the form $\phi\otimes x$ for different scalar functions ϕ, and hence this form also remains in the final result.

(3)–(4) follow by repeating the proof of (1)–(2) on the dual side. □

Remark 12.2.9. In the setting of Proposition 12.2.8, if we know *a priori* that $\pi_Q^{1,\alpha}x=\langle b_1(\cdot)x,h_Q^\alpha\rangle$ for some $b_1\in L^1_{\mathrm{loc,so}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, then our conclusion on b_1^x implies that $b_1\in\mathrm{BMO}_{\mathscr{D},\mathrm{so}}(\mathbb{R}^d;\mathscr{L}(X,Y))$ and

$$\|b_1\|_{\mathrm{BMO}^p_{\mathscr{D},\mathrm{so}}(\mathbb{R}^d;Y)}\leqslant\mathscr{T}_\Lambda.$$

According to Proposition 12.2.8, the following is a natural necessary condition for the L^p boundedness of paraproducts, even when restricted to very special functions only.

Definition 12.2.10. *We say that a paraproduct Λ as in (12.24) satisfies the* weak coefficient bound *if there is a finite constant C such that*

$$\|\pi_Q^{i,\alpha}\|_{\mathscr{L}(X,Y)}\leqslant C|Q|^{1/2}\tag{12.30}$$

for all $i=1,2$, $\alpha\in\{0,1\}^d\setminus\{0\}$ and $Q\in\mathscr{D}$.

While rather far from being a sufficient condition for any interesting boundedness results, this weak coefficient bound nevertheless allows us to make sense of the defining series of the paraproduct on a sufficiently rich class of functions for our subsequent purposes.

We have the following useful convergence result for *truncated* paraproducts:

Lemma 12.2.11. *Suppose that $\pi_Q^\alpha\in\mathscr{L}(X,Y)$ satisfy (12.30). Let $p\in(1,\infty)$ and $f\in L^p(\mathbb{R}^d;X)$ be boundedly supported, and consider the truncated paraproduct*

$${}_m\Pi f:=\sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)>2^{-m}\\ \alpha\in\{0,1\}^d\setminus\{0\}}}\pi_Q^\alpha\langle f\rangle_Q h_Q^\alpha.$$

(1) *For any* $m \in \mathbb{Z}$, *the series defining* ${}_m\Pi f$ *converges absolutely in* $L^p(\mathbb{R}^d;Y)$.
(2) *For* $2^{-m} \geqslant \operatorname{diam}(\operatorname{supp} f)$, *we have*

$$\|{}_m\Pi f\|_{L^p(\mathbb{R}^d;Y)} \leqslant c_{d,p} C \|E_m f\|_{L^p(\mathbb{R}^d;X)}, \tag{12.31}$$

and if $g \in L^{p'}(\mathbb{R}^d;Y^*)$, *then*

$$|\langle {}_m\Pi f, g\rangle| \leqslant c_{d,p} C \|E_m f\|_{L^p(\mathbb{R}^d;X)} \|E_m g\|_{L^{p'}(\mathbb{R}^d;Y^*)} \underset{m\to-\infty}{\longrightarrow} 0, \tag{12.32}$$

where C *is the constant in* (12.30) *and* $c_{d,p} = \dfrac{2^d-1}{1-2^{-d/p'}}$.

Proof. Let us first consider (2). When $2^{-m} \geqslant \operatorname{diam}(\operatorname{supp} f)$, the support $\operatorname{supp} f$ is contained in at most 2^d cubes $R_i \in \mathscr{D}$. Then in ${}_m\Pi f$, we only need to consider $Q \in \mathscr{D}$ with $Q \supsetneq R_i$ for some (not necessarily unique) $i = 1, \ldots, 2^d$. Then

$$\|{}_m\Pi f\|_{L^p(\mathbb{R}^d;Y)} = \sum_{i=1}^{2^d} \sum_{\substack{Q\in\mathscr{D} \\ Q\supsetneq R_i \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \|\pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha\|_{L^p(\mathbb{R}^d;Y)},$$

where

$$\begin{aligned}\|\pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha\|_{L^p(\mathbb{R}^d;Y)} &\leqslant \|\pi_Q^\alpha\|_{\mathscr{L}(X,Y)} \|\langle f\rangle_Q\|_X \|h_Q^\alpha\|_{L^p(\mathbb{R}^d)} \\ &= \|\pi_Q^\alpha\|_{\mathscr{L}(X,Y)} \frac{1}{|Q|} \Big\| \int_{R_i} f \Big\|_X \frac{|Q|^{1/p}}{|Q|^{1/2}} \leqslant \frac{C}{|Q|^{1/p'}} \Big\| \int_{R_i} f \Big\|_X,\end{aligned}$$

and hence

$$\begin{aligned}\|{}_m\Pi f\|_{L^p(\mathbb{R}^d;Y)} &\leqslant \sum_{i=1}^{2^d} (2^d-1) C \Big\| \int_{R_i} f \Big\|_X \sum_{Q\supsetneq R_i} \frac{1}{|Q|^{1/p'}} \\ &= \sum_{i=1}^{2^d} \frac{(2^d-1)C}{|R_i|^{1/p'}} \Big\| \int_{R_i} f \Big\|_X \sum_{k=1}^{\infty} 2^{-kd/p'} \\ &= \frac{2^d-1}{2^{d/p'}-1} C \sum_{i=1}^{2^d} |R_i|^{1/p} \Big\| \fint_{R_i} f \Big\|_X,\end{aligned}$$

where

$$\sum_{i=1}^{2^d} |R_i|^{1/p} \Big\| \fint_{R_i} f \Big\|_X \leqslant 2^{d/p'} \Big(\sum_{i=1}^{2^d} |R_i| \Big\| \fint_{R_i} f \Big\|_X^p \Big)^{1/p} = 2^{d/p'} \|E_m f\|_{L^p(\mathbb{R}^d;X)}.$$

This proves both the convergence of the series and the claimed bound (12.31).

Each term in the series defining ${}_m\varPi f$ is constant on cubes $R \in \mathscr{D}_m$. Hence ${}_m\varPi f = E_m({}_m\varPi f)$, and thus

$$|\langle {}_m\varPi f, g\rangle| = |\langle {}_m\varPi f, E_m g\rangle| \leqslant \|{}_m\varPi f\|_{L^p(\mathbb{R}^d;Y)} \|E_m g\|_{L^{p'}(\mathbb{R}^d;Y^*)},$$

so that (12.32) follows from (12.31).

Concerning (1), it only remains to consider the part of the series with $2^{-m} < \ell(Q) \leqslant \operatorname{diam}(\operatorname{supp} f)$. But there are only finitely many cubes Q of fixed side-length that intersect $\operatorname{supp} f$, and hence only finitely many cubes altogether that contribute to this remaining sub-series. Thus the absolute convergence is trivial. □

Corollary 12.2.12. *Suppose that $\pi_Q^\alpha \in \mathscr{L}(X,Y)$ satisfy (12.30). Then the series*

$$\sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big\langle \pi_Q^\alpha \langle f\rangle_Q, \langle h_Q^\alpha, g\rangle \Big\rangle$$

defining $\langle \varPi f, g\rangle$ converges absolutely for all $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$, and

$$\langle \varPi f, E_m g\rangle = \langle {}_m\varPi f, g\rangle.$$

Proof. Let $v := g - E_m g$. Then $v \in S_{00}(\mathscr{D};Y^*)$, and hence only finitely many of the terms $\langle h_Q^\alpha, v\rangle$ are non-zero. Hence it is enough to prove the convergence with $E_m g$ in place of g. Since $\langle h_Q^\alpha, E_m g\rangle = 0$ when $\ell(Q) \leqslant 2^{-m}$, this coincides with the series of $\langle {}_m\varPi f, E_m g\rangle$. Since $f \in S(\mathscr{D};X) \subseteq L^p(\mathbb{R}^d;X)$ is boundedly supported, the series defining ${}_m\varPi f$ converges absolutely in $L^p(\mathbb{R}^d;Y)$ by Lemma 12.2.11. Since $E_m g \in S(\mathscr{D};Y^*) \subseteq L^{p'}(\mathbb{R}^d;Y^*) \subseteq (L^p(\mathbb{R}^d;Y))^*$, the series of the bilinear form converges absolutely in $\mathbb{K}$.

The last identity follows by observing that

$$\langle h_Q^\alpha, E_m g\rangle = \begin{cases} \langle h_Q^\alpha, g\rangle, & \ell(Q) > 2^{-m}, \\ 0, & \ell(Q) \leqslant 2^{-m}, \end{cases}$$

and the proof is complete. □

Corollary 12.2.13. *Suppose that $\pi_Q^{i,\alpha} \in \mathscr{L}(X,Y)$ satisfy (12.30). Then the two series*

$$\sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big\langle \pi_Q^{1,\alpha} \langle f\rangle_Q, \langle h_Q^\alpha, g\rangle \Big\rangle + \sum_{\substack{Q\in\mathscr{D} \\ \alpha\in\{0,1\}^d\setminus\{0\}}} \Big\langle \pi_Q^{2,\alpha} \langle h_Q^\alpha, f\rangle, \langle g\rangle_Q \Big\rangle$$

defining $\langle \varLambda f, g\rangle$ converge absolutely for all $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$.

Proof. The convergence of the first series is the content of Corollary 12.2.12. The convergence of the second series follows by permuting the roles of $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$, and transposing $\pi_Q^{2,\alpha}$ to the dual side, since $(\pi_Q^{2,\alpha})^* \in \mathscr{L}(Y^*,X^*)$ satisfies the same estimate. □

12.2.b Sufficient conditions for boundedness

We will then turn to exploring conditions that ensure the boundedness of the full paraproduct Π. The obtained necessary conditions serve as a model for the type of sufficient conditions that we are looking for.

It is convenient to begin with a reduction to finite series. When Y is reflexive, we have

$$L^p(\mathbb{R}^d;Y) = L^p(\mathbb{R}^d;Y^{**}) \simeq (L^{p'}(\mathbb{R}^d;Y^*))^*.$$

Since $S_{00}(\mathscr{D};Y^*)$ is dense in $L^{p'}(\mathbb{R}^d;Y^*)$, it is enough to show that the action of Πf is bounded on $S_{00}(\mathscr{D};Y^*)$ with respect to the norm of $L^{p'}(\mathbb{R}^d;Y^*)$, uniformly for f in the unit ball of $L^p(\mathbb{R}^d;X)$. Since any fixed $g \in S_{00}(\mathscr{D};Y^*)$ only "sees" a finite part of Πf, it is enough to prove a uniform $L^p(\mathbb{R}^d;Y)$ estimate for the finite sums $\sum \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha$. A key initial estimate in this direction is the following:

Lemma 12.2.14. *Let X be a Banach space, Y be a UMD space, and $p \in (1,\infty)$. Let $\mathscr{F}$ be a finite collection of dyadic cubes. For all $f \in L^p(\mathbb{R}^d;X)$ and $\pi_Q^\alpha \in \mathscr{L}(X,Y)$, we then have*

$$\Big\| \sum_{Q\in\mathscr{F}} \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \beta_{p,Y}^- \beta_{p,Y}^+ \Big\| \Big(\sum_{Q\in\mathscr{F}} \varepsilon_Q \pi_Q^\alpha h_Q^0 \Big) f \Big\|_{L^p(\mathbb{R}^d;Y)}.$$

Proof. Since $(\pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha)_{Q\in\mathscr{F}}$ is a martingale difference sequence in $L^p(\mathbb{R}^d;Y)$, we have

$$\Big\| \sum_{Q\in\mathscr{F}} \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d;Y)} \leqslant \beta_{p,Y}^- \Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)}.$$

Rewriting the L^p norm on the product $\mathbb{R}^d \times \Omega$ with the help of Fubini's theorem, we observe that at each fixed $t \in \mathbb{R}^d$, the sequence of random variables

$$\varepsilon_Q \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha(t)$$

has the same joint distribution as

$$\varepsilon_Q \pi_Q^\alpha \langle f\rangle_Q h_Q^0(t) = \varepsilon_Q \mathbb{E}_Q(\pi_Q^\alpha f h_Q^0)(t),$$

since the possibly different sign of $h_Q^\alpha(t)$ and $h_Q^0(t)$ is invisible after multiplication by ε_Q. Using this and Stein's inequality (Theorem 4.2.23), we conclude that

$$\begin{aligned}\Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \pi_Q^\alpha \langle f\rangle_Q h_Q^\alpha \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)} &= \Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \mathbb{E}_Q(\pi_Q^\alpha f h_Q^0) \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)} \\ \leqslant \beta_{p,Y}^+ \Big\| \sum_{Q\in\mathscr{F}} \varepsilon_Q \pi_Q^\alpha f h_Q^0 \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)} &= \beta_{p,Y}^+ \Big\| \Big(\sum_{Q\in\mathscr{F}} \varepsilon_Q h_Q^0 \pi_Q^\alpha \Big) f \Big\|_{L^p(\mathbb{R}^d\times\Omega;Y)}.\end{aligned}$$

□

The previous lemma motivates the following. A background for the nomenclature will be discussed in the Notes.

Definition 12.2.15. *Let $p \in (1,\infty)$. For an indexed family $(\pi_Q)_{Q\in\mathscr{D}}$ in a Banach space Z, we define the* Carleson norm

$$\|(\pi_Q)\|_{\mathrm{Car}^p(\mathbb{R}^d;Z)} := \sup_{Q_0\in\mathscr{D}} \sup_{\substack{\mathscr{F}\subseteq\mathscr{D}\\ \text{finite}}} \frac{1}{|Q_0|^{1/p}} \Big\| \sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{F}}} \varepsilon_Q h_Q^0 \pi_Q \Big\|_{L^p(Q_0\times\Omega;Z)}.$$

With the help of Theorem 3.2.17 (the John–Nirenberg inequality for adapted sequences), one can check that any these Carleson norms are actually equivalent for different values of p. We will not need this observation, since the following proof directly shows that we can use any of these norms in our upper bound, as we like. Our first sufficient condition for paraproduct boundedness is stated in terms of this notion as follows:

Proposition 12.2.16 (Paraproducts vs. Carleson norms). *Let X be a Banach space, Y be a UMD space, and $p,q \in (1,\infty)$. Let Π be the paraproduct defined by an indexed family $(\pi_Q^\alpha)_{Q\in\mathscr{D},\alpha\in\{0,1\}^d\setminus\{0\}}$. In order that Π is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$, it is sufficient that $(\pi_Q^\alpha)_{Q\in\mathscr{D}}$ satisfies the* Car^p *condition for every $\alpha \in \{0,1\}^d \setminus \{0\}$. Moreover, we have the bound*

$$\|\Pi\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant 32\cdot 4^d pp' \beta_{q,Y}^- \beta_{q,Y}^+ \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \|(\pi_Q^\alpha)\|_{\mathrm{Car}^q(\mathbb{R}^d;\mathscr{L}(X,Y))}.$$

Proof. We are going to estimate

$$\langle \Pi^\alpha f, g\rangle = \sum_{Q\in\mathscr{D}} \Big\langle \pi_Q^\alpha \langle f\rangle_Q, \langle h_Q^\alpha, g\rangle \Big\rangle \tag{12.33}$$

for $f \in S_{00}(\mathbb{R}^d;X)$ and $g \in S_{00}(\mathbb{R}^d;Y^*)$; the latter guarantees that the sum is finitely nonzero. We may thus replace π_Q^α by $\mathbf{1}_{\mathscr{F}}(Q)\pi_Q^\alpha$ for some finite set $\mathscr{F} \subseteq \mathscr{D}$, but we do not indicate this explicitly in the notation.

Part I: Construction of principal cubes

Let $\mathscr{P}_0$ be the maximal cubes appearing in this sum. We then construct cube families $\mathscr{P}_n$ inductively as follows. For each $P \in \mathscr{P}_n$, let $\mathrm{ch}_{\mathscr{P}}(P)$ be the maximal dyadic subcubes P' of F such that either

$$\fint_{P'} \|f\|_X > 4\fint_P \|f\|_X \qquad \text{or} \qquad \fint_{P'} \|g\|_{Y^*} > 4\fint_P \|g\|_{Y^*}.$$

For each such P', we have

$$|P'| \leqslant \frac{1}{4}\max\Big\{ \frac{\int_{P'}\|f\|_X}{\fint_P \|f\|_X}, \frac{\int_{P'}\|g\|_{Y^*}}{\fint_P \|g\|_{Y^*}} \Big\}.$$

Since these P' are pairwise disjoint, we have

$$\sum_{P'\in \mathrm{ch}_{\mathscr{P}}(P)} \int_{P'} \|f\|_X \leqslant \int_P \|f\|_X = |P| \fint_P \|f\|_X$$

and similarly with g, and hence

$$\sum_{P'\in \mathrm{ch}_{\mathscr{P}}(P)} \leqslant \frac{1}{4}(|P|+|P|) = \frac{1}{2}|P|.$$

Thus

$$E_{\mathscr{P}}(P) := P \setminus \bigcup_{P'\in \mathrm{ch}_{\mathscr{P}}(P)} \qquad \text{satisfies} \quad |E_{\mathscr{P}}(P)| \geqslant \frac{1}{2}|P|.$$

Then we let

$$\mathscr{P}_{n+1} := \bigcup_{P\in\mathscr{P}_n} \mathrm{ch}_{\mathscr{P}}(P), \quad \mathscr{P} := \bigcup_{n=0}^{\infty} \mathscr{P}_n,$$

and the sets $E_{\mathscr{P}}(P)$, $P \in \mathscr{P}$, are seen to be pairwise disjoint.

Now every Q with a nonzero contribution to (12.33) will be contained in some $P \in \mathscr{P}_0 \subseteq \mathscr{P}$. Let $\mathrm{par}_{\mathscr{P}} P \in \mathscr{P}$ be the minimal such P. By construction, it follows that

$$\fint_Q \|f\|_X \leqslant 4\fint_P \|f\|_X, \quad \fint_Q \|g\|_{Y^*} \leqslant 4\fint_P \|g\|_{Y^*}, \quad \text{if} \quad \mathrm{par}_{\mathscr{P}} Q = P.$$

For $P \in \mathscr{P}$, let

$$\mathbb{P}_P h := \sum_{P'\in \mathrm{ch}_{\mathscr{P}}(P)} \mathbf{1}_{P'} \langle h\rangle_{P'} + \mathbf{1}_{E_{\mathscr{P}}(P)} h.$$

Let $h \in \{f, g\}$. If $u \in E_{\mathscr{P}}$ be a Lebesgue point of h, then all Q with $u \in Q \subseteq P$ fail the stopping criterion, and hence

$$\|\mathbb{P}_P h(u)\| = \|h(u)\| = \lim_{Q\to u} \|\langle h\rangle_Q\| \leqslant 4\langle \|h(\cdot)\|\rangle_P.$$

On the other hand, if $u \in P' \in \mathrm{ch}_{\mathscr{P}}(P)$, then its dyadic parent $\widehat{P'}$ fails the stopping criterion, and hence

$$\|\mathbb{P}_P h(u)\| = \|\langle h\rangle_{P'}\| \leqslant \langle \|h(\cdot)\|\rangle_{P'} \leqslant 2^d \langle \|h(\cdot)\|\rangle_{\widehat{P'}} \leqslant 2^d \cdot 4\langle \|h(\cdot)\|\rangle_P.$$

Hence we conclude that

$$\|\mathbb{P}_P h(u)\| \leqslant 4 \cdot 2^d \cdot \mathbf{1}_P(u) \langle \|h\|\rangle_P, \qquad h \in \{f, g\}.$$

If $\mathrm{par}_{\mathscr{P}} Q = P$ and $Q' \in \mathrm{ch}_{\mathscr{D}} Q$, then each $P' \in \mathrm{ch}_{\mathscr{P}} P$ is either disjoint from Q (thus *a fortiori* from Q'), or strictly contained in Q, hence contained in Q'. Thus

$$\int_{Q'} \mathbb{P}_P h = \sum_{\substack{P' \in \operatorname{ch}_{\mathscr{P}}(P) \\ P' \subsetneq Q'}} |P'| \langle h \rangle_{P'} + \int_{Q' \cap E_{\mathscr{P}}(P)} h = \int_{Q'} h, \quad \operatorname{par}_{\mathscr{P}} Q = P.$$

Since both $\mathbf{1}_Q$ and h_Q^α are linear combination of $Q' \in \operatorname{ch}_{\mathscr{D}} Q$, this implies in particular that

$$\langle f \rangle_Q = \langle \mathbb{P}_P f \rangle_Q, \quad \langle h_Q^\alpha, g \rangle = \langle h_Q^\alpha, \mathbb{P}_P g \rangle, \quad \operatorname{par}_{\mathscr{P}} Q = P.$$

Part II: Estimates under the principal cubes

With the principal cubes $P \in \mathscr{P}$ just constructed, we can now rearrange the sum (12.33) as

$$\langle \Pi^\alpha f, g \rangle = \sum_{P \in \mathscr{P}} \Big\langle \sum_{\substack{Q \in \mathscr{D} \\ \operatorname{par}_{\mathscr{P}} Q = P}} \pi_Q^\alpha \langle \mathbb{P}_P f \rangle_Q h_Q^\alpha, \mathbb{P}_P g \Big\rangle =: \sum_{P \in \mathscr{P}} I_P.$$

By Lemma 12.2.14 at the key step introducing the UMD constants, and applications of Hölder's inequality and the properties of the principal cubes elsewhere,

$$\begin{aligned}
I_P &\leqslant \Big\| \sum_{\substack{Q \in \mathscr{D} \\ \operatorname{par}_{\mathscr{P}} Q = P}} \pi_Q^\alpha \langle \mathbb{P}_P f \rangle_Q h_Q^\alpha \Big\|_{L^q(\mathbb{R}^d;Y)} \| \mathbb{P}_P g \|_{L^{q'}(\mathbb{R}^d;Y^*)} \\
&\leqslant \beta_{q,Y}^- \beta_{q,Y}^+ \Big\| \Big(\sum_{\substack{Q \in \mathscr{D} \\ \operatorname{par}_{\mathscr{P}} Q = P}} \varepsilon_Q \pi_Q^\alpha h_Q^0 \Big) \mathbb{P}_P f \Big\|_{L^q(\mathbb{R}^d \times \Omega;Y)} \| \mathbb{P}_P g \|_{L^{q'}(\mathbb{R}^d;Y^*)} \\
&\leqslant \beta_{q,Y}^- \beta_{q,Y}^+ \Big\| \sum_{\substack{Q \in \mathscr{D} \\ \operatorname{par}_{\mathscr{P}} Q = P}} \varepsilon_Q \pi_Q^\alpha h_Q^0 \Big\|_{L^q(\mathbb{R}^d \times \Omega;\mathscr{L}(X,Y))} \| \mathbb{P}_P f \|_{L^\infty(\mathbb{R}^d;X)} \times \\
&\qquad \times \| \mathbb{P}_P g \|_{L^\infty(\mathbb{R}^d;Y^*)} |P|^{1/q'} \\
&\leqslant \beta_{q,Y}^- \beta_{q,Y}^+ \| (\pi_Q^\alpha) \|_{\operatorname{Car}^q(\mathbb{R}^d;\mathscr{L}(X,Y))} |P|^{1/q} \times 4 \cdot 2^d \langle \| f \|_X \rangle_P \times \\
&\qquad \times 4 \cdot 2^d \langle \| g \|_{Y^*} \rangle_P |P|^{1/q'} \\
&= 16 \cdot 4^d \cdot \beta_{q,Y}^- \beta_{q,Y}^+ \| (\pi_Q^\alpha) \|_{\operatorname{Car}^2(\mathbb{R}^d;\mathscr{L}(X,Y))} \langle \| f \|_X \rangle_P \langle \| g \|_{Y^*} \rangle_P |P| \\
&=: 16 \cdot 4^d \cdot \beta_{q,Y}^- \beta_{q,Y}^+ \| (\pi_Q^\alpha) \|_{\operatorname{Car}^2(\mathbb{R}^d;\mathscr{L}(X,Y))} \times II_P.
\end{aligned}$$

(Note that, in the step that lead to the appearance of the Carleson norm, we made use of our implicit replacement of π_Q^α by $\mathbf{1}_{\mathscr{F}}(Q) \pi_Q^\alpha$, for some finite $\mathscr{F} \subseteq \mathscr{D}$, in the beginning of the proof.)

Finally,

$$\begin{aligned}\sum_{P\in\mathscr{P}} II_P &\leqslant 2\sum_{P\in\mathscr{P}}\langle\|f\|_X\rangle_P\langle\|g\|_{Y^*}\rangle_P|E_{\mathscr{P}}(P)|\\ &\leqslant 2\sum_{P\in\mathscr{P}}\int_{E_{\mathscr{P}}(P)} M_{\mathscr{D}}f\cdot M_{\mathscr{D}}g\\ &\leqslant 2\int_{\mathbb{R}^d} M_{\mathscr{D}}f\cdot M_{\mathscr{D}}g\\ &\leqslant 2\|M_{\mathscr{D}}f\|_{L^p(\mathbb{R}^d)}\|M_{\mathscr{D}}g\|_{L^{p'}(\mathbb{R}^d)}\\ &\leqslant 2\cdot p'\|f\|_{L^p(\mathbb{R}^d;X)}\cdot p\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)},\end{aligned}$$

by Doob's maximal inequality in the last step.

A combination of the estimates proves the proposition. □

To compare the necessary and sufficient conditions for paraproduct boundedness, we have the following relation between bounded mean oscillation and Carleson norms.

Proposition 12.2.17 (Carleson norms vs. BMO). *Let Z be a UMD space, and $p\in(1,\infty)$. If $b\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;Z)$, then $(\pi_Q^\alpha)_{Q\in\mathscr{D}}:=(\langle b,h_Q^\alpha\rangle)_{Q\in\mathscr{D}}$ satisfies the Car^p condition for each $\alpha\in\{0,1\}^d\setminus\{0\}$, and*

$$\max_{\alpha\in\{0,1\}^d\setminus\{0\}}\|(\pi_Q^\alpha)\|_{\mathrm{Car}^p(\mathbb{R}^d;Z)}\leqslant\beta_{p,Z}^+\|b\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Z)}.$$

This estimate also has a converse, but since it has no immediate use in the present discussion, we leave the details to an interested reader.

Proof. This is a direct computation

$$\begin{aligned}&\Big\|\sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{F}}}\varepsilon_Q h_Q^0\langle b,h_Q^\alpha\rangle\Big\|_{L^p(Q_0\times\Omega;Z)}\\ &\leqslant\inf_{c\in Z}\Big\|\sum_{\substack{Q\subseteq Q_0\\ Q\in\mathscr{F}}}\varepsilon_Q h_Q^\alpha\langle\mathbf{1}_{Q_0}(b-c),h_Q^\alpha\rangle\Big\|_{L^p(Q_0\times\Omega;Z)}\\ &\leqslant\inf_{c\in Z}\beta_{p,Z}^+\|\mathbf{1}_{Q_0}(b-c)\|_{L^p(\mathbb{R}^d;Z)}\qquad\text{by Proposition 12.1.5}\\ &\leqslant\beta_{p,Z}^+|Q_0|^{1/p}\|b\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d;Z)}.\end{aligned}$$

Taking the supremum over finite $\mathscr{F}\subseteq\mathscr{D}$ and $Q_0\in\mathscr{D}$, the claimed bound follows from the definition of Car^p. □

We can now formulate conditions for the boundedness of a paraproduct Π_b in terms of function space properties of b:

Theorem 12.2.18. *Let X be a Banach space, Y be a UMD space, and $p\in(1,\infty)$. Let $b\in L^1_{\mathrm{loc,so}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, and let Π_b be the paraproduct defined by the operators $\pi_Q^\alpha: x\mapsto\langle b(\cdot)x,h_Q^\alpha\rangle$. In order that Π_b is bounded from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$, it is*

(1) *necessary that* $b \in \mathrm{BMO}_{\mathscr{D},\mathrm{so}}(\mathbb{R}^d; \mathscr{L}(X,Y))$, *and*
(2) *sufficient that* $b \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d; Z)$ *for some subspace* $Z \hookrightarrow \mathscr{L}(X,Y)$ *with the UMD property.*

Moreover, we have the quantitative bounds

$$\begin{aligned}\|b\|_{\mathrm{BMO}_{\mathscr{D},\mathrm{so}}(\mathbb{R}^d;\mathscr{L}(X,Y))} &\leqslant \|\Pi\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ &\leqslant 32\cdot 8^d pp' \beta^-_{q,Y}\beta^+_{q,Y}\|j\|_{\mathscr{L}(Z,\mathscr{L}(X,Y))}\beta^+_{q,Z}\|b\|_{\mathrm{BMO}^q_{\mathscr{D}}(\mathbb{R}^d;Z)},\end{aligned}$$

where $j : Z \to \mathscr{L}(X,Y)$ *is the embedding map and* $q \in (1,\infty)$ *is arbitrary.*

Proof. The necessary condition and the lower bound for $\|\Pi\|$ are just restatements of Proposition 12.2.8 and Remark 12.2.9.

For the sufficient condition, from Proposition 12.2.16 we obtain

$$\|\Pi\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant 32\cdot 4^d pp'\beta^-_{q,Y}\beta^+_{q,Y}\sum_{\alpha\in\{0,1\}^d\setminus\{0\}}\|(\pi^\alpha_Q)\|_{\mathrm{Car}^q(\mathbb{R}^d;\mathscr{L}(X,Y))},$$

and the assumed embedding followed by Proposition 12.2.17 give us

$$\begin{aligned}\|(\pi^\alpha_Q)\|_{\mathrm{Car}^q(\mathbb{R}^d;\mathscr{L}(X,Y))} &\leqslant \|j\|_{\mathscr{L}(Z,\mathscr{L}(X,Y))}\|(\pi^\alpha_Q)\|_{\mathrm{Car}^q(\mathbb{R}^d;Z)} \\ &\leqslant \|j\|_{\mathscr{L}(Z,\mathscr{L}(X,Y))}\beta^+_{q,Z}\|b\|_{\mathrm{BMO}^q_{\mathscr{D}}(\mathbb{R}^d;Z)}.\end{aligned}$$

The estimate is concluded by noting that $\#(\{0,1\}^d\setminus\{0\}) = 2^d - 1 < 2^d$. □

For paraproducts defined by scalar-valued coefficients, we now obtain a complete characterisation of their boundedness on UMD spaces. For $p = q$, the equivalence (1) ⇔ (4) provides a partial solution of the L^p extension problem, discussed in Section 2.1, in the particular case of the paraproducts. Note, however, it does not exclude the possibility of $L^p(\mathbb{R}^d)$-bounded paraproducts extending boundedly to other classes of spaces besides UMD.

Corollary 12.2.19. *Let* X *be a UMD space, and* $p, q \in (1,\infty)$. *Let* Π_1, Π_2^* *and* $\Lambda := \Pi_1 + \Pi_2^*$ *be paraproducts with scalar coefficients* $\pi_Q^{1,\alpha}, \pi_Q^{2,\alpha} \in \mathbb{K}$. *Then the following are equivalent:*

(1) $\Lambda \in \mathscr{L}(L^p(\mathbb{R}^d;X))$;
(2) *both* $\Pi_1, \Pi_2^* \in \mathscr{L}(L^p(\mathbb{R}^d;X))$;
(3) *for some* $b_i \in \mathrm{BMO}(\mathbb{R}^d)$, *we have*

$$\pi_Q^{1,\alpha} = \langle b_1, h_Q^\alpha\rangle, \quad \pi_Q^{2,\alpha} = \langle b_2, h_Q^\alpha\rangle, \quad \forall Q \in \mathscr{D}, \alpha \in \{0,1\}^d\setminus\{0\};$$

(4) $\Lambda \in \mathscr{L}(L^q(\mathbb{R}^d))$.

Under these equivalent conditions, we have the estimates

$$\begin{aligned}\max_{i=1,2}\|b_i\|_{\mathrm{BMO}^{p_i}_{\mathscr{D}}(\mathbb{R}^d)} &\leqslant \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}, \\ \|\tilde{\Pi}_i\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &\leqslant 32\cdot 8^d\cdot pp'\cdot\beta^2_{q,X}\cdot\beta_{q,\mathbb{K}}\cdot\|b_i\|_{\mathrm{BMO}^{q_i}_{\mathscr{D}}(\mathbb{R}^d)}, \\ \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &\leqslant 64\cdot 8^d\cdot pp'\cdot\beta^2_{q,X}\cdot\beta_{q,\mathbb{K}}\cdot\|\Lambda\|_{\mathscr{L}(L^q(\mathbb{R}^d))}.\end{aligned}$$

where $\tilde{\Pi}_1 := \Pi_1$, $\tilde{\Pi}_2 := \Pi_2^*$, $p_1 := p$, $p_2 := p'$, $q_1 := q$, $q_2 := q'$.

Proof. (1) ⇒ (3): The assumed boundedness (1) and duality clearly implies the testing conditions

$$\|\Lambda(\mathbf{1}_Q \otimes x)\|_{L^p(\mathbb{R}^d,X)} \leqslant \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \|\mathbf{1}_Q \otimes x\|_{L^p(\mathbb{R}^d;X)},$$
$$\|\Lambda^*(\mathbf{1}_Q \otimes x^*)\|_{L^{p'}(\mathbb{R}^d,X^*)} \leqslant \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \|\mathbf{1}_Q \otimes x^*\|_{L^{p'}(\mathbb{R}^d;X^*)}.$$

Condition (3) then follows from Proposition 12.2.8, which also provides the bounds

$$\max\Big(\|b_1\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d)}, \|b_2\|_{\mathrm{BMO}^{p'}_{\mathscr{D}}(\mathbb{R}^d)}\Big) \leqslant \|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}.$$

(3) ⇒ (2): We use Theorem 12.2.18 with $Y = X$ and $Z = \mathbb{K} \cdot I_X$, which clearly embeds into $\mathscr{L}(X)$ with constant one. With this choice, the theorem shows that

$$\begin{aligned}\|\Pi_1\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &\leqslant 32 \cdot 8^d \cdot pp' \beta^-_{q,X} \beta^+_{q,X} \beta^+_{q,\mathbb{K}} \|b_1\|_{\mathrm{BMO}^q_{\mathscr{D}}(\mathbb{R}^d)}. \\ &\leqslant 32 \cdot 8^d \cdot pp' \beta^2_{q,X} \beta_{q,\mathbb{K}} \|b_1\|_{\mathrm{BMO}^p_{\mathscr{D}}(\mathbb{R}^d)},\end{aligned}$$

where we also used $\beta^{\pm}_{p,X} \leqslant \beta_{p,X}$. Similarly, we have

$$\|\Pi_2^*\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \|\Pi_2\|_{\mathscr{L}(L^{p'}(\mathbb{R}^d;X^*))} \leqslant 32 \cdot 8^d \cdot pp' \beta^2_{q,X} \beta_{q,\mathbb{K}} \|b_2\|_{\mathrm{BMO}^{q'}_{\mathscr{D}}(\mathbb{R}^d)},$$

using the same bound on the dual side and recalling that $\beta_{q',X^*} = \beta_{q,X}$.

(2) ⇒ (1): This is trivial by the triangle inequality.

(3) ⇔ (4): This is the already established equivalence (3) ⇔ (1) specialised to $X = \mathbb{K}$. The final quantitative bound follows by combining the bounds already established:

$$\begin{aligned}\|\Lambda\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &\leqslant \sum_{i=1}^2 \|\tilde{\Pi}_i\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \\ &\leqslant \sum_{i=1}^2 32 \cdot 8^d \cdot pp' \beta^2_{q,X} \beta_{q,\mathbb{K}} \|b_i\|_{\mathrm{BMO}^{q_i}_{\mathscr{D}}(\mathbb{R}^d)}, \\ &\leqslant \sum_{i=1}^2 32 \cdot 8^d \cdot pp' \beta^2_{q,X} \beta_{q,\mathbb{K}} \|\Lambda\|_{\mathscr{L}(L^q(\mathbb{R}^d))}\end{aligned}$$

and $\sum_{i=1}^2 32 = 64$. □

12.2.c Symmetric paraproducts

In this section, we will take a closer look at the special case of the symmetric paraproduct Λ_b with equal coefficients $\pi_Q^{i,\alpha} = \langle b, h_Q^\alpha \rangle$ for both $i = 1, 2$. Our goal is to obtain a qualitative improvement of the earlier Proposition 12.2.3. This will require developing modest prerequisites about the projective tensor product of Banach spaces, and we first turn to this task.

Definition 12.2.20. *For two Banach spaces X and Z, and a bilinear form $\lambda : X \times Z \to \mathbb{K}$, we define*

$$\|\lambda\|_{\mathscr{B}(X,Z)} := \sup\Big\{|\lambda(x,z)| : \|x\|_X \leqslant 1, \|z\|_Z \leqslant 1\Big\},$$
$$\mathscr{B}(X,Z) := \Big\{\lambda : X \times Z \to \mathbb{K} \text{ bilinear } \Big| \|\lambda\|_{\mathscr{B}(X,Z)} < \infty\Big\}.$$

Lemma 12.2.21. $\mathscr{B}(X,Z) \simeq \mathscr{L}(X,Z^*) \simeq \mathscr{L}(Z,X^*)$.

Proof. For $u \in \mathscr{L}(X,Z^*)$, we see that

$$\operatorname{Form}(u) : X \times Z \to \mathbb{K}, (x,z) \mapsto \langle ux, z\rangle$$

defines $\operatorname{Form}(u) \in \mathscr{B}(X,Z)$ of norm at most $\|u\|_{\mathscr{L}(X,Z^*)}$. For $\lambda \in \mathscr{B}(X,Z)$, we see that $\operatorname{Op}(\lambda) : X \to Z^* : x \mapsto \lambda(x,\cdot)$ defines $\operatorname{Op}(\lambda) \in \mathscr{L}(X,Z^*)$ of norm at most $\|\lambda\|_{\mathscr{B}(X,Z)}$. Both $\operatorname{Form} : \mathscr{L}(X,Z^*) \to \mathscr{B}(X,Z)$ and $\operatorname{Op} : \mathscr{B}(X,Z) \to \mathscr{L}(X,Z^*)$ are clearly linear and we just saw that they are contractive. Since both $\operatorname{Form}\circ\operatorname{Op}$ and $\operatorname{Op}\circ\operatorname{Form}$ are identities of the respective spaces, they must in fact be isometries. This proves the first identification, and $\mathscr{B}(X,Z) \simeq \mathscr{L}(Z,X^*)$ follows by symmetry, since clearly $\mathscr{B}(X,Z) \simeq \mathscr{B}(Z,X)$. □

Definition 12.2.22. *For two Banach spaces X and Z, and elements $x \in X$ and $z \in Z$, we define $x \otimes x \in \mathscr{B}(X,Z)^*$ by*

$$x \otimes z : \mathscr{B}(X,Z) \to \mathbb{K} : \lambda \mapsto \lambda(x,z).$$

Let further

$$X \otimes Z := \operatorname{span}\{x \otimes z : x \in X, z \in Z\} \subseteq \mathscr{B}(X,Z)^*,$$

and, for all $v \in X \otimes Z$,

$$\|v\|_{X\widehat{\otimes}Z} := \inf\Big\{\sum_{i=1}^n \|x_i\|_X\|z_i\|_Z : v = \sum_{i=1}^n x_i \otimes z_i\Big\},$$

where the infimum is over all possible representations of v of any length n. Finally, let $X\widehat{\otimes}Z$ be the completion of $X \otimes Z$ with respect to this norm.

Proposition 12.2.23. *For all Banach spaces X and Z, we have*

$$(X\widehat{\otimes}Z)^* = \mathscr{B}(X,Z),$$

in the following sense: For all $v \in X \otimes Z$ and $\lambda \in \mathscr{B}(X,Z)$, the pairing

$$\langle v, \lambda\rangle := \sum_{i=1}^n \lambda(x_i,z_i), \quad \text{if} \quad v = \sum_{i=1}^n x_i \otimes z_i,$$

is well defined and extends by continuity to all $v \in X\widehat{\otimes}Z$. Conversely, every element of $(X \otimes Z)^*$ has this form, and

$$\|\lambda\|_{(X\otimes Z)^*} = \|\lambda\|_{\mathscr{B}(X,Z)}.$$

Proof. To check that $\langle v, \lambda\rangle$ is well-defined, we need to verify that two different representations $v = \sum_{i=1}^{n_a} x_i^a \otimes z_i^a$, $a = 1, 2$, result in the same right-hand side. To see this, pick a basis $(x_j^0)_{j=1}^p$ for $\mathrm{span}\{x_i^a : 1 \leqslant i \leqslant n_a, a = 1, 2\}$ and a basis $(z_k^0)_{k=1}^q$ for $\mathrm{span}\{z_i^a : 1 \leqslant i \leqslant n_a, a = 1, 2\}$ and expand all x_i^a and z_i^a in the respective basis. With the help of the Hahn–Banach theorem, pick $x_m^* \in X^*$ and $z_n^* \in Z^*$ such that $\langle x_j^0, x_m^*\rangle = \delta_{j,m}$ and $\langle z_k^0, z_n^*, =\rangle\delta_{k,n}$, and consider the forms $\lambda_{m,n}(\,\cdot_1, \cdot_2) = \langle\,\cdot_1, x_m^*\rangle\langle\,\cdot_2, z_n^*\rangle \in \mathscr{B}(X, Z)$ to see that $x_j^0 \otimes z_k^0$ are linearly independent in $\mathscr{B}(X, Z)^*$. Hence their coefficients must be equal in the two expansions of v. Make the same expansions on the right-hand side, using the bilinearity of λ, to find that both expansions lead to linear combinations with equal coefficients of the values $\lambda(x_j^0, z_k^0)$.

