Master Thesis

Positive Energy Representations of Gauge Groups

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Abstract

Recent progress on the representation theory of certain infinite dimensional gauge groups has raised an interest in the strongly continuous unitary representations of the group $G = V \rtimes H \overset{d}{=} \mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{k}) \rtimes SL(2, \mathbb{C}) \times K$ that satisfy a certain positive energy condition[JN], where K is a semisimple compact Lie group with Lie algebra $\mathfrak{k}$.

An equivalent formulation of the positive energy condition is obtained, allowing for a geometrical interpretation of this condition and which yields necessary conditions for satisfying this condition. By the theory of the Mackey machine, the strongly continuous unitary representations of G that are of positive energy are classified by the corresponding stabilizers of the action of H on the dual group $\widehat{V}$. For the case of $K = SU(2)$, these are fully determined up to equivalence.

Finally, a method is developed that embeds homogeneous bundles as eigenspace subbundles of trivial bundles that in particular applies to the bundles obtained through the representation theory of G. The eigenspace subbundles thus obtained allow for a more detailed understanding of the induced representation and moreover resemble various theories in particle physics.

Preface

The subject of this master thesis has interested me from the start and in the past eight months, I have found myself becoming ever more intrigued by it, while slowly attaining a more comprehensive grasp of its underlying mathematical theory; a process which does not seem to end. I have definitely grown towards my mathematical aspirations, for which I am particularly grateful to my advisor Bas Janssens, who has excellently and patiently guided me through the process. Finally, my parents are deserving of gratitude and as such I would like to thank them, for being great parents.

I hope the reader finds himself interested in some part of this thesis.

Milan Niestijl,
August 29, 2019.

Contents

List of Notations

$\mathcal{L}(\mathcal{H})$	– The space of bounded operators on the Hilbert space $\mathcal{H}$.
$U(\mathcal{H})$	– The space of unitary operators on the Hilbert space $\mathcal{H}$.
$\boldsymbol{UR}(G)$	– The category of strongly continuous unitary representations of G.
$\Omega(\mathcal{A})$	– The spectrum of a commutative Banach- $*$ -algebra $\mathcal{A}$.
G^0	– The connected component of the group G.
$\widehat{N}$	– The dual group of an Abelian locally compact group.
$K, \mathfrak{k}$	– A compact semisimple Lie group K with real Lie algebra $\mathfrak{k}$.
$\mathcal{T}(V)$	– The tensor algebra of a vector space V.
$\mathcal{S}(V)$	– The subspace of $\mathcal{T}(V)$ consisting of symmetric tensors.
$\mathcal{S}^N(V)$	– The subspace of $V^{\otimes N}$ consisting of symmetric tensors.
$\mathcal{A}^N(V)$	– The subspace of $V^{\otimes N}$ consisting of alternating tensors.
$\bigvee^N V$	– The k-th symmetric power of a vector space V.
$\bigwedge^N V$	– The k-th alternating power of a vector space V.

Chapter 1

Introduction

This thesis is concerned with the mathematical formulation of various theories of particle physics. A recent result[JN] by B. Janssens and K.H. Neeb has made progress towards this goal, reducing the study of the representation theory of an infinite-dimensional symmetry group to that of a much simpler to understand group, for a subclass of representations that satisfy a so-called *positive energy* condition. The main purpose of this thesis is to understand the strongly continuous unitary representations of this simpler group that are of positive energy. A full understanding thereof could provide a significant stepping stone towards a mathematical formulation of various theories in particle physics, including the standard model.

Motivation and historical perspective

The primary motivation of this thesis lies in the attempt of physicists and mathematicians to formulate theories of particle physics in a mathematically rigorous way. Let us therefore consider a bit of the related history leading up to the main goal of this thesis.

Now, the standard model of particle physics is a theory describing three out of four of the fundamental interactions; electromagnetic, weak and strong interactions, thus leaving out gravity. In particular, the theory is compatible with both the theory of quantum mechanics and special relativity. A mathematical treatment of such a unification is often done using the language of Lie groups, which describe continuous symmetries, and the unitary representations thereof in some Hilbert space.

On the one hand, quantum mechanics postulates that the state space of a quantum system is a projective Hilbert space $\mathcal{P}(\mathcal{H}) = \mathcal{H}/\mathbb{C}^\times I$. Moreover, this state space comes with a symmetric map $p : \mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \to [0, 1]$ whose value represents a transition probability. If $\psi_1, \psi_2 \in \mathcal{H}$ have unit norm, then p is defined by

$$p([\psi_1], [\psi_2]) \mapsto |\langle \psi_1, \psi_2 \rangle|^2.$$

The quantum automorphism group of such a system is defined as the set of topological automorphisms of $\mathcal{P}(\mathcal{H})$ that preserve that map p, that is, that preserve the transition probabilities. Intuitively, these correspond to the transformations that have no qualitative effect on the quantum system. A key result in the history of quantum theory is Wigner's theorem [Wig39], which states that every element in the group of quantum automorphisms lifts to either a unitary or anti-unitary operator on the Hilbert space $\mathcal{H}$ and this element is unique up to a phase factor (if $\dim \mathcal{H} > 1$).

On the other hand, special relativity asserts that neither the distance between two points in space nor the time between two events is the same in all reference frames. That is, neither space nor time is separately invariant under arbitrary changes of reference frames. Instead, the quantity that is invariant under such changes of reference frames is the so-called space-time interval $(\Delta s)^2$

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2, \tag{1.1}$$

where c denotes the speed of light. This means that one should not consider space and time separately, but instead should consider the quadratic space $\mathbb{R}^4$ equipped with the *Minkowski form* η given by

$$\eta(x, y) = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 \tag{1.2}$$

The set of linear transformations of $\mathbb{R}^4$ that preserve η defines a Lie group $O(1, 3)$ called the *Lorentz group*. Since there is no preferred choice of origin (this is reflected by the Δ's in 1.1), any physical theory that is compatible with special relativity should not have qualitatively different behaviour under transformations by the *inhomogeneous Lorentz group* $\mathbb{R}^4 \rtimes O(1,3)$.

In particular, if we have any quantum system $\mathcal{P}(\mathcal{H})$ that is subject to the laws of special relativity, then by Wigner's theorem, there should be a projective unitary representation $SO(1,3)^0 \rtimes \mathbb{R}^4 \to \mathcal{P}U(\mathcal{H})$, where $SO(1,3)^0$ denotes the connected component of $SO(1,3)$. In 1939, Wigner classified such projective unitary representations [Wig39] in terms of *linear* unitary representations of the group $\widetilde{\mathcal{P}} = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ and in 1948, together with Bargmann, showed a strong connection of these representations with relativistic wave equations [BW48]. As an example, it was shown that the Dirac equation $i\hbar \sum_{k=0}^{3} \gamma_k \partial_k \psi = mc\psi$, which describes massive spin-$\frac{1}{2}$ particles, corresponds to an irreducible unitary representation of the group $\widetilde{P}$. This means in particular that the dynamical behaviour of such a particle is fully contained in the representation theory of $\widetilde{P}$.

More generally, if such a quantum system is known to be invariant under transformations in some symmetry group G, then in view of Wigner's theorem, if G is connected, there must be a projective unitary representation $G \to \mathcal{P}U(\mathcal{H})$ of G in $\mathcal{H}$. Often, such symmetry groups are finite dimensional semi-simple Lie groups over $\mathbb{R}$ or $\mathbb{C}$. In that case, Bargmann's theorem[Bar54], [HN91] implies that any continuous projective unitary representation of G lifts to a strongly continuous unitary representation of its universal covering group $\widetilde{G}$. Conversely, any irreducible representations of the covering group descends to a projective representation of the original group by Schur's lemma. Thus, the study of continuous *projective* unitary representations of G is reduced to the study of *linear* strongly continuous unitary representations of $\widetilde{G}$.

The success of Wigner in relating the relativistic wave equations to the representation theory of the corresponding symmetry group has led physicists to consider the unitary representations of the symmetry groups corresponding to various interactions in order to obtain information about quantum systems subject to such symmetries. For example, electromagnetism is known to contain $U(1)$ symmetry, isospin is subject to $SU(2)$ symmetry whereas flavour possesses $SU(3)$ symmetry. By choosing a symmetry group that includes multiple of the above three, physicists have attempted to unify the various forces into a single and larger symmetry group, yielding a single theory that encompasses each of the three forces above. There are many ways in which this can be done, but the standard model is based on their product $U(1) \times SU(2) \times SU(3)$.[BH09]

Now, these symmetries represent internal symmetries and are local, meaning that such symmetries may vary (smoothly) at different positions in space-time. In order to make these theories compatible with special relativity, one therefore has to consider a group that captures both the *global* symmetries of $\mathbb{R}^4 \rtimes SO(1,3)^0$ imposed by special relativity and the *local* symmetry K, for some simply connected semisimple compact Lie group K (such as $SU(2)$ and $SU(3)$).

Mathematically, this can be formulated in the language of principal fiber bundles. Explicitly, if $P \to \mathbb{R}^4$ is a principal K-bundle over $\mathbb{R}^4$, then one considers the associated group bundle $\mathcal{K} = P \times_{\mathrm{Ad}} K$ over $\mathbb{R}^4$ with typical fiber K. Denote by $\Gamma_c(\mathcal{K})$ the group of compactly supported sections of this bundle. Such a section assigns an element in K to each point in the manifold M in a continuous manner. Now, given an action of $\mathbb{R}^4 \rtimes SO(1,3)^0$ on $\mathbb{R}^4$ and a lift of this action to $\mathcal{K}$, one considers the symmetry group $\mathcal{G} = \Gamma_c(\mathcal{K}) \rtimes (\mathbb{R}^4 \rtimes SO(1,3)^0)$.

The representation theory of such groups is in general not well understood, so to simplify matters a bit, the Minkowski space $\mathbb{R}^4$ can be replaced by its conformal compactification Q [FLV07, section 2]. With this simplification, a recent result by B. Janssens and K.H. Neeb [JN] has reduced the study of a certain class of projective representations of $\mathcal{G}$ satisfying a so-called *positive energy condition* to the study of strongly continuous unitary representations of positive energy of the finite dimensional Lie group

$$\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K \tag{1.3}$$

for some suitable action of $SL(2,\mathbb{C}) \times K$ on $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4)$, where $\mathfrak{k}$ denotes the Lie algebra of K.

Having in mind the success of the study of the unitary representation theory of $SL(2,\mathbb{C}) \rtimes \mathbb{R}^4$ and the various symmetry groups $U(1), SU(2)$ and $SU(3)$ in relation to quantum systems subject to such symmetries, one is motivated to similarly study the unitary representations of the group 1.3.

Now, Wigner's classification of the strongly continuous unitary representations of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ has led to a general theory of *induced representations* for locally compact groups, developed largely by Mackey [Mac49, Mac52]. This theory considers the question of how the representation theory of a given group G is related to that of its closed subgroups. This theory is particularly nice and well-developed in the case that

G is of the form $G = N \rtimes H$ for some Abelian normal subgroup N. In this case, the strongly continuous unitary representations of G are completely classified by those of the various stabilizers subgroups H_ν of the action of H on the dual $\widehat{N}$ of N.

In particular the group 1.3 is of this form, so an obvious approach towards an understanding of the positive energy representations of this group is to try and apply the theory of induced representations to this group.

Goal of the thesis

The main purpose of this thesis is to follow up on the following result of B. Janssens and K.H. Neeb[JN]:

Theorem 1. *There exists some H in a maximal Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ such that there is a bijective correspondence between smooth projective positive energy representations of $\Gamma(Q,\mathcal{K})^0 \rtimes (\mathbb{R}^4 \rtimes SO(1,3)^0)$ and strongly continuous unitary representations of*

$$G = \mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K, \tag{1.4}$$

that are of positive energy with respect to the cone

$$C' = \{v \oplus (H \otimes v) \,:\, v_0 \geq 0 \text{ and } \eta(v,v) \geq 0\} \subset \mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \tag{1.5}$$

and where the action of $SL(2,\mathbb{C}) \times K$ on $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4)$ is given on simple tensors by

$$(w,k) \cdot v_1 \oplus (X \otimes v_2) = \phi(w)v_1 \oplus (Ad_k(X) \otimes \phi(w)v_2).$$

Here $\phi : SL(2,\mathbb{C}) \to SO(1,3)^0$ denotes the covering homomorphism.

Main goal

To develop an understanding of the strongly continuous unitary representations of the group G that are of positive energy with respect to the cone C'.

This problem is divided into three separate tasks:

1. **To understand the meaning and implications of the condition of the positive energy requirement.**
   In order to classify the representations of positive energy, a necessary first step is to determine which strongly continuous unitary representation of G do or do not satisfy this condition.

2. **To study the stabilizers of the action of $SL(2,\mathbb{C}) \times K$ on $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4)$ corresponding to representations of positive energy.**
   By the theory of induced representations, the strongly continuous representations of G are completely determined by the representation theory of these stabilizers. A first step is thus to determine all those stabilizers that correspond to a positive energy representation. These will then classify the strongly continuous unitary representations of G that are of positive energy.

3. **To understand the corresponding induced representations.**
   The induced representations are defined in abstract terms and therefore do not directly allow for a detailed understanding or interpretation, which would require a more concrete realization of the induced representations.

Structure of the thesis

With these tasks in mind, the thesis is structured as follows.

1. Part I, which consists of the first three chapters, introduces various preliminaries. Explicitly, chapter 2 introduces the language of fiber bundles, which plays a major role throughout the thesis. The second chapter discusses the theory of induced representations, which is the central tool used to classify the strongly continuous representations of G. Finally, the third chapter is concerned with Clifford algebras and spin groups. These spin groups are the universal covering group of the connected component $SO(V,q)^0$ of the group of isometries of some quadratic space (such as Minkowski space $(\mathbb{R}^4, \eta)$), and are constructed using Clifford algebras. The general construction of this covering homomorphism as well as some explicit cases will be relevant in both chapter 5 and chapter 6.

2. Part II is concerned with solving the three tasks mentioned above. Chapter 5 is devoted to the first two tasks; two understand the meaning of the positive energy condition and determine the stabilizers. To do so, an equivalent formulation of the positive energy condition is developed that admits a clear geometric interpretation and can be exploited to explore the implications of this condition. The second task is considered for the case $K = SU(2)$, yielding a full classification of the stabilizers corresponding to positive energy. Chapter 6 is concerned with the third task; to understand to corresponding induced representations. To do so, inspiration is taken from the early work of Wigner, who showed that relativistic wave equations can be obtained by realizing certain homogeneous bundles encountered through the induced representations as eigenspace subbundles of a suitable trivial bundle. An explicit method to realize such an embedding is first developed that applies in particular to the bundles obtained via the representation theory of G.

3. Finally, Part III consists of a conclusion and discussion of the obtained results, as well as the appendix. The latter includes in particular the historically important example of the unitary representations of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$, obtained using the theory of induced representations and their relation to relativistic wave equations.

Part I

Preliminaries

Chapter 2

Fiber Bundles

In this chapter, the notion of fiber bundles is introduced. These objects play an essential role in the theory of induced representations, which is most conveniently expressed in the language of fiber bundles. A good understanding of fiber bundles is therefore necessary to properly discuss induced representations. Moreover, a certain type of fiber bundles called homogeneous vector bundles, are the main object of study in chapter 6.

Now, the idea of fiber bundles is to generalize the notion of a product of two manifolds such that the resulting object is locally a product space, but globally may have a different topological structure. There are many variations of the notion depending on what additional structure is imposed. The most general notion is considered first, providing the general framework for various variations, after which several more specific structures are considered. In chapter 4, chapter 5 and chapter 6, the most relevant classes of fiber bundles will be principal fiber bundles and homogeneous vector bundles.

A brief overview of the theory is given, focusing mostly on the definitions and results needed in chapter 4 and chapter 6. For a more detailed exposure on the theory of fiber bundles, see [HH94]. All constructions are considered here in the smooth category. Nonetheless, the theory is similarly defined for topological spaces and manifolds.

2.1 General fiber bundles

To idea of a fiber bundle is that it is a smooth manifold that is locally a product, but globally may have a different topological structure. The precise definition is given below.

Definition 2. A **fiber bundle** with typical fiber F is a quadruple (E, M, π, F), where

— E and M are smooth manifolds, known as the **total space** and **base space**,

— $\pi : E \to M$ is a surjective submersion,

— F is a smooth manifold

such that for every $m \in M$, there exists some open neighborhood $U \subset M$ and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times F$ for which the following diagram commutes:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi_U} & U \times F \\ & \searrow^{\pi} & \downarrow \mathrm{Pr}_1 \\ & & U \end{array} .$$

For any $x \in M$, its fiber $\pi^{-1}(\{x\})$ is denoted E_x or $E(x)$. Since π is a surjective submersion, the fibers are embedded submanifolds of E and are diffeomorphic to F by the local triviality condition. We also say that $E \xrightarrow{\pi} M$ is a bundle over M with typical fiber F.

Definition 3. A morphism between two fiber bundles $E_1 \xrightarrow{\pi_1} M_1$ and $E_2 \xrightarrow{\pi_2} M_2$ is a pair $(f, \widetilde{f})$ of smooth maps $f : E_1 \to E_2$ and $\widetilde{f} : M_1 \to M_2$ making the diagram below commute:

$$\begin{array}{ccc} E_1 & \xrightarrow{\mathfrak{f}} & E_2 \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ M_1 & \xrightarrow{\widetilde{f}} & M_2 \end{array}$$

This morphism is an **isomorphism** if both f and $\widetilde{f}$ are diffeomorphisms. If $M = M_1 = M_2$ it is usually assumed that $\widetilde{f} = \mathrm{id}$, unless explicitly stated otherwise.

A fiber bundle that is isomorphic to a product of smooth manifolds $M \times F \xrightarrow{\mathrm{Pr}_1} M$ is called *trivial.* The last condition of a fiber bundle then simply states that the bundle should be locally trivial. The maps ϕ_U are called *local trivializations.*

Definition 4. A fiber bundle $S \xrightarrow{\pi_S} M$ is a **subbundle** of the vector bundle $E \xrightarrow{\pi_E} M$ if there exists a bundle map $f : S \to E$ that is also a smooth embedding.

Definition 5. A **section** of a fiber bundle $E \xrightarrow{\pi} M$ is a smooth map $s : M \to E$ that is a right-inverse to the projection π. That is, $\pi \circ s = \mathrm{id}_M$. A **local section** is a section of the sub bundle $\pi^{-1}(U) \to U$ for some open $U \subset M$. We denote the set of sections a fiber bundle $E \to M$ by $\Gamma(M; E)$ or by $\Gamma(E)$ if the base manifold M is clear.

Remark.

— Observe that the requirement $\pi \circ s = \mathrm{id}_M$ is equivalent to $s(x) \in E_x$ for every $x \in M$.

— Notice also that if the fiber bundle is trivial, say $E = M \times F$. Then there is a one-to-one correspondence between sections of the fiber bundle and smooth functions on M taking values in F:

$$\Gamma(E) \cong C^\infty(M; F).$$

Therefore, the notion of sections can be regarded as a generalization of functions on a smooth manifold.

2.2 Vector bundles

A common example of fiber bundles are that of vector bundles, where the typical fiber is a (usually finite-dimensional) vector space.

Definition 6. A **vector bundle** of rank k is a fiber bundle (E, M, π, V), where V is a vector space of dimension k, with the additional requirements that

— every fiber E_x is equipped with the structure of a real or complex vector space

— for every local trivialization $\phi_U : \pi^{-1}(U) \to U \times V$ and $x \in U$, the map $v \mapsto \phi_U^{-1}(x, v)$ is a linear isomorphism $V \to \pi^{-1}(U)$.

If each fiber is a Hilbert space and the map $v \mapsto \phi_U^{-1}(x, v)$ is unitary, then (E, M, π, V) is called a **Hilbert bundle**.

Definition 7. A morphism of vector bundles is a bundle map $(f, \widetilde{f})$ from a vector bundle $E_1 \to M_1$ to another vector bundle $E_2 \to M_2$ such that $f_x \stackrel{d}{=} f|_{E_1(x)} : E_1(x) \to E_2(x)$ is a linear map for every $x \in M_1$. For a morphism of Hilbert bundles, we require additionally that f_x is isometric.

Definition 8. A vector bundle $U \to M$ is a **vector subbundle** of the vector bundle $E \xrightarrow{\to} M$ if there exists a morphism of vector bundles $f : U \to E$ such that both f is a smooth embedding.

As with any fiber bundle, we can again consider the sections of a vector bundle. Since each fiber is equipped with the structure of a vector space sections can be added pointwise. Moreover, pointwise multiplication of a section by a smooth function on M again yields a smooth section. That is, under the pointwise operations

$$(s_1 + s_2)(p) \stackrel{d}{=} s_1(p) + s_2(p)$$
$$(f \cdot s)(p) \stackrel{d}{=} f(p)s(p),$$

$\Gamma(M; E)$ becomes a module over $C^\infty(M)$. Moreover, given a morphism $E_1 \xrightarrow{f} E_2$ of bundles over M, we obtain a map $\Gamma(E_1) \to \Gamma(E_2)$ given by $s \mapsto f \circ s$. This defines a functor Γ taking vector bundles over M to modules over $C^\infty(M)$.

Remark. Given a vector bundle $E \to M$, the space of compactly supported sections will be denoted by $\Gamma_c(E)$.

Proposition 9. *Let $f : E \to F$ be an injective morphism of vector bundles over X. Then f is a smooth embedding. If f is bijective then it is a diffeomorphism.*

Proof. Assume first that f is bijective. Then its inverse exists, so it remains to show it is smooth. In local coordinates f is given by $(u, v) \mapsto (u, \rho(u)v)$ for some smooth $\rho : U \to GL(\mathbb{R}^k)$. Locally, the inverse of f is then given by $(u, v) \mapsto (\rho(u)^{-1}v)$, which is smooth since $A \to A^{-1}$ is smooth in $GL(\mathbb{R}^k)$. Thus f is a diffeomorphism.

Assume next that f is injective. The previous argument now applies to the map $E \xrightarrow{f} F(E)$, where $F(E)$ is given the smooth structure of a submanifold of F. □

2.2.1 Operations

The vector space structure on each fiber of a vector bundle can be used to construct new vector bundles from old ones, imitating similar constructions on vector spaces. As these constructions will be particularly relevant in chapter 6, they are briefly discussed below.

Definition 10. A **pre-vector bundle** is a quadruple $(E, \pi, X, \mathcal{B})$ consisting of the following data

— A set E,

— a smooth manifold X of dimension n, say.

— a surjective map $E \xrightarrow{\pi} X$,

— a vector space structure on every fiber $E_x = \pi^{-1}(\{x\})$,

— a set, called the *pre-bundle atlas*, $\{(U_\alpha, \phi_\alpha)\}_\alpha$, where $\{U_\alpha\}_\alpha$ is an open covering of X such that each U_α is diffeomorphic to an open subset of $\mathbb{R}^n$ and $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ are bijective maps that restrict to linear isomorphisms on the fibers and such that all transition functions $U_\alpha \cap U_\beta \to GL(\mathbb{R}^n)$ of $\mathcal{B}$ are smooth.

Lemma 11. *Suppose that we are given a pre-vector bundle $(E, \pi, X, \mathcal{B})$. Then there is a unique smooth structure on E for which $\mathcal{B}$ is a smooth atlas. With respect to this smooth structure, $E \xrightarrow{\pi} X$ becomes a smooth vector bundle over X.*

Proof. We endow each $\pi^{-1}(U_\alpha)$ with the unique topology making the local trivializations ϕ_α into homeomorphisms. Next, the total space E is given the final topology with respect to all the injections $U_\alpha \hookrightarrow E$, that is, the finest topology for which all the injections $U_\alpha \hookrightarrow E$ are continuous. This makes E into a topological manifold for which $\mathcal{B}$ is an atlas (after identifying U_α with the open subset of $\mathbb{R}^n$ it is diffeomorphic to). Since the transition functions are smooth, $\mathcal{B}$ defines a smooth atlas on E, which is contained in a unique maximal smooth atlas $\widetilde{\mathcal{B}}$, making E into a smooth manifold. Equipped with this smooth structure, $E \to M$ becomes a smooth vector bundle.

□

This construction is usually applied in the case where a collection of vector spaces $\{E_x\}_{x \in M}$ indexed by M is given. The total space E, as a set, is then taken to be the disjoint union $E \stackrel{d}{=} \coprod_{x \in M} E_x$ and $\pi : E \to M$ is the associated projection.

The previous lemma is applied to define various algebraic operations between vector bundles. Let E and F be vector bundles over M of rank n and k, respectively. Let $\{(\pi_E^{-1}(U_\alpha), \phi_\alpha)\}_\alpha$ and $\{(\pi_F^{-1}(U_\alpha), \psi_\alpha)\}_\alpha$ be local trivializations of E and F.

Example (Tensor product of bundles).
As a concrete example, we define the tensor product $E \otimes F$ as bundle over M. Let $E = \coprod_{x \in M} E_x \otimes F_x$ and let $\pi : E \to M$ be the associated projection. For each $x \in U_\alpha$, the map $E_x \otimes F_x : \xrightarrow{\phi_{\alpha,x} \otimes \psi_{\alpha,x}} \{x\} \times \mathbb{R}^n \otimes \mathbb{R}^k$ is a linear isomorphism. Thus for each α we have a bijection $\chi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n \otimes \mathbb{R}^k$ that restricts to a linear isomorphism on each fiber. Moreover, choosing a basis for $\mathbb{R}^n \otimes \mathbb{R}^d$ shows that the elements of the matrix representing a transition function $U_\alpha \cap U_\beta \to GL(\mathbb{R}^n \otimes \mathbb{R}^k)$ are polynomial functions of the matrix elements of $\phi_\alpha(x)$ and $\psi_\alpha(x)$, which are smooth functions on $U_\alpha \cap U_\beta$ by assumption. Hence the transition functions are smooth, so that $(E, \pi, M, \mathcal{B})$ is a pre-vector bundle, where $\mathcal{B} = \{(U_\alpha, \chi_\alpha)\}_\alpha$.

One defines other algebraic operations in a similar fashion, defining the total space by applying the operations fiber wise and constructing the local trivializations out the individual ones. An overview of various common examples is given below.

— *Dual* E^*

— *Direct sum* $E \oplus F$,

— *Tensor product* $E \otimes F$,

— *Exterior powers* $\bigwedge^k E$.

— *Symmetric tensor products* $\bigvee^k E$.

2.3 Principal fiber bundles

In the following, the notion of principal fiber bundles is considered. These are fiber bundles that locally look like a product with some Lie group G and are equipped with a G-action that is locally simply multiplication in G. These objects are one of the main ingredients for constructing the induced representations in chapter 4 and are strongly connected to representation theory, the reason being that any representation of G gives rise to a vector bundle associated to the principal fiber bundle.

Definition 12. A **principal fiber bundle** is a fiber bundle (P, B, π, G) such that

— The typical fiber G is a Lie group,

— the total space P is equipped with a smooth right G-action,

— The local trivializations $\phi_U : \pi^{-1}(U) \to U \times G$ are G-equivariant, where G acts on $U \times G$ on the right factor by right multiplication.

From the definition, it follows that the typical fiber of a principal G-bundle is diffeomorphic to the group G and the action is locally just right-multiplication in G. In particular, the action of G preserves the fibers of P so the projection π is G-equivariant. Notice that these observations imply that G acts freely and transitively on each fiber. Together with the equivariance of π, this implies that π factors through the a bijection $P/G \xrightarrow{\widetilde{\pi}} B$:

$$\begin{array}{ccc} & P & \\ {}^{q}\swarrow & & \searrow^{\pi} \\ P/G & \xrightarrow{\widetilde{\pi}} & B \end{array} \tag{2.1}$$

Definition 13. A morphism from a principal G_1-bundles $P_1 \to B_1$ to a principal G_2 bundle $P_2 \to B_2$ is a pair (θ, λ) consisting of a smooth map $P_1 \xrightarrow{\theta} P_2$ and a Lie group homomorphism $G_1 \xrightarrow{\lambda} G_2$ such that

$$\theta(p \cdot g) = \theta(p) \cdot \lambda(g) \qquad \forall p \in P_1,\ g \in G_1. \tag{2.2}$$

Such a morphism is an isomorphism if θ is a diffeomorphism and λ is an isomorphism of Lie groups.

Remark.

1. The condition (2.2) implies that θ maps fibers to fibers and thus induces a map $\widetilde{\theta} : B_1 \to B_2$ such that $(\theta, \widetilde{\theta})$ is a morphism of fiber bundles:

$$\begin{array}{ccc} P_1 & \xrightarrow{\theta} & P_2 \\ \downarrow & & \downarrow \\ B_1 & \xrightarrow{\widetilde{\theta}} & B_2 \end{array}$$

2. If P_1 and P_2 are principal G-bundle, we write simply $\theta : P_1 \to P_2$ for the morphism (θ, id_G).

3. Since the action of G preserves the fibers of a principal G-bundle, a morphism between two principal G-bundles P_1 and P_2 is equivalently a smooth G-equivariant map $P_1 \to P_2$.

Definition 14.

- A continuous mapping of locally compact Hausdorff spaces is **proper** if the preimage of any compact set is compact.
- A smooth right action of a Lie group G on a manifold M is called **proper** if following map is proper:

$$M \times G \to M \times M, (m, g) \mapsto (m, m \cdot g).$$

Lemma 15. *Let $P \xrightarrow{\pi} B$ be a principal G-bundle. Then the G-action on P is proper.*

Proof. Let $K \subset P \times P$. To show its preimage under the map $(p, g) \mapsto (p, p \cdot g)$ is compact, we show it is sequentially compact, which is equivalent to being compact for manifolds. Let $\{(p_i, g_i)\}_{i \in \mathbb{N}}$ be a sequence in $P \times G$ such that $\{p_i, p_i \cdot g_i\} \subset K$. After passing to a subsequence if necessary, we may assume that $p_i \to p$ and $p_i \cdot g_i \to q$ for some $p, q \in K$. We need to show that $\{(p_i, g_i)\}_{i \in \mathbb{N}}$ has a convergent subsequence. Since $p_i \to p$, there exists a trivializing neighborhood U of p such that after discarding finitely many elements, $p_i \in U$ for all $i \in \mathbb{N}$. Let $\phi : \pi^{-1} \to U \times G$ be a local trivialization. Write

$$\begin{aligned}(m_i, x_i) &= \phi(p_i)\\ (m, x) &= \phi(p)\\ (m, k) &= \phi(q)\end{aligned}$$

Since $p_i \to p$ and $p_i \cdot g \to q$, we know that $(m_i, x_i) \to (m, x)$ and $(m_i, x_i \cdot g_i) \to (m, k)$.
Then $g_i = x_i^{-1}(x_i \cdot g_i) \to x^{-1}k$ and so $(p_i, g_i) \to (p, x^{-1}k)$. □

Corollary 16. *Let $P \xrightarrow{\pi} B$ be a principal G-bundle. There exists a unique smooth manifold structure on the orbit space P/G making the projection $P \to P/G$ a surjective submersion. Moreover, the orbit space P/G is diffeomorphic to B with respect to this smooth structure on P/G.*

Proof. Since the G-action on P is proper and free, the quotient manifold theorem implies that there exists a unique smooth structure on the orbit space P/G making the projection $P \xrightarrow{q} P/G$ into a surjective submersion. Since π and q are both surjective submersions, it follows that both $\widetilde{\pi}$ and $\widetilde{\pi}^{-1}$ are smooth. □

A converse of the previous corollary is given below, which completes a characterization of smooth principal G-bundles.

Lemma 17. *If G is a Lie group and P is a smooth manifold equipped with a right proper and free G-action, then $P \to P/G$ is a principal G-bundle.*

Proof. Since the action of G on P is proper and free, the quotient manifold theorem implies that there is a unique smooth structure on P/G for which the canonical projection $q : P \to P/G$ is a smooth submersion. It remains to show the equivariant local triviality condition. Since q is a surjective submersion, there exists a collection $\{(U_\alpha, s_\alpha)\}$ such that $\{U_\alpha\}$ cover P/G and $s : U_\alpha \to q^{-1}(U_\alpha)$ are smooth local sections of $P \to P/G$. Define for every α the smooth map

$$\begin{aligned}\phi_\alpha &: U_\alpha \times G \to q^{-1}(U_\alpha)\\ \phi_\alpha(u, x) &\mapsto s_\alpha(u) \cdot x\end{aligned}$$

Notice that this map is surjective and G-equivariant by construction. It is injective since the action of G on P preserves the fibers, so if $s_\alpha(u_1) \cdot x_1 = s_\alpha(u_2) \cdot x_2$, applying q yields $u_1 = u_2$. Since G acts freely on P, also $x_1 = x_2$ follows. It remains to show the tangent mapping of ϕ_α is bijective, which would imply that ϕ_α is a local diffeomorphism and therefore also a diffeomorphism, seeing as it is bijective. To do so, notice first that by the G-equivariance of ϕ_α, it suffices to show that $d(\phi_\alpha)_{(u,e)}$ is bijective for all $u \in U_\alpha$. As such, let $p = s_\alpha(u)$. Let $V_p q^{-1}(U_\alpha) \overset{d}{=} \ker dq_p$ be the *vertical* subspace of $T_p q^{-1}(U_\alpha)$ and define

$$\begin{aligned}A_p^{\#} &: \mathfrak{g} \to V_p q^{-1}(U_\alpha),\\ A_p^{\#}(\xi) &= \frac{d}{dt}\Big|_{t=0} p \cdot \exp(t\xi).\end{aligned}$$

Notice that $\frac{d}{dt}\big|_{t=0} p \cdot \exp(t\xi) \in V_p q^{-1}(U_\alpha)$ for any $\xi \in \mathfrak{g}$ because the action of G preserves the fibers of P. Now, we can identify $T_{(u,e)}(U_\alpha \times G) \cong T_u U_\alpha \times \mathfrak{g}$. Then the tangent map of ϕ_α at (u, e) is given by

$$\begin{aligned}d\phi_{(u,e)} &: T_u U_\alpha \times \mathfrak{g} \to T_p q^{-1}(U_\alpha)\\ d\phi_{(u,e)}(v, \xi) &= A_p^{\#}(\xi) + d(s_\alpha)_u(v),\end{aligned}$$

Now, the section s_α determines a splitting of the exact sequence

$$0 \to V_p q^{-1}(U_\alpha) \to T_p q^{-1}(U_\alpha) \xrightarrow{dq_p} T_u U_\alpha \to 0,$$

so that we obtain a direct sum decomposition of vector spaces:

$$T_p q^{-1}(U_\alpha) \cong V_p q^{-1}(U_\alpha) \oplus (ds_u)(T_u U_\alpha).$$

Thus $d\phi_{(u,e)}$ is bijective if and only if both $A_p^\# : \mathfrak{g} \to V_p q^{-1}(U_\alpha)$ and $d(s_\alpha)_u : TU_\alpha \to (ds_u)(T_u U_\alpha)$ are bijective. Notice that the latter is injective because $dq_p \circ ds_u = \mathrm{id}_{T_u U_\alpha}$ and surjective by construction. To see that $A_p^\#$ is bijective, observe first that the map $\theta_p : G \to q^{-1}(u)$ given by $g \mapsto p \cdot g$ is a diffeomorphism. Indeed, it is constant rank because it is G-equivariant and it is clearly smooth and bijective. Thus it is a diffeomorphism. Then for any $z \in V_p q^{-1}(U_\alpha)$ there exists a smooth curve $\gamma(t)$ in P satisfying $\gamma(0) = p$ and $\gamma'(0) = z$ that is completely contained in $q^{-1}(u)$ for some small enough interval containing 0. Using the diffeomorphism θ we obtain a smooth curve $x(t)$ in G and the element $\xi = \frac{d}{dt}\big|_{t=0} x(t) \in \mathfrak{g}$ satisfies $A_p^\#(\xi) = z$. Thus $A_p^\#$ is surjective. Finally, to see $A_p^\#$ is injective assume that $A_p^\#(\xi) = \frac{d}{dt}\big|_{t=0} p \cdot \exp(t\xi) = 0$. Then $p \cdot \exp(t\xi)$ is constant so in fact $p \cdot \exp(t\xi) = p$ for all t. Since the action on P is free, this implies $\xi = 0$. □

Corollary 18. *If G is a Lie group and H is a closed subgroup, then $G \to G/H$ is a principal H-bundle.*

Proof. It is clear that H acts freely on G from the right. To show the claim, it remains to show that this action is proper. We must show that the map

$$G \times H \to G \times G, \quad (x,h) \mapsto (x, xh)$$

is proper. Notice that this map is the following composition:

$$G \times H \hookrightarrow G \times G \xrightarrow{\phi} G \times G,$$

where $\phi(x,y) = (x, xy)$. Observe that ϕ is smooth and has a smooth inverse given by $(x,y) \mapsto (x, x^{-1}y)$. It is therefore a diffeomorphism and in particular proper. The inclusion $G \times H \hookrightarrow G \times G$ is proper because $H \times G$ is closed in $G \times G$. The conclusion follows. □

2.3.1 Associated fiber bundles

As mentioned previously, an important property of principal G-bundles is their tight connections with vector bundles via representations of G. Explicitly, suppose we are given a principal G-bundle $P \xrightarrow{\pi} B$ and a left action σ of G on some manifold F. The idea of an associated bundle is to define a fiber bundle E with typical fiber F in a suitable way. We will be mainly concerned with the case where F is a vector space and the resulting associated bundle is a vector bundle.

To construct the fiber bundle, consider first the product bundle $P \times F$ over B. The manifold $P \times F$ is has a smooth right G smooth given by $(p,f) \cdot g = (p \cdot g, \sigma(g)^{-1} f)$. Notice that the typical fiber of this bundle is $G \times F$, so we may attempt to define a map that is locally the action σ. First we need the following lemma.

Lemma 19. *Suppose G acts properly and freely on P from the right and F is a manifold equipped with a smooth left G-action σ. Then G acts properly and freely on $P \times G$ from the right.*

Proof. Let $p \in P$, $x \in G$ and $v \in F$. Suppose first that $(p,v) \cdot x = (p,v)$. Then in particular $p \cdot x = p$ so $x = e$ because G acts freely on P. Next, we must show that the map

$$\alpha : P \times F \times G \to (P \times F) \times (P \times F) \tag{2.3}$$
$$(p,v,x) \mapsto ((p,v), (p \cdot x, \sigma(x^{-1})v)) \tag{2.4}$$

is proper. As such, let K be a compact subset of $(P \times F) \times (P \times F)$. We will show $\alpha^{-1}(K)$ is compact by showing that its projection onto all separate factors is compact. Write

$$A_P = \mathrm{Pr}_P(\alpha^{-1}(K)),$$
$$A_F = \mathrm{Pr}_F(\alpha^{-1}(K)),$$
$$A_G = \mathrm{Pr}_G(\alpha^{-1}(K)).$$

If all these projections are compact, then $\alpha^{-1}(K)$ is compact, being a closed subset of the compact space $A_P \times A_F \times A_G$. Notice first that the projection of $\alpha^{-1}(K)$ onto $P \times G$ is compact because the action of G on P is proper. This means in particular that A_G is compact. Then $A_F \subset A_G \cdot B_F$ for some compact subset $B_F \subset F$. (In fact, B_F can be taken to be the projection of K onto the first F.) Since the group action on F is continuous and both A_G and B_F are compact, $A_G \cdot B_F$ is compact. Thus A_F is compact, being a closed subset thereof. □

It follows by the previous lemma and the quotient manifold theorem that we can endow the orbit space $E = P \times_G F \stackrel{d}{=} (P \times F)/G$ with a unique structure of a smooth manifold making the quotient map $P \times F \xrightarrow{q_E} E$ a smooth submersion. Since the fibers of P are preserved by the G-action, the projection $\pi \circ \mathrm{Pr}_1 : P \times F \to B$ induces a smooth surjective map $E \xrightarrow{\pi_E} B$ such that the following diagram commutes:

$$\begin{array}{ccc} P \times F & \xrightarrow{q_E} & E \\ {\scriptstyle \mathrm{Pr}_1}\big\downarrow & & \big\downarrow{\scriptstyle \pi_E} \\ P & \xrightarrow{\pi} & B \end{array} . \tag{2.5}$$

It turns out that $E \xrightarrow{\pi_E} B$ is a fiber bundle so that the above diagram defines a morphism of fiber bundles.

Lemma 20. *Let $P \xrightarrow{\pi} B$ be a principal G-bundle and let σ be a left action of G on some manifold F. Then $P \times_G F \to B$ is a fiber bundle with typical fiber F and equation* (2.5) *defines a morphism of fiber bundles.*

Proof. The local trivializations of E are defined such that the quotient map q_E is locally just $\mathrm{id} \times \sigma$. Explicitly, suppose that $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is a local trivialization of the principal bundle $P \to B$. Then the map

$$\widetilde{\psi}_\alpha : \pi^{-1}(U_\alpha) \times F \xrightarrow{\phi_\alpha \times \mathrm{id}_F} U_\alpha \times G \times F \xrightarrow{\mathrm{id} \times \sigma} U_\alpha \times F$$

is smooth and constant on G-orbits, where $\mathrm{id}_F(v) = v$ for $v \in F$. Indeed, suppose that $\phi_\alpha(p) = (m, x)$, then the map above is given by $\widetilde{\psi}_\alpha(p, v) = (m, \sigma(x)v)$ and we have $\widetilde{\psi}_\alpha(p \cdot g, \sigma(g)^{-1}v) = (m, \sigma(x)v)$. Since q_E is a surjective submersion it follows that this map induces a unique smooth map $\psi_\alpha : \pi_E^{-1}(U_\alpha) \to U_\alpha \times F$ such that $\widetilde{\psi_\alpha} = \psi_\alpha \circ q_E$. Notice that $\psi_\alpha([\phi_\alpha^{-1}(m, x), v]) = (m, \sigma(x)v)$. The situation is summarized in the following commutative diagram.

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) \times F & \xrightarrow{q_E} & \pi_E^{-1}(U_\alpha) \\ {\scriptstyle \phi_\alpha \times \mathrm{id}}\big\downarrow & \searrow^{\widetilde{\psi}_\alpha} & \big\downarrow{\scriptstyle \psi_\alpha} \\ U_\alpha \times G \times F & \xrightarrow[\mathrm{id} \times \sigma]{} & U_\alpha \times F \end{array} . \tag{2.6}$$

We proceed by showing that in fact ψ_α is a diffeomorphism, so that the vertical arrows in the above diagram are local trivializations and q_E is locally just $\mathrm{id} \times \sigma$.

Define χ_α to be the following composition of smooth maps:

$$\chi_\alpha : U_\alpha \times F \xrightarrow{\phi_\alpha^{-1}(\cdot, 1) \times \mathrm{id}_F} \pi^{-1} U_\alpha \times F \to \pi_E^{-1}(U_\alpha),$$

that is, $\chi_\alpha(m, v) = [\phi_\alpha^{-1}(m, 1), v]$. Suppose again that $\phi_\alpha(p) = (m, x)$. We compute

$$(\chi_\alpha \circ \psi_\alpha)([p, v]) = \chi_\alpha(m, \sigma(x)v) = [\phi_\alpha^{-1}(m, 1)\sigma(x)v] = [\phi_\alpha^{-1}(m, x), v] = [p, v],$$
$$(\psi_\alpha \circ \chi_\alpha)(m, v) = \psi_\alpha([\phi_\alpha^{-1}(m, 1), v]) = \psi_\alpha([\phi_\alpha^{-1}(m, x) \cdot x^{-1}, v]) = \psi_\alpha([p, \sigma(x)^{-1}v] = (m, v).$$

This shows that χ_α and ψ_α are smooth inverses of each other, so indeed ψ_α is a diffeomorphism. Next, observe that ψ_α also define local trivializations of the bundle $E \xrightarrow{\pi_E} B$. This follows from equation (2.5). Let $\widetilde{\pi_E}$ be the unique smooth map such that $\pi_E = \widetilde{\pi_E} \circ \psi_\alpha$:

$$\begin{array}{ccc} \pi_E^{-1}(U_\alpha) & & \\ {\scriptstyle \psi_\alpha}\big\downarrow & \searrow^{\pi_E} & \\ U_\alpha \times F & \xrightarrow[\widetilde{\pi_E}]{} & U_\alpha \end{array} .$$

We need to show that $\widetilde{\pi_E}$ is just projection onto the first coordinate. Now, in the local coordinates defined above, (2.5) becomes the commutative diagram

$$\begin{array}{ccc} U_\alpha \times G \times F & \xrightarrow{\mathrm{id} \times \sigma} & U_\alpha \times F \\ \big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\pi_E}} \\ U_\alpha \times G & \longrightarrow & U_\alpha \end{array} ,$$

where the arrows without label are the obvious projections. The assertion immediately follows.

Finally, observe that this also shows that the typical fiber of the constructed associated bundle $E \to B$ is F and that π_E is a submersion (seeing as $B \times F \xrightarrow{\mathrm{Pr}_1} B$ is). We conclude that $E \xrightarrow{\pi_E} F$ is indeed a fiber bundle with typical fiber F. □

Lemma 21. *Let P be a principal G-bundle and let σ be a finite dimensional representation of G on $\mathcal{H}_\sigma$. Then $E \stackrel{d}{=} P \times_G \mathcal{H}_\sigma$ is a vector bundle over B and* (2.5) *is a morphism of vector bundles. If $\mathcal{H}_\sigma$ is a Hilbert space and σ is a unitary representation, then $P \times_G \mathcal{H}_\sigma$ is a Hilbert bundle and* (2.5) *is a morphism of Hilbert bundles.*

Proof. For the first statement, the only thing that has not yet been shown is the fact that the local trivializations ψ_α restrict to linear maps on the fibers. This follows immediately from the observation that each $\widetilde{\psi}_\alpha : \pi^{-1}(U_\alpha) \times F \to U_\alpha \times \mathcal{H}_\sigma$ is linear on fibers. Recall that q_E is locally just $\mathrm{id} \times \sigma$, which is linear since σ is a representation. It is therefore clear that (2.5) is a morphism of vector bundles. Now, suppose that σ is a unitary representation. Then each fiber E_b is equipped with an inner product via $\langle [p, v_1], [p, v_2] \rangle_{E_b} \stackrel{d}{=} \langle v_1, v_2 \rangle_\sigma$, which is independent on the choice of $p \in P_b$ since σ is unitary. Notice that with respect to this inner product the local trivializations ψ_α become unitary on fibers, since

$$\langle \psi_\alpha^{-1}(b, v_1), \psi_\alpha^{-1}(b, v_2) \rangle_{E_b} = \langle [\phi_\alpha^{-1}(b,1), v_1], [\phi_\alpha^{-1}(b,1), v_2] \rangle_{E_b} = \langle v_1, v_2 \rangle_\sigma.$$

Thus ψ_α^{-1} restricts to an isometric and thence unitary map on fibers. It is then also clear that is a morphism of Hilbert bundles since q_E is locally $\mathrm{id} \times \sigma$, which is even unitary on fibers since σ is. □

Lemma 22. *For any fixed principal G-bundle P over B, the associated bundle construction defines a functor $F = P \times_G -$ from the category of finite dimensional representations of G to vector bundles over B and from the category of finite dimensional unitary representations of G to Hilbert bundles over B.*

Proof. suppose we are given two representations $\mathcal{H}_{\sigma_1}, \mathcal{H}_{\sigma_2}$ of G and an intertwining map $f \in \mathrm{Hom}_G(\mathcal{H}_{\sigma_1}, \mathcal{H}_{\sigma_2})$. The composition

$$P \times \mathcal{H}_{\sigma_1} \xrightarrow{\mathrm{id} \times f} P \times \mathcal{H}_{\sigma_2} \to P \times_G \mathcal{H}_{\sigma_2}$$

is smooth and constant on G-orbits. It therefore induces a unique smooth map $F(f) : P \times_G \mathcal{H}_{\sigma_1} \to P \times_G \mathcal{H}_{\sigma_2}$ that is a morphism of fiber bundles over B. Explicitly, it is given by $F(f)([p, v]) = [p, f(v)]$. A direct computation shows that in local coordinates, $F(f)$ is simply $U_\alpha \times \mathcal{H}_{\sigma_1} \xrightarrow{\mathrm{id} \times f} U_\alpha \times \mathcal{H}_{\sigma_2}$. Indeed, writing $\psi_{k,\alpha}$ for the local trivializations of $P \times_G \mathcal{H}_{\sigma_k}$ that were constructed above, we have

$$\begin{aligned}(\psi_{2,\alpha} \circ F(f) \circ \psi_{1,\alpha}^{-1})(m, v) &= \psi_{2,\alpha}([\phi_\alpha(m,1), f(v)]) \\ &= \psi_{2,\alpha} \circ \psi_{2,\alpha}^{-1} \\ &= (m, f(v)).\end{aligned}$$

It follows that $F(f)$ is linear on fibers so that it is a morphism of vector bundles.

If we consider instead the categories of unitary representations and Hilbert bundles, then an intertwining map f of unitary representations is additionally required to be isometric. Since $F(f)$ is locally $\mathrm{id} \times f$, it follows that also $F(f)$ is isometric on fibers and thence defines a morphism of Hilbert bundles.

Finally, this assignment of intertwining maps to morphisms of vector bundles over B is functorial. □

We mention here that the functor $P \times_G -$ preserves direct sums and tensor products. The precise statements are stated and proven in the next section, which discusses an equivariant setting. However, the result and proofs given there remain valid in this more general setting upon slight modifications.

