Van Kampen's Theorem and Fundamental Groupoids

by

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SUMMARY

In this thesis we will look at Van Kampen's Theorem, a classical result in the field of algebraic topology. The theorem gives a way to calculate fundamental groups of a topological space from the fundamental groups of a set of covering subspaces.

In chapter 1 we define the fundamental group, which is a topological property, mainly used for distinguishing topological spaces. The fundamental group of a topological space is defined by specifying a point in the space, the base point and considering all loops in the space at this point. We say two loops are homotopic if we can continuously transform one path into the other. The set of all homotopy classes of these loops will then form the fundamental group. The fundamental group is often explained as a tool to count holes in a topological space. This is because a loop around a hole is not homotopic to the constant path.

In chapter 2 we turn our attention to covering spaces. A particular well-behaving type of covering map we will encounter, which we call a G-covering, acts as a quotient map for a group action. Via these G-coverings we will see that covering space have a deep relation with fundamental groups. We will see that we can classify G-coverings by relating them to group homomorphisms from the fundamental group. As an example of this, we will construct all $(\mathbb{Z}/n\mathbb{Z})$-coverings of the circle.

We will use the classification of G-coverings to prove Van Kampen's Theorem in Chapter 3. The proof of the theorem boils down to proving the existence of a unique group homomorphism from given group homomorphisms. We will link these given group homomorphism to G-coverings, which we can then glue together. The result is a G-covering that is linked to the unique group homomorphism we are looking for. As an example of gluing, we will construct the torus as a covering space of the Klein bottle.

This version of Van Kampen's Theorem is unfortunately not able to calculate the fundamental group of the circle. The choice of just one base point lies in the way. To be able to calculate this basic example we need a generalisation of the fundamental group. A big part of this thesis is about this generalisation, the so-called fundamental groupoid, which we will define in Chapter 4. In this definition, we lose the requirement to choose one base point and we consider paths between an arbitrary number of points. The structure we get is not a group, but a groupoid, which is a term from category theory. We will therefore also spend some time giving a precise definition of the categorical terms we need.

We are able to upgrade Van Kampen's Theorem to calculate fundamental groupoids and we prove this generalisation in Chapter 5. This is not done via the theory of covering spaces and in Chapter 6 we will explore the idea of proving Van Kampen's Theorem for fundamental groupoids using covering spaces.

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The Fundamental Group

We begin with studying the fundamental group. This notion was first introduced by Poincaré in his *Analysis Situs* in 1895 [12]. In very basic terms, the fundamental group measures the number of "holes" a topological has. The plane $\mathbb{R}^2$ and the punctured plane $\mathbb{R}^2 \setminus (0,0)$ should intuitively not be considered the same; the punctured plane has a hole. However, many topological properties cannot distinguish the two spaces. With the help of paths we can capture this hole. In the plane any loop, which is a path that starts and ends at the same point, can be continuously deformed to a single point. A loop around the origin in the punctured plane cannot be deformed to a single point, as there is a "hole" in the middle. This leads us to the definition of the fundamental group. We will define it as the group consisting of all (homotopy classes) of loops at a base point.

1.1. Topological Spaces

To be able to introduce the fundamental group, we first have to define topological spaces. These can be interpreted as a generalisation of metric spaces, where we can measure "closeness", but we do not necessarily need a definition of distance. The only structure provided are the open sets.

A *topological space* is a pair (X, τ), where X is some set and $\tau \subseteq \mathscr{P}(X)$ is a family of subsets, called the *topology*, that satisfies the following:

- $\emptyset, X \in \tau$
- An arbitrary union of sets in τ is again in τ: If $\sigma \subseteq \tau$, then $\bigcup \sigma \in \tau$.
- A finite intersection of sets in τ is again in τ: If $A_1, \dots, A_n \in \tau$, then $\bigcap_{i=1}^{n} A_i \in \tau$.

The elements of τ are called *open* and complements of open sets are called *closed*. We call any open set containing the point $x \in X$ an open neighbourhood of x. If we specify a basepoint $x \in X$, we call the pair (X, x) a pointed topological space. We often omit τ and call X a topological space, if τ is clear from context, or specification is not necessary.

Example 1.1.1. All metric spaces are topological spaces by choosing τ as the collection of open sets according to the definition of open for metric spaces. In case one is not

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familiar with the notion of topological spaces, it suffices to think of metric spaces in the rest of this report.

It is often more useful to give a smaller family of open sets that generates all open sets in some way.

Definition 1.1.2. A *base* of a topological space (X,τ) is a subfamily $\mathscr{B} \subseteq \tau$, such that every open set of X is a union of elements of $\mathscr{B}$.

Example 1.1.3. The collection $\mathscr{B} = \{B_d(x,r) : x \in X, r > 0\}$ of open balls is a base for the metric space (X,d). In particular, the collection of open intervals in $\mathbb{R}$ is a base of the standard topology on $\mathbb{R}$.

We can naturally define topologies on new spaces that arise from old ones. We will show some of them here. Unless otherwise stated we will assume these topologies on the new spaces.

Let (X,τ) be a topological space and $Y \subseteq X$ a subset. Then $\tau_Y = \{O \cap Y : O \in \tau\}$ is a topology on Y, which is called the *subspace topology*. The pair (Y,τ_Y) is called a *subspace* of (X,τ).

Let (X,τ_X) and (Y,τ_Y) be two topological spaces. Then the *product topology* on $X \times Y$ is the topology generated by the base $\mathscr{B} = \{U \times V : U \in \tau_X, V \in \tau_Y\}$. This definition can be extended to a topology on an arbitrary Cartesian product of topological spaces.

Let (X,τ_X) be a topological space. Let $\sim$ be an equivalence relation on X and let $Q = X/\sim$ be the set of equivalence classes, with $q : X \to Q$ the quotient map, that sends each element to its equivalence class. Then $\tau_Q = \{U \subseteq Q : q^{-1}(U) \text{ is open in } X\}$ is a topology on Q called the *quotient topology*.

1.2. CONTINUOUS MAPS

In different settings we are concerned with different types of maps. When working with groups we prefer working with only group homomorphisms. In vector spaces we use linear maps. When working with topological spaces, just like metric spaces, we are mainly interested in continuous maps.

Definition 1.2.1. Let (X,τ_X) and (Y,τ_Y) be topological spaces. A map $f : X \to Y$ is *continuous* if the pre-image of every open set in Y is open in X. That is, for all $O \in \tau_Y$, we have $f^{-1}(O) \in \tau_X$.

For a map $f : (X,x) \to (Y,y)$ between pointed topological spaces we require that $f(x) = y$.

For metric spaces, this definition coincides with the ε-δ definition.

If a continuous map $f : X \to Y$ has a continuous inverse $f^{-1} : Y \to X$, we call f a *homeomorphism*. If a homeomorphism exists between X and Y we call the spaces *homeomorphic*. In topology we consider homeomorphic spaces to be "the same" and we are thus often looking at properties of spaces that are invariant under homeomorphisms. Topology is sometimes called "rubber-sheet geometry", as you can interpret a homeomorphism as stretching a rubber object, where you are not allowed to tear the object[1].

[1] Famously a coffee mug is homeomorphic to a donut.

The identity map $\mathrm{id} : X \to X$ that sends every element to itself is continuous and in fact a homeomorphism. The composition of two continuous maps is again continuous and the composition of two homeomorphisms is again a homeomorphism. From this we conclude that "being homeomorphic" forms an equivalence relation on the class of topological spaces.

Just like we can continuously transform topological spaces into each other using homeomorphisms, we can also continuously transform continuous maps into each other, using a homotopy.

Definition 1.2.2. Let X and Y be topological spaces and $f, g : X \to Y$ be two continuous maps. A *homotopy* from f to g is a continuous map $F : [0,1] \times X \to Y$, such that $F(0,x) = f(x)$ and $F(1,x) = g(x)$.

If there exists a homotopy from f to g, we call f and g *homotopic* and write $f \simeq g$.

Through these homotopies we can also compare two spaces in a weaker sense than the homeomorphism.

Definition 1.2.3. Let X and Y be two topological spaces. A *homotopy equivalence* between X and Y is a continuous map $f : X \to Y$, such that there exists a continuous map $g : Y \to X$, with $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. If such a homotopy equivalence exists, we call X and Y *homotopy equivalent.*

From the definition we can immediately see that any continuous map is homotopic to itself. This yields that any two homeomorphic spaces are automatically homotopy equivalent, as any homeomorphism f is a homotopy equivalence, with $g = f^{-1}$.

Let $f, g, h : X \to Y$ be continuous maps and suppose F and G are homotopies from resp. f to g and g to h. Then

$$F^{\leftarrow} : [0,1] \times X \to Y, \quad \text{given by } (t,x) \mapsto F(1-t,x)$$

is a homotopy from g to f and

$$F * G : [0,1] \times X \to Y, \quad \text{given by } (t,x) \mapsto \begin{cases} F(2t,x), & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ G(2t-1,x), & \text{for } \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

is a homotopy from f to h. This means "being homotopic" is an equivalence relation on the set of continuous maps from X to Y.

In similar fashion we see that "being homotopy equivalent" is an equivalence relation on the class of topological spaces.

Example 1.2.4. We call any topological space that is homotopy equivalent to the space consisting of a single point[2] *contractible.*

All $\mathbb{R}^n$ are contractible: Let $f : \mathbb{R}^n \to \{*\}$ be given by $\underline{x} \mapsto *$ and $g : \{*\} \to \mathbb{R}^n$ be given by $* \mapsto \underline{0} = (0,0,...,0)$. Then $f \circ g = id_{\{*\}}$, so we only need to show $g \circ f : \mathbb{R}^n \to \mathbb{R}^n$, given

[2] We can talk about *the* space consisting of a single point, as all spaces consisting of exactly one point are homeomorphic.

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by $\underline{x} \mapsto \underline{0}$ is homotopic to the identity on $\mathbb{R}^n$. To see this let $F : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$ be given by

$$(t, \underline{x}) \mapsto (1-t)\underline{x} = ((1-t)x_1, (1-t)x_2, ..., (1-t)x_n).$$

This is a continuous map and $F(0, \underline{x}) = \underline{x} = \mathrm{id}_{\mathbb{R}^n}(\underline{x})$ and $F(1, \underline{x}) = \underline{0} = (g \circ f)(\underline{x})$. We see that this is a homotopy from $g \circ f$ to the identity on $\mathbb{R}^n$ and we conclude $\mathbb{R}^n$ is contractible.

From this we can also conclude that the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are homotopy equivalent for $n, m \geqslant 1$.

The spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are only homeomorphic if $n = m$ (Fulton [5], Section 23c).

Example 1.2.5. We define the n-sphere S^n as the points in $\mathbb{R}^{n+1}$ with norm 1. Then S^n and S^m are homotopy equivalent if and only if $n = m$ (Fulton [5], Section 23c).

Special types of continuous maps we are particularly interested in and that will help us construct the fundamental group are paths. In topological spaces that are somewhat picturable this definition coincides with what one would expect to be a path.

Definition 1.2.6.
1. A *path* in X is a continuous map $\gamma : [0,1] \to X$. The points $x = \gamma(0)$ and $y = \gamma(1)$ are respectively called the *starting* and *ending points* and we call γ a path from x to y.
2. The *reversal* of a path $\gamma : [0,1] \to X$, is the path $\gamma^{-1} : [0,1] \to X$, given by $t \mapsto \gamma(1-t)$.
3. A path from x to itself is called a *loop* at x and x is called the *base point* of the path.
4. The loop $c_x : [0,1] \to X$, given by $t \mapsto x$ is called the *constant loop*.

We can interpret $[0,1]$ as a time interval. Then $\gamma(t)$ gives the point in space where the path is at time t.

The *concatenation* of two paths γ_1 and γ_2 is defined, if $\gamma_1(1) = \gamma_2(0)$, as

$$\gamma_1 \odot \gamma_2 : [0,1] \to X$$
$$t \mapsto \begin{cases} \gamma_1(2t), & \text{for } 0 \leqslant t \leqslant \frac{1}{2}, \\ \gamma_2(2t-1), & \text{for } \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

which is again a path.

Let γ_1 and γ_2 be two paths from x to y. A *path homotopy* from γ_1 to γ_2 is a homotopy $\Gamma : [0,1]^2 \to X$, where we also require $\Gamma(t, 0) = x$ and $\Gamma(t, 1) = y$ for all $t \in [0,1]$. This implies that for all fixed $t \in [0,1]$, $s \mapsto \Gamma(t, s)$ is a path from x to y. If such a path homotopy exists we call γ_1 and γ_2 *path homotopic* and again write $\gamma_1 \simeq \gamma_2$. This is an equivalence relation on the set of paths from x to y.

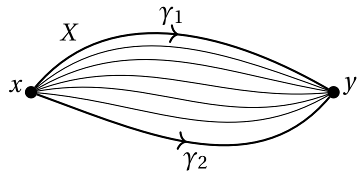

Figure 1.2.1: A path homotopy from γ_1 to γ_2 drawn for some values of t.

1.3. THE FUNDAMENTAL GROUP

We write $P(X;x,y)$ for the set of all paths in X from x to y. For the set of all loops at x we write $P(X;x)$ instead of $P(X;x,x)$. We can now take the quotient of the equivalence relation $\simeq$ on this set, which results in the fundamental group.

Theorem 1.3.1. *Let (X,x) be a pointed topological space. $P(X;x)/\simeq$ is a group under the operation $\cdot$, defined by $[\gamma]\cdot[\gamma'] = [\gamma\odot\gamma']$. The identity element is the equivalence class of the constant loop c_x. The inverse of $[\gamma]$ is given by $[\gamma]^{-1} = [\gamma^{-1}]$, the equivalence class of the reversal of γ.*

Proof. We first confirm that the operation $\cdot$ is well-defined.

Let $\gamma_1,\gamma_2,\gamma_1',\gamma_2' \in P(X;x)$ and suppose that $[\gamma_1] = [\gamma_1']$ and $[\gamma_2] = [\gamma_2']$. We want to prove that $[\gamma_1\odot\gamma_2] = [\gamma_1'\odot\gamma_2']$. Let Γ_1 be a path homotopy from γ_1 to γ_1' and let Γ_2 be a path homotopy from γ_2 to γ_2'.

We define Δ as follows:

$$\Delta : [0,1]\times[0,1] \to X$$
$$(t,s) \mapsto \begin{cases} \Gamma_1(t,2s), & \text{for } 0 \leqslant s \leqslant \frac{1}{2},\\ \Gamma_2(t,2s-1), & \text{for } \frac{1}{2} \leqslant s \leqslant 1.\end{cases}$$

Then $\Delta(0,s) = (\gamma_1\odot\gamma_2)(s)$ and $\Delta(1,s) = (\gamma_1'\odot\gamma_2')(s)$. Furthermore $\Delta(t,0) = \Gamma_1(t,0) = x$ and $\Delta(t,1) = \Gamma(t,1) = x$. As Δ is continuous, we find that Δ is a path homotopy from $\gamma_1\odot\gamma_2$ to $\gamma_1'\odot\gamma_2'$ and in particular $[\gamma_1\odot\gamma_2] = [\gamma_1'\odot\gamma_2']$ as we wanted.

We now prove the group axioms for $P(X;x)/\simeq$ under this operation.

Let $\gamma_1 \in P(X;x)$. To prove neutrality of $[c_x]$ we need to construct path homotopies Γ_1 from $\gamma_1\odot c_x$ to γ_1 and Γ_2 from $c_x\odot\gamma_1$ to γ_1.

To prove the existence of the inverse element, we need to construct path homotopies Γ_3 from $\gamma_1\odot\gamma_1^{-1}$ to c_x and Γ_4 from $\gamma_1^{-1}\odot\gamma_1$ to c_x.

Now let $\gamma_2,\gamma_3 \in P(X;x)$ be loops at x as well. To prove associativity we need to construct a path homotopy Γ_5 from $(\gamma_1\odot\gamma_2)\odot\gamma_3$ to $\gamma_1\odot(\gamma_2\odot\gamma_3)$.

We give five maps below and we claim they are path homotopies as specified above. This can be confirmed by carefully checking the requirements.

These prove all group axioms, which concludes our proof.

$$\Gamma_1 : [0,1]\times[0,1] \to X$$
$$(t,s) \mapsto \begin{cases} \gamma_1(\frac{2s}{1+t}), & \text{for } 0 \leqslant s \leqslant \frac{1+t}{2},\\ x, & \text{for } \frac{1+t}{2} \leqslant s \leqslant 1.\end{cases}$$

$$\Gamma_2 : [0,1]\times[0,1] \to X$$
$$(t,s) \mapsto \begin{cases} x, & \text{for } 0 \leqslant s \leqslant \frac{1-t}{2},\\ \gamma_1(\frac{s-(1-t)/2}{1-(1-t)/2}), & \text{for } \frac{1-t}{2} \leqslant s \leqslant 1.\end{cases}$$

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$$\Gamma_3 : [0,1] \times [0,1] \to X$$
$$(t,s) \mapsto \begin{cases} \gamma_1(2s(1-t)), & \text{for } 0 \leqslant s \leqslant \frac{1}{2}, \\ \gamma_1(2(1-s)(1-t)), & \text{for } \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

$$\Gamma_4 : [0,1] \times [0,1] \to X$$
$$(t,s) \mapsto \begin{cases} \gamma_1((1-2s)(1-t)), & \text{for } 0 \leqslant s \leqslant \frac{1}{2}, \\ \gamma_1((2s-1)(1-t)), & \text{for } \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

$$\Gamma_5 : [0,1] \times [0,1] \to X$$
$$(s,t) \mapsto \begin{cases} \gamma_1(\frac{4s}{t+1})), & \text{for } 0 \leqslant s \leqslant \frac{t+1}{4}, \\ \gamma_2(4s-1-t), & \text{for } \frac{t+1}{4} \leqslant s \leqslant \frac{t+2}{4}, \\ \gamma_3(\frac{4s-2-t}{2-t}), & \text{for } \frac{t+2}{4} \leqslant s \leqslant 1. \end{cases}$$

□

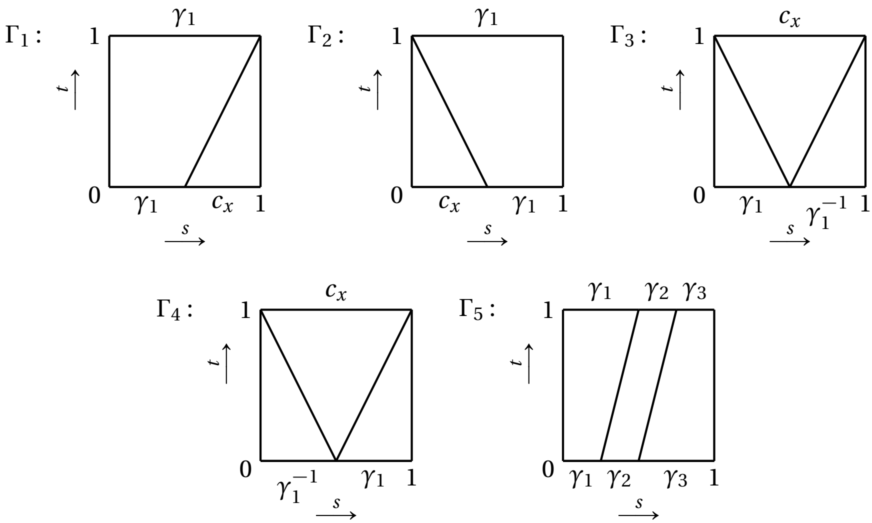


Figure 1.3.1: Visualisation in $[0,1]^2$ of the path homotopies used in the proof of Theorem 1.3.1.

Definition 1.3.2. We write $\pi_1(X,x) = P(X;x)/\simeq$ and call this the *fundamental group*.

The fundamental group is thus the set of all homotopy classes of loops in X at x. We can concatenate these loops, which gives it the structure of a group. The fundamental group would not be interesting if it were not invariant under homeomorphism. Luckily it is, and it is even invariant under homotopy equivalence. To prove this, we first have to show that a continuous map between topological spaces induces a map of fundamental groups.

Proposition 1.3.3. *Let (X,x) and (Y,y) be pointed topological spaces and let $f:(X.x)\to(Y,y)$ be a continuous map.*

(a) There is unique group homomorphism $f_*:\pi_1(X,x)\to\pi_1(Y,y)$, such that for all $\gamma\in P(X;x)$ we have $f_*([\gamma])=[f\circ\gamma]$.
(b) Let $g:(Y,y)\to(Z,z)$ be some other continuous map between pointed topological spaces. Then $g_*\circ f_*=(g\circ f)_*$ as group homomorphisms.