Having verified that the action of λ on $X \otimes Z$ is well defined, its linearity is clear. Moreover,

$$\sum_{i=1}^{n} |\lambda(x_i, z_i)| \leqslant \|\lambda\|_{\mathscr{B}(X,Z)} \sum_{i=1}^{n} \|x_i\|_X \|z_i\|_Z,$$

and taking the infimum over all representations of v shows that

$$|\langle v, \lambda\rangle| \leqslant \|\lambda\|_{\mathscr{B}(X,Z)} \|v\|_{X\widehat{\otimes}Z}$$

for all $v \in X \otimes Z$. From this estimate, we can uniquely extend the action of λ to all $v \in X\widehat{\otimes}Z$ by density, with the estimate

$$\|\lambda\|_{(X\otimes Z)^*} \leqslant \|\lambda\|_{\mathscr{B}(X,Z)}.$$

On the other hand, we also have

$$|\lambda(x, z)| = |\langle x \otimes z, \lambda\rangle| \leqslant \|x \otimes z\|_{X\widehat{\otimes}Z} \|\lambda\|_{(X\otimes Z)^*} \leqslant \|x\|_X \|z\|_Z \|\lambda\|_{(X\otimes Z)^*};$$

thus $\|\lambda\|_{\mathscr{B}(X,Z)} \leqslant \|\lambda\|_{(X\otimes Z)^*}$, and hence in fact there is equality.

Conversely, if $\xi \in (X \otimes Z)^*$, we can define $\lambda \in \mathscr{B}(X, Z)$ by $\lambda(x, z) := \langle x \otimes z, \xi\rangle$. From the previous construction, it is then clear that $\langle v, \lambda\rangle = \langle v, \xi\rangle$ for all $v \in (X \otimes Z)$, and hence every $\xi \in (X \otimes Z)^*$ arises from the previous construction. □

Corollary 12.2.24. $\|x \otimes z\|_{X\widehat{\otimes}Z} = \|x\|_X \|z\|_Z$.

Proof. We compute the norm by duality:

$$\begin{aligned} \|x \otimes z\|_{X\widehat{\otimes}Z} &= \sup\Big\{|\langle x \otimes z, \xi\rangle| : \|\xi\|_{(X\widehat{\otimes}Z)^*} \leqslant 1\Big\} \\ &= \sup\Big\{|\lambda(x, z)| : \|\lambda\|_{\mathscr{B}(X,Z)} \leqslant 1\Big\}. \end{aligned}$$

It is clear from the definition that $|\lambda(x, z)| \leqslant \|x\|_X \|z\|_Z$ for any λ as in the last supremum. On the other hand, the Hahn–Banach theorem guarantees the existence of $x^* \in X^*$ and $z^* \in Z^*$ of norm one such that $\langle x, x^*\rangle = \|x\|_X$ and $\langle z, z^*\rangle = \|z\|_Z$. Then clearly $\lambda(\,\cdot_1, \cdot_2) = \langle\,\cdot_1, x^*\rangle\langle\,\cdot_2, z^*\rangle$ has $\|\lambda\|_{\mathscr{B}(X,Z)} \leqslant 1$ and gives $\lambda(x, z) = \|x\|_X \|z\|_Z$. □

We are now ready to prove the following improvement of Proposition 12.2.3:

Theorem 12.2.25. *Let X and Y be UMD spaces and $p \in (1,\infty)$. For every function $b \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, the symmetric paraproduct Λ_b defines a bounded operator from $L^p(\mathbb{R}^d;X)$ to $L^p(\mathbb{R}^d;Y)$ of norm*

$$\begin{aligned}\|\Lambda_b\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} &\leqslant 6\cdot 2^d\cdot(pp'+\beta^+_{p,X}\beta^+_{p',Y^*})\|b\|_{\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;\mathscr{L}(X,Y))}\\ &\leqslant 30\cdot 2^d\cdot\beta_{p,X}\beta_{p,Y}\|b\|_{\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;\mathscr{L}(X,Y))}\end{aligned}$$

Proof. By density, it suffices to consider the action of Λ_b on $f\in S_{00}(\mathscr{D};X)$, paired with $g\in S_{00}(\mathscr{D};Y^*)$. We will rewrite this pairing with the help of the projective tensor product duality between $X\widehat{\otimes}Y^*$ and $\mathscr{B}(X,Y^*)\simeq\mathscr{L}(X,Y^{**})=\mathscr{L}(X,Y)$, recalling that UMD spaces are reflexive (Theorem 4.3.3). In the following computation, the summation is always over $Q\in\mathscr{D}$ and $\alpha\in\{0,1\}^d\setminus\{0\}$.

$$\begin{aligned}\langle\Lambda_b f,g\rangle &= \sum\Big\{\Big\langle\langle b,h_Q^\alpha\rangle\langle f\rangle_Q,\langle h_Q^\alpha,g\rangle\Big\rangle_{X,Y^*}+\Big\langle\langle b,h_Q^\alpha\rangle\langle f,h_Q^\alpha\rangle,\langle g\rangle_Q\Big\rangle_{X,Y^*}\Big\}\\ &=\sum\Big\langle\langle b,h_Q^\alpha\rangle,\langle f\rangle_Q\otimes\langle h_Q^\alpha,g\rangle+\langle f,h_Q^\alpha\rangle\otimes\langle g\rangle_Q\Big\rangle_{\mathscr{L}(X,Y),X\widehat{\otimes}_\pi Y^*}\\ &=\Big\langle b,\sum h_Q^\alpha\Big[\langle f\rangle_Q\otimes\langle h_Q^\alpha,g\rangle+\langle f,h_Q^\alpha\rangle\otimes\langle g\rangle_Q\Big]\Big\rangle=:\langle b,h\rangle.\end{aligned}$$

On the last line, we are using the H^1–BMO-duality from Theorem 11.1.30; for $f\in S_{00}(\mathscr{D};X)$ and $g\in S_{00}(\mathscr{D};Y^*)$, the summation is finite, and thus $h\in L^\infty_c(\mathbb{R}^d;X\widehat{\otimes}_\pi Y^*)$. Since $b\in\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d;\mathscr{L}(X,Y))\subseteq L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathscr{L}(X,Y))$, the pointwise duality product $\langle b(u),h(u)\rangle$ is integrable, and one find by dominated convergence in the defining formula of Theorem 11.1.30 that the duality can be computed simply as the integral of $\langle b(u),h(u)\rangle$ over $\mathbb{R}^d$. Thus, an application of Theorem 11.1.30 followed by Theorem 11.1.28 shows that

$$\begin{aligned}|\langle\Lambda_b f,g\rangle| &\leqslant\|b\|_{\mathrm{BMO}(\mathbb{R}^d;\mathscr{L}(X,Y))}\|h\|_{H^1_{\mathrm{at}}(\mathbb{R}^d;X\otimes Y^*)}\\ &\leqslant\|b\|_{\mathrm{BMO}(\mathbb{R}^d;\mathscr{L}(X,Y))}\cdot 6\cdot 2^d\cdot\|h\|_{H^1_{\max}(\mathbb{R}^d;X\otimes Y^*)},\end{aligned}$$

and it remains to estimate the H^1 norm here. Recall that

$$\|h\|_{H^1_{\max}(\mathbb{R}^d;X\otimes Y^*)}=\|M_{\mathscr{D}}h\|_{L^1(\mathbb{R}^d)}=\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|\langle h\rangle_R\|_{X\otimes Y^*}\Big\|_{L^1(\mathbb{R}^d)}.$$

By the properties of Haar functions, we find that

$$\begin{aligned}\langle h\rangle_R &=\sum_{Q\supsetneq R}\sum_{\alpha\in\{0,1\}^d\setminus\{0\}}\Big\langle h_Q^\alpha\Big[\langle f\rangle_Q\otimes\langle h_Q^\alpha,g\rangle+\langle f,h_Q^\alpha\rangle\otimes\langle g\rangle_Q\Big]\Big\rangle_R\\ &=\sum_{Q\supsetneq R}\Big[\langle f\rangle_Q\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q)+(\langle f\rangle_{Q_R}-\langle f\rangle_Q)\otimes\langle g\rangle_Q\Big],\end{aligned}$$

where Q_R is the unique dyadic child of Q that contains R.

Next, we make the following algebraic observation:

$$\begin{aligned}&\langle f\rangle_{Q_R}\otimes\langle g\rangle_{Q_R}-\langle f\rangle_Q\otimes\langle g\rangle_Q\\&=(\langle f\rangle_{Q_R}-\langle f\rangle_Q+\langle f\rangle_Q)\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q+\langle g\rangle_Q)-\langle f\rangle_Q\otimes\langle g\rangle_Q\\&=\langle f\rangle_Q\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q)+(\langle f\rangle_{Q_R}-\langle f\rangle_Q)\otimes\langle g\rangle_Q\\&\qquad+(\langle f\rangle_{Q_R}-\langle f\rangle_Q)\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q).\end{aligned}$$

Thus

$$\begin{aligned}\langle f\rangle_R=\sum_{Q\supsetneq R}&\Big[\langle f\rangle_{Q_R}\otimes\langle g\rangle_{Q_R}-\langle f\rangle_Q\otimes\langle g\rangle_Q\Big]\\&+\sum_{Q\supsetneq R}(\langle f\rangle_{Q_R}-\langle f\rangle_Q)\otimes(\langle g\rangle_{Q_R}-\langle g\rangle_Q)=:I_R+II_R.\end{aligned}$$

The sum I_R is telescopic and, since $f\in S_{00}(\mathscr{D};X)$ (we don't even need the similar property of g at this point), its terms vanish for all large enough Q. Thus in fact

$$I_R=\langle f\rangle_R\otimes\langle g\rangle_R,\qquad \|I_R\|_{X\widehat{\otimes}_\pi Y^*}=\|\langle f\rangle_R\|_X\|\langle g\rangle_R\|_{Y^*}$$

and

$$\begin{aligned}\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|I_R\|_{X\widehat{\otimes}_\pi Y^*}\Big\|_{L^1(\mathbb{R}^d)}&\leqslant\|M_{\mathscr{D}}f\cdot M_{\mathscr{D}}g\|_{L^1(\mathbb{R}^d)}\\&\leqslant\|M_{\mathscr{D}}f\|_{L^p(\mathbb{R}^d)}\|M_{\mathscr{D}}g\|_{L^{p'}(\mathbb{R}^d)}\\&\leqslant p'\|f\|_{L^p(\mathbb{R}^d;X)}\cdot p\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}\end{aligned}$$

by Doob's maximal inequality in the last step.

Turning to II_R, we note that $\langle f\rangle_{Q_R}-\langle f\rangle_Q$ is the constant value of $\mathbb{D}_Qf(u)$ for any $u\in R$, and similarly for g. As before, the summation in II_R is finitely non-zero, and we can disentangle it with the help of a Rademacher sequence $(\varepsilon_Q)_{Q\in\mathscr{D}}$ as

$$II_R=\mathbb{E}\Big(\sum_{P\supsetneq R}\varepsilon_P\mathbb{D}_Pf(u)\Big)\otimes\Big(\sum_{Q\supsetneq R}\bar{\varepsilon}_Q\mathbb{D}_Qg(u)\Big).$$

Thus

$$\begin{aligned}\|II_R\|_{X\widehat{\otimes}Y^*}&\leqslant\mathbb{E}\Big\|\sum_{P\supsetneq R}\varepsilon_P\mathbb{D}_Pf(u)\Big\|_X\Big\|\sum_{Q\supsetneq R}\bar{\varepsilon}_Q\mathbb{D}_Qg(u)\Big\|_{Y^*}\\&\leqslant\Big\|\sum_{P\supsetneq R}\varepsilon_P\mathbb{D}_Pf(u)\Big\|_{L^p(\Omega;X)}\Big\|\sum_{Q\supsetneq R}\bar{\varepsilon}_Q\mathbb{D}_Qg(u)\Big\|_{L^{p'}(\Omega;Y^*)}\\&\leqslant\Big\|\sum_{P\in\mathscr{D}}\varepsilon_P\mathbb{D}_Pf(u)\Big\|_{L^p(\Omega;X)}\Big\|\sum_{Q\in\mathscr{D}}\bar{\varepsilon}_Q\mathbb{D}_Qg(u)\Big\|_{L^{p'}(\Omega;Y^*)},\end{aligned}$$

where the last step was an application of the contraction principle. Thus

$$\sup_{R\ni u}\|II_R\|_{X\widehat{\otimes}Y^*}\leqslant\Big\|\sum_{P\in\mathscr{D}}\varepsilon_P\mathbb{D}_Pf(u)\Big\|_{L^p(\Omega;X)}\Big\|\sum_{Q\in\mathscr{D}}\overline{\varepsilon}_Q\mathbb{D}_Qg(u)\Big\|_{L^{p'}(\Omega;Y^*)}$$

and

$$\begin{aligned}\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|II_R\|_{X\widehat{\otimes}Y^*}\Big\|_{L^1(\mathbb{R}^d)}&\\ \leqslant\Big\|\sum_{P\in\mathscr{D}}\varepsilon_P\mathbb{D}_Pf\Big\|_{L^p(\mathbb{R}^d\times\Omega;X)}&\Big\|\sum_{Q\in\mathscr{D}}\overline{\varepsilon}_Q\mathbb{D}_Qg\Big\|_{L^{p'}(\mathbb{R}^d\times\Omega;Y^*)}\\ \leqslant\beta^+_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)}&\cdot\beta^+_{p',Y^*}\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}.\end{aligned}$$

A combination of the estimates of I_R and II_R shows that

$$\begin{aligned}\|h\|&_{H^1_{\max}(\mathbb{R}^d;X\widehat{\otimes}Y^*)}\\ &\leqslant\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|I_R\|_{X\widehat{\otimes}Y^*}\Big\|_{L^1(\mathbb{R}^d)}+\Big\|\sup_{R\in\mathscr{D}}\mathbf{1}_R\|I_R\|_{X\widehat{\otimes}Y^*}\Big\|_{L^1(\mathbb{R}^d)}\\ &\leqslant(pp'+\beta^+_{p,X}\beta^+_{p',Y^*})\|f\|_{L^p(\mathbb{R}^d;X)}\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)},\end{aligned}$$

and altogether we have proved the first estimate claimed in the theorem.

The final estimate is seen as follows: First, we have $\beta^+_{p,X}\leqslant\beta_{p,X}$ and $\beta^+_{p',Y^*}\leqslant\beta_{p',Y^*}=\beta_{p,Y}$ by the observation after Proposition 4.2.3, and Proposition 4.2.17(2). Second, denoting $p^*=\max(p,p')\geqslant 2$, we have $\beta_{p,Z}\geqslant\beta_{p,\mathbb{R}}=p^*-1\geqslant\frac{1}{2}p^*$ by Theorem 4.5.7, and hence $pp'\leqslant(p^*)^2\leqslant 4\beta_{p,X}\beta_{p,Y}$. □

12.2.d Mei's counterexample: no simple sufficient conditions

The following theorem shows the impossibility of obtaining simple upper bounds for operator-valued paraproducts in infinite-dimensional spaces, even by considering Hilbert spaces only, and even by replacing the bounded mean oscillation conditions by the stronger L^∞ norm.

Theorem 12.2.26 (Mei). *Let ϕ be a function such that*

$$\|\Pi_b\|_{\mathscr{L}(L^2(\mathbb{R};\ell^2_N))}\leqslant\phi(N)\|b\|_{L^\infty(\mathbb{R};\mathscr{L}(\ell^2_N))}\quad\text{for all } b\in L^\infty(\mathbb{R};\mathscr{L}(\ell^2_N)).$$

Then

$$\phi(N)\geqslant\|\triangle_N\|_{\mathscr{L}(\mathscr{L}(\ell^2_N))}\geqslant\frac{1}{\pi}(\log N-1),$$

where $\triangle_N:\mathscr{L}(\ell^2_N)\to\mathscr{L}(\ell^2_N)$ is the lower triangle projection defined by

$$\triangle_N(e_i\otimes e_j):=\begin{cases}e_i\otimes e_j, & \text{if } i>j,\\ 0, & \text{else}\end{cases}$$

and extended by linearity.

Proof. For $a \in \mathscr{L}(\ell^2_N)$ and $u, v \in \ell^2_N$, we have the tensor product $u \otimes v \in \mathscr{L}(\ell^2_N)$, and the trace duality $\langle a, u \otimes v \rangle = \langle au, v \rangle$.

Let $b \in L^\infty(\mathbb{R}; \mathscr{L}(\ell^2_N))$, and $f, g \in L^2(\mathbb{R}; \ell^2_N)$. We can then write

$$\langle \Pi_b f, g \rangle = \Big\langle \sum_{I \in \mathscr{D}} \langle b, h_I \rangle \langle f \rangle_I h_I, g \Big\rangle = \Big\langle b, \sum_{I \in \mathscr{D}} \langle f \rangle_I \otimes \langle h_I, g \rangle h_I \Big\rangle =: \langle b, \Pi_{\otimes g} f \rangle,$$

where suggestive notation $\Pi_{\otimes g}$ is defined by the last identity. In the two rightmost expressions, the duality is that between $L^\infty(\mathbb{R}; \mathscr{L}(\ell^2_N))$ and its predual $L^1(\mathbb{R}; \mathscr{C}^1(\ell^2_N))$. (We recall from Theorem D.2.6 that $(\mathscr{C}^1(H))^* = \mathscr{L}(H)$ for any Hilbert space H and from Theorem 1.3.10 that $(L^1(\mathbb{R}; X))^* = L^\infty(\mathbb{R}; X^*)$ when X^* has the Radon–Nikodým property, which the finite-dimensional (hence reflexive) $X = \mathscr{L}(\ell^2_N)$ does by Theorem 1.3.21.)

Thus we deduce that

$$\begin{aligned}
\|\Pi_{\otimes g} f\|_{L^1(\mathbb{R}; \mathscr{C}^1(\ell^2_N))} &= \sup \Big\{ |\langle b, \Pi_{\otimes g} f \rangle| : \|b\|_{L^\infty(\mathbb{R}; \mathscr{L}(\ell^2_N))} \leqslant 1 \Big\} \\
&= \sup \Big\{ |\langle \Pi_b f, g \rangle| : \|b\|_{L^\infty(\mathbb{R}; \mathscr{L}(\ell^2_N))} \leqslant 1 \Big\} \\
&\leqslant \phi(N) \|f\|_{L^2(\mathbb{R}; \ell^2_N)} \|g\|_{L^2(\mathbb{R}; \ell^2_N)}.
\end{aligned}$$

We now apply this to a special choice of $f, g \in L^2(\mathbb{R}; \ell^2_N)$. Let $(r_i)_{i=1}^N$ be the standard realisation of a Rademacher sequence on $[0,1)$, i.e., $r_i(t) := \mathbf{1}_{[0,1)}(t) \operatorname{sgn}(\sin(2^i \pi t))$. With $u, v \in \ell^2_N$, we take $f = \sum_{i=1}^N r_i \langle u, e_i \rangle e_i$ and $g = \sum_{i=1}^N r_i \langle v, e_i \rangle e_i$, where $(e_i)_{i=1}^N$ is the standard orthonormal basis of ℓ^2_N. Then

$$\begin{aligned}
\Pi_{\otimes g} f(t) &= \sum_{j=1}^N \sum_{i=1}^{j-1} r_i(t) \langle u, e_i \rangle e_i \otimes r_j(t) \langle v, e_j \rangle e_j \\
&= D_{r(t)} \Big(\sum_{1 \leqslant i < j \leqslant N} \langle u, e_i \rangle \langle v, e_j \rangle e_i \otimes e_j \Big) D_{r(t)} \\
&= D_{r(t)} \Big(T_N \sum_{i,j=1}^N \langle u, e_i \rangle \langle v, e_j \rangle e_i \otimes e_j \Big) D_{r(t)} = D_{r(t)} \big(T_N (u \otimes v) \big) D_{r(t)}
\end{aligned}$$

where $D_{r(t)} = \sum_{i=1}^N r_i(t) e_i \otimes e_i$ and $\tilde{\triangle}_N$ is the upper triangle projection defined by

$$\tilde{\triangle}_N (e_i \otimes e_j) := \begin{cases} e_i \otimes e_j, & \text{if } i < j, \\ 0, & \text{else} \end{cases}$$

and extended by linearity. Since $D_{r(t)}$ is unitary for every $t \in [0,1)$, it follows that

$$\|\Pi_{\otimes g} f\|_{L^1(\mathbb{R}; \mathscr{C}^1(\ell^2_N))} = \|\tilde{\triangle}_N (u \otimes v)\|_{L^1([0,1); \mathscr{C}^1(\ell^2_N))} = \|\tilde{\triangle}_N (u \otimes v)\|_{\mathscr{C}^1(\ell^2_N)}.$$

Dy Lemma D.1.1 and the definition of the Schatten class, every $s \in \mathscr{C}^1(\ell^2_N)$ has a singular value decomposition

$$s = \sum_{k=1}^{n} a_k(s) u_k \otimes v_k, \quad \|u_k\|_{\ell^2_N} = \|v_k\|_{\ell^2_N} = 1, \quad \sum_{k=1}^{n} a_k(s) = \|s\|_{\mathscr{C}^1(\ell^2_N)}$$

where $a_k(s) \geqslant 0$ are the approximation numbers of s. Letting $f_k, g_k \in L^2(\mathbb{R}; \ell^2_N)$ of norm one be the functions corresponding to u_k, v_k, we find that

$$\begin{aligned} \|\tilde{\triangle}_N s\|_{\mathscr{C}^1(\ell^2_N)} &\leqslant \sum_{k=1}^{n} a_k(s) \|\tilde{\triangle}_N(u_k \otimes v_k)\|_{\mathscr{C}^1(\ell^2_N)} \\ &= \sum_{k=1}^{n} a_k(s) \|\Pi_{\otimes g_k} f_k\|_{L^1(\mathbb{R}; \mathscr{C}^1(\ell^2_N))} \\ &\leqslant \sum_{k=1}^{n} a_k(s) \phi(N) = \phi(N) \|s\|_{\mathscr{C}^1(\ell^2_N)}. \end{aligned}$$

Noting that the lower triangle projection $\triangle_N$ on $\mathscr{L}(\ell^2_N) = (\mathscr{C}^1(\ell^2_N))^*$ is the adjoint of the upper triangle projection $\tilde{\triangle}_N$ on $\mathscr{C}^1(\ell^2_N))$, this implies that

$$\|\triangle_N\|_{\mathscr{L}(\mathscr{L}(\ell^2_N))} = \|\tilde{\triangle}_N\|_{\mathscr{L}(\mathscr{C}^1(\ell^2_N))} \leqslant \phi(N),$$

which is the first claimed inequality.

The final bound is essentially Lemma 7.5.12, where a variant

$$T_N(e_i \otimes e_j) := \begin{cases} e_i \otimes e_j, & \text{if } i \geqslant j, \\ 0, & \text{else} \end{cases}$$

was considered instead. However, the lower bound for the norm of this operator was achieved by testing with the Hilbert matrix $A_N = (\mathbf{1}_{\{i \neq j\}} (i-j)^{-1})_{i,j=1}^N$ with vanishing diagonal; hence $\Delta_N(A_N) = T_N(A_N)$, and the same lower bound follows for Δ_N as well. □

12.3 The $T(1)$ theorem for abstract bilinear forms

In Sections 11.2 and 11.3, the leading theme was extrapolating the boundedness of a singular integral operator from $L^{p_0}(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; X)$, with a different exponent p, or even to $L^p(w; X)$, with a different weight w. A question that was largely left open in these sections was how to verify the assumed boundedness on some $L^{p_0}(\mathbb{R}^d; X)$ to begin with. In the spirit of the L^p-extension problem discussed in Section 2.1, we here obtain the following useful answer that allows us to extrapolate the vast existing information about scalar-valued singular integrals to the UMD-valued situation:

Theorem 12.3.1. *Let $p_0 \in (1, \infty)$, and let $T \in \mathscr{L}(L^{p_0}(\mathbb{R}^d))$ be an operator associated with a Calderón–Zygmund standard kernel $K : \dot{\mathbb{R}}^{2d} \to \mathbb{K}$. Let X be a UMD space and $p \in (1, \infty)$. Then $T \otimes I_X$ extends to a bounded linear operator on $L^p(\mathbb{R}^d; X)$.*

In fact, this result will be obtained as a corollary of general criteria, known as "$T(1)$ theorems", for the boundedness of operators associated with Calderón–Zygmund kernels; and we will also obtain versions dealing with operator-valued kernels. However, the very statement of these results requires some preparations that we take up next. Concerning the proofs, we only mention at this point that the dyadic singular integral operators and paraproducts, whose boundedness we already studied in Sections 12.1 and 12.2, will play a significant role; indeed, our general strategy is to decompose a Calderón–Zygmund operator into a convergent series of dyadic singular integral operators and paraproducts. Thus, this final section brings together several of the themes developed in this chapter.

12.3.a Weakly defined bilinear forms

In order to make a non-tautological study of the question of boundedness of an operator, we need to give a meaning to the notion of an "operator" before its boundedness has been established. As usual, this will involve postulating the action of the operator on a dense class of test functions from which we wish to extend this action to the full space under consideration. For a dyadic analysis of singular integral operators, it is convenient to adopt the following framework:

Definition 12.3.2. *For a Banach space Z, a Z-valued bilinear form on $S(\mathscr{D})$ is a bilinear mapping*

$$\mathfrak{t} : S(\mathscr{D})^2 \to Z.$$

If $Z = \mathscr{L}(X, Y)$, we extend the action of such a mapping to

$$\mathfrak{t} : S(\mathscr{D}; X) \times S(\mathscr{D}; Y^*) \to \mathbb{K}$$

by letting

$$\mathfrak{t}(\phi \otimes x, \psi \otimes y^*) := \langle \mathfrak{t}(\phi, \psi)x, y^* \rangle \in \mathbb{K}, \qquad \phi, \psi \in S(\mathscr{D}), \quad x \in X, y^* \in Y^*,$$

and extending by bilinearity, observing that $S(\mathscr{D}; X) = S(\mathscr{D}) \otimes X$.

Remark 12.3.3 ($S(\mathscr{D})$ vs. $S_{00}(\mathscr{D})$ in the definition). Since $S_{00}(\mathscr{D}; X)$ is already dense in $L^p(\mathbb{R}^d; X)$, in order to construct a bounded bilinear form on $L^p(\mathbb{R}^d; X) \times L^{p'}(\mathbb{R}^d; Y)$, it would be sufficient to have an *a priori* estimate on $S_{00}(\mathscr{D}; X) \times S_{00}(\mathscr{D}; Y^*)$. However, for the type of theorems that we have in mind, we also like to make assumptions on the action of our bilinear forms on functions like $\mathbf{1}_Q \in S(\mathscr{D}) \setminus S_{00}(\mathscr{D})$, and hence we need to have our initial bilinear form defined on the larger product $S(\mathscr{D}; X) \times S(\mathscr{D}; Y^*)$. This gives rise to the following problem, where we take $X = Y = \mathbb{K}$ for simplicity, since the issue is already present in this case:

Suppose that we have a bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to \mathbb{K}$ that satisfies the estimate

$$|\mathfrak{t}(f,g)| \leqslant C\|f\|_p\|g\|_{p'} \quad \forall (f,g) \in S_{00}(\mathscr{D})^2.$$

Thus there exists $T \in \mathscr{L}(L^p(\mathbb{R}^d))$ such that $\mathfrak{t}(f,g) = \langle Tf, g\rangle$ whenever $(f,g) \in S_{00}(\mathscr{D})^2$. Does it follow that $\mathfrak{t}(f,g) = \langle Tf, g\rangle$ for all $(f,g) \in S(\mathscr{D})^2$?

Perhaps unexpectedly, the answer is "no": Consider the bilinear form

$$\mathfrak{t}(f,g) := \int_{\mathbb{R}^d} f \cdot \int_{\mathbb{R}^d} g, \quad (f,g) \in S(\mathscr{D})^2.$$

If $(f,g) \in S_{00}(\mathscr{D})^2$, we have the a priori bound $|\mathfrak{t}(f,g)| = 0$, and hence the unique operator $T \in \mathscr{L}(L^p(\mathbb{R}^d))$ is given by $T = 0$. But of course $\mathfrak{t}$ is not identically zero on $S(\mathscr{D})^2$. It is also clear that there cannot possible be any $T \in \mathscr{L}(L^p(\mathbb{R}^d))$ with $\langle Tf, g\rangle = \mathfrak{t}(f,g)$ for all $(f,g) \in S(\mathscr{D})^2$.

To avoid this problem, we make sure to get our *a priori* estimates on the full set $S(\mathscr{D}; X) \times S(\mathscr{D}; Y^*)$.

Definition 12.3.4. *A bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ is said to determine a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$ provided that this operator T satisfies*

$$\mathfrak{t}(f,g) = \langle Tf, g\rangle$$

for all $(f,g) \in S(\mathscr{D}; X) \times S(\mathscr{D}; Y^*)$.

In the case of reflexive spaces, the last-mentioned condition can be characterised by an *a priori* estimate. Finding sufficient conditions for such an estimate will be our primary concern below. The assumption of reflexivity is not a serious restriction at this stage, since the deeper related considerations that we shall encounter below will have much stronger assumptions, anyway.

Lemma 12.3.5. *Let X and Y be reflexive Banach spaces, and let $X_0 \subseteq X$ and $Y^0 \subseteq Y^*$ be dense. Consider a bilinear form*

$$\mathfrak{t} : S(\mathscr{D}; X_0) \times S(\mathscr{D}; Y^0) \to \mathbb{K}.$$

Let $C \geqslant 0$ be a constant and $p \in (1, \infty)$. Then the following conditions, each to hold for every choice of $(f,g) \in S(\mathscr{D}; X_0) \times S(\mathscr{D}; Y^0)$, are equivalent:

(1) *There is $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$ of norm at most C such that*

$$\langle Tf, g\rangle = \mathfrak{t}(f,g).$$

(2) *There is $T^* \in \mathscr{L}(L^{p'}(\mathbb{R}^d; Y^*), L^{p'}(\mathbb{R}^d; X^*))$ of norm at most C such that*

$$\langle f, T^*g\rangle = \mathfrak{t}(f,g).$$

(3) *There is a uniform estimate*

$$|\mathfrak{t}(f,g)| \leqslant C\|f\|_{L^p(\mathbb{R}^d; X_0)}\|g\|_{L^{p'}(\mathbb{R}^d; Y^0)}.$$

Proof. (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are immediate.

(3) $\Rightarrow$ (1): Fix $f \in \mathcal{Q}(\mathbb{R}^d; X_0)$. Then $g \mapsto \mathfrak{t}(f, g)$ defines a bounded linear functional on a dense subspace of $L^{p'}(\mathbb{R}^d; Y^*)$, and hence on $L^{p'}(\mathbb{R}^d; Y^*)$. Thus there is $\Lambda_f \in (L^{p'}(\mathbb{R}^d; Y^*))^*$ such that

$$\mathfrak{t}(f, g) = \langle \Lambda_f, g \rangle.$$

Moreover, since $Y = Y^{**}$ is reflexive, it has the Radon–Nikodým property by Theorem 1.3.21, and hence $\Lambda_f \in (L^{p'}(\mathbb{R}^d; Y^*))^* \simeq L^p(\mathbb{R}^d; Y)$ by Theorem 1.3.10.

From the linearity of the left side in f, one deduces that $f \mapsto \Lambda_f$ is a linear map from $S(\mathscr{D}; X) \subseteq L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; Y)$, and (3) shows that it is bounded. Hence there is a bounded extension $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$ with the required identity for $(f, g) \in S(\mathscr{D}; X_0) \times S(\mathscr{D}; Y^0)$.

(3) $\Rightarrow$ (2): This can be proved either similarly to the previous case, or using the already proven implication (3) $\Rightarrow$ (1) and the general existence result of an adjoint

$$T^* \in \mathscr{L}((L^p(\mathbb{R}^d; Y))^*, (L^p(\mathbb{R}^d; X))^*) \simeq \mathscr{L}(L^{p'}(\mathbb{R}^d; Y^*), L^{p'}(\mathbb{R}^d; X^*)),$$

where the identification of the spaces was again based on the assumed reflexivity via Theorems 1.3.21 and 1.3.10. By definition, the adjoint satisfies

$$\langle f, T^* g \rangle = \langle Tf, g \rangle$$

for all (f, g) in $L^p(\mathbb{R}^d; X) \times L^{p'}(\mathbb{R}^d; Y^*) \supseteq S(\mathscr{D}; X_0) \times S(\mathscr{D}; Y^0)$. □

The very formulation of the conditions that give rise to the name "$T(1)$ theorem" requires us to slightly extend the initial domain of weakly defined singular integral operators.

Definition 12.3.6. *For a bilinear* $\mathfrak{t} : S(\mathscr{D})^2 \to Z$, *we say that* $\mathfrak{t}(h_Q^\alpha, \mathbf{1})$ *is well-defined if the series*

$$\mathfrak{t}(h_Q^\alpha, \mathbf{1}) := \sum_{\substack{R \in \mathscr{D} \\ \ell(R) = \ell(Q)}} \mathfrak{t}(h_Q^\alpha, \mathbf{1}_R)$$

converges absolutely. We say that $\mathfrak{t}(\cdot, \mathbf{1})$ *is well-defined if* $\mathfrak{t}(h_Q^\alpha, \mathbf{1})$ *is well-defined for every* $Q \in \mathscr{D}$ *and* $\alpha \in \{0, 1\}^d \setminus \{0\}$.

We define $\mathfrak{t}(\mathbf{1}, h_Q^\alpha)$ *and* $\mathfrak{t}(\mathbf{1}, \cdot)$ *analogously.*

Lemma 12.3.7. *If* $\mathfrak{t}(h_Q^\alpha, \mathbf{1})$ *is well-defined, then*

(1) *for every* $k \in \mathbb{Z}$ *with* $2^{-k} \geqslant \ell(Q)$, *we have*

$$\mathfrak{t}(h_Q^\alpha, \mathbf{1}) = \sum_{R \in \mathscr{D}_k} \mathfrak{t}(h_Q^\alpha, \mathbf{1}_R),$$

where the series converges absolutely in the weak operator topology;

(2) *for every* $f \in S_{00}(\mathscr{D})$, *the series*

$$\mathfrak{t}(f, \mathbf{1}) := \sum_{R \in \mathscr{D}_k} \mathfrak{t}(f, \mathbf{1}_R)$$

converges absolutely at least for all sufficiently negative $k \in \mathbb{Z}$; *moreover, the value of the series is independent of* $k \in \mathbb{Z}$, *as long as it converges absolutely.*

The analogous statements hold for $\mathfrak{t}(\mathbf{1}, \cdot)$.

Proof. (1): Let $\ell(Q) = 2^{-j}$. For $k = j$, the claim of the lemma is just the definition. For $2^{-k} > 2^{-j}$ and $R \in \mathscr{D}_k$, we have

$$\mathfrak{t}(h_Q^\alpha, \mathbf{1}_R) = \mathfrak{t}\Big(h_Q^\alpha, \sum_{\substack{S \in \mathscr{D}_j \\ S \subseteq R}} \mathbf{1}_S\Big) = \sum_{\substack{S \in \mathscr{D}_j \\ S \subseteq R}} \mathfrak{t}(h_Q^\alpha, \mathbf{1}_S).$$

With $f = h_Q^\alpha$, we then have

$$\mathfrak{t}(f, \mathbf{1}) = \sum_{S \in \mathscr{D}_j} \mathfrak{t}(f, \mathbf{1}_S) = \sum_{R \in \mathscr{D}_k} \sum_{\substack{S \in \mathscr{D}_j \\ S \subseteq R}} \mathfrak{t}(f, \mathbf{1}_S) = \sum_{R \in \mathscr{D}_k} \mathfrak{t}(f, \mathbf{1}_R), \tag{12.34}$$

where the first equality holds by assumption, and the assumed absolute convergence allows to make the rearrangements and to get the absolute convergence also in the subsequent steps.

(2): Each $f \in S_{00}(\mathscr{D})$ is a linear combination of terms of the form $h_{Q_i}^{\alpha_i}$, where $i \in \mathscr{F}$ for some finite index set $\mathscr{F}$. If $Q_0 \in \mathscr{D}_{j_0}$ is the largest cube appearing here, then by the previous part of the lemma we know that

$$\sum_{R \in \mathscr{D}_k} \mathfrak{t}(h_{Q_i}^{\alpha_i}, \mathbf{1}_R)$$

converges absolutely for each $k \leqslant j_0$. Hence also

$$\sum_{R \in \mathscr{D}_k} \mathfrak{t}(f, \mathbf{1}_R) = \sum_{i \in \mathscr{F}} \langle f, h_{Q_i}^{\alpha_i} \rangle \sum_{R \in \mathscr{D}_k} \mathfrak{t}(h_{Q_i}^{\alpha_i}, \mathbf{1}_R)$$

converges absolutely. If the absolute convergence holds for some j and k, the equality of the corresponding series follows from (12.34).

The case of $\mathfrak{t}(\mathbf{1}, \cdot)$ is entirely analogous. □

As we shall see later, the forms $\mathfrak{t}(\mathbf{1}, \cdot)$ and $\mathfrak{t}(\cdot, \mathbf{1})$ are closely related to paraproducts. Since the boundedness of paraproducts is tricky, it is useful to be able identify situations, when they can be avoided, i.e., when $\mathfrak{t}(\mathbf{1}, \cdot) = 0 = \mathfrak{t}(\cdot, \mathbf{1})$.

With this goal in mind, we will now discuss an important case of *translation-invariant* bilinear forms. We first check that some natural candidates for the definition are equivalent:

Lemma 12.3.8. *Let Z be a Banach space. The following conditions are equivalent for a bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$:*

(1) $\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R) = \mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_{R\dot{+}m})$ *for all $Q, R \in \mathscr{D}$ with $\ell(Q) = \ell(R)$, and all $m \in \mathbb{Z}^d$, where $Q\dot{+}m := Q + m\ell(Q)$.*
(2) $\mathfrak{t}(f, g) = \mathfrak{t}(\tau_h f, \tau_h g)$ *for all $f, g \in S(\mathscr{D})$ and all dyadic rational vectors h, i.e., all h of the form $h = m2^{-k}$ for some $m \in \mathbb{Z}^d$ and $k \in \mathbb{Z}$, where $\tau_h f(s) := f(s - h)$.*

If $Z = \mathscr{L}(X, Y)$, these are also equivalent to a variant of (2) for all $f \in S(\mathscr{D}; X)$ and $g \in S(\mathscr{D}; Y^*)$ instead.

Proof. (2)⇒(1): This is immediate by taking $f = \mathbf{1}_Q$, $g = \mathbf{1}_R$ and $h = m\ell(Q)$, or $f = \mathbf{1}_Q \otimes x$, $g = \mathbf{1}_R \otimes y^*$ for arbitrary $x \in X$ and $y^* \in Y^*$ in the variant with $Z = \mathscr{L}(X, Y)$.

(1)⇒(2): By definition, each f, g is a linear combination of some indicators $\mathbf{1}_Q$ (or $\mathbf{1}_Q \otimes x$ resp. $\mathbf{1}_Q \otimes y^*$) with $Q \in \mathscr{D}$ (and $x \in X$, $y^* \in Y^*$), and we have $h = m_h 2^{-k_h}$ for some $m_h \in \mathbb{Z}^d$ and $k_h \in \mathbb{Z}$. Since any dyadic cube is an exact union of dyadic cubes of any given smaller size, and h can be expressed in a similar form $h = (2^{(k-k_h)})2^{-k}$ for any $k \geqslant k_h$, we may assume that we have $Q \in \mathscr{D}_k$ and $h = m2^{-k}$ for the same $k \in \mathbb{Z}$ to begin with. By bilinearity of both sides of the claim in (2), we thus need to verify that $\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R) = \mathfrak{t}(\tau_h \mathbf{1}_Q, \tau_h \mathbf{1}_R) = (\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_{R\dot{+}m})$ for each $Q, R \in \mathscr{D}_k$, but this is exactly what we assumed in (1). □

Definition 12.3.9. *A bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is called* translation-invariant, *if it satisfies the equivalent conditions of Lemma 12.3.8.*

Formally, it is easy to see that $\mathfrak{t}(\mathbf{1}, \cdot) = 0 = \mathfrak{t}(\cdot, \mathbf{1})$ if $\mathfrak{t}$ is translation invariant. Namely, if $Q \in \mathscr{D}$, and Q_1 is the "lower left quadrant" of Q, then

$$Q = \bigcup_{\gamma \in \{0,1\}^d} (Q_1 \dot{+} \gamma), \qquad h_Q^\alpha = \sum_{\gamma \in \{0,1\}^d} \langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma} \mathbf{1}_{Q_1 \dot{+} \gamma},$$

where the coefficients $\langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma}$ are equal to $\pm |Q|^{-1/2}$, with equally many of each sign. Now, *formally*, we have

$$\text{“} \quad \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1 \dot{+} \gamma}) = \mathfrak{t}(\tau_{\gamma\ell(Q_1)} \mathbf{1}, \tau_{\gamma\ell(Q)} \mathbf{1}_{Q_1}) = \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1}), \quad \text{”}$$

and hence

$$\begin{aligned} \text{“} \quad (\mathbf{1}, h_Q^\alpha) &= \sum_{\gamma \in \{0,1\}^d} \langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma} \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1 \dot{+} \gamma}) \\ &= \sum_{\gamma \in \{0,1\}^d} \langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma} \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1}) = 0 \cdot \mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1}) = 0. \quad \text{”} \end{aligned}$$

Problems with this computation are:

(1) While we defined $\mathfrak{t}(\mathbf{1}, h_Q^\alpha)$ for cancellative Haar functions h_Q^α, the expressions "$\mathfrak{t}(\mathbf{1}, \mathbf{1}_{Q_1 \dot{+} \gamma})$" above need not even be defined; i.e., even if the series defining the former converges, an analogous series for the latter need not.
(2) The assumption that $\mathfrak{t}$ is translation invariant was made on the class of functions $S(\mathscr{D})$ only, and the constant function $\mathbf{1}$ is not in this class.