2.3.2 Homogeneous Hilbert bundles

Finally, we consider the notion of a homogeneous Hilbert bundle, which is a Hilbert bundle equipped with additional structure. Namely, both its total and base spaces should be equipped with structure-preserving G-actions such that the projection map is equivariant. It is additionally required that the G acts transitively on the base space, so that for any two fibers there exists a group element that identifies these fibers via the group action. This type of equivariant bundles will play a prominent role in chapter 4 and chapter 6. In fact, the group action of G on both the base and total space induces an action on the of sections of this bundle, and it is *this* action that gives rise to the induced representation in chapter 4. As these objects will play a major role throughout the thesis, they are discussed in a bit more detail.

Definition 23. Let G be a Lie group and $E \xrightarrow{\pi} M$ be a vector bundle. Assume further that both E and M are equipped with smooth left G-actions α and β. The vector bundle $E \to M$ is called **homogeneous** for G if the action of G on M is transitive and $(\alpha(g), \beta(g))$ is a isomorphism of vector bundles for every $g \in G$. That is, the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{\alpha(g)} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ B & \xrightarrow{\beta(g)} & B \end{array}.$$

If $E \to M$ is a Hilbert bundle and $\alpha(g)$ is unitary for every $g \in G$, then $E \to M$ is called a **homogeneous Hilbert bundle.**

Notice that the diagram above is equivalent to the statement that the projection $E \xrightarrow{\pi} M$ is G-equivariant.

Definition 24. An **morphism** of homogeneous vector(Hilbert) bundles for G is a G-equivariant morphism of vector(Hilbert) bundles. It is an isomorphism of it is an isomorphism of vector(Hilbert) bundles.

Suppose now that G is a Lie group and H is a closed subgroup. We have already seen that $G \to G/H$ is a principal H-bundle. Now, left multiplication in G induces unique smooth left G actions on the coset space G/H and on any associated vector bundle $E \stackrel{d}{=} G \times_H \mathcal{H}_\sigma$ such that $\beta(g) \circ q = l_g \circ q$ and $\alpha(g) \circ q_E = q_E \circ l_g \times \mathrm{id}$:

$$\begin{array}{ccc} G & \xrightarrow{l_g} & G \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ G/H & \xrightarrow{\beta(g)} & G/H \end{array}, \qquad \begin{array}{ccc} G \times \mathcal{H}_\sigma & \xrightarrow{l_g \times \mathrm{id}} & G \times \mathcal{H}_\sigma \\ {\scriptstyle q_E}\downarrow & & \downarrow{\scriptstyle q_E} \\ E & \xrightarrow{\alpha(g)} & E \end{array}.$$

It turns out that associated Hilbert bundles of this form are homogeneous, and conversely all homogeneous vector bundles are of this form.

Lemma 25. *Let G be a Lie group with closed subgroup H and let σ be a unitary finite dimensional representation of H on the Hilbert space $\mathcal{H}_\sigma$. Then $G \times_H \mathcal{H}_\sigma \to G/H$ is a homogeneous Hilbert bundle.*

Proof. The left G-action on G/H is clearly transitive. Write $E = G \times_H \mathcal{H}_\sigma$. Denote the left G-action on E and B by α and β, respectively. Fix $g \in G$. It is clear that $\alpha(g)$ and $\beta(g)$ are diffeomorphisms of E and G/H and that $\alpha(g)$ is linear on fibers. It follows by the commutativity of the diagram below that they even define a morphism of vector bundles:

$$\begin{array}{ccccc} G \times \mathcal{H}_\sigma & \xrightarrow{l_g \times \mathrm{id}} & G \times \mathcal{H}_\sigma & \xrightarrow{q_E} & E \\ {\scriptstyle \mathrm{Pr}_1}\downarrow & & \downarrow{\scriptstyle \mathrm{Pr}_1} & & \downarrow{\scriptstyle \pi_E} \\ G & \xrightarrow{l_g} & G & \xrightarrow{q} & G/H \end{array},$$

where $l_g G \to G$ denotes left-multiplication by g in G. Notice that the right-hand square is just (2.5). Indeed, it follows that

$$\pi_E \circ \alpha(g) \circ q_E = \pi_E \circ q_E \circ l_g \times \mathrm{id} = q \circ l_g \circ \mathrm{Pr}_1 = \beta(g) \circ \pi_E.$$

Since q_E is surjective, this implies that $\pi_E \circ \alpha(g) = \beta(g) \circ \pi_E$. Now, $\alpha(g)$ is unitary on fibers because $\alpha(g) \circ q_E = q_E \circ l_g \times \mathrm{id}$ maps any fiber $\{x\} \times \mathcal{H}_\sigma$ above $x \in G$ unitarily to E_{gxH}. Since q_E is unitary as a map $\{x\} \times \mathcal{H}_\sigma \to E_{xH}$, it follows that also $\alpha(g)|_{E_x} : E_x \to E_{gx}$ must be unitary. $\square$

Lemma 26. *Let G be a Lie group with closed subgroup H and let σ be a unitary finite dimensional representation of H on the Hilbert space $\mathcal{H}_\sigma$. Then $G \times_H -$ defines a functor from the category of unitary representations of H to the category of G-homogeneous Hilbert bundles above G/H.*

Proof. In view of lemma 22, it remains to show that the map $F(f)$ as in the proof of that lemma is in fact G-equivariant. This is immediate from its definition. $\square$

Lemma 27. *Let $E \to B$ be a homogeneous Hilbert bundle under the action of some Lie group G. Then for any $b \in B$ we have an equivalence of homogeneous Hilbert bundles:*

$$\begin{array}{ccc} G \times_{G_b} E_b & \xrightarrow{\Phi} & E \\ \big\downarrow & & \big\downarrow \\ G/G_b & \xrightarrow{\phi} & B \end{array} .$$

Proof. To proof the claim, pick any $b \in B$. One checks that the map $\phi : G/G_b \to B, [g] \mapsto g \cdot b$ is well-defined, injective and smooth. Since G acts transitively on B, it is also surjective. The map is clearly G-equivariant and since G acts transitively on itself, it follows that it has constant rank. Thus, ϕ is a diffeomorphism. Similarly, the map $\Phi : G \times_{G_b} E_b \to E, [g, v] \mapsto g \cdot v$ is well defined and G-equivariant. Notice that Φ is a smooth map because the G-action on E is smooth and the map $G \times E_b \to E \times_{G_b} E_b$ is a surjective submersion. The smooth maps ϕ and Φ make the diagram above commute so that they do indeed define a bundle morphism. Since Φ is linear on fibers, it is even a morphism of vector bundles. Now, Φ is surjective by homogeneity: $E_{g \cdot b} = g \cdot E_b$ and G acts transitively on B. Injectivity follows since $g_1 \cdot v_1 = g_2 \cdot v_2 \iff g_2^{-1} g_1 \in G_b$ and therefore

$$g_1 \cdot v_1 = g_2 \cdot v_2 \implies [g_1, v_1] = [g_2(g_2^{-1} g_1), v_1] = [g_2, (g_2^{-1} g_1) \cdot v_1] = [g_2, v_2].$$

It follows by proposition 9 that Φ is a diffeomorphism. Finally, Φ is isometric and thence unitary on fibers by the following quick computation:

$$\langle \Phi([g, v_1]), \Phi([g, v_2]) \rangle_{E_{g \cdot b}} = \langle g \cdot v_1, g \cdot v_2 \rangle_{E_{g \cdot b}} = \langle [g, v_1], [g, v_2] \rangle.$$

Notice that the last equality holds by the definition of the inner product on the fibers of $G \times_{G_b} \mathcal{H}_\sigma$ and the fact that the fibers of E are all unitarily equivalent to $\mathcal{H}_\sigma$. $\square$

Suppose G be a Lie group with closed subgroup H. Then $G \xrightarrow{q} G/H$ be the canonical map and let $G \times_H \mathcal{H}_\sigma$ be a homogeneous vector bundle. We can equivalently consider F as a right G space by defining $v \cdot x \stackrel{d}{=} \sigma(x^{-1}) v$. The following lemma establishes a connection between sections of the associated bundle $G \times_G \mathcal{H}_\sigma$ and smooth H-equivariant maps $\mathrm{Hom}_H(G, \mathcal{H}_\sigma)$. The former admit a clear geometrical interpretation, but the latter can easier to use in certain proofs.

Lemma 28. *Let G be a Lie group with closed subgroup H. Let $G \xrightarrow{q} G/H$ be the canonical map. Let σ be a finite dimensional representation of H on $\mathcal{H}_\sigma$. Then there is a G-equivariant linear bijection*

$$\Gamma(G \times_H \mathcal{H}_\sigma) \xrightarrow{\Phi} Hom_H(G, \mathcal{H}_\sigma),$$

where G acts on the two spaces according to

$$\begin{aligned} (g \cdot s)(xH) &\stackrel{d}{=} g \cdot s(g^{-1} xH), && s \in \Gamma(G \times_H \mathcal{H}_\sigma), \\ (g \cdot f)(x) &\stackrel{d}{=} f(g^{-1} x), && f \in Hom_H(G, \mathcal{H}_\sigma). \end{aligned}$$

Moreover, $q(\operatorname{supp} \Phi(s)) = \operatorname{supp}(s)$.

Proof. Write $E = G \times_H \mathcal{H}_\sigma$. The idea is to use the group action on E to translate all fibers to the fiber above the identity coset, which is isomorphic to $\mathcal{H}_\sigma$ as a H-representation. Let $\phi : E_H \to \mathcal{H}_\sigma$ be a H-equivariant linear isomorphism. Define the maps

$$\begin{aligned} \Phi &: \Gamma(E) \to \mathrm{Hom}_H(G, \mathcal{H}_\sigma), && \Psi : \mathrm{Hom}_H(G, \mathcal{H}_\sigma) \to \Gamma(E) \\ (\Phi s)(x) &\stackrel{d}{=} \phi(x^{-1} \cdot s(xH)), && (\Psi f)(xH) = x \cdot \phi^{-1}(f(x)) \end{aligned}$$

Notice that all the maps involved are smooth as a composition of smooth maps and $\psi(f)$ is indeed well-defined by the H-equivariance of ϕ. Moreover, these two operations are each others inverse, thus showing bijectiviy. Next, Φ is indeed G-equivariant since

$$\Phi(g \cdot s)(x) = \phi(x^{-1} g \cdot s(g^{-1} xH)) = \phi((g^{-1} x)^{-1} \cdot s(g^{-1} xH)) = \phi(s)(g^{-1} x) = (g \cdot \phi(s))(x).$$

Finally, the last statement follows trivially since $x \in \operatorname{supp}(\Phi(s)) \iff xH \in \operatorname{supp}(s)$. $\square$

Finally, the following lemma is concerned with the behaviour of homogeneous bundles under the bundle operations defined in section 2.2.1 and will be heavily made use of in chapter 6.

Lemma 29. *Let G be a Lie group with closed subgroup H. Suppose that $\mathcal{F}$ and $\{\mathcal{F}_i\}_{i=1}^N$ are finite dimensional unitary representations of H. We have the following equivalences of G-homogeneous Hilbert bundles over G/H:*

$$\begin{aligned} G \times_H \bigoplus_i \mathcal{F}_i &\cong \bigoplus_{i=1}^N G \times_H \mathcal{F}_i, \\ G \times_H \bigotimes_i \mathcal{F}_i &\cong \bigotimes_{i=1}^N G \times_H \mathcal{F}_i, \\ G \times_H \bigvee^k \mathcal{F} &\cong \bigvee^k G \times_H \mathcal{F}, \\ G \times_H \bigwedge^k \mathcal{F} &\cong \bigwedge^k G \times_H \mathcal{F}, \\ G \times_H \mathcal{F}^* &\cong (G \times_H \mathcal{F})^*. \end{aligned}$$

Where the inner products on $\bigotimes_i \mathcal{F}_i$, $\bigvee^k \mathcal{F}$, $\bigwedge^k \mathcal{F}$ and $\mathcal{F}^*$ are defined by

$$\begin{aligned} \langle v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k \rangle &\stackrel{d}{=} \prod_{i=1}^k \langle v_i, w_i \rangle^k, \\ \langle v_1 \vee \cdots \vee v_k, w_1 \vee \cdots \vee w_k \rangle &\stackrel{d}{=} \prod_{i=1}^k \langle v_i, w_i \rangle^k, \\ \langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle &\stackrel{d}{=} \prod_{i=1}^k \langle v_i, w_i \rangle^k, \\ \langle \langle -, v \rangle, \langle -, w \rangle \rangle &= \langle v, w \rangle. \end{aligned}$$

Proof. Write $F = G \times_H \bigoplus_i \mathcal{F}_i$ and $E_i \stackrel{d}{=} G \times_H \mathcal{F}_i$. We proof the first equivalence, the others being completely similar. Define the map

$$\begin{aligned} t &: F \to \bigoplus_i E_i, \\ t &: [x, \oplus_i v_i] \mapsto (xH, \oplus_i [x, v_i]), \end{aligned}$$

which is a well-defined, bijective G-equivariant morphism of vector bundles over G/H. By virtue of proposition 9, it remains to show it is smooth and unitary on fibers, for which we consider local coordinates. Let $\phi : q^{-1}(U) \to U \times G$ be a local trivialization of $G \xrightarrow{q} G/H$. The corresponding trivialization of a general associated bundle $G \times_H \mathcal{H}_\sigma \xrightarrow{p} G/H$ is the inverse of $U \times \mathcal{H}_\sigma \to p^{-1}(U), (u, z) \mapsto [\phi^{-1}(u, 1), z]$. By construction, the local trivialization of a direct sum of vector bundles is locally just the direct sum of the individual trivializations. Therefore, denoting by ψ_F and ψ_E the corresponding local trivializations of E and F, we find

$$\begin{aligned} (\psi_E \circ t \circ \psi_F^{-1})(u, \oplus_i v_i) &= (\psi_E \circ t)[\phi^{-1}(u, 1), \oplus_i v_i] \\ &= \psi_E(u, \oplus_i [\phi^{-1}(u, 1), v_i]) \\ &= (u, \oplus_i v_i). \end{aligned}$$

Thus, t is in local coordinates simply the identity, which is obviously smooth and unitary on fibers. □

Chapter 3

Clifford Algebras, Spin Groups and their Representations

In geometry, one may consider a vector space V together with a non-degenerate symmetric bilinear (or quadratic) form q. In particular, Minkowski space $(\mathbb{R}^4, \eta)$ is of this form. Any vector space defines an exterior algebra and this algebra is of importance in fields such as physics, differential geometry and representation theory. However, it has no concern for the additional geometric data defined on a quadratic space; the bilinear form q. The idea of a Clifford algebra is to deform the exterior algebra by altering its multiplication in such a way that it encodes this geometric data in an algebraic structure and in a suitable manner. It is therefore no surprise that Clifford algebras have found an important role in geometry and physics; in particular in special relativity.

The Clifford algebra is constructed such that it gives a one-to-one correspondence between the automorphisms of the Clifford algebra leaving V invariant and elements in the orthogonal group $O(V, q)$. This correspondence suggests that Clifford algebras might play a prominent role in the study of these orthogonal groups and this is indeed the case. Of particular interest in this thesis is the fact that they can be used to construct the universal covering group of $\mathrm{Spin}(V, q)^0$ along with its covering homomorphism. The explicit construction of this homomorphism will be relevant in section 4.5 and chapter 6.

This chapter is restricted to precisely those definitions and results that are needed in other chapters. Moreover, most proofs are omitted for the sake of brevity. For a more detailed exhibition of these results, we refer to [LM89], on which this exposition is based. The chapter starts out by defining the Clifford algebra and examining some of its basic properties, after which the spin group $\mathrm{Spin}(V, q)$ is introduced. A main result of the chapter is the fact that $\mathrm{Spin}(V, q)^0$ is the universal covering group of $SO(V, q)^0$, along with the construction of the covering homomorphism. Next, the representation theory of a Clifford algebra and its spin group is considered to a very brief extent. Finally, the isomorphisms $\mathrm{Spin}(3) \cong SU(2)$ and $\mathrm{Spin}(1, 3)^0 \cong SL(2, \mathbb{C})$ are realized explicitly, which will be needed extensively in chapter 5 and chapter 6.

3.1 Clifford algebra

As mentioned in the introduction, the idea of a Clifford algebra is to deform the exterior algebra by altering its multiplication in such a way that it encodes this geometric data in an algebraic structure.

Before we dive into the definitions, we make a brief remark. Notice that the process of polarization yields a bijective correspondence between non-degenerate quadratic forms and non-degenerate symmetric bilinear forms on a real or complex vector space V. With this observation in mind, no distinction is made between the two and we write $q(v)$ for the quadratic form and $q(v, w)$ for the corresponding bilinear form.

Definition 30.
A **quadratic space** (V, q) is a real or complex vector space V together with a non-degenerate quadratic form q on V.

A morphism between two quadratic spaces $(V_1, q_1) \to (V_2, q_2)$ is a linear map $\lambda : V_1 \to V_2$ such that $\lambda^* q_2 = q_1$, that is, $q_2(\lambda(v)) = q_1(v)\ \forall v \in V_1$.

Given an quadratic space (V, q), define its orthogonal group $O(V, q)$ to be its group of automorphisms and $SO(V, q)$ to be the automorphisms of determinant one:

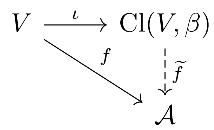
$$O(V,q) \overset{d}{=} \{\lambda \in GL(V) \,:\, \lambda^* q = q\}$$
$$SO(V,q) \overset{d}{=} O(V,q) \cap SL(V)$$

We define the Clifford algebra $\mathrm{Cl}(V, q)$ up to isomorphism by the universal property we want it to satisfy. Its existence and uniqueness are proven immediately after.

Definition 31. Let (V, q) be a quadratic space. Let $\mathrm{Cl}(V, q)$ be an associative unital algebra and $V \xrightarrow{\iota} \mathrm{Cl}(V, q)$ be a linear map such that

1. $\iota(V)$ generates $\mathrm{Cl}(V, q)$,
2. $\iota(v)^2 = q(v) \cdot 1$

and such that $V \xrightarrow{\iota} \mathrm{Cl}(V, q)$ is universal with these properties, in the sense that for any other unital associative algebra $\mathcal{A}$ together with a linear map $V \xrightarrow{f} \mathcal{A}$ satisfying these properties, there exists a unique unital algebra homomorphism $\mathrm{Cl}(V, q) \xrightarrow{\tilde{f}} \mathcal{A}$ making the diagram below commute.



Then $\mathrm{Cl}(V, q)$ is called the **Clifford algebra** associated to (V, q).

Proposition 32. *The Clifford algebra $Cl(V, q)$ as defined above exists and is unique up to isomorphism. Moreover, the map $V \xrightarrow{\iota} Cl(V, q)$ is injective.*

Proof. The proof given here is taken from [LM89, p. 8]. Let $\mathcal{J}$ be the two-sided ideal of the tensor algebra $\mathcal{T}(V)$ generated by all elements of the form
$v \otimes v - q(v) \cdot 1$. Define $\mathrm{Cl}(V, q)$ to be the quotient of the tensor algebra $\mathcal{T}(V)$ by $\mathcal{J}$:

$$\mathrm{Cl}(V,q) \overset{d}{=} \mathcal{T}(V)/\mathcal{J}.$$

Let $\iota : V \to \mathrm{Cl}(V, q)$ be the restriction of the canonical map $\mathcal{T}(V) \to \mathrm{Cl}(V, q)$ to $V = \bigotimes^1 V$. By construction $\mathrm{Cl}(V, q)$ is a unital algebra generated by $\iota(V)$ and subject to the relation $\iota(v)^2 = q(v) \cdot 1$.

We show ι is injective. It suffices so to show $V \cap \mathcal{J} = 0$. By definition of $\mathcal{J}$, any element $\phi \in \mathcal{J} \cap V$ can be written as a finite sum of the form $\phi = \sum_i a_i \otimes (v_i \otimes v_i - q(v_i)1) \otimes b_i$ for some tensors $a_i, b_i \in \mathcal{T}(V)$. We may assume that all a_i, b_i are of pure degree. Let $k = \max_i \deg a_i + \deg b_i < \infty$. Then since $\phi \in V$, we find that

$$\sum_{\deg a_i + \deg b_i = k} a_i \otimes v_i \otimes v_i \otimes b_i = 0.$$

Contracting with q this implies in particular that also

$$\sum_{\deg a_i + \deg b_i = k} a_i \otimes q(v_i)1 \otimes b_i = 0.$$

and therefore

$$\sum_{\deg a_i + \deg b_i = k} a_i \otimes (v_i \otimes v_i - q(v_i)1) \otimes b_i = 0.$$

Induction on k yields that $\phi = 0$, so ι is injective.

Next, we show that $\mathrm{Cl}(V,q)$ satisfies the required universal property. Let $\mathcal{A}$ be a unital associative algebra and $V \xrightarrow{f} \mathcal{A}$ be a linear map such that $f(V)$ generates $\mathcal{A}$ and $f(v)^2 = q(v)1$. By the universal property of the tensor algebra, the map f factors through the tensor algebra via a unique algebra homomorphism $\mathcal{T}(V) \xrightarrow{\overline{f}} \mathcal{A}$. The properties mean precisely that this $\overline{f}(v \otimes v - q(v)1) = 0$ for any $v \in V$ and therefore $\overline{f}$ is trivial on the ideal $\mathcal{J}$. This means that it factors through the quotient $\mathcal{T}(V)/\mathcal{J} = \mathrm{Cl}(V,q)$ via a unique algebra homomorphism $\widetilde{f} : \mathrm{Cl}(V,q) \to \mathcal{A}$. Thus we are done. ☐

Remark.

1. Notice that if $q = 0$, the ideal $\mathcal{J}$ from the previous proof is generated by all elements of the form $v \otimes v$. It follows that we have an algebra isomorphism $\mathrm{Cl}(V,0) \cong \Lambda(V)$.
2. By the above proposition, we may identify V as a linear subspace of $\mathrm{Cl}(V,q)$. Therefore, the embedding ι is usually omitted and we simply write $v \cdot w$ for $v, w \in V$ instead of the more precise notation $\iota(v) \cdot \iota(w)$.
3. Notice that the condition $v^2 = q(v)$ may equivalently be given by $v \cdot w + w \cdot v = 2q(v,w) \cdot 1$ by means of polarization.

Corollary 33. *The assignment $(V,q) \to Cl(V,q)$ defines a functor from the category of quadratic spaces to the category of unital associative algebras. In particular, the group of orthogonal transformations $O(V,q)$ extends to a group of automorphisms of the Clifford algebra.*

Corollary 34. *The automorphism α of $Cl(V,q)$ obtained by extending the map $V \to V, v \mapsto -v$ induces a decomposition of the Clifford algebra into its eigenspaces:*

$$Cl(V,q) = Cl^0(V,q) \oplus Cl^1(V,q),$$

where $Cl^k(V,q) \stackrel{d}{=} \{t \in Cl(V,q) \,:\, \alpha(t) = (-1)^k t\}$.
Moreover, this decomposition defines a $\mathbb{Z}_2$-grading on the Clifford algebra.

Definition 35. The subspace $\mathrm{Cl}^0(V,q)$ is called the **even part** of $\mathrm{Cl}(V,q)$ where $\mathrm{Cl}^1(V,q)$ is called the **odd** part.

Finally, there are some close relations between a Clifford algebra and the exterior algebra. These results are not needed in later chapters. Nonetheless, they show that the Clifford algebra is indeed an enhancement of the exterior algebra determined by the form q. In view of clarity, we mention them without proof.

The natural filtration on the tensor algebra descends to a filtration $\{\mathcal{F}^r\}$ on the Clifford algebra with the property that $\mathcal{F}^r \cdot \mathcal{F}^l \subseteq \mathcal{F}^{r+l}$. This makes the Clifford algebra into a filtered algebra and we can define its associated graded algebra by $\bigoplus_{r \geq 0} \mathcal{F}^{r+1}/\mathcal{F}^r$.

Proposition 36. *For any quadratic form q, the associated graded algebra of $Cl(V,q)$ is naturally isomorphic to $\Lambda(V)$.*

Proof. See [LM89, p. 10]. ☐

Proposition 37. *There is a vector space isomorphism $Cl(V,q) \cong \Lambda(V)$ that is compatible with the filtrations.*

Proof. See [LM89, p. 10]. ☐

Corollary 38. *The Clifford algebra $Cl(V,q)$ has dimension 2^n, where $n = \dim V$.*

3.2 Spin group

The Clifford algebra can be used to construct the universal covering group of $SO(V,q)^0$ and to construct explicitly the universal covering homomorphism onto $SO(V,q)^0$. Explicitly, the group of units of the Clifford algebra acts on the Clifford algebra by the adjoint action. It turns out that restricting this action to a suitable subgroup yields the covering homomorphism. The precise construction of this homomorphism, in particular the fact that it acts on $\mathrm{Cl}(V,q)$ via the adjoint action, will be relevant in chapter 6.

Let $\mathrm{Cl}^\times(V,q)$ be the multiplicative group of units of the Clifford algebra. This group acts on the Clifford algebra via the adjoint representation

$$\mathrm{Ad} : \mathrm{Cl}^\times(V,q) \to GL(\mathrm{Cl}(V,q)),$$
$$\mathrm{Ad}_\phi(x) = \phi x \phi^{-1}.$$

Proposition 39. *Let $v \in V$ be such that $q(v) \neq 0$. Then for all $w \in V$, we have*

$$-Ad_v(w) = w - 2\frac{q(v,w)}{q(v)}v.$$

In particular, $Ad_v(V) = V$, $Ad_v^* q = q$ and thus $Ad_v \in O(V,q)$.

Proof. The proof makes use of the trivial observation that for $q(v) \neq 0$ the requirement $v^2 = q(v)1$ implies that $v^{-1} = \frac{v}{q(v)}$. Secondly, recall the equation $vw + wv = 2q(v,w)1$. We compute

$$-Ad_v(w) = -vwv^{-1} = -\frac{vwv}{q(v)} = -\left(2q(v,w)1 - wv\right)\frac{v}{q(v)} = w - 2\frac{q(v,w)}{q(v)}v.$$

The last equation follows by expanding out the terms using bilinearity

$$q(\mathrm{Ad}_v(w)) = q\left(w - \frac{2q(v,w)}{q(v)}v, w - \frac{2q(v,w)}{q(v)}v\right) = q(w).$$

□

Definition 40.

1. Let $P(V,q)$ be the subgroup of $\mathrm{Cl}^\times(V,q)$ generated by those $v \in V$ for which $q(v) \neq 0$.
2. Define the **Pin group** $\mathrm{Pin}(V,q)$ to be the subgroup of $\mathrm{Cl}^\times(V,q)$ generated by all elements of $v \in V$ with $|q(v)| = 1$.
3. Define the **Spin group** by $\mathrm{Spin}(V,q) = \mathrm{Pin}(V,q) \cap \mathrm{Cl}^0(V,q)$.

Remark.

1. Proposition 39 implies that the adjoint action defines a homomorphism $P(v,q) \xrightarrow{\mathrm{Ad}} O(V,q)$.
2. The right hand side of proposition 39 is just a reflection across the hyperplane $\{w \in V : q(v,w) = 0\}$. The minus sign on the left-hand side implies that in general, Ad is not surjective onto $O(V,q)$. For example, if V is the one-dimensional real line and $v \neq 0$, proposition 39 implies that $-\mathrm{Ad}_v$ is reflection across the origin, so Ad_v is just the identity. More generally, for odd dimensions of V, we find that $\mathrm{Ad}_v \in SO(V,q)$.

To remedy the non-surjectivity mentioned in the preceding remark, it turns out that one should consider a *twisted* adjoint representation instead. Define

$$\widetilde{\mathrm{Ad}} : \mathrm{Cl}^\times(V,q) \to GL(\mathrm{Cl}(V,q)),$$
$$\widetilde{\mathrm{Ad}}_\phi(x) = \alpha(\phi) x \phi^{-1},$$

where α is the automorphism of $\mathrm{Cl}(V,q)$ induced by the map $v \mapsto -v$. Then for $v, w \in V$ with $q(v) \neq 0$ we have

$$\widetilde{\mathrm{Ad}}_v(w) = w - 2\frac{q(v,w)}{q(v)}v.$$

Remark. Notice that $\widetilde{\mathrm{Ad}}_\phi = \mathrm{Ad}_\phi$ for $\phi \in \mathrm{Cl}^0(V,q)$. In particular, the twisted and non-twisted adjoint actions coincide on $\mathrm{Spin}(V,q)$.

Suppose now that the vector space V is real. In that case we may find a basis of V such that $q(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$. We say that q has signature (r,s) and write $\mathrm{Cl}_{r,s}$ in place of $\mathrm{Cl}(V,q)$ and $O(r,s) = O(V,q)$. We come to the main result of this section.

Theorem 41. *There are short exact sequences*

$$0 \to \mathbb{Z} \to Spin_{r,s} \xrightarrow{Ad} SO(r,s) \to 1$$
$$0 \to \mathbb{Z} \to Pin_{r,s} \xrightarrow{\widetilde{Ad}} O(r,s) \to 1$$

for all (r,s). If $(r,s) \neq (1,1)$, these coverings are connected over each component of $O(r,s)$. Moreover, Ad is the universal covering homomorphism when restricted to the identity components in the following spacial cases, where $r \geq 3$:

$$0 \to \mathbb{Z}_2 \to Spin_r^0 \xrightarrow{Ad} SO(r)^0 \to 1$$
$$0 \to \mathbb{Z}_2 \to Spin_{r,1}^0 \xrightarrow{Ad} SO(r,1)^0 \to 1.$$

Proof. For the proof, see [CGLM08, p. 20]. □

3.3 Explicit isomorphisms in low dimensions

Some of these groups are known explicitly for small dimensions and the explicit isomorphisms and covering homomorphisms will be needed in chapter 5 and chapter 6, so they are discussed here. In particular, the universal covering homomorphism $SL(2,\mathbb{C}) \to SO(1,3)^0$ is constructed, which also yields the universal covering homomorphism $SU(2) \to SO(3)$.

Let us first consider the construction of the covering homomorphism $\phi : SL(2,\mathbb{C}) \to SO(1,3)^0$. Let C denote the real four-dimensional space of 2×2 complex Hermitian matrices. Let σ_1, σ_2 and σ_3 denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.1}$$

and let σ_0 be the identity. Then $\sigma_0, \sigma_1, \sigma_2$ and σ_3 form a orthogonal basis of C with respect to the inner product $\langle a, b\rangle = \mathrm{tr}(a^* b) = \mathrm{tr}(ab)$, so we can identify $\mathbb{R}^4$ with C using this choice of basis. We denote this identification by A so that

$$A : \mathbb{R}^4 \to C$$
$$A : x \mapsto \sum_{i=0}^{3} x_i \sigma_i$$

Observe that

$$\begin{aligned} \mathrm{tr}(A(x)) &= 2x_0 \\ \det(A(x)) &= \eta(x,x) = x_0^2 - x_1^2 - x_2^2 - x_3^2. \end{aligned} \tag{3.2}$$

There is a continuous representation of $SL(2,\mathbb{C})$ in C given by

$$\psi(m)\xi = m\xi m^*, \qquad m \in SL(2,\mathbb{C}), \xi \in C. \tag{3.3}$$

Under the identification A described above we thus obtain a continuous representation $\phi = A^{-1} \circ \psi \circ A$ of $SL(2,\mathbb{C})$ in $\mathbb{R}^4$. Notice that $\det(m) = 1$ for $m \in SL(2,\mathbb{C})$ and therefore we have $\det(m\xi m^*) = \det(\xi)$ for any $\xi \in C$. It follows that ϕ preserves the quadratic form induced by η so that ϕ maps into $O(1,3)$. This yields a continuous Lie group homomorphism $\phi : SL(2,\mathbb{C}) \to O(1,3)$.

Lemma 42. *The induced map on Lie algebras $\phi_* : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{so}(1,3)$ is an isomorphism.*

Proof. First, we claim that $\ker\phi = \{-1, 1\}$. Indeed, let $m \in SL(2,\mathbb{C})$. If $\phi(m) = I$, then $m\xi m^* = \xi$ for every $\xi \in C$. Thus, taking $\xi = I$, it follows that $m^* = m^{-1}$. Therefore, $m\xi = \xi m$ for every ξ, which implies that m is a multiple of the identity. Since $\det(m) = 1$, the claim follows. It follows that ϕ is locally injective and ϕ_* is injective. Since both $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{so}(1,3)$ are six-dimensional[Var07, p. 333], ϕ_* is an isomorphism. ☐
☐

Corollary 43. *The map $\phi : SL(2,\mathbb{C}) \to SO(1,3)^0$ is the universal covering homomorphism. In particular, $SL(2,\mathbb{C}) \cong Spin(1,3)^0$*

Proof. The previous lemma implies that ϕ maps surjectively onto $SO(1,3)^0$, seeing as the latter is connected. Any surjective Lie group homomorphism whose differential is an isomorphism is in fact a covering map. Since $SL(2,\mathbb{C})$ is simply connected, we are done. ☐

Lemma 44. *Identify $SO(1,2)^0$ with the subgroup of $SO(1,3)^0$ leaving the point $e_2 = (0,1,0,0)$ fixed, then $\phi^{-1}(SO(1,2)^0) = SL(2,\mathbb{R})$ and therefore $SO(1,2)^0 \cong SL(2,\mathbb{R})/\{\pm I\}$.*

Proof. Indeed, observe that $A(e_2) = -i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so for $m \in SL(2,\mathbb{C})$, we have

$$\phi(m) \in SO(1,2)^0 \iff m\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} m^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A direct computation shows that this happens if and only if $m \in SL(2,\mathbb{R})$. ☐

Remark.

— The previous lemma is *not* saying that $SL(2,\mathbb{R})$ is the universal covering group of $SO(1,2)^0$, which would be false since $SL(2,\mathbb{R})$ is not simply connected. Nonetheless, the result is useful when studying the representation theory of $SO(1,2)^0$. The lemma above is used in the classification of the strongly continuous unitary representations of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$, where the group $SL(2,\mathbb{R})$ occurs as one of the stabilizers of the action of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$. See also section 4.5.

Lemma 45. *Embed $\mathbb{R}^3 \hookrightarrow \mathbb{R}^4$ via $x \mapsto (0,x)$. Then $\phi(SU(2)) = SO(3)$ and $\phi^{-1}(SO(3)) = SU(2)$.*

$$\begin{array}{ccc} SU(2) & \xrightarrow{\phi} & SO(3) \\ \downarrow & & \downarrow \\ SL(2,\mathbb{C}) & \xrightarrow{\phi} & SO(1,3)_0 \end{array}.$$

Proof. Indeed, suppose $m \in SL(2,\mathbb{C})$. Then by equation (3.2) $\phi(m)$ is orthogonal on $\mathbb{R}^3$ if and only if $\psi(m)$ preserves both $\det(\cdot)$ and $\mathrm{tr}(\cdot)$. Since $\det(m) = 1$, $\psi(m)$ always preserves the determinant. If m is unitary, then by the conjugation invariance of the trace it is clear that $\phi(m)$ is orthogonal. Conversely, suppose that $\psi(m)$ is orthogonal so that it preserves $\mathrm{tr}(\cdot)$. Then for every $\xi \in C$ we have $\mathrm{tr}(\xi) = \mathrm{tr}(m\xi m^*) = \mathrm{tr}(m^* m\xi)$. That is, $\langle m^*m, \xi\rangle_C = \langle I, \xi\rangle_C$ for every $\xi \in C$. This implies $m^*m = I$, so $m^* = m^{-1}$ and m is unitary. ☐

Corollary 46. *The restriction of ϕ to $SU(2)$ is the the universal covering homomorphism onto $SO(3)$. Therefore $Spin(3)^0 \cong SU(2)$.*

Lemma 47. *The representation of $\psi|_{SU(2)}$ on C decomposes as $\mathbb{R}I \oplus \mathfrak{su}(2)$, where $SU(2)$ acts trivially on $\mathbb{R}I$ and via the adjoint representation Ad on $\mathfrak{su}(2)$.*

Proof. Notice first that $\psi(u)\xi = u\xi u^{-1}$ for $\xi \in C$. Let S be the linear space generated by the Pauli matrices $\{\sigma_1, \sigma_2, \sigma_3\}$. Notice that $S = \{X \in M_2(\mathbb{C}) : \mathrm{tr}(X) = 0, X^* = X\}$. Now, any element $u \in SU(2)$ acts (via ψ) trivially on the identity element and leaves S invariant. It follows that so that as a real $SU(2)$ representation, C decomposes as $C \cong \mathbb{R}I \oplus S$, where $SU(2)$ acts trivially on $\mathbb{R}I$. Finally, it is trivial that $S \cong \mathfrak{su}(2)$ as $SU(2)$ representations. ☐

Corollary 48. *The adjoint representation of $SU(2)$ on $\mathfrak{su}(2)$ becomes the covering homomorphism under the isomorphism $\mathfrak{su}(2) \cong \mathbb{R}^3$.*

3.4 Representations

In this final section we discuss to small extent the representation theory of Clifford algebras and their spin groups, mentioning only those results that are needed in chapter 6. Nonetheless, their representation theory is understood in great detail and much more can be said than is done here. Moreover, as certain results are quite standard they are given without proof. For a more detailed exposure, the interested reader may consult [LM89, Chapter 1.5].

Now, any representation of the Clifford algebra $\rho : \mathrm{Cl}(V,q) \to \mathrm{End}(\mathcal{F})$ gives rise to a representation $S : \mathrm{Spin}(V,q)^0 \to GL(\mathcal{F})$ of the connected component $\mathrm{Spin}(V,q)^0$ of the spin group by restriction $S = \rho|_{\mathrm{Spin}(V,q)^0}$. From the previous section we know that the adjoint action of $\mathrm{Spin}(V,q)^0$ on $\mathrm{Cl}(V,q)$ becomes the covering homomorphism ϕ when restricted to V:

$$\begin{array}{ccc} \mathrm{Spin}(V,q)^0 & \xrightarrow{\phi} & SO(V,q) \\ & \searrow^{\mathrm{Ad}} & \downarrow \\ & & GL(\mathrm{Cl}(V,q)) \end{array}$$

Therefore, the representation S of $\mathrm{Spin}(V,q)^0$ satisfies the following equivariance condition:

$$\rho(\phi(w)v) = \rho(\mathrm{Ad}_w(v)) = \rho(wvw^{-1}) = S(w)\rho(v)S(w)^{-1} \qquad w \in \mathrm{Spin}(V,q), \quad v \in V. \tag{3.4}$$

It is this equivariance condition that plays an essential role in the representation theory of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$, as studied by Wigner[Wig39], see also section 8.3 for more details. In a related fashion, it will also play a crucial role in chapter 6.

Definition 49. Given a representation ρ of $\mathrm{Cl}(V,q)$, the representation S of $\mathrm{Spin}(V,q)^0$ obtained by restriction of ρ is called the **spin representation** associated to ρ.

In physics literature, representations of the Clifford algebra are often given in terms of the so-called *gamma-matrices* or *dirac-matrices*, which are defined given a choice of basis for the vector space V. If $\{e_k\}_{k=1}^n$ is a basis of V, the gamma matrices are defined as $\gamma_k \stackrel{d}{=} \rho(e_k)$ and satisfy

$$\begin{cases} \gamma_k^2 & = q(e_k)1 \\ \gamma_k\gamma_r + \gamma_r\gamma_k & = 0 \end{cases} \tag{3.5}$$

Conversely, any set of endomorphisms $\{\gamma_k\}$ satisfying equation (3.5) determines a representation of $\mathrm{Cl}(V,q)$ by defining $\rho(e_r) \stackrel{d}{=} \gamma_r$ and extending it to $\mathrm{Cl}(V,q)$ using the universal property.

Finally, we consider the special case $\mathrm{Cl}_{1,3}$ and its unique irreducible representation ρ on $\mathbb{C}^4$[LM89, p. 32, theorem 5.7]. Let $S = \rho|_{SL(2,\mathbb{C})}$ be the spin representation associated to ρ. We make a specific choice of gamma matrices γ_r that exhibits S in a particularly nice fashion. Let σ_1, σ_2 and σ_3 be the Pauli-matrices defined in equation (3.1) and let γ_r be the 4×4 matrices defined by

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_r = \begin{pmatrix} 0 & -\sigma_r \\ \sigma_r & 0 \end{pmatrix}, \quad r = 1, 2, 3. \tag{3.6}$$

A computation on the level of Lie algebras [Var07, p. 201] shows that with this choice of gamma matrices, S is given by

$$S(w) = \begin{pmatrix} (w^*)^{-1} & 0 \\ 0 & w \end{pmatrix}, \quad w \in SL(2,\mathbb{C}). \tag{3.7}$$

Chapter 4

Induced Representations

This chapter discusses the theory of induced representations, which was largely developed by Mackey [Mac49, Mac52] and is concerned with the question of how the representation theory of a group G is connected to that of its closed subgroups. In particular, it will provide the main tool that is used to study the irreducible representations of the group $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2, \mathbb{C}) \times K$, reducing the classification thereof to the representation theory of various subgroups. A similar approach was originally taken by Wigner in his classification of the irreducible unitary representations of $\mathbb{R}^4 \rtimes SL(2, \mathbb{C})$[Wig39].

A brief overview of the theory is given below.

Any unitary representation of G restricts to a unitary representation of a closed subgroup H. It is then a natural question to ask whether or not there is a procedure in the opposite direction, that is, a way to construct a unitary representations of the larger group G from unitary representation of the smaller subgroup H. It turns out that such as an induction is indeed possible and is constructed in a very geometrical manner. Explicitly, one has a natural G-action on the continuous sections of a homogeneous bundle $E \stackrel{d}{=} G \times_H \mathcal{H}_\sigma \to G/H$. Using an invariant measure on G/H, an inner product can be defined on the space of such sections which can be used the extend the aforementioned action to a unitary representation.

After a moment of consideration, one observes that the induced representation comes with additional structure. Indeed, compactly supported sections can be multiplied pointwise by elements in $C_0(G/H)$, thus obtaining a non-degenerate $*$-representation M of the C^*-algebra $C_0(G/H)$ on the same space $\mathcal{F}$. At the same time, we have a natural action of G on $C_0(G/H)$ induced by the action of G on G/H. Moreover, M is G-equivariant in the sense that $\pi(x)M(\phi)\pi(x)^{-1} = M(x \cdot \phi)$. such an equivariant pair (π, M) consisting of a unitary representation of G and a non-degenerate $*$-representation of $C_0(G/H)$ is called a *system of imprimitivity* for G based on G/H and defines a category, whose morphisms are isometric linear maps that intertwine both the representations π and M.

It was proven by Mackey [Mac49, Mac58] that the induction procedure actually defines an equivalence of categories. It follows that the induction process described above is really a two-fold process:

$$\begin{array}{c} \text{Unitary representations of } H \\ \big\downarrow \text{induction} \\ \text{Systems of imprimitivity for } G \text{ based on G/H} \\ \big\downarrow \text{restriction} \\ \text{Unitary representations of } G \end{array} \cdot$$

It therefore remains to determine which representations of G actually *lift* to a non-trivial system of imprimitivity based on some homogeneous space G/H for some closed subgroup H. For those that do are induced from a representation of H.

One example in which this happens for *all* representations of G is if it is of the particular form $G = N \rtimes H$ for some Abelian group N, where the action of H on N is required to be sufficiently 'nice'. Moreover, the closed subgroups from which the representations are induced are precisely the stabilizers G_ν of the action of G on the dual group $\widehat{N}$. This means that in fact the whole representation theory of G is determined by the representation theory of these stabilizers, which, as it turns out, can be further reduced to the subgroup $G_\nu \cap H$. This method of classifying the unitary representations of a group $N \rtimes H$ is called the *Mackey Machine*.

In the following, the assumption is made that G/H has a G-invariant Radon measure, which allows for significant simplifications. By corollary 146 and [Var07, p. 342, theorem 9.2], this assumption will suffice for the purposes of this report. However, it is not needed for the theory of induced representations, which extends more generally after suitable modifications. Once again we refer to [Fol95].

The exposition of this theory given below is based on [Fol95]. Some technical proofs are omitted for the sake of brevity, which the interested reader may find in [Fol95]. Throughout this chapter, the following notation is used. For any Lie group X, we denote by $\boldsymbol{UR}(X)$ the category of strongly continuous unitary representations of X. Let G be a fixed Lie group and let H be a closed subgroup and let $G \xrightarrow{q} G/H$ be the canonical projection. Finally, let $\sigma : H \to U(\mathcal{H}_\sigma)$ be a unitary representation of H on a Hilbert space $\mathcal{H}_\sigma$ and let μ be an invariant measure on G/H.

4.1 Construction

We start with the construction of induced representations, treating two perspectives each starting its construction on either side of the G-equivariant bijection $\Gamma(G \times_H \mathcal{H}_\sigma) \cong \mathrm{Hom}_H(G, \mathcal{H}_\sigma)$. The first one, called the *induced picture*, is based on H-equivariant maps. This perspective is more convenient for proofs and moreover extends more directly to infinite dimensional representations of H. The second construction considers instead sections of the associated bundle $G \times_H \mathcal{H}_\sigma$. It therefore has a clear geometric interpretation, thence its name *the geometric picture*. The two are equivalent if $\mathcal{H}_\sigma$ is finite dimensional. Seeing as the induced picture applies applies more generally, henceforth we will make use of the induced picture.

Induced picture

As mentioned, this perspective starts its construction from H-equivariant maps $G \to \mathcal{H}_\sigma$. First, let us define the space of continuous H-intertwining maps $G \to \mathcal{H}_\sigma$:

$$C_H(G, \mathcal{H}_\sigma) = \{ \ f \in C(G, \mathcal{H}_\sigma) \ : \ f(x\xi) = \sigma(\xi)^{-1} f(x) \ \}.$$

Next, define

$$\mathcal{F}_0 \stackrel{d}{=} \{ f \in C_H(G, \mathcal{H}_\sigma) \ : \ q(\operatorname{supp} f) \text{ is compact } \}. \tag{4.1}$$

Consider $\mathcal{F}_0$ as a representation of G under the left-regular action. We aim to define an invariant inner product this space such that the left regular action becomes isometric and extends to a unitary representation on the Hilbert space completion of $\mathcal{F}_0$. The following result, taken from [Fol95, p. 152], describes the functions in $\mathcal{F}_0$ in more detail.

Lemma 50. *For $\alpha \in C_c(G, \mathcal{H}_\sigma)$, define*

$$f_\alpha(x) = \int_H \sigma(\eta)\alpha(x\eta)d\eta.$$

Then $f_\alpha \in \mathcal{F}_0$ and f_α is uniformly continuous on G.
Moreover, every element of $\mathcal{F}_0$ is of this form for some $\alpha \in C_c(G, \mathcal{H}_\sigma)$.

Proof. We first show that every element of $\mathcal{F}_0$ is of this form. Let $f \in \mathcal{F}_0$. By lemma 143 there exists $\psi \in C_c(G)$ such that $A\psi \equiv 1$ on $\operatorname{supp} f$, where $A : C_c(G) \to C_c(G/H)$ is the averaging map equation (8.2). Let $\alpha = \psi \cdot f$. Then

$$f_\alpha = \int_H \psi(x\eta)\sigma(\eta)f(x\eta)d\eta = \int_H \psi(x\eta)f(x)d\eta = A\psi f(x) = f(x)$$

Next, let $\alpha \in C_c(G, \mathcal{H}_\sigma)$. Then $q(\operatorname{supp} f_\alpha) \subset q(\operatorname{supp} \alpha)$ is compact and f_α is H-equivariant by the translation invariance of the left-Haar measure. It remains to show f_α is uniformly continuous. Let N be a compact neighborhood of 1 in G and let $K \subset G$ be a compact lift of $\operatorname{supp} f$ (using lemma 142). Define

$J = K^{-1}N(\operatorname{supp}\alpha) \cap H$, which is compact in H. Since α is continuous, there exists for any $\epsilon > 0$ a neighborhood N_ϵ such that $\|\alpha(x) - \alpha(y)\|_\sigma \leq \epsilon$ whenever $xy^{-1} \in N_\epsilon$. The for $x \in K$ and $xy^{-1} \in N_\epsilon$, we have

$$\|f_\alpha(x) - f_\alpha(y)\| = \| \int_J \sigma(\eta)(\alpha(x\eta) - \alpha(y\eta))d\eta \leq \epsilon|J|.$$

This means that f_α is uniformly continuous on K and hence on KH by H-equivariance. Since $\operatorname{supp} f_\alpha = KH$, we are done. □

For any two $f, g \in \mathcal{F}_0$ the smooth map $x \mapsto \langle f(x), g(x)\rangle$ factors through G/H, since the action on $\mathcal{H}_\sigma$ is unitary and functions in $\mathcal{F}_0$ are H-equivariant. Thus, we can define

$$\langle f, g\rangle = \int_{G/H} \langle f(x), g(x)\rangle d\mu(xH) \tag{4.2}$$

This is an inner product on $\mathcal{F}_0$, and since μ is invariant it is preserved by left translations. Let $\mathcal{F}_\sigma$ be the Hilbert space completion of $\mathcal{F}_0$, that is,

$$\mathcal{F}_\sigma = \overline{\mathcal{F}_0}^{\langle .,.\rangle} \tag{4.3}$$

Lemma 51. *Let G and $\mathcal{F}_\sigma$ be as above. The left translation operators L_x extend to a strongly continuous unitary representation on $\mathcal{F}_\sigma$.*

Proof. Consider first fixed $x \in G$. The translation-invariance of μ implies immediately that $\|L_x f\| = \|f\|$ for all $f \in \mathcal{F}_0$. By a standard approximation argument, it follows that L_x extends to an isometric operator on $\mathcal{F}_\sigma$. Since L_x is invertible (with inverse $L_{x^{-1}}$), it is unitary.