We will see a proof of this proposition in Theorem 4.4.2 in slightly different language. The induced group homomorphism f_* does not necessarily preserve injections or surjections. In particular an inclusion of topological spaces need not induce an inclusion of fundamental groups. We will see examples for this later.

We can now state and prove our main theorem, but we first need two lemmas, that can be found in (Hatcher [8], Section 1.1).

Lemma 1.3.4. *Let X be a topological space and $x,y\in X$ such that there exists a path δ from x to y. Then $\beta_\delta:\pi_1(X,x)\xrightarrow{\sim}\pi_1(X,y)$ given by $[\gamma]\mapsto[\delta^{-1}\circ\gamma\circ\delta]$ is an isomorphism.*

Proof. We firstly see that β_δ is a group homomorphism:

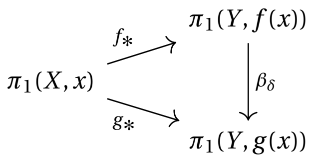
$$\begin{aligned}\beta_\delta[\gamma_1\odot\gamma_2]&=[\delta^{-1}\odot\gamma_1\odot\gamma_2\odot\delta]\\&=[\delta^{-1}\odot\gamma_1\odot\delta\odot\delta^{-1}\odot\gamma_2\odot\delta]\\&=[\delta^{-1}\odot\gamma_1\odot\delta]\cdot[\delta^{-1}\odot\gamma_2\odot\delta]\\&=\beta_\delta[\gamma_1]\cdot\beta_\delta[\gamma_2].\end{aligned}$$

Now the group homomorphism $\beta_{\delta^{-1}}$ gives an inverse:

$$\begin{aligned}(\beta_{\delta^{-1}}\circ\beta_\delta)[\gamma]&=\beta_{\delta^{-1}}[\delta^{-1}\odot\gamma\odot\delta]\\&=[\delta\odot\delta^{-1}\odot\gamma\odot\delta\odot\delta^{-1}]=[\gamma]\end{aligned}$$

and similarly $\beta_\delta\beta_{\delta^{-1}}[\gamma]=[\gamma]$. This proves our claim. □

Lemma 1.3.5. *Let $f,g:X\to Y$ be continuous maps. Let $F:[0,1]\times X\to Y$ be a homotopy from f to g and let $x\in X$. Let δ be the path in Y defined by $\delta(t)=F(t,x)$. Then the diagram*



commutes[3].

[3]Note that a commuting diagram means that following all paths in the diagram with the same starting and ending points result in the same map.

Proof. Write $F_t(y) = F(t, y)$ for fixed t as function from X to Y. Let δ_t be the restriction of δ to the interval $[0, t]$, $\delta_t(s) = \delta(ts)$.

Let γ be a loop at x. Then $\delta_t^{-1} \odot (F_t(\gamma)) \odot \delta_t$ is homotopic to $\delta_s^{-1} \odot (F_s(\gamma)) \odot \delta_s$ for $0 \leqslant s, t \leqslant 1$. The desired equality now follows from the case $s = 0$ and $t = 1$. □

Theorem 1.3.6. *Let X and Y be topological spaces and let $f : X \to Y$ be a homotopy equivalence. Then the induced group homomorphism $f_* : \pi_1(X, x) \xrightarrow{\sim} \pi_1(Y, f(y))$ is a group isomorphism.*

Proof. Let $g : Y \to X$ a continuous map, such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.

Now consider the maps

$$\pi_1(X, x) \xrightarrow{f_*} \pi_1(Y, f(x)) \xrightarrow{g_*} \pi_1(X, g(f(x))) \xrightarrow{f_*} \pi_1(Y, f(g(f(x))))$$

We have $g \circ f \simeq \mathrm{id}_X$, so there is a homotopy $F : X \to X$ from id_X to $g \circ f$. Now we see from Lemma 1.3.5 that for $\delta = F(t, x)$ we have $\beta_\delta \circ \mathrm{id}_x = \beta_\delta = g_* \circ f_*$. This means the composition of the first two maps is an isomorphism, by Lemma 1.3.4. In particular, we see that f_* is injective. Similar reasoning with the second and third maps show that g_* is injective.

We now use that $g_* \circ f_*$ is an isomorphism again, together with the fact that g_* is injective, to see that f_* is surjective. We now have that f_* is injective and surjective, so it is an isomorphism. □

As two homeomorphic spaces are also homotopy equivalent, the following result is obvious.

Corollary 1.3.7. *Let (X, x) and (Y, y) be homeomorphic pointed topological spaces and let $f : X \to Y$ be a homeomorphism with $f(x) = y$. Then $f_* : \pi_1(X, x) \xrightarrow{\sim} \pi_1(Y, y)$ is an isomorphism.*

The contrapositive is such an important statement that we will state it here for emphasis.

> Let (X, x) and (Y, y) be pointed topological spaces with non-isomorphic fundamental groups. Then (X, x) and (Y, y) are not homotopy equivalent, nor homeomorphic.

This statement is the main reason we are interested in calculating fundamental groups of topological spaces. Proving that two spaces are homeomorphic or homotopy equivalent usually consists of constructing an explicit homeomorphism or homotopy equivalence. Showing that two spaces are not homeomorphic or not homotopy equivalent would require excluding that any map can be a homeomorphism or homotopy equivalence, which is often much harder. By calculating the fundamental group we can draw this conclusion quite easily. Note that the converse of this statement is not true in general. We cannot conclude that two spaces are homeomorphic or homotopy equivalent by them having isomorphic fundamental groups.

1.4. PROPERTIES OF TOPOLOGICAL SPACES

We can use Theorem 1.3.6 to calculate the fundamental groups of some spaces. First of all note that the fundamental group of the space consisting of just one point is trivial, since the constant path is the only path in the space. We call any space with trivial fundamental group[4] *simply connected.*

Corollary 1.4.1. *Any contractible space is simply connected.*

This follows directly from the definition of a contractible space and Theorem 1.3.6. We saw in Example 1.2.4 that every $\mathbb{R}^n$ is contractible, so we conclude that $\pi_1(\mathbb{R}^n, \underline{0}) \cong \{e\}$, where e is the identity element of a group. We will see more exotic examples of fundamental groups in the next chapter.

Another property we can classify topological spaces by is (path-)connectedness, in a global and local setting. We usually want to work with spaces that have as many of these properties, as they tend to behave well.

Definition 1.4.2. A topological space X is *connected* if there are exactly two subsets of X that are both open and closed[5].

A non-empty topological space X is *path-connected* if for any two points $x, y \in X$ there exists a path in X from x to y.

If a space is not (path-)connected, but it consists of (path-)connected subsets, it is interesting to look at those. We define the *(path-)connected components* of a topological space to be the maximal subsets that are (path-)connected.

We can also look at these connected properties locally.

Definition 1.4.3. A topological space X is *locally (path-)connected* if it has a base consisting of (path-)connected subsets.

Any space that is (locally) path connected is automatically (locally) connected. The converse statement is not necessarily true. (Path-)connectedness does not imply locally (path-)connectedness. A connected and locally path-connected space is automatically path-connected. The quotient of a (path-)connected space is again (path-)connected. For proofs of the true and counterexamples of the false statements, see (Croom [2], Chapter 5) and (Engelking [4], Section 6.3).

We can also look at the simply connected property locally.

Definition 1.4.4. Suppose X is a connected and locally path-connected topological space. Then X is *semi-locally simply connected* if every point $x \in X$ has a open set U containing x, such that every loop at x completely contained in U is path homotopic in X with the constant loop at x.

[4] That is, every path is homotopic to the constant path.

[5] In a nonempty space X the subsets $\emptyset$ and X are alway both open and closed, so these are the only two such spaces allowed.

All spaces with a base consisting of simply connected sets are semi-locally simply connected, but not all semi-locally simply connected spaces need to have such a base.

Logically path-connectedness has a connection with the fundamental group.

Theorem 1.4.5. *Let X be a path-connected topological space and let $x, y \in X$ two points. Then $\pi_1(X, x) \cong \pi_1(X, y)$.*

This result follows directly from the definition of path-connectedness and Lemma 1.3.4. For this reason it makes sense to talk about *the* fundamental group $\pi_1(X)$, without specifying a basepoint, if X is a path-connected space.

2

Covering Spaces

We now move our attention to a particular class of maps, namely *covering maps*. A covering map has the fundamental property that for a small enough subspace of the codomain, the preimage looks like a stack of copies of that subspace. Some covering maps are even more well-behaved and act as the quotient map for a group action of a group G on the domain, which we then call G-coverings. It turns out that these maps have a deep connection with the fundamental groups we saw in chapter 1 and we can even use covering maps to calculate fundamental groups.

In this chapter we will also see the *classification of G-coverings*, which gives and explicit description of all G-coverings a space can have.

2.1. Covering maps

We begin by giving the definition of a covering map.

Definition 2.1.1. A *covering map* is a surjective continuous map $p : Y \to X$ with the following property:

There is an open cover $\mathscr{U}$ of X, such that for all $U \in \mathscr{U}$ the subspace $p^{-1}(U) \subseteq Y$ is a disjoint union of open subsets $V \subseteq Y$, for which the maps $p|_V : V \to U$ are homeomorphisms. For a point $x \in X$ we call the set $p^{-1}\{x\}$ the *fiber* of x.

The pair (Y, p) is called a *covering space* of X.

We call $p : (Y, y) \to (X, x)$ a *pointed covering map* if $p : Y \to X$ is a covering map and $p(y) = x$.

Intuitively, (Y, p) being a covering space of X means that for some small open subsets U of X the pre-image of the small set looks like a union of copies of U. From this we can immediately see that X is locally (path-)connected if and only if Y is locally (path-)connected.

The best way to get intuition for the definition is to look at some examples.

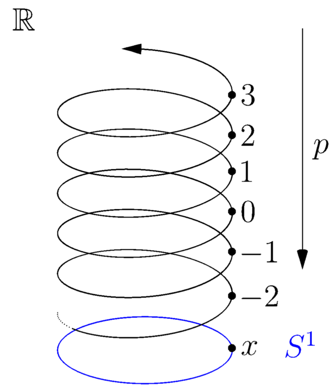


Figure 2.1.1: Covering of the circle S^1 by $\mathbb{R}$.

Example 2.1.2. Let T be any non-empty set with the discrete topology, meaning every subset is open. Then the projection on the second coordinate $p : T \times X \to X$ is a covering map of X, which we call the *trivial covering*. In this case we can take $\mathscr{U} = \{X\}$ in Definition 2.1.1. In paricular, the identity map $X \to X$ is a trivial covering.

Example 2.1.3. If we interpret the circle S^1 as the complex numbers with norm 1, we see a covering of S^1 by the real line $\mathbb{R}$ as follows:

$$\begin{aligned} p : \mathbb{R} &\to S^1, \\ t &\mapsto e^{2\pi i t}. \end{aligned}$$

We can view this covering as rolling up the real line as a spiral above the circle and projecting downwards. We see that $p^{-1}\{1\} = \mathbb{Z}$.

We also see that the circle covers itself. Let $n \in \mathbb{Z}_{\geqslant 1}$, then

$$\begin{aligned} p_n : S^1 &\to S^1, \\ z &\mapsto z^n. \end{aligned}$$

is a covering map. The fiber $p_n^{-1}\{1\}$ is given by the set of n-th roots of unity.

A special case of a covering map is when the covering space is simply connected. We do need some requirements for the space X.

Definition 2.1.4. Suppose X is connected and locally path-connected[1]. A covering map $u : \widetilde{X} \to X$ is called universal if $\widetilde{X}$ is simply connected. The pair $(\widetilde{X}, u)$ is called a universal covering space.

The name comes from the following universal property.

[1] Note that this implies X is path-connected as well.

Theorem 2.1.5. *Let $u : (\widetilde{X}, \widetilde{x}) \to (X, x)$ be a pointed universal covering map. Let $p : (Y, y) \to (X, x)$ be another pointed covering map. There exists exactly one continuous map $f : (\widetilde{X}, \widetilde{x}) \to (Y, y)$ such that the diagram*

$$\begin{array}{ccc} & & (Y, y) \\ & \overset{f}{\nearrow} & \big\downarrow p \\ (\widetilde{X}, \widetilde{x}) & \xrightarrow{u} & (X, x) \end{array}$$

commutes.

A proof can be found in (Fulton [5], Section 13c).

From this we can conclude universal coverings are unique.

Corollary 2.1.6. *Let $u_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x)$ and $u_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x)$ be two pointed universal covering maps. Then there is a unique isomorphism of pointed covering spaces $f : (\widetilde{X}_1, \widetilde{x}_1) \xrightarrow{\sim} (\widetilde{X}_2, \widetilde{x}_2)$, such that $u_1 = u_2 \circ f$.*

This result will be proven in Example 5.1.2.

Example 2.1.7. The covering $p : \mathbb{R} \to S^1$ from Example 2.1.3 is universal.

Not every topological space has a universal cover.

Theorem 2.1.8. *Suppose X is a connected and locally path-connected space. Then X has a universal covering if and only if it is semi-locally simply connected.*

The universal covering $\widetilde{X}$ can be constructed by defining $\widetilde{X}$ as the set of homotopy classes of paths in X that start at x. The covering is then defined as

$$\begin{aligned} p : \widetilde{X} &\to X \\ [\gamma] &\mapsto \gamma(1). \end{aligned}$$

The rest of the proof is putting a topology on $\widetilde{X}$ and showing p is indeed a covering map. This proof can be found in (Fulton [5], Theorem 13.20).

2.2. G-COVERINGS

Some covering maps $Y \to X$ behave exceedingly well. These coincide with a group action on Y and X being the quotient space of Y under this action. For this, we need the action to be even.

Definition 2.2.1. A group action $\cdot : G \times Y \to Y$ is called *even*, if every point in Y has an open neighbourhood $V \subset Y$, such that $g \cdot V$ and $h \cdot V$ are disjoint if $g, h \in G$ are not equal.

Note that we write the even action of a group on a topological space from the left.

Example 2.2.2. Let $\mathbb{Z}$ act on $\mathbb{R}$ by means of translation: $m \cdot x = m + x$. This action is even. For any point $x \in \mathbb{R}$, we can take the open neighbourhood $V = (x - \frac{1}{4}, x + \frac{1}{4})$, which satisfies the requirement.

For an even action of G on Y and $y, y' \in Y$ we set $y \sim y'$ if there is some $g \in G$ with $g \cdot y = y'$. We can then look at the quotient space, see Section 1.1.

We state two properties of quotient spaces.

Lemma 2.2.3. *(a) Let X be a topological space and $\sim$ an equivalence relation on X and let $q : X \to Q = X/\sim$ be the quotient map. Let $f : X \to Y$ be a continuous map, such that for all $x, x' \in X$ with $x \sim x'$ we have $f(x) = f(x')$. Then there is a unique continuous map $\overline{f} : Q \to Y$, with $f = \overline{f} \circ q$.*

(b) Any surjective and open continuous map is a quotient map for some equivalence relation. In particular a covering map is a quotient map.

See (Engelking [4], Section 2.4) for a proof.

Theorem 2.2.4. *Suppose G acts evenly on Y. Then the quotient map $p : Y \to Y/G$ is a covering map.*

Proof. The map p is clearly surjective and by definition continuous. We first prove the image of an open set in Y is open in Y/G. Let $V \subseteq Y$ open. Then $p^{-1}(p(V)) = \bigcup_{g \in G} gV$, which is the union of open subsets of Y, which is again open. By the definition of the quotient topology, it now follows that $p(V)$ is open in Y/G.

Now take $x \in Y/G$ and $y \in p^{-1}\{x\}$. Take an open neighbourhood $V \subseteq Y$ of y that satisfies the property in Definition 2.2.1. Then $U = p(V)$ is an open neighbourhood of x, as p is open, and by definition the set $p^{-1}(U)$ is the disjoint union of gV, where g runs through G. For each $g \in G$, we prove the map $p|_{gV} : gV \to U$ is a homeomorphism.

It is clearly surjective. Let $y, y' \in V$ and suppose $p(gy) = p(gy')$. Then there is some $h \in G$, such that $gy = hgy'$. By definition of the even action, it follows that $g = hg$, which implies h is the identity, so $y = y'$. As the map is also open, it follows that the inverse is also continuous, so we conclude it is a homeomorphism.

The subsets U now form an open cover of Y/G, that satisfies the criteria from Definition 2.1.1. □

Theorem 2.2.5. *Suppose $p : Y \to X$ is a continuous map and G acts evenly on Y. The following are equivalent:*

(1) The map $p : Y \to X$ is a covering map, where the fibers are exactly the orbits under the action of G.

(2) The map $p : Y \to X$ is a quotient map for the action of G.

Proof. (1) $\implies$ (2): this follows from Lemma 2.2.3(b). (2) $\implies$ (1): this follows from Theorem 2.2.4. □

Definition 2.2.6. We call $p : Y \to X$ a G*-covering* if p satisfies the equivalent conditions of Theorem 2.2.5.

Example 2.2.7. From Example 2.1.3 and 2.2.2 we see that $p : \mathbb{R} \to S^1$, $t \mapsto e^{2\pi i t}$ is a $\mathbb{Z}$-covering. This also means that the quotient space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to S^1.

We write $\mathrm{Aut}(Y)$ for the group of all homeomorphisms from Y to itself, with the composition as operation. If $p : Y \to X$ is a G-covering we can view G as a subgroup of $\mathrm{Aut}(Y)$.

2.3. MONODROMY ACTION

The defining property of a covering map makes that the pre-image of a path is a union of paths. We can see this as lifting the path to the covering space.

Theorem 2.3.1 (Path lifting)**.** *Let $p: Y \to X$ be a covering map and let γ be a path in X and write $\gamma(0) = x$. Let $y \in p^{-1}\{x\}$. Then there is a unique path $\widetilde{\gamma}_y$ with $\widetilde{\gamma}_y(0) = y$ and $p \circ \widetilde{\gamma}_y = \gamma$.*

2

A full proof can be found in (Fulton [5], Proposition 11.6). The construction of $\widetilde{\gamma}_y$ is done by splitting the path γ in smaller paths that all lie in an open subset that has the property that the pre-image is a disjoint union of homeomorphic sets. We can then glue the pre-images of the smaller paths together to get the path we need.

The path $\widetilde{\gamma}_y$ is called the *lift* of γ with starting point y.

Similarly we can lift homotopies. A full proof can be found in (Fulton [5], Proposition 11.8).

Theorem 2.3.2 (Homotopy lifting)**.** *Let X be a topological space and $x \in X$. Let $p: X \to Y$ be a covering and let $y \in p^{-1}\{x\}$.*

Suppose γ and γ' are homotopic paths with $\gamma(0) = \gamma'(0) = x$ and let $\Gamma: [0,1]^2 \to X$ be a path homotopy from γ to γ'. Then there is a unique path homotopy $\widetilde{\Gamma}: [0,1]^2 \to Y$ from $\widetilde{\gamma}_y$ to $\widetilde{\gamma}'_y$, such that $p \circ \widetilde{\Gamma} = \Gamma$.

As we have now linked covering spaces and paths, it is not very surprising that we can also link the fundamental group with covering spaces. It turns out that there is a group action of the fundamental group of a topological space on the fiber of the base point. We denote this action from the right.

We remind the reader that a group action of a group G on a set Y is transitive if for every y and y' in Y, there is some $g \in G$, such that $y \cdot g = y'$. The action is called free if $y \cdot g = y$ implies that $g = e$.

Theorem 2.3.3. *Let X be a topological space and let $x \in X$. Let $p: Y \to X$ be a covering map and write $Y_x = p^{-1}\{x\}$.*

(a) The map

$$\begin{aligned} *: Y_x \times \pi_1(X, x) &\to Y_x, \\ (y, [\gamma]) &\mapsto y * [\gamma] = \widetilde{\gamma}_y(1) \end{aligned}$$

defines a right action of the fundamental group $\pi_1(X, x)$ on the fiber set Y_x.

(b) If Y is path connected, the right action in (a) is transitive.

(c) If Y is simply connected, the right action in (a) is free.