Nevertheless, under a mild decay assumption, and some care with limits, we can bootstrap the above heuristics into a solid argument:

Proposition 12.3.10. *Suppose that $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is translation-invariant and satisfies the decay assumption, for all $Q \in \mathscr{D}$ and $m \geqslant M_Q$, that*

$$\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_{Q \dot{+} m})\| + \|\mathfrak{t}(\mathbf{1}_{Q \dot{+} m}, \mathbf{1}_Q)\| \leqslant c_Q |m|^{-d}. \tag{12.35}$$

Then $\mathfrak{t}(\mathbf{1}, \cdot) = 0 = \mathfrak{t}(\cdot, \mathbf{1})$.

Proof. We fix some $Q \in \mathscr{D}_k$ and $\alpha \in \{0,1\}^d \setminus \{0\}$. By definition, we have

$$\begin{aligned} \mathfrak{t}(\mathbf{1}, h_Q^\alpha) &= \sum_{m \in \mathbb{Z}^d} \mathfrak{t}(\mathbf{1}_{Q \dot{+} m}, h_Q^\alpha) = \lim_{M \to \infty} \sum_{\substack{m \in \mathbb{Z}^d \\ |m|_\infty \leqslant M}} \mathfrak{t}(\mathbf{1}_{Q \dot{+} m}, h_Q^\alpha) \\ &= \lim_{M \to \infty} \sum_{\beta, \gamma \in \{0,1\}^d} \langle h_Q^\alpha \rangle_{Q_1 \dot{+} \gamma} \sum_{\substack{m \in \mathbb{Z}^d \\ |m|_\infty \leqslant M}} \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} \beta \dot{+} 2m}, \mathbf{1}_{Q_1 \dot{+} \gamma}), \end{aligned}$$

where rearranging the order of the finite sums inside the limit presents no issues. Here

$$\mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} \beta \dot{+} 2m}, \mathbf{1}_{Q_1 \dot{+} \gamma}) = \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} (\beta - \gamma) \dot{+} 2m}, \mathbf{1}_{Q_1}),$$

and hence, noting that $\beta - \gamma \in \{-1, 0, 1\}^d$,

$$\begin{aligned} \sum_{\substack{m \in \mathbb{Z}^d \\ |m|_\infty \leqslant M}} \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} \beta \dot{+} 2m}, \mathbf{1}_{Q_1 \dot{+} \gamma}) &= \sum_{\substack{n \in \mathbb{Z}^d \\ n \in [-2M, 2M]^d + (\beta - \gamma)}} \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} n}, \mathbf{1}_{Q_1}) \\ &= \Big(\sum_{\substack{n \in \mathbb{Z}^d \\ n \in [-(2M-1), 2M-1]^d}} + \sum_{\substack{n \in \mathbb{Z}^d \\ n \in [-2M, 2M]^d + (\beta - \gamma) \\ n \notin [-(2M-1), 2M-1]^d}} \Big) \mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} n}, \mathbf{1}_{Q_1}) \\ &=: I_M + II_M^{\beta - \gamma}. \end{aligned}$$

In $II_M^{\beta-\gamma}$, we note that at least one component n_i of n must satisfy $|n_i| \geqslant 2M$, and hence the decay assumption (12.35) ensures that

$$\|\mathfrak{t}(\mathbf{1}_{Q_1 \dot{+} n}, \mathbf{1}_{Q_1})\| \leqslant c_{Q_1} (1 + 2M)^{-d}.$$

On the other hand, we have $n \in [-(2M+1), 2M+1]^d \setminus [-(2M-1), (2M-1)]^d$, and the total number of such $n \in \mathbb{Z}^d$ is

$$\begin{aligned}(1+2(2M+1))^d-(1+2(2M-1))^d &= (4M+3)^d-(4M-1)^d \\ &\leqslant 4d(4M+3)^{d-1},\end{aligned}$$

and hence

$$\|II_M^{\beta-\gamma}\| \leqslant 4d(4M+3)^{d-1} \times c_{Q_1}(1+2M)^{-d} \leqslant c_d c_{Q_1} M^{-1}.$$

Substituting back, it follows that

$$\begin{aligned}\mathfrak{t}(\mathbf{1},h_Q^\alpha) &= \lim_{M\to\infty} \sum_{\beta,\gamma\in\{0,1\}^d} \langle h_Q^\alpha\rangle_{Q_1\dot{+}\gamma}(I_M+II_M^{\beta-\gamma}) \\ &= \lim_{M\to\infty} \sum_{\beta,\gamma\in\{0,1\}^d} \langle h_Q^\alpha\rangle_{Q_1\dot{+}\gamma} II_M^{\beta-\gamma} = \lim_{M\to\infty} O(M^{-1}) = 0.\end{aligned}$$

The computation for $\mathfrak{t}(h_Q^\alpha,\mathbf{1})$ is entirely similar. □

Remark 12.3.11. It is easy to see from the proof that the decay assumption (12.35) could be somewhat weakened. We have not strived for maximal generality at this point, but stated a condition that is both relatively simple to formulate and easy to verify in our main application to Calderón–Zygmund singular integrals.

12.3.b The BCR algorithm and Figiel's decomposition

In order to analyse $\mathfrak{t}(f,g)$, we will use the auxiliary operators

$$E_k f = \sum_{Q\in\mathscr{D}_k} E_Q f = \sum_{Q\in\mathscr{D}_k} \langle f\rangle_Q 1_Q, \qquad \mathscr{D}_k = \{Q\in\mathscr{D} : \ell(Q)=2^{-k}\}.$$

$$D_k f = E_{k+1}f - E_k f = \sum_{Q\in\mathscr{D}_k}\Big(\sum_{Q'\in\mathrm{ch}(Q)} E_{Q'}f - E_Q f\Big) = \sum_{Q\in\mathscr{D}_k} D_Q f.$$

Our starting point for the analysis of a bilinear form is the following useful identity:

Lemma 12.3.12 (Beylkin–Coifman–Rokhlin (BCR) algorithm). *Let X,Y be Banach spaces, and let $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ be bilinear. Suppose that $f\in S(\mathscr{D};X)$ and $g\in S(\mathscr{D};Y^*)$ are constant on all $Q\in\mathscr{D}_M$. Then for all integers $m<M$,*

$$\begin{aligned}\mathfrak{t}(f,g) &= \sum_{m\leqslant k<M} \mathfrak{t}(D_k f, D_k g) + \sum_{m\leqslant k<M} \mathfrak{t}(D_k f, E_k g) \\ &\quad + \sum_{m\leqslant k<M} \mathfrak{t}(E_k f, D_k g) + \mathfrak{t}(E_m f, E_m g).\end{aligned} \tag{12.36}$$

Proof. That f is constant on all $Q \in \mathscr{D}_M$ means that $f = E_M f$, and similarly $g = E_M g$. Thus we have

$$\begin{aligned}\mathfrak{t}(f,g) - \mathfrak{t}(E_m f, E_m g) &= \mathfrak{t}(E_M f, E_M g) - \mathfrak{t}(E_m f, E_m g) \\ &= \sum_{m \leqslant k < M} (\mathfrak{t}(E_{k+1} f, E_{k+1} g) - \mathfrak{t}(E_k f, E_k g)),\end{aligned}$$

where

$$\begin{aligned}\mathfrak{t}(E_{k+1} f, E_{k+1} g) &= \mathfrak{t}((D_k + E_k)f, (D_k + E_k)g) \\ &= \mathfrak{t}(D_k f, D_k g) + \mathfrak{t}(D_k f, E_k g) + \mathfrak{t}(E_k f, D_k g) + \mathfrak{t}(E_k f, E_k g),\end{aligned}$$

and hence

$$\begin{aligned}&\mathfrak{t}(E_{k+1} f, E_{k+1} g) - \mathfrak{t}(E_k f, E_k g) \\ &\qquad = \mathfrak{t}(D_k f, D_k g) + \mathfrak{t}(D_k f, E_k g) + \mathfrak{t}(E_k f, D_k g).\end{aligned}$$

□

Remark 12.3.13. The upper bound $k < M$ imposed on the summation variables above is redundant: the condition that f and g are constant on all $Q \in \mathscr{D}_M$ implies that $D_k f = 0 = D_k g$ for $k \geqslant M$, so that the right side would remain unchanged if we allow the summations to run to infinity.

The final term in the expansion 12.36 is an error term, and can be controlled under the following mild conditions, which are obviously necessary for $\mathfrak{t}$ to define a bounded operator on L^p:

Definition 12.3.14. *We say that a bilinear* $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ *satisfies*

(1) *the* weak boundedness property *if*

$$\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_Q)\|_Z \leqslant \|\mathfrak{t}\|_{wbp} |Q| \qquad \forall Q \in \mathscr{D};$$

(2) *the* adjacent weak boundedness property *if*

$$\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_{Q \dot{+} n})\|_Z \leqslant \|\mathfrak{t}\|_{awbp} |Q| \quad \forall Q \in \mathscr{D}, \ \forall n \in \{-1,0,1\}^d. \tag{12.37}$$

Lemma 12.3.15. *Let X, Y be Banach spaces, and let a bilinear $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ satisfy the adjacent weak boundedness property. Then for all $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y)$, and all negative enough m, we have*

$$|\mathfrak{t}(E_m f, E_m g)| \leqslant 2^d \|\mathfrak{t}\|_{awbp} \|E_m f\|_{L^p(\mathbb{R}^d;X)} \|E_m g\|_{L^{p'}(\mathbb{R}^d;Y^*)} \underset{m \to -\infty}{\longrightarrow} 0.$$

Proof. We choose m so negative that the (bounded) supports of $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$ are both contained in the union of at most 2^d cubes $Q \in \mathscr{D}_m$ such that any two of them are related by $R = Q \dot{+} n$ for some $n \in \{-1,0,1\}^d$. We then have

$$\mathfrak{t}(E_m f, E_m g) = \sum_{Q,R\in\mathscr{D}_m} \mathfrak{t}(E_Q f, E_R g) = \sum_{Q,R\in\mathscr{D}_m} \mathfrak{t}(\langle f\rangle_Q \mathbf{1}_Q, \langle g\rangle_R \mathbf{1}_R),$$

and thus

$$\begin{aligned}
|\mathfrak{t}(E_m f, E_m g)| &\leqslant \sum_{Q,R\in\mathscr{D}_m} \|\mathfrak{t}(\mathbf{1}_Q,\mathbf{1}_R)\|_{\mathscr{L}(X,Y)} \|\langle f\rangle_Q\|_X \|\langle g\rangle_R\|_{Y^*} \\
&\leqslant \sum_{Q,R\in\mathscr{D}_m} \|\mathfrak{t}\|_{awbp} |Q| \|\langle f\rangle_Q\|_X \|\langle g\rangle_R\|_{Y^*} \\
&= \|\mathfrak{t}\|_{awbp} \sum_{Q\in\mathscr{D}_m} |Q|^{1/p} \|\langle f\rangle_Q\|_X \sum_{R\in\mathscr{D}_m} |R|^{1/p'} \|\langle g\rangle_R\|_{Y^*} \\
&\leqslant \|\mathfrak{t}\|_{awbp} 2^{d/p'} \Big(\sum_{Q\in\mathscr{D}_m} |Q| \|\langle f\rangle_Q\|_X^p \Big)^{1/p} 2^{d/p} \Big(\sum_{R\in\mathscr{D}_m} |R| \|\langle g\rangle_R\|_{Y^*}^{p'} \Big)^{1/p'} \\
&= 2^d \|\mathfrak{t}\|_{awbp} \|E_m f\|_{L^p(\mathbb{R}^d;X)} \|E_m g\|_{L^p(\mathbb{R}^d;Y^*)},
\end{aligned}$$

which is the claimed bound. □

The other terms in (12.36) can be identified with the various operators that we have studied in the previous sections:

Definition 12.3.16. *Let X, Y be Banach spaces, let $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ be a bilinear form, and let $\mathfrak{t}(\cdot,\mathbf{1})$ and $\mathfrak{t}(\mathbf{1},\cdot)$ be well-defined. We define the following operators associated with $\mathfrak{t}$:*

$$\mathfrak{H}_{\mathfrak{t}} := \sum_{\alpha,\gamma} \mathfrak{H}^{\alpha\gamma}_{\mathfrak{t}_0^{\alpha,\gamma}}, \quad \textit{where } \mathfrak{H}^{\alpha\gamma}_{\mathfrak{t}_0^{\alpha,\gamma}} \textit{ are Haar multipliers (12.3)},$$

$$T_{n,\mathfrak{t}} := \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} T^{\alpha\gamma}_{\phi_n,\mathfrak{t}_n^{\alpha,\gamma}}, \quad \textit{where } T^{\alpha\gamma}_{\phi_n,\mathfrak{t}_n^{\alpha,\gamma}} \textit{ are Figiel's operators (12.14)}$$

$$\textit{with } \begin{cases} \phi_n(Q) := Q\dot{+}n := Q + n\ell(Q), \\ \mathfrak{t}_n^{\alpha,\gamma}(Q) := \mathfrak{t}(h_Q^\alpha, h^\gamma_{Q\dot{+}n}), \end{cases}$$

$$U^i_{n,\mathfrak{t}} := \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} U^\alpha_{\phi_n,\mathfrak{u}_n^{i,\alpha}}, \quad \textit{where } U^\alpha_{\phi_n,\mathfrak{u}_n^{i,\alpha}} \textit{ are Figiel's operators (12.19)},$$

$$\textit{with } \mathfrak{u}_n^{i,\alpha}(Q) := \begin{cases} \mathfrak{t}_n^{1,\alpha}(Q)^* := \mathfrak{t}(h^0_{Q\dot{+}n}, h_Q^\alpha)^*, & i = 1, \\ \mathfrak{t}_n^{2,\alpha}(Q) \ \ := \mathfrak{t}(h_Q^\alpha, h^0_{Q\dot{+}n}), & i = 2. \end{cases}$$

We also define the related paraproducts:

$\varPi^1_{\mathfrak{t}}$:= *paraproduct with coefficients* $\mathfrak{t}(\mathbf{1}, h_Q^\alpha)$,
$\varPi^2_{\mathfrak{t}}$:= *paraproduct with coefficients* $\mathfrak{t}(h_Q^\alpha, \mathbf{1})^*$,
$\varLambda_{\mathfrak{t}}$:= *bi-paraproduct with coefficients* $\pi_Q^{\alpha,1} = \mathfrak{t}(\mathbf{1}, h_Q^\alpha)$ *and* $\pi_Q^{\alpha,2} = \mathfrak{t}(h_Q^\alpha, \mathbf{1})$,
$\mathfrak{l}_{\mathfrak{t}}$:= *the bilinear form of* $\varLambda_{\mathfrak{t}}$.

We may drop the subscript $\mathfrak{t}$ from these notations if it is obvious from the context.

Remark 12.3.17. Our indexing of the operators $U^i_{n,\mathfrak{t}}$ may appear counterintuitive at first sight, as one might like to think of the operators $U^2_{n,\mathfrak{t}}$, which act on $f \in L^p(\mathbb{R}^d;X)$ with coefficients $\mathfrak{t}(h^\alpha_Q, h^0_{Q\dotplus n}) \in \mathscr{L}(X,Y)$, as deserving to be the "primary" ones, rather than $U^1_{n,\mathfrak{t}}$, which act on the dual side $g \in L^{p'}(\mathbb{R}^d;Y^*)$ with adjoint coefficients $\mathfrak{t}(h^0_{Q\dotplus n}, h^\alpha_Q)^* \in \mathscr{L}(Y^*,X^*)$. However, this indexing is chosen, since the operators $U^i_{n,\mathfrak{t}}$ naturally arise in parallel with the paraproducts Π_i of the same index $i \in \{1,2\}$—see (12.42) and (12.43) below—, and it turns out to have some other advantages in the sequel.

With this notation, we can formula Figiel's decomposition of a bilinear form:

Proposition 12.3.18 (Figiel). *Let X, Y be Banach spaces, let $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ be a bilinear form, and let $\mathfrak{t}(\cdot,\mathbf{1})$ and $\mathfrak{t}(\mathbf{1},\cdot)$ be well-defined. For all*

$$f \in S(\mathscr{D};X), \qquad g \in S(\mathscr{D};Y^*), \qquad m \in \mathbb{Z},$$

denoting

$$u := (I - E_m)f \in S_{00}(\mathscr{D};X), \qquad v := (I - E_m)g \in S_{00}(\mathscr{D};Y^*),$$

we have the following identity with absolute convergence:

$$\begin{aligned}\mathfrak{t}(f,g) = \langle \mathfrak{H}_{\mathfrak{t}} u, g\rangle + \langle \Pi^1_{\mathfrak{t}} f, v\rangle + \langle u, \Pi^2_{\mathfrak{t}} g\rangle + \mathfrak{t}(E_m f, E_m g) +& \\ + \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \Big\{ \langle T_{n,\mathfrak{t}} u, g\rangle + \langle f, U^1_{n,\mathfrak{t}} v\rangle + \langle U^2_{n,\mathfrak{t}} u, g\rangle \Big\},&\end{aligned} \tag{12.38}$$

where the operators on the right are as in Definition 12.3.16. If these coefficients satisfy

$$\|\mathfrak{t}(\mathbf{1}, h^\alpha_Q)\|, \|\mathfrak{t}(h^\alpha_Q, \mathbf{1})\| \leqslant C|Q|^{1/2}, \tag{12.39}$$

then we have the further identity, with all terms below well defined:

$$\langle \Pi^1_{\mathfrak{t}} f, v\rangle + \langle u, \Pi^2_{\mathfrak{t}} g\rangle = \langle \Lambda_{\mathfrak{t}} f, g\rangle - \langle {}_m\Pi^1_{\mathfrak{t}} f, g\rangle - \langle f, {}_m\Pi^2_{\mathfrak{t}} g\rangle. \tag{12.40}$$

Remark 12.3.19. Since $\mathfrak{H}^{\alpha\gamma}_\lambda = T^{\alpha\gamma}_{\phi_0,\lambda}$, we could have incorporated the Haar multiplier into the second line of (12.38) as $\langle \mathfrak{H}_{\mathfrak{t}} u, g\rangle = \langle T_{0,\mathfrak{t}} u, g\rangle$. But we prefer to keep it separate, since its treatment will involve some differences compared to the rest of the $T_{n,\mathfrak{t}}$.

Proof of Proposition 12.3.18. We start with the identity (12.36) of Lemma 12.3.12. Since the sums are finitely nonzero, we are free rearrange as follows, observing that dyadic cubes Q, R of the same size are necessarily integer (times side-length) translates of each other:

$$\begin{aligned}\sum_{k\geqslant m} \mathfrak{t}(D_k f, D_k g) &= \sum_{k\geqslant m} \sum_{Q,R\in\mathscr{D}_k} \mathfrak{t}(D_Q f, D_R g) \\ &= \sum_{k\geqslant m} \sum_{Q\in\mathscr{D}_k} \sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, D_{Q\dotplus n} g) = \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, D_{Q\dotplus n} g)\end{aligned}$$

and we can also switch the order of the last two sums. Observing that $u = (I-E_m)f$ satisfies $D_Q u = D_Q f$ for $\ell(Q) \leqslant 2^{-m}$ and $D_Q u = 0$ for $\ell(Q) > 2^{-m}$, we find that, replacing f by u (and/or g by v) we can drop the restriction $\ell(Q) \leqslant 2^{-m}$ in the sum. Moreover, using the convention that summations over α and γ are always over the set $\{0,1\}^d \setminus \{0\}$,

$$\begin{aligned}\sum_{Q\in\mathscr{D}} \mathfrak{t}(D_Q u, D_{Q\dot{+}n} g) &= \sum_{\alpha,\gamma}\sum_{Q\in\mathscr{D}} \Big\langle \mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma)\langle h_Q^\alpha, u\rangle, \langle h_{Q\dot{+}n}^\gamma, g\rangle \Big\rangle \\ &= \sum_{\alpha,\gamma} \langle T^{\alpha,\gamma}_{\phi_n,\mathfrak{t}_n^{\alpha\gamma}} u, g\rangle = \langle T_n u, g\rangle.\end{aligned}$$

Hence

$$\sum_{k\geqslant m} \mathfrak{t}(D_k f, D_k g) = \sum_{n\in\mathbb{Z}^d} \langle T_n u, g\rangle = \langle \mathfrak{H} u, g\rangle + \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \langle T_n u, g\rangle \tag{12.41}$$

For the terms involving E_k, we begin in the same way but then introduce an additional twist to force some cancellation:

$$\begin{aligned}\sum_{k\geqslant m} \mathfrak{t}(D_k f, E_k g) &= \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, E_{Q\dot{+}n} g) \\ &= \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \sum_{n\in\mathbb{Z}^d} \Big(\mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}(\langle g\rangle_{Q\dot{+}n} - \langle g\rangle_Q)) + \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}\langle g\rangle_Q)) \Big).\end{aligned}$$

The assumption that $\mathfrak{t}(\cdot, \mathbf{1})$ is well-defined guarantees the absolute convergence of

$$\sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}\langle g\rangle_Q) =: \mathfrak{t}(D_Q f, \langle g\rangle_Q).$$

Recalling that only finitely many $D_Q f$ with $\ell(Q) \leqslant 2^{-m}$ are non-zero, we also get the absolute convergence of

$$\sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}\langle g\rangle_Q) = \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \mathfrak{t}(D_Q f, \langle g\rangle_Q) =: \mathfrak{p}_m(f,g),$$

and hence, by triangle inequality, that of

$$\sum_{n\in\mathbb{Z}^d} \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}(\langle g\rangle_{Q\dot{+}n} - \langle g\rangle_Q)).$$

Thus we can make the rearrangements

$$\sum_{k\geqslant m} \mathfrak{t}(D_k f, E_k g) = \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leqslant 2^{-m}}} \mathfrak{t}(D_Q f, \mathbf{1}_{Q\dot{+}n}(\langle g\rangle_{Q\dot{+}n} - \langle g\rangle_Q)) + \mathfrak{p}_m(f,g)$$

where adding the summation condition $n \neq 0$ was for free, since the factor $\langle g \rangle_{Q \dotplus n} - \langle g \rangle_Q$ evidently vanishes when $n = 0$. Again, replacing f by u allows us to drop the restrictions to $\ell(Q) \leqslant 2^{-m}$ both in the sum spelled out above and in $\mathfrak{p}_m(f,g)$. Moreover,

$$\begin{aligned} &\sum_{Q \in \mathscr{D}} \mathfrak{t}(D_Q u, \mathbf{1}_{Q \dotplus n}(\langle g \rangle_{Q \dotplus n} - \langle g \rangle_Q)) \\ &\qquad = \sum_{\alpha} \sum_{Q \in \mathscr{D}} \Big\langle \mathfrak{t}(h_Q^\alpha, h_{Q \dotplus n}^0) \langle h_Q^\alpha, u \rangle, \langle h_{Q \dotplus n}^0 - h_Q^0, g \rangle \Big\rangle \\ &\qquad = \sum_{\alpha} \langle U^\alpha_{\phi_n, \mathfrak{t}_n^{\alpha,0}} u, g \rangle = \langle U^2_{n,\mathfrak{t}} u, g \rangle. \end{aligned}$$

Directly from the definitions, we also have

$$\begin{aligned} \mathfrak{p}_m(f,g) &= \sum_{Q \in \mathscr{D}} \mathfrak{t}(D_Q u, \langle g \rangle_Q) \\ &= \sum_{Q \in \mathscr{D}} \sum_{\alpha \in \{0,1\}^d \setminus \{0\}} \mathfrak{t}(h_Q^\alpha \langle h_Q^\alpha, u \rangle, \langle g \rangle_Q) \\ &= \sum_{Q \in \mathscr{D}} \sum_{\alpha \in \{0,1\}^d \setminus \{0\}} \Big\langle \mathfrak{t}(h_Q^\alpha, \mathbf{1}) \langle h_Q^\alpha, u \rangle, \langle g \rangle_Q \Big\rangle \\ &= \sum_{Q \in \mathscr{D}} \sum_{\alpha \in \{0,1\}^d \setminus \{0\}} \Big\langle \langle h_Q^\alpha, u \rangle, \mathfrak{t}(h_Q^\alpha, \mathbf{1})^* \langle g \rangle_Q \Big\rangle = \langle u, \Pi^2_{\mathfrak{t}} g \rangle. \end{aligned}$$

In the computation above, the fact that $u \in S_{00}(\mathscr{D}; X)$ guarantees that all summations are finite, and the last step is simply the definition of the paraproduct via its action of the finitely non-zero Haar expansions in the dual space. Hence we have verified that

$$\sum_{k \geqslant m} \mathfrak{t}(D_k f, E_k g) = \sum_{\substack{n \in \mathbb{Z}^d \\ n \neq 0}} \langle U^2_{n,\mathfrak{t}} u, g \rangle + \langle u, \Pi^2_{\mathfrak{t}} g \rangle, \tag{12.42}$$

and the proof that

$$\sum_{k \geqslant m} \mathfrak{t}(E_k f, D_k g) = \sum_{\substack{n \in \mathbb{Z}^d \\ n \neq 0}} \langle f, U^1_{n,\mathfrak{t}} v \rangle + \langle \Pi^1_{n,\mathfrak{t}} f, v \rangle \tag{12.43}$$

is entirely analogous. Substituting the previous two identities and (12.41) into (12.36), we obtain the claimed (12.38).

Under the additional assumption (12.39), we know from Corollary 12.2.12 that $\langle \Pi^1_{\mathfrak{t}} f, g \rangle$ is well-defined and bilinear in $(f,g) \in S(\mathscr{D}; X) \times S(\mathscr{D}; Y^*)$, and hence

$$\langle \Pi^1_{\mathfrak{t}} f, v \rangle = \langle \Pi^1_{\mathfrak{t}} f, g \rangle - \langle \Pi^1_{\mathfrak{t}} f, E_m g \rangle = \langle \Pi^1_{\mathfrak{t}} f, g \rangle - \langle {}_m \Pi^1_{\mathfrak{t}} f, g \rangle.$$

Similarly, $\langle u, \Pi^2_{\mathfrak{t}} g \rangle = \langle f, \Pi^2_{\mathfrak{t}} g \rangle - \langle f, {}_m \Pi^2_{\mathfrak{t}} g \rangle$, and the previous two identities combine to give (12.40), noting that $\langle \Pi^1_{\mathfrak{t}} f, g \rangle + \langle f, \Pi^2_{\mathfrak{t}} g \rangle = \langle \Lambda_{\mathfrak{t}} f, g \rangle$. □

12.3.c Figiel's $T(1)$ theorem

The previous section culminated in Proposition 12.3.18, which established a decomposition of a generic bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X, Y)$ in terms of various fundamental operators. This is as far as it seems useful to proceed with identities, and we now turn to conditions that allow us to make meaningful estimates of the terms in the obtained decomposition. For this purpose, we introduce a certain family of norms. For a smooth discussion of a couple of closely related variants, it is convenient to adopt the following general framework.

Definition 12.3.20. *Let Z be a Banach space, and $\mathscr{P}(Z)$ the collection of all subsets of Z. We say that $\wp : \mathscr{P}(Z) \to [0, \infty]$ is a good set-bound on Z, if it satisfies the following properties for all $\mathscr{S}, \mathscr{T} \subseteq Z$:*

(1) *If $\mathscr{S} \subseteq \mathscr{T}$, then $\wp(\mathscr{S}) \leqslant \wp(\mathscr{T})$.*
(2) *$\wp(\mathscr{S} \cup \mathscr{T}), \wp(\mathscr{S} + \mathscr{T}) \leqslant \wp(\mathscr{S}) + \wp(\mathscr{T})$.*
(3) *If $\mathscr{Z} \subseteq \mathbb{K}$, then $\wp(\mathscr{Z}\mathscr{T}) \leqslant \sup_{z \in \mathscr{Z}} |z| \times \wp(\mathscr{T})$.*
(4) *$\wp(\mathscr{T}) = \wp(\operatorname{conv} \mathscr{T}) = \wp(\operatorname{abs\,conv} \mathscr{T})$.*
(5) *$\wp(\mathscr{T}) = \wp(\overline{\mathscr{T}})$, where $\overline{\mathscr{T}}$ denotes the norm-closure of $\mathscr{T}$.*

We primarily have in mind the following three cases:

Lemma 12.3.21. *Let X and Y be Banach spaces and $p \in [1, \infty)$. Then each of the following $\wp$ is a good set-bound on $Z = \mathscr{L}(X, Y)$:*

(a) *$\wp = \mathscr{U}$, where $\mathscr{U}(\mathscr{T}) := \sup\{\|T\| : T \in \mathscr{T}\}$,*
(b) *$\wp = \mathscr{R}_p$, the R-bound of order p,*
(c) *$\wp = \mathscr{R}_p^*$, the dual R-bound defined by*

$$\mathscr{R}_p^*(\mathscr{T}) := \mathscr{R}_p(\mathscr{T}^*), \qquad \mathscr{T}^* := \{T^* \in \mathscr{L}(Y^*, X^*) : T \in \mathscr{T}\}.$$

Proof. (a): The verification of the properties is immediate.

(b): Properties (1) and (2) for $\wp = \mathscr{R}_p$ are contained in the items with same numbers in Proposition 8.1.19. Property (3) follows from

$$\mathscr{R}_p(\mathscr{Z}\mathscr{T}) \leqslant \mathscr{R}_p(\mathscr{Z})\mathscr{R}_p(\mathscr{T}), \quad \mathscr{R}_p(\mathscr{Z}) = \sup_{z \in \mathscr{Z}} |z|,$$

where the first estimate is Proposition 8.1.19(3) and the second is immediate from Kahane's contraction principle (cf. the discussion right before Definition 8.1.1 of R-boundedness). Finally, properties (4) and (5) are contained in Propositions 8.1.21 and 8.1.22, respectively.

(c): All properties are direct corollaries of the corresponding properties in (b), since all set operations involved in these properties are well-behaved under the adjoint operation:

(1) $\mathscr{S} \subseteq \mathscr{T}$ if and only if $\mathscr{S}^* \subseteq \mathscr{T}^*$,

(2) $(\mathscr{S} \cup \mathscr{T})^* = \mathscr{S}^* \cup \mathscr{T}^*$ and $(\mathscr{S} + \mathscr{T})^* = \mathscr{S}^* + \mathscr{T}^*$,
(3) if $\mathscr{Z} \subseteq \mathbb{K}$, then $(\mathscr{Z}\mathscr{T})^* = \mathscr{Z}\mathscr{T}^*$,
(4) $(\operatorname{conv} \mathscr{T})^* = \operatorname{conv}(\mathscr{T}^*)$ and $(\operatorname{abs\,conv} \mathscr{T})^* = \operatorname{abs\,conv}(\mathscr{T}^*)$,
(5) $(\overline{\mathscr{T}})^* = \overline{\mathscr{T}^*}$.

□

Definition 12.3.22 (Figiel norms of a bilinear form). *For a bilinear form* $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X, Y)$, *let* $\mathfrak{t}_n^{\alpha\gamma}, \mathfrak{t}_n^{i,\alpha} : \mathscr{D} \to \mathscr{L}(X, Y)$ *be the associated functions appearing in Proposition 12.3.18. For* $s \geqslant 0$ *and a good set-bound* $\wp$ *on* $\mathscr{L}(X, Y)$, *we define*

$$\begin{aligned}
\|\mathfrak{t}^\theta\|_{\mathrm{Fig}^s(\wp)} &:= \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} (2+\log_2|n|)^s \wp(\mathfrak{t}_n^\theta), \qquad \theta \in \{(\alpha,\gamma),(i,\alpha)\},\\
\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^s(\wp)} &:= \sum_{\alpha,\gamma\in\{0,1\}^d\setminus\{0\}} \|\mathfrak{t}^{\alpha\gamma}\|_{\mathrm{Fig}^s(\wp)},\\
\|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^s(\wp)} &:= \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \|\mathfrak{t}^{i,\alpha}\|_{\mathrm{Fig}^s(\wp)}, \qquad i\in\{1,2\},\\
\|\mathfrak{t}\|_{\mathrm{Fig}^s(\wp)} &:= \sum_{i=0}^{2} \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^s(\wp)}.
\end{aligned}$$

When $\wp = \mathscr{U}$ *is as in Lemma 12.3.21(a), we write* $\mathrm{Fig}^s(\infty) := \mathrm{Fig}^s(\mathscr{U})$.

Remark 12.3.23. Referring to Proposition 12.3.18, one observes that the Figiel norms impose control on pairings $\mathfrak{t}(h_Q^\alpha, h_Q^\gamma)$, where at least one of the Haar functions is cancellative, i.e., $(\alpha,\gamma) \neq (0,0)$. This is in contrast to the decay condition (12.35), where $\alpha = \gamma = 0$.

Since we also encountered the adjoint function $\mathfrak{u}_n^{1,\alpha}(Q) := (\mathfrak{t}_n^{1,\alpha}(Q))^*$, we recall the following results from the previous volumes:

Proposition 12.3.24. *Let* X *and* Y *be Banach spaces,* $\mathscr{T} \subseteq \mathscr{L}(X, Y)$, *and* $p \in (1,\infty)$. *If* X *is* K*-convex (resp. a UMD space), then*

$$\mathscr{R}_{p'}^*(\mathscr{T}) \leqslant K_{p,X}\mathscr{R}_p(\mathscr{T})\Big(\leqslant \beta_{p,X}^+\mathscr{R}_p(\mathscr{T})\Big).$$

If Y *is* K*-convex (resp. a UMD space), then*

$$\mathscr{R}_p(\mathscr{T}) \leqslant K_{p,Y}\mathscr{R}_{p'}^*(\mathscr{T})\Big(\leqslant \beta_{p,Y}^+\mathscr{R}_{p'}^*(\mathscr{T})\Big).$$

In particular, if both X *and* Y *are* K*-convex (resp. UMD spaces), the set-bounds* $\mathscr{R}_p$ *and* $\mathscr{R}_{p'}^*$ *are equivalent on* $\mathscr{L}(X, Y)$.

Proof. The first inequalities in both chains are restatements of bounds in Proposition 8.4.1, and we have $K_{p,Z} \leqslant \beta_{p,Z}^+$ by Proposition 4.3.10. □

Thanks to Proposition 12.3.24, we would not need to distinguish (when working in UMD spaces) between direct and adjoint R-boundedness conditions, as such assumptions are actually equivalent. Nevertheless, we choose to do so, for twofold reasons. First, as far as quantitative conclusions are concerned, we would lose a constant each time we pass to the dual side, whereas in many applications, verifying the R-boundedness of concrete operators is just as easy (or difficult) directly on the dual side, so that applying the general duality result for R-boundedness is unnecessary. Second, writing the adjoint bounds explicitly, where they are relevant, will hopefully better clarify the role of the different assumptions in the estimates.

In the following lemma, we observe that Figiel norm estimates, of the type we will need to assume any way, will also guarantee the well-definedness of $\mathfrak{t}(\cdot,\mathbf{1})$ and $\mathfrak{t}(\mathbf{1},\cdot)$), which allows us to drop these as separate assumptions in the sequel.

Lemma 12.3.25. *Let X and Y be Banach spaces, and let $\mathfrak{t}: S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ be a bilinear form. If $\|\mathfrak{t}^{(2)}\|_{\mathrm{Fig}^0(\infty)} < \infty$ (resp. $\|\mathfrak{t}^{(1)}\|_{\mathrm{Fig}^0(\infty)} < \infty$), then $\mathfrak{t}(\cdot,\mathbf{1})$ (resp. $\mathfrak{t}(\mathbf{1},\cdot)$) is well defined, and*

$$\begin{aligned} \|\mathfrak{t}(h_Q^\alpha,\mathbf{1})\| &\leqslant \|\mathfrak{t}^{2,\alpha}\|_{\mathrm{Fig}^0(\infty)}|Q|^{1/2}, \\ \Big(\|\mathfrak{t}(\mathbf{1},h_Q^\alpha)\| &\leqslant \|\mathfrak{t}^{1,\alpha}\|_{\mathrm{Fig}^0(\infty)}|Q|^{1/2}\Big). \end{aligned} \tag{12.44}$$

Proof. For every $Q \in \mathscr{D}$ and $\alpha \in \{0,1\}^d \setminus \{0\}$, we have

$$\begin{aligned} \sum_{\substack{R\in\mathscr{D}\\ \ell(R)=\ell(Q)}} \|\mathfrak{t}(h_Q^\alpha,\mathbf{1}_R)\| &= \sum_{n\in\mathbb{Z}^d} \|\mathfrak{t}(h_Q^\alpha,h^0_{Q\dot{+}n})\|\,|Q|^{1/2} \\ &\leqslant \sum_{n\in\mathbb{Z}^d} \|\mathfrak{t}_n^{\alpha,0}(Q)\|\,|Q|^{1/2} = \|\mathfrak{t}^{2,\alpha}\|_{\mathrm{Fig}^0(\infty)}|Q|^{1/2} < \infty, \end{aligned}$$

which shows both that $\mathfrak{t}(\cdot,\mathbf{1})$ is well defined and the related bound. The case of $\mathfrak{t}(\mathbf{1},\cdot)$ is analogous. □

Theorem 12.3.26 ($T(1)$ theorem for bilinear forms). *Let $p \in (1,\infty)$ and $1 \leqslant t_i \leqslant p \leqslant q_i \leqslant \infty$, $i = 0,1,2$, where $q_1 = \infty$ and $t_2 = 1$. Consider the following conditions:*

(i) *X and Y are UMD spaces;*
(ii) *X has cotype q_i and Y has type t_i, or one of them has both, for each $i = 0,1,2$,*
(iii) *$\mathfrak{t}: S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ is a bilinear form with*

$$\sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha\gamma}) + \sum_{i=0}^{2} \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\mathscr{R}_p)} < \infty,$$

where $\sigma_i := 1/t_i - 1/q_i$,

(iv) $\mathfrak{t}$ *satisfies the adjacent weak boundedness property.*

Under assumptions (i) *through* (iv)*, the bilinear form* $\mathfrak{t}-\mathfrak{l}_{\mathfrak{t}}$ *defines a bounded operator* $T-\Lambda_{\mathfrak{t}} \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$ *that satisfies*

(a) *the norm estimate:*

$$\begin{aligned}\|T-\Lambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant \beta_{p,X}\beta_{p,Y}\Big\{ &\sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha,\gamma})+ \\ &+ A_d \min_{i=1,2} C_{0,i}\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{\sigma_0}(\wp_i)} + B_d \sum_{i=1}^{2} C_{i,i}\|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)}\Big\}\end{aligned}$$

where $A_d := 6\cdot(81)^d$, $B_d := 5\,200\cdot(81)^d$, $\wp_1 := \mathscr{R}^*_{p'}$, $\wp_2 := \mathscr{R}_p$, *and*

$$C_{i,2} := C_{(12.15)}(X,Y,p,q_i,t_i), \qquad C_{i,1} := C_{(12.15)}(Y^*,X^*,p',t_i',q_i'),$$

(b) *the representation formula, with absolute convergence for all* $f \in L^p(\mathbb{R}^d;X)$ *and* $g \in L^{p'}(\mathbb{R}^d;Y^*)$*:*

$$\langle (T-\Lambda_{\mathfrak{t}})f,g\rangle = \langle \mathfrak{H}_{\mathfrak{t}}f,g\rangle + \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \Big(\langle T_{n,\mathfrak{t}}f,g\rangle + \langle f,U^1_{n,\mathfrak{t}}g\rangle + \langle U^2_{n,\mathfrak{t}}f,g\rangle\Big), \tag{12.45}$$

where the operators on the right are as in Definition 12.3.16.

Under assumptions (i) *through* (iii)*, the following conditions are equivalent:*

(1) $\mathfrak{t}$ *defines a bounded operator* $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$*;*
(2) $\mathfrak{t}$ *satisfies* (iv)*, and* $\mathfrak{l}_{\mathfrak{t}}$ *defines a bounded* $\Lambda_{\mathfrak{t}} \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$.

Under these equivalent conditions, we have both (a) *and* (b)*.*

Remark 12.3.27. The assumptions of Theorem 12.3.26 allow a certain trade-off between the Figiel norms that one imposes on the bilinear form $\mathfrak{t}$ on the one hand, and (co)type assumptions (and the size of the related constants) on the spaces X and Y on the other hand. Indeed, the norms $\|\ \|_{\mathrm{Fig}^{\sigma_i}}$ become smaller with decreasing $\sigma_i = 1/t_i - 1/q_i$, thus with increasing type t_i or decreasing cotype q_i, but at the same time the related constants $C_{(12.15)}$ may increase.