It remains to show the strong continuity. Now, any $f \in \mathcal{F}_0$ is uniformly continuous so for any $\epsilon > 0$ we can find some open neighborhood N_ϵ of 1 such that $xy^{-1} \in N_\epsilon \implies \|f(x) - f(y)\|_\sigma < \epsilon$. By replacing N_ϵ with $N_\epsilon \cap N_\epsilon^{-1}$, we may assume that N_ϵ is symmetric (by the continuity of $x \mapsto x^{-1}$, N_ϵ^{-1} is open). For any $g, x, y \in G$, we have $(x^{-1}g)(y^{-1}g)^{-1} \in N_\epsilon \iff x^{-1}y \in N_\epsilon \iff xy^{-1} \in N_\epsilon$. Therefore, $xy^{-1} \in N_\epsilon \implies \|L_x f(g) - L_y f(g)\|_\sigma = \|f(x^{-1}g) - f(y^{-1}g)\|_\sigma < \epsilon$ and hence $\|L_x f - L_y f\| < \epsilon \cdot |q(\operatorname{supp} f)|$. This proves the asserted continuity for any $f \in \mathcal{F}_0$. As the operators L_x are all bounded and strongly continuous on $\mathcal{F}_0$, the map is strongly continuous on all of $\mathcal{F}$ by a standard 3-ϵ argument and thus define a unitary representation of G on $\mathcal{F}_\sigma$.

□

Definition 52. The representation of G on $\mathcal{F}_\sigma$ obtained in the previous lemma is called the **induced representation**, and is denoted $\operatorname{ind}_H^G(\sigma)$.

The Hilbert space $\mathcal{F}$ can actually be identified as the space $L^2_H(G, \mathcal{H}_\sigma; \mu)$ of $\mathcal{H}_\sigma$-valued functions that are measurable, square integrable and that satisfy $f(x\xi) = \sigma(\xi^{-1})f(x)$ μ-almost everywhere[Bla61]. This result will be of no consequence to us, so it is not discussed in detail.

Geometric picture

In the geometric picture, we assume that the H-representation $\mathcal{H}_\sigma$ is *finite* dimensional, and start instead from sections of the associated bundle $G \times_H \mathcal{H}_\sigma$. Other than that, we employ the same strategy.

Since H is a closed subgroup of the Lie group G, we know from corollary 18 that $G \xrightarrow{q} G/H$ is a principal H-bundle. We can therefore construct the homogeneous Hilbert bundle $G \times_H \mathcal{H}_\sigma \to G/H$. Write $E \stackrel{d}{=} G \times_H \mathcal{H}_\sigma$ and $E \xrightarrow{\pi_E} G/H$ for the corresponding surjective submersion. The homogeneity implies that G acts on the sections $\Gamma(E)$ of this bundle via

$$(g \cdot s)(xH) \stackrel{d}{=} g \cdot s(g^{-1}xH).$$

Observe that $g \cdot s$ is indeed a section. Now, we can use the invariant measure on G/H to define an inner product on $\Gamma(E)$:

$$\langle s_1, s_2\rangle \stackrel{d}{=} \int_{G/H} \langle s_1(xH), s_2(xH)\rangle_{E_{xH}} d\mu(xH) \tag{4.4}$$

let $\mathcal{G}_\sigma$ be the Hilbert space completion of compactly supported smooth sections with respect to this inner product $\mathcal{G}_\sigma \stackrel{d}{=} \overline{\Gamma_c(E)}^{\langle\cdot,\cdot\rangle}$. The following computation shows that this inner product is preserved by the action of G:

$$\begin{aligned}\langle g\cdot s_1, g\cdot s_2\rangle &= \int_{G/H} \langle g\cdot s_1(g^{-1}xH), g\cdot s_2(g^{-1}xH)\rangle_{E_{xH}} d\mu(xH)\\ &\overset{(1)}{=} \int_{G/H} \langle g\cdot s_1(xH), g\cdot s_2(xH)\rangle_{E_{gxH}} d\mu(xH)\\ &\overset{(2)}{=} \int_{G/H} \langle s_1(xH), s_2(xH)\rangle_{E_{xH}} d\mu(xH)\\ &= \langle s_1, s_2\rangle.\end{aligned}$$

Here, (1) follows by the invariance of the measure μ and (2) by the homogeneity of the Hilbert bundle.

Define

$$\mathcal{F}_0^\infty \overset{d}{=} \{f \in \mathrm{Hom}_H(G, \mathcal{H}_\sigma) \,:\, q(\operatorname{supp} f) \text{ is compact }\}.$$

Remark.

— Notice that by lemma 28, $\mathcal{F}_0^\infty$ is precisely the image of $\Gamma_c(E)$ under the map $\Phi : \Gamma(E) \to \mathrm{Hom}_H(G, \mathcal{H}_\sigma)$ defined in lemma 28.

Lemma 53. *The space $\Gamma_c(E)$ of compactly supported smooth sections is dense in the space of compactly supported continuous sections of $E \to G/H$, where both spaces are endowed with the topology defined by* (4.4)*.*

Proof. It is well-known that for an arbitrary open subset $\Omega \subseteq \mathbb{R}^n$ and finite-dimensional Hilbert space $\mathcal{H}_\sigma$, the space $C_c^\infty(\Omega, \mathcal{H}_\sigma)$ is dense in $C_c(\Omega, \mathcal{H}_\sigma)$ with respect to the sup-norm. Then for any Radon measure ν on Ω we find that $C_c^\infty(\Omega, \mathcal{H}_\sigma)$ is also dense in $C_c(\Omega, \mathcal{H}_\sigma)$ with respect to the norm inherited from $L^2(\Omega; \mathcal{H}_\sigma, \nu)$. Indeed, for $f \in C_c(\Omega, \mathcal{H}_\sigma)$ we can a compact set $K \subset U$ such that $\operatorname{supp} f$ is properly contained in K and a sequence $f_n \in C_c^\infty(\Omega, \mathcal{H}_\sigma)$ such that $\|f_n - f\|_\infty \to 0$ and $\operatorname{supp} f_n \subset K$. Then $\|f_n - f\|_{L^2} \le \nu(K)\|f_n - f\|_\infty \to 0$.

Now, using a partition of unity we can reduce to the case above. Explicitly, let $s : G/H \to E$ be an arbitrary compactly supported continuous section of the bundle $E \to G/H$. Since G/H is a smooth manifold and s is compactly supported, we can find a *finite* open covering $\{U_\alpha\}_\alpha$ of $\operatorname{supp} s$ and fiber-wise unitary local trivializations $\pi_E^{-1}(U_\alpha) \xrightarrow{\phi_\alpha} U_\alpha \times \mathcal{H}_\sigma$ such that each U_α is diffeomorphic to an open subset of $\mathbb{R}^n$. Let $\{\psi_\alpha\}$ be a smooth partition of unity subordinate to the covering $\{U_\alpha\}$. Let $s_\alpha = \psi_\alpha \cdot s$ so that $s = \sum_\alpha s_\alpha$ and $\operatorname{supp} s_\alpha \subset U_\alpha$. By the local triviality, there exists $f_\alpha \in C_c(U_\alpha, \mathcal{H}_\sigma)$ such that $\phi \circ s_\alpha = (\mathrm{id}_{U_\alpha}, f_\alpha)$. Then by the above argument, for fixed α we can find a sequence $f_\alpha^{(n_\alpha)} \in C_c^\infty(U_\alpha, \mathcal{H}_\sigma)$ such that $f_\alpha^{(n_\alpha)} \to f_\alpha$ in $L^2(U_\alpha; \mathcal{H}_\sigma, \mu)$ as $n_\alpha \to \infty$. Since ϕ is fiber-wise unitary this implies that $s_\alpha^{(n_\alpha)} \to s_n$ in $\mathcal{G}_\sigma$, where $s_\alpha^{(n_\alpha)} = \phi^{-1} \circ (\mathrm{id}_{U_\alpha}, f_\alpha^{(n_\alpha)})$. Because the covering $\{U_\alpha\}$ is finite, we find

$$\sum_\alpha s_\alpha^{(n_\alpha)} \to \sum_\alpha s_\alpha = s \text{ in } \mathcal{G}_\sigma.$$

□

Corollary 54. *Let $\Phi : \Gamma(E) \to Hom_H(G, \mathcal{H}_\sigma)$ be the G-equivariant bijection described in lemma 28. Then $\Phi|_{\Gamma_c(E)}$ extends to a unitary G-equivariant map $\Phi : \mathcal{G}_\sigma \to \mathcal{F}_\sigma$.*

Proof. Notice first that Φ is isometric if $\mathrm{Hom}_H(G, \mathcal{H}_\sigma)$ is endowed with the norm defined by equation (4.2). Indeed, we compute

$$\begin{aligned}\|\Phi(s)\|_{\mathcal{F}_\sigma}^2 &= \int_{G/H} \|\phi(x^{-1}\cdot s(xH))\|_\sigma^2 d\mu(xH)\\ &= \int_{G/H} \|x^{-1}\cdot s(xH))\|_{E_H}^2 d\mu(xH)\\ &= \int_{G/H} \|s(xH))\|_{E_{xH}}^2 d\mu(xH)\\ &= \|s\|_{\mathcal{G}_\sigma}^2.\end{aligned}$$

By the previous lemma Φ extends to a unique linear map Ψ from the space of continuous compactly supported sections of $E \to G/H$ into the closure of $\mathrm{Hom}_H(G, \mathcal{H}_\sigma)$ with respect to the norm (4.2). This extension is given by the map that is defined precisely as in lemma 28, but for continuous instead of smooth maps. Using the last statement of lemma 28, it is clear that the image of this map is precisely $\mathcal{F}_0$. Extending further to $\mathcal{G}_\sigma$ gives the result. □

Corollary 55. *If $\mathcal{H}_\sigma$ is finite-dimensional, the unitary representations defined via the geometric and induced picture are equivalent.*

Corollary 56. *Let G and E be as above. The G-action on $\Gamma_c(E)$ extends to a strongly continuous unitary representation of G on $\mathcal{G}_\sigma$.*

4.2 Properties

Next, let us consider some properties of the induction procedure. The first important observation is that it is actually functorial:

Lemma 57. *The assignment $\sigma \mapsto ind_H^G(\sigma)$ defines a functor $ind_H^G : \boldsymbol{UR}(H) \to \boldsymbol{UR}(G)$.*

Proof. Suppose that σ_1 and σ_2 are two unitary representations of H. Write F_{σ_1} and F_{σ_2} for the Hilbert spaces on which $\mathrm{ind}_H^G(\sigma_k)$ act $(k = 1, 2)$ and write $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ for the corresponding dense subspaces as in (4.1). Then any isometric intertwining map $T \in \mathrm{Hom}_H(\sigma_1, \sigma_2)$ induces an isometric map $\mathcal{F}_0^1 \xrightarrow{\widetilde{T}} \mathcal{F}_0^2$ defined by $f \mapsto T \circ f$. This map trivially intertwines the left-regular action and it is isometric with respect to the inner products defined by (4.2) because T is isometric. It therefore extends to an isometric map $\mathcal{F}_{\sigma_1} \xrightarrow{\widetilde{T}} \mathcal{F}_{\sigma_2}$. It is clear that the assignment $T \mapsto \widetilde{T}$ is functorial. □

Lemma 58. *Let σ be a unitary representation of H and $\pi = ind_H^G(\sigma)$.*
Then $\pi(n)f(x) = \sigma(x^{-1}nx)f(x)$.

Proof. Consider the induced picture. We compute $\pi(n)f(x) = f(n^{-1}x) = f(x(x^{-1}nx)) = \sigma(x^{-1}n^{-1}x)f(x)$. □

Lemma 59.

1. *$\sigma_1 \cong \sigma_2$ in $\boldsymbol{UR}(H) \implies ind_H^G(\sigma_1) \cong ind_H^G(\sigma_1)$ in $\boldsymbol{UR}(G)$*
2. *If $\{\sigma_i\}_i$ are unitary representations of H, then $ind_H^G(\bigoplus_i \sigma_i) \cong \bigoplus_i(ind_H^G(\sigma_i))$ in $\boldsymbol{UR}(G)$.*

Proof.

1. We have already seen that $\mathrm{ind}_H^G : \boldsymbol{UR}(H) \to \boldsymbol{UR}(G)$ is a functor and every functor preserves isomorphisms.
2. This follows from the observation that the map $\mathrm{Hom}_H(G, \bigoplus_i \mathcal{H}_{\sigma_i}) \to \bigoplus_i \mathrm{Hom}_H(G, \mathcal{H}_{\sigma_i}), f \mapsto (\mathrm{Pr}_i \circ f)_i$ is a G-equivariant isomorphism.

□

The next theorem shows that if G is compact, then ind_H^G is right-adjoint to the restriction functor.

Theorem 60 (The Frobenius Reciprocity Theorem)**.**
Let G be a compact Lie group and H a closed subgroup, π an irreducible unitary representation of G and σ an irreducible unitary representation of H. Then

$$Hom_G(\pi, ind_H^G(\sigma)) \cong Hom_H(\pi|_H, \sigma)$$

Proof. A sketch of the proof is given. It is well-known that every irreducible representation of a compact Lie group is finite-dimensional [Fol95, p. 126, Theorem 5.2]. As a consequence of this fact and theorem 145, the Hilbert space $\mathcal{F}$ on which $\mathrm{ind}_H^G(\sigma)$ acts can be identified with the subspace $L^2_H(G, \mathcal{H}_\sigma)$ of $L^2(G, \mathcal{H}_\sigma)$ consisting of those square-integrable functions f that satisfy $f(x\xi) = \sigma(\xi^{-1})f(x)$ for all $x \in G$, $\xi \in H$.

Define a map

$$\phi : \mathcal{H}_\pi \to \mathrm{Hom}_H(G, \mathcal{H}_\pi)$$
$$\phi_v(x) = \pi(x)^{-1}v$$

This map is G-equivariant, since

$$(g \cdot \phi_v)(x) = \pi(g^{-1}x)^{-1}v = \pi(x)^{-1}\pi(g)v = \phi_{\pi(g)v}(x).$$

Now, we have seen that any isometric $T \in \operatorname{Hom}_H(\pi|_H, \sigma)$ induces a G-equivariant map

$$\operatorname{Hom}_H(G, \mathcal{H}_\pi) \xrightarrow{T\circ -} \operatorname{Hom}_H(G, \mathcal{H}_\sigma).$$

Define for each isometric $T \in \operatorname{Hom}_H(\pi|, \mathcal{H}_\sigma)$ the map A_T as the following G-equivariant composition:

$$A_T : \mathcal{H}_\pi \xrightarrow{\phi} \operatorname{Hom}_H(G, \mathcal{H}_\pi) \xrightarrow{T\circ -} \operatorname{Hom}_H(G, \mathcal{H}_\sigma).$$

That is,

$$A_T(v)(xH) = T\pi(x)^{-1}v.$$

By the G-equivariance, it is clear that $A_T \in \operatorname{Hom}_G(\pi, \operatorname{ind}_H^G(\sigma))$. Moreover, if $A_{T_1} = A_{T_2}$, then in particular $A_{T_1}(v)(e) = A_{T_2}(v)(e)$ for all $v \in \mathcal{H}_\pi$, which just states that $T_1 v = T_2 v$ and thus $T_1 = T_2$. The assignment $T \mapsto A_T$ is therefore injective. It remains to show it is surjective. Let $A \in \operatorname{Hom}_G(\pi, \operatorname{ind}_H^G(\sigma))$. Suppose for a moment that every function in the range of A can be evaluated pointwise and let ev_1 be evaluation at the identity. Define $T = \operatorname{ev}_1 \circ A$:

$$T : \mathcal{H}_\pi \xrightarrow{A} \overline{\operatorname{Hom}_H(G, \mathcal{H}_\sigma)} \xrightarrow{\operatorname{ev}_1} \mathcal{H}_\sigma.$$

Notice that ev_1 is H equivariant, since

$$\operatorname{ev}_1(L_x f) = f(h^{-1}) = \sigma(h)f(1) = \sigma(h)\operatorname{ev}_1 f.$$

As A is G-equivariant by definition, it follows that $T \in \operatorname{Hom}_H(\pi|_H, \mathcal{H}_\sigma)$. Finally, $A = A_T$ since

$$(A_T v)(x) = (\operatorname{ev}_1 \circ A)(\pi(x)^{-1}v) = (\operatorname{ev}_1 \circ L_x^{-1} A)v = (Av)(x).$$

It therefore remains to prove that functions in the range of A can be evaluated pointwise. This is a consequence of Peter-Weyl theorem, but we refer to [Fol95, p. 133, 160] for the details.

□

Theorem 61 (Induction in stages)**.**
Let G be a Lie group. Suppose H is a closed subgroup of G and K is a closed subgroup of H. Let σ be a unitary representation of K. Then $ind_K^G(\sigma) \cong ind_H^G(ind_K^H(\sigma))$ in $\boldsymbol{UR}(G)$.

Proof. The proof is due to Mackey and can be found in e.g. [Fol95, p. 166].

□

4.3 Systems of imprimitivity

In this section, the so-called *systems of imprimitivity* are introduced. To motivate their study and definition, we first consider two different situations in which these objects naturally occur.

1. Firstly, let H be a closed subgroup of G and let σ be a finite dimensional unitary representation of H on $\mathcal{H}_\sigma$. Let $E = G \times_H \mathcal{H}_\sigma$ and $\mathcal{G} = \overline{\Gamma_c(E)}$ be the Hilbert space on which G acts according to the geometrical picture. Notice that $\Gamma_c(E)$ is closed under multiplications by elements in $C_0(G/H)$. Moreover, the latter space has a natural G-action induced by the left action of G on G/H. Write $\pi = \operatorname{ind}_H^G(\sigma)$. The following lemma shows that we obtain a non-degenerate $*$-representation of $C_0(G/H)$ on $\mathcal{G}$ that is compatible with the various actions of G.

   Lemma 62. *The linear operators $M(\phi)s \stackrel{d}{=} \phi \cdot s$ defined on $\Gamma_c(E)$ extend to a linear operator on $\mathcal{G}$ for any fixed $\phi \in C_0(G/H)$. This defines a non-degenerate $*$-representation of $C_0(G/H)$ on $\mathcal{G}$ that is G-equivariant in the following sense:*

   $$\pi(x)M(\phi)\pi(x)^{-1} = M(L_x\phi), \tag{4.5}$$

   where $L_x\phi(gH) = \phi(x^{-1}gH)$.

Proof. Notice first that for fixed $\phi \in C_0(G/H)$ we have $\|M(\phi)s\| \le \|\phi\|_\infty \|s\|$ for all $s \in \Gamma_c(E)$, since

$$\|M(\phi)s\|^2 = \int_{G/H} \|\phi(xH) \cdot s(xH)\|^2_{E_{xH}} d\mu(xH) \le \|\phi\|^2_\infty \|s\|^2.$$

This means that M extends to a bounded linear operator on $\mathcal{G}$. Next, we have $M(\overline{\phi}) = M(\phi)^*$, because

$$\langle \phi(xH) \cdot s_1(xH), s_2(xH) \rangle_{E_{xH}} = \langle s_1(xH), \overline{\phi(xH)} \cdot s_2(xH) \rangle_{E_{xH}}$$

for every $xH \in G/H$ and therefore $\langle M(\phi)s_1, s_2 \rangle = \langle s_1, M(\overline{\phi})s_2 \rangle$. It is clear that M also respects multiplication. We show M is non-degenerate. Suppose $s \in \mathcal{G}$ is such that $M(\phi)s = 0$ for every $\phi \in C_0(G/H)$. Then $\phi(xH)s(xH) = 0$ for every $xH \in G/H$ and $\phi \in C_0(G/H)$. This implies $s = 0$. Finally, we compute

$$(\pi(x)M(\phi)\pi(x)^{-1}s)(gH) = (M(\phi)\pi(x)^{-1}s)(x^{-1}gH) = \phi(x^{-1}gH) \cdot s(gH) = M(L_x\phi)s(gH).$$

□

Now, a completely similar construction also works if $\mathcal{H}_\sigma$ is infinite dimensional via the induced picture. In this case, the $*$-representation $C_0(G/H) \xrightarrow{M} \mathcal{L}(\mathcal{F}_\sigma)$ is given by $M(\phi)f = (\phi \circ q) \cdot f$ for $\phi \in C_0(G/H)$, where $G \xrightarrow{q} G/H$ is the quotient map. In this case, M satisfies the same equivariance condition (4.5).

2. Secondly, suppose that $G = N \rtimes H$, where N is a non-trivial closed Abelian normal subgroup of the Lie group G. Let π be a unitary representation of G on $\mathcal{H}_\pi$. Then π restricts to a unitary representation of N. Since N is Abelian, we know from theorem 152 that there exists a unique $\mathcal{H}_\pi$-projection-valued measure P on $\widehat{N}$ such that

$$\pi(n) = \int_{\widehat{N}} \langle n, \nu \rangle dP(\nu).$$

G acts on N by conjugation which induces a left action on $\widehat{N}$ defined by

$$\langle n, x \cdot \nu \rangle \overset{d}{=} \langle x^{-1} \cdot n, \nu \rangle = \langle x^{-1}nx, \nu \rangle.$$

We know that π is a representation of $G = N \rtimes H$ and so its restriction to N must be compatible with the action of G on N, which imposes a compatibility condition on the spectral measure P. Indeed, for every $x \in G$, $n \in N$ we have:

$$\begin{aligned}\pi(xnx^{-1}) = \int_{\widehat{N}} \langle xnx^{-1}, \nu \rangle dP(\nu) = \int_{\widehat{N}} \langle n, x^{-1}\nu \rangle dP(\nu) = \int_{\widehat{N}} \langle n, \nu \rangle dP(x \cdot \nu)\\ = \pi(x)\pi(n)\pi(x^{-1}) = \int_{\widehat{N}} \langle n, \nu \rangle \pi(x) dP(\nu) \pi(x)^{-1},\end{aligned}$$

from which it follows that $\pi(x)dP(\nu)\pi(x)^{-1} = dP(x \cdot \nu)$ is satisfied.

Motivated by these two examples, we make the following more general definition

Definition 63. A **system of imprimitivity** for G based on S is an ordered triple $\Sigma = (\pi, S, P)$ consisting of

1. a unitary representation π on $\mathcal{H}_\pi$
2. a topological space S equipped with a continuous left G-action.
3. a non-degenerate $*$-representation M of $C_0(S)$ on $\mathcal{H}_\pi$ that satisfies

$$\pi(x)M(\phi)\pi(x)^{-1} = M(L_x\phi).$$

Notice that instead of specifying the non-degenerate $*$-representation, on may equivalently specify a regular $\mathcal{H}_\pi$-projection-valued measure P on S that satisfies

$$\pi(x)P(E)\pi(x)^{-1} = P(xE) \qquad x \in G, E \subset S \text{ measurable }.$$

The two formulations are related by the fact that any non-degenerate $*$-representation M of $C_0(S)$ is given by $M(\phi) = \int_S \phi dP$ for some uniquely determined projection-valued measure P, see also corollary 153.

Definition 64.

- We call a representation π **imprimitive** if it belongs to a non-trivial system of imprimitivity (π, S, M) meaning that S is not a single point. Otherwise, it is called **primitive.**
- A system of imprimitivity (π, S, M) is called **transitive** if S is a homogeneous space, i.e., if $S = G/H$ for some closed subgroup H of G.
- A map $T : \mathcal{H}_{\pi_1} \to \mathcal{H}_{\pi_2}$ is said to **intertwine** two systems of imprimitivity $\Sigma_1 = (\pi_1, S, M_1)$ and $\Sigma_2 = (\pi_2, S, M_2)$ if it intertwines both the unitary representations π_1 and π_2 and the *-representations M_1 and M_2. Denote by $\mathrm{Hom}_G(\Sigma_1, \Sigma_2)$ the set of intertwining maps $\Sigma_1 \to \Sigma_2$. Similarly, we write $\mathrm{Hom}_G(\Sigma, \Sigma)$ by $\mathrm{Hom}_G(\Sigma)$.

Definition 65. The transitive system of imprimitivity obtained by the inducing construction that is described in lemma 62 is called the **canonical system of imprimitivity** associated to $\mathrm{ind}_H^G(\sigma)$. The corresponding spectral measure on G/H is $P(E)s = \chi_E s$ for $E \subset G/H$ Borel-measurable. In the induced picture we have instead $M(\phi)f = (\phi \circ q)f$ with spectral measure $P(E)f = (\chi_E \circ q)f$.

Lemma 66. *Let G be a Lie group with closed subgroup H. Let σ_1 and σ_2 be two representations of H and let $\Sigma_k = (ind_H^G(\sigma_k), G/H, M_k)$ be the canonical system of imprimitivity associated to $ind_H^G(\sigma_k)$ for $k = 1, 2$. If $T \in Hom_H(\sigma_1, \sigma_2)$, then $ind_H^G(T) \in Hom_G(\Sigma_1, \Sigma_2)$.*

Proof. We have already seen in lemma 57 that $\mathrm{ind}_H^G(T) \in \mathrm{Hom}_G(\mathrm{ind}_H^G(\sigma_1), \mathrm{ind}_H^G(\sigma_2))$, so it remains to show it intertwines M_1 and M_2. It suffices to consider the dense subspaces $\mathcal{F}_0^1$ and $\mathcal{F}_0^2$ defined by (4.1). Recall that $\widetilde{T} : \mathcal{F}_0^1 \to \mathcal{F}_0^2$ is simply given by $T \circ -$. Using the fact that T is linear, we have for any $x \in G$, $\phi \in C_0(G/H)$ and $f \in \mathcal{F}_0^1$:

$$(\widetilde{T} M_1(\phi) f)(x) = T(\phi \circ q)(x) \cdot f(x) = (\phi \circ q)(x) \cdot (T \circ f)(x) = (M_2(\phi)\widetilde{T} f)(x).$$

□

Now, we have a defined category with $\boldsymbol{TSOI}_H^G$ with transitive systems of imprimitivity $(\pi, G/H, M)$ as objects and intertwining maps as morphisms. In the preceding sections, a functor $\boldsymbol{UR}(H) \xrightarrow{\mathrm{ind}_H^G} \boldsymbol{TSOI}_H^G$ was constructed, which sends a unitary representation σ of H to the canonical system of imprimitivity Σ associated to $\mathrm{ind}_H^G(\sigma)$ and intertwining maps $T \in \mathrm{Hom}_H(\sigma_1, \sigma_2)$ to $\widetilde{T} \in \mathrm{Hom}_G(\Sigma_1, \Sigma_2)$.

The following two theorems state that this functor is an equivalence of categories, which further clarifies the importance of the notion of a system of imprimitivity:

$$\text{Unitary representations of H} \cong \text{Systems of imprimitivity for } G \text{ based on } G/H.$$

This result is really the crux of the matter. Nonetheless, for brevity and seeing as we will only need the result and not so much the methods used to prove them, we give these results without proof. Still, the proofs are interesting in their own right. The results are due to Mackey and can be found e.g. in [KT12, p. 125] and [Fol95, p. 178].

Theorem 67.
Suppose H is a closed subgroup of G and that σ_1, σ_2 are unitary representations of H.
Let $\Pi_k = ind_H^G(\sigma_k)$ and $\Sigma_k = (\Pi_k, G/H, M)$ be the associated system of imprimitivity, where $k = 1, 2$.
Then the map $T \mapsto \widetilde{T}$ is a bijection from $Hom_H(\sigma_1, \sigma_2)$ to $Hom_G(\Sigma_1, \Sigma_2)$.

Theorem 68 (The Imprimitivity Theorem)**.**
Let $\Sigma = (\pi, G/H, M)$ be a transitive system of imprimitivity on G.
There is a unitary representation σ of H such that Σ is equivalent to the system of imprimitivity associated to $ind_H^G(\sigma)$. Moreover, σ is uniquely determined up to equivalence.

It follows that the induction process described above is really a two-fold process:

$$\begin{array}{c}\text{Unitary representations of } H \\ \big\downarrow{\scriptstyle \text{induction}} \\ \text{Systems of imprimitivity for } G \text{ based on G/H } \cdot \\ \big\downarrow{\scriptstyle \text{restriction}} \\ \text{Unitary representations of } G\end{array}$$

Therefore, if a representation of G lifts to a non-trivial system of imprimitivity for G based on some non-trivial homogeneous space G/H, then it is the induction of a unitary representation of H. The question remains for which representations of G such a lift is possible.

4.4 Mackey machine

The previous section has resulted in a sufficient condition for a representation of G to be completely determined by its restriction to a non-trivial closed subgroup H. Namely, the representation should lift to a non-trivial transitive system of imprimitivity for G. It turns out that for Lie groups of the special form $G = N \rtimes H$ for some Abelian N, the situation is particularly nice and such a lift is always possible. The obtained method of studying the representation theory of groups of this form is called the *Mackey machine*, and can be used to obtain a full classification of the representation theory of the group in terms of induced representations. The aim of this section is three-fold:

1. Every irreducible unitary representation π of $G = N \rtimes H$ is part of a transitive system of imprimitivity for G based on some homogeneous space G/G_ν, where $\nu \in \widehat{N}$ and G_ν is its stabilizer. This means that it is induced by a representation σ of G_ν.
2. The restriction of the representation σ to N acts on $\mathcal{H}_\sigma$ simply according to the scalar action of the character ν and is unique up to equivalence.
3. The representations of G_ν that restrict to the scalar action of ν on N are up to equivalence uniquely characterized by the representations of the so-called *Little group* $H_\nu = G_\nu \cap H$.

It follows in particular that the representation theory of G is fully determined by the representation theory of the various stabilizers G_ν, by inducing them up to G. The Mackey machine is the main tool used to classify the irreducible representations of the group $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K$ in terms of the representation theory of various closed subgroups of this group.

The proofs given here are based on [Fol95, p. 182-187].

0. Regularity of the group action on $\widehat{N}$.

Now, the Mackey machine applies only if the action of G on $\widehat{N}$ satisfies a certain regularity condition. Before discussing the three points above, let us first say a few words regarding this regularity. The relevant notion is defined below.

Definition 69. Suppose G is a locally compact group acting on a locally compact space M. We say that the orbit space is **countably separated** if there is a countable family $\{E_j\}$ of G-invariant Borel sets in M such that each orbit in M is the intersection of all the E_j's that contain it.

The following lemma establishes an equivalent formulation and was proven by Glimm [Gli61, p. 124].

Lemma 70. *Suppose that a locally compact group G acts on a locally compact topological space M and both G and M are Hausdorff and second countable. Then the following are equivalent*

1. *For every $m \in M$, the map $G/G_m \to \mathcal{O}_m$ is a homeomorphism, where $\mathcal{O}_m = G \cdot m$ is given the subspace topology as a subspace of M.*
2. *The orbit space is countably separated.*

Proof. For the proof, we refer to [Gli61, Theorem 1]. □

Notice that $G = N \rtimes H$ is a Lie group (by assumption), which is in particular Hausdorff and second countable. The following two lemmas show sufficient conditions for the orbit space to be countably separated that will be enough for the purposes of this thesis.

Lemma 71. *Suppose G is a linear algebraic group defined over $\mathbb{R}$ and let V be an affine algebraic variety defined over $\mathbb{R}$. Let further $\pi : G \times V \to V$ be a real algebraic action defined by restriction to the real points of an algebraic action over $\mathbb{R}$. Then each orbit of π is locally closed and is an embedded submanifold of V. In particular, the orbit space is countably separated.*

Proof. See [FFB+98, p. 72, Corollary 4.9.3]. □

Remark.

— Consider the setting of lemma 71 and assume further that V is equipped with a non-degenerate G-invariant bilinear form β. Consider the action of G on $\widehat{V}$ determined by $\langle v, g \cdot \xi \rangle = \langle \pi(x^{-1})v, \xi \rangle$ and identify $\widehat{V} \cong V$ via
$$\langle v, w \rangle = e^{i\beta(v,w)}, \qquad v, w \in V.$$
Under this identification, the G-invariance of β implies that the action of G on $\widehat{V}$ transfers simply to the action π on V. The previous lemma now implies that the orbit space of $\widehat{V}$ is countably separated.

1. Every irreducible unitary representation is part of a transitive system of imprimitivity.

Assume that G contains a closed Abelian normal Lie subgroup N. Then G acts on N by conjugation, which induces an action of G on its dual group $\widehat{N}$ given by

$$\langle n, x \cdot \nu \rangle \stackrel{d}{=} \langle x^{-1}, nx, \nu \rangle, \qquad x \in G, n \in N, \nu \in \widehat{N}.$$

Denote for each $\nu \in \widehat{N}$ the stabilizer of ν by G_ν and the orbit by $\mathcal{O}_\nu$.

Suppose that π is a unitary representation of G. Notice that since N is Abelian, proposition 151 implies that we may identify $\widehat{N}$ with the spectrum $\Omega(L^1(G))$ of $L^1(G)$. This is the main reason for transferring the action of G on N over to $\widehat{N}$. Moreover, from theorem 152 we know that there is a projection-valued measure P on $\widehat{N}$ such that
$$\pi(n) = \int \langle n, \nu \rangle dP(\nu) \quad n \in N. \tag{4.6}$$
We have already seen that $(\pi, \widehat{N}, P)$ is a system of imprimitivity, so remains to show it is in fact a transitive system of imprimitivity. The strategy will be to show that P is supported on a single orbit, which by the previous lemma is a homogeneous space $\mathcal{O} \cong G/G_\nu$ for some $\nu \in \mathcal{O}$. Then $(\pi, G/G_\nu, P)$ is a **transitive** system of imprimitivity.

First, observe that if π is irreducible, then P is ergodic in the sense of the lemma below.

Lemma 72. *Let π be an irreducible representation of G. If $E \subset \widehat{N}$ is a G-invariant Borel-set, then either $P(E) = 0$ or $P(E) = I$.*

Proof. Since $(\pi, \widehat{N}, P)$ is a system of imprimitivity and E is G-invariant, we have

$$\pi(x)P(E)\pi(x)^{-1} = P(x \cdot E) = P(E).$$

This means that $P(E) \in \mathrm{Hom}_G(\pi)$. By Schur's lemma, it follows that $P(E)$ is a multiple of the identity, which implies the result since $P(E)$ is a projection. □

Proposition 73. *Suppose that the orbit space of $\widehat{N}$ under the action of G is countably separated. If π is irreducible, then there is an orbit $\mathcal{O}$ in $\widehat{N}$ such that $P(\mathcal{O}) = I$.*

Proof. Let $\{E_j\}_{j \in \mathbb{N}}$ be a countable family of G-invariant Borel-measurable sets, so that for every orbit $\mathcal{O} \subset \widehat{N}$, there is some $J \subset \mathbb{N}$ such that $\mathcal{O} = \bigcap_{j \in J} E_j$. In particular, $\mathcal{O}$ is measurable and $P(\mathcal{O})$ is a projection onto the intersection of all the ranges $P(E_j), j \in J$. Notice that $P(E_j) = 0$ or $P(E_j) = I$ for every j. If $P(E_j) = 0$ for some $j \in J$, then $P(\mathcal{O}) = 0$. Therefore, for every orbit $\mathcal{O}$ we either have $P(\mathcal{O}) = 0$ or $P(\mathcal{O}) = I$. If $P(\mathcal{O}) = 0$ for every orbit, this means that for every orbit $\mathcal{O}$ there exists some $E_{j_\mathcal{O}}$ containing $\mathcal{O}$ on which P is zero. Since $\widehat{N} = \bigcup_{\mathcal{O} \in G \backslash \widehat{N}} E_{j_\mathcal{O}}$, this implies $P(\widehat{N}) = 0$, contradiction the fact the $P(\widehat{N}) = I$. □

Notice that $P(\widehat{N}/\mathcal{O}) = 0$ since $I = P(\mathcal{O}) + P(\widehat{N}/\mathcal{O})$. Therefore, the orbit in the above proposition is also unique. We have completed the first goal; that every irreducible unitary representation of $G = N \rtimes H$ lifts to a transitive system of imprimitivity for G based on some homogeneous space G/G_ν. By the imprimitivity theorem 68, $(\pi, G/G_\nu, P)$ is unitarily equivalent to the canonical system of imprimitivity associated to $\mathrm{ind}_H^G(\sigma)$ for some unitary representation σ of H.

2. σ is unique up to equivalence and its restriction to N acts according to ν.

Lemma 74. *Let $\pi = ind_H^G(\sigma)$ be an induced representation and $(\pi, G/G_\nu, P)$ be the canonical system of imprimitivity associated to π. Then $\pi(n)f(x) = \langle n, x\cdot\nu\rangle f(x)$.*

Proof. We consider the induced picture. By equation (4.6) we have

$$\pi(n) = \int_{G/G_\nu} \langle n, x\cdot\nu\rangle dP(xG_\nu) = M(\phi),$$

where $\phi \in C_b(G/G_\nu)$ is given by $\phi : xG_\nu \mapsto \langle n, x\nu\rangle$. This follows by an application of the Borel-functional calculus, which holds for bounded measurable functions, so in particular for ϕ. Then

$$(\pi(n)f)(x) = (M(\phi)f)(x) = \langle n, x\nu\rangle f(x) = \langle x^{-1}nx, \nu\rangle f(x) = \langle n, x\cdot\nu\rangle f(x).$$

□

Lemma 75. *Let G be a Lie group with closed subgroup H. Suppose that σ is a unitary representation of H on some Hilbert space $\mathcal{H}_\sigma$. Let $\mathcal{F}_0$ be defined by equation (4.3). The set $\{f(1) : f \in \mathcal{F}_0\}$ is dense in $\mathcal{H}_\sigma$.*

Proof. The following proof is based on [Fol95, p. 158]. Let $\{U_n\}_{n\in\mathbb{N}}$ be a non-increasing sequence of open sets such that $U_n \downarrow \{1\}$ as $n\to\infty$. Using Tietze's extension theorem, there exists a family $\{\psi_n \in C_c(G)\}_{n\in\mathbb{N}}$ such that for all $n\in\mathbb{N}$ it holds that $\operatorname{supp}\psi_n$ is compact and contained in U_n, $\psi_n \geq 0$ and $\int_H \psi_n(\eta)d\eta = 1$. Notice that for any n, by lemma 50 the function

$$f_n(x) = \int_H \psi_n(x\eta)\sigma(\eta)v d\eta, \qquad x\in G$$

is in $\mathcal{F}_0$. Moreover, by continuity of σ, we can find for any $\epsilon > 0$ an open neighborhood N_ϵ of 1 in H such that for any $\eta \in N_\epsilon$ we have $\|\sigma(\eta)v - v\| < \epsilon$. Since $U_n \downarrow \{1\}$ there exists $N\in\mathbb{N}$ such that $U_n \subset N_\epsilon$ for all $n \geq N$. Then for all $n\geq N$ we have

$$\|f_n(1) - v\|_\sigma \leq \int_{U_n} \psi_n \|\sigma(\eta)v - v\|_\sigma d\eta < \epsilon \int_{U_n} \psi_n = \epsilon.$$

□

Lemma 76. *Suppose that $(\pi, G/G_\nu, P)$ is a transitive system of imprimitivity for some $\nu \in \widehat{N}$ with π irreducible. Then π is unitarily equivalent to $ind_{G_\nu}^G(\sigma)$ for some irreducible representation σ of G_ν that satisfies $\sigma(n) = \langle n,\nu\rangle I$ for all $n\in N$.*

Proof. We may assume that $(\pi, G/G_\nu, M)$ is the canonical system of imprimitivity associated to $\operatorname{ind}_{G_\nu}^G(\sigma)$ for some unitary representation σ of G_ν. Combining lemma 58 with the previous lemma, we have $\sigma(x^{-1}nx)f(x) = \langle x^{-1}nx, \nu\rangle f(x)$. In particular $\sigma(n)f(1) = \langle n,\nu\rangle f(1)$. By lemma 75, $\{f(1) | f\in\mathcal{F}_0\}$ is dense in $\mathcal{H}_\sigma$ and the result follows. □

It turns out that we also have the following converse statement.

Lemma 77.
Suppose that the orbit space of $\widehat{N}$ under the action of G is countably separated. If $\nu\in\widehat{N}$ and σ is an irreducible representation of G_ν such that $\sigma(n) = \langle n,\nu\rangle I$ for all $n\in N$, then $ind_{G_\nu}^G(\sigma)$ is irreducible. Moreover, if σ' is another such irreducible representation of G_ν such that $ind_{G_\nu}^G(\sigma') \cong ind_{G_\nu}^G(\sigma)$ in $\boldsymbol{UR}(G)$, then $\sigma'\cong\sigma$ in $\boldsymbol{UR}(G_\nu)$.

Proof. Let $\Sigma = (\pi, G/G_\nu, P)$ be the canonical system of imprimitivity associated to $\operatorname{ind}_{G_\nu}^G(\sigma)$. By the same argument as in the proof of lemma 74, we know that $\pi(n)f(x) = \langle n, x\cdot\nu\rangle f(x) = \int_{G/G_\nu}\langle n, x\cdot\nu\rangle dP(xG_\nu)$. It follows that P is the spectral measure for $\pi|_N$ after identifying $G/G_\nu \cong \mathcal{O}_\nu \hookrightarrow \widehat{N}$. This implies both statements of the lemma. Firstly, if $T\in\operatorname{Hom}_G(\pi)$ then by theorem 152, T also commutes with also $P(E)$ for every Borel set $E\subset\widehat{N}$. This means that $T\in\operatorname{Hom}_G(\Sigma)$ by corollary 153. The converse is trivially true. Thus, we have

$$\operatorname{Hom}_G(\pi) = \operatorname{Hom}_G(\Sigma) \cong \operatorname{Hom}_{G_\nu}(\sigma),$$

using the fact that $\operatorname{ind}_{G_\nu}^G : \boldsymbol{UR}(G_\nu) \to \boldsymbol{TSOI}(G)$ is an equivalence of categories for the latter equality. Since σ is irreducible, Schur's lemma implies that $\operatorname{Hom}_{G_\nu}(\sigma) = \mathbb{C}I$. Thus also $\operatorname{Hom}_G(\pi) = \mathbb{C}I$, which means that π is irreducible. Secondly, it follows in similar fashion that an equivalence between the unitary representations $\operatorname{ind}_{G_\nu}^G(\sigma)$ and $\operatorname{ind}_{G_\nu}^G(\sigma')$ actually defines an equivalence between their canonical systems of imprimitivity. The uniqueness in theorem 68 now implies that $\sigma\cong\sigma'$ in $\boldsymbol{UR}(G_\nu)$. □

Thus, we have completed the second aim of this section; that the irreducible representations of G are up to equivalence precisely given by representations of the stabilizer G_ν whose restriction to N is just the scalar action of ν on N.

3. The representations of G_ν that restrict to ν on N are determined by the Little group.

Now, let $H_\nu = G_\nu \cap H \cong G_\nu/N$. This group is called the **Little group** associated to ν. Observe that $G_\nu = N \rtimes H_\nu$. The following proposition completes the classification.

Proposition 78. *Let $\nu \in \widehat{N}$. Then every unitary representation σ of G_ν satisfying $\sigma(n) = \langle n, \nu\rangle I$ is of the form $\rho(n,h) = \langle n,\nu\rangle\rho(h)$ for some unitary representation ρ of H_ν. Moreover, σ is irreducible if and only if ρ is. Finally, writing $\nu\rho$ for such representations, we have*

$$\nu\rho_1 \cong \nu\rho_2 \text{ in } \boldsymbol{UR}(G_\nu) \iff \rho_1 \cong \rho_2 \text{ in } \boldsymbol{UR}(H_\nu).$$

Proof. If σ is a representation of $G_\nu = N \rtimes H_\nu$ on $\mathcal{H}_\sigma$ satisfying $\sigma(n) = \langle n,\nu\rangle I$, then we have

$$\sigma(n,m) = \sigma(n,1)\sigma(0,h) = \langle n,\nu\rangle\rho(h),$$

where $\rho(h) = \sigma(0,h)$ is a unitary representation of H_ν. Next, any subspace $W \subset \mathcal{H}_\sigma$ is σ-invariant if and only if it is ρ-invariant, since N just acts by scalars. This implies the statement on irreducibility. The last statement follows similarly. □

In summary, we have obtained the following:

Theorem 79. *Suppose $G = N \rtimes H$, where N is Abelian. Suppose further that the orbit space of $\widehat{N}$ under the action of G is countably separated. If $\nu \in \widehat{N}$ and ρ is an irreducible representation of H_ν, then $ind_{G_\nu}^G(\nu\rho)$ is an irreducible representation of G and every irreducible representation of G is of this form. Moreover,*

$$\nu\rho_1 \cong \nu\rho_2 \text{ in } \boldsymbol{UR}(G_\nu) \iff \rho_1 \cong \rho_2 \text{ in } \boldsymbol{UR}(H_\nu).$$

We make a final remark concerning the bundles that occur when applying the Mackey machine that is particularly relevant in chapter 6 and section 8.3.

Lemma 80. *Let $\nu \in N$ and $G_\nu = N \rtimes H_\nu$ be the corresponding stabilizer. Let σ be any representation of G_ν on $\mathcal{H}_\sigma$. There is an isomorphism of H-homogeneous Hilbert bundles:*

$$\begin{array}{ccc} H \times_{H_\nu} H_\sigma & \xrightarrow{\Phi} & G \times_{G_\nu} H_\sigma \\ \downarrow & & \downarrow \\ H/H_\nu & \xrightarrow{\phi} & G/G_\nu \end{array} .$$

Proof. The map Φ is defined by $\Phi : [v,h] \mapsto [v,(1,h)]$. This is well-defined and smooth. It is directly checked that this map is injective. To see surjectivity, notice that any element $[v,(n,h)] \in G \times_{G_\nu} \mathcal{H}_\sigma$ has a representative of the form $(v',(1,h))$ for some appropriate v'. Next, $\phi : H/H_\nu \to G/G_\nu$ is defined by $[h] \mapsto [(1,h)]$, which is also well-defined, smooth and bijective. Since it is H-equivariant and H acts transitively on H/H_ν, it has constant rank and is thus a diffeomorphism. The diagram above commutes and Φ is H-equivariant and linear on fibers, so Φ is a morphism of homogeneous Hilbert bundles for H. An application of proposition 9 shows that it is a diffeomorphism, so we are done

□

Corollary 81. *Consider the same setting as the previous lemma.*

$$ind_{G_\nu}^G(\sigma)\Big|_H \cong ind_{H_\nu}^H(\sigma|_{H_\nu}).$$

4.4.1 Regularity of the action of Spin$(r,s)^0 \times K$ on $V \oplus (\mathfrak{k} \otimes V)$.

The main goal of this thesis is to study the representation theory of the group $G = N \rtimes H \overset{d}{=} \mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K$ for some compact semisimple Lie group K with real lie algebra $\mathfrak{k}$. The strategy to do so is to apply the theory of the Mackey machine, so that every strongly continuous unitary representation of the group G is obtained by inducing a suitable representation of a Little group up to G. Now, the Mackey machine only applies if the orbit space of the action of $SL(2,\mathbb{C}) \times K$ on the dual space $\widehat{N}$ is countably separated. The aim of this section is to address this matter and justify the use of the Mackey machine in chapter 5 and chapter 6.

Now, in chapter 6 we will be concerned with a more general family of groups, namely groups of the form

$$N \rtimes H = V \oplus (\mathfrak{k} \otimes V) \rtimes \mathrm{Spin}(r,s)^0 \times K, \tag{4.7}$$

where (V,q) is a real quadratic space of signature (r,s) and the action of H on V is given on simple tensors by

$$(w,k) \cdot v_1 \oplus (X \otimes v_2) = \phi(w)v_1 \oplus (\mathrm{Ad}_k X \otimes \phi(w)v_2), \tag{4.8}$$

where $\phi : \mathrm{Spin}(r,s)^0 \to SO(r,s)^0$ denotes the covering homomorphism. Thus, we consider groups of this more general form (4.7). Notice that the non-degenerate bilinear form q is $\mathrm{Spin}(r,s)^0$-invariant and because K is semisimple and compact there exists a Ad-invariant inner product κ on $\mathfrak{k}$. It follows that the bilinear form $\beta = q \oplus (\kappa \otimes q)$ on N is H-invariant and non-degenerate. Thus, we may use it to identify $\widehat{N} \cong N$ via the pairing

$$\langle v_1 \oplus t_1, v_2 \oplus t_2 \rangle = e^{i\beta(v_1 \oplus t_1, v_2 \oplus t_2)}$$

so that under this identification, the action of H on $\widehat{N}$ transfers to the action (4.8) on N. We aim to apply lemma 71 to the action (4.8) of H on N.