Proof. (a)

We first prove that the map is well-defined. Suppose γ and γ' are two homotopic loops at x in X, so $[\gamma] = [\gamma']$. Then Theorem 2.3.2 tells us that $\widetilde{\gamma}_y$ and $\widetilde{\gamma}'_y$ are homotopic as well, and in particular that $\widetilde{\gamma}_y(1) = \widetilde{\gamma}'_y(1)$, so $*$ is well-defined.

2

We now prove that the map is a group action. The property $y*[c_x] = y$ follows directly from the fact that $(\widetilde{c_x})_y = c_y$, which can be easily seen from the definition of path lifting.

The other property we have to show is that for all $\gamma, \gamma' \in P(X;x)$ we have

$$(y*[\gamma])*[\gamma'] = y*([\gamma]\cdot[\gamma']).$$

Let us write $y_1 = \widetilde{\gamma}_y(1)$ and $y_2 = \widetilde{\gamma}'_{y_1}(1)$. Then we have by definition

$$\begin{aligned} y_1 &= y*[\gamma] \\ y_2 &= y_1*[\gamma'] \\ &= (y*[\gamma])*[\gamma'] \end{aligned}$$

On the other hand we can see that the ending point of $\widetilde{\gamma}_y$ is the same as the starting point of $\widetilde{\gamma}'_{y_1}$ and the the concatenation $\widetilde{\gamma}_y \odot \widetilde{\gamma}'_{y_1}$ is the unique lift of $\gamma \odot \gamma'$ with starting point y. From this we see that

$$\begin{aligned} y_2 &= (\widetilde{\gamma}_y \odot \widetilde{\gamma}'_{y_1})(1) \\ &= \widetilde{(\gamma \odot \gamma')}_y(1) \\ &= y*[\gamma \odot \gamma'] \\ &= y*([\gamma]\cdot[\gamma']) \end{aligned}$$

which is what we had to show.

(b)

p is surjective so Y_x is non-empty. Now let $y_1, y_2 \in Y_x$. Since Y is path-connected there is a path $\widetilde{\gamma}$ from y_1 to y_2. The path $\gamma = p \circ \widetilde{\gamma}$ is a loop at x and the lift of γ with initial point y_1 is exactly $\widetilde{\gamma}$. We now get that $y_1 * [\gamma] = \widetilde{\gamma}(1) = y_2$. From this we can conclude the right action is transitive.

(c)

Let $y_1 \in Y_x$. We have to show that the stabiliser of y_1 in $\pi_1(X,x)$ is trivial. Suppose $[\gamma]$ is an element of this stabiliser. Then by definition $\widetilde{\gamma}_{y_1}(1) = y_1$, so $\widetilde{\gamma}_{y_1}$ is a loop at y_1. As Y is simply connected, this loop is homotopic in Y to the constant loop at y_1. From this we get that γ is homotopic in X to the constant loop at x, so $[\gamma] = [c_x]$ and the stabiliser of y is trivial. From this we can conclude the right action is free. □

We call this right action of $\pi_1(X,x)$ on the fiber set Y_x the *monodromy action* of the fundamental group.

Let $p: Y \to X$ be a G-covering. The group G has a natural action on the fiber $Y_x = p^{-1}\{x\}$ as well. We sometimes denote $g(y)$ instead of $g \cdot y$ to avoid confusion with the group action of the fundamental group. We can now relate the right monodromy action of $\pi_1(X,x)$ on the fiber Y_x with the left action of G on that fiber.

First of all we note that the actions are compatible with each other. That is, for $y \in Y$ with $p(y) = x$, $g \in G$ and $\alpha \in \pi_1(X,x)$ we have that $g(y*\alpha) = g(y)*\alpha$, see (Fulton [5], Section 14a).

Furthermore, for every $\alpha \in \pi_1(X,x)$ there is a $g \in G$, such that $g(y) = y * \alpha$. This is because the action of G on Y_x is free (Definition 2.2.1) and transitive (Theorem 2.2.5),

This means we have a well-defined map $\rho = \rho_p : \pi_1(X,x) \to G$ that links every element $\alpha \in \pi_1(X,x)$ with the element $g \in G$, such that $g(y) = y * \alpha$. This map does depend on the choice of $y \in Y_x$.

Theorem 2.3.4. *Let $p : Y \to X$ be a G-covering.*

(a) The map $\rho : \pi_1(X,x) \to G$ is a group homomorphism.

(b) The kernel of ρ is equal to the stabiliser of y, $Stab_y = p_*(\pi_1(Y,y))$.

(c) The group homomorphism ρ is surjective if and only if the monodromy action is transitive. If Y is path connected, both conditions are satisfied.

Proof. (a)

Suppose γ and γ' are loops at x. From the definition of ρ and the compatibility of the group actions we see

$$\begin{aligned} \rho([\gamma])\rho([\gamma'])(y) &= \rho([\gamma])(y * \gamma') \\ &= \rho([\gamma])(y) * \gamma' \\ &= (y * \gamma) * \gamma' \\ &= y * (\gamma \odot \gamma') \\ &= \rho([\gamma \odot \gamma'])(y) \\ &= \rho([\gamma] \cdot [\gamma'])(y). \end{aligned}$$

From this it follows that ρ is a group homomorphism.

(b)

The equality $\mathrm{Ker}(\rho) = \mathrm{Stab}_y$ follows from the definition of ρ.

Let $[\zeta] \in p_*(\pi_1(Y,y))$, then $[\zeta] = [p \circ \gamma]$ for some loop γ at y in Y. Then we have that

$$y * [\zeta] = \widetilde{\zeta}_y(1) = \widetilde{p \circ \gamma}_y(1) = \gamma(1) = y,$$

so $[\zeta] \in \mathrm{Stab}_y$.

Now let $[\zeta] \in \mathrm{Stab}_y$. Then $\widetilde{\zeta}_y(1) = y * [\zeta] = y$. This means that $\widetilde{\zeta}_y$ is a loop at y, so $[\widetilde{\zeta}_y] \in \pi_1(Y,y)$. But then $p_*([\widetilde{\zeta}_y]) = [\zeta] \in p_*(\pi_1(Y,y))$.

From this we see that $\mathrm{Stab}_y \subseteq p_*(\pi_1(Y,y))$ and $p_*(\pi_1(Y,y)) \subseteq \mathrm{Stab}_y$, so we conclude $\mathrm{Stab}_y = p_*(\pi_1(Y,y))$.

(c)

The equivalence follows from the definitions. If Y is path-connected, we get from Theorem 2.3.3 that the monodromy action is transitive. □

Definition 2.3.5. We call the group homomorphism $\rho : \pi_1(X,x) \to G$ the *monodromy representation* belonging to the G-covering p.

The following lemma about the monodromy representation of the restriction of a G-covering will be useful in proving Van Kampen's Theorem.

Lemma 2.3.6. *Let G be a group, X a topological space and $x \in X$. Let $p : (Y, y) \to (X, x)$ be a pointed G-covering and let $\rho : \pi_1(X, x) \to G$ the corresponding monodromy representation. Let $X' \subset X$ a subspace such that $x \in X'$. Write $i_* : \pi_1(X', x) \to \pi_1(X, x)$ for the group homomorphism induced by the inclusion. Then*

(a) The restriction $p|_{p^{-1}X'} : (p^{-1}X', y) \to (X', x)$ is a pointed G-covering.

(b) The monodromy representation ρ' corresponding to $p|_{p^{-1}X'}$ satisfies the equality $\rho' = \rho \circ i_*$.

Proof. (a)

The proof that $p|_{p^{-1}X'}$ is a covering map follows from the definition and is left to the reader. The group G acts evenly on $p^{-1}X'$ in the natural way. As p is a G-covering the fibers are exactly the orbits under G. Let $x' \in X'$. Then $p^{-1}\{x'\}$ is exactly the orbit of $y' \in p^{-1}\{x'\}$. As $p^{-1}\{x'\} \subset p^{-1}X'$ this orbit is fully contained in $p^{-1}X'$ and the fibers of the covering map $p|_{p^{-1}X'}$ are exactly the orbits under G as well, so it is a G-covering.

(b)

Let $\alpha \in \pi_1(X', x)$. Then $\rho'(\alpha) = \varphi \in G$, such that $\varphi(y) = y * \alpha$, where $*$ is the monodromy action of $\pi_1(X, x)$ on the fiber $Y'_x = (p|_{p^{-1}X'})^{-1}\{x\}$. Let us denote $\star$ for the monodromy action of $\pi_1(X, x)$ on $Y_x = p^{-1}\{x\}$. We will prove that $\varphi(y) = y \star i_*(\alpha)$.

We have that

$$y * \alpha = \widetilde{\alpha}_y(1)$$
$$y \star i_*(\alpha) = \widetilde{i_*(\alpha)}_y(1)$$

Furthermore $p|_{p^{-1}X'}(\widetilde{\alpha}_y) = \alpha$, so $p(\widetilde{\alpha}_y) = [\alpha] = i_*(\alpha)$. By uniqueness of path-lifting we have

$$\widetilde{\alpha}_y(1) = \widetilde{i_*(\alpha)}_y(1),$$

so we conclude that $\rho' = \rho \circ i_*$. □

Using the monodromy representation we can finally calculate some non-trivial fundamental groups. First we state two corollaries of Theorem 2.3.4.

Corollary 2.3.7. *Let $p : (Y, y) \to (X, x)$ be a pointed G-covering and suppose Y is path-connected. Then the monodromy representation induces a group isomorphism*

$$\pi_1(X, x)/p_*(\pi_1(Y, y)) \xrightarrow{\sim} G.$$

This corollary follows from Theorem 2.3.4(c) and the isomorphism theorem for groups.

As $p_*(\pi_1(Y, y))$ is trivial when $\pi_1(Y, y)$ is trivial, we immediately see the following result.

Corollary 2.3.8. *Let $p : (Y, y) \to (X, x)$ a pointed G-covering and suppose Y is simply connected. Then the monodromy representation induces a group isomorphism*

$$\pi_1(X, x) \xrightarrow{\sim} G.$$

Example 2.3.9. The continuous map $p:\mathbb{R}\to S^1$, given by $t\mapsto e^{2\pi i t}$ is a $\mathbb{Z}$-covering, see Example 2.2.7. As $\mathbb{R}$ is simply connected, we have the isomorphism $\pi_1(S^1)\simeq\mathbb{Z}$.

The inclusion of the circle into $\mathbb{R}^2$ now shows that an inclusion of topological spaces does not induce an inclusion of fundamental groups.

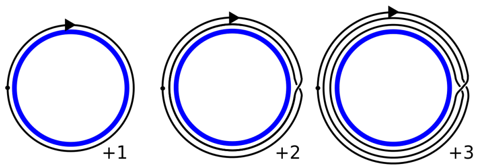


Figure 2.3.1: Intuitively, $\mathbb{Z}$ shows the number of times we wrapped a loop around the circle.

2.4. CLASSIFICATION OF G-COVERINGS

We have now constructed a group homomorphism $\rho_p:\pi_1(X,x)\to G$ for every pointed G-covering $p:Y\to X$. We can ask ourselves if we can also do the opposite; given a group homomorphism $\rho:\pi_1(X,x)\to G$, can we construct a G-covering $p_\rho:Y\to X$?

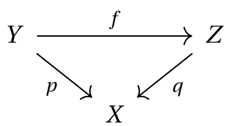
The answer is yes, under certain conditions. The constructions are even the inverse of each other. We show this in the Classification of G-coverings, Theorem 2.4.3.

First we have to rule out a technicality, to avoid counting G-coverings that can be considered the same twice. We can define what it means for coverings to be the same, via morphisms of coverings.

Definition 2.4.1. Let X be a topological space and let $p:Y\to X$ and $q:Z\to X$ be two covering maps. A *morphism of covering spaces over X* is a continuous map $f:Y\to Z$, such that $p=q\circ f$. If f is a homeomorphism, we call F an *isomorphism of covering spaces.*

Suppose p and q are G-coverings. Then f is a *morphism of G-coverings* (resp. *isomorphism*) if it is a morphism (resp. isomorphism) of covering spaces and it is compatible with the action of G. That is, if for all $g\in G$ and for all $y\in Y$, we have that $f(g\cdot y)=g\cdot f(y)$.

A morphism of covering spaces makes the diagram



commute.

From the definition we can clearly see that isomorphic G-coverings yield the same monodromy representation.

Next we look at the condition a topological space has to satisfy to be able to classify its G-coverings.

2

Definition 2.4.2. We call a topological space X *unloopable* if it is connected, locally path-connected and semi-locally simply connected.

The condition that a topological space must be unloopable seems like a rather strict requirement, but there are many often studied spaces that are unloopable. For example every connected topological manifold, a topological space where every point has an open neighbourhood that is homeomorphic to $\mathbb{R}^n$ for some n, is unloopable (Lee [11], Chapter 2).

Theorem 2.4.3 (Classification of G-coverings)**.** *Let X be an unloopable topological space and let $x \in X$. Let G be (fixed) group. The assignment of ρ_p to a G-covering map p yields a bijection of sets between the set of group homomorphisms from $\pi_1(X,x)$ to G and the set of isomorphism classes of pointed G-coverings:*

$$\mathrm{Hom}(\pi_1(X,x),G) \longleftrightarrow \{\textit{pointed } G\textit{-coverings}\}/\textit{ isomorphism}.$$

The full proof of this theorem is rather technical and long and can be found in (Fulton [5], section 14a). We will only show the construction of both bijective maps here and omit the proof that these are indeed bijective and are each other's inverse.

The construction of a group homomorphism from a G-covering is the monodromy representation, Definition 2.3.5.

Let $\rho : \pi_1(X,x) \to G$ be a group homomorphism. We will construct a G-covering $p_\rho : Y_\rho \to X$.

First we choose a pointed universal covering of $u : (\widetilde{X},\widetilde{x}) \to (X,x)$. This is possible as we assume X is unloopable. We give G the discrete topology, meaning any subset is open. Then the space $\widetilde{X} \times G$ is a stack of copies of $\widetilde{X}$. There is a group action from the left of $\pi_1(X,x)$ on this space. Let $[\gamma] \in \pi_1(X,x)$, $z \in \widetilde{X}$ and $g \in G$. We define

$$[\gamma] \bullet (z,g) = (z * [\gamma^{-1}], g \cdot \rho([\gamma]^{-1})).$$

Here $*$ is the monodromy action and $\cdot$ is the group operation of G. We then define Y_ρ to be the quotient of $\widetilde{X} \times G$ by this action

$$Y_\rho = (\widetilde{X} \times G)/\pi_1(X,x).$$

Write $\langle (z,g) \rangle$ for the equivalence class of $(z,g) \in \widetilde{X} \times G$ under this action. We set $y_\rho = \langle (\widetilde{x}, e) \rangle$.

We define the map $p_\rho : Y_\rho \to X$ by taking $\langle (z,g) \rangle$ to $u(z)$.

We omit the proof that this is indeed a covering map and that the action of G defined on Y_ρ makes it a G-covering and refer the interested reader once again to (Fulton [5], section 14a).

Example 2.4.4. Suppose ρ is the trivial homomorphism, which sends every element of $\pi_1(X,x)$ to the neutral element of G. Then

$$[\gamma] \bullet (z,g) = (z * [\gamma^{-1}], g \cdot \rho([\gamma]^{-1})) = (z * [\gamma^{-1}], g).$$

From this we see that the projection $\widetilde{X} \times G \to X \times G$, given by $(z,g) \mapsto (u(z),g)$ is a quotient map for the action of $\pi_1(X,x)$ on $\widetilde{X} \times G$, so we have a natural homeomorphism $Y_\rho \xrightarrow{\sim} X \times G$. But the covering $X \times G \to X$ is trivial, so p_ρ is as well.

Example 2.4.5. Suppose ρ is surjective. Then we claim that the continuous map $\widetilde{X} \to Y_\rho$ given by $w \mapsto \langle (w, e)\rangle$ is surjective. Let $(z, g) \in \widetilde{X} \times G$ and take $[\gamma] \in \pi_1(X.x)$ with $\rho([\gamma]) = g$. Then $\langle (z, g)\rangle = \langle (z * [\gamma^{-1}], g \cdot g^{-1})\rangle = \langle (z * [\gamma^{-1}], e)\rangle$, so $w = z * [\gamma^{-1}]$ works. Since $\widetilde{X}$ is connected, Y_ρ is connected as well.

This result is already very interesting at its own, but it will also help us with the proof of Van Kampen's Theorem.

There is also an classification of G-coverings without base points, see (Fulton [5], Exercise 14.3). In this there is a bijection between G-coverings modulo isomorphism and the set $\mathrm{Hom}(\pi_1(X, x).G)$ modulo conjugation. Two group homomorphisms $\rho, \rho \in \mathrm{Hom}(\pi_1(X, x).G)$ are conjugate if there is some $g \in G$, such that $\rho'(\alpha) = g \cdot \rho(\alpha) \cdot g^{-1}$ for all $\alpha \in \pi_1(X, x)$. If G is an abelian group a group homomorphism $\pi_1(X, x) \to G$ is only conjugate to itself, so $\mathrm{Hom}(\pi_1(X, x), G)/\{\text{conjugation}\} = \mathrm{Hom}(\pi_1(X, x), G)$.

2.5. $\mathbb{Z}_n$-COVERINGS OF THE CIRCLE

To give an example of the classification theorem, we will look at all pointed $\mathbb{Z}/n\mathbb{Z}$-coverings of the circle S^1 for $n \geqslant 2$. For convenience we will write $\mathbb{Z}_n$ for $\mathbb{Z}/n\mathbb{Z}$.

The fundamental group of the circle is isomorphic to $\mathbb{Z}$ and there are exactly n group homomorphisms from $\pi_1(S^1) \cong \mathbb{Z}$ to $\mathbb{Z}_n$. This is because

$$\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}_n) = \{\varphi_a : k \mapsto ka : a \in \mathbb{Z}_n\}.$$

The classification of G-coverings 2.4.3 now tells us there are exactly n different $\mathbb{Z}_n$-coverings of the circle. As $\mathbb{Z}_n$ is an abelian group, we can look at $\mathbb{Z}_n$-coverings without base point.

We will explicitly show all $\mathbb{Z}_4$-coverings of the circle and briefly state a generalisation for all $\mathbb{Z}_n$-coverings.

The group homomorphism $\varphi_{\overline{0}} : k \mapsto k\overline{0} = \overline{0}$ is the trivial homomorphism. By Example 2.4.4, this homomorphism corresponds to the trivial covering $p_0 : \mathbb{Z}_4 \times S^1 \to S^1$.

The group homomorphisms $\varphi_{\overline{1}}$ and $\varphi_{\overline{3}}$ are surjective, as $\overline{1}$ and $\overline{3}$ are both generators for $\mathbb{Z}_4$. Example 2.4.5 now tells us that the corresponding covering should be connected. We claim that in both cases the covering space is the circle itself.

We define two left actions λ_1 and λ_3 of $\mathbb{Z}_4$ on the circle S^1.

$$\begin{aligned}
\lambda_1 : \mathbb{Z}_4 \times S^1 &\to S^1, \\
(a, z) &\to i^a \cdot z, \\
\lambda_3 : \mathbb{Z}_4 \times S^1 &\to S^1, \\
(a, z) &\to i^{3a} \cdot z = (-i)^a \cdot z.
\end{aligned}$$

It follows directly that these actions are even.

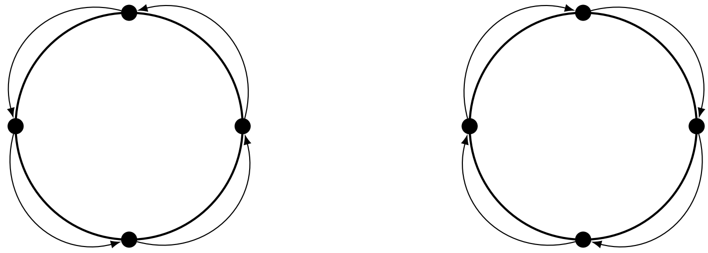


Figure 2.5.1: The actions λ_1 (left) and λ_3 (right) on the circle. The orbit of the point 1 is shown.