Let $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$ and suppose that X has cotype q and Y has type t, or one of them has both. In Theorem 12.3.26, we will then choose $(t_1,q_1) = (t,\infty)$ and $(t_2,q_2) = (1,q)$; thus $\sigma_1 = 1/t$ and $\sigma_2 = 1/q'$. However, there are three prominent choices of the exponents t_0 and q_0:

(0) With $(t_0,q_0) = (t,q)$, we have

$$\sigma_0 = \frac{1}{t} - \frac{1}{q} \leqslant \min_{i=1,2} \sigma_i,$$

with strict inequality if both t and q are chosen to be non-trivial (as one always can for UMD spaces X and Y by Proposition 7.3.15). This shows that a strictly weaker condition is required on $\mathfrak{t}^{(0)}$ than on $\mathfrak{t}^{(i)}$ with $i=1,2$, but this seems to be largely a curiosity.

(1) With $(t_0, q_0) = (t_1, q_1) = (t, \infty)$, we have $\sigma_0 = \sigma_1$. Thus, we impose a stronger norm of $\mathfrak{t}^{(0)}$ than in case (0), but we achieve the following better constants in Theorem 12.3.26(a) under this choice:

$$C_{0,1} = C_{(12.15)}(Y^*, X^*, p', t', 1) = C_{1,1},$$

while an inspection of (12.15) shows that $C_{0,1}$ is larger than $C_{1,1}$ in general.

(2) Similarly, with $(t_0, q_0) = (t_2, q_2) = (1, q)$, we get

$$C_{0,2} = C_{(12.15)}(X, Y, p, q, 1) = C_{2,2}.$$

Using either choice (1) or (2) in Theorem 12.3.26, its key norm estimate admits the following form, under the assumption (we recall) that X has cotype q and Y has type t, or one of them has both,

$$\|T - \Lambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))} \leqslant \beta_{p,X}\beta_{p,Y}\Big\{ \sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha,\gamma}) + \\ + \sum_{i=1}^{2} C_i \Big(A_d \|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} + B_d \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} \Big) \Big\},$$

where $\sigma_1 = 1/t$, $\sigma_2 = 1/q'$, and

$$C_1 := C_{(12.15)}(Y^*, X^*, p', t', 1), \quad C_2 := C_{(12.15)}(X, Y, p, q, 1).$$

Proof of Theorem 12.3.26. The core of the proof will consist of establishing claims (a) and (b) under the full set of assumptions (i) through (iv). Assuming that this is already done, let us see how to conclude the rest of the proof.

The equivalence of (1) and (2) is asserted under the assumptions (i) through (iii) only. However, the adjacent weak boundedness property (iv) is clearly necessary for (1) and it is explicitly assumed in (2), so we can assume that this condition is satisfied in any case, and so we are in fact working under the full set of assumptions (i) through (iv) also in this remaining part of the proof. Thus the consequences (a) and (b) of this assumption are valid. In particular, since the bilinear form $\mathfrak{t} - \mathfrak{l}$ defines a bounded operator under this assumption, it is clear that $\mathfrak{t}$ defines a bounded operator if and only if $\mathfrak{l}$ does.

We then turn to the actual proof of (a) and (b) under the assumptions (i) through (iv). From Lemma 12.3.25, we get that $\mathfrak{t}(\cdot, \mathbf{1})$ and $\mathfrak{t}(\mathbf{1}, \cdot)$, and hence the two paraproducts, are well defined, and their coefficients satisfy the bounds (12.44). For $f \in S(\mathscr{D}; X)$ and $g \in S(\mathscr{D}; Y^*)$, we then have both identities (12.38) and (12.40) provided by Proposition 12.3.18. Combined together, they read as

$$\mathfrak{t}(f, g) = \langle \mathfrak{H} u_m, g\rangle + \langle \Lambda f, g\rangle + \mathscr{E}_m(f, g) + \\ + \sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} \Big\{ \langle T_n u_m, g\rangle + \langle f, U_n^1 v_m\rangle + \langle U_n^2 u_m, g\rangle \Big\}, \tag{12.46}$$

where $u_m := (I - E_m)f \in S_{00}(\mathscr{D}; X)$, $v_m := (I - E_m)g \in S_{00}(\mathscr{D}; Y^*)$, and the error term

$$\mathscr{E}_m(f, g) = \langle {}_m\Pi_1 f, g\rangle + \langle f, {}_m\Pi_2 g\rangle + \mathfrak{t}(E_m f, E_m g)$$

satisfies

$$\begin{aligned} |\mathscr{E}_m(f, g)| \leqslant \Big(c_{d,p} \sum_{i=1}^{2} \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^0(\infty)} + 2^d \|\mathfrak{t}\|_{awbp}\Big) \times \\ \times \|E_m f\|_{L^p(\mathbb{R}^d;X)} \|E_m g\|_{L^p(\mathbb{R}^d;Y^*)} \underset{m\to-\infty}{\longrightarrow} 0 \end{aligned} \tag{12.47}$$

by Lemmas 12.2.11 and 12.3.25 for the paraproduct terms and Lemma 12.3.15 for both the final term and the limit.

Directly from Theorem 12.1.11, we deduce that

$$\begin{aligned} |\langle \mathfrak{H} u_m, g\rangle| &\leqslant \sum_{\alpha,\gamma} |\langle \mathfrak{H}^{\alpha\gamma}_{\mathfrak{t}_0^{\alpha,\gamma}} u_m, g\rangle| \\ &\leqslant \beta^+_{p,X} \beta^+_{p',Y^*} \sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha,\gamma}) \|u_m\|_p \|g\|_p, \end{aligned} \tag{12.48}$$

where, and in the rest of the proof, we abbreviate

$$\|\ \|_p := \|\ \|_{L^p(\mathbb{R}^d;X)}, \qquad \|\ \|_{p'} := \|\ \|_{L^{p'}(\mathbb{R}^d;Y^*)}.$$

Note that $\phi_n(Q) := Q\dot{+}n$ satisfies $\phi_n(Q) \subseteq 3Q^{(N)}$ provided that $|n| \leqslant 2^N$; thus in particular for $N = \lceil \log_2^+ |n| \rceil$; this is relevant in view of applying Corollary 12.1.27 and Theorem 12.1.28. From Corollary 12.1.27, we deduce that

$$\begin{aligned} |\langle T_n u_m, g\rangle| &\leqslant \sum_{\alpha,\gamma} |\langle T^{\alpha\gamma}_{\phi_n, \mathfrak{t}_n^{\alpha,\gamma}} u_m, g\rangle| \\ &\leqslant A_d \beta_{p,X} \beta_{p,Y} (2 + \log_2 |n|)^{1/t_0 - 1/q_0} \min_{i=1,2} C_{0,i} \wp_i(\mathfrak{t}_n^{\alpha,\gamma}) \|u_m\|_p \|g\|_{p'} \end{aligned}$$

using the notation of the statement of the theorem that we are proving. Hence

$$\sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} |\langle T_n u_m, g\rangle| \leqslant A_d \beta_{p,X} \beta_{p,Y} \min_{i=1,2} C_{0,i} \|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{1/t_0 - 1/q_0}(\wp_i)} \|u_m\|_p \|g\|_{p'}$$

Similarly, recalling that $t_2 := 1$, Theorem 12.1.28 guarantees that

$$\begin{aligned} |\langle U^2_n u_m, g\rangle| &\leqslant \sum_{\alpha} |\langle U^{\alpha}_{\phi_n, \mathfrak{t}_n^{2,\alpha}} u_m, g\rangle| \\ &\leqslant B_d \beta_{p,X} \beta_{p,Y} (2 + \log_2 |n|)^{1/t_2 - 1/q_2} \sum_{\alpha} C_{2,2} \wp_2(\mathfrak{t}_n^{2,\alpha}) \|u_m\|_p \|g\|_{p'} \end{aligned}$$

in the notation of the theorem, and hence

$$\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} |\langle U_n^2 u_m, g\rangle| \leqslant B_d \beta_{p,X}\beta_{p,Y} C_{2,2}\|\mathfrak{t}^{(2)}\|_{\mathrm{Fig}^{1/t_2-1/q_2}(\wp_2)}\|u_m\|_p\|g\|_{p'}.$$

For the term $\langle f, U_n^1 v_m\rangle$, we again apply Theorem 12.1.28 but on the dual side, with X, Y, p replaced by Y^*, X^*, p'. By assumption, Y has type $t_1 \leqslant p$, and hence Y^* has cotype $t_1' \geqslant p'$ by Proposition 7.1.13. So we can indeed apply Theorem 12.1.28 with X, Y, p, q replaced by Y^*, X^*, p', t_1'. Recalling that $q_1 := \infty$, and noting that $1 - 1/t_1' = 1/t_1 = 1/t_1 - 1/q_1$, this gives

$$\begin{aligned}|\langle f, U_n^1 v_m\rangle| &\leqslant \sum_\alpha |\langle f, U^\alpha_{\phi_n,(\mathfrak{t}_n^{1,\alpha})^*} v_m\rangle| \\ &\leqslant B_d\beta_{p',X^*}\beta_{p',Y^*}(2+\log_2|n|)^{1/t_1-1/q_1}\|f\|_p\|v_m\|_{p'}\times \\ &\qquad \times C(Y^*, X^*, p', t_1')\mathscr{R}_{p'}((\mathfrak{t}_n^{1,\alpha})^*),\end{aligned}$$

where $\beta_{p',X^*}\beta_{p',Y^*} = \beta_{p,X}\beta_{p,Y}$ and

$$C(Y^*, X^*, p', t_1')\mathscr{R}_{p'}((\mathfrak{t}_n^{1,\alpha})^*) = C_{1,1}\mathscr{R}^*_{p'}(\mathfrak{t}_n^{1,\alpha}) = C_{1,1}\wp_1(\mathfrak{t}_n^{1,\alpha})$$

in the notation of the theorem. Hence

$$\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} |\langle f, U_n^1 v_m\rangle| \leqslant B_d \beta_{p,X}\beta_{p,Y} C_{1,1}\|\mathfrak{t}^{(1)}\|_{\mathrm{Fig}^{1/t_1-1/q_1}(\wp_1)}\|f\|_p\|v_m\|_{p'}.$$

Noting that $\|u_m\|_p \leqslant 2\|f\|_p$ and $\|v_m\|_{p'} \leqslant 2\|g\|_{p'}$, and using the assumption about $\|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{1/t_i-1/q_i}(\mathscr{R}_p)}$ (combined with Proposition 12.3.24 in the case of $\mathscr{R}_{p'}((\mathfrak{t}_n^{1,\alpha})^*)$), it follows that the series in (12.46) are term-wise and uniformly in m dominated by absolutely convergent series. This allows us to pass to the limit $m \to -\infty$ in (12.46) with dominated convergence to deduce that

$$(\mathfrak{t}-\mathfrak{l})(f,g) = RHS(12.45) \qquad \forall\ f \in S(\mathscr{D}; X),\ g \in S(\mathscr{D}; Y^*). \tag{12.49}$$

Taking the same limit in the term-wise bounds above, we obtain

$$\begin{aligned}|(\mathfrak{t}-\mathfrak{l})(f,g)| &= |\mathfrak{t}(f,g) - \langle \Lambda f, g\rangle| \\ &\leqslant \beta_{p,X}\beta_{p,Y}\Big\{\sum_{\alpha,\gamma}\mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\alpha,\gamma}) + A_d\min_{i=1,2} C_{0,i}\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{1/t_0-1/q_0}(\wp_i)} \\ &\qquad + B_d\sum_{i=1}^2 C_{i,i}\|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{1/t_i-1/q_i}(\wp_i)}\Big\}\|f\|_{L^p(\mathbb{R}^d;X)}\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}\end{aligned} \tag{12.50}$$

again for all $f \in S(\mathscr{D}; X)$ and $g \in S(\mathscr{D}; Y^*)$, where A_d, B_d and C_i are as in the statement of the Theorem.

This estimate shows that the bilinear form $\mathfrak{t}-\mathfrak{l}$ satisfies a relevant *a priori* bound, and hence defines an operator $T - \Lambda \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$. By density, it is immediate that (12.50) remains valid with general $f \in L^p(\mathbb{R}^d; X)$ and $g \in L^{p'}(\mathbb{R}^d; Y^*)$, and this proves the claimed norm bound (a) for $T - \Lambda$.

We can then replace $(\mathfrak{t}-\mathfrak{l})(f,g)$ by $\langle (T-\Lambda)f,g\rangle$ in (12.49). Approximating general $f \in L^p(\mathbb{R}^d;X)$ and $g \in L^{p'}(\mathbb{R}^d;Y^*)$ by functions as in (12.49), and using dominated convergence and the term-wise bounds recorded above, this proves the representation (b). This completes the proof of the claims under the assumption that $\mathfrak{t}$ satisfies the adjacent weak boundedness property. □

12.3.d Improved estimates via random dyadic cubes

A feature of Theorem 12.3.26 is that it deals with a bilinear form adapted to a fixed system of dyadic cubes $\mathscr{D}$. This is an advantage in applications to questions of intrinsically dyadic nature. But it is also a certain limitation in view of applications to non-dyadic questions, in that the assumptions of Theorem 12.3.26 fail to take advantage of possible information about non-dyadic cubes. For example, with some effort, one could use Theorem 12.3.26 to re-derive the boundedness of the Hilbert transform on $L^p(\mathbb{R};X)$, which we proved in a different way in Theorem 5.1.13. However, the conclusion derived from Theorem 12.3.26 would be quantitatively weaker, in terms of the dependence on the UMD constant $\beta_{p,X}$, which was quadratic in Theorem 5.1.13. For $X=Y$, Theorem 12.3.26 also features the explicit factor $\beta_{p,X}^2$, but there is another $\beta_{p,X}$ implicit in the constants $C_{(12.15)}$. On the other hand, it is evident that, for $\mathfrak{t}(f,g) := \langle Hf,g\rangle$, there is no difference in estimating $\mathfrak{t}(h_I^\alpha, h_K^\gamma)$ for dyadic or non-dyadic intervals I,J. But Theorem 12.3.26, as formulated, makes no use of this additional information.

We now wish derive to variant of Theorem 12.3.26 to address these issues. First of all, we need a straightforward generalisation to $\mathbb{R}^d$ of the random dyadic systems that we used in the one-dimensional case in Section 5.1.

Lemma 12.3.28. *Let $\mathscr{D}$ be a fixed dyadic system on $\mathbb{R}^d$, in the sense of Definition 11.1.6.*

(1) *For every $\omega = (\omega_j)_{j\in\mathbb{Z}^d} \in (\{0,1\}^d)^{\mathbb{Z}}$,*

$$\mathscr{D}^\omega := \big\{Q \dot{+} \omega : Q \in \mathscr{D}\big\}$$

is another dyadic system on $\mathbb{R}^d$, where

$$Q\dot{+}\omega := Q + \ell(Q,\omega), \quad \ell(Q,\omega) := \sum_{j:2^{-j}<\ell(Q)} 2^{-j}\omega_j.$$

(2) *Conversely, every dyadic system $\mathscr{D}'$ has this form for some $\omega \in (\{0,1\}^d)^{\mathbb{Z}}$.*

Proof. Let $\mathscr{D}^0$ be the standard dyadic system, and consider a family of shifts $s_j + \mathscr{D}_j^0$. These clearly satisfy property (i) of Definition 11.1.6. A necessary and sufficient condition for them to satisfy (ii) of Definition 11.1.6 is that $s_j - s_{j+1} \in 2^{-j-1}\mathbb{Z}^d$.

If $\mathscr{D}$ is a dyadic system defined by shifts s_j, then $\mathscr{D}^\omega$ is defined by the shifts $s_j + \omega_{(j)}$, where

$$\omega_{(j)} := \sum_{k>j} \omega_k 2^{-k}.$$

These satisfy $(s_j+\omega_{(j)})-(s_{j+1}-\omega_{(j+1)}) = (s_j-s_{j+1})+\omega_{j+1}2^{-j-1} \in 2^{-j-1}\mathbb{Z}^d$, and hence $\mathscr{D}^\omega$ is also a dyadic system, as claimed in (1).

Then suppose that $\mathscr{D}$ and $\mathscr{D}'$ are two dyadic systems defined by shifts s_j and s'_j, respectively. It is clear that the family $\mathscr{D}_j = s_j + \mathscr{D}^0_j$ only depends on $s_j \mod 2^{-j}$, and hence we may assume without loss of generality that both $s_j \in [0, 2^{-j})^d$ and $t_j := s'_j - s_j \in [0, 2^{-j})^d$. Since both $s_j - s_{j+1} \in 2^{-j-1}\mathbb{Z}^d$ and $s'_j - s'_{j+1} \in 2^{-j-1}\mathbb{Z}^d$, it follows that also $t_j - t_{j+1} \in 2^{-j-1}\mathbb{Z}^d$. Together with the fact that $t_j \in [0, 2^{-j})^d$ and $t_{j+1} \in [0, 2^{-j-1})^d$, one finds that in fact $t_j - t_{j+1} \in 2^{-j-1}\{0,1\}^d$. Denoting $\omega_{j+1} := 2^{j+1}(t_j - t_{j+1}) \in \{0,1\}^d$, we obtain

$$t_j = t_{j+1} + 2^{-j-1}\omega_{j+1} = \ldots = \sum_{k>j} 2^{-k}\omega_k = \omega_{(j)},$$

and then

$$\mathscr{D}'_j = s'_j + \mathscr{D}^0_j = t_j + s_j + \mathscr{D}^0_j = \omega_{(j)} + \mathscr{D}_j = \mathscr{D}^\omega_j,$$

as claimed in (2), and this completes the proof. □

Definition 12.3.29. *For $\omega = (\omega_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}}$, let*

$$j_\omega := \sup\{j \in \mathbb{Z} : \omega_j \neq 0\} \in \mathbb{Z} \cup \{-\infty, \infty\},$$
$$(\{0,1\}^d)^{\mathbb{Z}}_0 := \Big\{\omega \in (\{0,1\}^d)^{\mathbb{Z}} : j_\omega < \infty\Big\}.$$

We say that $\omega \in (\{0,1\}^d)^{\mathbb{Z}}$ is eventually zero *if $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$.*

Lemma 12.3.30. *For every $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$, we have*

$$S(\mathscr{D}^\omega) = S(\mathscr{D}), \qquad S_0(\mathscr{D}^\omega) = S_0(\mathscr{D}).$$

Moreover, there exists an $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$ such that $S_{00}(\mathscr{D}^\omega) = S_0(\mathscr{D})$.

Proof. Recall that $S(\mathscr{D})$ is the span of indicators $\mathbf{1}_Q$ of $Q \in \mathscr{D}$. Since every $Q \in \mathscr{D}_j$ can be written as a union of smaller cubes $Q' \in \mathscr{D}_k$, for any $k > j$, we see that, for any given $j_0 \in \mathbb{Z}$, the space $S(\mathscr{D})$ only depends on $\bigcup_{j>j_0} \mathscr{D}_j$. On the other hand, if ω is eventually zero, and j_ω is as in the definition of this property, then $\mathscr{D}^\omega_j = \mathscr{D}_j$ for $j > j_\omega$. The first claimed identity thus follows.

The second identity follows by restricting to functions of vanishing integral on both sides.

Finally, it is easy to choose $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$ in such a way that $\mathscr{D}^\omega$ contains an increasing sequence of cubes that exhausts all $\mathbb{R}^d$. Then, given any $f \in S(\mathscr{D})$, we can find some $Q_0 \in \mathscr{D}^\omega$ that contains the support of f. If, in addition, $f \in S_0(\mathscr{D}) = S_0(\mathscr{D}^\omega)$, then f can be expanded in terms of finitely many Haar functions h^α_Q with $Q \subseteq Q_0$, and thus $f \in S_{00}(\mathscr{D}^\omega)$. Since this holds for every $f \in S_0(\mathscr{D})$, we obtain the final identity. □

Remark 12.3.31. Without the assumption of eventually zero, the conclusion of Lemma 12.3.30 fails in general. For instance, the indicator of the shifted dyadic interval $\frac{1}{3}+[0,1)$ cannot be expressed as a finite linear combination of standard dyadic intervals.

Thanks to Lemma 12.3.30, any bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ may also be regarded as a bilinear form $\mathfrak{t} : S(\mathscr{D}^\omega)^2 \to Z$ for every eventually zero ω. Although the objects in fact coincide, it will be convenient to denote the latter by $\mathfrak{t}^\omega$. This is particularly relevant when considering the various auxiliary objects derived from the bilinear form. In particular, extending the notation from Proposition 12.3.18, we have

$$\mathfrak{t}_n^{\omega;\alpha,\gamma}(R) := \mathfrak{t}(h_R^\alpha, h_{R\dot{+}n}^\gamma), \qquad R = Q\dot{+}\omega \in \mathscr{D}^\omega$$

$$\mathfrak{u}_n^{\omega;i,\alpha}(R) := \begin{cases} \mathfrak{t}_n^{\omega;1,\alpha}(R)^* := \mathfrak{t}(h_{R\dot{+}n}^0, h_R^\alpha)^*, & i=1, \\ \mathfrak{t}_n^{\omega;2,\alpha}(R) := \mathfrak{t}(h_R^\alpha, h_{R\dot{+}n}^0)^*, & i=2. \end{cases}$$

The advantage of considering several dyadic systems $\mathscr{D}^\omega$ is that this allows us to dispense with some of the cubes within each $\mathscr{D}^\omega$.

Definition 12.3.32. *For a dyadic system $\mathscr{D}$ and $k \in \mathbb{Z}_{\geqslant 2}$, a cube $Q \in \mathscr{D}$ is called k-good (in $\mathscr{D}$) if*

$$\operatorname{dist}(R, \complement R^{(k)}) \geqslant \frac{1}{4}\ell(R^{(k)}) = 2^{k-2}\ell(R),$$

where $R^{(k)}$ is the kth dyadic ancestor of R in $\mathscr{D}$.

Lemma 12.3.33. *Consider a random choice of $\omega \in (\{0,1\}^d)^{\mathbb{Z}_{\geqslant M}}$ with respect to the uniform probability on this space. For every $Q \in \mathscr{D}$ with $\ell(Q) \geqslant 2^{-M}$,*

(1) *the random set $Q\dot{+}\omega$ and the event $\{Q\dot{+}\omega$ is k-good in $\mathscr{D}^\omega\}$ are independent;*
(2) $\mathbb{P}(Q\dot{+}\omega$ is k-good in $\mathscr{D}^\omega) = 2^{-d}$.

Proof. (1) follows by observing that $Q\dot{+}\omega$ depends only on ω_j with $2^{-M} \leqslant 2^{-j} < \ell(Q)$, whereas $\{Q\dot{+}\omega$ is k-good in $\mathscr{D}^\omega\}$ depends on the relative position of $Q\dot{+}\omega$ with respect to cubes $R\dot{+}\omega$ with $\ell(R) = 2^k\ell(Q)$, which in turn depends on ω_j with $\ell(Q) \leqslant 2^{-j} < 2^k\ell(Q)$.

(2): When all ω_j with $\ell(Q) \leqslant 2^{-j} < 2^k\ell(Q)$ are independently chosen from $\{0,1\}^d$, it is easy to see that the probability of $\{Q\dot{+}\omega$ is k-good in $\mathscr{D}^\omega\}$ is equal to the geometric probability (i.e., the relative volume) of the "good region"

$$R_{\text{good}} := \Big\{ s \in R : \operatorname{dist}(s, \complement R) \geqslant \frac{1}{4}\ell(R) \Big\} = \frac{1}{2}\bar{R}$$

of the $\mathscr{D}^\omega$-ancestor R of Q, and this is simply

$$\frac{|R_{\text{good}}|}{|R|} = \frac{|\frac{1}{2}\bar{R}|}{|R|} = 2^{-d}.$$

□

Definition 12.3.34. *For $\theta \in \{(\alpha,\gamma),(i,\alpha)\}$, and $n \in \mathbb{Z}^d \setminus \{0\}$, we define*

$$\begin{aligned} \mathfrak{t}^{\omega;\theta}_{n,\mathrm{good}}(R) &:= \mathbf{1}_{\{R \textit{ is } k(n)\textit{-good in } \mathscr{D}^\omega\}} \mathfrak{t}^{\omega;\theta}_n(R), \\ k(n) &:= 2 + \lceil \log_2 |n| \rceil. \end{aligned}$$

We define Figiel's operators $T^{\mathrm{good}}_{n,\mathfrak{t}^\omega}$ and $U^{i,\mathrm{good}}_{m,\mathfrak{t}^\omega}$ as in Definition 12.3.16, but with $\mathfrak{t}^{\omega;\theta}_{n,\mathrm{good}}$ in place of the respective $\mathfrak{t}^\theta_n$

For $n \in \mathbb{Z}^d \setminus \{0\}$, we have $k(n) \geqslant 2$, and hence the notion of "$k(n)$-good" is well-defined. For $n = 0$ we would formally get $k(0) = -\infty$, and "$-\infty$-good" reduces to the triviality $\mathrm{dist}(R, \complement R) \geqslant 0$; accordingly, for definiteness, we let

$$\mathfrak{t}^{\omega;\theta}_{0,\mathrm{good}}(R) := \mathfrak{t}^{\omega;\theta}_0(R).$$

Replacing all quantities in Definition 12.3.22 by their "good" restrictions, we have a natural definition of the Figiel norms

$$\begin{aligned} &\|\mathfrak{t}^{\omega;\theta}_{\mathrm{good}}\|_{\mathrm{Fig}^s(\wp)}, \quad \theta \in \{(\alpha,\gamma),(i,\alpha)\}, \\ &\|\mathfrak{t}^{\omega;(i)}_{\mathrm{good}}\|_{\mathrm{Fig}^s(\wp)}, \quad i = 1,2, \qquad \|\mathfrak{t}^{\omega}_{\mathrm{good}}\|_{\mathrm{Fig}^s(\wp)}. \end{aligned}$$

As we are about to see, these good parts will suffice to control a bounded extension of the form $\mathfrak{t}$, and this also allows us to obtain a better dependence on the UMD constants. Here is the precise statement:

Theorem 12.3.35 ($T(1)$ theorem for bilinear forms, random version). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and consider the conditions:*

(i) *X and Y are UMD spaces,*
(ii) *X has cotype q and Y has type t, or one of them has both,*
(iii) *$\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ is a bilinear form with*

$$\sum_{\alpha,\gamma \in \{0,1\}^d \setminus \{0\}} \mathscr{D}\mathscr{R}_p(\mathfrak{t}^{\omega;\alpha,\gamma}_0) + \min_{i=1,2} \|\mathfrak{t}^{\omega;(0)}\|_{\mathrm{Fig}^{\sigma_i}(\mathscr{R}_p)} + \sum_{i=1}^2 \|\mathfrak{t}^{\omega;(i)}\|_{\mathrm{Fig}^{\sigma_i}(\mathscr{R}_p)} \leqslant C,$$

uniformly in $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$, where $\sigma_1 = 1/t$ and $\sigma_2 = 1/q'$.
(iv) *the forms $\mathfrak{t}^\omega$ satisfy the adjacent weak boundedness property $\|\mathfrak{t}^\omega\|_{awbp} \leqslant C$ uniformly in $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$,*

Under assumptions (i) through (iii), the following conditions are equivalent:

(1) $\mathfrak{t}$ *defines a bounded linear operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(2) $\mathfrak{t}$ *satisfies (iv) and the paraproducts $\Lambda_{\mathfrak{t}^\omega}$ are uniformly bounded.*

Under these equivalent conditions, we have:

(a) *the norm estimate:*

$$\begin{aligned}
&\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\
&\quad\leqslant \sup_{\omega} \|\varLambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} + \beta_{p,X}\beta_{p,Y}\Big\{ \sup_{\omega} \sum_{\alpha,\gamma} \mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\omega;\alpha,\gamma}) \\
&\qquad + 12\cdot 2^d \sup_{\omega}\Big(\min_{i=1,2} c_i \|\mathfrak{t}_{\mathrm{good}}^{\omega;(0)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} + \sum_{i=1}^{2} c_i \|\mathfrak{t}_{\mathrm{good}}^{\omega;(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)}\Big)\Big\},
\end{aligned}$$

where the suprema are over $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$, *and*

$$\begin{aligned}
&\wp_1 := \mathscr{R}^*_{p'}, \quad \wp_2 := \mathscr{R}_p, \quad \sigma_1 := 1/t, \quad \sigma_2 := 1/q', \\
&c_1 := \min_{Z=X,Y} c_{t',Z^*;p'}, \quad c_2 := \min_{Z=X,Y} c_{q,Z;p};
\end{aligned} \tag{12.51}$$

(b) *the representation formula*

$$\begin{aligned}
\langle Tf,g\rangle = \mathbb{E}\Big(\langle \mathfrak{H}_{\mathfrak{t}^\omega} f, g\rangle + \langle \varLambda_{\mathfrak{t}^\omega} f,g\rangle + 2^d \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \Big\{ \langle T^{\mathrm{good}}_{n,\mathfrak{t}^\omega} f, g\rangle + \\
+ \langle f, U^{1,\mathrm{good}}_{n,\mathfrak{t}^\omega} g\rangle + \langle U^{2,\mathrm{good}}_{n,\mathfrak{t}^\omega} f, g\rangle \Big\}\Big),
\end{aligned} \tag{12.52}$$

with absolute convergence for all $f \in S(\mathscr{D};X)$ *and* $g \in S(\mathscr{D};Y^*)$, *where* $\mathbb{E}$ *is the expectation over* $\omega \in (\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$, *and* $M \in \mathbb{Z}$ *is any large enough number such that* f *and* g *are constant on all* $Q \in \mathscr{D}_M$.

Proof. We begin by observing that, according to Lemma 12.3.30, assumptions (i) through (iii) of the present theorem imply assumption (i) through (iii) of Theorem 12.3.26 uniformly for every $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$. Thus the qualitative statement (1)⇔(2) is just an application of Theorem 12.3.26 to each $\mathscr{D}^\omega$ in place of $\mathscr{D}$, observing the uniformity just mentioned.

The more interesting part consist of the new quantitative conclusions that we obtain for the implication (2)⇒(1). This requires revisiting some details of the proof of Theorem 12.3.26.

Let $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$, and let us specifically assume that both f and g are constant on all $Q \in \mathscr{D}_M$ for some (in general large) $M \in \mathbb{Z}$. We identify $(\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$ with $\{\omega = (\omega_j)_{j\in\mathbb{Z}} \in (\{0,1\}^d)^{\mathbb{Z}} : \omega_j = 0 \text{ for } j > M\}$.

For each $\omega \in (\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$, we have $\mathscr{D}^\omega_M = \mathscr{D}_M$, and hence f and g have the same piecewise constancy property with respect to these dyadic systems. For each $m \leqslant M$ and $\omega \in (\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$, we then write an analogue of (12.46),

$$\begin{aligned}
\mathfrak{t}(f,g) = \langle \mathfrak{H}_{\mathfrak{t}^\omega} u^\omega_m, g\rangle + \mathfrak{l}_{\mathfrak{t}^\omega}(f,g) + \mathscr{E}^\omega_m(f,g) + \\
+ \sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \Big\{ \langle T_{n,\mathfrak{t}^\omega} u^\omega_m, g\rangle + \langle f, U^1_{n,\mathfrak{t}^\omega} v^\omega_m\rangle + \langle U^2_{n,\mathfrak{t}^\omega} u^\omega_m, g\rangle \Big\},
\end{aligned} \tag{12.53}$$

where all symbols have the same meaning as in (12.46), but with $\mathscr{D}^\omega$ in place of $\mathscr{D}$. In particular,

$$u_m^\omega = (I - E_m^\omega)f, \qquad v_m^\omega = (I - E_m^\omega)g,$$

where $E_m^\omega = \mathbb{E}(\ |\mathscr{D}_m^\omega)$ satisfy $\|u_m^\omega\|_p \leqslant 2\|f\|_p$ and $\|v_m^\omega\|_{p'} \leqslant 2\|g\|_{p'}$.

The first and third terms on the right of (12.53) are estimated as in the proof Theorem 12.3.26. As in (12.47), we have

$$|\mathscr{E}_m^\omega(f,g)| \leqslant \Big(c_{d,p}\sum_{i=1}^{2}\|\mathfrak{t}^{\omega;(i)}\|_{\mathrm{Fig}^0(\infty)} + 2^d\|\mathfrak{t}^\omega\|_{awbp}\Big)\|E_m^\omega f\|_p\|E_m^\omega g\|_{p'} \to 0$$

when $m \to -\infty$; note that this convergence is bounded by (iii), (iv), and the easy estimates $\|E_m^\omega f\|_p \leqslant \|f\|_p$ and $\|E_m^\omega g\|_{p'} \leqslant \|g\|_{p'}$. Then, as in (12.48), from Theorem 12.1.11 we get

$$|\langle \mathfrak{H}_{\mathfrak{t}^\omega} u_m^\omega, g\rangle| \leqslant \beta_{p,X}\beta_{p,Y}\sum_{\alpha,\gamma}\mathscr{D}\mathscr{R}_p(\mathfrak{t}_0^{\omega;\alpha,\gamma})\|u_m^\omega\|_p\|g\|_{p'}.$$

The second term on the right of (12.53) is directly estimated by the uniform boundedness of the paraproducts $\Lambda_{\mathfrak{t}^\omega}$.

We then turn to the more interesting part on the second line of (12.53), where we begin with some observations. Due to the presence of the truncation parameter m, all dyadic operators in (12.53) involve cubes of side-length at most 2^{-m}. On the other hand, due to the constancy of f and g on $Q \in \mathscr{D}_M = \mathscr{D}_M^\omega$, their martingale differences are non-zero only on cubes of side-length strictly larger than 2^{-M}. Hence the right-hand side of (12.53) actually depends on $(\omega_j)_{m<j\leqslant M}$ only, rather than the infinite sequence $(\omega_j)_{j\leqslant M}$. Nevertheless, it will be convenient to also refer to this latter sequence, as we are about to see.

We compute the expectation of (12.53) with respect to the choice of $\omega \in (\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$. As we just observed, this is actually just an arithmetic average over a finite set of $2^{d(M-m)}$ elements, so no integrability or measurability issues arise at this point.

We wish to manipulate this average a little. We note that each of the terms on the second line of (12.53) take the generic form

$$\sum_{Q\in\mathscr{D}}^{*}\Phi(Q\dot{+}\omega),$$

where

$$\Phi(R) \in \Big\{\sum_{\alpha,\gamma}\Big\langle \mathfrak{t}(h_R^\alpha, h_{R\dot{+}n}^\gamma)\langle f, h_R^\alpha\rangle, \langle g, h_{R\dot{+}n}^\gamma\rangle\Big\rangle,$$
$$\sum_{\alpha}\Big\langle \mathfrak{t}(h_R^\alpha, h_{R\dot{+}n}^0)\langle f, h_R^\alpha\rangle, \langle g, h_{R\dot{+}n}^0 - h_R^0\rangle\Big\rangle,$$

$$\sum_{\gamma}\Big\langle \mathfrak{t}(h^0_{R\dot{+}n},h^\gamma_R)\langle f,h^0_{R\dot{+}n}-h^0_R\rangle,\langle g,h^\gamma_R\rangle\Big\rangle\Big\},$$

and the notation $\sum^*$ suppresses not only the size condition that $2^{-M} < \ell(Q)\leqslant 2^{-m}$ but also an implicit restriction to a fixed finite family of cubes of each size, depending on the supports of f and g.

Inserting $1 = 2^d\cdot\mathbb{E}(\mathbf{1}_{\{Q\dot{+}\omega \text{ is } k\text{-good}\}})$, it hence follows, using in particular the independence property established in Lemma 12.3.33(1), that

$$\begin{aligned}\mathbb{E}\sum_{Q\in\mathscr{D}}^{*}\Phi(Q\dot{+}\omega) &= \sum_{Q\in\mathscr{D}}^{*}2^d\cdot\mathbb{E}(\mathbf{1}_{\{Q\dot{+}\omega \text{ is } k\text{-good}\}})\mathbb{E}\Phi(Q\dot{+}\omega)\\ &= 2^d\sum_{Q\in\mathscr{D}}^{*}\mathbb{E}\big(\mathbf{1}_{\{Q\dot{+}\omega \text{ is } k\text{-good}\}}\Phi(Q\dot{+}\omega)\big)\\ &= 2^d\cdot\mathbb{E}\sum_{\substack{Q\in\mathscr{D}:\\ Q\dot{+}\omega \text{ is } k\text{-good}}}^{*}\Phi(Q\dot{+}\omega).\end{aligned}$$

Thus, at the cost of the factor 2^d, we can reduce the summation to k-good cubes only.

Taking the expectation of (12.53) and applying the above observation to the terms on the second line, with $k = k(n)$ as in Definition 12.3.34, we obtain

$$\begin{aligned}\mathfrak{t}(f,g) = \mathbb{E}\Big(&\langle\mathfrak{H}_{\mathfrak{t}^\omega}u^\omega_m,g\rangle + \mathfrak{l}_{\mathfrak{t}^\omega}(f,g)+\mathscr{E}^\omega_m(f,g)+\\ &+2^d\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}}\Big\{\langle T^{\text{good}}_{n,\mathfrak{t}^\omega}u^\omega_m,g\rangle+\langle f,U^{1,\text{good}}_{n,\mathfrak{t}^\omega}v^\omega_m\rangle+\langle U^{2,\text{good}}_{n,\mathfrak{t}^\omega}u^\omega_m,g\rangle\Big\}\Big),\end{aligned}\tag{12.54}$$

where the various "good" operators are defined in Definition 12.3.34.

When $k = k(n)$ is as in Definition 12.3.34, and $R = Q\dot{+}\omega$ is k-good, it follows directly from Definition 12.3.32 that

$$\operatorname{dist}(R,\complement R^{(k,\omega)})\geqslant 2^{k-2}\ell(R)\geqslant |n|\ell(R),$$

and hence $R\dot{+}n\subseteq R^{(k,\omega)}$. Thus the operators on the right of (12.54) are in the scope of the sharper special cases of Figiel's estimates, Corollary 12.1.27(2) and Theorem 12.1.28(2).

An application of these estimates to (12.54), in the case of $U^{\omega,1}_n$ on the dual side and otherwise directly as in Corollary 12.1.27(2) and Theorem 12.1.28(2), gives

$$\begin{aligned}|\langle f,U^{1,\text{good}}_{n,\mathfrak{t}^\omega}v^\omega_m\rangle| &\leqslant \sum_{\alpha}6\beta_{p,X}\beta_{p,Y}c_1(1+k(n))^{\sigma_1}\wp_1(\mathfrak{t}^{\omega;1,\alpha}_{n,\text{good}})\|f\|_p\|v^\omega_m\|_{p'},\\ |\langle U^{2,\text{good}}_{n,\mathfrak{t}^\omega}u^\omega_m,g\rangle| &\leqslant \sum_{\alpha}6\beta_{p,X}\beta_{p,Y}c_2(1+k(n))^{\sigma_2}\wp_2(\mathfrak{t}^{\omega;2,\alpha}_{n,\text{good}})\|u^\omega_m\|_p\|g\|_{p'},\end{aligned}$$

$$|\langle T^{\text{good}}_{n,\mathfrak{t}^\omega} u^\omega_m, g\rangle| \leqslant \sum_{\alpha,\gamma} 3\beta_{p,X}\beta_{p,Y} \min_{i=1,2} c_i (1+k(n))^{\sigma_i} \wp_i(\mathfrak{t}^{\omega;\alpha,\gamma}_{n,\text{good}}) \|u^\omega_m\|_p \|g\|_{p'}.$$

It follows from Definition 12.3.34 that

$$k(n)+1 \leqslant 4+\log_2|n| \leqslant 2(2+\log_2|n|),$$

and hence

$$\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \sum_{\alpha} (1+k(n))^{\sigma_i} \wp_i(\mathfrak{t}^{\omega;i,\alpha}_{n,\text{good}}) \leqslant 2\|\mathfrak{t}^{\omega;(i)}_{\text{good}}\|_{\text{Fig}^{\sigma_i}(\wp_i)},$$

$$\sum_{\substack{n\in\mathbb{Z}^d\\ n\neq 0}} \sum_{\alpha,\gamma} (1+k(n))^{\sigma_i} \wp_i(\mathfrak{t}^{\omega;\alpha,\gamma}_{n,\text{good}}) \leqslant 2\|\mathfrak{t}^{\omega;(0)}_{\text{good}}\|_{\text{Fig}^{\sigma_i}(\wp_i)}.$$

We have thus estimated all terms on the right of (12.54). Let us further recall that $\|u^\omega_m\|_p \leqslant 2\|f\|_p$ and $u^\omega_m \to f$ in $L^p(\mathbb{R}^d;X)$ as $m\to-\infty$, with similar results for v^ω_m, g and p' in place of u^ω_m, f and p. We can thus pass to the limit $m\to-\infty$ in (12.54) and apply dominated convergence to deduce the claimed representation formula (12.52). Applying the same estimates above to (12.52) in place of (12.54), we deduce the claimed norm estimate (a). This completes the proof of Theorem 12.3.35. □

12.4 The $T(1)$ theorem for singular integrals

A natural question arising from the Theorems 12.3.26 and 12.3.35 above is whether their assumptions are verified by some familiar operators. In particular, what is the relation of these conditions to the Calderón–Zygmund operators discussed in Chapter 11? We will address this question in the present section. Recall from Definition 11.3.1 that

$$c_K := \sup\{|s-t|^d\|K(s,t)\| : (s,t)\in\dot{\mathbb{R}}^{2d}\}.$$

Definition 12.4.1 (Weakly defined singular integral operator). *Let Z be a Banach space, and $\mathscr{C}$ be a collection of bounded Borel subsets of $\mathbb{R}^d$. We say that a bilinear form $\mathfrak{t} : S(\mathscr{C})^2 \to Z$ is a weakly defined singular integral with associated kernel $K : \dot{\mathbb{R}}^{2d}\to Z$, if $c_K<\infty$ and*

$$\mathfrak{t}(\mathbf{1}_Q,\mathbf{1}_R) = \iint_{\mathbb{R}^{2d}} K(s,t)\mathbf{1}_Q(t)\mathbf{1}_Q(s)\,\mathrm{d}s\,\mathrm{d}t \tag{12.55}$$

whenever $Q,R\in\mathscr{C}$ are disjoint.