Observe that $\mathrm{Spin}(r,s)^0$ acts on V via $SO(r,s)^0$. Now, $SO(r,s)$ is a linear algebraic group defined over $\mathbb{R}$ and V is vacuously an affine variety defined over $\mathbb{R}$. Moreover, it is clear that the defining action of $SO(r,s)$ on V is also a morphism of algebraic varieties, meaning that it is defined by restriction of a polynomial map. Thus, lemma 71 yields that the action of $SO(r,s)$ on V is countably separated. However, we are interested in the action of the connected component of the identity $SO(r,s)^0$. It is not clear that this subgroup is also a subvariety of $SO(r,s)$ so that lemma 71 does not directly yield the required result. Instead we make use of the following lemma.

Lemma 82. *Suppose that $G = H \rtimes \mathbb{Z}_2$ is a Lie group that acts smoothly on some vector space V. Let $v \in V$ and write $\mathcal{O}_v^H$ and $\mathcal{O}_v^G$ for the H- and G-orbit of v, respectively. Endow both $\mathcal{O}_v^H$ and $\mathcal{O}_v^G$ with the subspace topology, as subspaces of V. If the orbit map $G/G_v \to \mathcal{O}_v^G$ is a homeomorphism, then also the orbit map $H/H_v \to \mathcal{O}_v^H$ is a homeomorphism.*

Proof. Define the map $\iota : H/H_v \to G/G_v$ by $\iota([h]) = [h,1]$. Notice that this map is smooth and well-defined. To proof the claim, it suffices to show that ι is actually an embedding. We distinguish the two possible cases.

1. Suppose first that $G_v = H_v \rtimes \mathbb{Z}_2$. Then $G/G_v = (H \rtimes \mathbb{Z}_2)/(H_v \rtimes \mathbb{Z}_2) \cong H/H_v$, where the latter diffeomorphism is given by $[h,1] \mapsto [h]$. Indeed, notice that any element in $(H \rtimes \mathbb{Z}_2)/(H_v \rtimes \mathbb{Z}_2)$ admits a unique representation of the form $[h,1]$. Moreover this map is of constant rank by H-equivariance and since it is also bijective, it is a diffeomorphism. It follows that $\iota : H/H_v \to G/G_v \cong H/H_v$ is just the identity, which is clearly an diffeomorphism and in particular an embedding.

2. Suppose instead that $G_v = H_v \times \{1\}$. In this case, $G/G_v = (H \rtimes \mathbb{Z}_2)/(H_v \rtimes \{1\}) \cong H/H_v \times \mathbb{Z}_2$, where the latter diffeomorphism is given by $A : [h,n] \mapsto ([h],n)$. Indeed, this map is well-defined and bijective because H_v acts trivially on $\mathbb{Z}_2$. Now, let $q_{H_v} : H \to H/H_v$ and $q_1 : H \rtimes \mathbb{Z}_2 \to (H \rtimes \mathbb{Z}_2)/(H_v \times \{1\})$ be the quotient maps. Define further $q_2 = q_{H_v} \times \mathrm{id} : H \rtimes \mathbb{Z}_2 \to H/H_v \times \mathbb{Z}_2$. Notice that A is a smooth map making the diagram below commute:

$$\begin{array}{ccc} & H \rtimes \mathbb{Z}_2 & \\ {}^{q_1}\swarrow & & \searrow^{q_2} \\ (H \rtimes \mathbb{Z}_2)/(H_v \times \{1\}) & \xrightarrow{\quad A \quad} & H/H_v \times \mathbb{Z}_2 \end{array} \; . \tag{4.9}$$

   Since both q_1 and q_2 are smooth submersions by corollary 18, it follows that both A and A^{-1} are smooth. We conclude that $\iota : H/H_v \to G/G_v \cong H/H_v \times \mathbb{Z}_2$ is given by $[h] \mapsto ([h],1)$, which is clearly an embedding.

□

Corollary 83. *Consider the setting of lemma 82 and consider the action of Spin*$(r,s)^0$ *on* $\widehat{V}$ *given by* $\langle v, x \cdot \xi\rangle \overset{d}{=} \langle x^{-1} \cdot v, \xi\rangle$. *If* $SO(r,s) \cong SO(r,s)^0 \rtimes \mathbb{Z}_2$, *then the orbit space of the action of Spin*$(r,s)^0$ *on* $\widehat{V}$ *is countably separated.*

Proof. Notice first that we may identify $\widehat{V} \cong V$ using the Spin(r,s)-invariant quadratic form q via the pairing

$$\langle v, w\rangle = e^{iq(v,w)}.$$

Under this identification, the action of Spin$(r,s)^0$ on $\widehat{V}$ transfers to the original action on V. By lemma 71, the orbit space of the action of $SO(r,s)$ on V is countably separated, which by lemma 70 is equivalent to the statement that every orbit map of $SO(r,s)$ is a homeomorphism. Then lemma 82 implies that also every orbit map of $SO(r,s)^0$ is a homeomorphism, which completes the proof, again by lemma 70. □

Remark.

— It is known [Kna02, p. 73, proposition 1.124] that for every $r,s > 0$ the group $SO(r,s)$ has two connected components. The question therefore remains for which $r,s > 0$ the following exact sequence splits:

$$1 \to SO(r,s)^0 \to SO(r,s) \to \mathbb{Z}_2 \to 1.$$

— In the case we are mostly interested in, namely that of $(r,s) = (1,3)$, this sequence is known to split [Var07, p. 333, theorem 9.1].

Lemma 84. *Suppose that K is a linear algebraic Lie group with real Lie algebra $\mathfrak{k}$. Then the adjoint action of its Lie algebra $\mathfrak{k}$ is an algebraic action defined over $\mathbb{R}$.*

Proof. Notice that the Lie algebra $\mathfrak{k}$ is vacuously an algebraic variety over $\mathbb{R}$. Since K is a linear algebraic group defined over $\mathbb{R}$, the adjoint action on its Lie algebra consists of multiplication and inversion in $GL(\mathbb{R}^n)$, where $n = \dim \mathfrak{k}$. Since $GL(\mathbb{R}^n)$ is an algebraic group, we are done. □

Remark. Observe that in particular $SU(N)$ is a linear algebraic group defined over $\mathbb{R}$.

Proposition 85. *Suppose that K is a linear algebraic Lie group defined over $\mathbb{R}$ with real Lie algebra $\mathfrak{k}$ and assume that $SO(r,s) \cong SO(r,s)^0 \rtimes \mathbb{Z}_2$. Then the orbit space of the action of Spin$(r,s)^0 \times K$ on $\mathbb{R}^d \oplus (\mathfrak{k} \otimes \mathbb{R}^d)$ is countably separated.*

Proof. Since K is a linear algebraic group defined over $\mathbb{R}$, also Spin$(r,s) \times K$ is a linear algebraic group defined over $\mathbb{R}$. Now, $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^d)$ is vacuously an algebraic variety defined over $\mathbb{R}$. Since both the action of Spin(r,s) on $\mathbb{R}^d$ and of K on $\mathfrak{k}$ are algebraic actions defined over $\mathbb{R}$, the same holds for the action of Spin$(r,s) \times K$ on $\mathbb{R}^d \otimes \mathfrak{k}$. Indeed, choosing a basis for $\mathbb{R}^d \times \mathfrak{k}$ reveals that the components of this action are given by the product of the corresponding components of the separate actions of Spin(r,s) on $\mathbb{R}^d$ and K on $\mathfrak{k}$. Therefore, lemma 71 implies that the orbit space under the action of Spin$(r,s) \times K$ is countably separated. Furthermore, because Spin$(r,s) =$ Spin$(r,s)^0 \rtimes \mathbb{Z}_2$, also the orbit space under the action of Spin$(r,s)^0 \times K$ is countably separated. Indeed, this follows by an application of lemma 82 after noting that $(\text{Spin}(r,s)^0 \rtimes \mathbb{Z}_2) \times K \cong (\text{Spin}(r,s)^0 \times K) \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts trivially on K. □

Corollary 86. *Suppose that K is a linear algebraic Lie group defined over $\mathbb{R}$ with real Lie algebra $\mathfrak{k}$. The orbit space of the action of $SL(2,\mathbb{C}) \times K$ on $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4)$ is countably separated.*

4.5 Projective unitary representations of the Poincaré group

In this section, the Mackey machine is applied to the Poincaré group, leading to a full classification of its continuous projective unitary representations in terms of the representation theory of the various Little groups. The results of this section are based in [Fol95, p. 190]. The Poincaré group is defined by

$$\mathcal{P} = \mathbb{R}^4 \rtimes SO(1,3)^0 \tag{4.10}$$

With the group law $(b_1, T_1) \cdot (b_2, T_2) = (b_1 + T_1 b_2, T_1 T_2)$. Now, Bargmann's theorem[Bar54] implies that any continuous projective unitary representation of $\mathcal{P}$ lifts to a strongly continuous unitary representation of its universal covering group $\widetilde{\mathcal{P}}$. The study of the projective unitary representations of the Poincaré group therefore leads us to study the strongly continuous irreducible unitary representations of its universal covering group, which is $\widetilde{\mathcal{P}} = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$. By corollary 83, we know that the orbit space of the action of $SL(2,\mathbb{C})$ on $\widehat{\mathbb{R}^4}$ is countably separated, so that the use of the Mackey machine is justified. In the following, we determine the orbits and corresponding stabilizers of this action. Recall that any irreducible induced representation of $\widetilde{\mathcal{P}}$ corresponds to a spectral measure that is supported on one of these orbits. Moreover, if the representation of the corresponding stabilizers is finite dimensional, then the induced representation is constructed out of sections of homogeneous Hilbert bundles over the respective orbit.

Let $SL(2,\mathbb{C}) \xrightarrow{\phi} SO(1,3)^0$ be the covering homomorphism constructed in equation (3.3). Let η be the Minkowski form and use it to identify $\widehat{\mathbb{R}^4}$ with $\mathbb{R}^4$ via the pairing

$$\langle x, p \rangle = e^{i\eta(x,p)}$$

By $SL(2,\mathbb{C})$-invariance of the Minkowski form η, the action of $SL(2,\mathbb{C})$ on $\widehat{\mathbb{R}^4}$ transfers under this identification to the action on $\mathbb{R}^4$ given by $w \cdot p = \phi(w)p$.

Let us first determine the of the action of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$ explicitly. Notice that for every $\lambda \in \mathbb{R}$, the level sets

$$M_\lambda = \left\{ p \in \widehat{\mathbb{R}^4} \,:\, \eta(p,p) = \lambda \right\}$$

are invariant under the action of $SL(2,\mathbb{C})$. For $s \in \mathbb{R}^4$ write $s = \begin{pmatrix} t \\ x \end{pmatrix}$ with $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$. Consider the case when $\lambda > 0$. Notice that

$$s \in M_\lambda \iff t^2 - |x|^2 = \lambda$$

M_λ is therefore a hyperboloid of two sheets $\mathcal{O}_\lambda^\pm$, where

$$\mathcal{O}_\lambda^\pm = \left\{ \begin{pmatrix} t \\ x \end{pmatrix} \in M_\lambda \,:\, \pm t > 0 \right\}.$$

Notice that these sheets are the connected components of M_λ. Since $SL(2,\mathbb{C})$ is connected any orbit must also be connected and it follows that each of the sheets $\mathcal{O}_\lambda^\pm$ must be invariant under the action of $SL(2,\mathbb{C})$.

Lemma 87. *For $\lambda \geq 0$, each of the sheets $\mathcal{O}_\lambda^\pm$ is an orbit of $SL(2,\mathbb{C})$. For $\lambda < 0$ the level set M_λ is an orbit.*

Proof. We need to show that $SL(2,\mathbb{C})$ acts transitively on each of these sheets. We consider first $\mathcal{O}_\lambda^+$. To prove the claim, we will show that for every point $s \in \mathcal{O}_\lambda^+$, there exists $A_s \in SO(1,3)^0$ such that A_s maps the point $e_\lambda = (\sqrt{\lambda}, 0, 0, 0)$ to s. In that case, if $s, z \in \mathcal{O}_\lambda^+$, then $z = A_z A_s^{-1} s$ and therefore the action of $\mathcal{P}$ on $\mathcal{O}_\lambda^+$ is transitive.

Notice first that by rescaling s if necessary, we may assume that $\lambda = 1$. Moreover, because the action of $SO(3) \subseteq SO(1,3)^0$ on S^2 is transitive, we may further assume that $x = (x_0, 0, 0)$, so $t^2 - x_0^2 = 1$. Let $\gamma \in \mathbb{R}$ be such that $t = \cosh(\gamma)$ and $x_0 = \sinh(\gamma)$. Define

$$A_\gamma = \begin{pmatrix} \cosh(\gamma) & \sinh(\gamma) & 0 & 0 \\ \sinh(\gamma) & \cosh(\gamma) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $A_\gamma e_1 = s$. Moreover, notice that A_γ preserves η and $[0,\gamma] \to O(1,3), t \mapsto A_t$ defines a continuous path connecting the identity and A_γ. Thus $A_\gamma \in SO(1,3)^0$ and completing the proof of the claim for $\mathcal{O}_\lambda^+$.

Next, observe that A_γ maps the point $-e_1$ to $(-\cosh(\gamma), -\sinh(\gamma), 0, 0) \in \mathcal{O}_1^-$ and the point e_2 to $(\sinh(\gamma), \cosh(\gamma), 0, 0) \in M_{-1}$, so a similar arguments also prove the transitivity of the action on $\mathcal{O}_\lambda^-$ for $\lambda > 0$ and on M_λ for $\lambda < 0$.

Finally, consider $\lambda = 0$. Since the origin is an invariant element, $\{0\}$ is a distinguished orbit. Moreover the subsets $\mathcal{O}_0^\pm$ are once again invariant. Observe that A_γ maps $\pm(e_1 + e_2)$ to $\pm(\cosh(\gamma) + \sinh(\gamma), \sinh(\gamma) + \cosh(\gamma)) \in \mathcal{O}_0^\pm$, so once more a similar argument also proves transitivity of the action on $\mathcal{O}_0^\pm$. □

We conclude that the orbits are M_λ for $\lambda < 0$, $\mathcal{O}_\lambda^\pm$ for $\lambda \geq 0$ and $\{0\}$. They are depicted in figure 4.1.

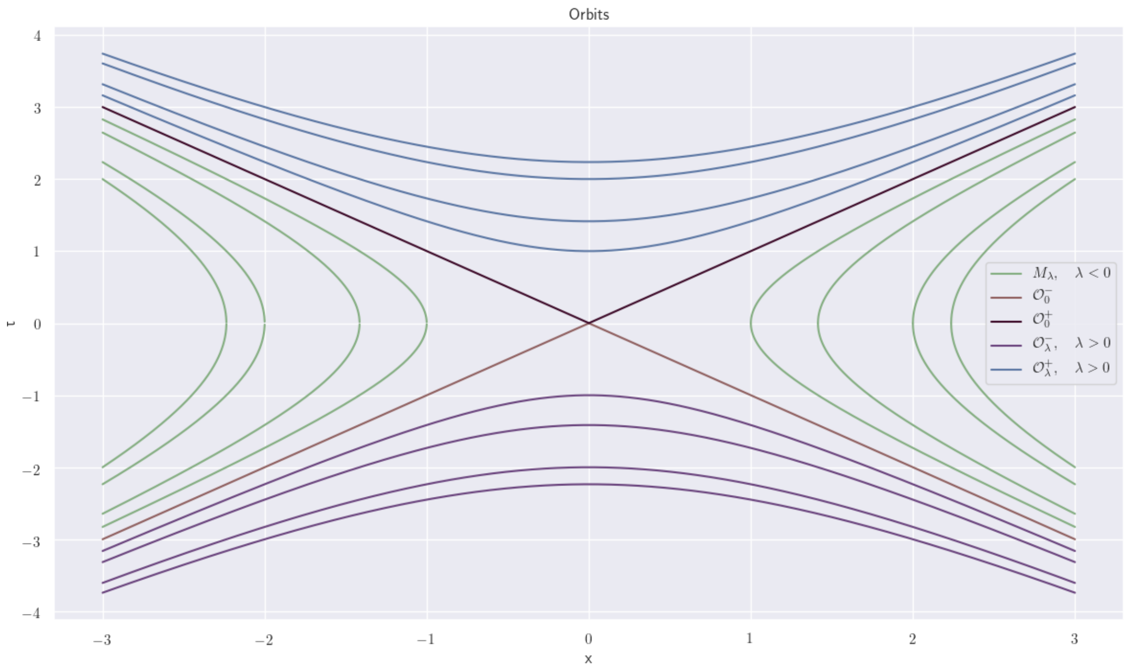


Figure 4.1: Two-dimensional representation of the orbits of the action of $\widetilde{\mathcal{P}}$ on $\widehat{\mathbb{R}^4}$.

Next, let us determine the stabilizers H_ν of the action of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$ corresponding to the various orbits $\mathcal{O}_\nu$. Notice for any point $\nu \in \mathbb{R}^4$, its stabilizer is simply the pre-image under ϕ of its stabilizer with respect to the $SO(1,3)^0$-action.

— First, consider M_λ for $\lambda < 0$. Take $a = (0, \sqrt{|\lambda|}, 0, 0)$. Choosing a basis for $\mathbb{R}^4$ that containing a, one finds that the stabilizer in $SO(1,3)^0$ is given by $SO(1,2)^0 \hookrightarrow SO(1,3)^0$. From lemma 44 we know that $H_a = \phi^{-1}(SO(1,2)^0) = SL(2,\mathbb{R})$.

— Next, consider $\mathcal{O}_\lambda^\pm$ with $\lambda > 0$. Choose the point $b = (\pm\lambda^{\frac{1}{2}}, 0, 0, 0)$. Then once again, after choosing an appropriate basis one finds that the stabilizer of b in $SO(1,3)^0$ is $SO(3) \hookrightarrow SO(1,3)^0$. By corollary 48, it follows that $H_b = SU(2)$.

— The origin is an invariant element which means that $H_0 = SL(2,\mathbb{C})$.

— Finally, consider the light-like orbits $\mathcal{O}_0^\pm$. Consider the points $\alpha_\pm = (\pm 1, 0, 0, \pm 1) \in \mathcal{O}_0^\pm$ and let $F_\pm = SL(2,\mathbb{C})_{\alpha_\pm}$ be the corresponding stabilizers. Let $A : \mathbb{R}^4 \to C, x \mapsto \sum_{i=0}^3 x_i \sigma_i$ be as in the construction of the covering homomorphism ϕ. Notice that $A(\alpha_\pm) = \pm 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and therefore

$$w \in F_\pm \iff \phi(w)\alpha_\pm = \alpha_\pm \iff w \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

A computation shows that

$$F_{\pm} = E \overset{d}{=} \left\{ \begin{pmatrix} z & a \\ 0 & z^{-1} \end{pmatrix} \,:\, z, a \in \mathbb{C}, \quad |z| = 1 \right\}.$$

Write $m_{z,a} = \begin{pmatrix} z & a \\ 0 & z^{-1} \end{pmatrix}$. Notice that $m_{z,1} \cdot m_{1,a} \cdot m_{z,1}^{-1} = m_{1,z^2 a}$. Thus if we identify $z \in U(1)$ with $m_{z,0}$ and $a \in \mathbb{C}$ with $m_{1,a}$, we find that $F_{\pm} \cong \mathbb{C} \rtimes_{\sigma} U(1)$ where $\mathbb{C}$ is considered as an Abelian group under addition and $\sigma(z)a = z^2 a$. Now, let $\theta : U(1) \to SO(2)$ be an isomorphism. Then the mapping $(a, z) \mapsto (a, \theta(z^2))$ is a surjective homomorphism from $\mathbb{C} \rtimes_{\sigma} U(1)$ to $\mathbb{R}^2 \rtimes SO(2)$ with kernel $\{\pm 1\}$. Therefore, we find that

$$\phi(E) \cong E/\{\pm I\} \cong \mathbb{C} \rtimes_{\sigma} U(1)/\{\pm I\} \cong \mathbb{R}^2 \rtimes SO(2) =: E(2).$$

We conclude that the irreducible strongly continuous unitary representations of $\widetilde{\mathcal{P}}$ are precisely those of $SL(2,\mathbb{R}), SU(2), SL(2,\mathbb{C})$ and E induced up to $\widetilde{\mathcal{P}}$ according to the theory of the Mackey machine. Not all of these are currently physically relevant. Wigner related the representations induced from $SU(2)$ and E to certain wave equations, see also section 8.3 for a detailed exhibition.

Part II

Chapter 5

Positive Energy Representations

This chapter finally begins addressing the main goal of this thesis; to obtain an understanding of the strongly continuous unitary representations of the group $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K$ that are of *positive energy*, in a specific sense. Before diving into the details, let us recall briefly the motivation leading up to this problem.

The main historical motivation lies in the attempt of physicists and mathematicians to formulate theories of particle physics in a mathematically rigorous way. Such theories should in particular be compatible with both the theory of quantum mechanics and that of special relativity and the connection between these two theories is mathematically found in representation theory.

Recall that the state space of a quantum system is a projective Hilbert space. If it is known that such a quantum system exhibits an invariance with respect to some connected symmetry group, then Wigner's theorem [Wig39] states that there must exist a projective unitary representations of this group on the state space of the quantum system.

In particular, there must be a projective unitary representation of the group $\mathbb{R}^4 \rtimes SO(1,3)^0$ in the state space of any quantum system that is consistent with the theory of special relativity. Moreover, various fundamental interactions are known to exhibit further symmetries. In particular isospin exhibits $SU(2)$ symmetry and flavour is subject to $SU(3)$ symmetry. These interactions are internal and the corresponding symmetries are *local*, meaning that they may vary (smoothly) at different positions in space-time. On the other hand, the symmetry of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ imposed by special relativity is *global*. A symmetry group capturing both these local and global symmetries can mathematically be described in terms if fiber bundles.

Explicitly, let K be a simply connected semisimple compact Lie group with real Lie algebra $\mathfrak{k}$. If $P \to \mathbb{R}^4$ is a principal K-bundle, then one considers the associated group bundle $\mathcal{K} = P \times_{\mathrm{Ad}} K$ over $\mathbb{R}^4$ with typical fiber K. Now, given the action of $\mathbb{R}^4 \rtimes SO(1,3)^0$ on $\mathbb{R}^4$ and a lift of this action to $\mathcal{K}$, one considers the symmetry group $\mathcal{G} = \Gamma_c(\mathcal{K}) \rtimes (\mathbb{R}^4 \rtimes SO(1,3)^0)$, where $\Gamma_c(\mathcal{K})$ denotes the group compactly supported sections of $\mathcal{K} \to \mathbb{R}^4$.

The representation theory of such groups is in general not well understood, but a simplification is obtained if the Minkowski space $\mathbb{R}^4$ is be replaced by its conformal compactification Q [FLV07, section 2]. With this simplification, a recent result[JN] by B. Janssens and K.H. Neeb has reduced the study of a certain class of projective representations of $\mathcal{G}$ satisfying a so-called *positive energy condition* to a much simpler problem. The precise result is given below.

Theorem 88. *There exists some H in a maximal Abelian subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ such that there is a bijective correspondence between smooth projective positive energy representations of $\Gamma(Q,\mathcal{K})^0 \rtimes (\mathbb{R}^4 \rtimes SO(1,3)^0)$ and strongly continuous unitary representations of*

$$V \rtimes H \stackrel{d}{=} \mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K,$$

that are of positive energy with respect to the cone

$$C' = \{v \oplus (H \otimes v) \, : \, v_0 \geq 0 \text{ and } \eta(v,v) \geq 0\} \subset \mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \tag{5.1}$$

and where the action of $SL(2,\mathbb{C}) \times K$ on $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4)$ is given on simple tensors by

$$(w,k) \cdot v_1 \oplus (X \otimes v_2) = \phi(w)v_1 \oplus (Ad_k(X) \otimes \phi(w)v_2).$$

Here $\phi : SL(2,\mathbb{C}) \to SO(1,3)^0$ denotes the covering homomorphism.

This chapter is devoted to an understanding of the strongly continuous unitary representations of the Lie group $V \rtimes H$ that are of positive energy at the cone (5.1).

This amounts firstly to an understanding of the positive energy condition, which is reformulated in a way that allows for a geometric interpretation. This leads in to a detailed understanding of the condition.

Secondly, as the group $V \rtimes H$ is of a form compatible with the Mackey machine, the strongly continuous unitary representations of $V \rtimes H$ are determined by the representation theory of the various Little groups H_ν. As such, the second half of the chapter is concerned with the classification of the various stabilizers H_ν corresponding to representations of positive energy for the special case that $K = SU(2)$. A full classification of these stabilizers is obtained.

5.1 The positive energy condition

In the following, the the notion of a representation of positive energy is introduced. After stating the definition given in [JN] its meaning and implications are examined in detail, yielding a further understanding of this condition.

Definition 89. Suppose $G = N \rtimes_\alpha H$ where N is Abelian. Let $\rho : G \to U(\mathcal{H})$ be a strongly continuous unitary representation. Then

1. The representation ρ is said to be of **positive energy** at $Z \in \mathrm{Lie}(N)$ if the spectrum of the infinitesimal generator $-i\, d\rho|_{\mathrm{Lie}(N)}(Z) \overset{d}{=} -i\, \frac{d}{dt}\big|_{t=0} \rho(\exp(tZ), 1)$ is bounded from below.
2. The **positive energy cone** $C \subseteq \mathrm{Lie}(N)$ associated to ρ is the cone of elements in $\mathrm{Lie}(N)$ at which ρ is of positive energy.

Remark.

1. By the theory of the Mackey machine we know that if the orbit space of $\widehat{N}$ under the action of H is countably separated, then for every irreducible unitary representation ρ there exists some $\nu \in \widehat{N}$ such that the spectral measure associated to ρ is supported on the orbit $\mathcal{O}_\nu$ and moreover, there is a representation of its stabilizer G_ν such that $\rho \cong \mathrm{ind}_{G_\nu}^G(\sigma)$. The Hilbert space $\mathcal{F}_\sigma$ on which this space acts is defined according (4.3). Now, the Abelian part acts as a multiplication operator $\rho(n)f(x) = \langle n, x\cdot\nu\rangle f(x)$ for $f \in \mathcal{F}_\sigma$. Recall that $\nu : N \to U(1)$ is a Lie group homomorphism. Then $\nu_* : \mathrm{Lie}(N) \to i\mathbb{R}$, and
$$-i\, d\rho|_{\mathrm{Lie}(N)}(Z)f(x) = -i(x\cdot\nu)_*(Z)f(x), \qquad Z \in \mathrm{Lie}(N),\ f \in \mathcal{F}_\sigma,\ x \in G.$$
2. If further N is a vector space equipped with a non-degenerate symmetric bilinear form β that is H-invariant, we can identify $\widehat{N} \cong N$ via the pairing $\langle v, w\rangle = e^{i\beta(v,w)}$. Under this identification, the action of H on $\widehat{N} \cong N$ becomes $h \cdot n = \alpha(h)n$. Notice also that $\mathrm{Lie}(N) \cong N$. We have
$$-i(x\cdot\nu)_*(n)f(x) = -i\frac{d}{dt}\langle tn, x\cdot\nu\rangle f(x) = -i\frac{d}{dt}e^{it\beta(n,x\cdot\nu)}f(x) = \beta(n, x\cdot\nu)f(x).$$
Therefore,
$$\mathrm{Spec}(-i\, d\rho|_N(n)) = \beta(n, \mathcal{O}_\nu).$$
In particular, we see that whether or not a representation is of positive energy at the point n only depends on the orbit $\mathcal{O}_\nu$ on which the spectral measure corresponding to ρ has its support.

In view of the second point in the previous remark, we make the following definition.

Definition 90. Suppose $G = N \rtimes_\alpha H$ where N is a real vector space equipped with a non-degenerate symmetric bilinear form that is H-invariant. Assume further that the orbit space of N is countably separated. We say that the orbit $\mathcal{O}_\nu \subset N$ is of positive energy at $n \in N$ if $\beta(n, \mathcal{O}_\nu)$ is bounded from below. A point $\nu \in N$ is said to be of positive energy at $n \in N$ if $\mathcal{O}_\nu$ is of positive energy at $n \in N$.

Proposition 91. *Consider $G = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$, where $\mathbb{R}^4$ is equipped with the Minkowski form η. The orbits in $\mathbb{R}^4$ that are positive at all elements in the open cone*

$$C = \{p \in \mathbb{R}^4 \,:\, p_0 > 0 \text{ and } \eta(p,p) > 0\} \tag{5.2}$$

are precisely the orbits that lie within the closure $\overline{C}$ of this cone.

Proof. Notice first that $\eta(p,\mathcal{O}_\nu) = \eta(\phi(w)p, \phi(w)\mathcal{O}_\nu) = \eta(\phi(w)p, \mathcal{O}_\nu)$ for $p \in C$, $w \in SL(2,\mathbb{C})$ and $\nu \in \mathbb{R}^4$. Therefore, $\eta(p,\mathcal{O}_\nu)$ is bounded from below if and only if $\eta(\phi(w)p,\mathcal{O}_\nu)$ is bounded from below. We know from section 4.5 that any point in C is in the orbit of me_0 for some $m > 0$, so by the previous observation an orbit $\mathcal{O}_\nu$ is of positive energy at all points in C if and only if $\eta(e_0,\mathcal{O}_\nu)$ is bounded from below. As such, consider the linear functional $\eta(e_0,-)$. The kernel of this map is $\mathrm{Span}\{e_1,e_2,e_3\}$. Moreover, $\eta(e_0,e_0) = 1$ is positive and therefore $\eta(p_0,\mathcal{O}_\nu)$ is bounded from below if and only if $\mathrm{Pr}_0(\mathcal{O}_\nu)$ is bounded from below, where Pr_0 is the projection onto the first coordinate. The various orbits have been determined explicitly in section 4.5 and the conclusion follows. □

5.1.1 The positive energy condition for $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K$

Let us examine in detail what it means for a strongly continuous unitary representation of the group $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K$ to be of positive energy at the cone (5.1).

Let K be a simply connected compact Lie group with semisimple real Lie algebra $\mathfrak{k}$. Consider the group $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K$, where $SL(2,\mathbb{C}) \times K$ acts on $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4)$ according to

$$(w,k) \cdot (v_1 \oplus (X \otimes v_2)) \stackrel{d}{=} \phi(w)v_1 \oplus (\mathrm{Ad}_k X \otimes \phi(w)v_2).$$

Here, $SL(2,\mathbb{C}) \xrightarrow{\phi} SO(1,3)^0$ denotes the covering homomorphism as in corollary 43. Assume further that the corresponding orbit space is countably separated, so that the theory of the Mackey machine applies. In view of corollary 86 this holds in particular if $K = SU(N)$.

Now, we have already observed that in this case, the positive energy requirement is really a condition on the *orbits* on which the projection-valued measure corresponding to some irreducible representation is supported. Thus, we will be concerned with the orbits of positive energy, in the sense of definition 90.

Obtaining an equivalent formulation

First, an equivalent formulation is determined for an orbit to be of positive energy at the cone C', which gives this definition a more geometrical and intuitive interpretation. Eventually, this leads to the conclusion that for points $\nu = p + A$ with $p \in \partial C$ the condition forces A to be a rank-one operator of the form $\eta(p,\cdot)X$ for some $X \in \mathfrak{k}$. On the other hand if p is in the interior $p \in C_0$, then the condition virtually imposes no restriction on the possible stabilizers corresponding to orbits of positive energy.

Identify $\mathbb{R}^{4^*} \cong \mathbb{R}^4$ using η, $\mathfrak{k}^* \cong \mathfrak{k}$ using an Ad-invariant positive definite inner product κ on $\mathfrak{k}$, which exists because K is compact so that the Killing form is negative definite on $\mathfrak{t}$. Furthermore, identify $\mathfrak{k} \otimes \mathbb{R}^4 \cong \mathbb{R}^{4^*} \otimes \mathfrak{k} \cong \mathrm{Hom}(\mathbb{R}^4,\mathfrak{k})$ according to equation (8.10). These identifications are discussed in more detail in section 8.4. In particular, under these identifications the action of $SL(2,\mathbb{C}) \times K$ transfers on $\mathrm{Hom}(\mathbb{R}^4,\mathfrak{k})$ to the action given by

$$(w,k) \cdot A = \mathrm{Ad}_k \circ A \circ \phi(w)^{-1}, \qquad (w,k) \in SL(2,\mathbb{C}) \times K,\ A \in \mathrm{Hom}(\mathbb{R}^4,\mathfrak{k}).$$

Thus, we are interested in the strongly continuous representations of the group

$$G = V \rtimes H = \mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4,\mathfrak{k}) \rtimes SL(2,\mathbb{C}) \times K \tag{5.3}$$

Denote by $(-)^\star : \mathrm{Hom}(\mathbb{R}^4,\mathfrak{k}) \to \mathrm{Hom}(\mathfrak{k},\mathbb{R}^4)$ the transpose obtained via the identifications $\mathbb{R}^{4^*} \cong \mathbb{R}^4$ and $\mathfrak{k}^* \cong \mathfrak{k}$. Then according to lemma 163, the bilinear on $\mathrm{Hom}(\mathbb{R}^4,\mathfrak{k})$ corresponding under these identifications to the bilinear form $\eta \otimes \kappa$ on $\mathbb{R}^4 \otimes \mathfrak{k}$ is given by $\beta(A,B) = \mathrm{tr}(A^\star B) = \mathrm{tr}(B^\star A)$. Finally, the cone (5.1) becomes

$$C'' = \{\, v \oplus \eta(\cdot,v)H \,:\, v \in C \,\}. \tag{5.4}$$

Lemma 92. *Let β be the bilinear form on Hom$(\mathbb{R}^4,\mathfrak{k})$ given by $\beta(A,B) = \mathrm{tr}(A^\star B)$. Fix $v \in C$ and define $M_v = \eta(\cdot,v)H \in$ Hom$(\mathbb{R}^4,\mathfrak{k})$ for some non-zero $H \in \mathfrak{t}$. Then for any $A \in$ Hom$(\mathbb{R}^4,\mathfrak{k})$, we have*

$$\beta(M_v,A) = \kappa(Av,H).$$

Proof. By lemma 160 we know that $M_v^\star$ is the unique linear map $\mathfrak{k} \to \mathbb{R}^4$ satisfying $\kappa(M_v p, X) = \eta(p, M_v^\star X)$ for all $p \in \mathbb{R}^4$ and $X \in \mathfrak{k}$. We compute that

$$\kappa(M_v p, X) = \eta(p, v)\kappa(H, X) = \eta(p, \kappa(X, H)v).$$

Therefore $M_v^\star = \kappa(\cdot, H)v$ and hence $M_v^\star Ap = \kappa(Ap, H)v$. Since $\eta(v, v) \neq 0$ and η is symmetric, we may extend v to a η-orthogonal basis of $\mathbb{R}^4$ and using this basis one finds that $\beta(M_v, A) = \mathrm{tr}(M_v^\star A) = \kappa(Av, H)$. □

Lemma 93. *Let $\nu = p + A \in \mathbb{R}^4 \oplus Hom(\mathbb{R}^4, \mathfrak{k})$, $H \in \mathfrak{t}$ be non-zero and let $v \in C$ be arbitrary. Define $M_v = \eta(\cdot, v)H$. Then $\mathcal{O}_\nu$ is of positive energy at $v \oplus M_v$ if and only if $\eta(v, \mathcal{O}_{p+A^\star X})$ is bounded from below for every X in the adjoint orbit of H, where $\mathcal{O}_{p+A^\star X}$ is the orbit of $p + A^\star X$ in $\mathbb{R}^4$ under the action of $SL(2,\mathbb{C})$.*

Proof. By definition, the orbit $\mathcal{O}_\nu$ is of positive energy at $v \oplus M_v$ if $(\eta \oplus \beta)(v \oplus M_v, \mathcal{O}_\nu)$ is bounded from below. By the previous lemma 92, this is equivalent to

$$\exists K_v \geq 0 \quad : \quad \eta(v, \phi(w)p) + \kappa(\mathrm{Ad}_k A\phi(w)^{-1}v, H) \geq -K_v \qquad \forall w \in SL(2,\mathbb{C}),\ k \in K.$$

Write $\mathcal{O}_H$ for the adjoint orbit of H in $\mathfrak{k}$. Fix $w \in SL(2,\mathbb{C})$. Then we have

$$\begin{aligned}
&\eta(v, \phi(w)p) + \kappa(A\phi(w)^{-1}v, \mathcal{O}_H) \\
&= \eta(v, \phi(w)p) + \eta(\phi(w)^{-1}v, A^\star\mathcal{O}_H) \\
&= \eta(v, \phi(w)p) + \eta(v, \phi(w)A^\star\mathcal{O}_H) \\
&= \eta(v, \phi(w)(p + A^\star\mathcal{O}_H)).
\end{aligned}$$

Therefore, $p + A$ is of positive energy at $v \oplus M_v$ if and only if $\eta(v, \mathcal{O}_{p+A^\star X})$ is bounded from below for every $X \in \mathcal{O}_H$. □

Corollary 94. *Let $\nu = p + A \in \mathbb{R}^4 \oplus Hom(\mathbb{R}^4, \mathfrak{k})$ and let $H \in \mathfrak{t}$ be non-zero. Then $\mathcal{O}_\nu \subset \mathbb{R}^4 \oplus Hom(\mathbb{R}^4, \mathfrak{k})$ is of positive energy at all points in the cone* (5.4) *if and only if $p + A^\star\mathcal{O}_H \subseteq \overline{C}$, where $\mathcal{O}_H$ denotes the adjoint orbit of H in $\mathfrak{k}$.*

Proof. For fixed $X \in \mathcal{O}_H$, we know from proposition 91 that $\eta(v, \mathcal{O}_{p+A^\star X})$ is bounded from below at *all* points $v \in C$ if and only if $p + A^\star X \in \overline{C}$. Therefore, ν satisfies the positive energy condition at all points $v \oplus M_v \in C''$ if and only if $p + A^\star\mathcal{O}_H \subseteq \overline{C}$. □

Remark.

— It holds that $(\mathrm{Ad}_k A\phi(w)^{-1})^\star = \phi(w)A^\star\mathrm{Ad}_k^{-1}$, see also section 8.4. Indeed, using lemma 160 we know that $A^\star$ is the unique linear map satisfying $\kappa(Ap, X) = \eta(p, A^\star X)$ for every $p \in \mathbb{R}^4$ and $X \in \mathfrak{k}$. Thus, using the invariance of η and κ we compute that

$$\kappa(\mathrm{Ad}_k A\phi(w)^{-1}p, X) = \kappa(A\phi(w)^{-1}p, \mathrm{Ad}_k^{-1}X) = \eta(\phi(w)^{-1}p, A^\star\mathrm{Ad}_k^{-1}X) = \eta(p, \phi(w)A^\star\mathrm{Ad}_k^{-1}X).$$

The condition $p + A^\star\mathcal{O}_H \subseteq \overline{C}$ can be visualized quite explicitly, as is shown in figure 5.1. The value of $A^\star H$ can be interpreted as a perturbation of p and the perturbation $p + A^\star H$ is not allowed to go outside the closed light-cone. Notice that in these images all $A^\star\mathcal{O}_H$ has the shape of an ellipsoid, which is not the case in general.

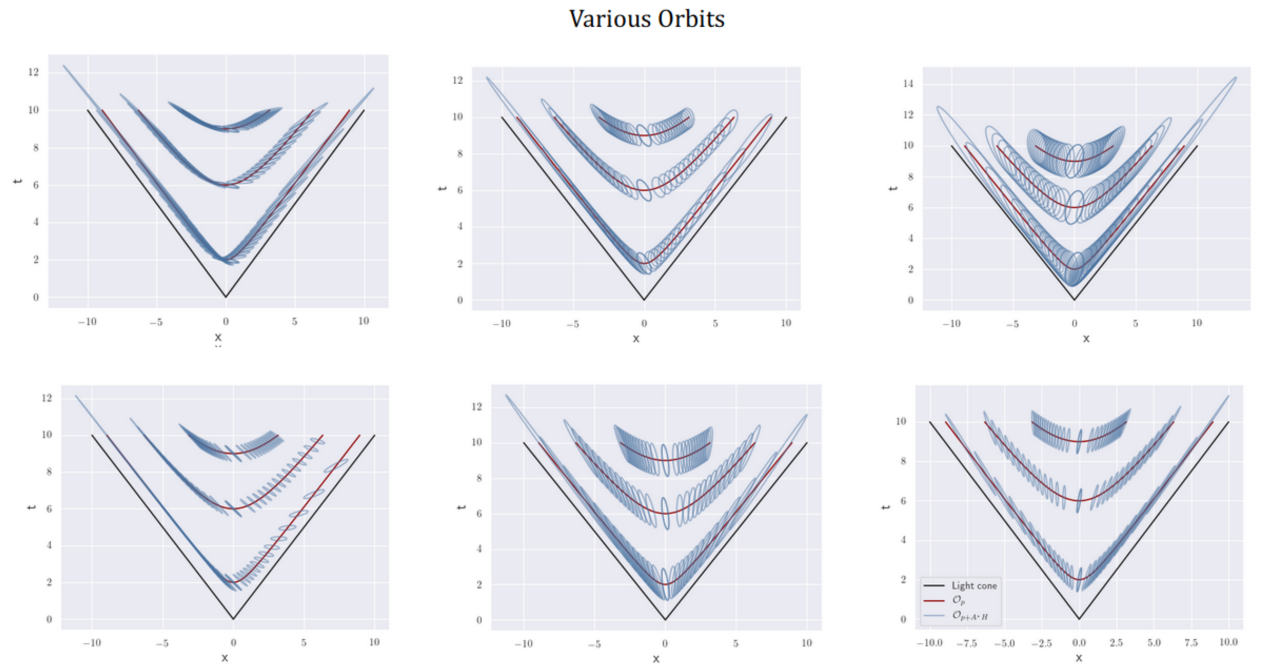


Figure 5.1: Two-dimensional representation of various orbits of $p + A^\star H$ that satisfy the positive energy condition at the cone (5.4).

Implications of the positive energy condition

Let us proceed with some further implications of the positive energy requirement. Notice first that the following are equivalent:

$$\begin{aligned} & p + A^\star \mathcal{O}_H \subseteq \overline{C}, \\ \iff \quad & A^\star \mathcal{O}_H \subseteq \overline{C} - p, \\ \iff \quad & A^\star \operatorname{conv} \mathcal{O}_H \subseteq \overline{C} - p, \end{aligned}$$

where $\operatorname{conv} S$ denotes the convex hull of S.

The Weyl group of the root system for $\mathfrak{t}$ may be defined as the quotient $W = N(\mathfrak{t})/Z(\mathfrak{t})$, where

$$\begin{aligned} Z(\mathfrak{t}) &= \{\, k \in K \,:\, \operatorname{Ad}_k(H) = H, \quad \forall H \in \mathfrak{t} \,\}, \\ N(\mathfrak{t}) &= \{\, k \in K \,:\, \operatorname{Ad}_k(H) \subset \mathfrak{t}, \quad \forall H \in \mathfrak{t} \,\}. \end{aligned}$$

The Weyl group acts on $\mathfrak{t}$ via $[k] \cdot X = \operatorname{Ad}_k(X)$. Moreover, the roots of $\mathfrak{t}$ in $\mathfrak{k}_{\mathbb{C}}$ are elements of $i\mathfrak{t}^*$. Because K is compact we may identify $i\mathfrak{t}^* \cong i\mathfrak{t} \cong \mathfrak{t}$ using κ. Thus, we will consider the roots of $\mathfrak{t}$ as elements in $\mathfrak{t}$ using these identifications. The fact will also be needed that if $\mathfrak{k}$ is simple, then the Weyl-group acts irreducibly on $\mathfrak{t}$, see also[Hum72, p. 53, 73]. In particular, this implies that the roots of $\mathfrak{t}$ span $\mathfrak{t}$.

Lemma 95. *Assume $\mathfrak{k}$ is simple and let $H \in \mathfrak{t}$ be arbitrary. Then the center of mass of the Weyl-orbit $\mathcal{O}_H^W \subset \mathfrak{t}$ is zero. That is,*

$$\frac{1}{|\mathcal{O}_H^W|} \sum_{Z \in \mathcal{O}_H^W} Z = 0.$$

Proof. Let Z_0 be the center of mass. Since every element of the Weyl-group maps $\mathcal{O}_H^W$ bijectively onto itself, Z_0 is an W-invariant element. In particular, for every root H_α the reflection s_α across the hyper plane $\ker \kappa(H_\alpha, \cdot)$ leaves Z_0 invariant. Then $Z_0 \perp H_\alpha$ for every root H_α. Since the roots span $\mathfrak{t}$, this implies that $Z_0 = 0$. □

Corollary 96. *Assume that $\mathfrak{k}$ is simple. The center of mass of any adjoint orbit $\mathcal{O}_H \subset \mathfrak{k}$ is zero. That is,*

$$\int_K Ad_k(H) d\mu(k) = 0,$$

where μ is the normalized Haar-measure on K.

Proof. Using the translation-invariance of the Haar-measure, we have

$$\int_K \mathrm{Ad}_k H d\mu(k) = \frac{1}{|\mathcal{O}_H^W|} \sum_{Z\in\mathcal{O}_H^W} \int_K \mathrm{Ad}_k Z d\mu(k) = \frac{1}{|\mathcal{O}_H^W|} \int_K \mathrm{Ad}_k \sum_{Z\in\mathcal{O}_H^W} Z d\mu(k) = 0.$$

□

As a consequence, we obtain the following necessary condition for a point $p + A \in \mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{k})$ to be of positive energy at the cone (5.4).

Corollary 97. *Assume $\mathfrak{k}$ is simple. If $p+A \in \mathbb{R}^4 \oplus Hom(\mathbb{R}^4, \mathfrak{k})$ is of positive energy at all points $v \oplus M_v \in C''$, then $p \in \overline{C}$.*

Proof. The center of mass of any adjoint orbit $\mathcal{O}_H$ is zero and therefore we must have $0 \in \mathrm{conv}\, \mathcal{O}_H$. From corollary 94 we know that $p + A$ is of positive energy at the cone $\in C''$ if and only if $A^\star \,\mathrm{conv}\, \mathcal{O}_H \subseteq \overline{C} - p$. Thus in particular we must have $0 \in \overline{C} - p$, or equivalently $p \in \overline{C}$. □

Lemma 98. *Assume $\mathfrak{k}$ is simple and let $H \in \mathfrak{t}$. Then either $H = 0$ or the interior of* $\mathrm{conv}\, \mathcal{O}_H$ *contains* 0.

Proof. Notice first $0 \in \mathrm{conv}\, \mathcal{O}_H$ because it is the center of mass of $\mathrm{conv}\, \mathcal{O}_H$, according to corollary 96.

Assume that $H \neq 0$. Observe that Span $\mathcal{O}_H^W$ is W-invariant. Since the Weyl group acts irreducibly on $\mathfrak{t}$, it follows that Span $\mathcal{O}_H^W = \mathfrak{t}$. Because K is compact, every adjoint orbit intersects $\mathfrak{t}$ in a Weyl-orbit, see also[ABH+80, p. 74]. This implies that Span $\mathcal{O}_H = \mathfrak{k}$. Indeed, any $X \in \mathfrak{k}$ may be written as $X = \sum_{Z\in\mathcal{O}_H^W} c_Z \mathrm{Ad}_k Z \in \mathrm{Span}\, \mathcal{O}_H$ for some $k \in K$.

It is well-known that a bounded closed convex set is the intersection of all the closed half-spaces containing it [HUL01, p. 56]. It follows that if 0 is on the boundary of $\mathrm{conv}\, \mathcal{O}_H$, then there exists a non-zero linear functional ϕ such that $\phi(\mathrm{conv}\, \mathcal{O}_H) \geq 0$. Let $V = (\ker \phi)^\perp$ and notice that V has codimension one. Let P_V be the κ-orthogonal projection onto V and let $E \in V$ be the unique element satisfying $\phi(E) > 0$ and $\kappa(E, E) = 1$. Notice that for any $X \in \mathcal{O}_H$, it holds that $\phi(X) = \kappa(X, E)\phi(E)$. Because $\phi(\mathrm{conv}\, \mathcal{O}_H) \geq 0$, it follows that for any $X \in \mathcal{O}_H$ we have $\kappa(X, E) \geq 0$. Observe further that Span $P_V\ \mathcal{O}_H = P_V$ Span $\mathcal{O}_H = P_V\mathfrak{k} = V$ and therefore there exists an element $k_0 \in K$ such that $\kappa(\mathrm{Ad}_{k_0}, E) > 0$. By continuity, there is an open neighborhood $U \subset K$ of k_0 such that for all $k \in U$ it holds that $\kappa(\mathrm{Ad}_k H, E) > 0$. Then we have

$$\int_K \kappa(\mathrm{Ad}_k H, E) d\mu(k) \geq \int_U \kappa(\mathrm{Ad}_k H, E) d\mu(k) > 0.$$

Extending E to a basis of $\mathfrak{k}$, this implies that the center of mass of $\mathcal{O}_H$ is nonzero. In view of corollary 96, we are done. □

Corollary 99. *Suppose that $\mathfrak{k}$ is semisimple and that $H \in \mathfrak{t}$ is not contained in any proper ideal of $\mathfrak{k}$. Then the interior of* $\mathrm{conv}\, H$ *contains* 0.