The set $A = \{e^{2\pi i t} : 0 \leqslant t \leqslant \frac{1}{4}\}$, which is homeomorphic to the interval $[0,1]$ via $f\colon A \to [0,1]$, given by $e^{2\pi i t} \mapsto 4t$ now has a representative of all equivalence classes under the equivalence relation induced by both λ_1 and λ_3. Furthermore the points $e^{2\pi i 0} = 1$ and $e^{2\pi i \frac{1}{4}} = i$ are in the same equivalence class under both relations. These points correspond to 0 and 1 under f. This means that $S^1/\lambda_1 = A/\lambda_1$ and $S^1/\lambda_3 = A/\lambda_3$ are both homeomorphic to the interval $[0,1]$ with endpoints identified. This is the same as the quotient space of $\mathbb{R}$ under the even action of $\mathbb{Z}$ from Example 2.2.2. This latter space is homeomorphic to the circle, so the spaces S^1/λ_1 and S^1/λ_3 are as well.

We write Y_1 for S^1 with the action λ_1 and Y_3 for S^1 with the action λ_3. The coverings $p_1 : Y_1 \to S^1$ and $p_3 : Y_3 \to S^1$ are then both $\mathbb{Z}_4$-coverings of the circle.

To show that these coverings do not represent the same element in the bijection of Theorem 2.4.3, we have to show the coverings are not isomorphic as $\mathbb{Z}_4$-coverings. We will prove this by contradiction.

Suppose $f : Y_a \to Y_b$ is an isomorphism of $\mathbb{Z}_4$-coverings. Then in particular, $f : S^1 \to S^1$ is a homeomorphism. By definition f is compatible with the actions λ_1 and λ_3, so $f(i^a \cdot z) = (-i)^a \cdot f(z)$.

By potentially adding a rotation of the circle, which is an isomorphism of G-coverings, we may without loss of generality assume that $z = 1$ is the (unique) element such that $f(z) = 1$.

Then from the compatibility we see that $f(i) = -i$, $f(-1) = -1$ and $f(-i) = i$. As 1 and -1 are resp. the unique points with $f(1) = 1$ and $f(-1) = -1$ and we have that $f(i) = -i$, we see that $\Im(f(z)) < 0$ whenever $\Im(z) > 0$. But $\Im(-i) = -1 < 0$ and $\Im(f(-i)) = \Im(i) = 1 \not< 0$. We have reached contradiction and conclude p_1 and p_2 are not isomorphic as G-coverings.

For the $\mathbb{Z}_4$-covering that corresponds to the element $\overline{2}$, we consider an action of $\mathbb{Z}_4$ on the disjoint union $S^1 \sqcup S^1$. To separate the two circles we write $S^1 \sqcup S^1 = S_1 \cup S_2$. For a $z \in S_1 \cup S_2$, we write $z = z_1$ if $z \in S_1$ and $z = z_2$ if $z \in S_2$.

We divide both these circles in two parts as follows:

$$S_{i,1} = \{e^{2\pi i t} : 0 \leqslant t < \frac{1}{2}\}$$
$$S_{i,2} = \{e^{2\pi i t} : \frac{1}{2} \leqslant t < 1\}.$$

The left action is then defined as follows:

$$\lambda_2 : \mathbb{Z}_4 \times (S_1 \cup S_2) \longrightarrow S_1 \cup S_2$$
$$(\overline{1}, z_1) \longmapsto \begin{cases} -z_1, & \text{if } z_1 \in S_{1,1}, \\ -z_2, & \text{if } z_1 \in S_{1,2}, \end{cases}$$
$$(\overline{1}, z_2) \longmapsto \begin{cases} -z_2, & \text{if } z_2 \in S_{2,1}, \\ -z_1, & \text{if } z_2 \in S_{2,2}. \end{cases}$$

The action by the other elements a of $\mathbb{Z}_n$ are deduced from the action of $\overline{1}$, by writing $a = \overline{1} + \overline{1} + \cdots + \overline{1}$. This action is even.

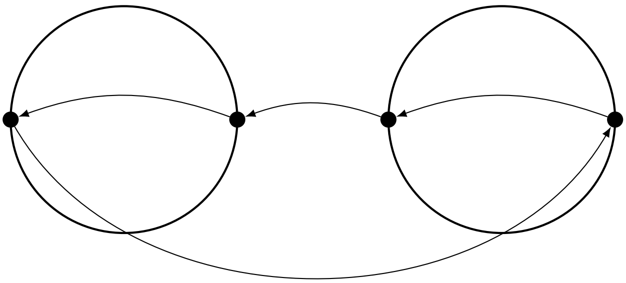


Figure 2.5.2: The action λ_2 on the disjoint union of two circles. The orbit of the point 1_1 is shown.

By an argument very similar to the case for λ_1 and λ_3 we can see that the quotient space of $S_1 \cup S_2$ under this action is also homeomorphic to the circle. We write Y_2 for the space $S_1 \cup S_2$ under the action λ_2 and p_2 for the $\mathbb{Z}_4$-covering $p_2 : Y_2 \longrightarrow S^1$. This $\mathbb{Z}_4$-covering is not isomorphic to p_i for $i = 0, 1, 3$ as the disjoint union of two circles is not homeomorphic to one circle nor to the disjoint union of four circles.

By the Classification of G-coverings 2.4.3 we can now conclude that the coverings p_i for $i = 0, 1, 2, 3$ are all four $\mathbb{Z}_4$-coverings of the circle S^1 up to isomorphism.

We can generalise the construction of these four coverings to construct all $\mathbb{Z}_n$-coverings of the circle. We will only state what the covering spaces and the actions of $\mathbb{Z}_n$ on them look like, without proofs.

We write $\zeta_k = e^{\frac{2\pi i}{k}}$.

2

Let $n \geqslant 2$ and let $a \in \mathbb{Z}_n$. The covering space corresponding to a is the disjoint union of $\gcd(a,n)$ circles

$$\bigsqcup_{i=1}^{\gcd(a,n)} S^1.$$

If $\gcd(a,n) = 1$, this is only one circle. Then the action λ_a of $\mathbb{Z}_n$ on the circle S^1 is defined by $(\overline{1}, z) \mapsto \zeta^a \cdot z$. We write Y_a for the space S^1 together with the action λ_a and $p_a : Y_a \to S^1$ for the corresponding $\mathbb{Z}_n$-covering.

For the case $\gcd(a,n) \neq 1$ we need some notation. Let us write $\tilde{a} := n/\gcd(a,n)$ and $\hat{a} := a/\gcd(a,n)$.

In this case the covering space consists of multiple copies of the circle, for which we will write $\bigcup_{i=1}^{\gcd(a,n)} S_i$. We again write $z = z_i$ if $z \in \bigcup_{i=1}^{\gcd(a,n)} S_i$ is an element of S_i. For convenience we may write $z = z_{\gcd(a.n)+1}$ instead of $z = z_1$. We divide each of these circles into $\tilde{a}$ parts $S_{i,j}$ as we did for the $\mathbb{Z}_4$-covering p_2.

The action λ_a of $\mathbb{Z}_n$ on this space is as follows:

$$(\overline{1}, z_i) = \begin{cases} \zeta_{\tilde{a}}^{\hat{a}} \cdot z_i, & \text{if } z_i \in S_{i,j} \text{ and } j + \hat{a} \leqslant \tilde{a} \\ \zeta_{\tilde{a}}^{\hat{a}} \cdot z_{i+1}, & \text{if } z_i \in S_{i,j} \text{ and } j + \hat{a} > \tilde{a}. \end{cases}$$

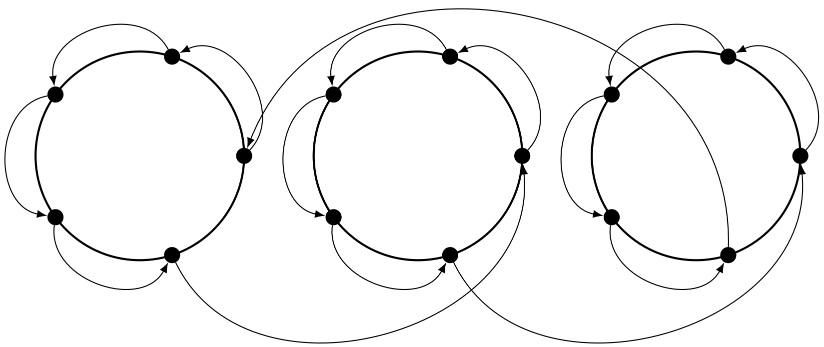


Figure 2.5.3: The action λ_3 of $\mathbb{Z}_{15}$ on the disjoint union of three circles. The orbit of the point 1_1 is shown.

We again write Y_a for the space $\bigcup_{i=1}^{\gcd(a,n)} S_i$ together with the action λ_a and $p_a : Y_a \to S^1$ for the corresponding $\mathbb{Z}_n$-covering.

One can prove that the maps p_a are indeed $\mathbb{Z}_n$-coverings of the circle and that they are not mutually isomorphic as $\mathbb{Z}_n$-coverings. As we have found n coverings p_n, Theorem 2.4.3 tells us we have found all.

3

Van Kampen's Theorem

We are now ready to state and prove Van Kampen's Theorem.

The first formulation of the theorem was given and proved by German mathematician Herbert Seifert in his dissertation in 1931 (Seifert [13]). The theorem is for this reason also known as the Seifert-Van Kampen Theorem. In 1933 Van Kampen published a more general version of this theorem (Van Kampen [10]). The proof we will give is due to Alexander Grothendieck and can be found in (Godbillon [6]).

The theorem states a relation between the fundamental group of a topological space X and the fundamental group of a collection of open subsets that cover X, which all have to satisfy certain conditions. The theorem is usually stated for the case of two open subsets covering X and a quick proof of that version can also be found in (Fulton [5], Section 14c). We will prove the statement for an arbitrary large cover of X.

3.1. Gluing coverings

Crucial in Grothendieck's proof of Van Kampen's Theorem is the gluing of topological spaces.

Suppose we are given a collection of spaces Y_a, for a in an index set $\mathscr{A}$ and suppose for each pair $a, b \in \mathscr{A}$ we are given an open subset $Y_{ab} \subset Y_a$ and a homeomorphism

$$\vartheta_{ab} : Y_{ab} \to Y_{ba}.$$

In case of the pair a, a we demand that $Y_{aa} = Y$ and ϑ_{aa} is the identity on Y.

These homeomorphisms should satisfy the *co-cycle condition*:
For any three $a, b, c \in A$ we must have

$$\vartheta_{ab}(Y_{ab} \cap Y_{ac}) \subseteq Y_{bc},$$
$$\vartheta_{ac} = \vartheta_{bc} \circ \vartheta_{ab} \qquad \text{on } Y_{ab} \cap Y_{ac}.$$

Note that this implies that $\vartheta_{ba} \circ \vartheta_{ab}$ is the identity on Y_{ab}.

Through these homeomorphisms we define an equivalence relation on the disjoint union $\bigsqcup_{a\in\mathscr{A}} Y_a$. Let $y, y' \in \bigsqcup_{a\in\mathscr{A}} Y_a$ and suppose $y \in Y_a$ and $y' \in Y_b$. Then we set $y \sim y'$ if $y \in Y_{ab}$ and $\vartheta_{ab}(y) = y'$.

As ϑ_{aa} is the identity on Y the relation is reflexive. Symmetry and transitivity follow from the cocycle condition. We write $Y^{\vartheta}_{\mathscr{A}} = \bigsqcup_{a\in\mathscr{A}} Y_a / \sim$. If $\mathscr{A} = \{a,b\}$ we also write $Y_a \cup_{\vartheta} Y_b$ for the quotient space.

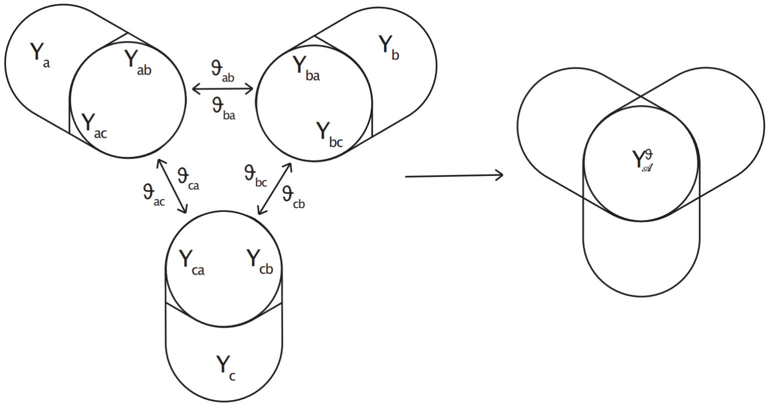


Figure 3.1.1: The gluing of the spaces Y_a, Y_b and Y_c.

Let $\varphi_a : Y_a \to Y$ be the map that takes a point in Y_a to its equivalence class. We will state some properties of this map. The proofs are mainly set-theoretic and can be found in (Fulton [5], Appendix A3).

Lemma 3.1.1. *(a) Each $\varphi_a(Y_a)$ is open in Y_a.*

(b) $\varphi_a : Y_a \to \varphi_a(Y_a)$ is a homeomorphism.

(c) Y is the union of the sets $\varphi(Y_a)$.

(d) $\varphi_a(Y_{ab}) = \varphi_b(Y_{ba})$.

(e) On Y_{ab}, $\vartheta_{ab} = \varphi_b^{-1} \circ \varphi_a$.

Let T be a topological space and let $\{f_a : Y_a \to T \,|\, a \in \mathscr{A}\}$ be a collection of continuous maps, that satisfies $f_a = f_b \circ \vartheta_{ab}$ on Y_{ab} for all $a, b \in \mathscr{A}$. We can then *glue* these maps together to a (unique) map

$$f^{\mathscr{A}}_{\vartheta} : Y^{\vartheta}_{\mathscr{A}} \to T,$$

with the property that $f^{\mathscr{A}}_{\vartheta} \circ \varphi_a = f_a$ on Y_a for all $a \in \mathscr{A}$. This follows from the universal property of quotient spaces, Lemma 2.2.3.

Let X be a topological space and let $\mathscr{O} = \{U_a : a \in \mathscr{A}\}$ be an open cover of X that is closed under finite intersections and suppose there is some $x \in X$, such that $x \in U_a$ for all $a \in \mathscr{A}$. Let G be a fixed group and suppose we have a pointed G-covering of every member of the open cover, $p_a : (Y_a, y_a) \to (U_a, x)$ for every $a \in \mathscr{A}$.

Suppose for every pair $a, b \in \mathscr{A}$ we have an isomorphism of pointed G-coverings (and in particular a homeomorphism)

$$\vartheta_{ab} : (p_a^{-1}(U_a \cap U_b), y_a) \xrightarrow{\sim} (p_b^{-1}(U_a \cap U_b), y_b).$$

Suppose again that ϑ_{aa} is the identity on Y_a and the homeomorphisms satisfy the co-cycle condition.

We call the tuple

$$(\{p_a : a \in \mathscr{A}\}, \{\vartheta_{ab} : a, b \in \mathscr{A}\})$$

gluing data over the open cover $\mathscr{O} = \{U_a : a \in \mathscr{A}\}$ with respect to the basepoint x.

We can now glue these coverings together as above, which results in the following map

$$p : (Y^{\vartheta}_{\mathscr{A}}, y) \to (X, x),$$

where y is the equivalence class of y_a. Since each ϑ_{ab} is an isomorphism of pointed G-coverings this definition of y is unambiguous.

This map satisfies the equality $p \circ \varphi_a = p_a$, where φ_a once again is the map that sends a point in Y_a to its equivalence class in $Y^{\vartheta}_{\mathscr{A}}$.

Lemma 3.1.2. *The map $p : (Y^{\vartheta}_{\mathscr{A}}, y) \to (X, x)$ is a G-covering of X.*

Proof. From Lemma 3.1.1 we know that φ_a is a homeomorphism from Y_a onto $p^{-1}(U_a)$. As $Y^{\vartheta}_{\mathscr{A}}$ is the union of $p^{-1}(U_a)$ for $a \in \mathscr{A}$ we can conclude that p is a covering map.

We define an action of G on $Y^{\vartheta}_{\mathscr{A}}$ in the following way: Let $y' \in Y^{\vartheta}_{\mathscr{A}}$. Then $y' = \varphi_a(y'_a)$ for some $a \in \mathscr{A}$ and $y'_a \in Y_a$. We set $g \cdot y' = g \cdot \varphi_a(y'_a) = \varphi_a(g \cdot y'_a)$.

This definition is well-defined. Suppose there is also some $b \in \mathscr{A}$ and $y'_b \in Y_b$ such that $y' = \varphi_b(y'_b)$. Then $y'_a \sim y'_b$, so $\vartheta_{ab}(y'_a) = y'_b$. But ϑ is an isomorphism of G-coverings, so

$$\vartheta_{ab}(g \cdot y'_a) = g \cdot \vartheta(y'_a) = g \cdot y'_b.$$

This means that $g \cdot y'_a \sim g \cdot y'_b$ and $\varphi_a(g \cdot y'_a) = \varphi_b(g \cdot y'_b)$.

Through this compatibility of φ_a with the action of G we can similarly see that the action of G on $Y^{\vartheta}_{\mathscr{A}}$ is even.

To show that p is indeed a G-covering, we will show that the orbits of G coincide with the fibers of p. Let $y' \in Y^{\vartheta}_{\mathscr{A}}$ and once again let $a \in \mathscr{A}$, such that $y' = \varphi_a(y'_a)$. Set $x' = p(y')$, then $x' = p_a(y'_a)$. The covering p_a is a G-covering, so $p_a^{-1}\{x'\}$ is the orbit of y'_a under the action of G. Then by definition of the action of G on $Y^{\vartheta}_{\mathscr{A}}$ we have that $\varphi_a(p_a^{-1}\{x'\})$ is the orbit of y' under the action of G.

We now immediately see that $\varphi_a(p_a^{-1}\{x'\}) \subseteq p^{-1}\{x'\}$. Suppose there is some $\tilde{y} \in p^{-1}\{x'\}$ that is not in $\varphi_a(p_a^{-1}\{x'\})$. Let $b \in \mathscr{A}$, such that $\tilde{y} \in Y_b$. Then $p_b(\tilde{y}) = x'$ and $x' \in U_a \cap U_b$. But then

$$\tilde{y} \in p_b^{-1}(U_a \cap U_b).$$

But then also $\vartheta_{ba}(\tilde{y}) \in p_a^{-1}(U_a \cap U_b)$ and $p_a(\vartheta_{ba}(\tilde{y})) = x'$, so $\vartheta_{ba}(\tilde{y}) \in p_a^{-1}\{x'\}$ and $\tilde{y} \in \varphi_a(p_a^{-1}\{x'\})$, which is a contradiction. We conclude that

$$\varphi_a(p_a^{-1}\{x'\}) = p^{-1}\{x'\}.$$

This means that p is a G-covering. □

Conversely, let $p : (Y, y) \to (X, x)$ be a given pointed G-covering of X. Then we can assign gluing data over $\mathscr{O}$ with respect to x to p:

$$(\{p|_{p^{-1}U_a} : a \in \mathscr{A}\}, \{\mathrm{id}_{p^{-1}(U_a \cap U_b)} : a, b \in \mathscr{A}\})$$

From the construction it is immediately clear that gluing a G-covering from this gluing data yields a G-covering that is isomorphic as pointed G-covering to p. We can also see from the construction that the gluing data assigned to the pointed G-covering $p : (Y^{\vartheta}_{\mathscr{A}}, y) \to (X, x)$ is isomorphic to the gluing data p was constructed from. This gives us the following result.

Lemma 3.1.3. *Let G be a fixed group. Let X a topological space and let $x \in X$. Let $\mathscr{O} = \{U_a : a \in \mathscr{A}\}$ be an open cover of X, such that $x \in U_a$ for all $a \in \mathscr{A}$.*

There is a bijection between the set of gluing data $(\{p_a\}, \{\vartheta_{ab}\})$ over $\mathscr{O}$ with the respect to x modulo isomorphism and pointed G-coverings $p : (Y, y) \to (X, x)$ modulo isomorphism.

The bijection is given by the assignment of the gluing data $(\{p|_{p^{-1}U_a}\}, \{\mathrm{id}_{p^{-1}(U_a \cap U_b)}\})$ to a pointed G-covering $p : Y \to X$ and the assignment of the G-covering $p : (Y^{\vartheta}_{\mathscr{A}}, y) \to (X, x)$ to the gluing data $(\{p_a\}, \{\vartheta_{ab}\})$

3.2. VAN KAMPEN'S THEOREM

From now we suppose that X is an unloopable space and $\mathscr{O} = \{U_a : a \in \mathscr{A}\}$ is an open cover that is closed under finite intersection. Suppose there is some $x \in X$ with $x \in U_a$ for all $a \in \mathscr{A}$ and suppose each U_a is unloopable.