As usual, the main case of interest will be $\mathscr{C}=\mathscr{D}$.

The following lemma, which will also play a role later, shows that the integral in (12.55) is well defined under the assumption that $c_K<\infty$: While in (12.55) we do not require the cubes to have equal size, we can always dominate the integral with such a case by passing to a dyadic ancestor of the smaller cube, if necessary.

Lemma 12.4.2. *For disjoint cubes $Q, R \subseteq \mathbb{R}^d$ of equal size $\ell(Q) = \ell(R)$, we have*

$$\iint_{Q\times R} \frac{1}{|s-t|^d} \,\mathrm{d}s\,\mathrm{d}t \leqslant (1 + \frac{dv_d}{2})|Q| < 18 \cdot |Q|,$$

where v_d is the volume of the unit ball in $\mathbb{R}^d$.

Proof. We first write

$$\begin{aligned}\iint_{Q\times R} \frac{1}{|s-t|^d} \,\mathrm{d}s\,\mathrm{d}t &= \iint_{Q\times R} d\int_{|s-t|}^{\infty} r^{-d-1}\,\mathrm{d}r\,\mathrm{d}s\,\mathrm{d}t \\ &= d\int_0^{\infty} |\{(s,t)\in Q\times R : |s-t|<r\}| r^{-d-1}\,\mathrm{d}r.\end{aligned}$$

Denoting by v_d is the volume of the unit ball in $\mathbb{R}^d$, we have

$$\begin{aligned}|\{(s,t)\in Q\times R : |s-t|<r\}| &= \int_{\{s\in Q:\mathrm{dist}(s,R)<r\}} |\{t\in R : |s-t|<r\}|\,\mathrm{d}s \\ \leqslant |\{s\in Q : \mathrm{dist}(s,R)<r\}|(\frac{v_d r^d}{2}\wedge |R|) &\leqslant (r\wedge \ell(Q))\frac{|Q|}{\ell(Q)}(\frac{v_d r^d}{2}\wedge |R|),\end{aligned}$$

where we used the geometric observation that, for $s \in Q \subseteq \complement R$, at least half of any ball of centre s lies in $\complement R$. Hence

$$\begin{aligned}\iint_{Q\times R} \frac{1}{|s-t|^d} \,\mathrm{d}s\,\mathrm{d}t &\leqslant d\int_0^{\ell(Q)} r\frac{|Q|}{\ell(Q)}\cdot \frac{v_d r^d}{2}\cdot r^{-d-1}\,\mathrm{d}r \\ &\quad + d\int_{\ell(Q)}^{\infty} |Q|\cdot|R| r^{-d-1}\,\mathrm{d}r = \frac{dv_d}{2}|Q| + |R|,\end{aligned}$$

where $|R| = |Q|$, since $\ell(R) = \ell(Q)$.

Finally, $dv_d/2 = \pi^{d/2}/\Gamma(d/2) =: f(d/2)$. From the functional equation $\Gamma(x+1) = x\Gamma(x)$, we find that $f(x+1)/f(x) = \pi/x$, so that $\max\{f(n) : n \in \mathbb{N}\} = f(4)$ and $\max\{f(n+\frac{1}{2}) : n \in \mathbb{N}\} = f(7/2)$. Computing these two values, one checks that $\max\{f(d/2) : d \in \mathbb{N}\} = f(7/2) = \frac{8}{15}\pi^3 < 17$. □

For weakly defined singular integrals, some properties imposed as assumptions on general bilinear forms are automatically satisfied:

Lemma 12.4.3. *Let Z be a Banach space and $\mathfrak{t} : \dot{\mathbb{R}}^{2d} \to Z$ a weakly defined singular integral operator with kernel K. Then $\mathfrak{t}$ satisfies the adjacent weak boundedness property if and only if it satisfies the weak boundedness property, and moreover*

$$\|\mathfrak{t}\|_{wbp} \leqslant \|\mathfrak{t}\|_{awbp} \leqslant \max\{\|\mathfrak{t}\|_{wbp}, 18\cdot c_K\}.$$

Proof. The "only if" part is obvious. For "if", it suffices to estimate $\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R)$ for $R = Q\dot{+}n$ and $n \in \{-1,0,1\}^d \setminus \{0\}$. Then $Q \cap R = \varnothing$, so that we have access to the kernel representation (12.55), and Lemma 12.4.2 provides us with the bound

$$\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_R)\| \leqslant \iint_{Q\times R} \frac{c_K}{|s-t|^d}\,\mathrm{d}s\,\mathrm{d}t \leqslant 18 \cdot |Q| \cdot c_K.$$

□

Proposition 12.4.4. *Let Z be a Banach space and $\mathfrak{t} : \dot{\mathbb{R}}^{2d} \to Z$ a weakly defined singular integral operator. If $\mathfrak{t}$ is translation-invariant (in the sense of Definition 12.3.9), then $\mathfrak{t}(\mathbf{1},\cdot) = 0 = \mathfrak{t}(\cdot,\mathbf{1})$.*

Proof. By Proposition 12.3.10, it suffices to verify that $\mathfrak{t}$ satisfies the decay condition (12.35). Let $Q \in \mathscr{D}$ and $m \in \mathbb{Z}^d \setminus \{-1,0,1\}^d$. Then, for $s \in Q$ and $t \in Q\dot{+}m$, and denoting by z_Q the centre of Q, we have

$$\begin{aligned}|s-t| \geqslant |s-t|_\infty &\geqslant |m\ell(Q)|_\infty - |s-z_Q|_\infty - |t-(z_Q+m\ell(Q))|_\infty \\ &\geqslant |m|_\infty\ell(Q) - \frac{1}{2}\ell(Q) - \frac{1}{2}\ell(Q) \geqslant \frac{1}{2}|m|_\infty\ell(Q) \geqslant \frac{|m|\ell(Q)}{2\sqrt{d}},\end{aligned}$$

and hence

$$\begin{aligned}\|\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_{Q\dot{+}m})\| &\leqslant \int_Q\int_{Q\dot{+}m} \frac{c_K}{|s-t|^d}\,\mathrm{d}s\,\mathrm{d}t \\ &\leqslant |Q|^2 c_K\Big(\frac{2\sqrt{d}}{|m|\ell(Q)}\Big)^d = |Q|c_K(2\sqrt{d})^d|m|^{-d}.\end{aligned}$$

This is one half of the decay condition (12.35). The estimate for $\mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_Q)$ is entirely similar. □

Despite the simple observations above, in order to make serious conclusions about weakly defined singular integrals, we will need the following elaboration of the earlier Definition 11.3.1:

Definition 12.4.5 ($\wp$–Calderón–Zygmund kernel). *Let Z be a Banach space, $\wp$ a good set-bound on Z, and $K : \dot{\mathbb{R}}^{2d} \to Z$. We define the quantities*

$$c_K(\wp) := \wp(\{|s-t|^d K(s,t) : s \neq t\}),$$

and, for $u \in [0, \frac{1}{2}]$,

$$\omega_K^1(\wp; u) := \wp\Big(\Big\{|s-t|^d\big(K(s,t) - K(s',t)\big) : |s-s'| \leqslant u|s-t|\Big\}\Big), \quad (12.56)$$

$$\omega_K^2(\wp; u) := \wp\Big(\Big\{|s-t|^d\big(K(s,t) - K(s,t')\big) : |t-t'| \leqslant u|s-t|\Big\}\Big). \quad (12.57)$$

Remark 12.4.6. (1) We recover Definition 11.3.1 by taking $\wp(\mathscr{T}) = \mathscr{U}(\mathscr{T}) := \sup\{\|T\| : T \in \mathscr{T}\}$. Our main interest now will be $\wp \in \{\mathscr{R}_p, \mathscr{R}^*_{p'}\}$.
(2) In analogy with Lemma 11.3.3, one can check that

$$\omega_K^i(\wp; \frac{1}{2}) \leqslant (1 + 2^d) c_K(\wp).$$

(3) If $K(s,t) = \mathfrak{K}(s-t)$ for some $\mathfrak{K} : \mathbb{R}^d \setminus \{0\} \to Z$, then

$$c_K(\wp) = \wp(\{|s|^d \mathfrak{K}(s) : s \neq 0\}) =: \tilde{c}_{\mathfrak{K}}(\wp),$$

and, for both $i \in \{1,2\}$,

$$\omega_K^i(\wp; u) = \wp\Big(\Big\{|s|^d\big(\mathfrak{K}(s) - \mathfrak{K}(s')\big) : |s - s'| \leqslant u|s|\Big\}\Big) =: \tilde{\omega}_{\mathfrak{K}}(\wp; u).$$

Such a K (or $\mathfrak{K}$) is referred to as a *convolution kernel.*

If $\mathfrak{t}$ is a weakly defined singular integral with $\wp$–Calderón–Zygmund kernel K, the conditions of Definition 12.4.5 only provide control away from the diagonal $s = t$. To compensate for this, we also need the following assumption directly on the bilinear form $\mathfrak{t}$:

Definition 12.4.7 (Weak $\mathscr{D}\mathscr{R}_p$-boundedness property). *Letting* $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y)$ *be a bilinear form, we define*

$$\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} := \mathscr{D}\mathscr{R}_p\Big(\Big\{\frac{\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_Q)}{|Q|}\Big\}_{Q\in\mathscr{D}}\Big)$$

Our goal in this section will be to use these assumptions to control the Haar coefficients $\mathfrak{t}(h_Q^\alpha, h_R^\gamma)$, where $R = Q + \ell(Q)n$, in the way that was assumed in the Theorems 12.3.26 and 12.3.35 on bilinear forms. Using the defining condition (12.55) and bilinearity (noting that h_Q^α is a linear combination of $\mathbf{1}_{Q'}$ for $Q' \in \mathrm{ch}(Q)$, and likewise h_R^γ), we have in particular that

$$\mathfrak{t}(h_Q^\alpha, h_R^\gamma) = \iint_{Q\times R} K(s,t)\,\mathrm{d}s\,\mathrm{d}t, \qquad Q \cap R = \varnothing.$$

If K is a $\wp$-Calderón–Zygmund kernel, we can establish the following estimates:

Lemma 12.4.8. *Let Z be a Banach space and $\wp$ a good set-bound on Z. Let $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ be a weakly defined singular integral with kernel $K : \dot{R}^{2d} \to Z$. Then for all $\alpha, \gamma \in \{0,1\}^d$, we have, for all $n \in \mathbb{Z}^d \setminus \{0\}$,*

$$\wp\Big\{\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma) : Q \in \mathscr{D}\Big\} \leqslant 18 \cdot 2^d \cdot c_K(\wp), \tag{12.58}$$

and, for $|n| \geqslant \frac{3}{2}\sqrt{d}$,

$$\wp\Big\{\mathfrak{t}(h^{\alpha}_{Q\dot{+}n},h^{\gamma}_{Q}):Q\in\mathscr{D}\Big\}\leqslant(\frac{3}{2})^d\cdot|n|^{-d}\cdot\omega^1_K(\wp;\frac{\frac{3}{4}\sqrt{d}}{|n|})\quad \textit{if}\quad \gamma\neq 0. \tag{12.59}$$

$$\wp\Big\{\mathfrak{t}(h^{\alpha}_{Q},h^{\gamma}_{Q\dot{+}n}):Q\in\mathscr{D}\Big\}\leqslant(\frac{3}{2})^d\cdot|n|^{-d}\cdot\omega^2_K(\wp;\frac{\frac{3}{4}\sqrt{d}}{|n|})\quad \textit{if}\quad \alpha\neq 0, \tag{12.60}$$

Proof. Including momentarily also $n=0$ for later use, we have the expansion

$$\begin{aligned}\mathfrak{t}(h^{\alpha}_{Q},h^{\gamma}_{Q\dot{+}n}) &= \sum_{\substack{R\in\mathrm{ch}(Q)\\ S\in\mathrm{ch}(Q\dot{+}n)}}\mathfrak{t}(\mathbf{1}_R,\mathbf{1}_S)\langle h^{\alpha}_{Q}\rangle_R\langle h^{\gamma}_{Q\dot{+}n}\rangle_S\\ &=\delta_{n,0}\sum_{R\in\mathrm{ch}(Q)}\mathfrak{t}(\mathbf{1}_R,\mathbf{1}_R)\langle h^{\alpha}_{Q}\rangle_R\langle h^{\gamma}_{Q\dot{+}n}\rangle_S\\ &\qquad+\sum_{\substack{R\in\mathrm{ch}(Q)\\ S\in\mathrm{ch}(Q\dot{+}n)\\ R\neq S}}\mathfrak{t}(\mathbf{1}_R,\mathbf{1}_S)\langle h^{\alpha}_{Q}\rangle_R\langle h^{\gamma}_{Q\dot{+}n}\rangle_S=:I_Q+II_Q.\end{aligned}\tag{12.61}$$

(The summation condition $R\neq S$ in II_Q is automatic for $n\neq 0$, but it makes no harm to include it). Since

$$\sum_{\substack{R\in\mathrm{ch}(Q)\\ S\in\mathrm{ch}(Q\dot{+}n)}}|R||\langle h^{\alpha}_{Q}\rangle_R\langle h^{\gamma}_{Q\dot{+}n}\rangle_S|=\sum_{\substack{R\in\mathrm{ch}(Q)\\ S\in\mathrm{ch}(Q\dot{+}n)}}|R|\frac{1}{|Q|}=\sum_{S\in\mathrm{ch}(Q\dot{+}n)}1=2^d,$$

we see that

$$II_Q\in 2^d\,\mathrm{abs\,conv}\Big(\Big\{\frac{\mathfrak{t}(\mathbf{1}_U,\mathbf{1}_V)}{|U|}:U,V\in\mathscr{D},\ U\cap V=\varnothing,\ \ell(U)=\ell(V)\Big\}\Big),$$

where

$$\begin{aligned}\mathfrak{t}(\mathbf{1}_U,\mathbf{1}_V)&=\iint_{U\times V}K(s,t)\,\mathrm{d}s\,\mathrm{d}t=\iint_{U\times V}|s-t|^dK(s,t)\frac{\mathrm{d}s\,\mathrm{d}t}{|s-t|^d}\\ &\in 18\cdot|U|\cdot\overline{\mathrm{abs\,conv}}\Big(\Big\{|u-v|^dK(u,v):(u,v)\in\dot{\mathbb{R}}^{2d}\Big\}\Big),\end{aligned}$$

by Proposition 1.2.12 and Lemma 12.4.2 in the last step. Combining the above inclusions with the defining properties of good set-bounds (Definition 12.3.20), we obtain

$$\wp(\{II_Q:Q\in\mathscr{D}\})\leqslant 18\cdot 2^d\cdot c_K(\wp), \tag{12.62}$$

which coincides with (12.58) when $n\neq 0$.

For large values of n, we want to obtain a decay, which is not present in the uniform estimate just established. In this case we apply the kernel representation combined with the vanishing mean of h^{α}_{Q} (when $\alpha\neq 0$), to the result that

$$\begin{aligned}\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma) &= \iint K(s,t) h_Q^\alpha(t) h_{Q\dot{+}n}^\gamma(s) \,\mathrm{d}s \,\mathrm{d}t \\ &= \iint [K(s,t) - K(s,z_Q)] h_Q^\alpha(t) h_{Q\dot{+}n}^\gamma(s) \,\mathrm{d}s \,\mathrm{d}t,\end{aligned}$$

where z_Q is the centre of Q. For $t \in Q$ and $s \in Q\dot{+}n$, we have $|t - z_Q| \leqslant \frac{1}{2}\sqrt{d}\ell(Q)$, whereas

$$|s - z_Q| \geqslant |z_{Q\dot{+}n} - z_Q| - |s - z_{Q\dot{+}n}| \geqslant (|n| - \frac{1}{2}\sqrt{d})\ell(Q),$$

and hence

$$\frac{|t - z_Q|}{|s - z_Q|} \leqslant \frac{\frac{1}{2}\sqrt{d}}{|n| - \frac{1}{2}\sqrt{d}} \leqslant \frac{1}{2} \quad \text{if} \quad |n| \geqslant \frac{3}{2}\sqrt{d}.$$

In this case we have

$$\begin{aligned}&\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma) \in \iint \frac{1}{|s - z_Q|^d} |h_Q^\alpha(t) h_{Q\dot{+}n}^\gamma(s)| \,\mathrm{d}s \,\mathrm{d}t \\ &\times \overline{\operatorname{abs\,conv}}\Big(\Big\{ |u - v|^d [K(u,v) - K(u,v')] : |v - v'| \leqslant \frac{\frac{1}{2}\sqrt{d}}{|n| - \frac{1}{2}\sqrt{d}} |u - v| \Big\}\Big),\end{aligned}$$

and hence, by estimate (12.56) of a Calderón–Zygmund kernel (Definition 12.4.5) and the defining properties of good set-bounds (Definition 12.3.20), we arrive at

$$\begin{aligned}&\wp\Big(\Big\{ \mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma) : Q \in \mathscr{D} \Big\}\Big) \\ &\qquad \leqslant \frac{1}{|Q|} \iint_{Q \times (Q\dot{+}n)} \frac{1}{|s - z_Q|^d} \,\mathrm{d}s \,\mathrm{d}t \times \omega_K^2\Big(\frac{\frac{1}{2}\sqrt{d}}{|n| - \frac{1}{2}\sqrt{d}}\Big) \\ &\qquad \leqslant \frac{1}{(|n| - \frac{1}{2}\sqrt{d})^d} \omega_K^2\Big(\frac{\frac{1}{2}\sqrt{d}}{|n| - \frac{1}{2}\sqrt{d}}\Big) \leqslant (\frac{3}{2})^d \cdot |n|^{-d} \omega_K^2\Big(\frac{\frac{3}{4}\sqrt{d}}{|n|}\Big)\end{aligned}$$

when $|n| \geqslant \frac{3}{2}\sqrt{d}$.

The estimate of $\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}n}^\gamma)$ with $\gamma \neq 0$ is entirely analogous to this, using regularity in the other variable instead. □

Concerning the diagonal $n = 0$, which was excluded in Lemma 12.4.8, we have the following estimate:

Lemma 12.4.9. *Let X and Y be Banach spaces and $p \in (1, \infty)$. Let $\mathfrak{t} : \dot{\mathbb{R}}^{2d} \to \mathscr{L}(X, Y)$ be a weakly defined singular integral with the weak $\mathscr{D}\mathscr{R}_p$-boundedness property. Then*

$$\mathscr{D}\mathscr{R}_p(\{\mathfrak{t}(h_Q^\alpha, h_Q^\gamma)\}_{Q \in \mathscr{D}}) \leqslant \|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} + 18 \cdot 2^d \cdot c_K(\wp), \quad \wp \in \{\mathscr{R}_p, \mathscr{R}_{p'}^*\}.$$

Proof. We use the expansion (12.61) with $n = 0$,

$$\mathfrak{t}(h_Q^\alpha, h_Q^\gamma) = I_Q + II_Q,$$

where we now need to consider also the term I_Q. We estimate the expression in the definition of $\mathscr{DR}_p(\{I_Q\}_{Q\in\mathscr{D}})$:

$$\begin{aligned}\sum_{Q\in\mathscr{D}} |Q||\langle I_Q x_Q, y_Q^*\rangle| &\leqslant \sum_{Q\in\mathscr{D}} |Q| \sum_{R\in\mathrm{ch}(Q)} |\langle \mathfrak{t}(\mathbf{1}_R, \mathbf{1}_R)x_Q, y_Q^*\rangle||\langle h_Q^\alpha\rangle_R \langle h_Q^\gamma\rangle_R| \\ &= \sum_{Q\in\mathscr{D}} \sum_{R\in\mathrm{ch}(Q)} |\langle \mathfrak{t}(\mathbf{1}_R, \mathbf{1}_R)x_Q, y_Q^*\rangle| \\ &= \sum_{R\in\mathscr{D}} |\langle \mathfrak{t}(\mathbf{1}_R, \mathbf{1}_R)x_{R^{(1)}}, y_{R^{(1)}}^*\rangle| \\ &\leqslant \|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)} \Big\| \sum_{R\in\mathscr{D}} \varepsilon_R x_{R^{(1)}} \mathbf{1}_R \Big\|_{L^p(\Omega\times\mathbb{R}^d;X)} \times \\ &\qquad \times \Big\| \sum_{R\in\mathscr{D}} \varepsilon_R y_{R^{(1)}}^* \mathbf{1}_R \Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}.\end{aligned}$$

Using the usual observation that, by Fubini's theorem and the fact that only one $R \in \mathscr{D}$ of each generation is "seen" at each fixed $s \in \mathbb{R}^d$, we can replace the random ε_R by $\varepsilon_{n(R)}$ depending on the generation of R only, or further by the equidistributed sequence of $\varepsilon_{n(R^{(1)})}$, we have

$$\begin{aligned}\Big\| \sum_{R\in\mathscr{D}} \varepsilon_R z_{R^{(1)}} \mathbf{1}_R \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} &= \Big\| \sum_{Q\in\mathscr{D}} \sum_{R\in\mathrm{ch}(Q)} \varepsilon_{n(Q)} z_Q \mathbf{1}_R \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} \\ = \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_{n(Q)} z_Q \mathbf{1}_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)} &= \Big\| \sum_{Q\in\mathscr{D}} \varepsilon_Q z_Q \mathbf{1}_Q \Big\|_{L^p(\Omega\times\mathbb{R}^d;Z)}\end{aligned}$$

for both choices of $z_Q \in \{x_Q, y_Q^*\}$ and $Z \in \{X, Y\}$. Hence

$$\mathscr{DR}_p(\{I_Q\}_{Q\in\mathscr{D}}) \leqslant \|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)},$$

and hence, by the obvious triangle inequality for $\mathscr{DR}_p$, and its domination by either $\wp \in \{\mathscr{R}_p, \mathscr{R}_{p'}^*\}$ according to Lemma 12.1.8, we have

$$\begin{aligned}\mathscr{DR}_p(\{\mathfrak{t}(h_Q^\alpha, h_Q^\gamma)\}_{Q\in\mathscr{D}}) &\leqslant \mathscr{DR}_p(\{I_Q\}_{Q\in\mathscr{D}}) + \mathscr{DR}_p(\{II_Q\}_{Q\in\mathscr{D}}) \\ &\leqslant \|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)} + \wp(\{II_Q\}_{Q\in\mathscr{D}}) \\ &\leqslant \|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)} + 18 \cdot c_K(\wp)\end{aligned}$$

by (12.62) in the last step. □

We can now give estimates for the Figiel norms featuring in the $T(1)$ theorems for bilinear forms:

Lemma 12.4.10. *Let Z be a Banach space and $\wp$ a good set-bound on Z. Let $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ be a weakly defined singular integral with kernel $K : \dot{\mathbb{R}}^{2d} \to Z$. Then for all $s \in [0,1]$, we have the estimates*

$$\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^s(\wp)}, \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^s(\wp)} \leqslant a_d c_K(\wp) + b_d \|\omega_K^i(\wp)\|_{\mathrm{Dini}^s}, \qquad i = 1, 2,$$

where a_d, b_d depend only on the dimension d, and

$$\|\omega\|_{\mathrm{Dini}^s} := \int_0^{1/2} \omega(u)(\log_2 \frac{1}{u})^s \frac{\mathrm{d}u}{u}. \tag{12.63}$$

Remark 12.4.11. For $u \in (0, \frac{1}{2})$, we have $\frac{1}{u} \in (2, \infty)$, thus $\log_2 \frac{1}{u} \in (1, \infty)$. Hence $(\log_2 \frac{1}{u})^s$ and therefore $\|\omega\|_{\mathrm{Dini}^s}$ are increasing in s.

Proof of Lemma 12.4.10. From Definition 12.3.22 and Lemma 12.4.8, it follows that

$$\begin{aligned}
\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^s(\wp)} &= \sum_{\alpha,\gamma \in \{0,1\}^d \setminus \{0\}} \|\mathfrak{t}^{\alpha\gamma}\|_{\mathrm{Fig}^s(\wp)}, \\
&= \sum_{\alpha,\gamma \in \{0,1\}^d \setminus \{0\}} \sum_{\substack{n \in \mathbb{Z}^d \\ n \neq 0}} (2 + \log_2 |n|)^s \wp(\{\mathfrak{t}(h_Q^\alpha, h_{Q \dot{+} n}^\gamma) : Q \in \mathscr{D}\}) \\
&\leqslant (2^d - 1)^2 \Big\{ \sum_{|n| < 3\sqrt{d}} (2 + \log_2(3\sqrt{d})) \cdot 18 \cdot 2^d \cdot c_K(\wp) + \\
&\qquad + \sum_{|n| \geqslant 3\sqrt{d}} (2 + \log_2 |n|)^s (\frac{3}{2})^d |n|^{-d} \omega_K^i \Big(\wp; \frac{\frac{3}{4}\sqrt{d}}{|n|}\Big) \Big\} \\
&=: (2^d - 1)^2 (I + II_i) \leqslant 4^d (I + II_i).
\end{aligned} \tag{12.64}$$

Since both $\alpha \neq 0 \neq \gamma$, one can apply either of the estimates (12.59) or (12.60) of Lemma 12.4.8, and thus take either $i \in \{1, 2\}$ above. Similarly,

$$\begin{aligned}
\|\mathfrak{t}^{(2)}\|_{\mathrm{Fig}^s(\wp)} &= \sum_{\alpha \in \{0,1\}^d \setminus \{0\}} \|\mathfrak{t}^{2,\alpha}\|_{\mathrm{Fig}^s(\wp)}, \\
&= \sum_{\alpha \in \{0,1\}^d \setminus \{0\}} \sum_{\substack{n \in \mathbb{Z}^d \\ n \neq 0}} (2 + \log_2 |n|)^s \wp(\{\mathfrak{t}(h_Q^\alpha, h_{Q \dot{+} n}^0) : Q \in \mathscr{D}\}) \\
&\leqslant (2^d - 1)(I + II_2) \leqslant 4^d (I + II_2),
\end{aligned} \tag{12.65}$$

where we only have access to estimate (12.60), but not (12.59), of Lemma 12.4.8, now that the second Haar function $h_{Q \dot{+} n}^0$ is non-cancellative. The very last step in (12.65) is of course wasteful, but we make it in order to treat the right-hand sides of both (12.64) and (12.65) at the same time.

Finally, in complete analogy with (12.65), we also have

$$\|\mathfrak{t}^{(1)}\|_{\mathrm{Fig}^s(\wp)} = \sum_{\alpha\in\{0,1\}^d\setminus\{0\}} \|\mathfrak{t}^{1,\alpha}\|_{\mathrm{Fig}^s(\wp)} \leqslant 4^d(I + II_1), \tag{12.66}$$

as we now have access to estimate (12.59), but not (12.60), of Lemma 12.4.8.

It is immediate that

$$4^d I = a_d \cdot c_K(\wp), \quad a_d := 4^d \sum_{|n|<3\sqrt{d}} (2+\log_2(3\sqrt{d}))\cdot 18\cdot 2^d. \tag{12.67}$$

For the other term, we partition the summation over dyadic annuli, in which the summand is roughly a constant:

$$4^d II_i \leqslant 6^d \sum_{k=0}^{\infty} \sum_{\substack{3\cdot 2^k\sqrt{d}\leqslant |n| \\ <3\cdot 2^{k+1}\sqrt{d}}} (2+\log_2(3\sqrt{d})+k)^s (3\cdot 2^k\sqrt{d})^{-d}\omega_K^i(\wp;2^{-k-2}).$$

The unit-cubes Q_n with centres $n\in\mathbb{Z}^d$ are disjoint, and for $|n|<3\cdot 2^{k+1}\sqrt{d}$, they are contained in $B(0,(3\cdot 2^{k+1}+\frac{1}{2})\sqrt{d})$. Thus

$$\sum_{|n|<3\cdot 2^{k+1}\sqrt{d}} 1 \leqslant v_d\Big((3\cdot 2^{k+1}+\frac{1}{2})\sqrt{d}\Big)^d \leqslant v_d(6.5)^d 2^{kd}\sqrt{d}^d, \tag{12.68}$$

where v_d is the volume of the unit ball, and hence

$$\begin{aligned} 4^d II_i &\leqslant 6^d \sum_{k=0}^{\infty} v_d(6.5)^d 2^{kd}\sqrt{d}^d (2+\log_2(3\sqrt{d})+k)^s(3\cdot 2^k\sqrt{d})^{-d}\omega_K^i(\wp;2^{-k-2}) \\ &\leqslant (13)^d v_d(2+\log_2(3\sqrt{d}))\sum_{k=0}^{\infty}(1+k)^s\omega_K^i(\wp;2^{-k-2}). \end{aligned}$$

Since $\omega_K^i(\wp;u)$ is non-decreasing, we can finally estimate

$$(1+k)^s\omega_K^i(\wp;2^{-k-2}) \leqslant \frac{1}{\log 2}\int_{2^{-k-2}}^{2^{-k-1}} (\log_2\frac{1}{u})^s\frac{\omega_K^i(\wp;u)}{\log 2}\frac{\mathrm{d}u}{u}, \quad k=0,1,\ldots,$$

and hence

$$4^d II_i \leqslant b_d\|\omega_K^i(\wp)\|_{\mathrm{Dini}^s}, \qquad b_d := \frac{(13)^d v_d(2+\log_2(3\sqrt{d}))}{\log 2}.$$

With (12.64), (12.65), (12.66), and (12.67), this concludes the proof. (An estimate similar to (12.68) could also be used to give a more explicit bound for the constant a_d in (12.67), if desired.) □

We have now everything prepared for proving the following:

Theorem 12.4.12 ($T(1)$ theorem for operator-valued kernels). *Let $p\in(1,\infty)$ and $1\leqslant t\leqslant p\leqslant q\leqslant\infty$, and suppose that:*

(i) *X and Y are UMD spaces.*
(ii) *X has cotype q and Y has type t, or one of them has both.*
(iii) *$\mathfrak{t} : S(\mathscr{D})^2 \to Z := \mathscr{L}(X, Y)$ is a weakly defined singular integral and the kernel $K : \dot{\mathbb{R}}^{2d} \to Z$ of $\mathfrak{t}$ satisfies the Calderón–Zygmund estimates*

$$c_K(\mathscr{R}_p) + \|\omega_K^1(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/t}} + \|\omega_K^2(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}} < \infty. \tag{12.69}$$

Then the following conditions are equivalent:

(1) *$\mathfrak{t}$ defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$;*
(2) *$\mathfrak{t}$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} < \infty$, and the associated bi-paraproduct $\Lambda_{\mathfrak{t}}$ is bounded in $\mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$;*
(3) *each $\mathfrak{t}^\omega$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant C$, and the associated bi-paraproduct $\Lambda_{\mathfrak{t}^\omega}$ defines a bounded operator in $\mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$, uniformly in $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$.*

Under these equivalent conditions, we have

(a) *the first norm estimate:*

$$\begin{aligned}
&\|T - \Lambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\\
&\quad\leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} + c_d\Big(C_1 c_K(\mathscr{R}_{p'}^*) + C_2 c_K(\mathscr{R}_p)\Big) +\\
&\qquad\qquad + c_d'\Big(C_1\|\omega_K^1(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/t}} + C_2\|\omega_K^2(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},
\end{aligned}$$

where c_d, c_d' are constants that depend only on d, and

$$C_1 := C_{(12.15)}(Y^*, X^*, p', t', 1), \qquad C_2 := C_{(12.15)}(X, Y, p, q, 1);$$

(b) *the second norm estimate:*

$$\begin{aligned}
&\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} - \sup_\omega \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\\
&\quad\leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d \sup_\omega \|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} + c_d^0\Big(c_1 c_K(\mathscr{R}_{p'}^*) + c_2 c_K(\mathscr{R}_p)\Big) +\\
&\qquad\qquad + c_d^1\Big(c_1\|\omega_K^1(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/t}} + c_2\|\omega_K^2(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},
\end{aligned}$$

where the suprema are over $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, the constants c_d, c_d' depend only on d, and

$$c_1 := \min_{Z=X,Y} c_{t',Z^*;p'}, \qquad c_2 := \min_{Z=X,Y} c_{q,Z;p}; \tag{12.70}$$

(c) *the representation formulas (12.45) and (12.52).*

Proof. The plan of the proof is to reduce the theorem at hand to Theorems 12.3.26 and 12.3.35 on abstract bilinear forms.

(1)⇔(2): This will be an application of Theorem 12.3.26 (and Remark 12.3.27). Assumption (i) is identical in both theorems. Next, as explained in

Remark 12.3.27, under the (co)type assumption (ii) of Theorem 12.4.12, the assumption (ii) of Theorem 12.3.26 are satisfied with

$$(t_1, q_1) := (t, \infty), \qquad (t_2, q_2) := (1, q),$$

and both choices of $(t_0, q_0) \in \{(t_i, q_i)\}_{i=1}^2$. Let $\sigma_1 := 1/t$ and $\sigma_2 := 1/q'$.

Concerning assumption (iii) on the bilinear form $\mathfrak{t}$, we need to check that the kernel assumptions (12.69) of the present theorem imply the assumptions on the Haar coefficients $\mathfrak{t}(h_Q^\alpha, h_R^\gamma)$ and the related Figiel norms of the bilinear form $\mathfrak{t}$. With the choices of (t_i, q_i) as just explained, and recalling that the set-bounds $\wp_1 := \mathscr{R}_{p'}^*$ and $\wp_2 := \mathscr{R}_p$ are equivalent in the spaces that we are considering, the assumption (12.69) can be equivalently written as

$$c_K(\wp_i) + \|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}} < \infty, \quad i \in \{1, 2\}. \tag{12.71}$$

By Example 12.1.10, we know that

$$\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant \|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))}, \tag{12.72}$$

so in particular the weak $\mathscr{D}\mathscr{R}_p$-boundedness property is either assumed, or implied by the assumptions, in each case of Theorem 12.4.12.

From Lemma 12.4.9, we then have

$$\mathscr{D}\mathscr{R}_p(\{\mathfrak{t}(h_Q^\alpha, h_Q^\gamma)\}_{Q \in \mathscr{D}}) \leqslant \|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)},$$

whereas Lemma 12.4.10 guarantees, for both $i \in \{1, 2\}$, that

$$\begin{aligned} \|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} &\leqslant a_d c_K(\wp_i) + b_d \|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}}, \\ \|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} &\leqslant a_d c_K(\wp_i) + b_d \|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}}, \end{aligned} \tag{12.73}$$

where both right-hand sides of are finite by (12.71). With either choice of $(t_0, q_0) \in \{(t_i, q_i)\}_{i=1}^2$, the resulting finiteness of the left-hand sides coincides with the assumption on these quantities in (iii) of Theorem 12.3.26.

Summarising, assumptions (i) through (iii) of Theorem 12.4.12, together with the weak $\mathscr{D}\mathscr{R}_p$-boundedness property of $\mathfrak{t}$, which is either assumed or implied by the assumptions of each case of Theorem 12.4.12, imply the corresponding assumptions (i) through (iii) of Theorem 12.3.26. Moreover, the condition of adjacent weak boundedness property appearing in Theorem 12.3.26 also follows from these assumptions by Lemma 12.4.3 and the domination of uniform bounds by either $\mathscr{D}\mathscr{R}_p$-bounds or $\wp_i$-bounds:

$$\|\mathfrak{t}\|_{awbp} \leqslant \max\{\|\mathfrak{t}\|_{wbp}, 18 \cdot c_K\} \leqslant \max\{\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)}, 18 \cdot c_K(\wp_i)\}.$$

Hence all assumptions, and thus all conclusions of Theorem 12.3.26 are valid under the assumptions of Theorem 12.4.12. This proves in particular the qualitative equivalence (1)$\Leftrightarrow$(2).

(a): For this quantitative estimate, we apply Remark 12.3.27, followed by (12.72) and (12.73), to get

$$\begin{aligned}
&\|T - \Lambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\
&\leqslant \beta_{p,X}\beta_{p,Y}\Big\{\sum_{\alpha,\gamma}\mathscr{DR}_p(\mathfrak{t}_0^{\alpha,\gamma}) + \sum_{i=1}^{2} C_i\Big(A_d\|\mathfrak{t}^{(0)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} + B_d\|\mathfrak{t}^{(i)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)}\Big)\Big\} \\
&\leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d\|\mathfrak{t}\|_{wbp(\mathscr{DR}_p)} + \sum_{i=1}^{2} C_i\Big(c_d c_K(\wp_i) + c_d'\|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}}\Big)\Big\},
\end{aligned}$$

where $c_d := (A_d + B_d)a_d$ and $c_d' := (A_d + B_d)b_d$. This is readily recognised to coincide with the bound asserted in (a) of the theorem.

(1)⇔(3): This will be an application of Theorem 12.3.35. Assumptions (i) and (ii) are identical in both theorems.

Concerning assumption (iii), we need to check that the kernel assumptions (12.69) of the present theorem imply the estimates on Figiel norms of each bilinear form $\mathfrak{t}^\omega$, uniformly in $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$. We already did this for $\mathfrak{t} = \mathfrak{t}^0$ above. However, all the lemmas of this section are stated for an arbitrary dyadic system $\mathscr{D}$, so we may in particular use them with any $\mathscr{D}^\omega$ in place of $\mathscr{D}$. Moreover, the constants in these estimates are explicit, and clearly independent of the particular ω. This proves the qualitative equivalence (1)⇔(3).

(b): For this quantitative estimate, we apply Theorem 12.3.35(a), followed by (12.72) and (12.73) with $\mathfrak{t}^\omega$ and $\mathscr{D}^\omega$ in place of $\mathfrak{t}$ and $\mathscr{D}$, to get

$$\begin{aligned}
&\Big(\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} - \sup_\omega \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\Big)\frac{1}{\beta_{p,X}\beta_{p,Y}} \\
&\quad\leqslant \sup_\omega \sum_{\alpha,\gamma}\mathscr{DR}_p(\mathfrak{t}_0^{\omega;\alpha,\gamma}) + 12\cdot 2^d \sup_\omega \sum_{i=1}^{2} c_i \sum_{j\in\{0,i\}} \|\mathfrak{t}_{\mathrm{good}}^{\omega;(j)}\|_{\mathrm{Fig}^{\sigma_i}(\wp_i)} \\
&\quad\leqslant 4^d \sup_\omega \|\mathfrak{t}^\omega\|_{wbp(\mathscr{DR}_p)} + \sup_\omega \sum_{i=1}^{2} c_i\Big(c_d^0 c_K(\wp_i) + c_d^1\|\omega_K^i(\wp_i)\|_{\mathrm{Dini}^{\sigma_i}}\Big),
\end{aligned}$$

where $c_d^0 = 24\cdot 2^d\cdot a_d$ and $c_d^1 = 24\cdot 2^d\cdot b_d$. This is readily recognised to coincide with the bound asserted in (a) of the theorem.

(c): The representation formulas are immediate from Theorems 12.3.26 and 12.3.35, since we already verified that the assumptions of the said theorems are valid in the present setting. □

12.4.a Consequences of the $T(1)$ theorem

We will now explore various consequences of Theorem 12.4.12 to more particular classes of operators. While Theorem 12.4.12 gives a complete characterisation of the boundedness of an operator T, a drawback is the fact that this characterisation involves the boundedness of another operator $\Lambda_{\mathfrak{t}}$ that is not necessarily easy to check, as we found in Section 12.2. Thus, the following special case, in which these paraproducts are completely avoided, will be useful:

Corollary 12.4.13 ($T(1)$ theorem for convolution kernels). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and suppose that:*

(i) *X and Y are UMD spaces;*
(ii) *X has cotype q and Y has type t, or one of them has both;*
(iii) *$\mathfrak{t} : S(\mathscr{D})^2 \to Z := \mathscr{L}(X,Y)$ is a weakly defined singular integral and the kernel $K : \dot{\mathbb{R}}^{2d} \to Z$ of $\mathfrak{t}$ has the convolution form $K(s,t) = \mathfrak{K}(s-t)$ and satisfies the Calderón–Zygmund estimates*

$$\tilde{c}_{\mathfrak{K}}(\mathscr{R}_p) + \|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_p)\|_{\mathrm{Dini}^{\sigma}} < \infty, \qquad \sigma := \max(\frac{1}{t}, \frac{1}{q'}), \tag{12.74}$$

where $\tilde{c}_{\mathfrak{K}}$ and $\tilde{\omega}_{\mathfrak{K}}$ are as in Remark 12.4.6(3);
(iv) *$\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_Q) = (\mathbf{1}_{Q\dot{+}m}, \mathbf{1}_{Q\dot{+}m})$ for all $Q \in \mathscr{D}$ and $m \in \mathbb{Z}^d$.*

Then the following conditions are equivalent:

(1) *$\mathfrak{t}$ defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(2) *$\mathfrak{t}$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} < \infty$;*
(3) *each $\mathfrak{t}^{\omega}$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}^{\omega}\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant C$, uniformly in $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$.*

Under these equivalent conditions, we have

(a) *the norm estimate*

$$\begin{aligned}
&\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\
&\quad\leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d \sup_{\omega} \|\mathfrak{t}^{\omega}\|_{wbp(\mathscr{D}\mathscr{R}_p)} + c_d^0\Big(c_1\tilde{c}_{\mathfrak{K}}(\mathscr{R}_{p'}^*) + c_2\tilde{c}_{\mathfrak{K}}(\mathscr{R}_p)\Big) + \\
&\qquad\qquad + c_d^1\Big(c_1\|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/t}} + c_2\|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},
\end{aligned}$$

where the supremum is over $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, the constants c_d, c_d' depend only on d, and c_1, c_2 are as in (12.70);
(b) *the representation formulas (12.45) and (12.52) with $\Lambda_{\mathfrak{t}} = \Lambda_{\mathfrak{t}^{\omega}} = 0$.*

Proof. We will check that $\mathfrak{t}$ is translation-invariant in the sense of Definition 12.3.9, i.e., that it satisfies the condition of Lemma 12.3.8(1). The very assumption (iv) of the corollary already takes care of the case $Q = R$. On the other hand, if $Q \neq R$ are dyadic cubes of the same size, then $Q \cap R = \varnothing = (Q\dot{+}m) \cap (R\dot{+}m)$, and hence we have access to the kernel representation

$$\begin{aligned}\mathfrak{t}(\mathbf{1}_Q,\mathbf{1}_R) &= \int_R\int_Q K(s,t)\,\mathrm{d}s\,\mathrm{d}t = \int_R\int_Q \mathfrak{K}(s-t)\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_R\int_Q \mathfrak{K}((s+m)-(t+m))\,\mathrm{d}s\,\mathrm{d}t \\ &= \int_{R\dot{+}m}\int_{Q\dot{+}m} \mathfrak{K}(s-t)\,\mathrm{d}s\,\mathrm{d}t = \mathfrak{t}(\mathbf{1}_{Q\dot{+}m},\mathbf{1}_{R\dot{+}m}),\end{aligned}$$

which proves the condition of Lemma 12.3.8(1) for arbitrary $Q,R\in\mathscr{D}$ of equal size. Thus indeed $\mathfrak{t}$ is translation-invariant.