Proof. Suppose $\mathfrak{k} = \bigoplus_i \mathfrak{k}_i$ decomposes $\mathfrak{k}$ into finitely many simple subalgebras and $K = \prod_i K_i$, where K_i is the unique simply connected Lie group integrating $\mathfrak{k}$. Similarly, let $t = \bigoplus t_i$ be the corresponding decomposition of $\mathfrak{t}$ into maximal Abelian Lie subalgebras of $\mathfrak{k}_i$. By assumption, the projection H_i of H onto $\mathfrak{t}_i$ is non-zero for every i. Now lemma 98 applies to each of the adjoint orbits of H_i of the action of K_i on $\mathfrak{k}_i$, so that 0 is contained in the interior convex hull of every adjoint orbit $\mathcal{O}_{H_i}$. This implies the result. □

Remark.

— Observe that we may assume without loss of generality that each H is not contained in any proper ideal of $\mathfrak{k}$ by considering the smallest ideal of $\mathfrak{k}$ containing H along with its unique integrating simply connected Lie group. **We make this assumption from now on.**

Lemma 100. *Let $p \in \partial C$ be non-zero and let $V = p^\perp$ be the orthogonal complement of Span$\{p\}$ with respect to the standard inner product on $\mathbb{R}^4$. Then for any $v \in V$, at most one of $p + v$ and $p - v$ is contained in $\overline{C}$.*

Proof. Write $p = (p_t, p_x)$ for $p_t \in \mathbb{R}$ and $p_x \in \mathbb{R}^3$. Let p' be the element $p' = (p_t, -p_x)$. Observe that

$$\begin{aligned} \eta(p, p') &= \|p\|^2, \\ \eta(p', p') &= 0, \\ \langle p, p' \rangle &= \eta(p, p) = 0. \end{aligned} \tag{5.5}$$

Define $A = \operatorname{Span}\{p'\} \subset V$ and let $B = \{p, p'\}^\perp$ be the orthogonal complement of A in V. We obtain an orthogonal decomposition $V = A \oplus B$ such that for any $a \in A$, $b \in B$ we have $\eta(a,a) = 0$ and $\eta(p,b) = \eta(a,b) = 0$. This latter observation holds in view of (5.5) because the codimension of both $\ker \eta(p,\cdot)$ and $\ker \eta(p',\cdot)$ is one. As a consequence, the equations

$$\begin{aligned} \eta(p,b) &= p_t b_t - \langle p_x, v_x \rangle = 0, \\ \langle p,b \rangle &= p_t b_t + \langle p_x, v_x \rangle = 0, \end{aligned}$$

imply that $p_t b_t = 0$ and therefore $b_t = 0$ for any $b \in B$. This means that $\eta(b,b) = -\|b_x\|^2 < 0$. Finally, fix $v = a + b \in A \oplus B$. Then we compute

$$\begin{aligned} \eta(p+(a+b), p+(a+b)) &= 2\eta(p,a+b) + \eta(a+b,a+b) = 2\eta(p,a) + \eta(b,b) = 2\eta(p,a) - \|b_x\|^2 \\ \eta(p-(a+b), p+(a+b)) &= -2\eta(p,a+b) + \eta(a+b,a+b) = -2\eta(p,a) + \eta(b,b) = -2\eta(p,a) - \|b_x\|^2 \end{aligned}$$

Thus at least one of these two must be negative, which completes the proof. □

Lemma 101. *Suppose that $p \in \partial C$. If $p + A$ is of positive energy at all points $v \oplus M_v \in C''$, then A is in the orbit of $\eta(p,\cdot)X$ for some $X \in \mathfrak{k}$.*

Proof. We have seen 0 is in the interior of $\operatorname{conv} \mathcal{O}_H$ so $\operatorname{conv} \mathcal{O}_H$ contains an open ball around the origin. This ball is mapped by $A^\star$ to some ellipsoid E centered at the origin. (This can be seen using e.g. the singular value decomposition.) Let $P^\perp$ be the orthogonal projection onto $\operatorname{Span}\{p\}^\perp$. Suppose $v \in E$. Notice that also $-v \in E$. If $P^\perp v \neq 0$, then by lemma 100 either $p + P^\perp v$ or $p - P^\perp v$ is not contained in $\overline{C}$. It follows that $\operatorname{Im} A^\star \subseteq \operatorname{Span}\{p\}$ is necessary. This means that $A^\star$ is a rank-one operator so there exists some $X \in \mathfrak{k}$ such that $A^\star = \kappa(X,\cdot)p$ and $A = \eta(p,\cdot)X$ is necessary. □

The condition above is not sufficient. Indeed, if $A = \eta(p,\cdot)H_A$ for some $H_A \in \mathfrak{t}$, then $A^\star = \kappa(H_A,\cdot)p$ so $A^\star \mathcal{O}_H = \kappa(H_A, \mathcal{O}_H)p$. Therefore,

$$A^\star \mathcal{O}_H \subseteq \overline{C} - p \iff \kappa(H_A, \mathcal{O}_H) \geq -1. \tag{5.6}$$

Thus, let us proceed to obtain a necessary and sufficient condition for points $\nu = p + A$ with $p \in \partial C$ on the boundary of the light cone.

Definition 102. For any $X \in \mathfrak{k}$, let $\mathfrak{k}_X = \{Y \in \mathfrak{k} : [X,Y] = 0\}$ be the centralizer of X and let $\mathfrak{k}^X = [X, \mathfrak{k}]$ be the image of $\operatorname{ad}(X)$.

Notice that $\mathfrak{k}_X$ and $\mathfrak{k}^X$ fit in an exact sequence:

$$0 \to \mathfrak{k}_X \to \mathfrak{k} \xrightarrow{\operatorname{ad}(X)} \mathfrak{k}^X \to 0.$$

We need the following few lemmas, which are standard results in the theory of compact Lie algebras. The proofs are taken from [ABH+80].

Lemma 103. *For any $X \in \mathfrak{k}$, the spaces $\mathfrak{k}_X$ and $\mathfrak{k}^X$ are perpendicular complements with respect to any invariant inner product $\langle \cdot, \cdot \rangle$.*

Proof. Notice that by definition of an invariant inner product, we have

$$\langle [X,Z], \mathfrak{k} \rangle = -\langle Z, [X, \mathfrak{k}] \rangle,$$

The inclusion $(\mathfrak{k}^X)^\perp \subseteq \mathfrak{k}_X$ follows immediately. Since $\ker \operatorname{ad}(X) = \mathfrak{k}_X$, the rank nullity theorem applied to $\operatorname{ad}(X)$ states that $\dim(\mathfrak{k}_X) + \dim(\mathfrak{k}^X) = \dim(\mathfrak{k})$. It follows that $\dim((\mathfrak{k}^X)^\perp) = \dim(\mathfrak{k}_X)$. Therefore, we must have $(\mathfrak{k}^X)^\perp = \mathfrak{k}_X$. □

Lemma 104. *The map $f : \mathcal{O}_H \to \mathbb{R}$, $Z \mapsto \kappa(Z, H_A)$ attains its minimum at the centralizer $\mathfrak{k}_{H_A}$.*

Proof. Since K is compact, so is the adjoint orbit $\mathcal{O}_H$. It follows by continuity that f attains its minimum at, say, Z_0. Let $X \in \mathfrak{k}$ be arbitrary. Then the function $t \mapsto f(\operatorname{Ad}(\exp^{tX}(Z_0)))$ has a minimum at $t = 0$, hence

$$0 = \frac{d}{dt}\bigg|_{t=0} f\left(e^{t\,\operatorname{ad}(X)}(Z_0)\right) = \kappa\left(\frac{d}{dt}\bigg|_{t=0} e^{t\,\operatorname{ad}(X)}(Z_0), H_A\right) = \kappa\left([X, Z_0], H_A\right).$$

Therefore, $\kappa([\mathfrak{k}, Z_0], H_A) = -\kappa(Z_0, -[H_A, \mathfrak{k}]) = 0$. Thus, $Z_0 \in (\mathfrak{k}^{H_A})^\perp = \mathfrak{k}_{H_A}$. □

Lemma 105. *If $H \in \mathfrak{t}$ is a regular element, then $\mathfrak{k}_H = \mathfrak{t}$.*

Proof. Suppose that $X \in \mathfrak{k}$ is such that $[H, X] = 0$. Then, in the complexification $\mathfrak{k}_{\mathbb{C}}$, we may decompose X as $X = X_0 + \sum_{\alpha \in R} X_\alpha$ with $X_0 \in \mathfrak{t} + i\mathfrak{t}$ and $X_\alpha \in \mathfrak{k}_{\mathbb{C}\alpha}$. Then

$$0 = [H, X] = \sum_{\alpha \in R} \alpha(H) X_\alpha.$$

Since H is regular, $X_\alpha = 0$ for every root α. That is, $X = X_0 \in \mathfrak{t} + i\mathfrak{t}$. Since $X \in \mathfrak{k}$ we must have $X \in \mathfrak{t}$. □

Corollary 106. *Suppose that $H \in \mathfrak{h}$ is a regular element.*
The map $f : \mathcal{O}_H \to \mathbb{R}$, $Z \mapsto \kappa(Z, H)$ attains its minimum at the Weyl orbit $\mathcal{O}_H^W$.

Using these standard results, we can formulate for $p \in \partial C$ the following necessary and sufficient condition on satisfying the positive energy condition at all elements in C', so long as the element $H \in \mathfrak{t}$ defining the positive energy condition is regular.

Corollary 107. *Suppose that $p \in \partial C$ and let $A = \eta(p, \cdot)H_A$ be such that $H_A \in \mathfrak{t}$ is a regular element. Then $\nu = p + A$ satisfies the positive energy condition at all elements in the cone C'' if and only if $\kappa(H_A, \mathcal{O}_{-H}^W) \leq 1$.*

Proof. We have seen in equation (5.6) that $p + A^\star \mathcal{O}_H \subset \overline{C} \iff \kappa(H_A, \mathcal{O}_H) \geq -1 \iff \kappa(H_A, \mathcal{O}_{-H}) \leq 1$. From corollary 106 we know that $\kappa(H_A, \mathcal{O}_H)$ achieves its minimum at the Weyl-orbit $\mathcal{O}_H^W$, or equivalently $\kappa(H_A, \mathcal{O}_{-H})$ achieves its maximum at the Weyl-orbit $\mathcal{O}_{-H}^W$. □

Remark.

— The requirement $\kappa(H_A, \mathcal{O}_{-H}^W) \leq 1$ defines a system of linear inequalities; one for each point on the Weyl-orbit of $-H$ and therefore defines a polytope in $\mathfrak{t}$. A point $p + A$ as in corollary 107 satisfies the positive energy condition at the cone C'' if and only if H_A is contained in the convex hull of this polytope.

We have found that for $p \in \partial C$, the positive energy condition is very restrictive and enforces A to be a rank one operator. On the other hand, if $p \in C_0$ in the open cone, the situation is quite the opposite.

Lemma 108. *Suppose $p \in C_0$. Then for any $A \in Hom(\mathbb{R}^4, \mathfrak{k})$, there exists some constant $c > 0$ such that the positive energy condition is satisfied by $p + c \cdot A$ at all elements in the cone C''.*

Proof. Since $p \in C_0$, there exists an open ball $B_r(0)$ around the origin such that $B_r(0) \subseteq \overline{C} - p$. Notice that $\mathcal{O}_H$ is bounded since K is compact and $A^\star$ is a bounded operator. It follows that there is some ball $B_R(0)$ around the origin in $\mathbb{R}^4$ such that $A^\star \mathcal{O}_H \subseteq B_R(0)$. Then

$$(\frac{r}{R} \cdot A)^\star = \frac{r}{R} \cdot A^\star \mathcal{O}_H \subseteq \frac{r}{R} B_R(0) = B_r(0) \subseteq \overline{C} - p$$

and therefore the positive energy condition is satisfied for $p + \frac{r}{R} \cdot A$. □

Remark. Notice that the stabilizers of $c \cdot A$ and A are the same. It follows that for $p \in C_0$ the positive energy condition does not impose a restriction on the stabilizers.

5.2 Stabilizers of the action of $SL(2,\mathbb{C})\times SU(2)$ on $\mathbb{R}^4\oplus\mathbf{Hom}(\mathbb{R}^4,\mathfrak{su}(2))$

Consider the group (5.3) for the special case that $K = SU(2)$. By the theory of the Mackey machine, the representation theory of $V \rtimes H$ is completely classified by the representation theory of the various Little groups H_ν, which are the stabilizers of the action of H on $V \cong \widehat{V}$. As such, the first step in an understanding of the full representation theory of $V \rtimes H$ is to determine these stabilizers. The following is concerned with the classification of these stabilizers, up to equivalence, corresponding to orbits of positive energy. A full classification is obtained. First, let us make precise what is meant by equivalent stabilizers.

Definition 109. Let G be a Lie group. We call two closed subgroups H_1 and H_2 **equivalent** if there exists a Lie group isomorphism $G \xrightarrow{\lambda} G$ that restricts to an isomorphism $H_1 \to H_2$. We say that λ defines an equivalence between H_1 and H_2.

Remark.

1. Notice that an equivalence between two closed subgroups H_1 and H_2 of G in the sense of definition 109 also defines an isomorphism $(\lambda, \lambda|_{H_1})$ of principal bundles from $G \to G/H_1$ to $G \to G/H_2$.
2. In particular, this implies that for any unitary representation σ of H_2 on $\mathcal{H}_\sigma$, we obtain also a unitary representation $\sigma \circ \lambda|_{H_1}$ of H_1 on the same space. Moreover, because λ is G-equivariant we obtain an isomorphism of homogeneous Hilbert bundles from $G \times_{H_1} \mathcal{H}_\sigma \to G/H_1$ to $G \times_{H_2} \mathcal{H}_\sigma \to G/H_2$ given by $[x, v] \mapsto [\lambda(x), v]$. Therefore the corresponding induced representations are also equivalent.
3. If two closed subgroups of G are conjugate, then they are equivalent. In particular, the stabilizers of different points in the same orbits of a smooth G-action are always equivalent. However, points in different orbits can also be equivalent.

Now, we have seen that the positive energy condition imposes very strong restrictions on elements $\nu = p + A$ for p in the light-cone $p \in \partial C$, forcing A to be rank-one, which makes a computation of the stabilizer much simpler. On the other hand, for p of positive mass $p \in C_0$, it does not impose any restriction on the corresponding isomorphism classes of stabilizers, but the action can be restricted to a unitary action which simplifies the computation of stabilizers, in particular allowing us to make use of lemma 159. This dichotomy is therefore very useful when classifying the isomorphism classes of stabilizers corresponding the orbits of positive energy.

Explicitly, for $p \in C_0$ and after restricting the action to $SU(2) \times SU(2)$, we can either restrict the action further to $SU(2)\times U(1)$ or reduce the problem to a classification up to equivalence of the stabilizers of $\mathrm{End}(\mathbb{R}^3)$ under the action of $SU(2) \times SU(2)$. The latter can be done explicitly, which is mainly a consequence of the fact that rotations, their invariant subspaces and (complex) eigenvectors are well-understood. The relevant results on this matter are described in section 8.5.

5.2.1 Points $p \in \partial C$ on the boundary of the light cone

Assume that $p \in \partial C$ and $\nu = p + A \in \mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{su}(2))$ satisfies the positive energy condition at the cone C' (5.1) so that $A = \eta(p, \cdot)X$ for some $X \in \mathfrak{su}(2)$. In view of the results of section 4.5 we may assume that $p = (1, 0, 0, 1)$ by acting with $SL(2, \mathbb{C})$ appropriately.

Lemma 110. *Consider the action of $SL(2, \mathbb{C}) \times SU(2)$ on $\mathbb{R}^4 \otimes \mathfrak{su}(2)$ defined by $(w, k) \cdot p \otimes X = \phi(w)p \otimes Ad_k(X)$. Let $t = p \otimes X$ be a simple tensor. Then the stabilizer of t is the product $SL(2, \mathbb{C})_p \times SU(2)_X$.*

Proof. Notice that t is stabilized by (w, k) if and only if $\phi(w)p = \lambda p$, $\mathrm{Ad}_k(X) = \mu X$ and $\lambda \cdot \mu = 1$ for some $\lambda, \mu \in \mathbb{R}$. Now, since the adjoint action Ad is orthogonal with respect to the Killing form, we find that $\mu = \pm 1$ is necessary. Every element in the restricted Lorentz group $SO(1, 3)^0$ preserves the direction of time so only $\lambda = \mu = 1$ is possible. □

Now, recall from section 4.5 that

$$SL(2, \mathbb{C})_p = E = \left\{ \begin{pmatrix} z & a \\ 0 & \overline{z} \end{pmatrix} : z, a \in \mathbb{C}, \quad |z| = 1 \right\}.$$

Therefore, by lemma 110 the stabilizer of $\nu = p + \eta(p, \cdot)X$ is simply given by

$$G_\nu = E \times K_X.$$

Remark.

— There are only two possibilities for the stabilizer $SU(2)_X$. Indeed, we have seen in corollary 48 that the adjoint map becomes the covering homomorphism $SU(2) \to SO(3)$ under the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$. Thus, we must either have $SU(2)_X \overset{conj}{=} U(1)$ or $SU(2)_X = SU(2)$. It follows that the stabilizers of ν are up to equivalence given by $E \times U(1)$ and $E \times SU(2)$.

5.2.2 Points $p \in C_0$ in the interior of the light cone

Let $\nu = p + A \in \mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{su}(2))$. For $p \in C_0$ we can assume that $p = me_0$ for some $m > 0$ and from section 4.5 we know that $SL(2, \mathbb{C})_p = SU(2)$. Thus, we may restrict the action further to $SU(2) \times SU(2)$ and the stabilizer of ν in $SL(2, \mathbb{C}) \times SU(2)$ is the stabilizer of A in $SU(2) \times SU(2)$.

Now, we know from lemma 47 that if we embed $\mathbb{R}^3 \hookrightarrow \mathbb{R}^4$ via $x \mapsto (0, x)$, then the restriction of the covering homomorphism $SL(2, \mathbb{C}) \to SO(1, 3)^0$ to $SU(2)$ is precisely the covering homomorphism $SU(2) \xrightarrow{\phi} SO(3)$ and so the action of $SU(2)$ on $\mathbb{R}^4$ given by $\psi \overset{d}{=} 1 \oplus \phi$. On the other hand, recall from corollary 48 that $\mathfrak{su}(2) \cong \mathbb{R}^3$ and under this isomorphism, the adjoint action becomes the covering homomorphism $SU(2) \xrightarrow{\phi} SO(3)$. Thus, we are concerned with the action of $G = SU(2) \times SU(2)$ on $\mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3)$ given by

$$(u_1, u_2) \cdot A = \phi(u_2) \circ A \circ 1 \oplus \phi(u_1)^{-1}, \qquad u_1, u_2 \in SU(2).$$

Observe that $\mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3)$ decomposes as a $SU(2) \times SU(2)$-representation into two irreducible components according to

$$\mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3) \cong \mathbb{R} \otimes \mathbb{R}^3 \oplus \mathrm{End}(\mathbb{R}^3) \cong \mathbb{R}^3 \oplus \mathrm{End}(\mathbb{R}^3),$$

where the left factor $SU(2)$ acts trivially on $\mathbb{R}^3$ and the right factor $SU(2)$ acts on $\mathbb{R}^3$ via the covering map ϕ. Indeed, the intertwining map is given by

$$\begin{pmatrix} a & A_0 \end{pmatrix} \mapsto a \oplus A_0.$$

It follows in particular that given an element $A = a \oplus A_0 \in \mathbb{R}^3 \oplus \mathrm{End}(\mathbb{R}^3)$, its stabilizer is $G_A = G_a \cap G_{A_0}$. Now this decomposition allows us to distinguish two cases.

1. If $a = 0$ we obtain $G_A = G_{A_0}$ so the problem is reduced to finding the stabilizers of elements in $\mathrm{End}(\mathbb{R}^3)$, which can be done because the invariant subspaces of rotations are well-understood. The strategy in this case is to restrict the action further to stabilizers of $A_0^T A_0$ and $A_0 A_0^T$ to simplify the computations.
2. If $a \neq 0$, then $SU(2)_a = U_a(1) \cong U(1)$ covers the rotations about a.
   We find that $G_A \subseteq G_a = SU(2) \times U_a(1)$. We may thus restrict the action further to G_a and the stabilizer G_A can be determined directly.

— **Case** $a = 0$.

Let us first determine the stabilizers in $G = SU(2) \times SU(2)$ of elements of the form $A = (0, A_0)$ for some $A_0 \in \mathrm{End}(\mathbb{R}^3)$. In this case, the stabilizer of A equals that of A_0. Notice that $SU(2)\times SU(2)$ acts on $\mathrm{End}(\mathbb{R}^3)$ via

$$(u_1, u_2) \cdot A = \phi(u_2) \circ A \circ \phi(u_1)^{-1}, \qquad u_1, u_2 \in SU(2).$$

Observe that ϕ is orthogonal on $\mathbb{R}^3$. In view of lemma 159 it follows that $G_{A_0} \subset G_{A_0^T A_0} \cap G_{A_0 A_0^T}$ so that we may restrict the action further to this latter subgroup to simplify the computations, where $SU(2)$ acts on the symmetric matrices $\mathrm{End}(\mathbb{R}^3)_{sa}$ via $u \cdot M = \phi(u) \circ M \circ \phi(u)^{-1}$. As such, let us first consider the possible stabilizers in $SU(2)$ of $A_0^T A_0$ (and thus also of $A_0 A_0^T$).

Denote by $U_a(1) \subset SU(2)$ the subgroup of $SU(2)$ covering all rotations about $a \in S^2$. Similarly, denote by $U_a^t(1)$ the subgroup of $SU(2)$ covering all rotations $R_v(\pi)$ for some $v \in \mathrm{Span}\{a\}^\perp \subset \mathbb{R}^3$. From lemma 169 we know in that

$$U_{e_3}(1) = \left\{ \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} : z \in U(1) \right\},$$
$$U_{e_3}^t(1) = \left\{ \begin{pmatrix} 0 & u \\ \overline{u} & 0 \end{pmatrix} : u \in U(1) \right\}.$$

Lemma 111. *There is an isomorphism of groups*

$$U(1) \rtimes_\gamma \mathbb{Z}_2 \to U_{e_3}(1) \cup U_{e_3}^t(1),$$

is an isomorphism of groups, where $\gamma([n])z = z^{((-1)^n)}$.

Proof. Notice first that the set on the right is indeed closed under the group operations and that its subgroup $U_{e_3}(1)$ is normal. It is moreover clear that any element on the right-hand side admits a unique decomposition of the form $\begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^n$, so that the group on the right is indeed a semi-direct product $U(1) \rtimes \mathbb{Z}_2$. Finally, a direct computation shows the correct action of $\mathbb{Z}_2$ on $U(1)$. □

Remark.

— From now on, we will write $U(1) \rtimes_\gamma \mathbb{Z}_2$ for the subgroup $U_{e_3}(1) \cup U_{e_3}^t(1)$ of $SU(2)$.

Lemma 112. *Consider the action of* $SU(2)$ *on the symmetric linear maps* $\mathrm{End}(\mathbb{R}^3)_{sa}$ *given by*

$$u \cdot M = \phi(u) \circ M \circ \phi(u)^{-1}.$$

Let $M \in \mathrm{End}(\mathbb{R}^3)_{sa}$. *Then the stabilizer* $SU(2)_M$ *is in the sense of definition 109 equivalent to one of the following subgroups:*

$$SU(2),$$
$$U(1) \rtimes_\gamma \mathbb{Z}_2,$$
$$Q_8,$$

where $Q_8 = \{\pm I, \pm\sigma_1, \pm\sigma_2, \pm\sigma_3\}$.

Proof. A rotation $R \in SO(3)$ stabilizers M if and only if $[M, R] = 0$, which by lemma 157 is equivalent to the statement that R leaves all the eigenspaces of M invariant. The invariant subspaces of rotations are known precisely, see also corollary 168. This allows us to determine all rotations that stabilize M.

Let $u \in SU(2)$ be arbitrary and write $\phi(u) = R \in SO(3)$. The identity element is always in the stabilizer so we assume that $u \neq \pm I$ and thus $R \neq I$. By acting with $SU(2)$, we may assume that the $\{e_1, e_2, e_3\}$ are eigenvectors of M, which affects the stabilizer $SU(2)_M$ only by a conjugation. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of M.

— Case I: $\lambda_1 \neq \lambda_2 \neq \lambda_3$.
  By corollary 168, the only rotations that preserve all three one-dimensional eigenspaces are I and $R_{e_k}(\pi)$ for $k = 1, 2, 3$. By lemma 169, the elements of $SU(2)$ covering these rotations are $\phi^{-1}(I) = \pm I$ and $\phi^{-1}(R_{e_k}(\pi)) = \pm\sigma_k$ which form a subgroup of $SU(2)$ is isomorphic to quaternion group Q_8.

— Case II: $\lambda_1 = \lambda_2 \neq \lambda_3$.
In this case, a rotation is covered by an element in the stabilizer of M if and only if it preserves both the one- and two-dimensional eigenspaces $V_{\lambda_3} = \text{Span}\{e_3\}$ and $W \stackrel{d}{=} \text{Span}\{e_3\}^{\perp}$. Using corollary 168, we find that the only possible such rotations are $R_{e_3}(\theta)$ for some $\theta \in [0, 2\pi)$ or $R_w(\pi)$ with $w \in W$.

Now the stabilizer $SU(2)_M$ is generated by the elements covering these rotations, which we know from lemma 169 to be $U(1) \rtimes_{\gamma} \mathbb{Z}_2$.

— Case III: $\lambda_1 = \lambda_2 = \lambda_3$.
In this case the corresponding eigenspace is all of $\mathbb{R}^3$ which is trivially kept invariant by R. Thus in this case stabilizer of M is all of $SU(2)$.

□

Remark.

— Given a subgroup H of $SU(2)$, We will denote any subgroup of $G = SU(2) \times SU(2)$ of the form $\{(u, \pm u) : u \in H\}$ by $H \rtimes_{\beta} \mathbb{Z}_2$. This is justified by lemma 113.

— Similarly, we write $(U(1) \times U(1)) \rtimes_{\tau} \mathbb{Z}_2$ for the subgroup of $SU(2) \times SU(2)$ on the right-hand side of lemma 114

Lemma 113. *If H is a group, define $G = \{(h, \pm h) : h \in H\}$. Then we have an isomorphism of groups*

$$H \rtimes_{\beta} \mathbb{Z}_2 \cong G,$$

where $\beta([n])h = (-1)^n h$.

Proof. The isomorphism is given by $(h, [n]) \mapsto (h, (-1)^n h)$. □

Lemma 114. *There is an isomorphism of groups*

$$(U(1) \times U(1)) \rtimes_{\tau} \mathbb{Z}_2 \to U_{e_3}(1) \times U_{e_3}(1) \cup U^t_{e_3}(1) \times U^t_{e_3}(1),$$

where $\tau([n])(z_1, z_2) = (z_1^{((-1)^n)}, z_2^{((-1)^n)})$.

Proof. The proof is completely similar to that of lemma 111. □

Theorem 115. *Let $G = SU(2) \times SU(2)$ and consider its action on End$(\mathbb{R}^3)$ given by*

$$(u_1, u_2) \cdot A_0 = \phi(u_2) \circ A_0 \circ \phi(u_1)^{-1}.$$

Then for any $A_0 \in$ End$(\mathbb{R}^3)$, its stabilizer G_{A_0} is in the sense of definition 109 equivalent to one of the following subgroups:

$$\begin{aligned}
&SU(2) \times SU(2),\\
&(U(1) \times U(1)) \rtimes_{\tau} \mathbb{Z}_2,\\
&(U(1) \rtimes_{\gamma} \mathbb{Z}_2) \rtimes_{\beta} \mathbb{Z}_2,\\
&Q_8 \rtimes_{\beta} \mathbb{Z}_2,
\end{aligned}$$

where $Q_8 = \{\pm I, \pm\sigma_1, \pm\sigma_2, \pm\sigma_3\} \subset SU(2)$ and $\mathbb{Z}_2 = \{\pm I\} \subset SU(2)$.

Proof. Let $(e_i)_{i=1}^3$ be an orthonormal basis of $\mathbb{R}^3$. According to the singular value decomposition, there exists $u_0, u_1 \in SU(2)$ such that

$$A_0 = \phi(u_0) \circ \Sigma \circ \phi(u_1)^{-1},$$

where the matrix of Σ with respect to the basis $(e_i)_{i=1}^3$ is given by $\text{diag}(\sigma_1, \sigma_2, \sigma_3)$. Thus, by acting with $SU(2) \times SU(2)$ if necessary we may assume that with respect to the standard basis $(e_i)_{i=1}^3$ of $\mathbb{R}^3$, A_0 is given by

$$A_0 = \Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3),$$

affecting the stabilizer of A only by a conjugation. Then the matrix of both $A_0^T A_0$ and $A_0 A_0^T$ is the diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\lambda_k = \sigma_k^2$. In particular, their stabilizers in $SU(2)$ are the same. Moreover, notice that the permutation matrix

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is in $SO(3)$ and the matrix of $P\Sigma P^{-1}$ is given by $\mathrm{diag}(\sigma_3, \sigma_1, \sigma_2)$. Thus, by conjugating with P we may further permute the singular values in this cyclic fashion, which affects the stabilizer of A only by a conjugation.

Because the stabilizer of A must be contained in $F = SU(2)_{A_0^T A_0} \times SU(2)_{A_0 A_0^T}$, we restrict the action of G on $\mathrm{Hom}(\mathbb{R}^4, \mathbb{R}^3)$ further to F. The strategy is to determine the stabilizer A according to lemma 159:

$$G_T = \left\{ (x, y) \in F \,:\, \phi(y) \circ A_0|_{V_k} = A_0 \circ \phi(x)|_{V_k} \quad \forall k \right\}, \tag{5.7}$$

where all eigenspaces V_k of $A_0^T A_0$ are considered that correspond to a *non-zero* singular value σ_k. Seeing as we have assumed $A_0 = \Sigma$, this simplifies to

$$G_T = \left\{ (x, y) \in F \,:\, \phi(y)|_{V_k} = \phi(x)|_{V_k} \quad \forall k \right\}. \tag{5.8}$$

Now, the stabilizers of $SU(2)_{A_0^T A_0}$ and $SU(2)_{A_0 A_0^T}$ have already been determined in the section above. They are given by lemma 112. From section 8.4 we know that for elements in F the subspaces U_σ and V_σ spanned by the corresponding singular vectors are reducing subspaces for the action of F. Denote $V = \ker A_0^\perp = \mathrm{Im}(A_0)$.

— Case I: A rank 1.
We may assume that $\sigma_1 = \sigma_2 = 0$ and $\sigma_3 \neq 0$. In this case, $SU(2)_{A^T A} = SU(2)_{AA^T} = U(1) \rtimes_\gamma \mathbb{Z}_2$, $V = \mathrm{Span}\{e_3\}$. According to lemma 169, the rotations covered by this subgroup are $R_{e_3}(\theta)$ and $R_w(\pi)$ for some $w \in \mathrm{Span}\{e_1, e_2\}$. Now, we need to determine all $(x, y) \in F$ such that $\phi(y)e_3 = \phi(x)e_3$. Notice that $R_{e_3}(\theta)e_3 = e_3$ for all θ and $R_w(\pi)e_3 = -e_3$ for all $w \in \mathrm{Span}\{e_1, e_2\}$. We thus find that

$$G_A = (U(1) \times U(1)) \rtimes_\tau \mathbb{Z}_2.$$

— Case II: A rank 2.
We distinguish the cases when the two non-zero singular values are distinct and when they are not.

- Case A: $0 \neq \lambda_1 = \lambda_2$ and $\lambda_3 = 0$.
Then $V = \mathrm{Span}\{e_1, e_2\}$ and we once again have $SU(2)_{A^T A} = SU(2)_{AA^T} = U(1) \rtimes_\gamma \mathbb{Z}_2$. Notice that

$$\begin{aligned} R_{v_1}(\pi)|_V = R_{v_2}(\pi)|_V &\iff v_1 = \pm v_2 \qquad v_1, v_2 \in V, \\ R_{e_3}(\theta_1)|_V = R_{e_3}(\theta_2)|_V &\iff \theta_1 = \theta_2. \end{aligned}$$

It follows that $\phi(x)|_V = \phi(y)|_V$ if and only if $x = \pm y$. Thus,

$$G_A = (U(1) \rtimes_\gamma \mathbb{Z}_2) \rtimes_\beta \mathbb{Z}_2.$$

- Case B: $\lambda_1, \lambda_2 \neq 0$, $\lambda_3 = 0$ and $\lambda_1 \neq \lambda_2$.
Now $V = \mathrm{Span}\{e_1, e_2\}$ and $SU(2)_{A^T A} = SU(2)_{AA^T} = \{\pm I\} \cup \{\pm i\sigma_k\}_{k=1}^3 \cong Q_8$. For $(x, y) \in F$ we again have $\phi(x)|_V = \phi(y)|_V$ if and only if $x = \pm y$ and thus

$$G_A \cong Q_8 \rtimes_\beta \mathbb{Z}_2.$$

— Case III: A rank 3.
A computation completely similar to the previous two yields

$$\begin{aligned} \lambda_1 \neq \lambda_2 \neq \lambda_3 &\implies G_A \cong Q_8 \rtimes_\beta \mathbb{Z}_2, \\ \lambda_1 = \lambda_2 \neq \lambda_3 &\implies G_A \cong (U(1) \rtimes_\gamma \mathbb{Z}_2) \rtimes_\beta \mathbb{Z}_2, \\ \lambda_1 = \lambda_2 = \lambda_3 &\implies G_A \cong SU(2) \rtimes_\beta \mathbb{Z}_2. \end{aligned}$$

□

— **Case $a \neq 0$.**

Next, we proceed with the case that $A = \begin{pmatrix} a & A_0 \end{pmatrix}$ for some non-zero $a \in \mathbb{R}^3$ and some $A_0 \in \mathrm{End}(\mathbb{R}^3)$. In this case, the stabilizer $SU(2)_a = U_a(1) \cong U(1)$ covers the rotations about a. Therefore,

$$G_A = G_a \cap G_{A_0} \subseteq G_a = SU(2) \times U_a(1)$$

so we may restrict the action further to G_a. The various possible stabilizers of A_0 in G_a can be determined directly, which leads to the following theorem.

Theorem 116. *Let $G = SU(2) \times SU(2)$ and consider its action on $Hom(\mathbb{R}^4, \mathbb{R}^3)$ given by*

$$(u_1, u_2) \cdot A = \phi(u_2) \circ A \circ (1 \oplus \phi(u_1)^{-1}).$$

Then for any $A \in End(\mathbb{R}^3)$ of the form $A = (a, A_0)$ for some $A_0 \in End(\mathbb{R}^3)$ and non-zero $a \in \mathbb{R}^3$, its stabilizer G_A is in the sense of definition 109 equivalent to one of the following subgroups.

$$\begin{aligned} & SU(2) \times U(1), \\ & U(1) \times U(1), \\ & U(1) \times \{1\}, \\ & \left\{ (z, z^2) \in U_{e_3}(1) \times U(1) \right\}, \\ & \left\{ (z, \overline{z^2}) \in U_{e_3}(1) \times U(1) \right\}, \\ & \mathbb{Z}_2 \times \{1\}. \end{aligned}$$

where $\mathbb{Z}_2 = \{\pm I\} \subset SU(2)$.

Proof. By first acting with the right factor $SU(2)$, we may assume that $a = e_3$. Since the stabilizer of A must in particular stabilize a, we may restrict the action of G to $G_a = SU(2) \times U(1)$, where embed $U(1) \hookrightarrow SU(2)$ via $z \mapsto \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}$.

It remains to determine which elements in G_a also stabilize $A_0 \in \mathrm{End}(\mathbb{R}^3)$. Notice that

$$\mathrm{End}(\mathbb{R}^3) \cong \mathbb{R}^3 \otimes \mathbb{R}^3$$

is an equivalence of G_a-representations, where ${\mathbb{R}^3}^* \cong \mathbb{R}^3$ using the standard inner product on $\mathbb{R}^3$ and where G_a acts on a simple tensor $x \otimes y$ according to

$$(u, z) \cdot \phi(u)x \otimes \phi(z)y.$$

Moreover, because $U(1)$ leaves both the subspace $\mathrm{Span}\{e_3\}$ and its orthogonal complement invariant, $\mathbb{R}^3 \otimes \mathbb{R}^3$ decomposes as a G_a-representation into two irreducible components:

$$\mathbb{R}^3 \otimes \mathbb{R}^3 \cong (\mathbb{R}^3 \otimes \mathbb{R}^2) \oplus (\mathbb{R}^3 \otimes \mathbb{R}) \cong (\mathbb{R}^3 \otimes \mathbb{R}^2) \oplus \mathbb{R}^3.$$

Moreover, $\mathbb{R}^2 \cong \mathbb{C}$ as a $U(1)$-representation and therefore we obtain another equivalence of G_a-representations:

$$\mathbb{R}^3 \otimes \mathbb{R}^2 \cong \mathbb{R}^3 \otimes \mathbb{C} \cong \mathbb{C}^3.$$

Thus, we consider the action of G_a on $\mathbb{C}^3 \oplus \mathbb{R}^3$ given by

$$(u, z) \cdot (v, p) = (z\phi(u)v, \phi(u)p), \qquad (u, z) \in G_a, \quad (v, p) \in \mathbb{C}^3 \oplus \mathbb{R}^3.$$

It remains to determine the stabilizer in G_a of a general element $(v, p) \in \mathbb{C}^3 \oplus \mathbb{R}^3$.
Write $H = G_a = SU(2) \times U(1)$. Observe that $H_{(v,p)} = H_v \cap H_p$. Thus, let us determine both H_v and H_p separately. Fix $(v, p) \in \mathbb{C}^3 \oplus \mathbb{R}^3$. Notice that $H_p = SU(2)_p \times U(1)$ so that there are two possibilities:

$$\begin{aligned} p = 0 &\implies H_p = SU(2) \times U(1), \\ p \neq 0 &\implies H_p = U_p(1) \times U(1). \end{aligned}$$

On the other hand, an element $(u, z) \in H$ stabilizes $v \in \mathbb{C}^3$ if and only if $\phi(u)v = \overline{z}v$. Corollary 172 now yields the various possibilities. Suppose that $b \in \mathbb{R}^3$ and that w is a fixed eigenvector of the rotations covered by $U_b(1)$. Define the homomorphism $\lambda_w : U_b(1) \to U(1)$ by the equation:

$$\phi(u)w = \lambda_w(u)w, \qquad u \in U_b(1).$$

Using corollary 172, we find that we have the following possibilities:

$$\begin{aligned} v = 0 &\implies H_v = SU(2) \times U(1), \\ v \in \mathbb{R}^3 &\implies H_v = U_v(1) \times \{1\}, \\ v \perp \overline{v} &\implies H_v = \left\{ (u, \overline{\lambda_v(u)} \in U_b(1) \times U(1)) \right\}, \quad b \in \operatorname{Span}\{v, \overline{v}\}^{\perp}, \\ v \notin \mathbb{R}^3, v \not\perp \overline{v} &\implies H_v = \mathbb{Z}_2 \times \{1\}. \end{aligned}$$

Now, notice that $U_b(1)$ is conjugate to $U_{e_3}(1)$ and λ_v is conjugation invariant. We know from corollary 167 and lemma 169 that $\phi(z)e_3 = z^2 e_3$ for $z \in U(1)$. Therefore if $b \in \operatorname{Span}\{v, \overline{v}\}^{\perp}$, then $\left\{ (u, \overline{\lambda_v(u)} \in U_b(1) \times U(1) \right\}$ is conjugate to one of

$$\begin{aligned} &\left\{ (z, z^2) \in U_{e_3}(1) \times U(1) \right\}, \\ &\left\{ (z, \overline{z^2}) \in U_{e_3}(1) \times U(1) \right\}. \end{aligned}$$

Notice further that if $b \notin \operatorname{Span}\{e_3\}$, then $U_{e_3}(1) \cap U_b(1) = \{\pm I\}$. Taking all possible intersections $H_p \cap H_v$ now yields the result.

□

In summary, we have proven the following.

Theorem 117. *Let $G = SL(2, \mathbb{C}) \times SU(2)$ and consider its action on $\mathbb{R}^4 \oplus Hom(\mathbb{R}^4, \mathbb{R}^3)$ given by*

$$(w, u) \cdot p \oplus A = \phi(w)p \oplus (\phi(u) \circ A \circ (1 \oplus \phi(w)^{-1})).$$

Suppose $\nu = p \oplus A \in \mathbb{R}^4 \oplus Hom(\mathbb{R}^4, \mathbb{R}^3)$ satisfies the positive energy condition in the sense of definition 90 at the cone C'' given by equation (5.4). Write $A = \begin{pmatrix} a & A_0 \end{pmatrix}$ for some $a \in \mathbb{R}^3$ and $A_0 \in End(\mathbb{R}^3)$.

Then $p \in \overline{C}$. Moreover, if $p \in \partial C$, then $A = \eta(p, \cdot)X$ for some $X \in \mathfrak{su}(2)$ and the stabilizer G_ν is equivalent, in the sense of definition 109, to one of the following subgroups:

$$\begin{aligned} &E \times U(1), \\ &E \times SU(2). \end{aligned}$$

If $p \in C_0$ and $a = 0$ then G_ν is equivalent to one of the following subgroups:

$$\begin{aligned} &SU(2) \times SU(2), \\ &(U(1) \times U(1)) \rtimes_\tau \mathbb{Z}_2, \\ &(U(1) \rtimes_\gamma \mathbb{Z}_2) \rtimes_\beta \mathbb{Z}_2, \\ &Q_8 \rtimes_\beta \mathbb{Z}_2, \end{aligned}$$

If $p \in C_0$ and, $a \neq 0$ then G_ν is equivalent to one of the following subgroups:

$$\begin{aligned} &SU(2) \times U(1), \\ &U(1) \times U(1), \\ &U(1) \times \{1\}, \\ &\left\{ (z, z^2) \in U_{e_3}(1) \times U(1) \right\}, \\ &\left\{ (z, \overline{z^2}) \in U_{e_3}(1) \times U(1) \right\}, \\ &\mathbb{Z}_2 \times \{1\}. \end{aligned}$$

where $\mathbb{Z}_2 = \{\pm I\} \subset SU(2)$ and $Q_8 = \{\pm I, \pm\sigma_1, \pm\sigma_2, \pm\sigma_3\} \subset SU(2)$.

Recall that according to the theory of the Mackey machine, the representation theory of the group $V \rtimes H$ (5.3) for the case $K = SU(2)$ is completely determined by that of the various stabilizers H_ν of the action of H on $V \cong \widehat{V}$, meaning that every irreducible strongly continuous unitary representation of $V \rtimes H$ is obtained by inducing irreducible representations from the stabilizers H_ν up to the full group $V \rtimes H$. Theorem 117 completes a full classification, up to equivalence, of the stabilizers that correspond to representations of $V \rtimes H$ that are of positive energy at the cone C'' (5.4) and therefore of the positive energy representations of $V \rtimes H$. Moreover, the representation theory of the various stabilizers is completely known so that in fact theorem 117 completely solves the problem of classifying all irreducible strongly continuous unitary representations of $V \rtimes H$ that are of positive energy at the cone C'' for the special case of $K = SU(2)$.

Chapter 6

Homogeneous Bundles as Embedded Subbundles of Trivial Bundles

The homogeneous bundles that arise when constructing induced representations are defined in abstract fashion and are thus not easy to understand in more detail. When considering the representation theory of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$, Wigner showed[BW48] that the obtained bundles could be realized as subbundles of trivial bundles over $\mathbb{R}^4$, see also the appendix section 8.3 for a detailed exhibition. The fibers of these bundles are described by certain eigenvalue equations that exposed the fact that the Fourier transform (in an appropriate sense) of sections of these bundles satisfy particular wave equations. An example of an equation that can be realized in this way is the famous Dirac equation, which describes relativistic spin $\frac{1}{2}$ massive particles:

$$i\hbar \sum_{k=0}^{4} \gamma_k \partial_k \psi = mc\psi. \tag{6.1}$$

Embedding the bundle in a trivial bundle has played an important role in the understanding these homogeneous bundles, and therefore also of the unitary representations of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$. Nonetheless, despite the success in physics of the theory developed by Mackey and Wigner, it does not directly provide a method to embed other homogeneous bundles in trivial bundles. In fact, the method uses à priori the physical knowledge that solutions of the Dirac equation should define a representation of the group $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ to explicitly construct a homogeneous Hilbert bundle. One then proceeds to show that this bundle is equivalent to one of the bundles of the form $SL(2,\mathbb{C}) \times_{SU(2)} \mathcal{H}_\sigma$.

One could try to understand the unitary positive energy representations of the group

$$V \oplus (\mathfrak{k} \otimes V) \rtimes \mathrm{Spin}(r,s)^0 \times K$$

of positive energy in analogous fashion to Wigner's analysis for case of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$. This could potentially yield a description or differential equation of relativistic particles that takes into account notions such as electric charge or color charge. As a stepping stone towards such a similar analysis, a more direct method is developed that realizes homogeneous bundles as eigenspace subbundles of trivial bundles.

In more detail, using the examples obtained via representations of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ as guidance, the observation is made that in a particular equivariant setting, homogeneous bundles can be realized as embedded eigenspace subbundles of trivial bundles. Several examples are given of bundles that can be embedded in trivial bundles using this technique, including bundles encountered when applying the Mackey machine to representations of $V \rtimes \mathrm{Spin}(r,s)^0$ and $\mathfrak{k} \rtimes_{\mathrm{Ad}} K$ for some semisimple Lie group K with Lie algebra $\mathfrak{k}$. Finally, the bundles are considered that are encountered via the unitary representations of $V \oplus (\mathfrak{k} \otimes V) \rtimes \mathrm{Spin}(r,s)^0 \times K$.

6.1 Homogeneous bundles as eigenspace subbundles

6.1.1 First observations

Using as guidance the results obtained by Wigner on the representation theory of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ and its relation to wave equations, which are described in section 8.3, let us make some observations that provide further understanding on how one could try to realize homogeneous bundles as eigenspace subbundles.

Denote by $\phi : SL(2,\mathbb{C}) \to SO(3,1)^0$ the covering homomorphism described in equation (3.3) and let η denote the Minkowski bilinear form, as usual. Consider the homogeneous Hilbert bundle bundle

$$E = SL(2,\mathbb{C}) \times_{SU(2)} \mathbb{C}^2 \to \mathcal{O}_m$$

for some $m > 0$, where

$$\mathcal{O}_m = \{p \in \mathbb{R}^4 \,:\, \eta(p,p) = m^2, \quad p_0 > 0\}$$

is an orbit of the action of $SL(2,\mathbb{C})$ on $\mathbb{R}^4$. Let us first briefly describe the result obtained in section 8.3. Let $\{\gamma_r\}$ be the Dirac matrices defined by equation (3.6). Recall that these matrices define a representation ρ of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^4,\eta)$ and the restriction of this representation to $SL(2,\mathbb{C})$ defines a representation S of $SL(2,\mathbb{C})$ on the same space that satisfies the equivariance condition (6.2):

$$\rho(\phi(w)p) = S(w)\rho(p)S(w)^{-1}. \tag{6.2}$$

We know from section 8.3 that there exists an isomorphism of $SL(2,\mathbb{C})$-homogeneous Hilbert bundles over $\mathcal{O}_m$:

$$\begin{aligned} SL(2,\mathbb{C}) \times_{SU(2)} \mathbb{C}^2 \xrightarrow{\Phi} & \left\{ (p,v) \in \mathcal{O}_m \times \mathbb{C}^4 \,:\, \sum_{k=0}^{3} p_k \gamma_k v = mv \right\} \\ &= \left\{ (p,v) \in \mathcal{O}_m \times \mathbb{C}^4 \,:\, \rho(p)v = mv \right\}, \end{aligned}$$

where the latter is endowed with the $SL(2,\mathbb{C})$ action given by $w \cdot (p,v) = (\phi(w)p, S(w)v)$ and with the smoothly varying $SL(2,\mathbb{C})$-invariant hermitian bilinear form given by

$$v \mapsto m^{-1}\langle \gamma_0 v, v\rangle. \tag{6.3}$$

This form defines a positive definite inner product on the fibers of $\left\{ (p,v) \in \mathcal{O}_m \times \mathbb{C}^4 \,:\, \rho(p)v = mv \right\}$, see also section 8.3 for more details.

Now, observe the following:

1. The equivariance condition (3.4) implies that if Φ maps the fiber above some p_0 onto an eigenspace of $\rho(p_0)$, then in fact Φ maps the fiber above any $p \in \mathcal{O}_m$ onto the eigenspace of $\rho(p)$ with the same eigenvalue.

2. The restriction of the action of $SL(2,\mathbb{C})$ to $SL(2,\mathbb{C})_p \cong SU(2)$ on an arbitrary fiber E_p is equivalent to the representation σ of $SU(2)$ on $\mathbb{C}^2$. By the equivariance of Φ, the same is true for the action of $SL(2,\mathbb{C})_p$ on $\Phi(E)_p$. This means that we have $SU(2)$-equivariant maps $\mathbb{C}^2 \cong E_p \cong \Phi(E)_p \hookrightarrow \mathbb{C}^4$. Thus, the representation of $SU(2)$ on $\mathbb{C}^4$ contains σ as a subrepresentation.