Then we can make use of the Classification of G-coverings 2.4.3 to translate Lemma 3.1.3 into group-theoretic terms.

Let $a, b \in \mathscr{A}$ and suppose that $U_a \subseteq U_b$. We then write $i_{ab} : U_a \hookrightarrow U_b$ for the inclusion. We write $i_a : U_a \hookrightarrow X$ for the inclusion of a member of the open cover in X. These inclusions form a commutative diagram of pointed topological spaces. In the standard case of $\mathscr{O} = \{U, V, U \cap V\}$, this diagram is as follows:

$$\begin{array}{ccc} (U \cap V, x) & \xrightarrow{i_{U \cap V, U}} & (U, x) \\ \downarrow{\scriptstyle i_{U \cap V, V}} & & \downarrow{\scriptstyle i_U} \\ (V, x) & \xrightarrow{i_V} & (X, x)\,. \end{array}$$

This commutative diagram induces a commutative diagram of fundamental groups, see Proposition 1.3.3. In the case of $\mathscr{O} = \{U, V, U \cap V\}$ this looks as follows:

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \xrightarrow{i_{U\cap V,U*}} & \pi_1(U, x) \\ \Big\downarrow{\scriptstyle i_{U\cap V,V*}} & & \Big\downarrow{\scriptstyle i_{U*}} \\ \pi_1(V, x) & \xrightarrow{i_{V*}} & \pi_1(X, x)\,. \end{array}$$

Van Kampen's theorem now gives a relation between $\pi_1(X, x)$ and the other fundamental groups and maps in between them in the diagram. We first concretely define a diagram of groups

Definition 3.2.1. Let $\{G_a : a \in \mathscr{A}\}$ a collection of groups and suppose that for some pairs $a, b \in \mathscr{A}$ a group homomorphism $f_{ab} : G_a \to G_b$ is given, such that if f_{ab} and f_{bc} are defined, f_{ac} is defined as well and $f_{ac} = f_{bc} \circ f_{ab}$.

We write

$$\mathscr{A} \times_f \mathscr{A} = \{(a, b) \in \mathscr{A} : f_{ab} \text{ is defined}\}.$$

We then call the tuple

$$(\{G_a : a \in \mathscr{A}\}, \{f_{ab} : (a, b) \in \mathscr{A} \times_f \mathscr{A}\})$$

a *diagram of groups*.

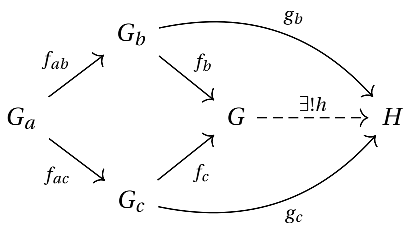
If $\mathscr{A}$ is finite, we can draw a diagram of groups by writing down all groups G_a and drawing arrows between G_a and G_b if f_{ab} is defined.

Definition 3.2.2. Suppose $(\{G_a : a \in \mathscr{A}\}, \{f_{ab} : (a, b) \in \mathscr{A} \times_f \mathscr{A}\})$ is a diagram of groups. Let G be a group and suppose for every $a \in \mathscr{A}$ a group homomorphism $f_a : G_a \to G$ is given, such that $f_a = f_b \circ f_{ab}$ for all $(a, b) \in \mathscr{A} \times_f \mathscr{A}$.

We call the tuple $(G, \{f_a : a \in \mathscr{A}\})$ a *pushout* of the diagram of groups $(\{G_a\}, \{f_{ab}\})$ if the following property holds:

Let H be an arbitrary group and let $g_a : G_a \to H$ be a group homomorphism for all $a \in \mathscr{A}$ and suppose that $g_a = g_b \circ f_{ab}$ for all $(a, b) \in \mathscr{A} \times_f \mathscr{A}$. Then there is a unique map $h : G \to H$, such that $g_a = h \circ f_a$ for all $a \in \mathscr{A}$.

In short, the definition says the diagram



should commute, if $(a, b), (b, c) \in \mathscr{A} \times_f \mathscr{A}$.

Remark. Pushouts are unique up to unique isomorphism if they exist, see Example 5.1.6.

Corollary 3.2.3. *A pushout is generated by the images of G_a under the group homomorphisms f_a.*

Proof. Suppose $(G, \{f_a\})$ is the pushout of the diagram of groups consisting of the groups G_a and homomorphisms f_{ab}. Let G' be the group generated by $f_a(G_a)$. Then $G' \subseteq G$ and we write $h' = h|_{G'} : G' \to H$. Let $a \in \mathscr{A}$. Then $g_a = h \circ f_a$ by assumption. But $f_a(G_a) \subset G'$, so $h \circ f_a = h' \circ f_a$ and we see that

$$g_a = h' \circ f_a.$$

As pushouts are unique we must have $G = G'$. □

Remark. Usually we call G together with the group homomorphisms $\{f_a\}$ a *colimit* of the diagram and reserve the term pushout for the case

$$\begin{array}{ccc} G_a & \xrightarrow{f_{ab}} & G_b \\ \downarrow{\scriptstyle f_{ac}} & & \downarrow{\scriptstyle f_b} \\ G_c & \xrightarrow{f_c} & G\,. \end{array}$$

If $(G, \{f_b, f_c\})$ is a pushout, we call the diagram above a *pushout diagram*.

We will see a slightly different definition for colimit in Definition 5.1.5 and use pushout for now for the general case as well.

Let X be a topological space and $\mathscr{O} = \{U_a : a \in \mathscr{A}\}$ be an open cover. Let $a, b \in \mathscr{A}$ and write $U_c = U_a \cap U_b$. Then the diagram

$$\begin{array}{ccccc} & & \pi_1(U_a, x) & & \\ & \overset{i_{ca*}}{\nearrow} & & \overset{i_{a*}}{\searrow} & \\ \pi_1(U_c, x) & & & & \pi_1(X, x) \\ & \underset{i_{cb*}}{\searrow} & & \underset{i_{b*}}{\nearrow} & \\ & & \pi_1(U_b, x) & & \end{array}$$

is commutative, so it satisfies the condition in Definition 3.2.2

It now may not be a surprise that $\pi_1(X, x)$ together with the group homomorphisms $\{i_{a*}\}$ is the pushout of the diagram consisting of the groups $\pi_1(U_a, x)$ and the group homomorphisms i_{ab*}. This is exactly what Van Kampen's Theorem says.

Theorem 3.2.4 (Van Kampen's Theorem)**.** *Suppose X is a topological space and let $x \in X$. Let $\mathscr{O} = \{U_a : a \in \mathscr{A}\}$ be an open cover of X that is closed under finite intersection and suppose that $x \in U_a$ for every $a \in \mathscr{A}$. Furthermore, suppose that X and U_a are all unloopable.*

Then $\pi_1(X, x)$ together with the group homomorphisms $\{i_{a*}\}$ is the pushout of the diagram consisting of the groups $\pi_1(U_a, x)$ for $a \in \mathscr{A}$ and the group homomorphisms i_{ab*} for $a, b \in \mathscr{A}$, such that $U_a \subseteq U_b$.

Corollary 3.2.5 (Van Kampen's Theorem for the usual covering)**.** *Suppose X is a topological space and let U,V be two open subspaces, such that $X = U \cup V$. Let $x \in U \cap V$ and suppose that $X, U, V, U \cap V$ are all unloopable.*

Then $\pi_1(X,x)$ together with the group homomorphisms i_{U*} and i_{V*} is the pushout of the diagram

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \xrightarrow{i_{U\cap V,U*}} & \pi_1(U,x) \\ \downarrow{\scriptstyle i_{U\cap V,V*}} & & \\ \pi_1(V,x) & & \end{array}$$

Proof of Theorem 3.2.4. Let G be an arbitrary fixed group. Let for all $a \in \mathscr{A}$ a group homomorphism $g_a : \pi_1(U_a, x) \to G$ be given and suppose that $g_a = g_b \circ i_{ab*}$ for every pair $a, b \in \mathscr{A}$, with $U_a \subseteq U_b$.

Via the Classification of G-coverings we can link these group homomorphisms g_a to pointed G-coverings $p_a : (Y_a, y_a) \to (U_a, x)$.

Let $a, b \in \mathscr{A}$ and let $c \in \mathscr{A}$, such that $U_c = U_a \cap U_b$, so $U_c \subset U_a$ and $U_c \subset U_b$. Then by assumption we have the equality $g_a \circ i_{ca*} = g_b \circ i_{cb*}$. By Lemma 2.3.6 $g_a \circ i_{ca*}$ is the monodromy representation belonging to the restriction

$$p_a|_{p_a^{-1}U_c} : (p_a^{-1}U_c, y_a) \to (U_c, x).$$

Similarly $g_b \circ i_{cb*}$ is the monodromy representation belonging to the restriction

$$p_b|_{p_b^{-1}U_c} : (p_b^{-1}U_c, y_b) \to (U_c, x).$$

The equality $g_a \circ i_{ca*} = g_b \circ i_{cb*}$ now induces an isomorphism of pointed G-coverings

$$\vartheta_{ab} : (p_a^{-1}U_c, y_a) \xrightarrow{\sim} (p_b^{-1}U_c, y_b).$$

In this way we obtain gluing data

$$(\{p_a : a \in \mathscr{A}\}, \{\vartheta_{ab} : a, b \in \mathscr{A}\}).$$

Lemma 3.1.3 now gives us a (unique) G-covering

$$p : (Y_{\mathscr{A}}^{\vartheta}, y) \to (X, x),$$

with the property that $p_a = p \circ \varphi_a$ We write $h : \pi_1(X,x) \to G$ for the monodromy representation belonging to this covering. The group homomorphism h is now such that $g_a = h \circ i_{a*}$, which follows from Lemma 2.3.6. The only thing left to show is that this h is unique.

Suppose there is some $h' : \pi_1(X,x) \to G$ that also satisfies $g_a = h' \circ i_{a*}$ for all $a \in \mathscr{A}$ and suppose $h \neq h'$. Let $p_{h'} : (Y_{h'}, y_{h'}) \to (X,x)$ the pointed G-covering corresponding to h' from the bijection in Theorem 2.4.3. Then p and $p_{h'}$ are not isomorphic as G-coverings as $h \neq h'$.

Now let

$$(\{p_{h'}|_{p^{-1}U_a} : a \in \mathscr{A}\}, \{\mathrm{id}_{p_{h'}^{-1}(U_a \cap U_b)} : a, b \in \mathscr{A}\})$$

be the gluing data corresponding to the pointed G-covering $p_{h'}$ from the bijection in Lemma 3.1.3. Then this gluing data is not isomorphic to the gluing data $(\{p_a\}, \{\vartheta_{ab}\})$ from which we constructed the G-covering p, as p and $p_{h'}$ are not isomorphic as G-coverings.

The covering map p_a was constructed from the monodromy representation g_a using the bijection from Theorem 2.4.3. From Lemma 2.3.6 we know that the monodromy representation that corresponds to $p_{h'}|_{p^{-1}U_a}$ is equal to $h' \circ i_{a*}$. But the equality $g_a = h' \circ i_{a*}$ gives us that these coverings are isomorphic as G-coverings. This means the gluing data $(\{p_a\}, \{\vartheta_{ab}\})$ is isomorphic to the gluing data $(\{p_{h'}|_{p^{-1}U_a}\}, \{\mathrm{id}_{p_{h'}^{-1}(U_a \cap U_b)}\})$, which is a contradiction to what we have proven before.

We conclude that our assumption was wrong, so $h' = h$ and h is the unique group homomorphism that satisfies $g_a = h \circ i_{a*}$ for all $a \in \mathscr{A}$. □

Van Kampen's Theorem is intended to calculate fundamental groups from spaces consisting of smaller spaces that we know the fundamental group of. We will show some examples of this.

Example 3.2.6. We write

$$S^n = \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}$$

for the n-dimensional unit sphere. Let $n \geqslant 2$ and let $x \in S^n$ be the point $x = (0, ..., 0, 1)$. Note that S^1 is the circle and we already saw that $\pi_1(S^1, x) \cong \mathbb{Z}$. We consider the open subsets of S^n

$$U = S^n \setminus (1, 0, ..., 0)$$
$$V = S^n \setminus (-1, 0, ..., 0)$$

The spaces U and V are the unit spheres with opposite points missing. Through stereographical projection, both spaces are homeomorphic to $\mathbb{R}^n$, see (Lee [11], Example 3.21). $\mathbb{R}^n$ is contractible and thus simply connected, see Example 1.2.4 and Corollary 1.4.1. We then get the following diagram of groups

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \xrightarrow{i_{U \cap V, U*}} & \{e\} \\ {\scriptstyle i_{U \cap V, V*}} \downarrow & & \\ \{e\} & & \end{array}$$

As U, V and $U \cap V$ are all unloopable, Van Kampen's Theorem now says that $\pi_1(S^n, x)$ together with the group homomorphisms i_{U*} and i_{V*} is the pushout of this diagram. The group homomorphisms i_{U*} and i_{V*} must be trivial as their domain is the trivial group. By Corollary 3.2.3, $\pi_1(S^n, x)$ is now generated by the images of $\{e\}$ under i_{U*} and i_{V*}, which are trivial, so $\pi_1(S^n, x)$ must be trivial itself as well.

We conclude that the unit sphere S^n is simply connected for $n \geqslant 2$.

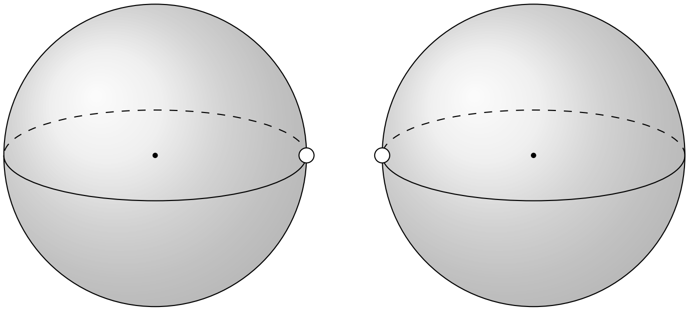


Figure 3.2.1: The subspaces U (left) and V (right) for S^n with $n = 2$.

Example 3.2.7. Next we will consider the bouquet of n circles. This space is the disjoint union of n circles which are glued together at one point. We write $S = \bigcup_{i=1}^n S_i^1$ for the disjoint union of n circles. On each circle we specify one point $z_i \in S_i^1$. We then identify these points z_i under the equivalence relation $\sim$. Then the bouquet of n circles is $B_n = S/\sim$ and we set the equivalence class z of these z_i as basepoint.

The bouquet is the union of n open subspaces $U_1, ..., U_n$ that are all homotopy equivalent to the circle. Let a_i be the loop that generates the fundamental group of the subspace U_i. The intersection of two or more of the subspaces U_i is contractible, so the fundamental group is trivial.

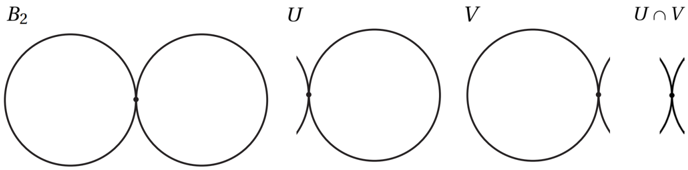


Figure 3.2.2: The covering of B_n for $n = 2$.

For the case $n = 2$, we have the following diagram of groups

$$\begin{array}{ccc} \{e\} & \xrightarrow{i_{1*}} & \langle a_1 \rangle \\ {\scriptstyle i_{2*}}\downarrow & & \downarrow{\scriptstyle j_{1*}} \\ \langle a_2 \rangle & \xrightarrow{j_{2*}} & \pi_1(B_2, x)\,. \end{array}$$

The group $\pi_1(B_n, z)$, which is the pushout of the diagram consisting of the groups $\langle a_i \rangle$ and the homomorphisms induced by inclusions, is called the *free group* on the generators $a_1, ..., a_n$.

As each $\langle a_i \rangle \cong \mathbb{Z}$ we also say

$$\pi_1(B_n, z) \cong \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} = \mathop{\Large *}_{i=1}^{n} \mathbb{Z},$$

which we call the *n-fold free product* of $\mathbb{Z}$.

There also other definitions of the free product of spaces or the free group on the generators. An extensive description can be found in (Lee [11], Chapter 9). The free group on the generators $a_1, ..., a_n$ is then defined as a group consisting of so-called words, an ordered tuple of finite length containing elements of the groups $\langle a_1 \rangle$, $\langle a_2 \rangle$, ..., $\langle a_n \rangle$. The operation is concatenation of the tuple and there is an equivalence relation, so-called elementary reduction, on these tuples.

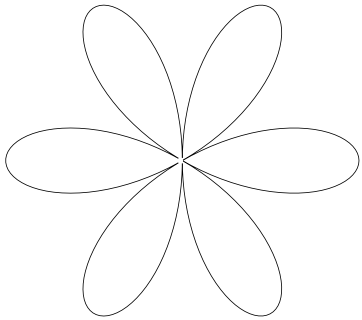


Figure 3.2.3: The bouquet of 6 circles.

Though this theorem is very useful to calculate fundamental groups of very complex spaces, it falls short for the most fundamental example in algebraic topology: the circle. If we choose an open cover of the circle consisting of unloopable subsets that all contain the base point, these are all homeomorphic to an open interval. The intersections of the subsets are in all cases homeomorphic to the union of multiple open intervals. This means that the intersection is not unloopable, since it is not (path-)connected. We cannot use Van Kampen's Theorem to calculate the fundamental group of the circle.

The main problem is the requirement to choose one base point. If we were able to choose multiple base points and consider the homotopy classes of all paths inbetween the base points, we have a much bigger overview of the situation. We can then cover the circle with two open subsets, each homeomorphic to an open interval, and consider base points in both path-connected components of the intersection. This motivates the definition of the fundamental groupoid, which will be introduced in Chapter 4. We can then generalise Van Kampen's Theorem to the case of fundamental groupoids and calculate the fundamental group of the circle through this upgraded theorem.

3.3. Covering of the Klein bottle

As an example of the stitching of covering spaces we used in the proof of Van Kampen's Theorem, we will construct the torus as an $\mathbb{Z}_2$-covering of the Klein bottle.

The spaces we will look at can all be described as a quotient space of the plane $\mathbb{R}^2$. For this we define $a, b : \mathbb{R}^2 \to \mathbb{R}^2$ as follows:

$$a : (x, y) \mapsto (x, y + 1)$$
$$b : (x, y) \mapsto (x + 1, -y).$$

These maps are both homeomorphisms of the plane with itself, so we can view them as elements of $\mathrm{Aut}(\mathbb{R}^2)$. We can then let the subgroups generated by a and b act on $\mathbb{R}^2$ in the natural way and take the quotient to these actions. Using this we can give the following definitions.

Definition 3.3.1.
- The *Klein bottle* is the quotient space $K = \mathbb{R}^2/\langle a, b\rangle$.
- The *Möbius band* is the quotient space[1][2] $M = (\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2}))/\langle b\rangle$.
- The *torus* is the quotient space $\mathbb{T} = \mathbb{R}^2/\langle a, b^2\rangle$.
- The *cylinder* is the quotient space $C = (\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2}))/\langle b^2\rangle$.

It is left to the reader to verify that these definitions yield homeomorphic topological spaces as other definitions of these spaces that may be found in literature. Note these four spaces are all unloopable.

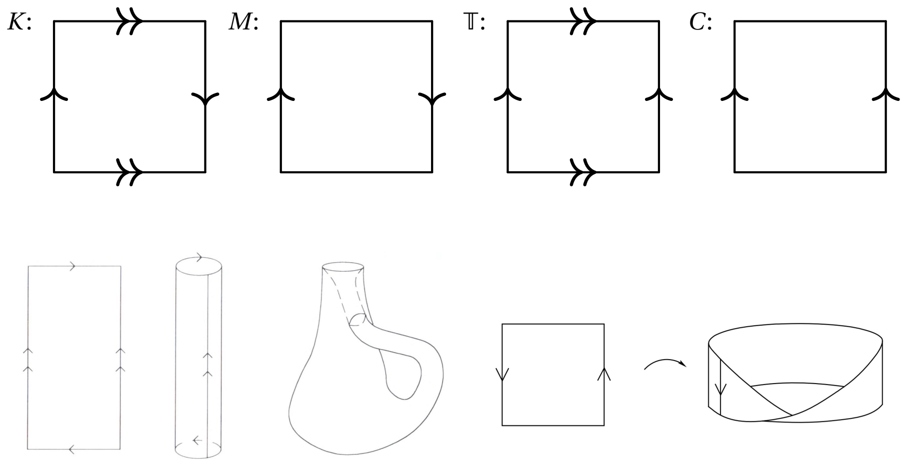


Figure 3.3.1: The Klein bottle can be constructed in four-dimensional space (Fulton [5], Section 8b). The Möbius band can be constructed in three dimensional space (Lee [11], Chapter 6).