Next, we wish to have the same property for $\mathfrak{t}^\omega$, for every $\omega\in(\{0,1\}^d)_0^{\mathbb{Z}}$, and requires verifying the identity $\mathfrak{t}(\mathbf{1}_{Q'},\mathbf{1}_{R'}) = \mathfrak{t}(\mathbf{1}_{Q'\dot{+}m},\mathbf{1}_{R'\dot{+}m})$ for each $Q',R'\in\mathscr{D}^\omega$ of equal size. By Lemma 12.3.30, we have $S(\mathscr{D}^\omega)=S(\mathscr{D})$ whenever $\omega\in(\{0,1\}^d)_0^{\mathbb{Z}}$. If $Q'\in\mathscr{D}^\omega$, then clearly $f=\mathbf{1}_{Q'}\in S(\mathscr{D}^\omega)=S(\mathscr{D})$, and similarly with $g=\mathbf{1}_{R'}$ where $R'\in\mathscr{D}^\omega$ has the same size. Thus Lemma 12.3.8 guarantees that

$$\mathfrak{t}(\mathbf{1}_{Q'\dot{+}m},\mathbf{1}_{R'\dot{+}m}) = \mathfrak{t}(\tau_{m\ell(Q')}f,\tau_{m\ell(Q')}g) = \mathfrak{t}(f,g) = \mathfrak{t}(\mathbf{1}_{Q'},\mathbf{1}_{R'})$$

for all $Q',R'\in\mathscr{D}^\omega$ of the same size, and hence also $\mathfrak{t}^\omega$ is translation-invariant.

By Proposition 12.4.4, it then follows that $\mathfrak{t}^\omega(\mathbf{1},\cdot)=0=\mathfrak{t}^\omega(\cdot,\mathbf{1})$, for every $\omega\in(\{0,1\}^d)_0^{\mathbb{Z}}$. Thus the conclusions of the corollary are immediate from Theorem 12.4.12 by setting all $\Lambda_{\mathfrak{t}}$ and $\Lambda_{\mathfrak{t}^\omega}$ to be zero. □

Lemma 12.4.14. *Let $Z=\mathscr{L}(X,Y)$ and $\Phi\in C_{\mathrm{b}}([0,\infty);Z)\cap C^1((0,\infty);Z)$, and suppose that*

(i) $\mathfrak{K}(u):=\mathbf{1}_{(0,\infty)}(u)\Phi'(u)$ *satisfies the Calderón–Zygmund estimate* (12.74)*;*
(ii) *the range of Φ is R-bounded, $\mathscr{R}_p(\Phi):=\mathscr{R}_p(\{\Phi(u):u\in[0,\infty)\})<\infty$;*
(iii) *a bilinear form $\mathfrak{t}:S(\mathscr{D})^2\to Z$ is defined, for all $f,g\in S(\mathscr{D})$, by*

$$\mathfrak{t}(f,g) := \lim_{\varepsilon\to 0}\iint_{|u-v|>\varepsilon}\mathfrak{K}(u-v)f(v)g(u)\,\mathrm{d}v\,\mathrm{d}u.$$

Then

(1) $\mathfrak{t}$ *is well-defined as a weakly defined singular integral with convolution kernel $K(u,v)=\mathfrak{K}(u-v)$;*
(2) $\mathfrak{t}^\omega$ *satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property*

$$\|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant \|\Phi(0)\| + \min\{\mathscr{R}_p(\Phi),\mathscr{R}^*_{p'}(\Phi)\};$$

(3) $\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)=(\mathbf{1}_{I\dot{+}m},\mathbf{1}_{I\dot{+}m})$ *for all $I\in\mathscr{D}$ and $m\in\mathbb{Z}$.*

Proof. (1): Clearly the integral inside the limit is well-defined, since we are cutting away the singularity. To show the existence of the limit, let first $f=1_I$ and $g=1_J$ for some intervals $I=[a_I,b_I)$ and $J=[a_J,b_J)$. Then

$$\begin{aligned}\int_{|u-v|>\varepsilon} \mathfrak{K}(u-v)f(v)\,\mathrm{d}v &= \int_{a_I}^{b_I} \mathbf{1}_{(\varepsilon,\infty)}(u-v)\Phi'(u-v)\,\mathrm{d}v \\ &= \mathbf{1}_{(a_I+\varepsilon,\infty)}(u)\int_{a_I}^{b_I\wedge(u-\varepsilon)} \Phi'(u-v)\,\mathrm{d}v \\ &= \mathbf{1}_{(a_I+\varepsilon,\infty)}(u)[\Phi((u-b_I)\vee\varepsilon)-\Phi(u-a_I)].\end{aligned}$$

Since Φ is continuous on $[0,\infty)$, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{|u-v|>\varepsilon} \mathfrak{K}(u-v)f(v)\,\mathrm{d}v &= \mathbf{1}_{(a_I,\infty)}(u)[\Phi((u-b_I)_+)-\Phi(u-a_I)], \\ &= \Phi((u-b_I)_+)-\Phi((u-a_I)_+).\end{aligned}$$

Since Φ is bounded on $[0,\infty)$, we can apply dominated convergence to obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\iint_{|u-v|>\varepsilon} &\mathfrak{K}(u-v)\mathbf{1}_I(v)\mathbf{1}_J(u)\,\mathrm{d}v\,\mathrm{d}u \\ &= \int_J [\Phi((u-b_I)_+)-\Phi((u-a_I)_+)]\,\mathrm{d}u.\end{aligned}\tag{12.75}$$

In particular, the limit defining $\mathfrak{t}(f,g)$ exists for all f,g of the form $f=\mathbf{1}_I$ and $g=\mathbf{1}_J$. By (bi)linearity, it exists for all $f,g\in S(\mathscr{D})$.

If $f,g\in S(\mathscr{D})$ are disjointly supported, then $\mathfrak{K}(u-v)f(v)g(u)$ is integrable. Hence

$$\mathfrak{t}(f,g)=\iint \mathfrak{K}(u-v)f(v)g(u)\,\mathrm{d}v\,\mathrm{d}u$$

by dominated convergence, and thus $\mathfrak{t}$ is a weakly defined singular integral with kernel $K(u,v)=\mathfrak{K}(u-v)$.

(2): With $J=I\in\mathscr{D}^\omega$, noting that $a_I\leqslant u<b_I$ for all $u\in I$, the identity (12.75) shows that

$$\begin{aligned}\frac{\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)}{|I|} &= \fint_I \big(\Phi(0)-\Phi(u-a_I)\big)\,\mathrm{d}u \\ &= \fint_0^{\ell(I)} \big(\Phi(0)-\Phi(u)\big)\,\mathrm{d}u \in \Phi(0)+\overline{\mathrm{abco}}(\Phi).\end{aligned}\tag{12.76}$$

Thus, by Lemma 12.1.8, we find that

$$\begin{aligned}\|t^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} &:= \mathscr{D}\mathscr{R}_p\Big(\Big\{\frac{\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)}{|I|}\Big\}_{I\in\mathscr{D}^\omega}\Big) \\ &\leqslant \min_{i=0,1}\wp_i\Big(\Big\{\frac{\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)}{|I|}\Big\}_{I\in\mathscr{D}^\omega}\Big), \qquad \wp_0:=\mathscr{R}_p, \quad \wp_1:=\mathscr{R}^*_{p'}, \\ &\leqslant \|\Phi(0)\|+\min_{i=0,1}\wp_i(\Phi).\end{aligned}$$

(3): From (12.76) it is evident that $\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)$ depends only on $\ell(I)$; since $\ell(I)=\ell(I\dot{+}m)$, it follows that $\mathfrak{t}(\mathbf{1}_I,\mathbf{1}_I)=\mathfrak{t}(\mathbf{1}_{I\dot{+}m},\mathbf{1}_{I\dot{+}m})$, as claimed. □

It often happens that kernels that we encounter satisfy standard Calderón–Zygmund estimates with the best possible Lipschitz modulus of continuity $\omega(u) = O(u)$ as $u \to 0$, but the implies constant in this estimate can be very large. At the same time, we also have a trivial bound $\omega(u) = O(1)$, where the implied constant may be much smaller. The following lemma provides a useful estimate of the Dini norms of ω in such cases, showing that the larger constant enters the estimates only via its logarithm:

Lemma 12.4.15. *Let* $0 < A \leqslant B < \infty$ *and* $\sigma \in [0, 1]$*. If* $\omega(u) \leqslant \min(A, Bu)$*, then*

$$\|\omega\|_{\mathrm{Dini}^\sigma} \leqslant 3A\Big(1 + \log^{\sigma+1}\frac{B}{A}\Big).$$

Proof.

$$(\log 2)^\sigma \|\omega\|_{\mathrm{Dini}^\sigma} \leqslant \int_0^{A/B} B\Big(\log\frac{1}{u}\Big)^\sigma \,\mathrm{d}u + \int_{A/B}^1 A\Big(\log\frac{1}{u}\Big)^\sigma \frac{\mathrm{d}u}{u} =: I + II,$$

where

$$I \leqslant -B\int_0^{A/B} \log u \,\mathrm{d}u = -B(u\log u - u)\Big|_0^{A/B} = A(\log\frac{B}{A} + 1)$$

and

$$II = A\int_{A/B}^1 (-\log u)^\sigma \frac{\mathrm{d}u}{u} = -A\frac{(-\log u)^{\sigma+1}}{\sigma+1}\Big|_{A/B}^1 = \frac{A}{\sigma+1}\Big(\log\frac{B}{A}\Big)^{\sigma+1}.$$

Let $G := \log(B/A)$. Since

$$G = (G^{\sigma+1})^{1/(\sigma+1)} \cdot 1^{\sigma/(\sigma+1)} \leqslant \frac{1}{\sigma+1}G^{\sigma+1} + \frac{\sigma}{\sigma+1},$$

we obtain

$$I + II \leqslant \frac{2A}{\sigma+1}G^{\sigma+1} + A\Big(1 + \frac{\sigma}{\sigma+1}\Big) \leqslant 2A(G^{\sigma+1} + 1).$$

Since $(\log 2)^{-\sigma} \leqslant (\log 2)^{-1} < 3/2$, the claim follows. □

Example 12.4.16. Let $\omega \in [0, \pi/2]$, $\sigma \in [0, 1]$, and suppose that

$$\varPhi \in C([0,\infty), Z) \cap H^\infty(\varSigma_\omega; Z)$$

has an R-bounded range. Then $\varPhi|_{[0,\infty)}$ and $\mathfrak{K}(u) = \mathbf{1}_{(0,\infty)}(u)\varPhi'(u)$ satisfy the assumptions of Lemma 12.4.14 with

$$\tilde{c}_{\mathfrak{K}}(\wp) \leqslant \frac{\wp(\varPhi)}{\sin\omega}, \quad \|\tilde{\omega}_{\mathfrak{K}}(\wp)\|_{\mathrm{Dini}^\sigma} \leqslant \frac{3\wp(\varPhi)}{\sin\omega}\Big(1 + \log^{1+\sigma}\frac{4}{\sin\omega}\Big), \quad \wp \in \{\mathscr{R}_p, \mathscr{R}^*_{p'}\}.$$

A particular instance of such a $\varPhi$ is (the negation of) an R-bounded holomorphic semigroup $\varPhi(z) = -e^{-zA}$, in which case $\mathfrak{K}(u) = Ae^{-uA}$ is the kernel of the so-called maximal regularity operator.

Remark 12.4.17. The role of the parameter $\sigma \in [0,1]$ in Example 12.4.16 is relatively insignificant and only recorded for curiosity. First, it only affects the power of the logarithm. Second, for applying Lemma 12.4.14, it is necessary to take $\sigma \geqslant \max(1/t, 1/q') \geqslant \frac{1}{2}$, and it is always sufficient to take $\sigma = 1$, so that the power of the logarithm will always be in the range $[\frac{3}{2}, 2]$.

Proof of Example 12.4.16. Let $\wp \in \{\mathscr{R}_p, \mathscr{R}_p^*\}$. It is evident that

$$\wp(\{\Phi(u) : u \in [0,\infty)\}) = \wp(\overline{\{\Phi(u) : u \in (0,\infty)\}}) \leqslant \wp(\{\Phi(z) : z \in \Sigma_\omega\}).$$

By Cauchy's formula, we have

$$\Phi^{(j)}(u) = \frac{j!}{2\pi i} \oint_{|z-u|=u\sin\omega} \frac{f(z)}{(u-z)^{j+1}} |\mathrm{d}z|, \qquad u > 0.$$

Denoting $\wp(\Phi) := \wp(\Phi(z) : z \in \Sigma_\omega)$, we hence have

$$\wp(t^j \Phi^{(j)}(t) : t > 0) \leqslant \frac{j!}{2\pi} \wp(\Phi) \sup_{t>0} \oint_{|z-t|=t\sin\omega} \frac{t^j |\mathrm{d}z|}{(t\sin\omega)^{j+1}} = \frac{j!\wp(\Phi)}{(\sin\omega)^j}.$$

With $\mathfrak{K}(u) = \mathbf{1}_{(0,\infty)}(u)\Phi'(u)$, it follows that

$$\tilde{c}_{\mathfrak{K}}(\wp) = \wp(|u|\mathfrak{K}(u) : u \neq 0) = \wp(u\Phi'(u) : u > 0) \leqslant \frac{\wp(\Phi)}{\sin\omega}.$$

Moreover,

$$\begin{aligned}
\tilde{\omega}_{\mathfrak{K}}(\wp; s) &= \wp(|u|[\mathfrak{K}(u) - \mathfrak{K}(u')] : |u-u'| \leqslant s|u|) \\
&= \wp\Big(u \int_{u'}^{u} \Phi''(v)\,\mathrm{d}v : |u-u'| \leqslant su\Big) \\
&\leqslant \frac{2\wp(\Phi)}{(\sin\omega)^2} \sup_{|u-u'|\leqslant su} \Big| \int_{u'}^{u} \frac{u}{v^2}\,\mathrm{d}v \Big|,
\end{aligned}$$

where

$$\Big| \int_{u'}^{u} \frac{u}{v^2}\,\mathrm{d}v \Big| = u\Big|\frac{1}{u} - \frac{1}{u'}\Big| = \frac{|u-u'|}{u'} \leqslant \frac{su}{(1-w)u} = \frac{s}{1-s} \leqslant 2s$$

for $|u-u'| \leqslant wu$ and $s \in [0, \frac{1}{2}]$. Thus $\tilde{\omega}_{\mathfrak{K}}(\wp; s) \leqslant 4\wp(\Phi)(\sin\omega)^{-2}s$.

By Remark 12.4.6(2), we also have $\tilde{\omega}_{\mathfrak{K}}(\wp; s) \leqslant \tilde{c}_{\mathfrak{K}}(\wp) \leqslant \wp(\Phi)(\sin\omega)^{-1}$. Thus, an application of Lemma 12.4.15 with $0 < A = \wp(\Phi)(\sin\omega)^{-1} < 4\wp(\Phi)(\sin\omega)^{-2} = B < \infty$, we deduce that

$$\|\tilde{\omega}_{\mathfrak{K}}\|_{\mathrm{Dini}^\sigma} \leqslant \frac{3\wp(\Phi)}{\sin\omega}\Big(1 + \log^{1+\sigma} \frac{4}{\sin\omega}\Big).$$

This completes the proof. □

We proceed to further corollaries of Theorem 12.4.12.

Corollary 12.4.18 ($T(1)$ theorem for antisymmetric kernels). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and suppose that:*

(i) *X and Y are UMD spaces.*
(ii) *X has cotype q and Y has type t, or one of them has both.*
(iii) *$K : \dot{\mathbb{R}}^{2d} \to Z := \mathscr{L}(X,Y)$ is an antisymmetric kernel, i.e.,*

$$K(s,t) = -K(t,s) \quad \textit{for all } (s,t) \in \dot{\mathbb{R}}^{2d},$$

which satisfies the Calderón–Zygmund estimates

$$c_K(\mathscr{R}_p) + \|\omega_K^1(\mathscr{R}_p)\|_{\mathrm{Dini}^{\max(1/t,1/q')}} < \infty. \tag{12.77}$$

(iv) *A bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is defined for all $f,g \in S(\mathscr{D})$ by*

$$\mathfrak{t}(f,g) := \frac{1}{2}\iint K(s,t)\big(f(t)g(s) - f(s)g(t)\big)\,\mathrm{d}t\,\mathrm{d}s. \tag{12.78}$$

Then $\mathfrak{t}$ is well-defined as a weakly defined singular integral with kernel K, and the following conditions are equivalent:

(1) *$\mathfrak{t}$ defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(2) *$\Lambda_{\mathfrak{t}}$ defines a bounded operator in $\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(3) *each $\Lambda_{\mathfrak{t}^\omega}$ defines a bounded operator in $\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$, uniformly in $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$.*

Under these equivalent conditions, we have

(a) *the norm estimates as in parts (a) and (b) of Theorem 12.4.12, with $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} = \|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} = 0$;*
(b) *the representation formulas (12.45) and (12.52).*

Proof. To check that $\mathfrak{t}$ is well-defined, we need to verify that the integrals in (12.78) make sense. By linearity, it is enough to consider $f = \mathbf{1}_Q$ and $g = \mathbf{1}_R$ for some $Q, R \in \mathscr{D}$. If $Q \cap R = \varnothing$, then each of the two terms under the integral is separately integrable by Lemma 12.4.2, and hence so is their difference. Otherwise, we may assume by the nestedness of dyadic cubes and symmetry that, e.g., $Q \subseteq R$. We can then split

$$\begin{aligned} f(t)g(s) - f(s)g(t) &= \mathbf{1}_Q(t)\mathbf{1}_R(s) - \mathbf{1}_Q(s)\mathbf{1}_R(t) \\ &= \mathbf{1}_Q(t)\big(\mathbf{1}_Q(s) + \mathbf{1}_{R\setminus Q}(s)\big) - \mathbf{1}_Q(s)\big(\mathbf{1}_Q(t) + \mathbf{1}_{R\setminus Q}(t)\big) \\ &= \mathbf{1}_Q(t)\mathbf{1}_{R\setminus Q}(s) - \mathbf{1}_Q(s)\mathbf{1}_{R\setminus Q}(t), \end{aligned}$$

observing the cancellation of the two equal terms $\mathbf{1}_Q(s)\mathbf{1}_Q(t)$. We can divide $R \setminus Q$ into finitely many cubes $P \in \mathscr{D}$ of the same size as Q, and then the integrability of each of the terms on the left against $K(s,t)$ follows from

Lemma 12.4.2. Thus the formula defining $\mathfrak{t}$ as a bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is meaningful.

To show that $\mathfrak{t}$ has associated kernel K, let $f, g \in S(\mathscr{D})$ be disjointly supported. As we already observed, in this case both terms under the integral are separately integrable, and we can write

$$\begin{aligned}\mathfrak{t}(f,g) &= \frac{1}{2}\iint K(s,t)\big(f(t)g(s) - f(s)g(t)\big)\,\mathrm{d}t\,\mathrm{d}s \\ &= \frac{1}{2}\iint K(s,t)f(t)g(s)\,\mathrm{d}t\,\mathrm{d}s - \frac{1}{2}\iint K(s,t)f(s)g(t)\,\mathrm{d}t\,\mathrm{d}s =: \frac{I - II}{2}.\end{aligned}$$

Using the antisymmetry of K and interchanging the names of the variables, and applying Fubini's theorem, we find that

$$-II = \iint K(t,s)f(s)g(t)\,\mathrm{d}t\,\mathrm{d}s = \iint K(s,ty)f(t)g(s)\,\mathrm{d}s\,\mathrm{d}t = I.$$

Hence

$$\mathfrak{t}(f,g) = \frac{I - II}{2} = I = \iint K(s,t)f(t)g(s)\,\mathrm{d}t\,\mathrm{d}s,$$

as required for $\mathfrak{t}$ to be a weakly defined singular integral with kernel K.

From the defining formula (12.78) it is immediate that $\mathfrak{t}(\mathbf{1}_Q, \mathbf{1}_Q) = 0$, and hence the quantities featuring in the weak boundedness property of $\mathfrak{t}$ vanish. With $Q \in \mathscr{D}^\omega$ (which still satisfies $\mathbf{1}_Q \in S(\mathscr{D})$ for $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, by Lemma 12.3.30, the same conclusion extends to $\mathfrak{t}^\omega$ for all $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$. The rest of the corollary is then a direct consequence of Theorem 12.4.12, simply setting $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} = \|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} = 0$. We only need to note that $\omega_K^1(\wp) = \omega_K^2(\wp)$ when K is antisymmetric, which is why a seemingly weaker assumption suffices in (12.77). □

Corollary 12.4.19 ($T(1)$ theorem for antisymmetric convolutions). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and suppose that:*

(i) *X and Y are UMD spaces.*
(ii) *X has cotype q and Y has type t, or one of them has both.*
(iii) *$K : \dot{\mathbb{R}}^{2d} \to Z := \mathscr{L}(X,Y)$ is an antisymmetric convolution kernel, i.e.,*

$$K(s,t) = \mathfrak{K}(s-t) = -\mathfrak{K}(t-s) \quad \textit{for all } (s,t) \in \dot{\mathbb{R}}^{2d},$$

which satisfies the Calderón–Zygmund estimates (12.74).

(iv) *A bilinear form $\mathfrak{t} : S(\mathscr{D})^2 \to Z$ is defined for all $f, g \in S(\mathscr{D})$ by*

$$\mathfrak{t}(f,g) := \frac{1}{2}\iint \mathfrak{K}(s-t)\big(f(t)g(s) - f(s)g(t)\big)\,\mathrm{d}t\,\mathrm{d}s.$$

Then $\mathfrak{t}$ is well-defined as a weakly defined singular integral with kernel K, which defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$ and satisfies

(a) *the norm estimate*

$$\begin{aligned}&\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\\&\quad\leqslant \beta_{p,X}\beta_{p,Y}\Big\{c_d^0\Big(c_1\tilde{c}_{\mathfrak{K}}(\mathscr{R}_{p'}^*)+c_2\tilde{c}_{\mathfrak{K}}(\mathscr{R}_p)\Big)+\\&\qquad+c_d^1\Big(c_1\|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/t}}+c_2\|\tilde{\omega}_{\mathfrak{K}}(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},\end{aligned}$$

where the supremum is over $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$*, the constants* c_d, c_d' *depend only on* d*, and* c_1, c_2 *are as in* (12.70).

(b) *the representation formulas* (12.45) *and* (12.52) *with* $\Lambda_{\mathfrak{t}} = \Lambda_{\mathfrak{t}^\omega} = 0$.

Proof. This is straightforward by combining (the proofs of) Corollaries 12.4.13 and 12.4.18. In particular, in the proof of Corollary 12.4.18 we observed that any bilinear form defined as in (iv) of the present corollary will satisfy $\mathfrak{t}(\mathbf{1}_Q,\mathbf{1}_Q)=0$ for all $Q\in\mathscr{D}$, and hence also $\mathfrak{t}(\mathbf{1}_{Q\dot{+}m},\mathbf{1}_{Q\dot{+}m})=0=\mathfrak{t}(\mathbf{1}_Q,\mathbf{1}_Q)$ for all $m\in\mathbb{Z}^d$. This is condition (iv) of Corollary 12.4.13 that was not explicitly assumed in the corollary that we are proving. □

Remark 12.4.20. As an immediate consequence of Corollary 12.4.13, we obtain another proof of the essence of Theorem 5.1.13 on the boundedness of the Hilbert transform H on $L^p(\mathbb{R};X)$ whenever $p\in(1,\infty)$ and X is a UMD space. Indeed, take $X=Y$, $t=1$, and $q=\infty$, so that the constants in (12.70) are simply $c_1=c_2=1$. Clearly the kernel $K(u,v)=\pi^{-1}(u-v)^{-1}$ of the Hilbert transform is an antisymmetric convolution kernel, and it is easy to check the Calderón–Zygmund estimates (12.74) with Dini^1 norms. Thus we obtain the estimate

$$\|H\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))}\leqslant c\cdot\beta_{p,X}^2,$$

with the same quantitative form as (5.24), aside from the unspecified numerical factor above, in contrast to the explicit constant 2 in (5.24). This is quite natural, considering that (5.24) was obtained by an argument tailored for the very Hilbert transform, whereas the argument that we just sketched was a specialisation of a much more general argument to the particular case of H.

The following corollary provides a solution to the L^p extension problem from Section 2.1 for the important class of Calderón–Zygmund operators:

Theorem 12.4.21 ($T(1)$ theorem for scalar-valued kernels). *Let* $p,s\in(1,\infty)$ *and* $1\leqslant t\leqslant p\leqslant q\leqslant\infty$*, and suppose that:*

(i) X *is a UMD space with cotype* q *and type* t*,*

(ii) $\mathfrak{t}:S(\mathscr{D})^2\to\mathbb{K}$ *is a weakly defined singular integral, whose kernel* $K:\dot{\mathbb{R}}^{2d}\to\mathbb{K}$ *satisfies the Calderón–Zygmund estimates*

$$c_K+\sum_{i=1}^{2}\|\omega_K^i\|_{\mathrm{Dini}^{\sigma_i}}<\infty, \tag{12.79}$$

where $\sigma_1=1/t$ *and* $\sigma_2=1/q'$.

Then the following conditions are equivalent:

(1) $\mathfrak{t}$ *defines a bounded operator* $T \in \mathscr{L}(L^p(\mathbb{R}^d; X))$*;*
(2) $\mathfrak{t}$ *defines a bounded operator* $T \in \mathscr{L}(L^s(\mathbb{R}^d))$*;*
(3) $\|\mathfrak{t}^\omega\|_{wbp} \leqslant C$ *uniformly in* $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$*, and for some* $b_i \in \mathrm{BMO}(\mathbb{R}^d)$*,*

$$\mathfrak{t}(\mathbf{1}, g) = \langle b_1, g\rangle, \quad \mathfrak{t}(f, \mathbf{1}) = \langle f, b_2\rangle \tag{12.80}$$

for all $f, g \in S_{00}(\mathscr{D}^\omega)$ *and* $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$*;*
(4) $\|\mathfrak{t}\|_{wbp} < \infty$*, and* (12.80) *for some* $b_i \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$ *and all* $f, g \in S_{00}(\mathscr{D})$*.*

Under these equivalent conditions, we have

$$\begin{aligned}\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &\leqslant \tilde{c}_d \beta_{p,X}^2 (c_1 + c_2) c_K + \\ &+ \tilde{c}_d\Big(\beta_{p,X}^2 + pp'\beta_{s,X}^2\beta_{s,\mathbb{K}}\Big)\Big(\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))} + \sum_{i=1}^2 c_i \|\omega_K^i\|_{\mathrm{Dini}^{\sigma_i}}\Big),\end{aligned} \tag{12.81}$$

with a dimensional constant $\tilde{c}_d$ *and cotype constants*

$$c_1 = c_{t', X^*; p'}, \qquad c_2 = c_{q, X; p}.$$

In particular, every $L^p(\mathbb{R}^d)$*-bounded Calderón–Zygmund operator having kernel bounds* (12.80) *with* $\sigma_1 = \sigma_2 = 1$*, extends boundedly to* $L^p(\mathbb{R}^d; X)$ *for every UMD space* X*, and one can take* $c_1 = c_2 = 1$ *in the estimate* (12.81)*.*

Proof. (1) $\Rightarrow$ (2): For $s = p$, this is evident by restricting the action of the operator to a one-dimensional subspace of X. The case of general $s \in (1, \infty)$ follows from the Calderón–Zygmund Theorem 11.2.5 (or even just its classical scalar-valued version).

(2) $\Rightarrow$ (3): The weak boundedness property follows from Example 12.1.10:

$$\|\mathfrak{t}^\omega\|_{wbp} \leqslant \|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))}, \tag{12.82}$$

and we turn to the construction of the functions b_i.

The operator $T \in \mathscr{L}(L^s(\mathbb{R}^d))$ is a Calderón–Zygmund operator with kernel K that satisfies in particular the Dini conditions in both variables, and hence both direct and dual (operator-)Hörmander conditions by Lemma 11.3.4. (The qualifier "operator" is redundant for scalar-valued kernels.) By (just the scalar-valued version of) Theorem 11.2.9, T has an extension $\widetilde{T} \in \mathscr{L}(L^\infty(\mathbb{R}^d), \mathrm{BMO}(\mathbb{R}^d)/\mathbb{K})$. By Theorem 11.2.9(b), for functions $\mathbf{1} \in L^\infty(\mathbb{R}^d)$ and $g \in S_{00}(\mathscr{D}^\omega) \subseteq L^\infty_{c,0}(\mathbb{R}^d)$, we have

$$\begin{aligned}\langle \widetilde{T}(\mathbf{1}), g\rangle &= \lim_{M\to\infty} \langle T(\mathbf{1}_{(1+2M)Q}), g\rangle \\ &= \lim_{M\to\infty} \sum_{\substack{m\in\mathbb{Z}^d \\ |m|_\infty \leqslant M}} \mathfrak{t}(\mathbf{1}_{Q\dot{+}m}, g) = \mathfrak{t}(\mathbf{1}, g).\end{aligned}$$

This is one of the claimed identities with $b_1 := \widetilde{T}(\mathbf{1}) \in \mathrm{BMO}(\mathbb{R}^d; Y)$, and Theorem 11.2.9, followed by Lemma 11.3.4, provide us with the estimates

$$\begin{aligned} \|b_1\|_{\mathrm{BMO}^s(\mathbb{R}^d;Y)} &= \|\widetilde{T}(\mathbf{1})\|_{\mathrm{BMO}^s(\mathbb{R}^d)} \\ &\leqslant (c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d)} + \|K\|_{\mathrm{Hör}^*})\|\mathbf{1}\|_{L^\infty(\mathbb{R}^d)} \\ &\leqslant (c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d)} + \sigma_{d-1}\|\omega_K^1\|_{\mathrm{Dini}}). \end{aligned} \tag{12.83}$$

The identity involving $b_2 := \widetilde{T^*}(\mathbf{1})$, and the estimate

$$\begin{aligned} \|b_2\|_{\mathrm{BMO}^{s'}(\mathbb{R}^d)} &= \|\widetilde{T^*}(\mathbf{1})\|_{\mathrm{BMO}^{s'}(\mathbb{R}^d)} \\ &\leqslant \big(c_d\|T^*\|_{\mathscr{L}(L^{s'}(\mathbb{R}^d))} + \\ &\qquad + \|(u,v) \mapsto K(u,v)^*\|_{\mathrm{Hör}^*}\big)\|\mathbf{1}\|_{L^\infty(\mathbb{R}^d)} \\ &\leqslant (c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))} + \sigma_{d-1}\|\omega_K^2\|_{\mathrm{Dini}}) \end{aligned} \tag{12.84}$$

are entirely analogous on the dual side.

(3) $\Rightarrow$ (4): This is obvious by restricting to $\omega = 0$ and noting that $\mathrm{BMO}(\mathbb{R}^d) \subseteq \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$.

(4) $\Rightarrow$ (1): Under assumption (4), we see that the paraproducts related to $\mathfrak{t}$ are in fact $\Pi_{\mathfrak{t}}^i = \Pi_{b_i}$, where $b_i \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$ by assumption. Thus Corollary 12.2.19 guarantees that

$$\begin{aligned} \|\Lambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &= \|\Pi_{b_1} + \Pi_{b_2}^*\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \\ &\leqslant 64 \cdot 8^d \cdot pp'\beta_{s,X}^2\beta_{s,\mathbb{K}}(\|b_1\|_{\mathrm{BMO}_{\mathscr{D}}^s(\mathbb{R}^d)} + \|b_2\|_{\mathrm{BMO}_{\mathscr{D}}^{s'}(\mathbb{R}^d)}). \end{aligned} \tag{12.85}$$

Our assumption (4) also involves $\|\mathfrak{t}\|_{wbp} < \infty$, and Corollary 12.1.9 guarantees that this coincides with the finiteness of $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} = \|\mathfrak{t}\|_{wbp}$, when $\mathfrak{t}$ is scalar-valued. Thus both assumptions $\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)} < \infty$ and $\|\Lambda_{\mathfrak{t}}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} < \infty$ of Theorem 12.4.12(2) are satisfied, hence also the equivalent condition of Theorem 12.4.12(1), and this coincides with condition (1) of the corollary that we are proving.

The quantitative estimates: While we have already closed the chain of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1), the claimed quantitative bounds require a direct analysis of the implication (3) $\Rightarrow$ (1), which relates to the implication (3) $\Rightarrow$ (1) of Theorem 12.4.12.

As in the proof of "(4) $\Rightarrow$ (1)", under assumption (3), we see that the paraproducts related to $\mathfrak{t}^\omega$ are in fact $\Pi_{\mathfrak{t}^\omega}^i = \Pi_{b_i}^\omega$; while the function $b_i \in \mathrm{BMO}(\mathbb{R}^d) \subseteq \mathrm{BMO}_{\mathscr{D}^\omega}(\mathbb{R}^d)$ is independent of ω, the superscript of the paraproduct signifies the fact that the defining series involves Haar functions and averages related to $Q \in \mathscr{D}^\omega$. Thus, imitating (12.85) and substituting the bounds (12.83) and (12.84), we obtain, with $s_1 := s$ and $s_2 := s'$,

$$\begin{aligned} \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &= \|\Pi^\omega_{b_1} + (\Pi^\omega_{b_2})^*\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \\ &\leqslant 64\cdot 8^d \cdot pp'\beta^2_{s,X}\beta_{s,\mathbb{K}}\sum_{i=1}^2 \|b_i\|_{\mathrm{BMO}^{s_i}(\mathbb{R}^d)} \\ &\leqslant 64\cdot 8^d \cdot pp'\beta^2_{s,X}\beta_{s,\mathbb{K}}\sum_{i=1}^2 \Big(c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))} + \sigma_{d-1}\|\omega^i_K\|_{\mathrm{Dini}}\Big). \end{aligned} \tag{12.86}$$

where we implicitly dominated $\|b_i\|_{\mathrm{BMO}^{s_i}_{\mathscr{D}^\omega}(\mathbb{R}^d)} \leqslant \|b_i\|_{\mathrm{BMO}^{s_i}(\mathbb{R}^d)}$ in the first estimate. We now substitute (12.86) and (12.82) into the second norm estimate in Theorem 12.4.12(b), noting that all R-bounds and $\mathscr{D}\mathscr{R}_p$-bounds may be omitted, since they simply reduce to uniform bounds for scalar-valued functions:

$$\begin{aligned} \|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leqslant \sup_\omega \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} &+ \\ + \beta^2_{p,X}\Big\{4^d \sup_\omega \|\mathfrak{t}^\omega\|_{wbp} + c^0_d(c_1+c_2)c_K &+ \\ + c^1_d\Big(c_1\|\omega^1_K\|_{\mathrm{Dini}^{1/t}} + c_2\|\omega^2_K\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\}&. \end{aligned}$$

This gives the bound asserted in the corollary. □

Remark 12.4.22. If $b_1 = b_2$, the term $pp'\beta^2_{s,X}\beta_{s,\mathbb{K}}$ can be omitted in (12.81). This applies in particular if T is translation-invariant.

Proof. By inspection of the proof of Theorem 12.4.21, the said term only arises in the estimate of $\Lambda_{\mathfrak{t}^\omega}$ in (12.86). Under the assumption that $b_1 = b_2$, we have $\Lambda_{\mathfrak{t}^\omega} = \Lambda^\omega_{b_1}$, and we may replace (12.86) by an application of Theorem 12.2.25:

$$\|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} = \|\Lambda^\omega_{b_1}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X))} \leqslant 30\cdot 2^d\cdot \beta^2_{p,X}\|b_1\|_{\mathrm{BMO}(\mathbb{R}^d)},$$

where

$$\|b_1\|_{\mathrm{BMO}(\mathbb{R}^d)} \leqslant \|b_1\|_{\mathrm{BMO}^s(\mathbb{R}^d)} \leqslant c_d\|T\|_{\mathscr{L}(L^s(\mathbb{R}^d))} + \sigma_{d-1}\|\omega^1_K\|_{\mathrm{Dini}}.$$

Substituting this alternative estimate into the proof of Theorem 12.4.21, we obtain the claimed modification of (12.81).