3. In fact, suppose that G is a Lie group with closed subgroup H and let σ be a finite dimensional representation of H on $\mathcal{H}_\sigma$. If we have any homogeneous vector bundle $E = G \times_H \mathcal{H}_\sigma \to G/H$ and an injective G-equivariant morphism of vector bundles

   $$\Psi : G \times_H \mathcal{H}_\sigma \hookrightarrow G/H \times \mathcal{F}_\delta$$

   for some finite dimensional representation δ of G on $\mathcal{F}_\delta$, then there must be an injective H-equivariant map $\theta : \mathcal{H}_\sigma \to \mathcal{F}_\delta$. That is, $\delta|_H$ must contain σ as a subrepresentation.
   Indeed, the representation of H on any fiber E_{xH} is equivalent to σ, so there are H-equivariant maps

   $$\mathcal{H}_\sigma \cong E_{xH} \cong \Psi(E)_{xH} \hookrightarrow \{xH\} \times \mathcal{F}_\delta.$$

   It can therefore be concluded that for the existence of such a G-equivariant injective morphism of vector bundles $E \xrightarrow{\Psi} G/H \times \mathcal{F}_\delta$, it is a necessary condition that δ contains σ as subrepresentation.

6.1.2 Construction

Having in mind the last few observations in the previous section, we proceed with the converse, namely a construction of an isomorphism between a homogeneous vector bundles and an eigenspace subbundle of a trivial bundle, that works in a specific equivariant setting.

Suppose that G is a Lie group G with a closed subgroup H so that quotient space G/H is a smooth manifold and $G \to G/H$ is a principal H-bundle. Let σ be a finite dimensional unitary representation of H on $\mathcal{H}_\sigma$ and let $E = G \times_H \mathcal{H}_\sigma$ be the associated Hilbert bundle over G/H.

Suppose that we are given a finite dimensional representation δ of G on $\mathcal{F}_\delta$ and an injective H-equivariant linear map
$\theta : \mathcal{H}_\sigma \hookrightarrow \mathcal{F}_\delta$. Then the following bundle map is well defined

$$\begin{aligned} \Phi : E &\hookrightarrow G/H \times \mathcal{F}_\delta \\ [x, z] &\mapsto (xH, \delta(x)\theta(z)) \end{aligned}$$

Moreover, this map is smooth and injective and linear on fibers. Injectivity and linearity are clear. To see that it is smooth notice first that the map $(x, z) \mapsto \delta(x)\theta(z), G \times \mathcal{H}_\sigma \to \mathcal{F}_\delta$ is smooth, being the following composition of smooth mappings:

$$G \times \mathcal{H}_\sigma \xrightarrow{\mathrm{id}\times\theta} G \times \mathcal{F}_\delta \xrightarrow{\delta} \mathcal{F}_\delta.$$

It follows that $\phi : G \times \mathcal{H}_\sigma \to G/H \times \mathcal{F}_\delta, (x, z) \mapsto (xH, \delta(x)\theta(z))$ is smooth. Since the quotient mapping $q : G \times \mathcal{H}_\sigma \to E$ is a smooth submersion (see section 2.3.1), it follows that Φ is smooth.

Lemma 118. *The smooth map $\Phi : E \to G/H \times \mathcal{F}_\delta$ is a smooth embedding.*

Proof. Since Φ is an injective and smooth morphism of vector bundles, proposition 9 yields the result immediately. □

We endow $G/H \times \mathcal{F}_\delta$ with a G-action given by

$$g \cdot (xH, v) = (gxH, \delta(g)v).$$

Corollary 119. *The map Φ is a morphism of G-homogeneous vector bundles.*

Proof. It remains only to show Φ is G-equivariant. As such, let $x, g \in G$ and $z \in \mathcal{H}_\sigma$. We compute

$$\Phi([gx, z]) = (gxH, \delta(gx)\theta(z)) = g \cdot (xH, \delta(x)\theta(z)) = g \cdot \Phi([x, z]).$$

□

Next, we use this bundle morphism to transfer the inner product on the fibers of E to a smoothly varying G-invariant positive definite inner product on the image $\Phi(E) \subset G/H \times \mathcal{F}_\delta$. Notice that Φ is an isomorphism of G-homogeneous vector bundles $E \to \Phi(E)$, so by pulling back along Φ^{-1} we obtain a smoothly varying inner product on $\Phi(E)$. Explicitly, define

$$\langle \Phi([x, z_1]), \Phi([x, z_2]) \rangle_{\Phi(E)_{xH}} \stackrel{d}{=} \langle [x, z_1], [x, z_2] \rangle_{E_{xH}} = \langle z_1, z_2 \rangle_H. \tag{6.4}$$

Corollary 120. *With respect to the inner product* (6.4) *on the fibers of $\Phi(E)$, the map Φ is an injective morphism of G-homogeneous Hilbert bundles. In particular E is equivalent to $\Phi(E)$ as Hilbert bundles.*

Remark.

1. If it so happens that $\mathcal{F}_\delta$ is itself equipped with an inner product, the restriction of this inner product to $\Phi(E)$ in general does not equal the inner product defined by equation (6.4).

2. This is true, however, if the inner product on $\mathcal{F}_\delta$ is G-invariant and coincides with (6.4) on any fiber of $\Phi(E)$. In fact, it is enough to have a G-invariant bilinear form on $\mathcal{F}_\delta$ that agrees with (6.4) on any fiber of $\Phi(E)$, as is the case for the form (6.3).

3. Suppose that a G-invariant bilinear form restricts to an inner product on $\theta(\mathcal{H}_\sigma)$ and $\theta : \mathcal{H}_\sigma \to \theta(\mathcal{H}_\sigma)$ is isometric. Then the inner product on $\mathcal{F}_\delta$ automatically coincides with (6.4) on the fiber above the identity $\Phi(E)_H$, so by G-invariance the restriction of the bilinear form on $\mathcal{F}_\delta$ to $\Phi(E)$ agrees with (6.4) everywhere.

Endow End($\mathcal{F}_\delta$) with a G-action given by

$$g \cdot T \stackrel{d}{=} \delta(g) \circ T \circ \delta(g)^{-1}. \tag{6.5}$$

Lemma 121. *Suppose that $\rho : G/H \to End(\mathcal{F}_\delta)$ is a G-equivariant map and that $\theta(H_\sigma)$ is an eigenspace of $\rho(H)$ corresponding to some eigenvalue λ. Then*

$$\Phi(E) = \{ \, (xH, v) \in G/H \times \mathcal{F}_\delta \, : \, \rho(xH)v = \lambda v \, \} \, .$$

Proof. The last assumption implies that the fiber $\Phi(E)_H$ above the identity coset is precisely the eigenspace of $\rho(H)$. The G-equivariance of ρ allows us to translate this eigenvalue equation to all other fibers. Explicitly, let $x \in G$. We show that $\Phi(E)_{xH}$ is precisely the eigenspace of $\rho(xH)$ corresponding to the eigenvalue λ. As such, let $v = \delta(x)\theta(z) \in \Phi(E)_{xH}$. Then we indeed have

$$\rho(xH)v = \delta(x)\rho(H)\delta(x)^{-1}v = \lambda v.$$

To show the converse, notice that by homogeneity of $G \times_H \mathcal{H}_\sigma$, any fiber $\Phi(E)_{xH}$ is of the same dimension as $\Phi(E)_H$. Now, using the G-equivariance of ρ and the fact that conjugating by $\delta(x)$ preserves the dimension of eigenspaces, it follows by reasons of dimensionality that $\Phi(E)_{xH}$ is precisely the eigenspace of $\rho(xH)$ corresponding to eigenvalue λ. □

We summarize the result in a theorem.

Theorem 122. *Suppose H is closed subgroup of a Lie group G. Let δ be a finite dimensional representation of G on $\mathcal{F}_\delta$ and suppose that we are given a G-equivariant map $\rho : G/H \to End(\mathcal{F}_\delta)$. If an eigenspace $\mathcal{H}_\lambda$ of $\rho(H)$ with eigenvalue λ is invariant under the H-representation $\delta|_H$, then there is an equivalence of homogeneous vector bundles over G/H given by*

$$G \times_H \mathcal{H}_\lambda \cong \{ \, (xH, v) \in G/H \times \mathcal{F}_\delta \, : \, \rho(xH)v = \lambda v \, \} \, ,$$

where $G/H \times \mathcal{F}_\delta$ is equipped with the G-action given by $g \cdot (xH, v) = (gxH, \delta(g)v)$.

If $\mathcal{H}_\lambda$ is a Hilbert space and the action of H on $\mathcal{H}_\lambda$ is unitary with respect to this Hilbert space structure, then the above is an equivalence of homogeneous Hilbert bundles, where the fiber of the bundle on the right are endowed with the inner product given by equation (6.4)*.*

Remark.

— Consider once more the bundle $SL(2,\mathbb{C}) \times_{SU(2)} \mathbb{C}^2 \to \mathcal{O}_m$. Recall from section 4.5 that $SU(2)$ is the stabilizer $SL(2,\mathbb{C})_{p_0}$ of the point $p_0 = me_0$. Let $\rho : \mathrm{Cl}(1,3) \to \mathrm{End}(\mathbb{C}^4)$ be the representation of the Clifford algebra defined by the gamma matrices equation (3.6). Equation (6.2) states that ρ is $SL(2,\mathbb{C})$-equivariant. Now, $\rho(p_0) = m\gamma_0$ has two two-dimensional eigenspaces $V_{\pm m}$ of eigenvalues $\pm m$ given by $V_{\pm 1} = \left\{ \begin{pmatrix} v \\ \pm v \end{pmatrix} \, : \, v \in \mathbb{C}^2 \right\}$. Moreover, the restriction of the $SL(2,\mathbb{C})$-representation S to $SU(2)$ is given by

$$S(u) = \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}, \quad u \in SU(2).$$

In particular, both eigenspaces $V_{\pm m}$ are $SU(2)$-invariant so that $\mathbb{C}^4$ decomposes as $SU(2)$-representation into two the two irreducible components

$$\mathbb{C}^4 \cong V_m \oplus V_{-m}.$$

The $SU(2)$-representation on both V_m and V_{-m} is equivalent to the fundamental representation σ_1 of $SU(2)$ on $\mathbb{C}^2$. Theorem 122 now implies that we have equivalences of homogeneous Hilbert bundles

$$\begin{aligned} SL(2,\mathbb{C}) \times_{SU(2)} \mathbb{C}^2 &\cong \left\{ (p, v) \in \mathcal{O}_m \times \mathbb{C}^4 \, : \, \rho(p)v = mv \, \right\} \\ &\cong \left\{ (p, v) \in \mathcal{O}_m \times \mathbb{C}^4 \, : \, \rho(p)v = -mv \, \right\}. \end{aligned}$$

— In the upcoming sections, we will see that there is a natural way to apply theorem 122 to bundles that are encountered via the Mackey machine in the representation theory of the groups $V \rtimes \mathrm{Spin}(r,s)$ for some quadratic space (V, q) of signature (r, s), $\mathfrak{k} \rtimes_{\mathrm{Ad}} K$ for some semisimple compact Lie group K with Lie algebra $\mathfrak{k}$ and finally $V \oplus (\mathfrak{k} \otimes V) \rtimes \mathrm{Spin}(r,s) \times K$. In particular, theorem 122 can be applied to the bundles that occur in the representation theory of $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2,\mathbb{C}) \times K$, which is the main group of interest in this thesis.

6.1.3 Operations

When considering the category of unitary representations of a fixed group G, there are multiple ways to form a new representations from a given set of representations. In section 2.2.1 it was established that the functor $G \times_H -$ from finite dimensional unitary representations of H to homogeneous Hilbert bundles over G/H is compatible with various such operations. In this section we consider how a homogeneous bundle, obtained via one of the operations of vector bundles described in section 2.2.1, can be realized as a subbundle of a trivial bundle if the individual bundles are à priori known to be subbundles of trivial bundles.

The following lemma is a main tool to complete the aim of this section, which then follows by an application of lemma 29.

Lemma 123. *Let V, V_i be a vector spaces. Let $\rho \in End(V), \{\rho_i \in End(V_i)\}_{i=1}^N$ be finitely many operators on these spaces and let $E_\lambda, E_{\lambda_i} \subset V$ be the eigenspaces of these operators corresponding to some eigenvalues $\lambda, \lambda_i \in \mathbb{C}$. Then the following are linearly isomorphic*

$$\bigotimes_i E_{\lambda_i} \cong \left\{ t \in \bigotimes_i V_i \,:\, \rho_i^i t = \lambda_i t \quad \forall i \right\}, \tag{6.6}$$

$$\bigwedge^N E_\lambda \cong \left\{ t \in \mathcal{A}^N(V) \,:\, \rho^i t = \lambda t \quad \forall i \right\}, \tag{6.7}$$

$$\bigvee^N E_\lambda \cong \left\{ t \in \mathcal{S}^N(V) \,:\, \rho^i t = \lambda t \quad \forall i \right\}. \tag{6.8}$$

where $\mathcal{A}^N(V)$ and $\mathcal{S}^N(V)$ denote the subspaces of $V^{\otimes N}$ consisting of the alternating and symmetric tensors of order N and where

$$\rho_i^i = 1 \otimes \cdots 1 \otimes \rho_i \otimes 1 \cdots \otimes 1, \qquad (i^{th} \text{ factor}),$$
$$\rho^i = 1 \otimes \cdots 1 \otimes \rho \otimes 1 \cdots \otimes 1, \qquad (i^{th} \text{ factor}).$$

Moreover, of ρ is a normal operator, then the algebraic dual space E_λ^* is linearly isomorphic to $\{ \xi \in V^* \,:\, \rho^* \xi = \lambda \xi \}$.

Proof. For the latter two equations, notice that we can make the following identifications:

$$\bigwedge^N E_\lambda \cong \mathcal{A}^N(E_\lambda) \hookrightarrow \mathcal{A}^N(V),$$
$$\bigvee^N E_\lambda \cong \mathcal{S}^N(E_\lambda) \hookrightarrow \mathcal{S}^N(V).$$

After making this observation, the proof for first the three equations is completely similar, so only the first one is proven. Moreover, we consider for simplicity only the case $N = 2$. The argument extends without problems to the general case.

Let first $t \in E_{\lambda_1} \otimes E_{\lambda_2}$ be a simple tensor, say $v = v_1 \otimes v_2$. Then

$$\rho_1^1 t = \rho_1(v_1) \otimes v_2 = \lambda_1 v_1 \otimes v_2 = \lambda_1 t.$$

The same computation also shows $\rho_2^2 t = \lambda_2 t$. By linearity $\rho_i^i t = \lambda_i t$ follows also for general tensors.

Conversely, suppose that $t \in V_1 \otimes V_2$ is some arbitrary tensor satisfying $\rho_i^i t = \lambda t$ for $i = 1, 2$. Let $\{f_k\}$ be basis of V_2. Taking a basis expansion in the second component, we may write $t = \sum_k v_k \otimes f_k$ for some sequence $(v_k) \in V_1$. We have

$$0 = \rho_1^1 t - \lambda_1 t = \sum_k (\rho_1(v_k) - \lambda_1 v_k) \otimes f_k.$$

Since $\{f_k\}$ is a basis for V_2, this implies $\rho_1(v_k) = \lambda_i v_k$ for all k and thus $t \in E_{\lambda_1} \otimes V_2$. A similar argument shows $t \in E_{\lambda_1} \otimes E_{\lambda_2}$.

Finally, for the last statement observe that ρ is diagonalizable because it is normal. Thus, we may decompose V into its eigenspaces $V \cong \bigoplus_\mu V_\mu$. Let $v \in V_\mu$ and $\xi \in V^*$. Then it holds that

$$\langle \rho v, \xi \rangle = \mu \langle v, \xi \rangle = \lambda \langle v, \xi \rangle \iff \mu = \lambda \text{ or } \langle v, \xi \rangle = 0.$$

The conclusion follows.

□

Lemma 124. *Suppose δ is a continuous representation of a Lie group G on $\mathcal{F}$. Consider the representations of G on $\bigwedge^N \mathcal{F}$ and $\bigvee^N \mathcal{F}$ given by*

$$g \cdot v_1 \wedge \cdots \wedge v_N \stackrel{d}{=} \delta(g)v_1 \wedge \cdots \wedge \delta(g)v_N,$$
$$g \cdot v_1 \vee \cdots \vee v_N \stackrel{d}{=} \delta(g)v_1 \vee \cdots \vee \delta(g)v_N$$

If we endow $\mathcal{A}^N(\mathcal{F})$ and $\mathcal{S}^N(\mathcal{F})$ with the G-action $\delta^{\otimes N}$, then the linear isomorphisms $\bigwedge^N \mathcal{F} \xrightarrow{\text{Alt}} \mathcal{A}^N(\mathcal{F})$ and $\bigvee^N \mathcal{F} \xrightarrow{\text{Sym}} \mathcal{S}^N(\mathcal{F})$ induced by the symmetrization and skew-symmetrization maps are G-equivariant.

If moreover $\mathcal{F}$ is a finite dimensional Hilbert space, δ is unitary and the various spaces are equipped with the following inner products:

$$\langle v_1 \otimes \cdots \otimes v_N, v_1 \otimes \cdots \otimes v_N \rangle \stackrel{d}{=} \prod_{i=1}^{N} \langle v_i, w_i \rangle_{\mathcal{F}},$$
$$\langle v_1 \wedge \cdots \wedge v_N, v_1 \wedge \cdots \wedge v_N \rangle \stackrel{d}{=} \prod_{i=1}^{N} \langle v_i, w_i \rangle_{\mathcal{F}},$$
$$\langle v_1 \vee \cdots \vee v_N, v_1 \vee \cdots \vee v_N \rangle \stackrel{d}{=} \prod_{i=1}^{N} \langle v_i, w_i \rangle_{\mathcal{F}}.$$

Then the maps Alt *and* Sym *are unitary.*

Proof. Notice that the inverses of the linear isomorphisms Alt and Sym are simply given by

$$\mathcal{A}^N(\mathcal{F}) \to \bigwedge^N \mathcal{F}, \qquad \mathcal{S}^N(\mathcal{F}) \to \bigvee^N \mathcal{F},$$
$$v_1 \otimes \cdots \otimes v_N \mapsto v_1 \wedge \cdots \wedge v_N, \qquad v_1 \otimes \cdots \otimes v_N \mapsto v_1 \vee \cdots \vee v_N.$$

In both cases, it is clear that these maps are equivalences of G-representations and moreover are isometric with respect to the inner products as defined above, in the case that $\mathcal{F}$ is a Hilbert space. □

Proposition 125. *Let G be a Lie group with closed subgroup H. Let $\mathcal{H}$ and $\{\mathcal{H}_i\}_{i=1}^N$ be (unitary) finite dimensional representations of H and let δ, δ_i be (unitary) finite dimensional representations of G on $\mathcal{F}$ and $\mathcal{F}_i$. Let ρ and ρ_i be maps*

$$\rho : G/H \to End(\mathcal{F}),$$
$$\rho_i : G/H \to End(\mathcal{F}_i).$$

and assume they are G-equivariant, where $End(\mathcal{F})$ and $End(\mathcal{F}_i)$ are equipped with the G action as in equation (6.5). Suppose further that we have the following equivalences of homogeneous vector/Hilbert bundles:

$$G \times_H \mathcal{H} \cong \{(xH, v) \in G/H \times \mathcal{F} \,:\, \rho(xH)v = \lambda v\}, \qquad \lambda \in \mathbb{C}$$
$$G \times_H \mathcal{H}_i \cong \{(xH, v_i) \in G/H \times \mathcal{F}_i \,:\, \rho_i(xH)v_i = \lambda_i v_i\}, \qquad \lambda_i \in \mathbb{C},$$

where the right-hand side is equipped with a left G action as in theorem 122. Then we have the following equivalences of vector/Hilbert bundles:

$$G \times_H \bigoplus_{i=1}^{N} \mathcal{H}_i \cong \left\{ (xH, v) \in G/H \times \bigoplus_{i=1}^{N} \mathcal{F}_i \,:\, \rho_i(xH)v_i = \lambda_i v_i \quad \forall i = 1, \cdots, N \right\},$$
$$G \times_H \bigotimes_{i=1}^{N} \mathcal{H}_i \cong \left\{ (xH, t) \in G/H \times \bigotimes_{i=1}^{N} \mathcal{F}_i \,:\, \rho_i^i(xH)t = \lambda_i t \quad \forall i = 1, \cdots, N \right\},$$
$$G \times_H \bigwedge^N \mathcal{H} \cong \{ (xH, t) \in G/H \times \mathcal{A}^N(\mathcal{F}) \,:\, \rho^i(xH)t = \lambda t \quad \forall i = 1, \cdots, N \},$$
$$G \times_H \bigvee^N \mathcal{H} \cong \{ (xH, t) \in G/H \times \mathcal{S}^N(\mathcal{F}) \,:\, \rho^i(xH)t = \lambda t \quad \forall i = 1, \cdots, N \},$$

where

$$\rho_i^i(xH) = 1 \otimes \cdots 1 \otimes \rho_i(xH) \otimes 1 \cdots \otimes 1, \qquad (i^{th} \textit{ factor}),$$
$$\rho^i(xH) = 1 \otimes \cdots 1 \otimes \rho(xH) \otimes 1 \cdots \otimes 1, \qquad (i^{th} \textit{ factor})$$

and where the spaces on the right are equipped with the following G-actions:

$$g \cdot (xH, v) = (gxH, \oplus_i \delta_i(g) v_i), \qquad v \in \bigoplus_{i=1}^{N} \mathcal{F}_i,$$
$$g \cdot (xH, t) = (gxH, \delta_1(g) \otimes \cdots \otimes \delta_N(g) t), \qquad t \in \bigotimes_{i=1}^{N} \mathcal{F}_i,$$
$$g \cdot (xH, t) = (gxH, \delta(g)^{\otimes N} t), \qquad t \in \mathcal{A}^N(\mathcal{F}) \ \textit{ or } \ t \in \mathcal{S}^N(\mathcal{F}).$$

Moreover, if $\rho(xH)$ is a normal operator for every $xH \in G/H$, then the following is an isomorphism of vector/Hilbert bundles:

$$G \times_H \mathcal{H}^* \cong \{(xH, \xi) \in G/H \times \mathcal{F}^* \, : \, \rho(xH)^* \xi = \lambda \xi \}, \tag{6.9}$$

where the bundle on the right is equipped with the G-action given by $g \cdot (xH, \xi) = (gxH, \delta^\vee(g)\xi)$.

Proof. It is immediate from lemma 29 and lemma 123 that the above are isomorphisms of vector/Hilbert bundles and moreover the first two are G-equivariant and unitary on fibers in case of Hilbert bundles. By lemma 124 also the vector bundle isomorphisms regarding symmetric and exterior powers two are G-equivariant and unitary on fibers in the case of Hilbert bundles.

To be a bit more concrete, let us show the case of $G \times_H \bigwedge^N \mathcal{H}$ in detail. Write

$$E \stackrel{d}{=} \{(xH, v) \in G/H \times \mathcal{F} \, : \, \rho(xH) v = \lambda v \} \cong G \times_H \mathcal{H}.$$

There is a G-equivariant injective morphism of Hilbert bundles $E \hookrightarrow G/H \times \mathcal{F}$ and therefore also $\bigwedge^N E \hookrightarrow G/H \times \bigwedge^N \mathcal{F}$. Moreover, $G/H \times \bigwedge^N \mathcal{F} \cong G/H \times \mathcal{A}^N(\mathcal{F})$ as Hilbert bundles over G/H. By lemma 124 this latter equivalence is G-equivariant so that E is isomorphic as a Hilbert bundle to the image of the composition $\bigwedge^N E \hookrightarrow G/H \times \bigwedge^N \mathcal{F} \cong G/H \times \mathcal{A}^N(\mathcal{F})$. This image is computed fiber-wise in lemma 123. The conclusion follows.

□

Remark.

— Consider the setting as in proposition 125. Identify $\mathcal{F} \cong \mathbb{C}^N$ for some $N \in \mathbb{N}$. We may identify $\mathbb{C}^N$ with the dual space $(\mathbb{C}^N)^*$ using the linear isomorphism $v \mapsto \langle -, \overline{v} \rangle$. Under these identifications, equation (6.9) becomes

$$G \times_H \mathcal{H}^* \cong \left\{(xH, v) \in G/H \times \mathbb{C}^N \, : \, \rho(xH)^t v = \lambda v \right\}. \tag{6.10}$$

6.2 Bundles arising via $V \rtimes \mathbf{Spin}(r,s)^0$

The example $SL(2,\mathbb{C}) \times_{SU(2)} \mathbb{C}^2$ discussed in section 6.1.2 makes use of the fact that a representation ρ of the Clifford algebra $\mathrm{Cl}(\mathbb{R}^4,\eta)$ on some vector space satisfies the equivariance condition (3.4) and the fact that this representation is reducible to an eigenspace of $\rho(H)$. This construction can be extended to bundles encountered in the representation theory of $V \rtimes \mathrm{Spin}(r,s)^0$ via the Mackey machine, for some finite dimensional real quadratic space (V,q) of signature (r,s). We assume further that $SO(r,s) \cong SO(r,s)^0 \rtimes \mathbb{Z}_2$ so that according to corollary 83 the orbit space of $\widehat{V}$ is countably separated and so the theory of the Mackey machine does indeed apply.

Write $G = \mathrm{Spin}(r,s)^0$. We can use the quadratic form q to identify V with its dual V^*. Under this identification, the action of G on V^* is simply given by $w \cdot v = \phi(w)v$, where $\phi : \mathrm{Spin}(r,s) \to SO(r,s)$ denotes the covering homomorphism defined in theorem 41. Let α be any point in the orbit $\mathcal{O}$ and let $G_\alpha \subset G$ be the corresponding stabilizer subgroup. Identify $G/G_\alpha \cong \mathcal{O}_\alpha$ via $[w] \mapsto w \cdot \alpha$. We consider homogeneous Hilbert bundles of the form $G \times_{G_\alpha} \mathcal{H}_\sigma$, where σ is a finite dimensional representation of G_α on $\mathcal{H}_\sigma$.

Notice that $\mathcal{O} \subseteq V \hookrightarrow \mathrm{Cl}(r,s)$. So any finite dimensional representation ρ of the Clifford algebra on some vector space $\mathcal{F}$ allows us to interpret elements of V as operators on this space. Moreover, if we restrict this representation to G we obtain a representation $S \stackrel{d}{=} \rho|_G$ of the latter group satisfying the required equivariance condition (3.4). The following shows that for any finite dimensional representation ρ of $\mathrm{Cl}(r,s)$, we obtain a representation of G_α satisfying the necessary conditions of theorem 122.

Lemma 126. *Let $\rho : Cl(r,s) \to End(\mathcal{F})$ be a representation of $Cl(r,s)$ and set $S = \rho|_{Spin(r,s)^0}$. Let $\alpha \in V$ be arbitrary. Then $S|_{G_\alpha}$ leaves all eigenspaces of $\rho(\alpha)$ invariant.*

Proof. This follows by the $\mathrm{Spin}(r,s)^0$ by equivariance of ρ. Indeed, we have for $k \in G_\alpha$:

$$\rho(\alpha) = \rho(\phi(k)\alpha) = S(k)\rho(\alpha)S(k)^{-1} \qquad w \in G_\alpha \tag{6.11}$$

And therefore $\rho(\alpha)$ intertwines the representation $S|_{G_\alpha}$. □

Lemma 127. *For any finite dimensional representation ρ of $Cl(r,s)$ on some vector space $\mathcal{F}$ and for any $\alpha \in V$ satisfying $q(\alpha) \neq 0$, there is an eigenspace of $\rho(\alpha)$ with eigenvalue a square root of $q(\alpha)$ that is invariant under the G_α-representation $\rho|_{G_\alpha}$.*

Proof. Notice that any orbit $\mathcal{O}$ is contained in a level set $\mathcal{O} \subset q^{-1}(\{\lambda\})$ for some $\lambda \in \mathbb{C}$. Since $v^2 = q(v)I$ in $\mathrm{Cl}(r,s)$ we have for any $v \in \mathcal{O}:\ \rho(v)^2 = q(v)I = \lambda I$. It follows that $\rho(v)$ is diagonalizable and the spectrum of $\rho(v)$ is contained in $\{\pm m\}$ where $m := \sqrt{\lambda}$ is any choice of a (complex) square root of λ. (Diagonalizability of $\rho(v)$ follows since the minimal polynomial of $\rho(v)$ divides $x^2 - \lambda = (x-m)(x+m)$ and hence must be a product of distinct linear factors). The corresponding eigenspaces are invariant with respect to the action of the stabilizer subgroup G_α by lemma 126. Therefore, $\mathcal{F}$ decomposes as G_α-representation as $\mathcal{F} = \mathcal{F}_m \oplus \mathcal{F}_{-m}$. We obtain by restriction a non-trivial G_α-representation on at least one of the eigenspaces $\mathcal{F}_{\pm m}$. □

Proposition 128. *Suppose ρ is a finite dimensional representation of $Cl(r,s)$ on some vector space $\mathcal{F}$ and set $S = \rho|_{Spin(r,s)^0}$. Let $\alpha \in V$. Suppose that $E_\lambda \subset \mathcal{F}$ is a non-trivial eigenspace of $\rho(\alpha)$. In view of lemma 126, we may consider E_λ as a G_α-representation by restricting S to G_α. There is an equivalence of homogeneous vector bundles*

$$Spin(r,s)^0 \times_{G_\alpha} E_\lambda \cong \left\{ (v,\xi) \in \mathcal{O}_\alpha \times \mathcal{F} \ :\ \rho(v)\xi = \lambda\xi \right\}, \tag{6.12}$$

where $Spin(r,s)^0$ acts on the bundle on the right according to $w \cdot (v,\xi) = (\phi(w)v, S(w)\xi)$. If $\mathcal{F}$ is a Hilbert space and the action of G_α on E_λ is unitary, then this is an equivalence of homogeneous Hilbert bundles, where the fibers of the bundle on the right are endowed with the inner product given by equation (6.4).

Proof. The map $\rho : \mathrm{Cl}(r,s) \to \mathrm{End}(\mathcal{F})$ is $\mathrm{Spin}(r,s)^0$-equivariant, so theorem 122 yields the result immediately. □

Remark.

— The Clifford algebra $\mathrm{Cl}(r,s)$ has either one or two inequivalent irreducible representations [LM89, p. 32, theorem 5.7]. In the case of $\mathrm{Cl}(1,3)$, there is a single irreducible representation, which acts on $\mathbb{C}^4$ and is determined by the gamma matrices given in equation (3.6).

— Notice that in general, the representation of G_α on E_λ as in proposition 128 is not irreducible.

6.3 Bundles arising via $\mathfrak{k} \rtimes_{\mathbf{Ad}} K$

There is another family of homogeneous vector bundles for which theorem 122 applies, namely those encountered via the representation theory of $\mathfrak{k} \rtimes_{\mathrm{Ad}} K$ for a semisimple compact Lie group K with real Lie algebra $\mathfrak{k}$. These bundles are of independent interest; in particular they are relevant when considering the representation theory corresponding to the symmetry groups $SU(2)$ and $SU(3)$ related to the weak and strong interactions in particle physics. However, they also provide a useful stepping stone towards understanding bundles obtained via the representation theory of $V \oplus (\mathfrak{k} \otimes V) \rtimes \mathrm{Spin}(r,s)^0 \times K$.

The idea is that the Killing form allows us to identify the dual space $\mathfrak{k}^*$ of $\mathfrak{k}$ with $\mathfrak{k}$. Then for any representation of K, elements $\mathfrak{k}$ can be interpreted as operators on the same space using the infinitesimal representation and this satisfies the equivariance required to apply theorem 122.

Explicitly, let $\alpha \in \mathfrak{k}$ be any point and let K_α be the stabilizer of α under the adjoint action Ad of K on $\mathfrak{k}$. Identify K/K_α and $\mathcal{O}_\alpha$ via $[k] \mapsto \mathrm{Ad}_k(\alpha)$. We are interested in homogeneous Hilbert bundles over $\mathcal{O}_\alpha$ of the form $K \times_{K_\alpha} \mathcal{H}_\sigma$, where σ is a finite dimensional representation of K_α on $\mathcal{H}_\sigma$, with the aim of realizing such bundles as eigenspace subbundles of trivial bundles using theorem 122. As such, let δ be a finite dimensional representation of K on some vector space $\mathcal{F}_\delta$.

Since K is compact and semisimple, the Killing form defines an Ad-invariant inner product on $\mathfrak{k}$, allows us to identify $\widehat{\mathfrak{k}} \cong \mathfrak{k}^* \cong \mathfrak{k}$. Under these identifications, the action of K on the dual space $\widehat{\mathfrak{k}}$ transfers simply to the adjoint action on $\mathfrak{k}$. Moreover, $\mathcal{O} \subseteq \mathfrak{k}$ so we can interpret elements in the orbit as operators on $\mathcal{F}_\delta$ using the infinitesimal representation

$$\delta_*(X)v \overset{d}{=} \frac{d}{dt}\bigg|_{t=0} \delta(\exp(tX))v.$$

Then δ_* satisfies the required K-equivariance, because

$$\delta(k)\delta_*(\alpha)\delta(k)^{-1}v = \frac{d}{dt}\bigg|_{t=0} \delta(k\exp(t\alpha)k^{-1})v = \frac{d}{dt}\bigg|_{t=0} \delta(\exp(t\mathrm{Ad}_k(\alpha)))v = \delta_*(\mathrm{Ad}_k(\alpha))v. \tag{6.13}$$

If an eigenspace V_λ of $\delta_*(\alpha)$ is K_α-invariant under the representation $\delta|_{K_\alpha}$, then theorem 122 implies that there is an equivalence of homogeneous Hilbert bundles:

$$\mathcal{O}_\alpha \times_{K_\alpha} \mathcal{H}_\sigma \cong \Big\{ (X,v) \in \mathcal{O}_\alpha \times V \,:\, \delta_*(X)v = \lambda v \Big\}. \tag{6.14}$$

Remark.

- From lemma 84 we know that if K is a linear algebraic group defined over $\mathbb{R}$, then the adjoint action on its Lie algebra is an algebraic action defined over $\mathbb{R}$. Thus, lemma 71 implies that the orbit space of $\mathfrak{k}$ is countably separated. It follows that the strongly continuous representations of $\mathfrak{k} \rtimes_{\mathrm{Ad}} K$ are completely classified by that of the various stabilizers K_α via the induction procedure described in section 4.4. Observe that in particular $SU(N)$ is a linear algebraic group defined over $\mathbb{R}$.
- Consider now the special case in which $K = SU(N)$ for some N, a case which is of particular importance in gauge theories. The Lie algebra $\mathfrak{su}(N)$ consists of skew-Hermitian matrices with trace zero. In particular, all elements in $\mathfrak{su}(N)$ are diagonalizable so that lemma 157 implies that stabilizer of α are precisely those elements in $SU(N)$ that leave every eigenspace of α invariant.
- In particular, suppose that δ is the defining representation of $SU(N)$ on $\mathbb{C}^N$ so that $\delta_*(X) = X$. Then, seeing as any $X \in \mathfrak{su}(N)$ is diagonalizable, $\mathbb{C}^N$ decomposes as a G_α-representation according to

$$\mathbb{C}^N \cong \bigoplus_\lambda E_\lambda,$$

where E_λ is the eigenspace of α corresponding to the eigenvalue λ. By theorem 122, this implies that for any such E_λ we have an isomorphism of homogeneous Hilbert bundles

$$K \times_{K_\nu} E_\lambda \cong \big\{ (X,v) \in \mathcal{O}_\alpha \times \mathbb{C}^N \,:\, Xv = \lambda v \big\},$$

where the bundle on the right is equipped with the $SU(N)$-action given by $u \cdot (X,v) = (\mathrm{Ad}_u X, uv)$ and its fibers are endowed with the inner product inherited from $\mathbb{C}^N$.

6.3.1 Adjoint orbits

The bundles discussed in section 6.3 are bundles over adjoint orbits of K on $\mathfrak{k}$. A detailed understanding of these bundles requires in particular further understanding of these adjoint orbits. It turns out that they can be classified by elements in a positive Weyl-chamber and that they are an algebraic variety. This means in particular that sections of bundles over the orbits satisfy $p(X)s(X) = 0$ for some polynomials $p \in \mathcal{S}(\mathfrak{k}^*)$. If we can take a Fourier transform, in an appropriate sense, this would mean that the sections satisfy $D_p\widehat{s} = 0$ for some differential operators D_p. This section is merely a collection of relevant results. We thus mention them without proof, giving references instead.

Theorem 129. *Let $Y \in \mathfrak{k}$ and fix a maximal Abelian subalgebra $\mathfrak{t}$. Let $\mathcal{O}_Y$ be the adjoint orbit of Y in $\mathfrak{k}$. That is, $\mathcal{O}_Y = Ad(G)Y$. Then the intersection $\mathcal{O}_Y \cap \mathfrak{t}$ is an orbit of the Weyl group.*

Proof. A proof can be found in [ABH+80, p. 74] □

Since the Weyl group acts transitively on the set of Weyl-chambers [Hum72, p.51], it follows that every Weyl-orbit intersects the closure $\overline{C}$ of a fixed Weyl-chamber in some point. By theorem 129 the same holds for a general adjoint orbit $\mathcal{O}_Y$, which means that these orbits are classified by points in $\overline{C}$. Orbits that intersect $\overline{C}$ in its interior are called **non-degenerate** or **generic**, whereas ones that intersect $\overline{C}$ on the boundary are called **degenerate**. See also figure 6.1, below.

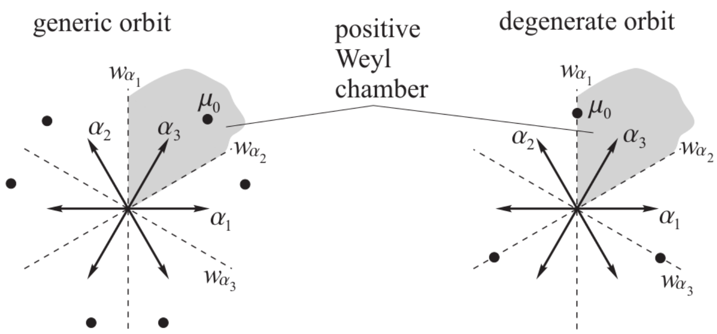


Figure 6.1: Root diagram for $SU(3)$ and illustration of degenerate and generic orbits. The black dots indicate the Weyl orbit of μ_0. Reprinted from [BH08, p. 4].

Because K is compact, the adjoint orbits are an algebraic subvariety of $\mathfrak{k}$ [Cro17, corollary 5.5]. This variety can be made explicit using the invariant polynomials $\mathcal{S}(\mathfrak{k}^*)^K$. Notice first that by K-invariance, any $p \in \mathcal{S}(\mathfrak{k}^*)^K$ is constant on the adjoint orbits. We begin with a definition.

Definition 130. A polynomial function of the form $x \mapsto \mathrm{tr}(\pi(x)^k)$ for some irreducible finite dimensional representation π of $\mathfrak{k}$ and some integer k is called a **trace polynomial**.

Theorem 131. *The map*

$$\gamma : \mathcal{S}(\mathfrak{k}^*)^K \to S(\mathfrak{h}^*)^W,$$
$$\gamma : p \mapsto p|_{\mathfrak{h}}$$

is an algebra isomorphism. Moreover, $\mathcal{S}(\mathfrak{k}^*)^K$ is generated by $\dim \mathfrak{h}$ *algebraically independent trace polynomials (and the unit).*

Proof. A proof of the statement that γ is an algebra isomorphism and $\mathcal{S}(\mathfrak{k}^*)^K$ is generated by trace polynomials can be found in [Hum72, p. 127-128]. Because W is a finite reflection group, a theorem by Chevalley[Che55] implies that the $\mathfrak{h}$-algebra $\mathcal{S}(\mathfrak{h})^*)^W$ is generated by $\dim \mathfrak{h}$ algebraically independent homogeneous polynomials, which completes the proof. □

Lemma 132. *Let $H_1, H_2 \in \mathfrak{h}$ lie in distinct Weyl-orbits.*
Then there exists $p_{\mathfrak{h}} \in \mathcal{S}(\mathfrak{h}^*)^W$ such that $p_h(H_1) \neq p_h(H_2)$.

Proof. For a proof of this statement, see [Hum72, p. 131]. □

The previous two lemmas imply that the trace polynomials separate the adjoint orbits of $\mathfrak{k}$ and it follows immediately that

$$\mathcal{O}_H = \left\{ X \in \mathfrak{k} : p(X) = p(H) \quad \forall p \in \mathcal{S}(\mathfrak{k}^*)^K \right\}.$$

6.4 Concrete examples

6.4.1 $\mathfrak{su}(2) \rtimes_{\mathbf{Ad}} SU(2)$

In this section, we apply the techniques developed in the previous sections to the homogeneous bundles encountered when applying the Mackey machine to $\mathfrak{su}(2) \rtimes_{\mathrm{Ad}} SU(2)$, where $\mathfrak{su}(2)$ is equipped with the Killing form κ. Before starting, we make a the observation this case falls under both the settings considered in section 6.2 and section 6.3.

Lemma 133. *There is an isomorphism of Lie algebras*

$$\mathfrak{su}(2) \cong \mathbb{R}^3,$$

where $\mathbb{R}^3$ is considered as a Lie algebra equipped with the cross product $\times$ as Lie bracket.

Proof. The identification is given by

$$\mathfrak{su}(2) \to \mathbb{R}^3$$
$$v_x = \sum_{k=1}^{3} x_k i\sigma_k \mapsto \sum_{k=1}^{3} x_k e_k = x,$$

where $\{\sigma_i\}$ are the Pauli matrices described in equation (3.1). $\square$

Now, recall from corollary 48 that $\mathrm{Spin}(3) \cong SU(2)$ and that covering homomorphism ϕ extends to the adjoint action on $\mathrm{Cl}(\mathbb{R}^3)$. Therefore, the previous lemma implies that there is an isomorphism of groups

$$\mathbb{R}^3 \rtimes_\phi SU(2) \cong \mathfrak{su}(2) \rtimes_{\mathrm{Ad}} SU(2).$$

Notice that the standard inner product on $\mathbb{R}^3$ is invariant under the cross-product (in the sense that $\langle x \times y, z\rangle + \langle y, x \times z\rangle = 0$) and $\mathbb{R}^3$ is simple, so that the standard inner product must be a multiple of the Killing form on $\mathbb{R}^3$. Identify $\widehat{\mathbb{R}^3}$ with $\mathbb{R}^3$ according to

$$\langle x, p\rangle = e^{i\langle x,p\rangle_{\mathbb{R}^3}}.$$

Under this identification and by the invariance of the inner product, $SU(2)$ acts on $\widehat{\mathbb{R}^3}$ by $u \cdot x = \phi(u)x$.

The line spanned by e_3 is a maximal Abelian subalgebra (as is any other one-dimensional subspace of $\mathbb{R}^3$). A positive Weyl chamber is $\overline{C} = \mathbb{R}_{\geq 0}$ and the points in $\overline{C}$ parametrize the adjoint orbits. The origin corresponds to the only degenerate case and non-zero multiples elements represent non-degenerate orbits.

Since $SO(3)$ acts by rotations, we know that the orbits are just spheres mS^2 for some $m > 0$. We can obtain the same result using the procedure described in section 6.3.1. Indeed, consider the defining representation of $\mathfrak{su}(2)$ on $\mathbb{C}^2$ and let $X = \sum_{k=1}^{3} x_k i\sigma_k$ be an element in $\mathfrak{su}(2)$, then $\mathrm{tr}(X) = 0$ and $\mathrm{tr}(X^2) = -2(x_1^2 + x_2^2 + x_3^2)$ so that the orbits are described by level sets of the invariant polynomial $x_1^2 + x_2^2 + x_3^2$.

Now, fix the point $\alpha = (0, 0, m) \in \overline{C}$. By the last remark in section 6.3, the stabilizer of α are those elements of $SU(2)$ that leave invariant all the eigenspaces of $im\sigma_3$. Thus, the stabilizer is the one-dimensional torus $U(1) \hookrightarrow SU(2)$, where the latter embedding is given by $z \mapsto \begin{pmatrix} z & 0 \\ 0 & \overline{z} \end{pmatrix}$.

The induced representations are constructed using associated homogeneous Hilbert bundles of the form

$$SU(2) \times_{U(1)} \mathcal{H} \to \mathcal{O}_m,$$

for some finite dimensional representation of $U(1)$ on $\mathcal{H}$. Recall that the irreducible representations of $U(1)$ are all one-dimensional and are given by $\sigma_n : z \mapsto z^n$ with $n \in \mathbb{Z}$.

In case of the trivial representation σ_0, the above bundle becomes trivial

$$SU(2) \times_{U(1)} \mathbb{C}_0 \cong SU(2)/U(1) \times \mathbb{C}_0 \cong \mathcal{O}_m^2 \times \mathbb{C}_0,$$

so the corresponding induced representation is the left regular representation on $L^2(S^2, \mathbb{C}_0)$.

Next, for any $n \in \mathbb{N}$ we have $\sigma_n \cong \sigma_1^{\otimes n}$ and $\sigma_{-n} \cong \sigma_{-1}^{\otimes n}$. Therefore lemma 29 implies that for any $n \in \mathbb{N}$:

$$SU(2) \times_{U(1)} \mathbb{C}_n \cong (SU(2) \times_{U(1)} \mathbb{C}_1)^{\otimes n},$$
$$SU(2) \times_{U(1)} \mathbb{C}_{-n} \cong (SU(2) \times_{U(1)} \mathbb{C}_{-1})^{\otimes n}.$$

By 125 it suffices to consider to the bundles $SU(2) \times_{U(1)} \mathbb{C}_1$ and $SU(2) \times_{U(1)} \mathbb{C}_{-1}$. Now, consider the the defining representation δ of $SU(2)$ on $\mathbb{C}^2$ so that $\delta_*(X) = X$ for $X \in \mathfrak{su}(2)$. Its restriction to $U(1)$ decomposes as $U(1)$-representation according to

$$\mathbb{C}_2 \cong \mathbb{C}_1 \oplus \mathbb{C}_{-1}.$$

Moreover, these invariant subspaces are precisely the eigenspaces of α corresponding to the eigenvalues $\pm im$. We find that we have equivalences of homogeneous Hilbert bundles:

$$SU(2) \times_{U(1)} \mathbb{C}_1 \cong \left\{ \ (X, v) \in \mathcal{O}_\alpha \times \mathbb{C}^2 \ : \ Xv = imv \ \right\},$$
$$SU(2) \times_{U(1)} \mathbb{C}_{-1} \cong \left\{ \ (X, v) \in \mathcal{O}_\alpha \times \mathbb{C}^2 \ : \ Xv = -imv \ \right\},$$

where both these bundles are equipped with the $SU(2)$-action given by $u \cdot (X, v) = \mathrm{Ad}_u(X, uv)$. Notice further that the inner product on these bundles inherited from $\mathbb{C}^2$ is $SU(2)$-invariant and coincides with the inner product induced from $SU(2) \times_{U(1)} \mathbb{C}_1$ on the fiber above α, because θ is isometric. Therefore, the two are equal on every fiber. Moreover, under the identification $\mathfrak{su}(2) \cong \mathbb{R}^3$ above, these become

$$SU(2) \times_{U(1)} \mathbb{C}_1 \cong \left\{ \ (X, v) \in \mathcal{O}_\alpha \times \mathbb{C}^2 \ : \ \sum_{k=1}^{3} p_k \sigma_k v = mv \ \right\},$$
$$SU(2) \times_{U(1)} \mathbb{C}_{-1} \cong \left\{ \ (X, v) \in \mathcal{O}_\alpha \times \mathbb{C}^2 \ : \ \sum_{k=1}^{3} p_k \sigma_k v = mv \ \right\}.$$

Now, using 125 we find that there are equivalences of homogeneous Hilbert bundles:

$$SU(2) \times_{U(1)} \mathbb{C}_n \cong \left\{ (p, t) \in \mathcal{O}_m \times \mathbb{C}^{2^{\otimes n}} \ : \ \sum_{k=1}^{3} p_k \sigma_k^\nu t = mt, \quad \nu = 1, \cdots, n \right\},$$
$$SU(2) \times_{U(1)} \mathbb{C}_{-n} \cong \left\{ \ (p, t) \in \mathcal{O}_m \times \mathbb{C}^{2^{\otimes n}} \ : \ \sum_{k=1}^{3} p_k \sigma_k^\nu t = -mt, \quad \nu = 1, \cdots, n \ \right\},$$

where $\sigma_k^\nu = 1 \otimes \cdots \otimes 1 \otimes \sigma_k \otimes 1 \otimes \cdots \otimes 1$.

Finally, notice that the $U(1)$-representation $\mathbb{C}_{-1}$ is equivalent to the dual representation $\mathbb{C}_{-1} \cong \mathbb{C}_1^*$. Thus, we may as well have applied equation (6.10) to find

$$SU(2) \times_{U(1)} \mathbb{C}_{-1} \cong \left\{ \ (X, v) \in \mathcal{O}_\alpha \times \mathbb{C}^2 \ : \ X^t v = imv \ \right\},$$

where the bundle on the right is now equipped with the $SU(2)$-action $u \cdot (X, v) = (\mathrm{Ad}_u(X), \overline{u} v)$.