Before we begin with covering the Klein bottle, we first note some interesting properties of the spaces we defined, which we can use in the covering of the Klein bottle.

Lemma 3.3.2. *The Möbius band M and the cylinder C are homotopy-equivalent to the circle S^1. This implies that $\pi_1(M) \cong \pi_1(C) \cong \mathbb{Z}$.*

Proof. Let $a' : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x+1$. We see that the natural action of $\langle a'\rangle$ on $\mathbb{R}$ is the same as the action of $\mathbb{Z}$ on $\mathbb{R}$ in Example 2.2.2. In example 2.2.7 we saw that the quotient space $\mathbb{R}/\mathbb{Z}$ is homeomorphic to the circle S^1, so the quotient space $\mathbb{R}/\langle a'\rangle$ is as well.

[1]We only look at the restriction of b to $\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$, but still denote it by b.
[2]Normally the Möbius band and the cylinder are defined with $[0,1]$ instead of $(0,1)$, but for the purpose of this example we look at this definition.

3

Similarly as in Example 1.2.4 we can retract the interval $(-\frac{1}{2},\frac{1}{2})$ to $\{0\}$, which is a homotopy equivalence. Then $\mathbb{R}\times(-\frac{1}{2},\frac{1}{2})$ is homotopy equivalent to $\mathbb{R}\times\{0\}$, so

$$M = (\mathbb{R}\times(-\frac{1}{2},\frac{1}{2}))/\langle b\rangle \simeq (\mathbb{R}\times\{0\})/\langle b\rangle,$$
$$C = (\mathbb{R}\times(-\frac{1}{2},\frac{1}{2}))/\langle b^2\rangle \simeq (\mathbb{R}\times\{0\})/\langle b^2\rangle,$$

Note that retracting the interval $(-\frac{1}{2},\frac{1}{2})$ to any other point than 0 would have been a problem for the Möbius band as (x,b) and $(x+1,-b)$ represent the same point. The fact that $0=-0$ makes that retracting to 0 is well-defined.

We will show explicit homeomorphisms $(\mathbb{R}\times\{0\})/\langle b\rangle \cong \mathbb{R}/\langle a'\rangle$ and $(\mathbb{R}\times\{0\})/\langle b^2\rangle \cong \mathbb{R}/\langle a'\rangle$, which show the Möbius band M and cylinder C are homotopy-equivalent to the circle. It is left to the reader to verify that these are indeed homeomorphisms.

We write $\overline{(x,0)}$ for the equivalence class of $(x,0)\in\mathbb{R}\times\{0\}$ under the action of $\langle b\rangle$, resp. $\langle b^2\rangle$ and $\widetilde{x}$ for the equivalence of $x\in\mathbb{R}$ under the action of $\langle a'\rangle$. We then define

$$f:(\mathbb{R}\times\{0\})/\langle b\rangle \to \mathbb{R}/\langle a'\rangle,$$
$$\overline{(x,0)} \mapsto \widetilde{x},$$

and

$$g:(\mathbb{R}\times\{0\})/\langle b^2\rangle \to \mathbb{R}/\langle a'\rangle,$$
$$\overline{(x,0)} \mapsto \frac{\widetilde{x}}{2}.$$

□

We do not want to lose the geometry of the situation, so we usually keep writing $\pi_1(M)$ and $\pi_1(C)$ instead of immediately replacing it with $\mathbb{Z}$. From the homotopy equivalence we can also see that the path γ_C which follows the *central circle* $(\mathbb{R}\times\{0\})/\langle b\rangle$ is the loop that generates $\pi_1(M)$.

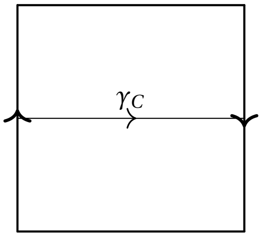


Figure 3.3.2: The path γ_C that follows the central line

As we now know the fundamental group of the Möbius band, we also now the number of $\mathbb{Z}_2$-coverings of M. There are exactly two group homomorphisms from $\pi_1(M)=\mathbb{Z}$ to $\mathbb{Z}_2$, the trivial and surjective homomorphisms. From Example 2.4.4 and 2.4.5 we know that these are the trivial covering and a covering by a connected space. We will show that the connected covering space is the cylinder.

We describe an even group action by $\mathbb{Z}_2$ on the cylinder $C = (\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2}))/\langle b^2 \rangle$. We then prove that the quotient space of this action is homeomorphic to the Möbius band, which proves that the cylinder covers the Möbius band.

We again write $\overline{(x,y)}$ for the equivalence class of $(x,y) \in \mathbb{R} \times (0,1)$ under the action of $\langle b^2 \rangle$. Then we define

$$\begin{aligned} \cdot : \mathbb{Z}_2 \times C &\to C \\ \overline{0} \cdot \overline{(x,y)} &= \overline{(x,y)} \\ \overline{1} \cdot \overline{(x,y)} &= \overline{(x+1,-y)} \end{aligned}$$

The action is clearly even. Note that

$$\overline{1} \cdot (\overline{1} \cdot \overline{(x,y)}) = \overline{1} \cdot \overline{(x+1,-y)} = \overline{(x+2,y)} = \overline{(x,y)}.$$

Lemma 3.3.3. *The quotient space $C/\mathbb{Z}_2$ for the action $\cdot$ is homeomorphic to the Möbius band M. This means that the cylinder is $\mathbb{Z}_2$-covering space of the Möbius band.*

Proof. This is the same as proving the equivalence relation on $\mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$ induced by $(\langle b^2 \rangle / \mathbb{Z}_2)$ is the same as the equivalence relation induced by $\langle b \rangle$. We will note two equivalent elements under the first equivalence by $\approx$ and under the latter by $\sim$.

Let $(x,y), (x',y') \in \mathbb{R} \times (-\frac{1}{2}, \frac{1}{2})$ and suppose $(x,y) \approx (x',y')$. Then $(x,y) \sim (x',y')$ as $\overline{1} \cdot \overline{(x,y)} = \overline{b(x,y)}$.

Now suppose that $(x,y) \sim (x',y')$. Then there is a $n \in \mathbb{Z}$ with $b^n(x,y) = (x',y')$. If n is even then $\overline{b^n(x,y)} = \overline{0} \cdot \overline{(x,y)}$. If n is odd, then $\overline{b^n(x,y)} = \overline{1} \cdot \overline{(x,y)}$. In either case we have $(x,y) \approx (x',y')$. □

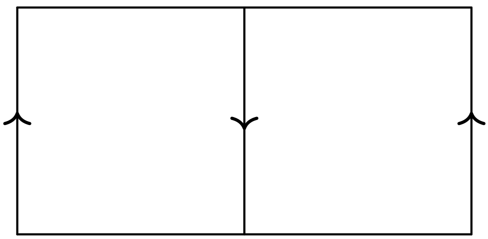


Figure 3.3.3: The pre-image of the Möbius band in the cylinder.

We now turn our attention to the Klein bottle and use the proof of Van Kampen's Theorem 3.2.4 to construct the torus as a $\mathbb{Z}$-covering of the Klein bottle.

We consider the open cover $\{U, V\}$ of K, where

$$\begin{aligned} U &= \left(\mathbb{R} \times (-\frac{1}{4}, \frac{1}{4})\right) / \langle a, b \rangle \cong \left(\mathbb{R} \times (-\frac{1}{4}, \frac{1}{4})\right) / \langle b \rangle, \\ V &= \left(\mathbb{R} \times (\frac{1}{6}, \frac{5}{6})\right) / \langle a, b \rangle \cong \left(\mathbb{R} \times (\frac{1}{6}, \frac{5}{6})\right) / \langle b \rangle. \end{aligned}$$

We get the homeomorphisms as the automorphism a does not affect U or V anymore as there are no two distinct points (x, y) and (x', y') in U or V with $a(x, y) = (x', y')$.

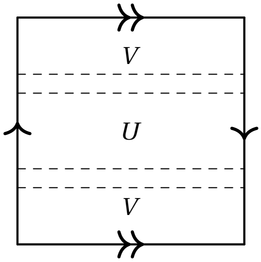


Figure 3.3.4: The open cover $\{U, V\}$ of the Klein bottle.

The intersection of U and V is

$$\begin{aligned} U \cap V &= \left(\mathbb{R} \times ((-\frac{1}{4}, -\frac{1}{6}) \cup (\frac{1}{6}, \frac{1}{4}))\right) / \langle a, b \rangle \\ &\cong \left(\mathbb{R} \times ((-\frac{1}{4}, -\frac{1}{6}) \cup (\frac{1}{6}, \frac{1}{4}))\right) / \langle b \rangle \cong \left(\mathbb{R} \times (\frac{1}{6}, \frac{1}{4})\right) / \langle b^2 \rangle. \end{aligned}$$

The last homeomorphism comes from the fact that $\{x\} \times (-\frac{1}{4}, -\frac{1}{6})$ is identified with $\{x+1\} \times (\frac{1}{6}, \frac{1}{4})$ for every $x \in \mathbb{R}$.

We see that U and V are both Möbius bands and $U \cap V$ is a cylinder. This yields the following commutative diagram of set inclusions and the induced commutative diagram of fundamental groups:

$$\begin{array}{ccc} C & \xrightarrow{i_1} & M \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ M & \xrightarrow{j_2} & K \end{array} \qquad\qquad \begin{array}{ccc} \pi_1(C) & \xrightarrow{i_{1*}} & \pi_1(M) \\ {\scriptstyle i_{2*}}\downarrow & & \downarrow{\scriptstyle j_{i*}} \\ \pi_1(M) & \xrightarrow{j_{2*}} & \pi_1(K) \end{array}$$

According to Van Kampen's Theorem 3.2.4, the latter is a pushout diagram.

We saw that the generating loop of the fundamental group of the Möbius band is the path γ_C that follows the central circle. The generating loop γ of the cylinder $U \cap V$ is homotopic to $\gamma_C \odot \gamma_C$ as it walks the length of U twice. This can also be seen from the well-known fact that the boundary of the Möbius band wraps around the center circle twice. This means that the map $i_{1*} : \pi_1(C) \to \pi_1(M)$ is given by $\gamma \mapsto \gamma_C \odot \gamma_C$. If we use the isomorphism $\pi_1(C) \cong \pi_1(M) \cong \mathbb{Z}$, this corresponds to the map $x \mapsto 2x$. By symmetry we have $i_{2*} = i_{1*}$.

We will now glue covering spaces of U and V to get a covering space of the Klein bottle. We first specify a basepoint $x \in X$ with $x \in U \cap V$. We let $p_1 : (C_1, y_1) \longrightarrow (U, x)$ and

$p_2 : (C_2, y_2) \to (V, x)$ be the covering maps of the cylinder, denoted by C_1 and C_2 onto the Möbius band from Lemma 3.3.3.

The monodromy representations ρ_1 and ρ_2 belonging to these coverings are the surjective homomorphisms $\mathbb{Z} \to \mathbb{Z}_2$. The monodromy representation belonging to the restriction

$$p_1|_{p_1^{-1}(U\cap V)} : (p_1^{-1}(U\cap V), y_1) \to (U\cap V, x) \text{ and } p_2|_{p_2^{-1}(U\cap V)} : (p_2^{-1}(U\cap V), y_2) \to (U\cap V, x)$$

are $\rho_1 \circ i_{1*}$ and $\rho_2 \circ i_{2*}$, which are the same. The kernels of ρ_1 and ρ_2 are $2\mathbb{Z}$, which are exactly the images of i_{1*} and i_{2*}. The monodromy representations belonging to $p_1|_{p_1^{-1}(U\cap V)}$ and $p_2|_{p_2^{-1}(U\cap V)}$ are therefore trivial.

The $\mathbb{Z}_2$-covering of $U \cap V$ in this construction should thus be the trivial covering $p' : C \sqcup C \to C$.

This corresponds to the fact that

$$\begin{aligned} p_1^{-1}(U\cap V) &= p_1^{-1}\left(\left(\mathbb{R} \times ((-\frac{1}{4}, -\frac{1}{6}) \cup (\frac{1}{6}, \frac{1}{4}))\right) / \langle b \rangle\right) \\ &= \left(\mathbb{R} \times ((-\frac{1}{4}, -\frac{1}{6}) \cup (\frac{1}{6}, \frac{1}{4}))\right) / \langle b^2 \rangle \\ &= \left(\mathbb{R} \times (-\frac{1}{4}, -\frac{1}{6})\right) / \langle b^2 \rangle \cup \left(\mathbb{R} \times (\frac{1}{6}, \frac{1}{4})\right) / \langle b^2 \rangle \\ &\cong C \sqcup C. \end{aligned}$$

The same holds for $p_2^{-1}(U\cap V)$. These are isomorphic as $\mathbb{Z}_2$-coverings in the natural sense, with homeomorphism $\vartheta : p_1^{-1}(U\cap V) \xrightarrow{\sim} p_2^{-1}(U\cap V)$.

We now show that in the $\mathbb{Z}_2$-covering $p : C_1 \cup_\vartheta C_2 \to K$, we have an homeomorphism $C_1 \cup_\vartheta C_2 \cong \mathbb{T}$.

We note the following homeomorphisms:

$$\begin{aligned} U &\cong (\mathbb{R} \times (-\frac{1}{2}, \frac{1}{6}))/\langle b \rangle, \quad \text{and } V \cong (\mathbb{R} \times (0, \frac{2}{3}))/\langle b \rangle, \\ C_1 &\cong (\mathbb{R} \times (-\frac{1}{2}, \frac{1}{6}))/\langle b^2 \rangle, \text{ and } C_2 \cong (\mathbb{R} \times (0, \frac{2}{3}))/\langle b^2 \rangle. \end{aligned}$$

Then

$$U \cap V \cong \left(\mathbb{R} \times ((-\frac{1}{2}, -\frac{1}{3}) \cup (0, \frac{1}{6}))\right) / \langle b \rangle$$

is covered by

$$C_1' \sqcup C_2' = (\mathbb{R} \times (-\frac{1}{2}, -\frac{1}{3}))/\langle b^2 \rangle \sqcup (\mathbb{R} \times (0, \frac{1}{6}))/\langle b^2 \rangle.$$

Stitching C_1 and C_2 along the identity on C_2' gives the cylinder $C' = (\mathbb{R} \times (-\frac{1}{2}, \frac{2}{3}))/\langle b^2 \rangle$. Identifying $C_1' \subseteq C_1$ and $C_1' \subseteq C_2$ is exactly identifying points on C' where the y-coordinates differ by one, which is taking the quotient to $\langle a \rangle$. This results in the homeomorphism

$$\begin{aligned} C_1 \cup_\vartheta C_2 &\cong (\mathbb{R} \times (-\frac{1}{2}, \frac{2}{3}))/\langle a, b^2 \rangle \\ &\cong \mathbb{R}^2/\langle a, b^2 \rangle \cong \mathbb{T}. \end{aligned}$$

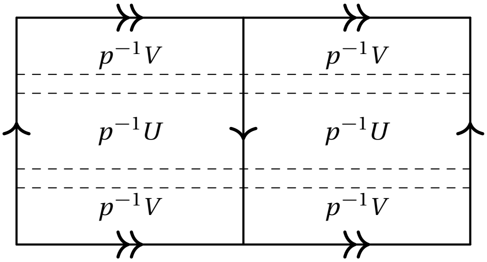


Figure 3.3.5: The embedding of two Klein bottles in a torus, made by glueing $p^{-1}U$ and $p^{-1}V$

We have now constructed the torus $\mathbb{T}$ as a $\mathbb{Z}_2$-covering of the Klein bottle K by gluing coverings of subsets of K that satisfy the requirements of Van Kampen's Theorem.

4

The Fundamental Groupoid

In this section we will look at a generalisation of the fundamental group, the so-called fundamental groupoid. Here we lose the need to choose just one basepoint from which we form loops and instead focus on multiple points — possibly all points — and the paths between them. Mathematician Alexander Grothendieck wrote the following words about this in his *Esquisse d'un Programme* [7]:

> " [...] people still obstinately persist, when calculating with fundamental groups, in fixing a single base point, instead of cleverly choosing a whole packet of points which is invariant under the symmetries of the situation, which thus get lost on the way. In certain situations (such as descent theorems for fundamental groups à la Van Kampen Theorem) it is much more elegant, even indispensable for understanding something, to work with fundamental groupoids with respect to a suitable packet of base points, [...] "

In general a groupoid looks like a group, except that the group operation is not defined on every pair of elements. To rigorously define the notion of a groupoid we turn to category theory.

4.1. Categories

Many concepts and structures in different branches of mathematics are formulated very precisely in short terms by category theory. Normally when we do mathematics we first specify what we are working with - groups, topological spaces, sets. In category theory these are all objects and the interesting maps between them are morphisms.

Definition 4.1.1. A *category* $\mathscr{C}$ consists of:

1. a collection of *objects*, $\mathrm{ob}(\mathscr{C})$,
2. for every two objects A and B of $\mathscr{C}$ a set $\mathrm{Hom}_{\mathscr{C}}(A,B)$, whose elements are called *morphisms* or *arrows*,

3. for every object A of $\mathscr{C}$ a specified element $\mathrm{id}_A \in \mathrm{Hom}_{\mathscr{C}}(A,A)$ called the *identity element*,

4. for every three objects A, B and C of $\mathscr{C}$ a map $\circ : \mathrm{Hom}_{\mathscr{C}}(B,C) \times \mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{C}}(A,C)$, called the *composition*,

such that the following axioms hold:

1. (Neutrality of the identity) For every pair of objects A and B of $\mathscr{C}$ and every $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$ we have $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$,

2. (Associativity of the composition) For objects A, B, C and D of $\mathscr{C}$ and morphisms $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$, $g \in \mathrm{Hom}_{\mathscr{C}}(B,C)$, $h \in \mathrm{Hom}_{\mathscr{C}}(C,D)$ we have $(h \circ g) \circ f = h \circ (g \circ f)$.

Remark. We often write $\mathrm{Hom}(A,B)$ instead of $\mathrm{Hom}_{\mathscr{C}}(A,B)$ if it is clear in which category we are working.

Remark. Note that the objects form a collection and not necessarily a set. If the objects of a category do form a set, we call it a *small category*.

The abstract definition of a category is best explained by giving some examples. Often morphisms are some sort of map, but the definition gives enough freedom to allow some not so obvious morphisms.

Example 4.1.2. The category **Set** is the category of sets. The objects are all sets and for two sets A and B, $\mathrm{Hom}(A,B)$ is given by the set B^A of all set-theoretic maps from A to B. The composition is the usual composition of maps. Set-theoretic maps are associative and the identity possesses the properties of axiom 1, so all axioms are satisfied.

We can also interpret a set S as a category itself, called **S**. The objects of **S** are all elements of S. For $s, t \in S$ we have $\mathrm{Hom}(s,t) = \varnothing$ if $s \neq t$. The set $\mathrm{Hom}(s,s)$ is given by one element, which then must be the identity. This is an example of a category, where the morphisms are not the usual maps.

The category **Grp** is the category of groups. The objects are all groups. For G and H groups, the set $\mathrm{Hom}(G,H)$ is the set of group homomorphisms from G to H. The composition is the usual composition, which yields another group homomorphism. The identity element is the usual identity map, which is also a group homomorphism. As we can view **Grp** as some sort of subcategory of **Set**, the axioms follow naturally.

By restricting **Grp** to only abelian groups, we get the category **Ab** of abelian groups.

Just like a set, a group G can be interpreted as a category itself as well, the category **BG**. This category exists of just one element, $*$. The set $\mathrm{Hom}(*,*) = G$ consists of all elements of G and composition of two elements is the result of the group operation on these elements. The identity element is the neutral element in the group. We sometimes just write G instead of **BG**, when it is clear from context we are looking at the group as a category.