If T is translation-invariant, the paraproduct terms vanish, and hence we can take $b_1 = b_2 = 0$, which is indeed a special case of $b_1 = b_2$. Of course, in this case, we do not even need to use Theorem 12.2.25. □

12.4.b The dyadic representation theorem

The randomised dyadic representation (12.52) underlying the proof of $T(1)$ Theorem 12.3.26 can be further reorganised into a form that has proven to be useful for various extensions. Recalling Definition 12.3.34 of the good parts of Figiel's operators, and in particular the quantity $k(n) := 2 + \lceil \log_2 |n| \rceil$, we regroup the sum over $n \in \mathbb{Z}^d \setminus \{0\}$ in (12.52) according to a constant value of $k(n)$ as

$$\sum_{n\in\mathbb{Z}^d\setminus\{0\}} = \sum_{k=2}^{\infty} \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{k-3}<|n|\leqslant 2^{k-2}}} .$$

We denote by $\mathrm{ch}^{(k)}(P)$ the collection of dyadic descendants of P of generation k, and define the operators

$$\begin{aligned}
\mathbb{D}_P^{(k)} &:= \sum_{Q\in\mathrm{ch}^{(k)}(P)} \mathbb{D}_Q, \quad \mathbb{E}_P^{(k)} := \sum_{Q\in\mathrm{ch}^{(k)}(P)} \mathbb{E}_Q, \\
\mathbb{D}_P^{[0,k)} &:= \mathbb{E}_P^{(k)} - E_P^{(0)} = \sum_{j=0}^{k-1} \mathbb{D}_P^{(j)}.
\end{aligned}$$

Lemma 12.4.23. *If* $\mathfrak{t} : S(\mathscr{D})^2 \to \mathscr{L}(X,Y) =: Z$ *is a weakly defined singular integral with kernel* $K : \dot{\mathbb{R}}^{2d} \to Z$*, then*

$$\begin{aligned}
\mathscr{T}_k &:= \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{k-3}<|n|\leqslant 2^{k-2}}} \langle T_{n,\mathfrak{t}^\omega}^{\mathrm{good}} f, g\rangle = \langle S^{(0,k)} f, g\rangle, \\
\mathscr{U}_k^1 &:= \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{k-3}<|n|\leqslant 2^{k-2}}} \langle f, U_{n,\mathfrak{t}^\omega}^{1,\mathrm{good}} g\rangle = \langle S^{(1,k)} f, g\rangle, \\
\mathscr{U}_k^2 &:= \sum_{\substack{n\in\mathbb{Z}^d \\ 2^{k-3}<|n|\leqslant 2^{k-2}}} \langle U_{n,\mathfrak{t}^\omega}^{2,\mathrm{good}} f, g\rangle = \langle S^{(2,k)} f, g\rangle,
\end{aligned}$$

where

$$S^{(i,k)} f = \sum_{P\in\mathscr{D}} A_P^{(i,k)} f, \qquad A_P^{(i,k)} f(s) = \fint_P a_P^{(i,k)}(s,t) f(t)\,\mathrm{d}t,$$

and these satisfy the identities

$$\begin{aligned}
A_P^{(0,k)} &= \mathbb{D}_P^{(k)} A_P^{(0,k)} \mathbb{D}_P^{(k)}, \\
A_P^{(1,k)} &= \mathbb{D}_P^{(k)} A_P^{(0,k)} \mathbb{D}_P^{[0,k)}, \\
A_P^{(2,k)} &= \mathbb{D}_P^{[0,k)} A_P^{(0,k)} \mathbb{D}_P^{(k)}.
\end{aligned} \tag{12.87}$$

For $i = 1, 2$*, we have the further splitting*

$$A_P^{(i,k)} f = A_{P;P}^{(i,k)} f - \sum_{R\in\mathrm{ch}^{(k)}} A_{P;R}^{(i,k)} f$$

where

$$A^{(i,k)}_{P;R}f(s)=\fint_R a^{(i,k)}_{P;R}(s,u)f(u)\,\mathrm{d}u,\qquad R\in\{P\}\cup\mathrm{ch}^{(k)}(P),$$

and these kernels have the bounds

$$\begin{aligned}\wp(\{a^{(0,k)}_P(s,u),a^{(i,k)}_{P,R}(s,u):s,u\in R\in\{P\}\cup\mathrm{ch}^{(k)}(P),P\in\mathscr{D}\})\\ \leqslant c_d\begin{cases}c_K(\wp), & \text{if } 2^k\leqslant 12\sqrt{d},\\ \omega^i_K(\wp;\frac{6\sqrt{d}}{2^k}), & \text{if } 2^k\geqslant 12\sqrt{d},\end{cases}\end{aligned}$$

Proof. By definition, the left-hand side of the claim is equal to

$$\mathscr{T}_k=\sum_{\substack{n\in\mathbb{Z}^d\\ 2^{k-3}<|n|\leqslant 2^{k-2}}}\ \sum_{\substack{Q\in\mathscr{D}_{k\text{-good}}\\ \alpha,\gamma\in\{0,1\}^d\setminus\{0\}}}\Big\langle \mathfrak{t}(h^\alpha_Q,h^\gamma_{Q\dot{+}n})\langle f,h^\alpha_Q\rangle,\langle g,h^\gamma_{Q\dot{+}n}\rangle\Big\rangle,$$

where the k-goodness of Q guarantees that $R:=Q\dot{+}n$, for $|n|\leqslant 2^{k-2}$, shares with Q the same kth dyadic ancestor $R^{(k)}=Q^{(k)}=:P\in\mathscr{D}$. Thus we can regroup this series under the ancestors P to get

$$\mathscr{T}_k=\sum_{P\in\mathscr{D}}\ \sum_{\substack{(Q,R)\in\mathscr{C}_k(P)\\ \alpha,\gamma\in\{0,1\}^d\setminus\{0\}}}\Big\langle \mathfrak{t}_{\mathrm{good}}(h^\alpha_Q,h^\gamma_R)\langle f,h^\alpha_Q\rangle,\langle g,h^\gamma_R\rangle\Big\rangle,$$

where

$$\mathscr{C}_k(P):=\Big\{(Q,R):\quad Q,R\in\mathrm{ch}^{(k)}(P),\quad \frac{1}{8}\ell(P)<|z_Q-z_R|\leqslant\frac{1}{4}\ell(P)\Big\}.$$

The subseries under each $P\in\mathscr{D}$ takes the asserted form $\langle A^{(k)}_P f,g\rangle$ if we define

$$a^{(0,k)}_P(s,u):=|P|\sum_{\substack{(Q,R)\in\mathscr{C}_k(P)\\ \alpha,\gamma\in\{0,1\}^d\setminus\{0\}}}\mathfrak{t}_{\mathrm{good}}(h^\alpha_Q,h^\gamma_R)h^\alpha_Q(u)h^\gamma_R(s).$$

The cases of $\mathscr{U}^i_k$ are analogous, and lead to representations of the same form with

$$a^{(1,k)}_P(s,u):=|P|\sum_{\substack{(Q,R)\in\mathscr{C}_k(P)\\ \gamma\in\{0,1\}^d\setminus\{0\}}}\mathfrak{t}_{\mathrm{good}}(h^0_Q,h^\gamma_R)[h^0_Q(u)-h^0_R(u)]h^\gamma_R(s),$$

and

$$a^{(2,k)}_P(s,u):=|P|\sum_{\substack{(Q,R)\in\mathscr{C}_k(P)\\ \alpha\in\{0,1\}^d\setminus\{0\}}}\mathfrak{t}_{\mathrm{good}}(h^\alpha_Q,h^0_R)h^\alpha_Q(u)[h^0_R(s)-h^0_Q(s)],$$

The further splitting is then naturally defined with

$$a^{(1,k)}_{P;P}(s,u) := |P| \sum_{\substack{(Q,R)\in\mathscr{C}_k(P)\\ \gamma\in\{0,1\}^d\setminus\{0\}}} \mathfrak{t}_{\text{good}}(h^0_Q, h^\gamma_R) h^0_Q(u) h^\gamma_R(s),$$

$$a^{(1,k)}_{P;R}(s,u) := |R| \sum_{\substack{Q:(Q,R)\in\mathscr{C}_k(P)\\ \gamma\in\{0,1\}^d\setminus\{0\}}} \mathfrak{t}_{\text{good}}(h^0_Q, h^\gamma_R) h^0_R(u) h^\gamma_R(s), \quad R\in \operatorname{ch}^k(P),$$

where the last summation runs over all relevant $Q \in \operatorname{ch}^k(P)$, for fixed R. Observe that $a^{(1,k)}_{P;R}$ has the factor $|R|$ in front, instead of $|P|$, due to our definition of $A^{(1,k)}_{P;R} f(s)$ as the average integral $\fint_R a^{(1,k)}_{P;R}(s,u) f(u)\,\mathrm{d}u$.

The splitting of $a^{(2,k)}_P$ is entirely analogous; in particular,

$$a^{(2,k)}_{P;Q}(s,u) := |Q| \sum_{\substack{R:(Q,R)\in\mathscr{C}_k(P)\\ \alpha\in\{0,1\}^d\setminus\{0\}}} \mathfrak{t}_{\text{good}}(h^\alpha_Q, h^0_R) h^\alpha_Q(u) h^0_Q(s), \quad Q\in \operatorname{ch}^k(P).$$

It remains to verify that these operators and their kernels satisfy the asserted properties. The identity $A^{(0,k)}_P = \mathbb{D}^{(k)}_P A^{(0,k)}_P \mathbb{D}^{(k)}_P$ is immediate from the orthogonality of the Haar functions, and the invariance of $A^{(i,k)}_P$ under composition by $\mathbb{D}^{(k)}_P$ on the side, where the cancellative Haar function appear in $a^{(i,k)}_P$ is justified similarly. Concerning the factors $\mathbb{D}^{[0,k)}_P$, we note that the are orthogonal projections onto functions supported on P, constant on each $Q \in \operatorname{ch}^{(k)}(Q)$, and integrating to zero. Noting the functions $h^0_Q - h^0_R$ belong to this class then justifies the remaining parts of the claimed identities.

Concerning the claimed bounds, we note that any given $(s,u) \in P\times P$ is contained in exactly one $Q\times R$ with $Q, R\in \operatorname{ch}^{(k)}(P)$, and moreover,

$$|h^\alpha_Q \otimes h^\gamma_R| = \frac{1_{Q\times R}}{|Q|^{1/2}|R|^{1/2}} = \frac{2^{kd}}{|P|} 1_{Q\times R}.$$

The claimed $\wp$-bounds for $a^{(0,k)}_P(s,u)$, as well as for $a^{(i,k)}_{P;P}(s,u)$, then follow from Lemma 12.4.8, noting that the factor $|P|$ in the definition of these kernels cancels with the $\frac{1}{|P|}$ above.

For $a^{(1,k)}_{P;R}$ with $R\in \operatorname{ch}^{(k)}(P)$, all terms in the defining sum are supported on the same $1_{R\times R}$, and each individual summand can be estimates by Lemma 12.4.8. We now have the smaller factor $|R|$ in front, but at the same time there are up to 2^{kd} terms in the sum, all of which accumulate on the same support now. Since $2^{kd}|R| = |P|$, we get the same final bound as before. The case of $a^{(1,k)}_{P;Q}$ with $Q\in \operatorname{ch}^{(k)}(P)$ in entirely analogous, and completes the proof. □

Definition 12.4.24. *An operator $S : S_{00}(\mathscr{D};X) \to S_{00}(\mathscr{D};Y)$ is called a dyadic shift of type (i,k), where $i\in\{0,1,2\}$ and $k\in\{2,3,\dots\}$, if*

$$S = \sum_{P\in\mathscr{D}} A_P, \qquad A_P f(s) = A_{P;P} f(s) - \sum_{Q\in \mathrm{ch}^{(k)}(P)} A_{P;Q} f(s),$$

where

$$A_{P;R} f(s) = \fint_R a_{P;R}(s,u) f(u)\,\mathrm{d}u, \qquad R \in \{P\}\cup \mathrm{ch}^{(k)}(P),$$

$\operatorname{supp} a_{P;R} \subseteq R\times R$,

$$\|S\|_{\mathrm{Shift}(\wp)} := \wp\Big(\Big\{a_{P;R}(s,u) : s,u\in R\in\{P\}\cup\mathrm{ch}^{(k)}(P), P\in\mathscr{D}\Big\}\Big) < \infty$$

for $\wp = \mathscr{R}_2$, *and moreover, for every* $P\in\mathscr{D}$,

(0) *if* $i=0$, *then* $A_P = \mathbb{D}_K^{(k)} A_P \mathbb{D}_K^{(k)}$, *and* $A_{P;Q}=0$ *for all* $Q\in\mathrm{ch}^{(k)}(P)$;
(1) *if* $i=1$, *then* $A_P = \mathbb{D}_K^{(k)} A_P \mathbb{D}_K^{[0,k)}$;
(2) *if* $i=2$, *then* $A_P = \mathbb{D}_K^{[0,k)} A_P \mathbb{D}_K^{(k)}$.

We say that a shift has type $i\in\{0,1,2\}$, *if it has type* (i,k) *with some* k.

Remark 12.4.25. In the language of Definition 12.4.24, the operators $S^{(i,k)}$ of Lemma 12.4.23 are dyadic shifts of type (i,k), and we may further write

$$\|S^{(i,k)}\|_{\mathrm{Shift}(\wp)} \leqslant c_d \begin{cases} c_K(\wp), & \text{if } 2^k \leqslant 12\sqrt{d},\\ \omega_K^i(\wp; \frac{6\sqrt{d}}{2^k}), & \text{if } 2^k \geqslant 12\sqrt{d}.\end{cases}$$

The key boundedness properties of these dyadic shifts are contained in the following:

Theorem 12.4.26. *Let* X *and* Y *be UMD spaces, and* $p\in(1,\infty)$. *Suppose that* X *has cotype* q *and* Y *has type* t *for some* $1\leqslant t\leqslant p\leqslant q\leqslant\infty$.

Then for all $i\in\{0,1,2\}$ *and* $k\in\{2,3,\dots\}$, *all dyadic shifts* S *of type* (i,k) *extends to a bounded operator from* $L^p(\mathbb{R}^d;X)$ *to* $L^p(\mathbb{R}^d;Y)$. *Moreover, they satisfy the norm estimates*

$$\|S\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant 4\cdot\beta_{p,X}\beta_{p,Y}\times\begin{cases}\|S\|_{\mathrm{Shift}(\mathscr{R}_p)} c_{t',Y^*;p'}\cdot k^{1/t}, & i=1,\\ \|S\|_{\mathrm{Shift}(\mathscr{R}^*_{p'})} c_{q,X;p}\cdot k^{1/q'}, & i=2;\end{cases}$$

and the norm of a shift of type $(0,k)$ *is bounded by the minimum of these two bounds, but with* 6 *in place of* 4.

Proof. We divide the proof into case according to the type of the shift under consideration.

Shifts of type 1

Let us start with the case $i = 1$. For $f \in S_{00}(\mathbb{R}^d; X) \subseteq L^p(\mathbb{R}^d; X)$ and $g \in S_{00}(\mathbb{R}^d; Y) \subseteq L^{p'}(\mathbb{R}^d; Y^*)$, we expand the pairing $\langle Sf, g\rangle$ by separating the scales according to $\log_2 \ell(P) \mod k$:

$$
\begin{aligned}
|\langle Sf, g\rangle| &= \Big|\sum_{j=0}^{k-1} \sum_{\substack{P\in\mathscr{D}\\ \log_2 \ell(P)\equiv j\\ \mathrm{mod}\ k}} \Big\langle \mathbb{D}_P^{(k)} A_P f, \mathbb{D}_P^{(k)} g\Big\rangle\Big| \\
&\leqslant \sum_{j=0}^{k-1} \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2 \ell(P)\equiv j\\ \mathrm{mod}\ k}} \varepsilon_P \mathbb{D}_P^{(k)} A_P f\Big\|_{L^p(\Omega\times\mathbb{R}^d;Y)} \\
&\qquad \times \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2 \ell(P)\equiv j\\ \mathrm{mod}\ k}} \varepsilon_P \mathbb{D}_P^{(k)} g\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)} =: \sum_{j=0}^{k-1} I_j \times II_j.
\end{aligned}
$$

In I_j, we write out $\mathbb{D}_P^{(k)} = \sum_{Q\in \mathrm{ch}^{(k)}(P)} \mathbb{D}_Q$ and note that, in a randomised sum like here, we are free to replace ε_Q by ε_P, since the difference is invisible to the $L^p(\Omega; Y)$ at a fixed $s \in \mathbb{R}^d$. This gives

$$
I_j = \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2 \ell(P)\equiv j\\ \mathrm{mod}\ k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q \mathbb{D}_Q A_P f\Big\|_{L^p(\mathbb{R};Y)}.
$$

Using the splitting of A_P, it then follows that

$$
\begin{aligned}
I_j &\leqslant \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2 \ell(P)\equiv j\\ \mathrm{mod}\ k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q \mathbb{D}_Q A_{P;P} f\Big\|_{L^p(\Omega\times\mathbb{R};Y)} \\
&\qquad + \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2 \ell(P)\equiv j\\ \mathrm{mod}\ k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q \mathbb{D}_Q A_{P;Q} f\Big\|_{L^p(\Omega\times\mathbb{R};Y)} \\
&=: III_j + IV_j.
\end{aligned}
$$

We first consider IV_j. Denoting by Q_s the unique dyadic child of Q that contains a given $s \in Q$, and with the understanding that $\mathbb{D}_Q$ acts in the s variable, we have

$$
\begin{aligned}
\mathbb{D}_Q A_{P;Q} f(s) &= \fint_Q \mathbb{D}_Q a_{P;Q}(s,u) \mathbb{D}_P^{[0,k)} f(u)\,\mathrm{d}u \\
&= \mathbf{1}_Q(s) \fint_Q \Big(\langle a_{P;Q}(\cdot,u)\rangle_{Q_s} - \langle a_{P;Q}(\cdot,u)\rangle_Q\Big)(\langle f\rangle_Q - \langle f\rangle_P)\,\mathrm{d}u
\end{aligned}
$$

$$=: \alpha_{P;Q}(s)\mathbf{1}_Q(s)(\langle f\rangle_Q - \langle f\rangle_P) = \alpha_{P;Q}(s)\mathbf{1}_Q(s)\mathbb{D}_P^{[0,k)}(s),$$

where

$$\alpha_{P;Q}(s) := \fint_Q \Big(\langle a_{P;Q}(\cdot,u)\rangle_{Q_s} - \langle a_{P;Q}(\cdot,u)\rangle_Q\Big)\,\mathrm{d}u$$

belongs to the two-fold multiple of the absolute convex hull of the set appearing in the definition of $\|S\|_{\mathrm{Shift}(\wp)}$. Thus

$$\begin{aligned} IV_j &= \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q \alpha_{P;Q}\mathbf{1}_Q \mathbb{D}_P^{[0,k)} f\Big\|_{L^p(\Omega\times\mathbb{R};Y)} \\ &\leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)} \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \sum_{Q\in\mathrm{ch}^{(k)}(P)} \varepsilon_Q \mathbf{1}_Q \mathbb{D}_P^{[0,k)} f\Big\|_{L^p(\Omega\times\mathbb{R};X)} \\ &\leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)} \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \varepsilon_P \mathbb{D}_P^{[0,k)} f\Big\|_{L^p(\Omega\times\mathbb{R};X)}, \end{aligned}$$

using the identity $\sum_{Q\in\mathrm{ch}^{(k)}(P)}\mathbf{1}_Q = \mathbf{1}_P$ and the interchangeability of ε_P and ε_Q in the random sum in the last step.

Observing that $(\mathbb{D}_P^{[0,k)} f)_{\log_2\ell(P)\equiv j \bmod k}$ is a martingale difference decomposition of f for each $j\in\{0,\dots,k-1\}$ to deduce directly from the definition of the UMD constants that

$$IV_j \leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\beta^+_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)}.$$

We then turn to term III_j. By the exchangeability of ε_P and ε_Q again, this can be written as

$$III_j = \Big\| \sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \bmod k}} \varepsilon_P \mathbb{D}_P^{(k)} A_{P;P} f\Big\|_{L^p(\Omega\times\mathbb{R};Y)},$$

where

$$\mathbb{D}_P^{(k)} A_{P;P} f(s) = \fint_P \mathbb{D}_P^{(k)} a_{P;P}(s,u)\mathbb{D}_P^{[0,k)} f(u)\,\mathrm{d}u,$$

and it is understood that $\mathbb{D}_P^{(k)}$ acts with respect to the s variable.

We will now make use of the tangent martingale construction as in Corollary 4.4.15 and explained just before the statement of the said result: For every $P\in\mathscr{D}$, let T_P be a copy of P equipped with the normalised measure $\nu_P := |P|^{-1}m|_P$, where m is the Lebesgue measure, and consider the product space $T := \prod_{P\in\mathscr{D}} T_P$ with probability measure $\nu := \otimes_{P\in\mathscr{D}}\nu_P$. Writing a typical element of T as $\mathbf{t} = (t_P)_{P\in\mathscr{D}}$, we then have

$$\mathbb{D}_P^{(k)} A_{P;P} f(s) = \int_T \mathbb{D}_P^{(k)} a_{P;P}(s, t_P) \mathbb{D}_P^{[0,k)} f(t_P) \, \mathrm{d}\nu(\mathbf{t}).$$

Hence (suppressing, as usual, the dependence of random functions on $\omega \in \Omega$),

$$\begin{aligned} III_j &= \Big\| s \mapsto \int_T \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \varepsilon_P \mathbb{D}_P^{(k)} a_{P;P}(s, t_P) \mathbb{D}_P^{[0,k)} f(t_P) \, \mathrm{d}\nu(\mathbf{t}) \Big\|_{L^p(\Omega \times \mathbb{R}^d; Y)} \\ &\leqslant \Big\| (s, \mathbf{t}) \mapsto \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \varepsilon_P \mathbf{1}_P(s) \mathbb{D}_P^{(k)} a_{P;P}(s, t_P) \mathbb{D}_P^{[0,k)} f(t_P) \Big\|_{L^p(\Omega \times \mathbb{R}^d \times T; Y)}. \end{aligned}$$

Here, $\mathbb{D}_P^{(k)} a_{P;P}(s, t_P)$ is the difference of two averages $\langle a_{P;P}(\cdot, t_P) \rangle_Q$, and hence in twice the absolute convex hull of the set in the definition of $\|S\|_{\mathrm{Shift}(\wp)}$. Thus, the definition of R-boundedness implies that

$$III_j \leqslant 2 \|S\|_{\mathrm{Shift}(\mathscr{R}_p)} \Big\| (s, \mathbf{t}) \mapsto \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \varepsilon_P \mathbf{1}_P(s) \mathbb{D}_P^{[0,k)} f(t_P) \Big\|_{L^p(\Omega \times \mathbb{R}^d \times T; X)}.$$

We are now in a position to apply Corollary 4.4.15. Indeed, the functions $\mathbb{D}_P^{[0,k)} f$ are "atoms" in the sense defined before that corollary: $\mathbb{D}_P^{[0,k)} f$ is supported on P, of average 0, and constant on all $P' \in \mathrm{ch}^{(k)}(P)$, which are the next smaller cubes in the scales-separated dyadic system $\{P \in \mathscr{D} : \log_2 \ell(P) \equiv j \mod k\}$. Thus, a direct application of Corollary 4.4.15 to

$$f = \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \mathbb{D}_P^{[0,k)} f$$

shows that

$$\Big\| (s, \mathbf{t}) \mapsto \sum_{\substack{P \in \mathscr{D} \\ \log_2 \ell(P) \equiv j \\ \bmod k}} \varepsilon_P \mathbf{1}_P(s) \mathbb{D}_P^{[0,k)} f(t_P) \Big\|_{L^p(\Omega \times \mathbb{R}^d \times T; X)} \leqslant \beta_{p,X} \|f\|_{L^p(\mathbb{R}^d; X)},$$

and hence

$$III_j \leqslant 2 \|S\|_{\mathrm{Shift}(\mathscr{R}_p)} \beta_{p,X} \|f\|_{L^p(\mathbb{R}^d; X)}.$$

Combining this with the estimate for term IV_j (and estimating the one-sided UMD constant by the basic UMD constant), we deduce that

$$I_j \leqslant III_j + IV_j \leqslant 4 \|S\|_{\mathrm{Shift}(\mathscr{R}_p)} \beta_{p,X} \|f\|_{L^p(\mathbb{R}^d; X)}.$$

Hence

$$|\langle Sf,g\rangle| \leqslant \sum_{j=0}^{k-1} I_j \times II_j \leqslant 4\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\beta_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)}\sum_{j=0}^{k-1} II_j,$$

where

$$\begin{aligned}\sum_{j=0}^{k-1} II_j &\leqslant k^{1/t}\Big(\sum_{j=0}^{k-1} II_j^{t'}\Big)^{\frac{1}{t'}} \\ &= k^{1/t}\Big(\sum_{j=0}^{k-1}\Big\|\sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \mathrm{mod}\ k}}\varepsilon_P\mathbb{D}_P^{(k)}g\Big\|^{t'}_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)}\Big)^{\frac{1}{t'}} \\ &\leqslant k^{1/t}\cdot c_{t',Y^*;p'}\Big\|\sum_{P\in\mathscr{D}}\varepsilon_P\mathbb{D}_P^{(k)}g\Big\|_{L^{p'}(\Omega\times\mathbb{R}^d;Y^*)} \\ &\leqslant k^{1/t}\cdot c_{t',Y^*;p'}\cdot\beta^+_{p',Y^*}\|g\|_{L^{p'}(\mathbb{R}^d;Y^*)}.\end{aligned}$$

Here $\beta^+_{p',Y^*}\leqslant\beta_{p',Y^*}=\beta_{p,Y}$ by Proposition 4.2.17(2), and $c_{t',Y^*;p'}\leqslant\tau_{t,Y;p}$ by Proposition 7.1.13 (or its easy extension to deal with the third index in these constants). This completes the proof for shift of type $(1,k)$.

Shifts of type 2

For a shift of type $(2,k)$, we note that its adjoint S^* is a shift of type $(1,k)$, and hence

$$\begin{aligned}\|S\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} &= \|S^*\|_{\mathscr{L}(L^{p'}(\mathbb{R}^d;Y^*),L^{p'}(\mathbb{R}^d;X^*))} \\ &\leqslant 4\|S^*\|_{\mathrm{Shift}(\mathscr{R}_{p'})}\beta_{p',Y^*}\beta_{p',X^{**}}c_{q,X;p}k^{1/q'} \\ &= 4\|S\|_{\mathrm{Shift}(\mathscr{R}^*_{p'})}\beta_{p,Y}\beta_{p,X}c_{q,X;p}k^{1/q'},\end{aligned}$$

which is the asserted bound in this case.

Shifts of type 0

Let finally S be a shift of type $(0,k)$. We can then proceed as in the case of type $(1,k)$ with slight modifications: In view of the eventual application of the tangent martingale estimate of Corollary 4.4.15, we now separate scales by $k+1$ levels instead of k, since $\mathbb{D}_P^{(k)}f$ is only guaranteed to be constant on $Q\in\mathrm{ch}^{(k+1)}(P)$. On the other hand, we now have $IV_j=0$, and hence $I_j=III_j$.

Following the argument in the case of type $(1,k)$ leads to

$$III_j\leqslant 2\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\Big\|(s,\mathbf{t})\mapsto\sum_{\substack{P\in\mathscr{D}\\ \log_2\ell(P)\equiv j\\ \mathrm{mod}\ k+1}}\varepsilon_P\mathbf{1}_P(s)\mathbb{D}_P^{(k)}f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)}.$$

To complete the estimate, we will need a little additional trick compared to the previous cases. First, we observe that

$$\mathbb{D}_P^{(k)} = (I - \mathbb{E}_P^{(k)})\mathbb{D}_P^{[0,k+1)}.$$

Second, we have

$$\begin{aligned}\mathbb{E}_P^{(k)} f(t_P) &= \mathbb{E}(f|\sigma(\mathrm{ch}^{(k+1)}(P)))(t_P) \\ &= \mathbb{E}\Big(\mathbf{t} \mapsto f(t_P)\Big| \bigotimes_{Q\in\mathscr{D}} \sigma(\mathrm{ch}^{(k+1)}(Q))\Big) =: \mathbb{E}\Big(\mathbf{t} \mapsto f(t_P)\Big|\mathscr{G}_{k+1}\Big),\end{aligned}$$

where on the right-hand side we take a conditional expectation with respect to a product σ-algebra on the product probability space T, of a function that only depends on the "coordinate" t_P of $\mathbf{t} \in T$. The importance of this last formula comes from the fact that only the function inside the conditional expectation, but not the conditional expectation operator itself, depends on the dyadic cube P. Using the previous two formulas, it follows that

$$\begin{aligned}&\Big\| \sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{(k)} f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)} \\ &\leqslant \Big\| \sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{[0,k+1)} f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)} \\ &\qquad + \Big\|\mathbb{E}\Big(\sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{[0,k+1)} f(t_P)\Big|\mathscr{G}_{k+1}\Big)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)} \\ &\leqslant 2\Big\| \sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{[0,k+1)} f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)}\end{aligned}$$

by the contractivity of the conditional expectation in the last step. This last expression has the same form as what we encountered with shifts of type $(1,k)$, only with $k+1$ in place of k. Thus, by an application of the tangent martingale inequality of Corollary 4.4.15, we have

$$\Big\| \sum_{\substack{P\in\mathscr{D} \\ \log_2 \ell(P)\equiv j \\ \mathrm{mod}\ k+1}} \varepsilon_P \mathbf{1}_P(s)\mathbb{D}_P^{[0,k+1)} f(t_P)\Big\|_{L^p(\Omega\times\mathbb{R}^d\times T;X)} \leqslant \beta_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)}.$$

Thus,

$$I_j = III_j \leqslant 4\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\beta_{p,X}\|f\|_{L^p(\mathbb{R}^d;X)},$$

which is the same bound as for the corresponding terms in the estimate of shifts of type $(1,k)$. The rest of the argument is exactly the same, only with $k+1$ in place of k, and leads to the conclusion that

$$\|S\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant 4\|S\|_{\mathrm{Shift}(\mathscr{R}_p)}\beta_{p,X}\beta_{p,Y}c_{t',Y^*;p'}(k+1)^{1/t}.$$

Since the adjoint of a shift of type $(0,k)$ is another shift of the same type, we also obtain

$$\|S\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \leqslant 4\|S\|_{\mathrm{Shift}(\mathscr{R}^*_{p'})}\beta_{p,X}\beta_{p,Y}c_{q,X;p}(k+1)^{1/q'},$$

and we can take the minimum of the two bounds. Since $k \geqslant 2$, we can also make the trivial estimates $k+1 \leqslant \frac{3}{2}k$ and $4\cdot(\frac{3}{2})^{1/v} \leqslant 6$ for $v \in \{t,q'\}$ so that in case $v \geqslant 1$. ☐

With the help of the shifts, we can represent any weakly defined singular integral with appropriate kernel bounds as follows:

Theorem 12.4.27 (Dyadic Representation Theorem). *Let $p \in (1,\infty)$ and $1 \leqslant t \leqslant p \leqslant q \leqslant \infty$, and suppose that:*

(i) *X and Y are UMD spaces,*
(ii) *X has cotype q and Y has type t,*
(iii) *$\mathfrak{t} : S(\mathscr{D})^2 \to Z := \mathscr{L}(X,Y)$ is a weakly defined singular integral and the kernel $K : \dot{\mathbb{R}}^{2d} \to Z$ of $\mathfrak{t}$ satisfies the Calderón–Zygmund estimates*

$$c_K(\mathscr{R}_p) + \|\omega^1_K(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/t}} + \|\omega^2_K(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/q'}} < \infty.$$

Then the following conditions are equivalent:

(1) *$\mathfrak{t}$ defines a bounded operator $T \in \mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$;*
(2) *each $\mathfrak{t}^\omega$ satisfies the weak $\mathscr{D}\mathscr{R}_p$-boundedness property $\|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} \leqslant C$, and the associated bi-paraproduct $\Lambda_{\mathfrak{t}^\omega}$ defines a bounded operator in $\mathscr{L}(L^p(\mathbb{R}^d;X), L^p(\mathbb{R}^d;Y))$, uniformly in $\omega \in (\{0,1\}^d)^{\mathbb{Z}}_0$.*

Under these equivalent conditions, we have

(a) *the dyadic representation formula*

$$\langle Tf,g\rangle = \mathbb{E}\Big(\langle \mathfrak{H}_{\mathfrak{t}^\omega} f,g\rangle + \langle \Lambda_{\mathfrak{t}^\omega} f,g\rangle + \sum_{\substack{k=2\\ i\in\{0,1,2\}}}^{\infty} \langle S^{(i,k)}_\omega f,g\rangle\Big)$$

with absolute convergence for all $f \in S(\mathscr{D};X)$ and $g \in S(\mathscr{D};Y^*)$, where $\mathbb{E}$ is the expectation over $\omega \in (\{0,1\}^d)^{\mathbb{Z}_{\leqslant M}}$, and $M \in \mathbb{Z}$ is any large enough number such that f and g are constant on all $Q \in \mathscr{D}_M$; the operators $\mathfrak{H}_{\mathfrak{t}^\omega}$ and $\Lambda_{\mathfrak{t}^\omega}$ are a Haar multiplier and a paraproduct as in (12.52), and each $S^{(i,k)}_\omega$ is a dyadic shift of type (i,k) (Definition 12.4.24) with respect to the dyadic system $\mathscr{D}^\omega$ and with shift norms estimated by

$$\|S^{(i,k)}_\omega\|_{\mathrm{Shift}(\wp)} \leqslant c_d \begin{cases} c_K(\wp), & \text{if } 2^k \leqslant 12\sqrt{d}, \\ \omega^i_K(\wp; \frac{6\sqrt{d}}{2^k}), & \text{if } 2^k \geqslant 12\sqrt{d}; \end{cases}$$

(b) *the resulting norm estimate:*

$$\begin{aligned}\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} - \sup_{\omega} \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ \leqslant \beta_{p,X}\beta_{p,Y}\Big\{4^d \sup_{\omega} \|\mathfrak{t}^\omega\|_{wbp(\mathscr{D}\mathscr{R}_p)} + c_d^0\Big(c_K(\mathscr{R}_p) + c_K(\mathscr{R}^*_{p'})\Big) + \\ + c_d^1\Big(c_{t',Y^*;p'}\|\omega_K^1(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/t}} + c_{q,X;p}\|\omega_K^2(\mathscr{R}^*_{p'})\|_{\mathrm{Dini}^{1/q'}}\Big)\Big\},\end{aligned}$$

where the suprema are over $\omega \in (\{0,1\}^d)_0^{\mathbb{Z}}$, *and the constants* c_d^0, c_d^1 *depend only on* d

Proof. We note that the present assumptions coincide with those of Theorem 12.4.12, except that (ii) of the present theorem is slightly stronger than (ii) of Theorem 12.4.12. Thus the equivalence of (1) and (2) is just repetition from Theorem 12.4.12.

The first new claim is the dyadic representation formula (a). To see this, recall that Theorem 12.4.12 gave the representation formula (12.52), repeated for convenience as

$$\begin{aligned}\langle Tf,g\rangle = \mathbb{E}\Big(\langle \mathfrak{H}_{\mathfrak{t}^\omega} f,g\rangle + \langle \Lambda_{\mathfrak{t}^\omega} f,g\rangle + 2^d \sum_{\substack{n\in\mathbb{Z}^d \\ n\neq 0}} \Big\{\langle T^{\mathrm{good}}_{n,\mathfrak{t}^\omega} f,g\rangle + \\ + \langle f, U^{1,\mathrm{good}}_{n,\mathfrak{t}^\omega} g\rangle + \langle U^{2,\mathrm{good}}_{n,\mathfrak{t}^\omega} f,g\rangle\Big\}\Big),\end{aligned}$$

where f, g, and $\mathbb{E}$ have the same meaning as in the claimed formula (a). On the other hand, Lemma 12.4.23 and Remark 12.4.25 inform us that the summation of the three types of terms over $n \in \mathbb{Z}^d \setminus \{0\}$ can be rearranged into a sum over $k \geqslant 2$ and $i \in \{0,1,2\}$ exactly as in the assertion.

From the representation (a), we can then estimate

$$\begin{aligned}\|T\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} - \sup_{\omega} \|\Lambda_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ \leqslant \sup_{\omega}\Big(\|\mathfrak{H}_{\mathfrak{t}^\omega}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\ + \sum_{\substack{k=2 \\ i\in\{0,1,2\}}}^{\infty} \|S^{(i,k)}_\omega\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}\Big).\end{aligned}$$

The first term here is estimated as in the proof of Theorem 12.4.12 by $4^d\|\mathfrak{t}\|_{wbp(\mathscr{D}\mathscr{R}_p)}$. For the remaining sum over shifts, we obtain from Theorem 12.4.26 (using this theorem with trivial type $t = 1$ for small k, and as stated for large k) that

$$\sum_{k=2}^{\infty}\|S^{(1,k)}_\omega\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))}$$

$$\begin{aligned}
&\leqslant 4 \cdot \beta_{p,X}\beta_{p,Y}\Big(\sum_{k:1\leqslant 2^k \leqslant 12\sqrt{d}} \|S_\omega^{(1,k)}\|_{\mathrm{Shift}(\mathscr{R}_p)} \cdot k \\
&\qquad\qquad + \sum_{k:2^k\geqslant 12\sqrt{d}} \|S_\omega^{(1,k)}\|_{\mathrm{Shift}(\mathscr{R}_p)} c_{t',Y^*;p'} \cdot k^{1/t}\Big) \\
&\leqslant c_d \beta_{p,X}\beta_{p,Y}\Big(\sum_{k:1\leqslant 2^k \leqslant 12\sqrt{d}} c_K(\mathscr{R}_p) k \\
&\qquad\qquad + c_{t',Y^*;p'} \sum_{k:2^k\geqslant 12\sqrt{d}} \omega_K^1(\mathscr{R}_p; \frac{6\sqrt{d}}{2^k}) k^{1/t}\Big) \\
&\leqslant c_d' \beta_{p,X}\beta_{p,Y}\Big(c_K(\mathscr{R}_p) + c_{t',Y^*;p'} \|\omega_K^1(\mathscr{R}_p)\|_{\mathrm{Dini}^{1/t}}\Big)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{k=2}^{\infty} \|S_\omega^{(2,k)}\|_{\mathscr{L}(L^p(\mathbb{R}^d;X),L^p(\mathbb{R}^d;Y))} \\
&\qquad \leqslant c_d' \beta_{p,X}\beta_{p,Y}\Big(c_K(\mathscr{R}_{p'}^*) + c_{q,X;p} \|\omega_K^2(\mathscr{R}_{p'}^*)\|_{\mathrm{Dini}^{1/q'}}\Big).
\end{aligned}$$

Finally, The sum over shifts of type $(0,k)$ may be estimated by either of the two bounds above (the different numerical constant in Theorem 12.4.26 is in any case absorbed into the unspecified dimensional constant). □

Remark 12.4.28. The norm estimate obtained in Theorem 12.4.27 via the representation in terms of dyadic shifts is essentially the same as that in Theorem 12.4.12 obtained via Figiel's representation. While the proof of Theorem 12.4.27 partially relied on the proof of Theorem 12.4.12 to avoid repetition, a larger part of the machinery behind the proof of Theorem 12.4.12, relying in particular on Figiel's Theorems 12.1.25 and 12.1.28 concerning his elementary operators, was replaced in the proof of Theorem 12.4.27 by Theorem 12.4.26 on the dyadic shifts, which in turn was based on the tangent martingale bounds of Corollary 4.4.15.

12.5 Notes

Section 12.1

The Haar multipliers $\mathfrak{H}_\lambda = \mathfrak{H}_\lambda^{\alpha\alpha}$ are special cases of martingale transforms discussed extensively in Volume I; see in particular Sections 3.5 and 4.2.e. In this framework, the predictable sequences multiplying the martingale differences

$$\mathbb{D}_k^\alpha f := \sum_{Q\in\mathscr{D}_k} \langle f, h_Q^\alpha\rangle h_Q^\alpha, \qquad \mathbb{D}_k^{-\alpha} f := \sum_{Q\in\mathscr{D}_k} \mathbb{D}_Q^{-\alpha} f$$

are then

$$v_k^\alpha = \sum_{Q\in\mathscr{D}_k} \lambda_k \mathbf{1}_{Q_k} \in L^\infty(\sigma(\mathscr{D}_k); \mathscr{L}(X,Y)), \qquad v_k^{-\alpha} \equiv 0.$$

On the other hand, the Haar multipliers $\mathfrak{H}_\lambda^{\alpha\gamma}$ with $\alpha \neq \gamma$ already take a departure from the general theory, and this is even more so with the general operators of Figiel.

(Note that the conventional indices of dyadic analysis and martingale theory are off by one from each other. In martingale theory, it is customary to emphasise measurability, and hence the indices of martingale differences agree with those of the σ-algebra that makes them measurable, while predictable multipliers are measurable with respect to the "previous" σ-algebra with index $k-1$. In dyadic analysis, the emphasis is on the supporting dyadic cubes, and hence the "kth" martingale difference $\mathbb{D}_k f$ is the sum of the local martingale differences $\mathbb{D}_Q$ supported, and averaging to zero, on the dyadic cubes $Q \in \mathscr{D}_k$, but then they are actually measurable only with respect to the "next" $\sigma(\mathscr{D}_{k+1})$; at the same time, the "predictable" multipliers are then measurable with respect to the σ-algebra indicated by their index.)

The relaxed R-boundedness notion $\mathscr{D}\mathscr{R}_p$ of Definition 12.1.6 seems to be new, but the slightly stronger $\mathscr{E}\mathscr{R}_p$ appears implicitly in Di Plinio, Li, Martikainen, and Vuorinen [2020b, Remark 6.29], where it is shown that the family $|Q|^{-1}\langle T\mathbf{1}_Q, \mathbf{1}_Q\rangle$ of Example 12.1.10 has this property when $T \in \mathscr{L}(L^p(\mathbb{R}^d; X), L^p(\mathbb{R}^d; Y))$ and X and Y are UMD spaces; this also follows by combining our Example 12.1.10 (on the $\mathscr{D}\mathscr{R}_p$ property of this family) and Corollary 12.1.17 (the equivalence of $\mathscr{D}\mathscr{R}_p$ and $\mathscr{E}\mathscr{R}_p$ for UMD spaces). An advantage of the new $\mathscr{D}\mathscr{R}_p$ is that it allows Example 12.1.10 without any assumptions on the Banach spaces.

The exact characterisation of the boundedness of the Haar multipliers $\mathfrak{H}_\lambda$ in Theorem 12.1.11 is new; by Lemma 12.1.8 and Propositions 12.1.13 and 12.1.14, the characterising condition is strictly more general than the R-boundedness condition $\|x \mapsto \mathscr{R}(\{\lambda_Q : x \in Q \in \mathscr{D}\})\|_\infty < \infty$. This seems at first to contradict Girardi and Weis [2005], where the necessity of uniform pointwise R-boundedness for operator-valued martingale transforms is established. This apparent contradiction is resolved by observing that, in order to obtain this necessity of R-boundedness, Girardi and Weis [2005] actually assume that their transforming sequence $(v_k)_{k\geqslant 1}$ is allowed to multiply any subsequence $(df_{n_k})_{k\geqslant 1}$ of the martingale difference sequence $(df_k)_{k=1}^\infty$, i.e., they assume the boundedness of the family of operators $f \mapsto \sum_{k\geqslant 1} v_k df_{n_k}$ instead of just $f \mapsto \sum_{k\geqslant 1} v_k df_k$. In the case of Haar multipliers, this would mean that, for a given sequence $\lambda = (\lambda_Q)_{Q\in\mathscr{D}}$ we would consider a family of operators including in particular all

$$f \mapsto \sum_{Q\in\mathscr{D}} \lambda_{Q^{(k)}} \langle f, h_Q^\alpha\rangle h_Q^\alpha,$$

where $k \in \mathbb{N}$ and $Q^{(k)}$ is the k generations larger dyadic ancestor of Q. However, in particular situations like that of Propositions 12.1.13, each coefficient λ_Q is naturally associated to a unique cube Q only.