6.4.2 $\mathfrak{su}(3) \rtimes_{\mathbf{Ad}} SU(3)$

Next, we consider the case where $K = SU(3)$ and $\mathfrak{k} = \mathfrak{su}(3)$. Notice that the complexification of $\mathfrak{su}(3)$ is $\mathfrak{sl}(3,\mathbb{C})$. The adjoint action of $SU(3)$ on $\mathfrak{su}(3)$ uniquely extends to an action on $\mathfrak{sl}(3,\mathbb{C})$. Since $\mathfrak{su}(3)$ is invariant under the adjoint action of $SU(3)$ on $\mathfrak{sl}(3,\mathbb{C})$, we can perform the computations in $\mathfrak{sl}(3,\mathbb{C})$ and consider the $SU(3)$-orbits in $\mathfrak{sl}(3,\mathbb{C})$ that lie within $\mathfrak{su}(3)$. A maximal Abelian subalgebra $i\mathfrak{t}$ of $\mathfrak{su}(3)$ is given by the span of iH_1 and iH_2, where

$$H_1 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_3 \end{pmatrix}.$$

Define the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} \overset{d}{=} \mathfrak{sl}(3,\mathbb{C})$ by $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$. It is clear that the orbits that lie in $\mathfrak{su}(3)$ are precisely classified by elements of $i\mathfrak{t}$. Let $T = \exp(i\mathfrak{t}) \cong U(1) \times U(1) \cong \mathbb{T}^2$ be a maximal torus.

We use the inner product defined by $\langle X, Y \rangle = \mathrm{tr}(X^* Y)$ to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. Notice firstly that this is indeed a multiple of the Killing form and secondly that when restricted to $\mathfrak{h}$, this inner product coincides with the one obtained by identifying $\mathfrak{h} \hookrightarrow \mathbb{C}^3$.

Now, $\mathfrak{g}$ has two positive simple roots

$$\alpha(H_1) = 2, \qquad \beta(H_1) = -1,$$
$$\alpha(H_2) = -1, \qquad \beta(H_1) = 2.$$

Under the identification $\mathfrak{h} \cong \mathfrak{h}^*$, these correspond to $\alpha = H_1$ and $\beta = H_2$. The fundamental weights are given by

$$\mu_1 = \frac{2}{3}H_1 + \frac{1}{3}H_2 \qquad \langle \mu_1, H_1 \rangle = 1 \qquad \langle \mu_2, H_1 \rangle = 0,$$
$$\mu_2 = \frac{1}{3}H_1 + \frac{2}{3}H_2 \qquad \langle \mu_1, H_2 \rangle = 0 \qquad \langle \mu_2, H_2 \rangle = 1.$$

It follows that the fundamental Weyl chamber is $\overline{C} = i\mathbb{R}_+\mu_1 + i\mathbb{R}_+\mu_2$. For general $\mu \in \overline{C}$, write

$$\begin{aligned} \mu &= 3ia\mu_1 + 3ib\mu_2 \\ &= (2a+b)iH_1 + (a+2b)iH_2 \\ &= i\ \mathrm{diag}\,(2a+b, b-a, -(2b+a)) \end{aligned}$$

for some $a, b \geq 0$.

Let us now give an explicit description of the adjoint orbits in $\mathfrak{su}(3)$. From theorem 131 we know that $\mathcal{S}(\mathfrak{k}^*)^K$ is generated by two algebraically independent trace polynomials. Notice further that all elements in $\mathfrak{su}(3)$ have trace zero. Therefore, these generators can not be of degree one. Consider the defining representation of $\mathfrak{su}(3)$ on $\mathbb{C}^3$. Let

$$X = \begin{pmatrix} ix & u & v \\ -\overline{u} & iy & w \\ -\overline{v} & \overline{w} & -i(x+y) \end{pmatrix}, \qquad u, v, w \in \mathbb{C},\ x, y \in \mathbb{R}$$

be a generic element of $\mathfrak{su}(3)$. Then we compute

$$\begin{aligned} \mathrm{tr}(X^2) &= -2p_2(X) \\ \mathrm{tr}(X^3) &= 3ip_3(X), \end{aligned}$$

where

$$\begin{aligned} p_2(X) &= |u|^2 + |v|^2 + |w|^2 + (x+y)^2, \\ p_3(X) &= x^2y + y^2x - (x+y)(|v|^2 - |u|^2) - 2\,\mathrm{Im}(u\overline{v}w). \end{aligned}$$

So p_2 and p_3 are the generators of $\mathcal{S}(\mathfrak{k}^*)^K$ and the orbit corresponding to μ is given explicitly by

$$\mathcal{O}_\mu = \left\{\ X \in \mathfrak{k}\ :\ p_2(X) = c_1^2 + c_2^2, \quad p_3(X) = c_1^2 c + c_1 c_2^2\ \right\}, \tag{6.15}$$

where $c_1 = 2a + b$ and $c_2 = a + 2b$.

Next, let us proceed with realizing the homogeneous Hilbert bundles obtained via the Mackey machine as eigenspace subbundles of trivial bundles.

We consider first a non-degenerate orbit, which means that $a, b > 0$. Notice that the stabilizer K_ν is precisely the subspace of $SU(3)$ for which all eigenspaces of μ are reducing subspaces. That is, $K_\nu = \mathbb{T}^2 = U(1) \times U(1)$, which is considered as subgroup of $SU(3)$ via $(z_1, z_2) \mapsto z_1 \oplus z_2 \oplus \overline{z_1 z_2}$.

In view of lemma 154, the irreducible representations of $U(1) \times U(1)$ are given by $\mathbb{C}_{n,m} \overset{d}{=} \mathbb{C}_n \otimes \mathbb{C}_m$ for $n, m \in \mathbb{Z}$, where $\mathbb{C}_i$ denotes the irreducible representation $z \mapsto z^i$ of $U(1)$. Notice that $\mathbb{C}_n^* \cong \mathbb{C}_{-n}$ and $\mathbb{C}_{a,b} \otimes \mathbb{C}_{c,d} \cong \mathbb{C}_{a+c,b+d}$.

First, let us consider the trivial representation. We have

$$SU(3) \times_{K_\mu} \mathbb{C}_{0,0} \cong O_\mu \times \mathbb{C}_{0,0}.$$

Consider next the case $|n| + |m| > 0$. By lemma 29 we have

$$\begin{aligned} SU(3) \times_{K_\mu} \mathbb{C}_{n,m} &= SU(3) \times_{K_\mu} (\mathbb{C}_{n,0} \otimes \mathbb{C}_{0,m}) \\ &= (SU(3) \times_{K_\mu} \mathbb{C}_{\operatorname{sgn} n,0})^{\otimes n} \otimes (SU(3) \times_{K_\mu} \mathbb{C}_{0,\operatorname{sgn} m})^{\otimes m}. \end{aligned}$$

In view of proposition 125 it suffices to consider the bundles $SU(3) \times_{K_\mu} \mathbb{C}_{1,0}$ and $SU(3) \times_{K_\mu} \mathbb{C}_{0,1}$. Now, let δ be the defining representation of $SU(3)$ on $\mathbb{C}^3$ so that $\delta_*(X) = X$ for $X \in \mathfrak{su}(3)$. The restriction of the defining representation to K_μ decomposes into irreducible K_μ-representations as follows:

$$\mathbb{C}^3 \cong \mathbb{C}_{1,0} \oplus \mathbb{C}_{0,1} \oplus \mathbb{C}_{-1,-1}.$$

These invariant subspaces are precisely the eigenspaces of μ corresponding to the eigenvalues $i(2a+b), i(b-a)$ and $-(2b+a)$. By theorem 122 we obtain the following equivalences of homogeneous Hilbert bundles:

$$\begin{aligned} SU(3) \times_{K_\mu} \mathbb{C}_{1,0} &\cong \left\{(X, v) \in \mathcal{O}_\mu \times \mathbb{C}^3 \,:\, Xv = i(2a+b)v\right\}, \\ SU(3) \times_{K_\mu} \mathbb{C}_{0,1} &\cong \left\{(X, v) \in \mathcal{O}_\mu \times \mathbb{C}^3 \,:\, Xv = i(b-a)v\right\}. \end{aligned}$$

Using equation (6.10), we also obtain

$$\begin{aligned} SU(3) \times_{K_\mu} \mathbb{C}_{-1,0} &\cong \left\{(X, v) \in \mathcal{O}_\mu \times \mathbb{C}^3 \,:\, X^t v = i(2a+b)v\right\}, \\ SU(3) \times_{K_\mu} \mathbb{C}_{0,-1} &\cong \left\{(X, v) \in \mathcal{O}_\mu \times \mathbb{C}^3 \,:\, X^t v = i(b-a)v\right\}, \end{aligned}$$

where the bundles on the right are equipped with the $SU(3)$-action given by $u \cdot (X, v) = (\mathrm{Ad}_u(X), \overline{u}v)$.
An application of proposition 125 yields the general case $\mathbb{C}_{n,m}$ with $|n| + |m| > 0$. As an example, consider the case where $n < 0$ and $m > 0$. Using proposition 125 we find that there is an equivalence of homogeneous Hilbert bundles

$$SU(3) \times_{K_\mu} \mathbb{C}_{n,m} \cong \left\{ \begin{aligned} (X, q) \in \mathcal{O}_\mu \times \mathbb{C}^{3\otimes(n+m)} \,:\, (X^q)^\nu q &= i(2a+b)q \quad \nu = 1, \cdots, n \\ X^\nu q &= i(b-a)q \quad \nu = n+1, \cdots, n+m \end{aligned} \right\}, \tag{6.16}$$

where $X^\nu = 1 \otimes \cdots 1 \otimes X \otimes 1 \cdots \otimes 1$ with X at the ν-th location and where the bundle on the right is equipped with the $SU(3)$-action given by $u \cdot (X, q) = (\mathrm{Ad}_u(X), (\overline{u}^{\otimes n} \otimes u^{\otimes m})q$.

Next, consider the degenerate case. Suppose that $b = 0$, so $\mu = i\mathrm{diag}(2a, -a, -a)$. The stabilizer of μ is precisely the subspace of $SU(3)$ for which all eigenspaces of μ are reducing subspaces. That is, $K_\mu = S(U(1) \times U(2))$. Similarly, if $a = 0$, then $\mu = i\mathrm{diag}(b, b, -2b)$ so $K_\mu = S(U(2) \times U(1))$.

Return to the case where $b = 0$, $a > 0$ and thus $K_\mu = S(U(1) \times U(2))$.
Observe first that $U(2) \cong S(U(1) \times U(2)) = K_\mu$ via the isomorphism $u \mapsto \det(u)^{-1} \oplus u$ and secondly that $U(2) \cong U(1) \times SU(2)/\{\pm(I, I)\}$. It follows that the irreducible representations of K_μ are precisely the irreducible representations of $U(1) \times SU(2)$ that are trivial on $(-1, -I)$.

Denote $\bigvee^m \mathbb{C}^2$ the m^{th} symmetric power of $\mathbb{C}^2$. It is known that $SU(2)$ acts irreducibly on $\bigvee^m \mathbb{C}^2$ for every $m \in \mathbb{N}$ and all irreducible unitary representations of $SU(2)$ are obtained in this manner, see also section 8.2.

Denote by σ_m the irreducible representation of $SU(2)$ on $\bigvee^m \mathbb{C}^2$. By lemma 154, the irreducible unitary representations of K_μ are precisely the representations $\pi_{n,m}$ of $U(1)\times SU(2)$ acting on $W_{n,m} = \mathbb{C}_n \otimes \bigvee^m \mathbb{C}^2$ and $W_{n,0} \overset{d}{=} \mathbb{C}_n$ that are trivial on $(-1,-I)$. Now, for $s\otimes v_1\cdots v_m \in W_{n,m}$ we have

$$\pi_{n,m}(-1,-I)(s\otimes v^{\otimes m}) = (-1)^n s\otimes(-v_1)\cdots(-v_m) = (-1)^{n+m} s\otimes v^m.$$

Therefore, the representations of K_μ are precisely $\pi_{n,m}$ where $n\in\mathbb{Z}$ and $m\geq 0$ are such that $n+m$ is even. Notice that the inner product on $W_{n,m}$ for $m>0$ is defined by

$$\langle z_1\otimes v_1\cdots v_m, z_2\otimes w_1\cdots w_m\rangle = z_1\overline{z_2}\prod_{i=1}^m\langle v_i, w_i\rangle.$$

Consider $n+m$ even and $m>0$. By lemma 29 we have

$$\begin{aligned} SU(3)\times_{K_\mu}(\mathbb{C}_n\otimes\bigvee^m\mathbb{C}^2) &\cong (SU(3)\times_{K_\mu}\mathbb{C}_n)\otimes(SU(3)\times_{K_\mu}\bigvee^m\mathbb{C}^2)\\ &\cong (SU(3)\times_{K_\mu}\mathbb{C}_{\operatorname{sgn} n})^{\otimes n}\otimes\bigvee^m(SU(3)\times_{K_\mu}\mathbb{C}^2)\end{aligned}$$

In view of proposition 125 it suffices to consider the bundles $SU(3)\times_{K_\mu}\mathbb{C}_1$ and $SU(3)\times_{K_\mu}\mathbb{C}^2$. Now, consider the defining representation δ of $SU(3)$ on $\mathbb{C}^3$ so that $\delta_*(X)=X$. Its restriction to K_μ decomposes as $\mathbb{C}_1\oplus\mathbb{C}^2$, where $U(1)\times SU(2)$ acts on $\mathbb{C}^1$ via $U(1)$ and $SU(2)$ acts trivially and vice-versa for the action of $U(1)\times SU(2)$ on $\mathbb{C}^2$. Moreover, $\mathbb{C}_1$ is precisely the eigenspace of $\mu = i\mathrm{diag}(2a,-a,-a)$ corresponding to the eigenvalue $i2a$ and $\mathbb{C}^2$ is the eigenspace of μ corresponding to eigenvalue $-ia$. By theorem 122 we obtain the following equivalences:

$$\begin{aligned} SU(3)\times_{K_\mu}\mathbb{C}_1 &\cong \left\{(X,v)\in\mathcal{O}_\mu\times\mathbb{C}^3 \,:\, Xv = 2iav\right\},\\ SU(3)\times_{K_\mu}\mathbb{C}^2 &\cong \left\{(X,v)\in\mathcal{O}_\mu\times\mathbb{C}^3 \,:\, Xv = -iav\right\}.\end{aligned}$$

An application of 125 now yields the general case. For $n+m$ even and $n,m>0$ we have

$$SU(3)\times_{K_\mu}W_{n,m}\cong\Big\{(X,q)\in\mathcal{O}_\mu\times{\mathbb{C}^3}^{\otimes n}\otimes S^m(\mathbb{C}^3)\,:X^\nu q = i2aq\quad \nu=1,\cdots,n$$
$$X^\nu q = -iaq\quad \nu = n+1,\cdots,n+m\Big\},$$

where the action of $SU(3)$ on this bundle is given by $u\cdot(X,t) = \mathrm{Ad}_u(X), u^{\otimes n}t$. If on the other hand $n<0$, we obtain

$$SU(3)\times_{K_\mu}W_{n,m}\cong\Big\{(X,q)\in\mathcal{O}_\mu\times{\mathbb{C}^3}^{\otimes n}\otimes S^m(\mathbb{C}^3)\,:(X^t)^\nu q = i2at\quad \nu=1,\cdots,n$$
$$X^\nu q = -iat\quad \nu = n+1,\cdots,n+m\Big\},$$

where now $SU(3)$ acts according to $u\cdot(X,q) = (\mathrm{Ad}_u(X),(\overline{u}^{\otimes n}\otimes u^{\otimes m})q)$. Finally, For $m=0$ and $0\neq n$ even, we have

$$\begin{aligned} SU(3)\times_{K_\mu}\mathbb{C}_0 &\cong \mathcal{O}_\mu\times\mathbb{C}_0\\ SU(3)\times_{K_\mu}\mathbb{C}_n &\cong \left\{(X,v)\in\mathcal{O}_\mu\times\mathbb{C}^3 \,:\, X^\nu v = 2iav\quad \nu=1,\cdots,n\right\},\\ SU(3)\times_{K_\mu}\mathbb{C}_{-n} &\cong \left\{(X,v)\in\mathcal{O}_\mu\times\mathbb{C}^3 \,:(X^t)^\nu v = 2iav\quad \nu=1,\cdots,n\right\},\end{aligned}$$

with the appropriate $SU(3)$-action on these bundles.

Choosing a basis reveals more explicitly what differential equations the Fourier transform of sections of these bundles would satisfy, assuming that the Fourier transform can indeed be suitably defined. Consider for example the degenerate case with $n,m>0$ and $n+m$ even. Let $\{E_k\}_{k=1}^8$ be a basis of $\mathfrak{su}(3)$ (such as the Gell-Mann matrices) and identify $\mathfrak{su}(3)\cong\mathbb{R}^8$ using this choice of basis. Then we obtain

$$SU(3)\times_{K_\mu}W_{n,m}\cong\Big\{(p,q)\in\mathcal{O}_\mu\times{\mathbb{C}^3}^{\otimes n}\otimes S^m(\mathbb{C}^3)\,:\,\sum_{k=1}^8 p_kE_k^\nu q = i2aq\quad \nu=1,\cdots,n$$
$$\sum_{k=1}^8 p_kE_k^\nu q = -iaq\quad \nu = n+1,\cdots,n+m\Big\}.$$

6.5 Bundles arising via $V \oplus (\mathfrak{k} \otimes V) \rtimes \mathbf{Spin}(r,s)^0 \times K$

In this final section, we consider the embedding of the homogeneous bundles that occur in representations of the group $G = V \oplus \mathfrak{k} \otimes V \rtimes \mathrm{Spin}(r,s) \times K$ in eigenspace subbundles of trivial bundles. The main motivation for doing so is to pursue an analysis of the representation theory of G in analogous fashion to Wigner's analysis of the representation theory of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$, where differential equations describing certain relativistic particles are recovered by embedding the corresponding bundles in eigenspace subbundles of trivial bundles. It turns out that the method developed in theorem 122 can be successfully applied to associated bundles over arbitrary orbits in $V \oplus \mathrm{Hom}(V,\mathfrak{k})$ for an appropriate representation of the stabilizer.

We assume that the orbit space of $V \oplus (\mathfrak{k} \otimes V)$ under the action of $\mathrm{Spin}(r,s)^0 \times K$ is countably separated. In particular, this is true for the case where $K = SU(N)$ and (V,q) is the Minkowski space $(\mathbb{R}^4, \eta)$.

From section 8.4 we know that under the identifications $V^* \cong V$ using η and $\mathfrak{k} \otimes V \cong \mathfrak{k} \otimes V^* \cong \mathrm{Hom}(V,\mathfrak{k})$, the action on $V \oplus \mathrm{Hom}(V,\mathfrak{k})$ becomes

$$(w,k) \cdot p \oplus A = \phi(w)p + \mathrm{Ad}_k \circ A \circ \phi(w)^{-1}. \tag{6.17}$$

As in chapter 5, we use the bilinear form q on V and an Ad-invariant inner product κ on $\mathfrak{k}$ to identify $V^* \cong V$ and $\mathfrak{k}^* \cong \mathfrak{k}$. Denote by $(-)^\star : \mathrm{Hom}(V,\mathfrak{k}) \to \mathrm{Hom}(\mathfrak{k}, V)$ the transpose map under these identifications. See also section 8.4 for more details on these identifications. In particular, from section 8.4 we know that $(\mathrm{Ad}_k A\phi(w)^{-1})^\star = \phi(w)A^\star \mathrm{Ad}_k^{-1}$.

In this setting, there are two natural ways in which the requirements of theorem 122 can be satisfied. That is, in which there exist a representation of G on some vector space $\mathcal{F}$ and a way to interpret elements in $V \oplus \mathrm{Hom}(V,\mathfrak{k})$ as operators $\mathcal{F}$ in an equivariant manner such that the eigenspaces of these operators are invariant under the restricted action of the stabilizer. The idea is to consider representations of G on which either $\mathrm{Spin}(r,s)^0$ or K acts trivially and to define equivariant projections of $V \oplus \mathrm{Hom}(\mathbb{R}^4,\mathfrak{k})$ into either V or $\mathfrak{k}$. In section 6.2 and section 6.3 we have already encountered suitable ways to interpret elements in these latter two spaces as operators acting on the vector space of some $\mathrm{Spin}(r,s)^0$ or K-representation, so by composition we obtain an equivariant map $V \oplus \mathrm{Hom}(V,\mathfrak{k}) \to \mathrm{End}(\mathcal{F})$.

Lemma 134. *Let $\nu = p_0 + A_0 \in Hom(V,\mathfrak{k})$ be arbitrary.*

1. *$\mathcal{O}_\nu$ is a fiber bundle over the $Spin(r,s)^0$-orbit $\mathcal{O}_{p_0}$, where the projection map is $Spin(r,s)^0$-equivariant and given by*
$$\xi : \mathcal{O}_\nu \to \mathcal{O}_{p_0}, \qquad \xi : p \oplus A \mapsto p. \tag{6.18}$$
2. *Write $X_0 = A_0 p_0$. Then $\mathcal{O}_\nu$ is a fiber bundle over the adjoint orbit $\mathcal{O}_{X_0}$, where the projection map is K-equivariant and given by*
$$\chi : \mathcal{O}_\nu \to \mathcal{O}_{X_0}, \qquad \chi : p \oplus A \mapsto Ap. \tag{6.19}$$
3. *Write $p_A = A_0^\star A_0 p_0$. Then $\mathcal{O}_\nu$ is a fiber bundle over the $Spin(r,s)^0$-orbit $\mathcal{O}_{p_A}$, where the projection map is $Spin(r,s)^0$-equivariant and given by*
$$\psi : \mathcal{O}_\nu \to \mathcal{O}_{p_A}, \qquad \psi : p \oplus A \mapsto A^\star Ap. \tag{6.20}$$

Proof.

1. Notice first that there are diffeomorphisms $\mathcal{O}_\nu \cong G/G_\nu$ and $\mathcal{O}_{p_0} \cong G/G_{p_0}$, so by corollary 18 G is a principal fiber bundle over both these orbits. In particular the quotient maps $q_\nu : G \to G/G_\nu$ and $q_p : G \to G/G_{p_0}$ are smooth submersion and locally trivial. Since $G_\nu \subset G_{p_0}$, the quotient q_p factors through a map $G/G_\nu \xrightarrow{f} G/G_{p_0}$. This map is clearly surjective and its differential df is a surjective bundle map because $dq_p = df \circ dq_\nu$ is surjective. Therefore f is a smooth submersion. Finally, f is locally trivial. To see why, notice that q_p is locally the projection $U \times G_{p_0} \to U$ and q_ν is in these local coordinates given by the projection $U \times G_{p_0} \to U \times G_{p_0}/G_\nu$. The result follows from $q_p = f \circ q_\nu$. Finally, notice that under the isomorphisms $\mathcal{O}_\nu \cong G/G_\nu$ and $\mathcal{O}_p \cong G/G_{p_0}$, the map $\mathcal{O}_\nu \to \mathcal{O}_{p_0}$ is given by $g \cdot p \oplus A \mapsto g \cdot p$ for $g \in G$. It is in particular clear that this map is equivariant.

2. We employ a completely similar argument. Notice that $\mathcal{O}_{X_0} \cong K/K_{X_0} \cong G/G_{X_0}$, where $\mathrm{Spin}(r,s)^0$ acts trivially on $\mathfrak{k}$. Moreover, if $(w,k) \in G_\nu$, then $\phi(w)p = p$ and $\mathrm{Ad}_k \circ A \circ \phi(w)^{-1} = A$. Thus,

$$\mathrm{Ad}_k X = \mathrm{Ad}_k Ap = \mathrm{Ad}_k A\phi(w)^{-1}\phi(w)p = A\phi(w)p = Ap = X.$$

Therefore, $G_\nu \subset G_{X_0}$. Using the same argument as in the first point, we obtain a locally trivial smooth submersion $\mathcal{O}_\nu \to \mathcal{O}_{X_0}$ given by $g \cdot \nu \mapsto g \cdot X$. Unraveling the definitions with $g = (w,k) \in G$, this means

$$\phi(w)p \oplus \mathrm{Ad}_k \circ A \circ \phi(w)^{-1} \mapsto \mathrm{Ad}_k(Ap) = (\mathrm{Ad}_k \circ A \circ \phi(w)^{-1})(\phi(w)p).$$

3. Similarly, notice that $\mathcal{O}_{p_A} \cong G/G_{p_A}$. This implies $G_\nu \subset G_{p_A}$, because if $(w,k) \in G_\nu$, then

$$\phi(w)p_A = \phi(w)A^\star Ap = (\mathrm{Ad}_k A\phi(w)^{-1})^\star(\mathrm{Ad}_k A\phi(w)^{-1})\phi(w)p = A^\star A\phi(w)p = A^\star Ap = p_A.$$

Thus we obtain a locally trivial smooth submersion $\mathcal{O}_\nu \to \mathcal{O}_{p_A}$ given by $g \cdot \nu \mapsto g \cdot p_A$. Unraveling the definitions, we similarly find that this map is given by $p \oplus A \mapsto A^\star Ap$.

□

Corollary 135. *Let $\nu = p_0 \oplus A_0$, $X_0 = A_0 p_0$, ξ, χ and ψ be as in lemma 134. Let $E \to \mathcal{O}_\nu$ be a vector bundle. Then any section $s \in \Gamma(\mathcal{O}_\nu, E)$ satisfies the following three equations, where $\mu = p + A \in \mathcal{O}_\nu$ is arbitrary:*

$$\begin{aligned}(q \circ \xi)(\mu)s(\mu) &= q(p_0)s(\mu),\\ (q \circ \psi)(\mu)s(\mu) &= q(p_A)s(\mu),\\ (P \circ \chi)(\mu)s(\mu) &= P(X_0)s(\mu), \qquad \forall P \in \mathcal{S}(\mathfrak{k}^*)^K.\end{aligned}$$

Remark.

— If we assume that a suitable Fourier transform can be defined on the space of sections $\Gamma(\mathcal{O}_\nu, E)$, then each of the equations in corollary 135 would yield a partial differential equation satisfied by the Fourier transform $\widehat{f}$ of such a section f. For example, if we consider the case of Minkowski space $(V,q) = (\mathbb{R}^4, \eta)$, then the first equation would correspond to $(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2})\widehat{f} = \eta(p_0,p_0)\widehat{f}$.

Proposition 136. *Consider a finite dimensional representation $\rho : Cl(V,q) \to End(\mathcal{F})$ of the Clifford algebra $Cl(V,q)$ and set $S = \rho|_{Spin(r,s)^0}$ be its restriction to the connected component of the spin group. Let $\nu = p_0 + A_0 \in V \oplus Hom(\mathbb{R}^4, \mathfrak{k})$. Let $c_1, c_2 \in \mathbb{R}$ be arbitrary. Suppose that E_λ is an eigenspace of $c_1\rho(p_0) + c_2\rho(A_0^\star A_0 p_0)$ with eigenvalue λ. Then there is an equivalence of homogeneous vector bundles over $\mathcal{O}_\nu$:*

$$G \times_{G_\nu} E_\lambda \cong \{\, (p \oplus A, v) \in \mathcal{O}_\nu \times \mathcal{F} \,:\, c_1\rho(p)v + c_2\rho(A^\star Ap)v = \lambda v \,\}, \tag{6.21}$$

where G_ν acts on E_λ via $(w,k) \cdot v = S(w)v$ and where the bundle on the right is equipped with the G-action given by

$$(w,k) \cdot (p \oplus A, v) = (\phi(w)p \oplus Ad_k \circ A \circ \phi(w)^{-1}, S(w)v).$$

Proof. Let ξ, ψ be defined by (6.18) and (6.20). We have already seen in equation (3.4) that the assignment $V \xrightarrow{\rho} \mathrm{End}(\mathcal{F})$ is equivariant with respect to the action of $\mathrm{Spin}(r,s)^0$:

$$\rho(\phi(w)p) = S(w)\rho(p)S(w)^{-1}.$$

Since both ξ and ψ are $\mathrm{Spin}(r,s)^0$-equivariant maps, it follows that the map

$$\theta : \mathcal{O}_\nu \to GL(\mathcal{F}), \qquad \theta = c_1 \cdot (\rho \circ \xi) + c_2 \cdot (\rho \circ \psi)$$

is G-equivariant, where K acts trivially on the latter space. Furthermore, we have seen in the proof of lemma 134 that $G_\nu \subset G_{p_0} \cap G_{p_A}$, where $p_A = A_0^\star A_0 p_0$. It follows that for any $g \in G_\nu$, we have

$$\theta(\nu) = \theta(g \cdot \nu) = S(w)\theta(\nu)S(w)^{-1}$$

and therefore $S|_{G_\nu}$ leaves the E_λ invariant. Thus, if we consider the representation δ of G on $\mathcal{F}$ on which $\mathrm{Spin}(r,s)^0$ acts via S and K acts trivially, then $\delta|_{G_\nu}$ leaves the eigenspace E_λ of $(\theta)(\nu)$ invariant. The result now follows by theorem 122. □

Corollary 137. *Consider the setting of proposition 136 and let ξ, ψ be as in* (6.18) *and* (6.20). *Define $\theta = c_1 \cdot \xi + c_2 \cdot \psi$. Write*

$$E^1 \stackrel{d}{=} \{ \ (p \oplus A, v) \in \mathcal{O}_\nu \times \mathcal{F} \ : \ c_1 \rho(p) v + c_2 \rho(A^\star A p) v = \lambda v \ \} \, ,$$
$$E^2 \stackrel{d}{=} \left\{ \ (p, v) \in \mathcal{O}_{\theta(\nu)} \times \mathcal{F} \ : \ \rho(p) v = \lambda v \ \right\} .$$

Then $(\theta \times id, \theta)$ is a surjective morphism of G-homogeneous Hilbert bundles $(E_1, \mathcal{O}_\nu) \to (E_2, \mathcal{O}_{\theta(\nu)})$, where K acts trivially on E^2 and $\mathcal{O}_{\theta(\nu)}$, and $Spin(r,s)^0$ acts on E_2 by $w \cdot (p, v) = (\phi(w)p, S(w)v)$.

Remark.

— The attentive reader may recognize that E_1 is precisely the pullback bundle of E_2 by θ.

— Any section $s \in \Gamma(\mathcal{O}_\nu, E_1)$ satisfies $\rho(p)s(\mu) = \lambda s(\mu)$ for all $\mu \in \theta^{-1}(\{p\})$.

— Suppose in particular that $V = \mathbb{R}^{1,3}$ and consider the point $\nu = me_0 \oplus A_0$ with $m > 0$. Let $\rho : \mathrm{Cl}(1,3) \to \mathrm{End}(\mathbb{C}^4)$ be the representation defined by equation (3.6) and write $S = \rho|_{SL(2,\mathbb{C})}$.
Notice that $\rho(me_0) = m\gamma_0$ has two non-trivial two-dimensional eigenspaces $E_{\pm m}$ with eigenvalues $\pm m$. Then proposition 136 yields that

$$G \times_{G_\nu} E_{\pm m} \cong \left\{ \ (p \oplus A, v) \in \mathcal{O}_\nu \times \mathbb{C}^4 \ : \ \rho(p) v = \pm m v \ \right\} ,$$

which recovers the eigenvalue equation corresponding to the Dirac equation for spin $\frac{1}{2}$-particles. However, we obtain *more* information, namely the orbit of ν in $\mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{k})$. In view of corollary 135, this corresponds to a set of equations satisfied by sections of homogeneous bundles over $\mathcal{O}_\nu$.

Now, an application of 125 yields

$$G \times_{G_\nu} \bigvee^N E_{\pm m} \cong \left\{ \ (p \oplus A, t) \in \mathcal{O}_\nu \times \mathcal{S}^N(\mathbb{C}^4) \ : \ \rho^\nu(p) t = \pm m t, \quad \forall \nu = 1, \cdots, N \ \right\} ,$$

where $\rho^\nu = 1 \otimes \cdots \otimes \rho \otimes 1 \cdots \otimes 1$. This similarly recovers the Dirac equation corresponding to higher spin, with the information of the orbit of A_0.

— Consider the same setting as in the previous point. Notice that for any $p \in \mathcal{O}_{me_0}$, the eigenspaces of $\rho(p)$ are two-dimensional. Taking $c_2 = 0$ in proposition 136 can thus never directly apply to homogeneous bundles over $\mathcal{O}_\nu$ of the form $G \times_{G_\nu} \mathbb{C}$. Yet, we have seen in theorem 117 that many of the stabilizers G_ν have one-dimensional irreducible representations. Taking $c_2 \neq 0$ may resolve the matter. To see why, consider

$$\Gamma \stackrel{d}{=} i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

In view of equation (3.7), Γ commutes with $S(w)$ for every $w \in SL(2,\mathbb{C})$. Suppose that $p_A = e_0 - e_3$. A direct computation shows that Γ commutes with $\rho(p_A) = \gamma_0 - \gamma_3$. By equivariance of ρ, it leaves the eigenspaces of $\rho(p)$ invariant for all $p \in \mathcal{O}_{p_A}$. Now, $\rho(p_A)$ has a single two-dimensional eigenspace E_0 with eigenvalue zero that is spanned by e_1 and e_2. The eigenspaces of Γ decompose E_0 further according to $E_0 = E_0^+ \oplus E_0^-$ where $\gamma v = \pm v$ on $E_0^\pm$ and because Γ intertwines S, the stabilizer G_{p_A} leaves both subspaces $E_0^\pm$ invariant. Recalling from the proof of lemma 134 that $G_\nu \subset G_{p_A}$, we find using proposition 136 and lemma 27 that

$$G \times_{G_\nu} E_0^\pm \cong \left\{ \ (p \oplus A, v) \in \mathcal{O}_\nu \times \mathbb{C}^4 \ : \ \rho(A^\star A p) v = 0, \quad \Gamma v = \pm v \ \right\} . \tag{6.22}$$

Notice that because Γ anti-commutes with every γ_k it does *not* leave the eigenspaces of $\rho(p)$ invariant whenever $\eta(p,p) \neq 0$. Therefore, a similar idea works only if $c_1, c_2 \in \mathbb{R}$ can be chosen such that $\eta(\theta(\nu), \theta(\nu)) = 0$, where $\theta = c_1 \cdot \xi + c_2 \cdot \psi : \mathcal{O}_\nu \to V$.

Proposition 138. *Let $\nu = p_0 + A_0 \in V \oplus Hom(V,\mathfrak{k})$ be arbitrary. Suppose that $\pi : K \to \mathcal{F}$ is a finite dimensional representation of K and let $Spin(r,s)^0$ act trivially on this space. Then for any eigenvalue λ of $\pi_*(A_0p_0)$ with eigenspace E_λ, there is an equivalence of homogeneous vector bundles*

$$G \times_{G_\nu} E_\lambda \cong \{ (p \oplus A, v) \in \mathcal{O}_\nu \times \mathcal{F} \,:\, \pi_*(Ap)v = \lambda v \}\,, \tag{6.23}$$

where G_ν acts on E_λ according to $(w,k)\cdot v = \pi(k)v$ and where the bundle on the right is equipped with the G-action given by

$$(w,k)\cdot(p\oplus A, v) = (\phi(w)p \oplus Ad_k \circ A \circ \phi(w)^{-1}, \pi(k)v).$$

Proof. Let χ be as in (6.19). We have already seen in equation (6.13) that π_* is K-equivariant. In view of lemma 134 the map $\pi_* \circ \chi$ is G-equivariant. It follows that the eigenspaces of $(\pi_* \circ \chi)(\nu)$ are invariant under the action of the stabilizer. Explicitly, we have for any $g = (w,k) \in G_\nu$:

$$\pi(k)(\pi_* \circ \chi)(\nu)\pi(k)^{-1} = \pi_*(\chi(g\cdot\nu)) = (\pi_* \circ \chi)(\nu).$$

Thus, we obtain a non-trivial G_ν-representation on each of the eigenspaces E_λ of $(\pi_* \circ \chi)(\nu)$. The result now follows by an application of theorem 122. □

Corollary 139. *Consider the setting of proposition 138, let χ be as in lemma 134 and let $X_0 = A_0p_0$. Write*

$$E^1 \overset{d}{=} \{ (p \oplus A, v) \in \mathcal{O}_\nu \times \mathcal{F} \,:\, \pi_*(Ap)v = \lambda v \}\,, \tag{6.24}$$

$$E^2 \overset{d}{=} \{ (X, v) \in \mathcal{O}_{X_0} \times \mathcal{F} \,:\, \pi_*(X)v = \lambda v \}\,. \tag{6.25}$$

Then $(\chi \times id, \chi)$ is a surjective morphism of G-homogeneous Hilbert bundles $(E_1, \mathcal{O}_\nu) \to (E_2, \mathcal{O}_{X_0})$, where $Spin(r,s)^0$ acts trivially on E^2 and $\mathcal{O}_{X_0}$, and K acts on E_2 by $k\cdot(X,v) = (Ad_k(X), \pi(k)v)$.

Remark.

— Consider $K = SU(N)$ and its defining representation π on $\mathbb{C}^N$. Then any element $X = Ap$ in the Lie algebra $\mathfrak{su}(N)$ is diagonalizable. The proof of proposition 138 shows that every eigenspace E_λ of X is G_ν-invariant and therefore, $\mathbb{C}^N$ decomposes as a G_ν-representation according to

$$\mathbb{C}^N \cong \bigoplus_\lambda E_\lambda.$$

Now, proposition 138 applies to each of the G_ν-representations E_λ.

— Corollary 139 states the sense in which all eigenvalue equations encountered in section 6.3 are contained in the representation theory of G. Indeed, any section $s \in \Gamma(\mathcal{O}_\nu, E^1)$ satisfies the equation $\pi_*(X)s(\mu) = \lambda s(\mu)$ for all $\mu \in \chi^{-1}(X)$ in the fiber of $X \in \mathcal{O}_{X_0}$. In particular, this applies to the case where $K = SU(N)$ and π is the defining representation on $\mathbb{C}^n$. Now, all of the eigenvalue equations encountered in section 6.4.1 and section 6.4.2 were obtained by restricting the defining representation to the eigenspaces of some $X \in \mathfrak{su}(N), N = 2,3$ and applying various bundle operations. Therefore all these equations are contained in the representation theory of $V \oplus \mathfrak{k} \otimes V \rtimes \mathrm{Spin}(r,s)^0 \times K$, in the sense described above. However, we obtain *additionally* the information of the orbit $\mathcal{O}_\nu$ the associated bundle $G \times_{G_\nu} \mathcal{H}$ is defined over.

— For massive points and $K = SU(2)$, the values of Ap have a direct relationship with the stabilizers obtained in theorem 117 and the method that was used to prove this result.

Indeed, suppose that $V = \mathbb{R}^{1,3}$ and consider the point $\nu = p_0 \oplus A_0 \in V + \mathrm{Hom}(V,\mathfrak{k})$ with $p_0 = m_p e_0$ for some $m_p > 0$. Write $A_0 = \begin{pmatrix} X & B \end{pmatrix}$ for some $X \in \mathfrak{k}$ and $B \in \mathrm{Hom}(\mathbb{R}^3, \mathfrak{k})$. Then A_0p_0 is simply given by $A_0p_0 = m_pX$. Now, for $K = SU(2)$ there are two possibilities. If $X = 0$, then $A_0p_0 = 0$ has only one eigenspace, namely all of $\mathbb{C}^2$ and $Ap = 0$ for every $p \oplus A \in \mathcal{O}_\nu$. If on the other hand $X \neq 0$, then X has two one-dimensional eigenspaces $\mathbb{C}_-$ and $\mathbb{C}_+$ corresponding to some eigenvalues $\pm m_s \neq 0$. Notice that this is precisely the distinction $a = 0$ or $a \neq 0$ made in section 5.2. In both cases, proposition 138 immediately yields isomorphisms of homogeneous vector bundles:

$$X = 0 \implies G \times_{G_\nu} \mathbb{C}^2 \cong \mathcal{O}_\nu \times \mathbb{C}^2, \tag{6.26}$$

$$X \neq 0 \implies G \times_{G_\nu} \mathbb{C}_\pm \cong \left\{ (p \oplus A, v) \in \mathcal{O}_\nu \times \mathbb{C}^2 \,:\, (Ap)v = \pm m_p m_s v \right\}. \tag{6.27}$$

— On the one hand, proposition 136 realizes bundles $G \times_{G_\nu} \mathcal{H}^S$ as eigenspace subbundles of trivial bundles, where $G_\nu \cap K$ acts trivially. On the other hand, proposition 138 does the opposite; it considers bundles $G \times_{G_\nu} \mathcal{H}^K$ where $G_\nu \cap \mathrm{Spin}(r,s)^0$ acts trivially. Proposition 125 can be applied to obtain similar results for representations of G_ν where both $G_\nu \cap \mathrm{Spin}(r,s)^0$ and $G_\nu \cap K$ act non-trivially.

6.5.1 The case of $K = SU(2)$

In the following, an example is considered of how the results of the previous section may be applied to study the representation theory of $G = V \rtimes H \stackrel{d}{=} \mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{su}(2)) \rtimes SL(2,\mathbb{C}) \times SU(2)$. By corollary 86 we know that the orbit space of the action of H on V is countably separated. Thus, according to the theory of the Mackey machine, the irreducible representations of G are classified by that of the various stabilizers of the action of H on $\widehat{V}$. As in the previous section, we identify $\mathbb{R}^{4^*} \cong \mathbb{R}^4$ using η and $\mathfrak{su}(2)^* \cong \mathfrak{su}(2)$ using κ. Denote by $(-)^\star : \mathrm{Hom}(\mathbb{R}^4, \mathfrak{k}) \to \mathrm{Hom}(\mathfrak{k}, \mathbb{R}^4)$ the corresponding transpose map. We may use the G-invariant bilinear form $\beta(x \oplus A, p \oplus B) = \eta(x,p) + \mathrm{tr}(A^\star B)$ to identify $\widehat{V} \cong V$ via the pairing

$$\langle x \oplus A, p \oplus B \rangle \stackrel{d}{=} e^{i\beta(x \oplus A, p \oplus B)}.$$

Under these identifications, the action of H on $\widehat{V}$ transfers to the action

$$(w,k) \cdot p \oplus A = \phi(w)p + \mathrm{Ad}_k \circ A \circ \phi(w)^{-1}.$$

In section 5.2, the stabilizers of this action where determined up to equivalence. They are given by theorem 117.

Identify $\mathfrak{su}(2) \cong \mathbb{R}^3$ using the Pauli matrices as in lemma 133 and consider a basis of $\mathbb{R}^4$ in which η corresponds to the matrix $D = \mathrm{diag}(1,-1,-1,-1)$. Let us consider a non-trivial example corresponding to the case $a \neq 0$ in the proof of theorem 117 whose stabilizer is a product of compact subgroups, namely consider the case that $H_\nu = U(1) \times U(1)$, where $U(1)$ embeds in $SU(2)$ and $SL(2,\mathbb{C})$ via $z \mapsto z \oplus \overline{z}$. In view of the proof of theorem 117, the corresponding element in $\mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{k})$ is $\nu = me_0 + A_0$, where

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ r & 0 & 0 & s \end{pmatrix}$$

for some $r, s \neq 0$. Now, by lemma 154, the irreducible representations of $H_\nu = U(1) \times U(1)$ are given by $\mathbb{C}_n \otimes \mathbb{C}_m$ with $n, m \in \mathbb{Z}$. We consider the representations $\mathbb{C}_\pm \otimes \mathbb{C}_0$ and $\mathbb{C}_0 \otimes \mathbb{C}_\pm$, noticing that the general case follows by means of suitable bundle operations using proposition 125.

First, let us consider the representations of H_ν where K acts trivially. Consider the representation $\rho : \mathrm{Cl}(1,3) \to \mathrm{End}(\mathbb{C}^3)$ defined by the Dirac matrices γ_k, which are given in equation (3.6) and set $S = \rho|_{SL(2,\mathbb{C})}$. We aim to choose the constant $c_1, c_2 \in \mathbb{R}$ in proposition 136 appropriately, following the last remark succeeding corollary 137.

From lemma 160 we know that $A^\star$ is the unique map satisfying $\kappa(Ap, x) = \eta(p, A^\star x)$ for all $p \in \mathbb{R}^4$ and $x \in \mathbb{R}^3$. In our choice of basis, this becomes $p^T A^T x = p^T D A^\star x$, where $(-)^T$ denotes the usual transpose of vectors and matrices. Thus $DA^\star = A^T$ so that the matrix of $A^\star$ is given by DA^T, seeing as $D^{-1} = D$.

Using the preceding observation, one computes that $A_0^\star A_0 p_0$ is given by $r^2 m \cdot e_0 - rsm \cdot e_3$. Let ξ, ψ be as in lemma 134 and define $\theta : \mathcal{O}_\nu \to \mathbb{R}^4$ by $\theta = r(s-r) \cdot \xi + \psi$. In that case, it holds that $\theta(\nu) = mrs(e_0 - e_3)$. We have seen in the last remark succeeding corollary 137 that the element $\Gamma = i\gamma_0\gamma_1\gamma_2\gamma_3$ intertwines the $SL(2,\mathbb{C})$-representation S and leaves the eigenspace of $\gamma_0 - \gamma_3$ corresponding to eigenvalue 0 invariant. This eigenspace is spanned by e_1 and e_2. In view of equation (3.7) we find that the action $U(1) \subset SL(2,\mathbb{C})$ on $\mathrm{Span}\{e_1\}$ and $\mathrm{Span}\{e_2\}$ is equivalent to $\mathbb{C}_{-1}$ and $\mathbb{C}_1$, respectively. Thus using proposition 136 and lemma 27, we find:

$$G \times_{G_\nu} (\mathbb{C}_{\pm 1} \otimes \mathbb{C}_0) \cong \left\{ \, (p \oplus A, v) \in \mathcal{O}_\nu \times \mathbb{C}^4 \, : \, r(s-r)\rho(p)v + \rho(A^\star A p)v = 0, \quad \Gamma v = \mp v \, \right\}.$$

On the other hand, because $A_0 p_0 = mre_3$ equation (6.26) implies that

$$G \times_{G_\nu} (\mathbb{C}_0 \otimes \mathbb{C}_{\pm 1}) \cong \left\{ \, (p \oplus A, v) \in \mathcal{O}_\nu \times \mathbb{C}^2 \, : (Ap)v = \pm mr \cdot v \, \right\}.$$

Part III

Chapter 7

Discussion and Conclusion

Following recent developments by B. Janssens and K.H. Neeb [JN], the strongly continuous unitary representations of groups of the form $G = \mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{k}) \rtimes SL(2, \mathbb{C}) \times K$ were studied that satisfy a certain positive energy condition.

For an element $p + A \in \mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{k})$, the of positive energy requirement was found to be equivalent to the condition that $A^\star$ maps the adjoint orbit of H into the closed future-pointing light cone: $A^\star \mathcal{O}_H \subseteq \overline{C}$. This equivalent formulation could be used to obtain the result that $p \in \overline{C}$ is a necessary requirement for satisfying the positive energy condition and that for $p \in \partial C$ in the future-pointing light cone, it is even necessary that A is a rank-one operator of the form $A = \eta(p, \cdot)X$ for some $X \in \mathfrak{k}$.

By the theory of the Mackey machine, the irreducible strongly continuous representations of G that satisfy the positive energy condition are classified by the corresponding stabilizers of the action of $SL(2, \mathbb{C}) \times K$ on $\mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{k})$. These stabilizers were completely determined up to equivalence for the case of $K = SU(2)$. The result is given in theorem 117. Now, the representation theory of these stabilizers is completely understood, so that a full classification of the strongly continuous representations of G satisfying the positive energy condition is obtained.

The condition $A^\star \mathcal{O}_H \subseteq \overline{C}$ admits a physical interpretation. Namely, the projection of a point $p \in C$ to the time-axis is usually interpreted as its total energy, whereas the value of $\eta(p, p)$ is interpreted as its mass. It is therefore tempting to interpret these values similarly for the perturbed point $p + A^\star H$. In this case, the positive energy condition states precisely that the point $p + A^\star H$ can not have negative mass nor can it be in the orbit of point with negative total energy.

In order to understand to induced representations of positive energy in more detail, a method was developed that embeds homogeneous bundles as eigenspace subbundles of trivial bundles that in particular applies to the bundles obtained in the representation theory of $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes SL(2, \mathbb{C}) \times K$. The eigenspace subbundles thus obtained recover in particular the Dirac equation as well as various eigenvalue equations corresponding to the intrinsic symmetries imposed by K.

Moreover, these eigenspace subbundles resemble various theories in particle physics. For example, elements $X \in \mathfrak{su}(N)$ in the adjoint representation of $SU(N)$ are commonly considered to act on vectors in a representation of $SU(N)$ via the infinitesimal representation, examples being pions and gluons. Such an interplay between the $SU(N)$, its adjoint representation on $\mathfrak{su}(N)$ and the actions of these two on some vector space is precisely how the eigenspace subbundles are obtained. Moreover, degenerate orbits in the adjoint representation $\mathfrak{su}(3)$ of $SU(3)$ may be connected to the phenomenon known to physicists as symmetry breaking.

In the case of $K = SU(2)$, a clear connection was found between the obtained eigenvalue subbundles and the method that was used to determine the stabilizers in theorem 117. Namely, the distinction made in the proof corresponds precisely to whether or not the value of Ap in $\mathfrak{su}(2)$ is zero.

Future Research

The results obtained in this thesis leave a number of open ends and possible directions for future research.

Firstly, in order to allow for a clear physical interpretation of the obtained eigenspace subbundles, it would be beneficial to show that the sections of these bundles can be interpreted as tempered distributions whose Fourier transform satisfy corresponding wave equations. This would complete the analogy to Wigner's analysis of the representation theory of $\mathbb{R}^4 \rtimes SL(2,\mathbb{C})$[BW48].