The category **Top** is the category of topological spaces. The objects are topological spaces and $\mathrm{Hom}(X,Y)$ consists of all continuous maps from X to Y. Composition and the identity element are again given by the usual composition and the identity map. The axioms follow again from the axioms in **Set**.

We can also construct the categories **Top$_*$** and **Top'** which are respectively the categories of pointed topological spaces and the categories of topological spaces with a specified subset. In the latter the objects are pairs (X, A), with $A \subset X$. The set of morphisms $\text{Hom}((X, A), (Y, B))$ are continuous maps $f : X \to Y$, with $f(A) \subset B$.

Another useful category is **Cov**(X) for a topological space X. This is the category with objects (Y, p), with $p : Y \to X$ a covering map. The morphisms are given by morphisms of covering spaces, Definition 2.4.1.

Categories can also be less obvious than the ones below. We consider the category $(\mathbb{N}, \leqslant)$. The objects are all elements of $\mathbb{N}$ and there is (exactly) one morphism from n to m, if $n \leqslant m$. This definition can be generalized to any preordered set.

4.2. GROUPOIDS

The concepts of bijections, group isomorphisms, homeomorphisms etc. fall in categorical terms under the term isomorphism. With this definition, we can also define groupoids.

Definition 4.2.1. A morphism $f : A \to B$ is called an *isomorphism* if there exists an inverse, i.e. if there exists a morphism $g : B \to A$, such that $g \circ f = \text{id}_A$ and $f \circ g = \text{id}_B$.

An isomorphism $f : A \to A$ is called an *automorphism* of A. The set of all automorfisms of A, $\text{Aut}_{\mathscr{C}}(A)$, forms a group under composition.

Definition 4.2.2. A *groupoid* is a category where all morphisms are isomorphisms. The category of groupoids is **Gpd**. The morphisms between groupoids are given by functors, see Definition 4.3.1.

Every group G can be viewed as a groupoid, via the category **BG**. All elements of G have an inverse, so all morphisms in **BG** has an inverse. In particular, every groupoid with only one object is a group.

Though the definition of a groupoid is very concise, the structure of a groupoid is not immediately clear. We can give another, equivalent definition of a groupoid, in the case that it is small[1], that resembles the definition of a group more, see (Szamuely [14], Chapter 2).

A groupoid consists of a set X, which we may view as objects, and a set A, which we may view as arrows. There are also the following maps:

$$\begin{aligned} \text{The source map } s :&\quad A \to X, \\ \text{The target map } t :&\quad A \to X, \\ \text{The identity } e :&\quad X \to A, \\ \text{The inverse } i :&\quad A \to A, \\ \text{A partial operation } m :&\quad A \times_X A \to A. \end{aligned}$$

The set $A \times_X A$, called the *fiber product* over X, is the subspace

$$\{(a, b) \in A \times A : t(a) = s(b)\}$$

[1] That is, the objects form a set.

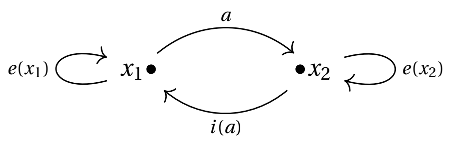
of $A \times A$.

We can interpret the source and target map as the points where the arrow respectively start and end. The partial operation on two arrows is then only defined when the target and source coincide.

As in the definition of groups, these maps should satisfy three axioms:

1. Suppose $(a,b),(b,c) \in A \times_X A$. Then $(m(a,b),c),(a,m(b,c)) \in A \times_X A$ and
$$m(m(a,b),c) = m(a,m(b,c)).$$
2. The map e is a section of s and t. That is, $s \circ e = \mathrm{id}_X$, $t \circ e = \mathrm{id}_X$. Furthermore
$$m(e(s(a)),a) = a \text{ and } m(a,e(t(a))) = a.$$
3. We have $(a,i(a)) \in A \times_X A$ and $(i(a),a) \in A \times_X A$. Furthermore
$$m(a,i(a)) = e(s(a)) \text{ and } m(i(a),a) = e(t(a)).$$

It is helpful to check these axioms for the groupoid with
$$X = \{x_1, x_2\}, \quad A = \{a, i(a), e(x_1), e(x_2)\},$$
of the following form



We get a group out of this definition - again - by looking at a groupoid with one object, $X = \{x\}$. Then $s(a) = t(a) = x$ for all $a \in A$, so $A \times_X A$ is just $A \times A$ and the partial operation m is just the group operation.

It is left to the reader to verify that these two definitions of a groupoid are equivalent.

4.3. FUNCTORS

Just like we can connect objects with each other within a category via morphisms, we can also connect categories with each other. We use functors for this. It is important that the structure is preserved, so a functor must satisfy some axioms.

Definition 4.3.1. Let $\mathscr{C}$ and $\mathscr{D}$ categories. A *functor* $F : \mathscr{C} \to \mathscr{D}$ consists of

1. For every object A of $\mathscr{C}$ an object $F(A)$ of $\mathscr{D}$,
2. For every pair of objects A and B of $\mathscr{C}$ a map
$$F(-) : \mathrm{Hom}_{\mathscr{C}}(A,B) \to \mathrm{Hom}_{\mathscr{D}}(F(A),F(B))$$
,

such that the following axioms are satisfied:

1. For every object A of $\mathscr{C}$ we have $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$,
2. For each three objects A, B and C of $\mathscr{C}$ and all $f \in \mathrm{Hom}(B,C), g \in \mathrm{Hom}(A,B)$ we have $F(f \circ g) = F(f) \circ F(g)$.

Note that isomorphisms are preserved by functors.
It is again a good idea to look at some examples of functors to get a feeling for it.

Example 4.3.2. The identity functor $id_{\mathscr{C}} : \mathscr{C} \to \mathscr{C}$ is the functor that sends every object and every morphism to itself. This is also the identity element in **Gpd**. The composition of the functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{E}$ is given by $(G \circ F)(A) = G(F(A))$ for objects A of $\mathscr{C}$ and $(G \circ F)(f) = G(F(f))$ for morphisms f in $\mathscr{C}$.

For categories such as **Top**, **Grp**, **Gpd**, etc. there is the forgetful functor to **Set**, that sends objects to itself as a set and morphisms to the same morphism as a set-theoretic function. This functor "forgets" the original structure of the objects.

The power set functor $\mathscr{P}$: **Set** → **Set** sends a set to its power set and a map $f : A \to B$ to the map $F(f) : \mathscr{P}(X) \to \mathscr{P}(Y)$ that sends $U \subseteq X$ to $f(U) \subset Y$.

A group action of G on a set X is a functor F : **BG** → **Set**, with $F(*) = X$ and $F(g) = gX$, where $gX : X \to X$ is given by $x \mapsto g \cdot x$. Note that since all elements of G are isomorphisms in **BG**, all gX are bijections. In fact every functor **BG** → **Set** can be interpreted as a group action. We call these functors G-*sets* or permutation group representations of G and denote the category of G-sets by $\boldsymbol{Set^G}$.

The fundamental group is a functor $\mathbf{Top}_*$ → **Grp**. We will prove this in Theorem 4.4.2.

We can also go one step further and connect functors with each other.

Definition 4.3.3. Let $\mathscr{C}$ and $\mathscr{D}$ categories and $F, G : \mathscr{C} \to \mathscr{D}$ functors. A *natural transformation* $\alpha : F \to G$ consists for each object A of $\mathscr{C}$ of an element $\alpha_A : \mathrm{Hom}_{\mathscr{D}}(F(A), G(A))$, such that for every two elements A and B of $\mathscr{C}$ and for every $f \in \mathrm{Hom}_{\mathscr{C}}(A,B)$ the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\ F(B) & \xrightarrow{\alpha_B} & G(B) \end{array}$$

in $\mathscr{D}$ commutes.
If every α_A is an isomorphism, we call α a *natural isomorphism*.

4.4. FUNDAMENTAL GROUPOID

We can now define a generalisation of the fundamental group, the fundamental groupoid. Instead of choosing just one basepoint and considering the loops at this basepoint, we choose multiple points and consider the paths between them. As we cannot concatenate all paths, the result is not a group, but a groupoid, as the name would suggest.

Definition 4.4.1. Let X a topological space and $A \subseteq X$. The *fundamental groupoid* $\Pi(X, A)$ is the category with the points $x \in A$ as objects. The morphisms $x \to y$ are given by the path homotopy classes of paths from x to y.

We write $\Pi(X)$ instead of $\Pi(X, X)$ and call it the *full fundamental groupoid*.

Remark. Viewing the fundamental group $\pi_1(X, x)$ as the category $\mathbf{B\pi_1(X,x)}$ is a special case of the fundamental groupoid, with $A = \{x\}$.

If we know the fundamental groupoid $\Pi(X, A)$ we can extract the fundamental group $\pi_1(X, x)$ from that if $x \in A$, by taking the automorphism group $\mathrm{Aut}_{\Pi(X,A)}(x)$. This yields the path homotopy classes from x to itself, which are just the loops at x.

When defining the fundamental group, we already saw that this is a way to assign a group to a pointed topological space. In category theory, this is captured in the definition of a functor, so it is natural to ask if the fundamental group(oid) is a functor. This is indeed the case as we will see in the theorem below.

Theorem 4.4.2. *The fundamental groupoid is a functor* $\Pi : \textbf{\textit{Top'}} \to \textbf{\textit{Gpd}}$.

Proof. We define $\Pi : \mathbf{Top'} \to \mathbf{Gpd}$ as follows: Let (X, A) be an object of **Top'**, then $\Pi : (X, A) \mapsto \Pi(X, A)$. For $f \in \mathrm{Hom}((X, A), (Y, B))$, the functor $\Pi(f) : \Pi(X, A) \to \Pi(Y, B)$ is given by $\Pi(f)(x) = f(x)$ for $x \in \mathrm{ob}(\Pi(X, A)) = A$ and $\Pi(f)[\gamma] = [f \circ \gamma]$ for $\gamma : x \to y$, $x, y \in A$.

We have seen that $\Pi(X, A)$ indeed is an object of **Gpd**, so Π is well-defined on objects. Let (X, A) and (Y, B) be objects of **Top'** and $f \in \mathrm{Hom}((X, A), (Y, B))$. We only need to check that $\Pi(f)$ is well-defined and that the axioms hold.

As f satisfies $f(A) \subseteq B$, $f(x)$ is an object of $\Pi(Y, B)$ for $x \in A$, so $\Pi(f)$ is well-defined on objects. Let $[\gamma]$ a morphism of $\Pi(X, A)$ and γ' a path in X that is path homotopic to γ. We have to show that $[f \circ \gamma] = [f \circ \gamma']$. Let Γ be a path homotopy from γ to γ'. Then $f \circ \Gamma : [0, 1]^2 \to Y$ is a path homotopy from $f \circ \gamma$ to $f \circ \gamma'$.

Next we show that $\Pi(f)$ satisfies the axioms of a functor from $\Pi(X, A)$ to $\Pi(Y, B)$. The identity element of $x \in \mathrm{ob}(\Pi(X, A)) = A$ is the path homotopy class of the constant path c_x. We see that

$$\Pi(f)[c_x] = [f \circ c_x] = [c_{f(x)}] = \mathrm{id}_{f(x)} = \mathrm{id}_{\Pi(f)(x)}.$$

Furthermore for γ and γ' two arbitrary paths in X with starting and endig points in A and $\gamma(1) = \gamma'(0)$ we have

$$\Pi(f)[\gamma \odot \gamma'] = [f \circ (\gamma \odot \gamma')] = [(f \circ \gamma) \odot (f \circ \gamma')] = [f \circ \gamma] \circ [f \circ \gamma'] = \Pi(f)[\gamma] \circ \Pi(f)[\gamma'].$$

The equalities all follow from carefully checking the definition of $\odot$ and $\circ$ in the context provided.

This proves that $\Pi(f) : \Pi(X, A) \to \Pi(Y, B)$ is a functor. We now only need to show that the functor axioms hold for Π.

The functor $\Pi(\mathrm{id}_{(X,A)}) : \Pi(X, A) \to \Pi(X, A)$ should be the identity functor. Let $x, y \in A$ and $\gamma : x \to y$. Then $\Pi(\mathrm{id}_{(X,A)})(x) = \mathrm{id}_{(X,A)}(x) = x$ and $\Pi(\mathrm{id}_{(X,A)})[\gamma] = [\mathrm{id}_{(X,A)} \circ \gamma] = [\gamma]$, so this is indeed the identity functor.

Next composition should be preserved.

Let $f \in \mathrm{Hom}((X, A), (Y, B))$ and let $g \in \mathrm{Hom}((Y, B), (Z, C))$. Note that

$$\Pi(g \circ f)(x) = (g \circ f)(x) = g(f(x)) = \Pi(g)(\Pi(f)(x)) = (\Pi(g) \circ \Pi(f))(x)$$

and

$$\Pi(g \circ f)[\gamma] = [(g \circ f) \circ \gamma] = [g \circ (f \circ \gamma)] = \Pi(g)(\Pi(f)[\gamma]) = (\Pi(g) \circ \Pi(f))[\gamma].$$

This proves the axioms and thus we can conclude that Π is indeed a functor. □

Corollary 4.4.3. *The fundamental group is a functor* $\pi_1 : \textbf{\textit{Top}}_* \to \textbf{\textit{Grp}}$.

Proof. This follows from Theorem 4.4.2 and the fact that $\pi_1(X, x) \cong \Pi(X, \{x\})$ is a group. □

5

Generalised Van Kampen's Theorem

One of the main reasons to use category theory in this report is because the structure described in Van Kampen's Theorem, the pushout, is defined very naturally in categorical terms. We will see this in the definition of the colimit. With use of this very general definition, we can upgrade Van Kampen's Theorem to the case of fundamental groupoids. In this chapter we will state and prove the theorem.

5.1. Colimits

We start with defining initial and final objects.

Definition 5.1.1. Let $\mathscr{C}$ be a category. An object I of $\mathscr{C}$ is an *initial object* if for every object A of $\mathscr{C}$ the set $\mathrm{Hom}(I, A)$ consists of exactly one element.

Initial objects are unique up to a unique isomorphism. Suppose A and B are both initial objects of a category $\mathscr{C}$. Then $\mathrm{Hom}(A,B) = \{f\}$ and $\mathrm{Hom}(B,A) = \{g\}$, so $\mathrm{Hom}(A,A) = \{g \circ f\}$ and $\mathrm{Hom}(B,B) = \{f \circ g\}$. But the identity element is always an automorphism, so we must have $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, so A and B are isomorphic.

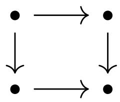
Example 5.1.2. Let X be a unloopable topological space. Then we can reformulate Theorem 2.1.5 as "The universal cover is an initial object in $\mathbf{Cov}(X)$". Corollary 2.1.6 now follows from the fact that initial objects are unique up to a unique isomorphism.

In the category $(\mathbb{N}, \leqslant)$ the object 0 is initial. The only object in the category $\mathbf{S}$ of a set S with one element is initial.

Initial objects need not exist. In the category $(\mathbb{R}, \leqslant)$ there does not exist an initial object. In fact, in any preorder, an initial object only exists if it also a well-order.

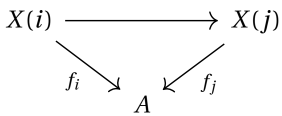
Definition 5.1.3. Let $\mathscr{C}$ a category and $\mathscr{I}$ a small category. A diagram in $\mathscr{C}$ is a functor $X : \mathscr{I} \to \mathscr{C}$.

In frequently occuring examples of diagrams, $\mathscr{I}$ only consists of a finite number of objects and morphisms, so we can draw them. We then frequently leave out the identity arrows, as they are always present. A typical case of a diagram is if $\mathscr{I}$ is of the form



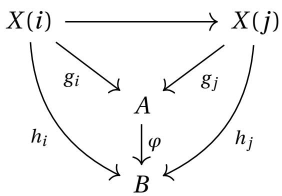
We can then interpret the diagram as filling in the bullets with objects of $\mathscr{C}$ and the arrows with morphisms in $\mathscr{C}$, such that the diagram commutes.

Definition 5.1.4. Let $\mathscr{C}$ be a category and $X : \mathscr{I} \to \mathscr{C}$ a diagram. A co-cone of X is an object A of $\mathscr{C}$, together with a collection of maps $f_i : X(i) \to A$ for $i \in \mathscr{I}$, such that for every arrow $i \to j$ in $\mathscr{I}$ the diagram



commutes.

Given a diagram $X : \mathscr{I} \to \mathscr{C}$ we have the corresponding category of co-cones. The objects are co-cones $(A, \{f_i\}_{i\in\mathscr{I}})$. A morphism between the co-cones $(A, \{f_i\}_{i\in\mathscr{I}})$ and $(B, \{g_i\}_{i\in\mathscr{I}})$ is given by a morphism $\varphi : A \to B$ in $\mathscr{C}$, such that the diagram



commutes.

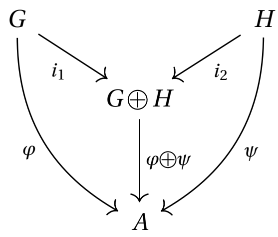
Definition 5.1.5. Let $X : \mathscr{I} \to \mathscr{C}$ be a diagram. An initial object in the category of co-cones of X is called a *colimit*, which we denote by $\operatorname{colim} X$.

Example 5.1.6. As the colimit is an initial object, it is unique up to unique isomorphism, if it exists.

This definition covers in one short definition many frequently occurring structures. We will give some examples below, that immediately give some intuition to the definition.

Example 5.1.7. Consider the diagram $X : \mathscr{I} \to \mathbf{Ab}$ for $\mathscr{I}$ of the form:

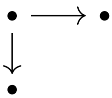
and label the bullets 1 and 2. The colimit of this diagram is the direct sum of $X(1)$ and $X(2)$. Suppose $X(1) = G$ and $X(2) = H$ and for an arbitrary third abelian group A the homomorphisms $\varphi : G \to A$ and $\psi : H \to A$ are given. In other words, $(A, \{\varphi, \psi\})$ is a co-cone of X. Then the diagram



commutes, where i_1 and i_2 are the inclusions $i_1 : g \mapsto (g, 1_H)$ and $i_2 : h \mapsto (1_G, h)$ and $\varphi \oplus \psi$ the homomorphism given by $(g, h) \mapsto \varphi(g) * \psi(h)$. This homomorphism is unique, so we conclude that the direct sum $G \oplus H$ indeed is the colimit of this diagram.

In **Set** the colimit of this diagram is given by the disjoint union of $X(1)$ and $X(2)$.

Example 5.1.8. If $\mathscr{I}$ is of the form



we call $\operatorname{colim} X$ a *pushout*. For groups this definition coincides with the definition of pushout of groups, Definition 3.2.2.

5.2. GENERALISED VAN KAMPEN'S THEOREM

We are now ready to finally state and prove the Generalised Van Kampen's Theorem.

Theorem 5.2.1 (Generalised Van Kampen's Theorem)**.** *Let X a topological space and $\mathscr{O}$ an open cover of X which is closed under finite intersection. Write $U = \bigcup_{j \in J} U^j$ for the decomposition of U into path-connected components.*

Let $A \subset X$, such that for every $U \in \mathscr{O}$ and every path-connected component U^j we have that $A \cap U^j \neq \varnothing$[1]*. Write $A_U = A \cap U$. Let $\mathscr{O}_A$ the category, with pairs (U, A_U) as objects and inclusions $U \subset V$ as morphisms.*

*Then $\Pi|_{\mathscr{O}_A} : \mathscr{O}_A \to$ **Gpd** is a diagram of groupoids. The fundamental groupoid $\Pi(X, A)$ is the colimit of this diagram. That is, $\Pi(X, A) \cong \operatorname{colim}_{\Pi|_{\mathscr{O}_A}}$*

Before we give the full proof, we will first briefly outline the idea. Since we are working with colimits, we first give a co-cone $(\mathscr{C}, \{\eta_U : U \in \mathscr{O}\})$ of the diagram $\Pi|_{\mathscr{O}_A}$. The groupoid morphisms $\Pi(U, A_U) \to \Pi(X, A)$ induced by the inclusion $U \hookrightarrow X$ make $\Pi(X, A)$ a co-cone of the diagram. We construct a map $\eta : \Pi(X, A) \to \mathscr{C}$ from the maps η_U, which is unique by construction.

[1] That is, A meets every path connected component of every member of the cover.