The underlying ideas of Section 12.1.b come from Figiel [1988], and they have been developed further by Hytönen [2006], but substantial details of the present treatment are new. Figiel [1988] also introduced the elementary operators T and U and proved the first versions of Theorems 12.1.25 and 12.1.28. A novelty of the present treatment, also reflected in the auxiliary considerations in Section 12.1.b, is to set up the argument in such a way as to obtain a reasonably efficient dependence of the estimates on the UMD constants, although we make no claims concerning sharpness. A technical point was to use the decomposition of Lemma 12.1.22 in such a way that the parts of the decomposition contribute additively, rather than multiplicatively, to the operator norms in Theorems 12.1.25 and 12.1.28; while this seems only natural in retrospect, it was not the case with earlier treatments of the analogous bounds by Figiel [1988] and Hytönen [2006]. This proof detail only affects the constants in the final estimates, which was not a concern in these earlier works.

Besides the "dyadic singular integrals" studied in this section, there are related classes of operators that might be regarded as "dyadic pseudo-differential operators", in that their symbol depends on both the spatial variable $s \in \mathbb{R}^d$ and the "dyadic frequency variable" $I \in \mathscr{D}$. These are the generalised Haar multipliers

$$\mathfrak{H}_{\lambda(s)} f(s) = \sum_{I \in \mathscr{D}} \lambda_I(s) \langle f, h_I \rangle h_I(s),$$

where each coefficient $\lambda_I(\cdot)$ is a function. A primary example considered by Katz and Pereyra [1999] consists of

$$\lambda_I(s) = w_I^t(s) := \Big(\frac{w(s)}{\langle w \rangle_I} \Big)^t,$$

where $t \in \mathbb{R}$ and w is in a (dyadic) A_p or (dyadic) reverse Hölder class. Given the close relation of their techniques to those of the present section, it seems likely that some of the results concerning the operators $\mathfrak{H}_{\lambda(\cdot)}$ could be generalised to functions taking values in a UMD space, but this line of research seems not to have been pursued so far.

Section 12.2

In analogy with the quote of Stein [1982] on square functions at the beginning of Chapter 9, also the concept of paraproduct is "not an idea in its pure form, but rather takes various shapes depending on the uses it is put to". A friendly overview to this variety of "shapes and uses" of paraproducts can be found in Bényi, Maldonado, and Naibo [2010]. Paraproducts were systematically introduced by Bony [1981], but Bényi et al. [2010] convincingly argue that their first version is already implicit in the treatment of commutators of singular integrals by Calderón [1965].

Our treatment concentrates on *dyadic* paraproducts. We are uncertain about the earliest appearance of this notion in the literature but it was certainly known to Figiel [1990]; according to this paper, the $L^p(\mathbb{R}^d;X)$-boundedness of the dyadic paraproduct with a scalar-valued $b \in \mathrm{BMO}_{\mathscr{D}}(\mathbb{R}^d)$ "relies on an estimate due to Jean Bourgain (October 1987, unpublished)". This argument was only presented in print much later by Figiel and Wojtaszczyk [2001]. In particular, Corollary 12.2.19 goes back to these works. The first results on the boundedness of operator-valued paraproducts on UMD spaces were obtained by Hytönen and Weis [2006b] for a Fourier-analytic cousin of the dyadic paraproduct that we have treated. A sufficient condition similar to Proposition 12.2.16, in terms of a version of the Carleson norm, was identified there under the name of "Littlewood–Paley–BMO" norm. The condition of Theorem 12.2.18, in terms of $\mathrm{BMO}(\mathbb{R}^d;Z)$ with values in a UMD subspace $Z \hookrightarrow \mathscr{L}(X,Y)$, is also implicit in Hytönen and Weis [2006b], and explicitly formulated by Hytönen [2006]. However, both Hytönen and Weis [2006b] and Hytönen [2006] also required an additional R-boundedness condition, most easily formulated by the requirement that the unit ball $\bar{B}_Z$ of Z should be an R-bounded subset of $\mathscr{L}(X,Y)$. This condition was found to be superfluous by Hytönen [2014] when revising the argument for an extension to non-doubling measures, a generality that we have not considered here. The details of the present approach are largely borrowed from Hänninen and Hytönen [2016], where several simplifications were found when specialising the considerations back to the case of the Lebesgue measure. A particular novelty of Hänninen and Hytönen [2016], which we have followed, was to estimate the vector-valued paraproduct directly in $L^p(\mathbb{R}^d;Y)$, in contrast to earlier arguments that achieved the L^p bounds only via interpolation from auxiliary end-point estimates between the Hardy space $H^1(\mathbb{R}^d;X)$ and $L^1(\mathbb{R}^d;Y)$ on the one hand, and between $L^\infty(\mathbb{R}^d;X)$ and $\mathrm{BMO}(\mathbb{R}^d;Y)$ on the other hand.

Theorem 12.2.25 on the boundedness of the symmetric paraproduct Λ_b is from Hytönen [2021]. The case when $p=2$ and $X=Y$ is a Hilbert space was obtained earlier by Blasco and Pott [2008], and extended to any $p \in (1,\infty)$ and any non-commutative $L^p(\mathcal{M})$ space (with the same p) by Mei [2010]. (Recall that $L^p(\mathcal{M})$ is a UMD space for $p \in (1,\infty)$—the case of Schatten classes, due to Bourgain [1986], is treated in Proposition 5.4.2, while the general case can be found in Berkson et al. [1986b]—so the mentioned result of Mei [2010] is indeed a special case of Theorem 12.2.25.) The auxiliary material on projective tensor products is classical; much more on this topic can be found in Ryan [2002].

Theorem 12.2.26 on the dimensional growth of the norms of operator-valued paraproducts is from Mei [2006]. The optimal dimensional dependence in the estimate

$$\|\Pi_b\|_{\mathscr{L}(L^2(\mathbb{R};\ell^2_N))} \leqslant \psi(N)\|b\|_{\mathrm{BMO}^{\mathrm{so}}_{\mathscr{D}}(\mathbb{R};\mathscr{L}(\ell^2_N))} := \psi(N) \sup_{u \in \bar{B}_{\ell^2_N}} \|b(\cdot)u\|_{\mathrm{BMO}_{\mathscr{D}}(\mathbb{R};\ell^2_N)}.$$

had been settled some years earlier: Independently, Katz [1997] and Nazarov, Treil, and Volberg [1997b] proved that $\psi(N) \lesssim 1 + \log N$, and the latter authors also obtained the preliminary lower bound $\psi(N) \gtrsim (1 + \log N)^{1/2}$. This was improved to $\psi(N) \gtrsim 1+\log N$ by Nazarov, Pisier, Treil, and Volberg [2002a]. For a while, there were hopes in the air of obtaining a dimension-free estimate with $\mathrm{BMO}_{\mathscr{D}}(\mathbb{R}; \mathscr{L}(\ell^2_N))$ in place of $\mathrm{BMO}^{\mathrm{so}}_{\mathscr{D}}(\mathbb{R}; \mathscr{L}(\ell^2_N))$ on the right. Some indications that made this plausible are discussed in the introduction of Mei [2006] who, however, destroyed such hopes were by the main result of that paper, reproduced as Theorem 12.2.26. In combination with the upper bound by Katz [1997] and Nazarov et al. [1997b] just mentioned, it shows that $1 + \log N$ is the optimal upper bound for $\|\Pi_b\|_{\mathscr{L}(L^2(\mathbb{R};\ell^2_N))}/\|b\|_{F(\mathbb{R};\mathscr{L}(\ell^2_N))}$ for any of the choices $F \in \{\mathrm{BMO}^{\mathrm{so}}_{\mathscr{D}}, \mathrm{BMO}_{\mathscr{D}}, L^\infty\}$.

Further relations between various BMO-type quantities and the norms of related transformations in infinite-dimensional Hilbert spaces have been studied by Blasco and Pott [2008, 2010]. Analogous results in the context of the operator-valued BMOA space of analytic functions are due to Rydhe [2017].

Section 12.3

We refer the reader to the Notes of the following section for an account of the $T(1)$ theorem in its more traditional meaning as a boundedness criterion for Calderón–Zygmund operators (as in the title of David and Journé [1984]). The section under discussion presents a rather non-canonical approach to this theory, introduced and described by Figiel [1990] as follows:

> Our approach is indirect in the following sense. Rather than trying to prove that some “classical” operators are bounded, we start from considering certain rather new operators, which in our opinion have a basic nature. (All the “singularities” which can occur in our context are neatly packaged inside the basic operators.) Having established precise estimates for the norms of those basic operators, we can take up the “general case”. We just look at the class of those operators which can be realised as the sum of an absolutely convergent (in the operator norm) operator series whose summands are simple compositions of our basic operators. Then it turns out that the choice was sufficiently efficient for that class to contain so-called generalised Calderón–Zygmund operators and much more.

A large part of this section, up to and including $T(1)$ Theorem 12.3.26, is an updated review of Figiel [1990], incorporating a few elaborations:

- the trade-off between the type and cotype properties of the underlying spaces and the minimal rate of convergence of the coefficients of the bilinear form, as in Theorem 12.3.26(ii) (which is implicit in the combination of Figiel [1988, 1990]);

- conditions involving R-boundedness to deal with operator-valued versions (first introduced into the context of $T(1)$ theorems at large by Hytönen and Weis [2006b] and into Figiel's approach by Hytönen [2006]);
- keeping track of, and optimising the argument for, the quantitative dependence on parameters like the UMD constants (which seems new for this "non-random" version of the $T(1)$ theorem, involving—in contrast to Theorem 12.3.35—one dyadic system $\mathscr{D}$ only).

The decomposition (12.36) of $\mathfrak{t}(f,g)$ into three *one-parameter* series, in contrast to the perhaps more obvious *two-parameter* decomposition

$$\mathfrak{t}(f,g) = \sum_{i,j} \mathfrak{t}(D_i f, D_j g),$$

was already used by Figiel [1990], but it is frequently referred to as the "BCR algorithm" after Beylkin, Coifman, and Rokhlin [1991]. They explored its advantages for the numerical evaluation of singular integrals, also making a connection with the $T(1)$ theorem but apparently independently of Figiel [1990]. A decade later in 2002, when two of the present authors started to investigate a Banach space valued $T(1)$ theorem (eventually published in Hytönen and Weis [2006b]), they were also initially unaware of the work of Figiel [1990], which was first brought to their attention by Hans-Olav Tylli. Ever since, the approach of Figiel [1990] has been highly influential for the development of the theory of Banach space valued singular integrals.

The second $T(1)$ Theorem 12.3.35, which makes use of a random choice of the dyadic system $\mathscr{D}^\omega$, has a history of its own. This method, referred to by its inventors as "pulling ourselves by hair", was introduced by Nazarov, Treil, and Volberg [1997a] to tackle the difficulties in estimating singular integrals with respect to a *non-doubling* measure μ, thus going beyond the established theory in spaces of homogeneous type due to Coifman and Weiss [1971]. Their original idea consisted of splitting a function into its "good" and "bad" parts, according to the "good" and "bad" cubes supporting the martingale differences $\mathbb{D}_Q f$:

$$f^\omega_{\text{good}} := \sum_{Q\in\mathscr{D}^\omega_{\text{good}}} \mathbb{D}_Q f, \quad f^\omega_{\text{bad}} := \sum_{Q\in\mathscr{D}^\omega_{\text{bad}}} \mathbb{D}_Q f,$$

and showing that the latter is small, "on average", with respect to a random choice of ω:

$$\mathbb{E}\|f^\omega_{\text{bad}}\|_{L^2(\mu)} \leqslant \varepsilon\|f\|_{L^2(\mu)}.$$

As a result, it is enough to estimate (an *a priori* bounded) operator T of "good" functions only. Namely, if

$$|\langle Tf^\omega_{\text{good}}, g^\omega_{\text{good}}\rangle| \leqslant C\|f^\omega_{\text{good}}\|_2\|g^\omega_{\text{good}}\|_2 \leqslant C\|f\|_2\|g\|_2,$$

then

$$\begin{aligned}|\langle Tf,g\rangle| &\leqslant |\langle Tf^{\omega}_{\text{good}},g^{\omega}_{\text{good}}\rangle|+|\langle Tf^{\omega}_{\text{good}},g^{\omega}_{\text{bad}}\rangle|+|\langle Tf^{\omega}_{\text{bad}},g\rangle|\\ &\leqslant C\|f\|_2\|g\|_2+\|T\|\,\|f\|_2\|g^{\omega}_{\text{bad}}\|_2+\|T\|\,\|f^{\omega}_{\text{bad}}\|_2\|g\|_2.\end{aligned}$$

Taking the expectations of both sides, it follows that

$$|\langle Tf,g\rangle| \leqslant C\|f\|_2\|g\|_2+2\varepsilon\|T\|\,\|f\|_2\|g\|_2,$$

hence

$$\|T\|\leqslant C+2\varepsilon\|T\|,\qquad \|T\|\leqslant\frac{C}{1-2\varepsilon}.$$

This method was successfully applied and further developed by Nazarov, Treil, and Volberg [2002b, 2003]. The latter work was extended to Banach space valued singular integrals with respect to non-doubling measures by Hytönen [2014]. The first arXiv version of this paper was posted already in 2008, and hence it was available to provide the backbone for the proof of the A_2 theorem in Hytönen [2012] (arXiv 2010); see the Notes of Chapter 11 for more on the latter. It was for the purposes of the A_2 theorem that a technical elaboration of the averaging method of Nazarov, Treil, and Volberg [1997a, 2002b, 2003] had to be invented: "on average", the bad part is not only small but completely absent. This allows the replacement of the estimates above by *identities* of the type

$$\langle Tf,g\rangle=\mathbb{E}\langle T^{\omega}_{\text{good}}f,g\rangle.$$

The observation that one can combine this averaging method with Figiel's decomposition of singular integrals in order to simplify the latter, and thereby obtain sharper quantitative conclusions (notably, a quadratic dependence on the UMD constant), was then made in Hytönen [2012] (arXiv 2011), where a version of Theorem 12.3.35 (for scalar kernels and under vanishing paraproduct conditions) was first established. The question of obtaining a linear dependence on the UMD constant is an outstanding open problem already in the special case of the Hilbert transform (see Problem O.6); but of course a possible counterexample could be more feasible within the larger class of operators covered by Theorem 12.3.35. A positive answer has been obtained for sufficiently smooth *even* singular integrals on $L^p(\mathbb{R};X)$ by Pott and Stoica [2014]; their result depends on the same averaging trick and the resulting dyadic representation theorem, but then applies different techniques to complete the estimate.

While our approach to the "random" $T(1)$ Theorem 12.3.35 took a detour via the "non-random" $T(1)$ Theorem 12.3.26, we should emphasise that this is by no means necessary; rather, in many recent extensions of the $T(1)$ theorem, one starts with the randomised set-up from the beginning, and it is often not even clear whether this could be avoided. We will say more about some of these extensions later in these Notes. The reasons that we have chosen to present also the non-random $T(1)$ Theorem 12.3.26 are (at least) two-fold: On the one hand, we feel that there is some historical documentary value in providing (probably) the first detailed exposition of the original Banach space valued $T(1)$ theorem of Figiel [1990], considering also the number of other results in the literature relying on this in their proofs (although, in many cases, one could alternatively apply one or several of the more recent variants). On the other hand, the non-random $T(1)$ Theorem 12.3.26 is not in all respects subsumed by the random $T(1)$ Theorem 12.3.35, which makes the first one applicable in some situations where the latter one is not, and it might hence be useful for the reader to keep the original $T(1)$ Theorem 12.3.26 in their toolbox.

While we are not aware of many such applications, here is at least one: *Pseudo-localisation* of singular integrals refers to estimates of the form

$$\|\mathbf{1}_{\complement \Sigma_{f,s}} Tf\|_{L^p(\mathbb{R}^d;X)} \leqslant \phi(s)\|f\|_{L^p(\mathbb{R}^d;X)}, \qquad s \in \mathbb{N},$$

where

$$\Sigma_{f,s} := \bigcup\{9Q : Q \in \mathscr{D}, \mathbb{D}_Q^{(s)} f \not\equiv 0\}, \quad \mathbb{D}_Q^{(s)} f := \sum_{R \in \mathrm{ch}^s Q} \mathbb{D}_R f,$$

and the point is obtaining a quantitative decay $\phi(s) \to 0$ as $s \to \infty$. Case $p = 2$ was considered by Parcet [2009] for $X = \mathbb{K}$ and by Mei and Parcet [2009] for a Hilbert space X, with applications to non-commutative Calderón–Zygmund and Littlewood–Paley theory, respectively. An extension to $p \in (1, \infty)$ and a UMD space X was obtained by Hytönen [2011] using a version of the $T(1)$ Theorem 12.3.26. This leads to studying a bilinear form whose Haar coefficients satisfy a non-standard estimate of the form

$$|\mathfrak{t}(h_Q^\alpha, h_{Q\dot{+}m}^\gamma)| \lesssim |m|^{-(d+\varepsilon)}\mathbf{1}_{(2\cdot 2^s,\infty)}(|m|) + |m|^{-d}\mathbf{1}_{(4\cdot 2^s-2, 4\cdot 2^s+2)}(|m|).$$

The first term on the right with decay $d + \varepsilon$ is typical, but the second one, without any ε, is not. However, this term is only supported in a relatively narrow region of values of the parameter $m \in \mathbb{Z}^d$, which still allows one to make favourable estimates of the Figiel norms of $\mathfrak{t}$.

A notable aspect of this application is that the construction of the set $\Sigma_{f,s}$ refers to a fixed dyadic system $\mathscr{D}$, which calls for a Haar expansion of the operator in terms of this same $\mathscr{D}$, as in the non-random $T(1)$ Theorem 12.3.26, and seems to prevent any effective application of the random systems $\mathscr{D}^\omega$, as in the random $T(1)$ Theorem 12.3.35. This suggests that, even after the successful recent (and very likely future) development of $T(1)$ theorems and other results based on random dyadic systems, the non-random $T(1)$ Theorem 12.3.26 might not become completely obsolete.

Section 12.4

The classical theory of Calderón and Zygmund [1952] had its focus on convolution operators. Their $L^2(\mathbb{R}^d)$ boundedness is amenable to methods of Fourier analysis, which then serves as a starting point for extrapolation to other $L^p(\mathbb{R}^d)$ and different function spaces, as discussed at length in Chapter 11. It was observed quite early, notably by Coifman and Weiss [1971], that these extrapolation aspects of the theory could be extended to much greater generality, certainly including non-convolution operators on $\mathbb{R}^d$ and much more. On the other hand, the boundedness of some prominent non-convolution operators was obtained by different methods over the years, including the commutators of Calderón [1965, 1977], and the *Cauchy integral on a Lipschitz graph*, which we give in the parametrised form

$$\mathscr{C}_A f(s) := \text{p.v.} \int_{-\infty}^{\infty} \frac{f(t)\,\mathrm{d}t}{s - t + i(A(s) - A(t))}.$$

The boundedness of $\mathscr{C}_A$ was first established, in the case of a small Lipschitz constant $\|A\|_{\mathrm{Lip}}$, by Calderón [1977], and eventually in full generality by Coifman, McIntosh, and Meyer [1982]. However, a general criterion for verifying the $L^2(\mathbb{R}^d)$ boundedness of any given Calderón–Zygmund operators was missing.

The first such general criterion was provided by the "$T(1)$ theorem" of David and Journé [1984]. In its original formulation, this theorem stated that an operator $T : \mathscr{S}(\mathbb{R}^d) \to \mathscr{S}'(\mathbb{R}^d)$, with a Calderón–Zygmund standard kernel, extends to a bounded operator on $L^2(\mathbb{R}^d)$, if and only if it satisfies the following three conditions, from which the name of the theorem (also introduced by David and Journé [1984] in the title of the first section of their paper) is derived:

(i) $T(1) \in \mathrm{BMO}(\mathbb{R}^d)$,
(ii) $T^*(1) \in \mathrm{BMO}(\mathbb{R}^d)$,
(iii) T has the weak boundedness property.

Despite being a complete and elegant characterisation, giving, e.g., the results of Calderón [1977] as a quick corollary, it turned out that it is not always feasible to use this theorem for some operators. As a prime example, the theorem of Coifman, McIntosh, and Meyer [1982] could not be directly recovered by David and Journé [1984], since $\mathscr{C}_A(1)$ does not admit an expression whose BMO norm could be easily estimated.

This shortcoming was fixed by the more general "$T(b)$ theorem" of David, Journé, and Semmes [1985], which replaced (i) and (ii) by the more flexible conditions

(i') $T(b_1) \in \mathrm{BMO}(\mathbb{R}^d)$,
(ii') $T^*(b_2) \in \mathrm{BMO}(\mathbb{R}^d)$,

where one is free to choose the pair of functions $b_i \in L^\infty(\mathbb{R}^d)$ subject only to the restriction that they be *accretive* (meaning $\Re b_i \geqslant \delta > 0$ almost everywhere) or just *para-accretive* (a technical generalisation, for which we refer the interested reader to the original paper). In particular, one can take $b_i = 1 + iA'$ for which the computation of (any finite truncations of) $\mathscr{C}_A(1 + iA')$ is easy.

While also this $T(b)$ theorem has been extended to UMD spaces by Hytönen [2006], the need for this is perhaps not as great as in the scalar-valued case, at least as far as the extension of the boundedness of scalar-valued Calderón–Zygmund operators to $L^p(\mathbb{R}^d; X)$ is concerned. The reason for this is that, while it might be difficult to check the $T(1)$ conditions (i) and (ii) *directly*, they can nevertheless be verified by the converse direction of the $T(1)$ theorem, provided that the $L^2(\mathbb{R}^d)$ boundedness of T is already known by some other method (such as the $T(b)$ theorem). This is, in essence, the point of the scalar-kernel $T(1)$ Theorem 12.4.21.

Corollary 12.5.1. *Let X be a UMD space, $p \in (1, \infty)$, and $A : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then the Cauchy integral on a Lipschitz graph $\mathscr{C}_A$ extends to a bounded operator on $L^2(\mathbb{R}; X)$ and*

$$\|\mathscr{C}_A\|_{\mathscr{L}(L^p(\mathbb{R};X))} \leqslant c_A pp' \cdot \beta_{2,X}^2,$$

where c_A is a constant that depends on A only.

Sketch of proof. By the theorem of Coifman, McIntosh, and Meyer [1982], the operator $\mathscr{C}_A$ is bounded on $L^2(\mathbb{R})$. It is straightforward to verify that the kernel of $\mathscr{C}_A$ is a standard kernel, and hence verifies the assumptions of Theorem 12.4.21 with Dini[1] conditions (and associated constants depending only on A), in which case only trivial type and cotype is needed. Thus Theorem 12.4.21, with $s = p = 2$, proves the corollary for $p = 2$. While we could apply Theorem 12.4.21 with $s = 2$ and any $p \in (1, \infty)$, a better quantitative conclusion for $p \neq 2$ is obtained by using case $p_0 = 2$ as input to the Calderón–Zygmund theorem 11.2.5, which then yields the asserted bound for all $p \in (1, \infty)$. □

Corollary 12.5.1 seems to have been first stated in Hytönen [2006]; however, given that it is essentially a concatenation of its scalar case due to Coifman, McIntosh, and Meyer [1982], and the $T(1)$ theorem of Figiel [1990], it was probably "known to experts" much earlier. The case when X is a UMD lattice was established by a different method already by Rubio de Francia [1986].

In a similar way, the extension of the non-homogeneous $T(1)$ theorem of Nazarov, Treil, and Volberg [2003] to UMD spaces has the following consequence:

Theorem 12.5.2. *Let μ be a positive non-atomic Radon measure on $\mathbb{C}$. Then the following conditions are equivalent*

(1) *There is a constant $c < \infty$ such that, for every disk $D = D(z, r) \subseteq \mathbb{C}$, the measure μ satisfies*
   (a) *the* linear growth condition $\mu(D(z, r)) \leqslant cr$, *and*
   (b) *the* local curvature condition

$$\iiint_{D\times D\times D} \frac{\mathrm{d}\mu(u)\,\mathrm{d}\mu(v)\,\mathrm{d}\mu(z)}{R(u, v, z)} \leqslant c\mu(D),$$

where $R(u,v,z)$ is the radius of the circle through u,v,z (understood as ∞, if the points are collinear).

(2) *The Cauchy integral*

$$\mathscr{C}_\mu f(u) := \int_{\mathbb{C}} \frac{f(v)\,\mathrm{d}\mu(v)}{u-v}$$

defines a bounded operator on $L^2(\mu)$.

(3) *For every UMD space X and every $p \in (1,\infty)$, the Cauchy integral $\mathscr{C}_\mu$ defines a bounded operator on $L^p(\mu;X)$.*

Note that $\mathscr{C}_A$ is (equivalent to) the special case, where μ is the arc-length measure on the graph $\{(t,A(t)) : t \in \mathbb{R}\}$.

Sketch of proof. The implication (2)⇒(1a) is due to David [1991] and (2)⇒(1b) due to Melnikov and Verdera [1995] and Mattila, Melnikov, and Verdera [1996]. The sufficiency of these geometric conditions, (1)⇒(2), was proved by Tolsa [1999].

The implication ((1a) and (2))⇒(3) follows from an analogue of Theorem 12.4.21 for measures on $\mathbb{R}^d$ with the power growth bound $\mu(B(s,r)) \leqslant cr^n$ $(0 < n \leqslant d)$, which is one of the main results of Hytönen [2014]. The implication (3)⇒(2) is trivial. □

This proof sketch highlights the role of $T(1)$ theorems as a device for extending deep results about the boundedness of specific operators from scalar-valued to vector-valued spaces, without the need to revisit the details of the original arguments. Indeed, by using the scalar-valued result (2) as an intermediate step, the equivalence of (1) and (3) is obtained without ever having to deal with the local curvature condition (1b) in the context of vector-valued functions!

Our operator-kernel $T(1)$ Theorem 12.4.12 is the outcome of a line of evolution starting with the first such results obtained by Hytönen and Weis [2006b] and Hytönen [2006], and continued with several variants and extensions addressing

- non-homogeneous measures (Hytönen [2014] (arXiv 2008), Martikainen [2012a] (arXiv 2010), Hytönen and Vähäkangas [2015]);
- simplifications of the underlying decomposition of the operator (Hytönen [2012], Hänninen and Hytönen [2016]);
- sharper conclusions under additional symmetry assumptions (Pott and Stoica [2014], Hytönen [2021]);
- product-space/multiparameter singularities (Di Plinio and Ou [2018], Hytönen, Martikainen, and Vuorinen [2019a]);
- multilinear operators (Di Plinio, Li, Martikainen, and Vuorinen [2020b], Airta, Martikainen, and Vuorinen [2022]).

While these papers extend the theory into several directions that we have not considered here, many of them also provide valuable pieces of insight into the basic case of linear Calderón–Zygmund operators on $\mathbb{R}^d$ with the Lebesgue measure, which we have tried to incorporate into the present treatment. Despite this extensive background material, some aspects of our present $T(1)$ Theorem 12.4.12 appear to be new:

(1) For the first time, we are able to state an operator-valued $T(1)$ theorem that gives a *characterisation* (as in the scalar-valued $T(1)$ theorem of David and Journé [1984]), and not just a *sufficient condition* (as in all operator-valued papers cited above), for the boundedness of a Calderón–Zygmund operator with an operator-valued kernel. This depends on two recent ideas, the combination of which appears here for the first time:
   (a) Replacing the (sufficient but not necessary) weak R-boundedness property of most of the previous contributions by the correct weak $\mathscr{D}\mathscr{R}_p$-boundedness property. As discussed in the Notes of Section 12.1, this idea is from Di Plinio, Li, Martikainen, and Vuorinen [2020b].
   (b) Treating the bi-paraproduct $\Lambda = \Pi_{T(1)} + \Pi^*_{T^*(1)}$ as a single object, and making its boundedness into a condition in its own right, rather than trying (in vain) to force it into a form involving some operator-valued BMO space. This is implicit in Hytönen [2021].

(2) Recording the quantitative dependence of the estimate in terms of both the UMD and the (co)type constants, and optimising the argument for what seems to be the best possible bound currently available. This was available in important special cases (notably in Hytönen [2012]), and arguably implicit in some other works, but seems to be original as an explicit statement in the present generality.

Consequences of the $T(1)$ theorem

The "$T(1)$ theorem for convolution kernels", Corollary 12.4.13, is a somewhat untypical statement, in that convolution kernels have been usually treated by more traditional Fourier-analytic methods, rather than the $T(1)$ technology. As such, this very formulation seems to be new. However, essentially the same class of operators was considered with Fourier methods by Hytönen and Weis [2007]. (Despite the publication year, this paper was actually the first joint project of its authors, which they completed and submitted in 8/2002, before starting their follow-up work on the $T(1)$ theorem, Hytönen and Weis [2006b], later in the same year.) In place of the combinatorial estimates for Figiel's operators from Sections 12.1.b and 12.1.c, this proof employed analogous Fourier-analytic estimates due to Bourgain [1986]. Just like the combinatorial details of the $T(1)$ theorem can be simplified with the random dyadic systems, the proof of the key lemma of Bourgain [1986] was later simplified in Hytönen [2012] by the same technology.

While the direct comparison of Corollary 12.4.13 with the results of Hytönen and Weis [2007] is complicated by the presence in Corollary 12.4.13 of the (untypical in the classical theory) weak boundedness property, Corollary 12.4.19 on antisymmetric kernels comes rather close to some results of Hytönen and Weis [2007]. Indeed, in this special situation, one can completely avoid both paraproducts and the weak boundedness property, obtaining a boundedness criterion in terms of the Calderón–Zygmund kernel bounds alone.

Corollary 12.4.18 on antisymmetric but non-convolution kernels (where the weak boundedness is automatic but a paraproduct is present) is probably new in the operator-valued setting, but a rather straightforward adaptation of similar statements that are well known in the scalar-valued theory.

On minimal smoothness conditions

As one can see from $T(1)$ Theorem 12.4.12 and its corollaries, the minimal smoothness of the kernel involves a modulus of continuity $\|\omega\|_{\mathrm{Dini}^\sigma}$, where $\sigma = \max(1/t, 1/q')$ if X has cotype q and Y has type t, or one of them has both. In the scalar-valued (or more generally Hilbert space) case, this reduces to $\sigma = \frac{1}{2}$. Incidentally, this appears to be the minimal condition required to run any known proof of the $T(1)$ theorem, even in the scalar case. As Figiel [1990] puts it,

> it was a nice surprise that such austere methods could in fact lead to some results which were not less general than their counterparts established earlier with no restrictions on the range of admissible methods.

While the original $T(1)$ theorem of David and Journé [1984] and most of its successors are formulated for Calderón–Zygmund standard kernels, an extension to Dini-type conditions was obtained shortly after by Yabuta [1985], who proved the theorem under the condition that $\|\omega^{\frac{1}{3}}\|_{\mathrm{Dini}} < \infty$. It is not obvious at first sight how this compared to Figiel's condition $\|\omega\|_{\mathrm{Dini}^{\frac{1}{2}}} < \infty$. However, we may observe that any non-decreasing ω on $[0,1]$ satisfies

$$\begin{aligned}
\int_0^1 \omega(t)\Big(\log\frac{1}{t}\Big)^\alpha \frac{\mathrm{d}t}{t} &= \int_0^1 \omega(t)^{\frac{1}{1+\alpha}}\omega(t)^{\frac{\alpha}{1+\alpha}}\Big(\int_t^1 \frac{\mathrm{d}s}{s}\Big)^\alpha \frac{\mathrm{d}t}{t} \\
&\leqslant \int_0^1 \omega(t)^{\frac{1}{1+\alpha}}\Big(\int_t^1 \omega(s)^{\frac{1}{1+\alpha}}\frac{\mathrm{d}s}{s}\Big)^\alpha \frac{\mathrm{d}t}{t} \\
&\leqslant \int_0^1 \omega(t)^{\frac{1}{1+\alpha}}\frac{\mathrm{d}t}{t}\Big(\int_0^1 \omega(s)^{\frac{1}{1+\alpha}}\frac{\mathrm{d}s}{s}\Big)^\alpha \\
&= \Big(\int_0^1 \omega(s)^{\frac{1}{1+\alpha}}\frac{\mathrm{d}s}{s}\Big)^{1+\alpha}.
\end{aligned}$$

With $\frac{1}{1+\alpha} = \frac{1}{3}$, we see that Yabuta's $\|\omega^{\frac{1}{3}}\|_{\mathrm{Dini}}$ dominates $\|\omega\|_{\mathrm{Dini}^\alpha}$ with $\alpha = 2$. (While the Dini^s norms were previously defined with $\log_2$ in place of log, and integrating over $[0,\frac{1}{2}]$ instead of $[0,1]$, the reader may easily verify that, extending ω from $[0,\frac{1}{2}]$ to $[0,1]$ by $\omega(t) := \omega(\min(t,\frac{1}{2}))$, these details affect at most the constants in the final conclusions.)

Subsequently, Meyer [1986] (according to Han and Hofmann [1993], but we have not been able to verify the original reference) relaxed the assumption to $\alpha = 1$ (plus a further weakening of the pointwise bounds to integral conditions rather closer to the Figiel conditions for bilinear forms as in our abstract $T(1)$ Theorems 12.3.26 and 12.3.35). Han and Hofmann [1993] obtained a further slight relaxation of the conditions of Meyer [1986], and Yang, Yan, and Deng [1997] proved the $T(1)$ theorem with assumptions essentially matching the special case $\alpha = \frac{1}{2}$ of the conditions of Figiel [1990] in the scalar-case. Later attempts to relax this condition were made by Grau de la Herrán and Hytönen [2018], who found that the same regularity is sufficient also for the non-homogeneous $T(1)$ theorem, but did not succeed in relaxing it even in the standard case. Thus, various different proof strategies all seem to meet this same threshold.

At the same time, it seems to remain unknown whether even the much weaker Hörmander conditions of Definition 11.2.1 could in principle be enough for a $T(1)$ theorem. A positive result in this direction seems out of reach with the presently available methods, but there does not seem to be any definitive counterexample to rule out this possibility. As very partial evidence for a counterexample, Yang, Yan, and Deng [1997] show that the $T(1)$ conditions for a Hörmander kernel are insufficient to guarantee the boundedness in some end-point spaces.

The dyadic representation theorem

A dyadic representation formula resembling Theorem 12.4.27 was first obtained by Hytönen [2012] as a key component of the original proof of the A_2 Theorem 11.3.26 for all standard Calderón–Zygmund operators in the scalar-valued case. Subsequent refinements and simplifications of the original representation were obtained by Hytönen, Pérez, Treil, and Volberg [2014], and Hytönen [2017]. The first version of both Theorems 12.4.26 and 12.4.27 for dyadic shifts and singular integrals on $L^p(\mathbb{R}^d; X)$ with operator-valued kernels were obtained by Hänninen and Hytönen [2016], by essentially the same techniques (notably, the tangent martingale estimates of Corollary 4.4.15) that we have followed. In all these contributions, like several other contemporary ones, the notion of dyadic shift was essentially that of Hytönen [2012], which is somewhat different from the present Definition 12.4.24. In the shifts of Hytönen [2012], the components A_K take the form

$$A_P f = \sum_{\substack{Q \in \mathrm{ch}^{(i)}(P) \\ R \in \mathrm{ch}^{(j)}(P)}} \alpha^P_{Q,R} \langle f, h^\alpha_Q \rangle h^\gamma_R,$$

with two independent complexity parameters $(i, j) \in \mathbb{N}^2$ in place of the single $k \geqslant 2$ in Theorem 12.4.27. The "new shifts" of Definition 12.4.24 were first introduced by Grau de la Herrán and Hytönen [2018]. Their Banach space valued theory, including Theorems 12.4.26 and 12.4.27 in essentially their present form, as well as multilinear extensions, has been developed by Airta, Martikainen, and Vuorinen [2022].

As far as proving the $T(1)$ theorem for Calderón–Zygmund operators on $L^p(\mathbb{R}^d;X)$ is concerned, the advantages of the Dyadic Representation Theorem 12.4.27 over (the randomised version of) Figiel's representation may be considered a question of mathematical taste (depending, among other things, on one's preference for the tangent martingales methods of Section 4.4 over the dyadic singular integrals of Section 12.1 or vice versa). However, these advantages become prominent in extensions of the $T(1)$ theory to other situations that we have not treated here. Roughly speaking, the decomposition of Figiel is essentially based on multi-scale versions of *translations*—reasonably well-behaved objects as far as translation-invariant spaces like $L^p(\mathbb{R}^d;X)$ are concerned, but somewhat unstable in more general situations. In contrast, the basic building block A_K of the dyadic shifts are essentially *averages*, which are much more stable operations. In particular, the averages $f \mapsto \mathbf{1}_Q\langle f\rangle_Q$ over arbitrary cubes $Q \subseteq \mathbb{R}^d$ are uniformly bounded on $L^p(w)$ if and only if $w \in A_p$, which partially explains the usefulness of such objects in the original context of proving the A_2 theorem. Averages are somewhat well-behaved even when taken with respect to non-doubling measures, which is the context in which a certain precursor of the dyadic representation of Hytönen [2012] (arXiv 2010) was established by Hytönen [2014] (arXiv 2008) in order to extend the non-homogeneous $T(1)$ theorem of Nazarov, Treil, and Volberg [2003] to the Banach space valued setting. Conversely, after the discovery of the Dyadic Representation Theorem, it was used by Volberg [2015] to give a new proof of the non-homogeneous $T(1)$ theorem.

An adaptation of the Dyadic Representation Theorem 12.4.27, by Hytönen, Li, H., and Vuorinen [2022], was instrumental in extending the $T(1)$ theory to singular integral operators adapted to so-called Zygmund dilations $(x_1,x_2,x_3) \mapsto (sx_1,tx_2,stx_3)$, where $s,t > 0$ are two independent parameters. Variants of the Dyadic Representation Theorem 12.4.27, with the Haar functions replaced by smoother wavelets, have been explored by Hytönen and Lappas [2022], Di Plinio, Wick, and Williams [2023c], and Di Plinio, Green, and Wick [2023b,a].

$T(1)$ theorems on other function spaces

The original $T(1)$ theorem of David and Journé [1984] was a characterisation of boundedness on $L^2(\mathbb{R}^d)$, while we have dealt with extensions of such results to $L^p(\mathbb{R}^d;X)$. However, the boundedness of a given (singular integral) operator is basic question arising in several other function spaces as well, and the $T(1)$ theorem has served as a model for similar results in other spaces. (See Chapter 14 for information about the functions spaces appearing in this discussion.) Extensions of the $T(1)$ theorems to Besov spaces $\dot{B}^s_{p,q}$ were obtained by Lemarié [1985] and to Triebel–Lizorkin spaces $\dot{B}^s_{p,q}$ and $\dot{F}^s_{p,q}$ by Frazier, Han, Jawerth, and Weiss [1989]. In these results, $p,q \in [1,\infty]$, and the smoothness parameter s was restricted by the Hölder exponent of the standard kernel of T. In order to cover a broader range of Besov and Triebel–Lizorkin spaces, where the smoothness index can take any value $s \in \mathbb{R}$, it is necessary to consider higher order Calderón–Zygmund estimates such as

$$|\partial^\alpha K(s,t)| \leqslant C|s-t|^{-d-|\alpha|}.$$

With appropriate assumptions of this type in place, Frazier, Torres, and Weiss [1988] and Torres [1991] obtained $T(1)$ criteria for the boundedness of Calderón–Zygmund operators on any Triebel–Lizorkin space $\dot{F}^s_{p,q}$, where $s \in \mathbb{R}$ and $p, q \in (0, \infty]$. The precise assumptions are necessarily somewhat technical, and the result splits into three cases, where $s < 0$; or $s \geqslant 0$ and $p, q \in [1, \infty]$, or $s \geqslant 0$ and $\min(p, q) \in (0, 1)$.

In a limited range of s again, $T(1)$ theorems on (scalar-)weighted Triebel–Lizorkin spaces $\dot{F}^s_{p,q}(w)$ were obtained by Han and Hofmann [1993], and on matrix-weighted Besov spaces $\dot{B}^s_{p,q}(W)$ by Roudenko [2003]. The full scale of both matrix-weighted Besov and Triebel–Lizorkin spaces $\dot{B}^s_{p,q}(W)$ and $\dot{F}^s_{p,q}(W)$ (as well as further generalisations with a fourth index) was covered by Bu, Hytönen, Yang, and Yuan [2023]. When restricted to the unweighted case, this last work even slightly simplifies the assumptions of Frazier, Torres, and Weiss [1988] and Torres [1991].

In all these mentioned works on $T(1)$ theorems beyond L^p spaces, the focus has been on special $T(1)$ theorems providing sufficient conditions for boundedness under vanishing paraproduct assumptions. General $T(1)$ theorems, providing a characterisation of boundedness on a given space, were obtained on Besov spaces $\dot{B}^s_{p,q}$ of positive smoothness $s > 0$ by Youssfi [1989], in terms of the weak boundedness property and the boundedness of higher order paraproducts. A far-reaching extension to Triebel–Lizorkin and other function spaces, including versions on quite general domains $\mathcal{O} \subseteq \mathbb{R}^d$, is due to Di Plinio, Green, and Wick [2023a].

For Banach space valued functions, special $T(1)$ theorems (i.e., with vanishing paraproduct assumptions) on Riesz potential spaces $\dot{H}^{s,p}(\mathbb{R}^d; X)$ and Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^d; X)$ were proved by Kaiser [2007, 2009], respectively. The results in $\dot{H}^{s,p}(\mathbb{R}^d; X)$ need the UMD property of X, but those in $\dot{B}^s_{p,q}(\mathbb{R}^d; X)$ do not. While we are not going to discuss these specific results in any further detail, the reader can witness a similar dichotomy—that the UMD property is needed to obtain results in certain function spaces, but not for analogous results in certain others—in our discussion of the theory of Banach space valued function spaces in Chapter 14.