For a path that fully lies in some member of the cover, it is clear how we can define η on this path, but for a path that does not fully lie in one member, we have to define the mapping of η differently. We divide such a path into smaller pieces that all fully lie in a member of the cover. The problem now is that the starting and ending points of these smaller paths in general are not elements of A. We can fix this by replacing the smaller paths with homotopic paths that do have starting and ending points in A. This is done in the following way.

Let γ be a path in X and write $\gamma = \Pi_{i=1}^{n}\gamma_i$, where every γ_i fully lies in some $U_i \in \mathcal{O}$. For $1 \leqslant i \leqslant n-1$ we have that $\gamma_i(1) = \gamma_{i+1}(0) \in U_i \cap U_{i+1} \in \mathcal{O}$. Write W_i for the path connected component of $U_i \cap U_{i+1}$ that contains $\gamma_i(1) = \gamma_{i+1}(0)$. Write W_0 for the path component of U_1 that contains $\gamma_1(0) = \gamma(0)$ and W_n for the path components of U_n that contains $\gamma_n(1) = \gamma(1)$. Choose a $z_i \in A_{W_i}$ for $0 \leqslant i \leqslant n$, which is possible, as the set is by assumption non-empty. As W_i is path connected, there is a path

$$\beta_i : \gamma_i(1) \to z_i,$$

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that lies fully in $U_i \cap U_{i+1}$ for $1 \leqslant i \leqslant n-1$, in U_1 for $i = 0$ and in U_n for $i = n$.

We then define

$$\hat{\gamma}_i = \beta_{i-1}^{-1} \odot \gamma_i \odot \beta_i$$

for $1 \leqslant i \leqslant n$ and set $\hat{\gamma} = \Pi_{i=1}^{n}\hat{\gamma}_i$. We see that

$$\hat{\gamma} = \prod_{i=1}^{n} \hat{\gamma}_i = \beta_0^{-1} \odot \gamma_1 \odot \beta_1 \odot \beta_1^{-1} \odot \gamma_2 \odot \beta_2 \odot \beta_2^{-1} \odot \cdots \odot \beta_{n-1} \odot \beta_{n-1}^{-1} \odot \gamma_n \odot \beta_n$$

is homotopic to $\gamma = \prod_{i=1}^{n} \gamma_i$,if the starting and ending points agree, as all $\beta_i \odot \beta_i^{-1}$ are homotopic to the constant path at $\gamma_i(1)$.

We can then define $\eta(\gamma)$ through the mappings of $\hat{\gamma}_i$ by η_{U_i}.

It is not immediately clear that the map η is well-defined and the main part of the proof is to check that it is indeed well-defined. This is mainly a problem for homotopic paths that do not fully lie in one of the members of the cover.

For these paths γ and γ' we divide a path homotopy into smaller pieces that all map into a member of the cover. Then we construct a sequence of paths, including γ and γ' and show these are all mapped to the same element by η.

In this proof we use a property of compact metric spaces, the Lebesgue Number Lemma, which we will state for completeness. The proof can be found in (Engelking [4], Section 4.3).

Lemma 5.2.2 (Lebesgue Number Lemma)**.** *Let (X, d) be a compact metric space and let an open cover of X be given. There exists a $\delta > 0$, such that every subset of X of diameter δ is contained in some member of the cover. Any δ with this property is called a Lebesgue number of the cover.*

Proof of Theorem 5.2.1. For $U, V \in \mathscr{O}$ with $U \subseteq V$, write $i_{U,V*} : \Pi(U, A_U) \to \Pi(V, A_V)$ for the morphism induced by the inclusion. Note that this morphism is well-defined as also $A_U \subseteq A_V$. For the morphism induced by the inclusion $U \hookrightarrow X$ we write $\iota_U : \Pi(U, A_U) \to \Pi(X, A)$. By functoriality of Π the diagram with the inclusion-induced morphisms commutes. Thus $(\Pi(X, A), \{\iota_U : U \in \mathscr{O}\})$ is a co-cone of $\Pi|_{\mathscr{O}_A}$.

Now let $\mathscr{C}$ be an arbitrary groupoid and let $\eta_U : \Pi(U, A_U) \to \mathscr{C}$ be functors such that $(\mathscr{C}, \{\eta_U : U \in \mathscr{O}\})$ is a co-cone of $\Pi|_{\mathscr{O}_A}$. We will construct a unique morphism of co-cones $\eta : \Pi(X, A) \to \mathscr{C}$, proving that $\Pi(X, A)$ is the colimit of the diagram.

The morphism of co-cones is a morphism of groupoids and hence a functor. Let $x \in A = \mathrm{ob}(\Pi(X, A))$ and let $U \in \mathscr{O}$, such that $x \in A_U$ ($\mathscr{O}$ is a cover of X, so there always exists such a U). Then we set $\eta(x) := \eta_U(x)$. Now let $\gamma : x \to y$ a path that is fully contained in some $U \in \mathscr{O}$. This means that also $x, y \in \mathrm{ob}(\Pi(U, A_U))$. Then we set $\eta[\gamma] = \eta_U[\gamma]$.

Let $\gamma : x \to y$ be an arbitrary path in X. As $[0,1]$ is a compact metric space and $\gamma^{-1}(\mathscr{O})$ forms an open cover of $[0,1]$ we can use the Lebesgue Number Lemma 5.2.2 to write $\gamma = \prod_{i=1}^{n} \gamma_i$, with each γ_i completely contained in some $U_i \in \mathscr{O}$. We can now look at the path $\hat{\gamma}$, as defined above. Since γ and $\hat{\gamma}$ are homotopic we see that $[\gamma] = [\hat{\gamma}]$ as elements of $\Pi(X, A)$.

We then define $\eta[\gamma] = \eta[\hat{\gamma}] = \eta[\prod_{i=1}^{n} \hat{\gamma}_i] = \prod_{i=1}^{n} \eta_{U_i}(\hat{\gamma}_i)$.

From the construction it is clear that (if the map is well-defined) this is a morphism of co-cones (the diagram commutes) and that η is unique. The only thing we need to show is that η is well-defined.

Let $U, V \in \mathscr{O}$ with $U \subseteq V$ and let $x \in U$. By assumption, $\mathscr{C}$ is a co-cone of $\Pi|_{\mathscr{O}_A}$ in **Gpd**, so we have that $\eta_U = \eta_V \circ i_{U,V*}$. Then $\eta_U(x) = \eta_V(i_{U,V*}(x)) = \eta_V(x)$, so η is well-defined on objects.

Similarly for $y \in U$ and $\gamma : x \to y$ a path fully in U we have

$$\eta_U[\gamma]_U = (\eta_V \circ i_{U,V*})[\gamma]_U = \eta_V[\gamma]_V,$$

so paths fully in some $U \in \mathscr{O}$ are also mapped well-defined by η.

Now let $\gamma : x \to y$ be an arbitrary path in X with endpoints in A and let γ' be path homotopic to γ with path homotopy Γ. Note that the collection $\Gamma^{-1}(\mathscr{O})$ is an open cover of $[0,1]^2$. As $[0,1]^2$ is a compact metric space, we can use the Lebesgue Number Lemma 5.2.2 again. Let δ be a Lebesgue number of this covering and let $k \in \mathbb{Z}_{\geqslant 1}$ be such that $\frac{1}{k} \leqslant \delta$. We can then divide $[0,1]^2$ into k^2 squares that all fully map into some $U \in \mathscr{O}$. We can now number these squares per row from the bottom ($[0,1] \times \{0\}$) to the top ($[0,1] \times \{1\}$) and in each row from left to right.

Now let ζ_r be the path in $[0,1]^2$, with starting point $(0,0)$ and ending point $(1,1)$, that seperates the first r squares from the rest. Then $\Gamma(\zeta_0) = \gamma \odot c_y \simeq \gamma$ and $\Gamma(\zeta_{k^2}) = c_x \odot \gamma' \simeq \gamma'$.

The paths ζ_r and ζ_{r+1} only differ on the $(r+1)$-th square, which is mapped fully into $U \in \mathscr{O}$. The path components that differ, ζ'_r and ζ'_{r+1} are clearly homotopic in $[0,1]^2$, so

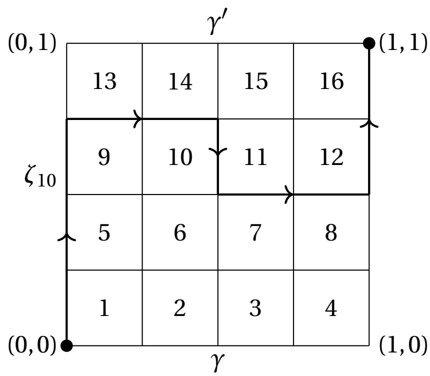


Figure 5.2.1: Numbered division of $[0,1]^2$ into squares, with the path ζ_{10} drawn.

$\Gamma(\zeta'_r)$ and $\Gamma(\zeta'_{r+1})$ in X are as well. The starting and ending point of ζ'_r and ζ'_{r+1} need not lie in A, so in similar fashion as earlier in the proof, we can make paths $\widehat{\Gamma(\zeta'_r)}, \widehat{\Gamma(\zeta'_{r+1})} : x' \to y'$ out of these, with $x', y' \in A_U$, by concatenating paths from the begin/endpoint to respectively x' and y'. These paths are homotopic as well and thus represent the same element in $\Pi(U, A_U)$. As both paths fully lie in U, they are mapped in a well-defined way by η: $\eta[\widehat{\Gamma(\zeta'_r)}] = \eta[\widehat{\Gamma(\zeta'_{r+1})}]$. As the rest of the paths ζ_r and ζ_{r+1} are the exact same, we can extract from this that $\eta[\Gamma(\zeta_r)] = \eta[\Gamma(\zeta_{r+1})]$. We can now apply this result multiple times to get

$$\eta[\gamma] = \eta[\Gamma(\zeta_0)] = \cdots = \eta[\Gamma(\zeta_{k^2})] = \eta[\gamma'],$$

as we wanted.

We conclude that η maps morphisms in $\Pi(X, A)$ in a well-defined way and thus that $\Pi(X, A)$ is the colimit of the diagram $\Pi|_{\mathcal{O}_A}$. □

5.3. FUNDAMENTAL GROUPOID OF THE CIRCLE

Now we have upgraded Van Kampen's Theorem to fundamental groupoids, we can finally calculate the fundamental group of the circle with Van Kampen.

We cover S^1 with two open subsets that are homeomorphic to an open interval. Then the intersection has two path-connected components that are both homeomorphic to an open interval. We let $A = \{x, y\}$ be the set of base points, where x and y each lie in a different path-connected component of $U \cap V$.

The fundamental groupoid $\Pi(U \cap V, A)$ then consists of two objects x and y, with the idenitity morphism at both objects. There is no path between x and y and every loop at x (resp. y) is homotopic to the constant loop at x (resp. y).

The fundamental groupoids $\Pi(U, A)$ and $\Pi(V, A)$ both consist of two objects x and y as well. This time there is a path from x to y in both sets. We call this path γ_U in U and γ_V in V. These paths have an inverse and obviously the groupoid consists also of the identity morphisms on x and y. These are the only morphisms in the groupoids, as

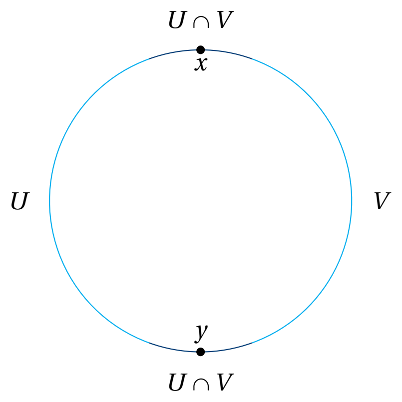
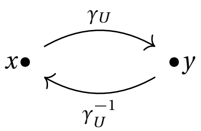
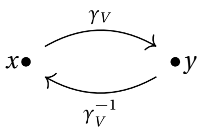


Figure 5.3.1: Covering of the circle by U and V.

every other path is homotopic to one of these. The groupoids $\Pi(U,A)$ and $\Pi(V,A)$ thus look like





Note that we leave out the identity morphisms in drawing the diagram.

The fundamental groupoid $\Pi(S^1,A)$ is now generated by the images ξ_1 and ξ_2 of γ_U and γ_V respectively under the maps $\Pi(U,A) \to \Pi(S^1,A)$ and $\Pi(V,A) \to \Pi(S^1,A)$. We now retract to the vertex x by looking at $\mathrm{Aut}_{\Pi(S^1,A)}(x)$. We then see that $\zeta = \xi_1 \circ \xi_2^{-1}$ generates the automorphism group. As $\pi_1(X,x) = \mathrm{Aut}_{\Pi(S^1,A)}(x)$ we see (just like we have seen in Example 2.3.9) that $\pi(X,x)$ is freely generated by one element, i.e. it is an infinite cyclic group. This means

$$\pi_1(X,x) \cong \mathbb{Z}.$$

6

GROUPOIDS AND COVERINGS

The proof we have given in Chapter 5 of Van Kampen's Theorem in the case of fundamental groupoids follows very different lines than the proof of Van Kampen's Theorem for fundamental groups in 3. One can ask if the approach of translating group homomorphisms into covering spaces and gluing these together also would work for fundamental groupoids. This question was also asked by Brown in [1] and he notes that such a proof is not available in literature.

6.1. GROUPOID COVERINGS

In the case of fundamental groups the proof of Van Kampen's theorem consisted of constructing a unique map $h : \pi_1(X,x) \to G$ for an arbitrary group G. In the case of fundamental groupoids we had to give a unique functor $\eta : \Pi(X,A) \to \mathscr{C}$ for an arbitrary groupoid $\mathscr{C}$.

We connected the group homomorphism h to a G-covering of X, so we can ask ourselves if there is also a notion of a $\mathscr{C}$-covering of a topological space, where $\mathscr{C}$ is a groupoid. This does exist and the notion of a groupoid cover can be found in (Szamuely [14], Chapter 2). It is a map $p : \Pi \to X \times X$.

In Theorem 2.1.8 we saw that we could construct a universal covering $p : \widetilde{X} \to X$ by considering $\widetilde{X}$ the set of homotopy classes of paths in X that start at x, if X is semi-locally simply connected. The map was then defined by mapping $[\gamma]$ to $\gamma(1)$. In similar fashion we have a cover $\hat{p} : \hat{X} \to X \times X$, where $\hat{X}$ now is the set of homotopy classes of all paths in X. The map $\hat{p}$ maps $[\gamma]$ to $(\gamma(0), \gamma(1))$. This map is a groupoid cover and it also has a universal property. For every groupoid cover $p : \Pi \to X \times X$ there is a unique morphism of groupoid covers $\hat{X} \to \Pi$.

6.2. FUNDAMENTAL THEOREM OF COVERING SPACES

The next question that arises is if we can upgrade the Classification of G-coverings 2.4.3. The answer is that we indeed can and the result is known as *the fundamental theorem of covering spaces*. We first need two new definitions to understand the theorem.

Definition 6.2.1. Let $\mathscr{C}$ and $\mathscr{D}$ be categories. We say a functor $F:\mathscr{C}\to\mathscr{D}$ is an equivalence of categories if there exists a functor $G:\mathscr{D}\to\mathscr{C}$ such that there are natural isomorphisms $\eta: G\circ F\to id_{\mathscr{C}}$ and $\eta: id_{\mathscr{D}}\to F\circ G$.

Example 6.2.2. Let X be a path-connected topological space and let $A\subset X$ and $x\in A$. If we consider $\Pi(X,A)$ and $\pi_1(X,x)$ as categories, the inclusion functor $\pi_1(X,x)\to\Pi(X,A)$ is an equivalence of categories. Let $y\in A$ and choose an isomorphism $\alpha_y: y\to x$ in $\Pi(X,A)$. In the case $y=x$ we choose $\alpha_x: x\to x$ the identity morphism. We then define the functor $G:\Pi(X,A)\to\pi_1(X,x)$ by $y\mapsto x$ for an object $y\in A$ and

$$[\gamma: y_1\to y_2]\mapsto[\alpha_{y_2}\circ\gamma\circ\alpha_{y_1}].$$

It is left to the reader to verify that there are now natural isomorphisms $\eta: GF\to id_{\pi_1(X,x)}$ and $\eta: id_{\Pi(X,A)}\to FG$.

Definition 6.2.3. A representation of a category $\mathscr{C}$ is a functor $\varphi:\mathscr{C}\to$ **Set**. The representations of $\mathscr{C}$ form a category $\mathbf{Set}^{\mathscr{C}}$, with natural transformations as morphisms.

We can now state the fundamental theorem of covering spaces.

6

Theorem 6.2.4 (Fundamental Theorem of Covering Spaces)**.** *Let X be a locally path-connected and semi-locally simply connected topological space. The categories **Cov(X)** and $\mathbf{Set}^{\Pi(X)}$ are equivalent through the functor Mon: **Cov(X)** $\to\mathbf{Set}^{\Pi(X)}$ and the functor Rec: $\mathbf{Set}^{\Pi(X)}\to$ **Cov(X)**.*

A full proof can be found in (tom Dieck [3], Theorem 3.3.2) and formulated differently in (Higgins [9], Chapter 13). Here we will only give the construction of the functors.

We start with the *monodromy* Mon: **Cov(X)** $\to\mathbf{Set}^{\Pi(X)}$.

Let (Y,p) be an object of **Cov(X)**. We construct a functor $\mathrm{Mon}_Y:\Pi(X)\to$ **Set** as follows:

Let $x\in X$ be an object of $\Pi(X)$. Then $\mathrm{Mon}_Y(x)=p^{-1}\{x\}$, so we send x to its fiber. Now let $\gamma: x\to y$ be a path in X. We define

$$\begin{aligned}\mathrm{Mon}_Y([\gamma]): p^{-1}\{x\}&\to p^{-1}\{y\},\\ x'&\mapsto\tilde{\gamma}_{x'}(1),\end{aligned}$$

where $\tilde{\gamma}_{x'}$ is the unique lift of γ with initial point x'.

Next we construct the functor called the *reconstruction of a covering space* Rec: $\mathbf{Set}^{\Pi(X)}$ $\to$ **Cov(X)**.

Let ρ an object of $\mathbf{Set}^{\Pi(X)}$, so $\rho:\Pi(X)\to$ **Set** is a functor. We construct a covering space $(Y_\rho,\mathrm{Rec}_\rho)$ as follows:

We set $Y_\rho=\bigsqcup_{x\in X}\rho(x)$ the disjoint union of all sets appearing in the representation ρ. Let $U\subset X$ open and path connected, such that every element of $\pi_1(U,x)$ is trivial in $\pi_1(X,x)$, for $x\in U$. This is possible as we assume X is semi-locally path-connected. For $\hat{x}\in\rho(x)$ with $x\in U$ we define

$$V_{U,\hat{x}}=\{\rho(\gamma)(\hat{x}): x'\in U\text{ and }\gamma: x\to x'\}$$

We claim without proof that these sets $V_{U,\hat{x}}$ form the base of a topology, which we call τ_ρ. The pair (Y_ρ, τ_ρ) is then a topological space.

We define the covering map

$$\begin{aligned} \mathrm{Rec}_\rho : Y_\rho &\to X \\ \rho(x) \ni \hat{x} &\mapsto x. \end{aligned}$$

If $f : \rho_1 \to \rho_2$ is a morphism of representations of $\Pi(X)$ we define the morphism of covering spaces:

$$\begin{aligned} \mathrm{Rec}(f) : Y_{\rho_1} &\to Y_{\rho_2}, \\ \rho_1(x) \ni \hat{x} &\mapsto f_x(\hat{x}) \in \rho_2(x). \end{aligned}$$

The composition of functors Rec $\circ$ Mon: **Cov(X)** $\to$ **Cov(X)** is naturally isomorphic to the identity functor on **Cov(X)**. We also get that the composition of functors Mon $\circ$ Rec: $\mathbf{Set}^{\Pi(X)} \to \mathbf{Set}^{\Pi(X)}$ is naturally isomorphic to the identity functor on $\mathbf{Set}^{\Pi(X)}$.

This equivalence of categories is thus even stronger than the ordinary equivalence of categories.

Though we have upgraded the classification theorem to the case of fundamental groupoids, we have lost the connection with G-coverings and there is also no immediately clear connection to groupoids coverings. The question whether we can prove Van Kampen's Theorem for fundamental groupoids in terms of covering spaces remains open.

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