Secondly, now that the strongly continuous unitary representations satisfying the positive energy condition of the group $G = \mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \mathfrak{k}) \rtimes SL(2,\mathbb{C}) \times K$ have completely been classified for the case of $K = SU(2)$, it remains to apply the results of section 6.5 to realize the corresponding homogeneous bundles as eigenspace subbundles. If we assume that a suitable Fourier transform can be defined on the space of sections of these bundles, this would yield various differential equations that correspond to irreducible representations and these could possibly be related to elementary particles.

It would be interesting to study the case of $K = SU(3)$ in a similar fashion, namely by determining the stabilizers corresponding to positive energy representations and realizing the resulting homogeneous bundles as eigenspace subbundles using section 6.5. Because $SU(3)$ is related to the symmetry group of the strong interaction, this case is of particular physical interest and could provide a further understanding of elementary particles.

Now, in order to determine these stabilizers, it is noteworthy to mention that the main idea of the proof of theorem 117 can be done more generally; namely to recursively restrict the action of $SL(2,\mathbb{C}) \times K$ to smaller subgroups, decompose $\mathbb{R}^4 \oplus \mathrm{Hom}(\mathbb{R}^4, \otimes\mathfrak{k})$ into irreducible representations of the restricted group and consider the various cases in this decomposition. There is good hope that such an approach works more generally because the adjoint action of $SU(N)$ on $\mathfrak{su}(N)$ takes values in $SO(\mathfrak{su}(N), \kappa)$, which are simply higher-dimensional rotations. In particular, the invariant subspaces of such transformations are well-understood so that the stabilizers of $AA^* \in \mathrm{End}(\mathfrak{k})$ can be determined.

Furthermore, as physicists are often interested in invariant quantities that distinguish or characterize various particles, it is of interest to determine the center of the universal enveloping algebra of $\mathbb{R}^4 \oplus (\mathfrak{k} \otimes \mathbb{R}^4) \rtimes \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{k}$. Such elements will act by scalars on any representation of the corresponding Lie algebra.

Finally, besides the physical relevance of the results obtained in this thesis, the method for embedding homogeneous bundles in trivial bundles could possibly be applied more generally and could therefore shed light on the representation theory of other groups, as well as on homogeneous bundles obtained through other means than representation theory.

Chapter 8

Appendix

8.1 Analysis on locally compact groups

This section consists of various results on locally compact groups that are needed throughout the text. Explicitly, two matters are addressed:

1. the construction of the induced representation given in section 4.1 makes the assumption that of the existence of an invariant measure on a quotient space G/H for some closed subgroup H of G. The first section gives some relevant results justifying this assumption.

2. Secondly, the Mackey machine heavily makes use of spectral theory for representations of an Abelian locally compact group. Specifically relevant is the fact that such a representation corresponds to a spectral measure on its dual group. This spectral measure plays an essential role in the Mackey machine. The second section is devoted to a summary of the relevant results.

The results in this section, including the proofs, are taken from [Fol95].

8.1.1 Invariant measures on homogeneous spaces

Let G be a locally compact group and let λ be a left Haar measure. This section is concerned with the existence of a G-invariant measure on homogeneous spaces G/H.

We first introduce the so called *modular function* Δ, which measures the extent to which the left Haar measure fails to be right-invariant.

Define for every $x \in G$ a new left Haar measure λ_x by $\lambda_x(E) = \lambda(Ex)$. Since the left Haar measure on G is unique up to a positive constant, we obtain for every $x \in G$ a number $\Delta(x) > 0$ such that $\lambda_x = \Delta(x)\lambda$. Moreover, this number $\Delta(x)$ does not depend on the original choice of λ, since

$$(c\lambda)_x = c\lambda_x = c\Delta(x)\lambda = \Delta(x)(c\lambda).$$

This yields a function $\Delta : G \to \mathbb{R}_\times$ called the **modular function** of G, where $\mathbb{R}_\times$ denotes the multiplicative group of positive real numbers. Notice that λ is both a left and right Haar measure if and only if $\Delta \equiv 1$. In this case, G is called **unimodular**.

Proposition 140. *The map $\Delta : G \to \mathbb{R}_\times$ is a continuous homomorphism. Moreover, for any $f \in L^1(G)$,*

$$\int R_y f d\lambda = \Delta(y^{-1}) \int f d\lambda. \tag{8.1}$$

Proof. It is trivially verified that $\lambda_{xy} = (\lambda_x)_y$ which implies that $\Delta(xy)\lambda = \Delta(y)\Delta(x)\lambda = \Delta(x)\Delta(y)\lambda$. The equality 8.1 is clear on simple functions since $\lambda(Ey^{-1}) = \Delta(y^{-1})\lambda(E)$. The general case follows by approximation. Continuity of Δ follows since for any fixed $f \in L^1(g)$, the map $g \mapsto \int R_g f d\lambda$ is continuous and by 8.1 we have

$$\Delta(g) = \frac{\int f d\lambda}{\int R_g f d\lambda},$$

which is continuous in g. □

The following proposition gives some classes of unimodular locally compact groups.

Proposition 141. *Let G be a locally compact group and let Δ denote its modular function.*

1. *If K is any compact subgroup of G then $\Delta|_K \equiv 1$.*
2. *If G is compact, then G is unimodular.*
3. *If $G/[G,G]$ is compact, then G is unimodular*
4. *If G is a connected semi-simple Lie group, then G is unimodular.*

Proof.

1. Since Δ is continuous, $\Delta(K)$ must be compact and since Δ is a homomorphism, it is a subgroup of $\mathbb{R}_\times$. The only compact subgroup of $\mathbb{R}_x$ is $\{1\}$, so that $\Delta|_K \equiv 1$.
2. This is immediate from the previous assertion.
3. Since $\mathbb{R}_x$ is Abelian, we have $\Delta([x,y]) = [\Delta(x), \Delta(y)] = 1$. It follows that Δ factors trough $G/[G,G]$. The conclusion now follows from the first assertion.
4. For a semi-simple Lie algebra $\mathfrak{g}$ it holds that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. But $[\mathfrak{g},\mathfrak{g}]$ is the Lie algebra of $[G,G]$. Since G is connected, it follows that $[G,G] = G$. Then $G/[G,G] = \{1\}$, which is clearly compact.

□

Now, consider a locally compact group G with closed subgroup H. Let $G \xrightarrow{q} G/H$ be the quotient map. It will be of particular interest to know when the quotient space G/H possesses a G-invariant Radon measure. This question is addressed in the following.

Let ξ be a left Haar measure of H and denote by Δ_G and Δ_H the modular functions of G and H, respectively. To define a G-invariant measure on G/H, the idea is to construct a surjective map $C_c(G) \xrightarrow{A} C_c(G/H)$ and attempt to define

$$\int_{G/H} Af dx = \int_G f,$$

which would define a G-invariant linear functional on $C_c(G/H)$ and thence a G-invariant Radon measure on G/H. Of course, one still needs to address to question of whether or not such an integral is actually well-defined, as well as define such a surjective map A.

Define the following map, which we call the *averaging* map:

$$\begin{aligned} A &: C_c(G) \to C_c(G/H) \\ (Af)(xH) &\overset{d}{=} \int_H f(x\xi) d\xi. \end{aligned} \tag{8.2}$$

Notice that the resulting map is indeed well-defined by the left invariance if ξ. Moreover, it satisfies

$$A((\phi \circ q) \cdot f) = \phi \cdot Af$$

for every $\phi \in C_c(G/H)$ and $f \in C_c(G)$.

Lemma 142 (Lifting compact sets)**.**
Let $E \subset G/H$ be compact. Then there exists a compact set $K \subset G$ such that $q(K) = KH = E$.

Proof. Let V be an open neighborhood of 1 with compact closure. Since q is an open map, $\{q(xV)\}_{x\in G}$ is an open cover of E so by compactness there exists a finite sub cover $\{x_j V\}_{j=1}^n$. Take $K = q^{-1}(E) \cap \bigcup_{j=1}^n x_j \overline{V}$. □

Lemma 143. *If $E \subset G/H$ is compact, then there exists $f \geq 0$ in $C_c(G)$ such that $Af = 1$ on E.*

Proof. Let $F \supset E$ be compact in G/H and let $K \subset G$ be a compact lift of F. Let $\phi \in C_c(G/H)$ be such that $\phi \geq 0$, $\operatorname{supp} \phi \subset E$ and $\phi|_F = 1$. (Such a function exists e.g. by Tietze's extension theorem.) Let $g \in C_c(G)$ be such that $g \geq 0$ on K. Take

$$f = \frac{\phi \circ q}{Ag \circ q} \cdot g.$$

□

Lemma 144 (Surjectivity of A).
Let $\phi \in C_c(G/H)$. Then there exists $f \in C_c(G)$ such that $\phi = Af$ and $q(\operatorname{supp} f) = \operatorname{supp}\phi$. Moreover, if $\phi \geq 0$ we can take $f \geq 0$.

Proof. Let $g \geq 0$ in $C_c(G)$ be such that $Ag \equiv 1$ on $\operatorname{supp}\phi$. Let $f = (\phi \circ q) \cdot g$. Then $Af = \phi$. The other properties are obvious. □

Finally, the main result of this section can be stated, which comes with a sort of Fubini theorem:

Theorem 145. *Suppose G is a locally compact group and H is a closed subgroup. There is a G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$. In this case, μ is unique up to a constant factor and if this factor is suitably chosen, we have*

$$\int_{G/H} Af d\mu = \int_G f(x)dx = \int_{G/H}\int_H f(x\xi)d\xi d\mu(xH). \tag{8.3}$$

Proof.
A proof can be found in [Fol95, p. 57]. □

Corollary 146.
Let G be a locally compact group and H a closed subgroup of G.
Then G/H has a G-invariant Radon measure of one of the following is satisfied

1. *H is compact,*
2. *G is a Lie group and both G and H are semisimple and connected.*

Proof. Both assertions follow by proposition 141

1. If H is compact, then both $\Delta_G|_H \equiv 1$ and $\Delta_H \equiv 1$.
2. If G is a semisimple Lie group, then since H is closed, it is a semisimple Lie group of its own right. By assumption they are both connected, so proposition 141 yields $\Delta_G \equiv 1$ and $\Delta_H \equiv 1$.

□

8.1.2 The group algebra

This section is mainly concerned with the results needed in the Mackey machine (section 4.4). Particularly important is the result that any representation of such a group corresponds to a unique projection-valued measure on the dual group $\widehat{G}$. The group algebra $L^1(G)$ associated to a locally compact group is first introduced. This algebra has very strong connections with the original group G, but being a Banach-$*$-algebra has more structure and is thus easier to handle. In particular, the Gelfand representation is available and the image of non-degenerate $*$-representation of $L^1(G)$ yields a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$, so that the spectral theorem is available.

A proof of the results mentioned here can be found e.g. in [Fol95]. We assume that G is σ-compact, which is in particular true if G is connected. In this case, the Haar measure of G is σ-finite and $L^1(G)^* \cong L^\infty(G)$ holds.

Let G be a locally compact group, let λ denote its left Haar measure and Δ the modular function. Write dx for $d\lambda(x)$. Then $L^1(G)$ is a Banach $*$-algebra with an approximation of the identity with the operations [Fol95, p.49-54]

$$(f * g)(x) = \int_G f(y)g(y^{-1}x)dy$$
$$f^*(x) = \Delta(x^{-1})\overline{f(x^{-1})}.$$

This algebra is called the **group algebra** of G. The theory of unitary representation G is strongly related to non-degenerate $*$-representations of its group algebra $L^1(G)$. Indeed, suppose that π is a unitary representation of G on $\mathcal{H}_\pi$. Define for $f \in L^1(G)$ the operator $\pi(f)$ on $\mathcal{H}_\pi$ by

$$\pi(f) = \int_G f(x)\pi(x)dx,$$

where the integral is to be interpreted in the weak sense, i.e, $\langle\pi(f)u, v\rangle = \int_G f(x)\langle\pi(x)u, v\rangle dx$. Notice that $|\langle\pi(f)u, v\rangle| \leq \|f\|_1\|u\|\|v\|$ so $\pi(f)$ is a bounded linear operator on $\mathcal{H}_\pi$ with norm $\|\pi(f)\| \leq \|f\|_1$.

Theorem 147. *Let π be a unitary representation of G. Then $\Phi : f \mapsto \pi(f)$ defines a non-degenerate $*$-representation of $L^1(G)$ on $\mathcal{H}_\pi$*

Proof. It is clear that Φ is linear. It is a a routine calculation to show Φ is an $*$-homomorphism $L^1(G) \to \mathcal{L}(\mathcal{H}_\pi)$, making use of Fubini-Tonelli and the substitution rule $d\lambda(x^{-1}) = \Delta(x^{-1})d\lambda(x)$. To see it is non-degenerate, we show that for any $u \neq 0$ there exits some $f \in L^1(G)$ such that $\pi(f)u \neq 0$. Fix $u \neq 0$. Since π is (strongly) continuous and G is locally compact, we can find a compact neighborhood V of $1 \in G$ such that $\|\pi(x)u - f\| \leq \|u\|$. taking $f = |V|^{-1}\mathbb{1}_V$ we find $\|\pi(f)u - u\| \leq \|u\|$ so in particular $\pi(f)u \neq 0$. □

Lemma 148. *The assignment $F : \pi(x) \to \pi(f)$ defines a functor from the category of unitary representation of G to the that of non-degenerate $*$-representations of $L^1(G)$.*
*Moreover, this functor is fully faithful. That is, for any two unitary representations $\pi_1, \pi_2 \in$ **UR**(G) there is a bijection between the Hom sets:*

$$Hom_G(\pi_1, \pi_2) \cong Hom_{L^1(G)}(F(\pi_1), F(\pi_2)).$$

Proof. It is clear that any linear map intertwining two unitary representations π_1, π_2 of G also intertwines the representations $F(\pi_1)$ and $F(\pi_2)$. Therefore F is indeed a faithful functor. It remains to show it is full. Suppose π_i acts on $\mathcal{H}_i$ and $T \in \text{Hom}_G(\pi_1, \pi_2)$. Notice that

$$\begin{aligned} & \int_G f(x)\langle T\pi_1(x)u, v\rangle dx = \int_G f(x)\langle \pi_2(x)Tu, v\rangle dx \qquad \forall f \in L^1(G), \quad u, v \in \mathcal{H}_1 \\ \iff \quad & \langle T\pi_1(x)u, v\rangle = \langle \pi_2(x)Tu, v\rangle, \qquad \forall x \in G, \quad u, v \in \mathcal{H}_1 \\ \iff \quad & T\pi_1(x) = \pi_2(x)T, \qquad \forall x \in G. \end{aligned}$$

□

The following theorem shows that this functor is essentially surjective on objects and therefore establishes an equivalence of categories between unitary representations of G and non-degenerate $*$-representations of $L^1(G)$.

Theorem 149. *Suppose π is a non-degenerate $*$-representation of $L^1(G)$ on the Hilbert space $\mathcal{H}_\pi$. Then π arises from a unique unitary representation of G on $\mathcal{H}_\pi$.*

Proof. A proof can be found in [Fol95, p. 74, Theorem 3.11]. □

Definition 150. Denote by $\widehat{G}$ the **dual space**, that is, the set of equivalence classes of irreducible unitary representations of G.

— Suppose that G is Abelian. By Schur's lemma, all irreducible representations of G are one-dimensional. Thus, if π is any irreducible representation of G, we may take $\mathcal{H}_\pi = \mathbb{C}$ and $\pi(x)(z) = \xi(x)z$ for some character ξ, that is, a continuous group homomorphism $\xi : G \to U(1)$. Therefore, we may identify $\widehat{G}$ with the set of characters of G. We use the notation $\xi(x) = \langle x, \xi\rangle$. Now, by theorem 147 any character $\xi \in \widehat{G}$ determines a non-degenerate $*$-representation of $L^1(G)$ on $\mathcal{B}(\mathbb{C}) \cong \mathbb{C}$:

$$\xi(f) = \int_G \langle x, \xi\rangle f(x)dx \tag{8.4}$$

— It turns out that every element in the spectrum of $L^1(G)$ arises in this manner so that $\Omega(L^1(G)) \cong \widehat{G}$. This fact is essential in the theory of the Mackey machine described in section 4.4.

Proposition 151. *$\widehat{G}$ can be identified with the spectrum of $L^1(G)$ via 8.4. That is, 8.4 determines a multiplicative functional on $L^1(G)$ and every multiplicative functional of $L^1(G)$ is of the form 8.4.*

Proof. Notice that this is *not* an application of 149, since elements in the spectrum of $L^1(G)$ are not necessarily $*$-homomorphisms. The proof given here is taken from [Fol95, p. 88].
Now, let $\tau : L^1(G) \to \mathbb{C}$ be a non-trivial algebra homomorphism. Since $L^1(G)^* \cong L^\infty(G)$, there exists some $\phi \in L^\infty(G)$ such that $\tau(f) = \int_G \phi f dx$. Let $f \in L^1(G)$ be such that $\tau(f) \neq 0$. Then for any $g \in L^1(G)$ we have

$$\begin{aligned} \tau(f) \int \phi(y)g(y)dy &= \tau(f)\tau(g) = tau(f * g) \\ &= \int\int \psi(x)f(xy^{-1})g(y)dydx \\ &= \int \tau(L_y f)g(y)dy. \end{aligned}$$

Since this holds for all g we obtain $\phi(y) = \frac{\tau(L_y f)}{\tau(f)}$ almost everywhere. By replacing ϕ by this expression if necessary, we may assume equality holds everywhere. Then ϕ is continuous and $\tau(f) = \int f\phi dx$ so it remains to show it is a character. Notice that

$$\phi(xy)\tau(f) = \tau(L_{xy}f) = \tau(L_x L_y f) = \phi(x)\phi(y)\tau(f).$$

Thus ϕ is multiplicative. Since ϕ is bounded, this implies that $|\phi(x)| = 1$. □

Remark. Notice that $\widehat{G}$ is an Abelian group under pointwise multiplication. If we endow $\widehat{G}$ with the weak-$*$-topology then it is compact by the Banach Alaoglu theorem, being a closed subset of the unit ball in $L^1(G)^*$. Thus, $\widehat{G}$ is a locally compact Abelian group.

Now, the non-degenerate $*$-representations of an Abelian Banach-$*$-algebra are very well understood, because they are decomposed by a regular spectral measure on its spectrum. This spectral measure simultaneously decomposes unitary representation of G, which one may expect given the equivalence of categories. The general idea followed by the precise statements are given below. For more details, see [Fol95, p. 26].

Assume for simplicity that $\mathcal{A}$ is unital, an assumption that is not necessary but simplifies matters. Suppose that we are given a $*$-representation $\phi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$. the idea is to obtain a spectral measure on $\mathcal{B}$ and pull this back to $\mathcal{A}$ using ϕ.

Let $\mathcal{B} = \overline{\phi(\mathcal{A})}$. Then $\mathcal{B}$ is a C^*-subalgebra of $\mathcal{L}(\mathcal{H})$. The Gelfand representation [Mur90, p. 15] $b \mapsto \widehat{b}$ realizes $\mathcal{B}$ as the continuous functions on its spectrum: $\mathcal{B} \cong C(\Omega(\mathcal{B}))$. This can be used to construct a functional calculus on the bounded measurable functions $B(\Omega(\mathcal{B})) \to \mathcal{B}$ that sends $\widehat{b} \mapsto b$. Applying this functional calculus to indicator functions on the spectrum yields a projection valued measure P on $\Omega(\mathcal{B})$ that satisfies $b = \int_{\Omega(\mathcal{B})} \widehat{b} dP$ for any $b \in \mathcal{B}$. Now, ϕ induces an injective continuous map $\Omega(\mathcal{B}) \xrightarrow{\phi^*} \Omega(\mathcal{A})$. This map can be used to pull the projection-valued measure P back to a projection-valued measure P_A on $\Omega(\mathcal{A})$ via $P_A = P \circ \phi^*$. Moreover, it then holds that $\phi(a) = \int_{\Omega(\mathcal{A})} \widehat{a} dP_A$.

In particular, given a non-degenerate $*$-representation of $L^1(G)$ there exists a unique regular projection-valued measure P on the spectrum $\Omega(L^1(G)) \cong \widehat{G}$ such that

$$\pi(f) = \int_{\widehat{G}} \xi(f) dP(\xi), f \in L^1(G).$$

Where $\xi(f)$ is given by 8.4. Now, the following theorem asserts that this projection-valued measure simultaneously decomposes the corresponding unitary representation π of G:

Theorem 152. *Let π be a unitary representation of the locally compact Abelian group G. There is a unique regular $\mathcal{H}_\pi$-projection-valued measure P on $\widehat{G}$ such that*

$$\pi(x) = \int_{\widehat{G}} \langle x, \xi \rangle dP(\xi) \qquad x \in G,$$
$$\pi(f) = \int_{\widehat{G}} \xi(f) dP(\xi) = \int_{\widehat{G}} \int_G \langle x, \xi \rangle f(x) dx dP(\xi) \qquad f \in L^1(G)$$

Moreover, an operator $T \in \mathcal{B}(\mathcal{H}_\pi)$ belongs to the commutant $Hom(\pi)$ of π if and only if T commutes with $P(E)$ for every Borel set $E \subset \widehat{G}$.

Corollary 153. *Let S be a locally compact Hausdorff space and let ϕ be a non-degenerate $*$-representation on $\mathcal{H}$. There is a unique regular projection-valued measure P on S such that $\phi(f) = \int_S f dP$for all $f \in C_0(S)$. A linear operator $T \in \mathcal{L}(\mathcal{H})$ belongs to the commutant $Hom(\phi)$ if and only if T commutes with $P(E)$ for every Borel set E.*

8.2 Some facts from representation theory of Lie groups.

Irreducible representations of a product of compact Lie groups

Lemma 154. *Suppose G_1, G_2 are two compact Lie groups. Then any irreducible representation of $G = G_1 \times G_2$ can be written as $\rho_1 \otimes \rho_2$ for some irreducible representations ρ_1, ρ_2 of G_1 and G_2, respectively.*

Proof. Suppose that π is a representation of $G_1 \times G_2$ on V. The embeddings $G_i \hookrightarrow G_1 \times G_2$ induce representations of G_1 and G_2 on V that intertwine each others action. Now, V decomposes into irreducible G_1 representations $V \cong \bigoplus_k V_k$ and by Schur's lemma, G_2 acts by a scalar on each component V_k. This implies that the action of G_2 on each V_k takes values in $\mathbb{C}I$ so that the action is irreducible. It follows that $\pi \cong \bigoplus \pi_1|_{V_k} \otimes \pi_2|_{\mathbb{C}I}$ is a decomposition into irreducible representations of $G_1 \times G_2$. Thus, if π is irreducible, this direct sum decomposition contains only one term, proving the claim. $\square$

Representations of $SU(2)$

We first describe the irreducible representations of $SU(2)$. These are well-known, see e.g. [Hal03].
For $n \in \mathbb{N}$, let $P_n(\mathbb{C}^2)$ denote the space of homogeneous complex-valued polynomials of degree n. Then the left regular representation σ_n of $SU(2)$ on $P_n(\mathbb{C}^2)$ given by $(\sigma_n(u)p)(z) = p(u^{-1}z)$ is an irreducible representation. Moreover, these are all inequivalent and exhaust all up to equivalence all finite dimensional irreducible representations of $SU(2)$. Notice that $SU(2)$ is compact and therefore its representations are unitarizable by averaging the action over the whole group.

Lemma 155. *The representation σ_n on $P_n(\mathbb{C}^2)$ is equivalent to the representation $\widetilde{\sigma}_n$ on $\bigvee^n \mathbb{C}^2$, where the action $\widetilde{\sigma}_n$ is given by $\widetilde{\sigma}_n(u)v_1 \cdots v_n = (uv_1) \cdots (uv_2)$.*

Proof. The claim is trivial for $n = 1$. Assume $n > 1$. A basis for the n^{th} symmetric power $\bigvee^n \mathbb{C}^2$ is given by $\{e_{i_1} \cdots e_{i_n} : i_1 \leq \cdots i_n\}$. Notice such a basis element only depends on the number of occurrences of each i in $i_1, \cdots i_n$. Therefore, this basis can be equivalently be written as $\{e_1^{p_1} e_2^{p_1} : p_1 + p_2 = n\}$. Define the map

$$\begin{aligned} \Phi : \bigvee^n \mathbb{C}^2 &\to P_n(\mathbb{C}^2) \\ \Phi(e_1^{p_1} e_2^{p_2}) &\mapsto x_1^{p_1} x_2^{p_2} \end{aligned}$$

It is clear that this is a linear bijection. However, this map is not $SU(2)$-equivariant. Let $u \in SU(2)$. We have

$$\begin{aligned} \Phi((ue_1)^{p_1}(ue_2)^{p_2}) &= \Phi\left((u_{11}e_1 + u_{21}e_2)^{p_1}(u_{12}e_1 + u_{22}e_2)^{p_2}\right) \\ &= (u_{11}x_1 + u_{21}x_2)^{p_1}(u_{12}x_1 + u_{22}x_2)^{p_2} \\ &= \Phi(e_1^{p_1} e_2^{p_2})(u^t x) \end{aligned}$$

Thus, pre-composing Φ with the linear automorphism of $\bigvee^n \mathbb{C}^2$ given by $v_1 \cdots v_n \mapsto \overline{v_1} \cdots \overline{v_n}$ yields an $SU(2)$-equivariant isomorphism $\bigvee^n \mathbb{C}^k \to P_n(\mathbb{C}^k)$. $\square$

Representations of the group $\mathbb{C} \rtimes_\sigma U(1)$

The group $G \stackrel{d}{=} \mathbb{C} \rtimes_\sigma U(1)$ is of the form suitable for the Mackey machine. Therefore, all irreducible unitary representations of G can be obtained by inducing representations of the stabilizer of the action of $U(1)$ on $\mathbb{C}$ up to G. Notice that the inner product $\mathrm{Re}(b^*a)$ on $\mathbb{C}$ is invariant under the action of $U(1)$. Therefore, the pairing

$$\langle a, b \rangle = e^{i\,\mathrm{Re}(b^*a)}$$

identifies $\widehat{\mathbb{C}}$ with $\mathbb{C}$ and under this identification, the action of $U(1)$ on $\widehat{\mathbb{C}}$ is becomes just $z \cdot b = z^2 b$. The orbits in $\widehat{\mathbb{C}}$ under this action are $\{0\}$ and $\rho \cdot U(1)$ for $\rho > 0$. The corresponding stability subgroups are $U(1)$ and $\{\pm 1\}$, respectively. Thus, the irreducible unitary representations of G are those of $U(1)$ and $\{\pm 1\}$ lifted up to G.

Since $U(1)$ is Abelian, its irreducible representations are its one-dimensional characters. They are given by $z \mapsto z^n$, $n \in \mathbb{Z}$. Finally, $\{\pm 1\}$ has only two representations, namely the trivial one and the one which sends ± 1 to itself.

8.3 Relation between projective representations of the Poincarè group and wave equations

There is a strong connection between the projective unitary representations of the Poincaré group $\mathcal{P} = \mathbb{R}^4 \rtimes SO(1,3)^0$ and some early results in the theory of relativistic quantum mechanics, which was first discovered by Wigner[Wig39, BW48]. A detailed exhibition of these results is given in this section, which is mainly based on [Var07, p.356-362].

It is well-known that the solution spaces of certain partial differential equations that occur in physics, such as the Maxwell or Dirac equations, are invariant under transformation by the Poincaré group. Moreover, quantum mechanics tells us that the state space of a particle subject to such an equation is a projective Hilbert space. In view of Wigners theorem[Wig39], there must be a projective unitary representation of the Poincaré group on this state space, so that we should be able to give a physical interpretation of some of the obtained representations. Now, Bargmann's theorem[Bar54] implies that any continuous projective unitary representation of $\mathcal{P}$ lifts to a strongly continuous unitary representation of its universal covering group $\widetilde{\mathcal{P}} = \mathbb{R}^4 \rtimes SL(2,\mathbb{C})$ and thus we are led to consider the representation theory of $\widetilde{\mathcal{P}}$.

It is shown in corollary 83 that the orbit space of $\widehat{\mathbb{R}^4}$ under the action of $SL(2,\mathbb{C})$ is countably separated. As a consequence, the theory of the Mackey machine applies (section 4.4) so that the irreducible strongly continuous unitary representations of $\widetilde{\mathcal{P}}$ are obtained by inducing a suitable representation up to $\widetilde{\mathcal{P}}$ from one stabilizers of the action of $SL(2,\mathbb{C})$ on $\widehat{\mathbb{R}^4}$. In section 4.5, the various stabilizers and orbits are determined. Recall further from section 4.1 that for finite dimensional unitary representations of such stabilizers, the induced representations are obtained as section of a homogeneous Hilbert bundle. In order to relate such induced representations to wave equations, these homogeneous bundles are realized as eigenvalue subbundles that correspond to these equations.

By lemma 27, we know that every Hilbert bundle $E \to B$ that is homogeneous with respect to the action of some group G is equivalent to $G \times_{G_b} E_b \to G/G_b$ for any $b \in B$. Therefore, the strategy in the succeeding is as follows: First, with the help of physical knowledge we obtain explicitly a Hilbert bundle $E \to B$ that is homogeneous with respect to a suitable $\widetilde{\mathcal{P}}$-action. If it is shown that the stabilizer $\widetilde{\mathcal{P}}_b$ at any point b is isomorphic to a certain Little group H of interest and the the representation of P_b^* on E_b is equivalent to a representation σ of H, then it follows that $E \cong G \times_H \mathcal{H}_\sigma$ and the induced representation $\mathrm{ind}_H^{\widetilde{\mathcal{P}}}(\sigma)$ is constructed out of the sections of this bundle.

Before we begin, a brief remark on how these Hilbert bundles might be obtained. The various orbits of the action of $(2,\mathbb{C})$ on $\widehat{\mathbb{R}^4}$ are subspaces of $\widehat{\mathbb{R}^4}$, which is physically interpreted as the momentum space. The differential equations are to be satisfied in the position-domain and so, via the Fourier transform we obtain a corresponding equation in the momentum-domain. Now, the idea is to consider a bundle such that sections of this bundle correspond to solutions to this latter equation. The Fourier transform of these sections, in the sense of tempered distributions then satisfies the corresponding differential equation. The following result makes the connection precise, the proof of which can be found in [Var07]:

Lemma 156. *Denote by α_λ the invariant measure on $\mathcal{O}_\lambda^+$, $\lambda \geq 0$. Let $s \geq 0$ and suppose that f is a complex-valued measurable function on $\widehat{\mathbb{R}^4}$ such that*

$$\int_{\mathcal{O}_\lambda^+} |f(p)|^2 p_0^{-s} d\alpha_\lambda(p) < \infty,$$

Then the distribution

$$T_f : \phi \mapsto \int_{\mathcal{O}_\lambda^+} f(p)\phi(p) d\alpha_\lambda, \quad \phi \in C_c^\infty(\widehat{\mathbb{R}^4}),$$

is well-defined, tempered and its Fourier transform $\widehat{T_f}$ satisfies the wave equation:

$$(\Box + m^2)\widehat{T_f} = 0$$

Where $\Box = \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$.

Representations corresponding to $\mathcal{O}_\lambda^+$

We consider the representations corresponding to the orbits $\mathcal{O}_\lambda^+$ and the Little group $SU(2)$. Physically, these orbits correspond to particles with half-integer spin and mass $m = \lambda^{\frac{1}{2}}$.

We start describing the Hilbert bundle corresponding to the defining representation σ_1 of $SU(2)$, which corresponds to spin $\frac{1}{2}$ particles and mass m. These are known to satisfy the Dirac wave equation of the electron, which is given by $\sum_{r=0}^{3} i\gamma_r \frac{\partial}{\partial x_r}\Phi = m\Phi$, where $\{\gamma_r\}$ are the Dirac matrices defined by equation (3.6). Because the orbits $\mathcal{O}_\lambda^+ \subset \widehat{N}$ and $\widehat{N}$ is interpreted as momentum space, we are motivated to take the Fourier transform of this equation and define the corresponding bundle by

$$B_m^{+,\frac{1}{2}} = \left\{ (p,v) \in \mathcal{O}_\lambda^+ \times \mathbb{C}^4 \mid \sum_{k=0}^{3} p_k\gamma_k v = mv \right\} \tag{8.5}$$

$$\pi : B_m^{+,\frac{1}{2}} \to \mathcal{O}_\lambda^+, \quad (p,v) \mapsto p. \tag{8.6}$$

Now, the Dirac matrices $\{\gamma_k\}$ satisfy (3.5) and thus define a representation $\rho : \mathrm{Cl}(\mathbb{R}^4,\eta) \to \mathbb{C}^4$ of the Clifford algebra of $(\mathbb{R}^4,\eta)$ in $\mathbb{C}^4$ such that $\rho(e_k) = \gamma_k$. In terms of this representation, the bundle $B_m^{+,\frac{1}{2}}$ can be written as

$$B_m^{+,\frac{1}{2}} = \left\{ (p,v) \in \mathcal{O}_\lambda^+ \times \mathbb{C}^4 \mid \rho(p)v = mv \right\}.$$

The restriction of the representation to the connected component of the identity of the spin group defines a representation S of $SL(2,\mathbb{C})$ on the same space that satisfies that equivariance condition (3.4)

$$\rho(\phi(w)p) = S(w)\rho(p)S(w)^{-1}.$$

Notice that as a consequence of this equivariance, the fibers above all points are of the same dimension and equal the dimension of the eigenspace of $\rho(p)$ corresponding to the eigenvalue m. Above the point $(m,0,0,0)$, this is precisely the eigenspace of γ_0 corresponding to the eigenvalue 1, which is the two-dimensional subspace spanned by $e_0 + e_2$ and $e_1 + e_3$.

Next, let us define an action of $SL(2,\mathbb{C})$ on $B_m^{+,\frac{1}{2}}$ such that the bundle projection π is equivariant. Notice that by the above equivariance, if $(p,v) \in B_m^{+,\frac{1}{2}}$, then $(\phi(w)p, S(w)v) \in B_m^{+,\frac{1}{2}}$ for any $w \in SL(2,\mathbb{C})$ so that we can endow $B_m^{+,\frac{1}{2}}$ with the smooth $SL(2,\mathbb{C})$-action given by $(w,b)\cdot(p,v) \stackrel{d}{=} (\phi(w)p, S(w)v)$. This makes the bundle $B_m^{+,\frac{1}{2}}$ $SL(2,\mathbb{C})$-homogeneous. Indeed, the action of $SL(2,\mathbb{C})$ on the orbit $\mathcal{O}_\lambda^+$ is transitive and it is clear that the map $(p,v) \mapsto g\cdot(p,v)$ is a linear isomorphism from the fiber above p to the fiber above $g\cdot p$.

We proceed by defining a family of inner products on the fibers of the bundle $B_m^{+,\frac{1}{2}}$ such that the action G becomes unitary on fibers, making the bundle into a homogeneous Hilbert bundle. A computation based on equation (3.6) and equation (3.7) shows that $S(w)^*\gamma_0 S(w) = \gamma_0$, thus the representation S leaves the Hermitian bilinear form $v \mapsto m^{-1}\langle\gamma_0 v, v\rangle$ invariant. This form is seen to be positive definite on all fibers by noting that γ_0 is self-adjoint and γ_k is skew-hermitian for $k = 1,2,3$. Therefore, the equation $\sum_{k=0}^{3} p_k\langle\gamma_k v, v\rangle = m\langle v,v\rangle$ decomposes into its real and imaginary part, from which it follows that $\sum_{k=1}^{3} p_k\langle\gamma_k v, v\rangle = 0$. Thus $p_0\langle\gamma_0 v, v\rangle = m\langle v,v\rangle$ and $m^{-1}\langle\gamma_0 v, v\rangle = p_0^{-1}\langle v,v\rangle$ for all $v \in B_m^{+,\frac{1}{2}}(p)$, which shows the assertion. This also shows that the form is equivalently given by $v \mapsto p_0^{-1}\langle v,v\rangle$. By the invariance of this form, the action of $SL(2,\mathbb{C})$ on $B_m^{+,\frac{1}{2}}$ is unitary on fibers so we do indeed obtain a homogeneous Hilbert bundle.

It remains to check that the representation of any stabilizer subgroup $SL(2,\mathbb{C})_p$ on the fiber above p is equivalent to the fundamental representation σ_1 of $SU(2)$.
Consider the point $p = (m,0,0,0)$. We have already seen in section 4.5 that the fiber above p is the eigenspace of γ_0 corresponding to eigenvalue one, which is the subspace of vectors in $\mathbb{C}^4$ of the form (z,z) for some $z \in \mathbb{C}^2$. Moreover, we know from section 4.5 that $SU(2)$ is the stabilizer of p. From equation (3.7), it follows directly that the representation of $SU(2)$ on E_p is given by $u\cdot(z,z) = (uz,uz)$, which is unitarily equivalent to the representation σ_1.

In view of lemma 27 and lemma 80, it can be concluded that the bundles $B_m^{+,\frac{1}{2}}$ and $\widetilde{\mathcal{P}} \times_{SU(2)} \mathbb{C}^2$ are isomorphic as Hilbert bundles and thus, the representation $\mathrm{ind}_{SU(2)}^{\widetilde{\mathcal{P}}}(\sigma_1)$ is constructed out of the sections of the bundle $B_m^{+,\frac{1}{2}}$.

Next, we consider σ_n for $n > 1$. We know from lemma 155 that this representation acts on the n^{th} symmetric power $\bigvee^n \mathbb{C}^2$. Define the following bundle using the symmetric product of bundles

$$B_m^{+,\frac{n}{2}} \stackrel{d}{=} \bigvee^n B_m^{+,\frac{1}{2}}$$

with the natural projection π_n. If we identify $\bigvee^n \mathbb{C}^2$ with the subspace $S^n(\mathbb{C}^2)$ of $\mathbb{C}^{2^{\otimes n}}$ consisting of symmetric tensors, the fiber $B_m^{+,\frac{n}{2}}(p)$ is precisely the subspace of symmetric tensors satisfying

$$\sum_{r=0}^{3} p_r \gamma_r^\nu t = mt, \quad \nu = 1, 2, ..., n \tag{8.7}$$

where $\gamma_r^\nu = 1 \otimes \cdots \otimes \gamma_r \otimes 1 \cdots \otimes 1$. Indeed, it is clear that any symmetric tensor in the fiber satisfies these equations. Conversely, if a symmetric tensor satisfies all these equations, then expanding the tensor out in bases in all except the ν^{th} factor shows that this factor is in the eigenspace of $\rho(p)$ with eigenvalue m. Since ν is arbitrary, it follows that $t \in B_m^{+,\frac{n}{2}}(p)$. Now, the fiber $B_m^{+,\frac{n}{2}}(p)$ is equipped with the inner product given by

$$\langle v_1 \otimes \cdots v_n, v_1' \otimes \cdots \otimes v_n' \rangle = p_0^{-n} \prod_{i=1}^{n} \langle v_i, v_i' \rangle^n \qquad v_i, v_i' \in B_m^{+,\frac{1}{2}}(p). \tag{8.8}$$

Define $S_n(w) \stackrel{d}{=} S(w)^{\otimes n}$. Then S_n leaves the space of symmetric tensors invariant and so $SL(2,\mathbb{C})$ acts on $B_m^{+,\frac{n}{2}}$ by $(w, b) \cdot (p, v_1 \otimes \cdots \otimes v_n) = (\phi(w)p, S_n(w)v_1 \otimes \cdots \otimes v_n)$. This action projects to a transitive action of $SL(2,\mathbb{C})$ on $\mathcal{O}_\lambda^+$ and S_n leaves the positive definite inner product defined above invariant, making the bundle into a homogeneous Hilbert bundle.

It remains to show that the representation of any stabilizer subgroup $SL(2,\mathbb{C})_p$ on the fiber above p is equivalent to the representation σ_n of $SU(2)$. We have for any $p \in \mathcal{O}_\lambda^+$ the following equivalence of unitary $SU(2)$-representations.

$$B_m^{+,\frac{n}{2}}(p) \cong \bigvee^n B_m^{+,\frac{1}{2}}(p) \cong \bigvee^n \mathbb{C}^2.$$

In view of lemma 155 and by lemma 27, we conclude that we have an equivalence of homogeneous Hilbert bundles

$$B_m^{+,\frac{n}{2}} \cong SL(2,\mathbb{C}) \times_{SU(2)} \bigvee^n \mathbb{C}^2.$$

Observe that by lemma 156, the Fourier transform of the sections of the bundle $B_m^{+,\frac{n}{2}}$ are tempered distributions, and they satisfy the equation

$$\sum_{r=0}^{3} i\gamma_r^\nu \frac{\partial}{\partial x_r} \Phi = m\Phi, \quad \nu = 1, 2, \cdots, n.$$

Representations corresponding to $\mathcal{O}_0^+$

We consider the representations corresponding to the orbits $\mathcal{O}_0^+$ and the Little group $E(2)$. We only consider the representations $z \mapsto z^n$ of $U(1)$ induced up to $E(2)$, as the others are not of much physical interest.

Consider the representations $z \mapsto z^{\pm 1}$. The corresponding Hilbert bundles can be obtained from $B_m^{+,\frac{1}{2}}$ by taking the limit $m \downarrow 0$. Explicitly, consider the bundle

$$B_0^{+,} = \left\{ (p, v) \in \mathcal{O}_0^+ \times \mathbb{C}^4 \; : \; \sum_{r=0}^{3} p_r \gamma_r v = 0 \right\}$$

with the projection map $\pi_0 : B_0^{+,} \to \mathcal{O}_0^+$. This bundle is a homogeneous vector bundle with respect to the $SL(2,\mathbb{C})$ action given by $w \cdot (p, v) = (\phi(m)p, S(w)v)$. That is, $SL(2,\mathbb{C})$ acts transitively on $\mathcal{O}_0^+$, the projection map is equivariant and the map $\pi_0^{-1}(p) \to \pi_0^{-1}(g \cdot p)$ is a linear isomorphism.

Using elementary linear algebra, one finds that for any $m > 0$, the fiber $B_m^{+,\frac{1}{2}}(p^{(m)})$ above the point $p^{(m)} = (\sqrt{1+m^2}, 0, 0, 1)$ is spanned by the vectors

$$\begin{aligned} v_1^{(m)} &= \frac{1}{2} m e_1 + \frac{1}{2}\left(1 + \sqrt{1+m^2}\right) e_3 \\ v_2^{(m)} &= \frac{1}{2}\left(1 + \sqrt{1+m^2}\right) e_2 + \frac{1}{2} m e_4 \end{aligned}$$

which converge in $\mathbb{R}^4 \times \mathbb{C}^4$ to e_3 and e_2 respectively under the limit $m \downarrow 0$. These span the fiber $B_0^{+,}(p_0)$ at $p_0 = (1, 0, 0, 1)$.

By homogeneity, it follows that for any point $(p, v) \in B_0^{+,}$, there exists a sequence of points $(p^{(m)}, v^{(m)}) \in B_m^{+,\frac{1}{2}}$ such that $(p^{(m)}, v^{(m)}) \to (p, v)$ in $\mathbb{R}^4 \times \mathbb{C}^4$ as $m \downarrow 0$. A consequence of this fact is that the Hermitian form $v \mapsto p_0^{-1}\langle v, v\rangle$ is $S(w)$-invariant for every $w \in SL(2,\mathbb{C})$ and positive definite on each fiber. Indeed, we know that these statements holds true for on $B_m^{+,\frac{1}{2}}$ for any $m > 0$ and therefore, by continuity, also for $B_0^{+,}$.

Now, the same argument can be used to exhibit $B_0^{+,}$ as the limit of the bundles $B_m^{+,\frac{1}{2}}$ as $m \uparrow 0$. One observes that

$$\Gamma \overset{d}{=} i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since Γ anti-commutes with all γ_r, one finds that it transforms $B_m^{+,\frac{1}{2}}$ to $B_{-m}^{+,\frac{1}{2}}$ for any $m > 0$. Thus, in the limit it leaves the fibers invariant. Moreover, since Γ commutes with $S(w)$ for every $w \in SL(2,\mathbb{C})$, the eigenspaces coincide. This means that the representation of the stability group $E(2)$ at any point $p \in \mathcal{O}_0^+$ in the fiber above p can be reduced by the eigenspaces of Γ. Therefore, we define

$$B_0^{+,\pm\frac{1}{2}} \overset{d}{=} \left\{(p, v) \in B_0^{+,} \,:\, \Gamma v = \mp v\right\}$$

By the preceding discussion, $E(2)$ leaves $B_0^{+,\pm\frac{1}{2}}$ both invariant. It remains to show that the representation of any stabilizer subgroup $SL(2,\mathbb{C})_p$ on the fiber above p is equivalent to the representation $z \mapsto z^{\pm 1}$. To do so, consider the point $p_0 = (1, 0, 0, 1)$. Recall that the fiber $B_0^{+,}(p_0)$ is spanned by e_2 and e_3, and $E(2)$ acts on this fiber by $m_{z,a} \mapsto S(m_{z,a})$. A computation based on equation (3.7) shows that

$$S(m_{z,a})(c_2e_2 + c_3e_3) = (z^{-1}c_2e_2 + zc_3e_3).$$

Thus, since e_1, e_2 span the eigenspace of Γ corresponding to the eigenvalue $+1$ and e_3, e_4 that corresponding to the eigenvalue -1, it follows directly that the representation of $E(2)$ on $B_0^{+,\pm\frac{1}{2}}(p_0)$ is $m_{z,a} \mapsto z^{\pm 1}$, which is equivalent to the representation $z \mapsto z^{\pm 1}$ of $U(1)$ on $\mathbb{C}$.

Next, we consider the representations $z \mapsto z^{\pm n}, n \in \mathbb{N}$. We proceed as in the case of $m > 0$. Define the bundle

$$\begin{aligned} B_0^{+,\pm\frac{n}{2}} &\overset{d}{=} B_0^{+,\frac{\pm 1}{2}} \otimes \cdots \otimes B_0^{+,\frac{\pm 1}{2}} \quad \text{(n factors)} \\ &= \left\{(p, t) \in \mathcal{O}_0^+ \times \mathbb{C}^{4^{\otimes n}} \,:\, \left(\sum_{r=0}^{3} p_r \gamma_r^\nu\right) t = 0, \quad \Gamma^\nu t = \mp t, \quad \nu = 1, 2, \cdots, n\right\} \end{aligned}$$

Where $\Gamma^\nu = i\gamma_0^\nu\gamma_1^\nu\gamma_2^\nu\gamma_3^\nu$. Now, $SL(2,\mathbb{C})$ acts on $B_0^{+,\pm\frac{n}{2}}$ by $w \cdot (p, t) = (\phi(w)p, S_n(w)t)$ and each fiber is given the inner product defined by

$$\langle v_1 \otimes \cdots v_n, v_1' \otimes \cdots \otimes v_n'\rangle = p_0^{-n} \prod_{i=1}^{n} \langle v_i, v_i'\rangle^n \qquad v_i, v_i' \in B_0^{+,\pm\frac{n}{2}}(p). \tag{8.9}$$

making $B_0^{+,\pm\frac{n}{2}}$ into a Homogeneous Hilbert bundle. The action of $E(2)$ on $B_0^{+,\pm\frac{n}{2}}$ is equivalent to the representation $z \mapsto z^{\pm 1} \otimes \cdots \otimes z^{\pm 1}$ on $\mathbb{C}^{\otimes n}$, which is equivalent to the representation $z^{\pm n}$ on $\mathbb{C}$.

Observe that by lemma 156, the Fourier transform of sections of the bundle $B_m^{+,\pm\frac{n}{2}}$ are tempered distributions and they satisfy the following equations, which are related to Majorana fermions.

$$\begin{aligned} &\sum_{r=0}^{3} i\gamma_r^\nu \frac{\partial}{\partial x_r}\Phi = 0, \\ &i\gamma_0^\nu\gamma_1^\nu\gamma_2^\nu\gamma_3^\nu\Phi = -\Phi, \quad \nu = 1, 2, \cdots, n. \end{aligned}$$

8.4 External tensor product of group actions

In order to understand It is beneficial to reformulate theorem 88 in a more familiar setting. As such, an understanding of an external tensor product of group actions is first developed in a general setting. Now, in section 5.2 a setting is encountered similar to that of theorem 88, in which all actions considered are in fact unitary. This case is a bit simpler to understand, so that it also provides a decent stepping stone towards the case in which only an invariant non-degenerate bilinear form is available; as is the case for the action of $SL(2,\mathbb{C})$ on Minkowski space $(\mathbb{R}^4, \eta)$.

Suppose that $G = G_1 \times G_2$ is a Lie group and G_i has a smooth representation π_i on V_i for $i = 1, 2$. Let us consider the action $\pi = \pi_1 \otimes \pi_2$ of G on $V = V_1 \otimes V_2$ defined on simple tensors by

$$\pi(g) \cdot v_1 \otimes v_2 \overset{d}{=} \pi_1(g_1)v_1 \otimes \pi_2(g_2)v_2.$$

Assume that we are given a G-invariant inner product κ on V_1 and a G-invariant non-degenerate symmetric bilinear form η on V_2. Notice that we have the following vector spaces are linearly isomorphic:

$$V_1 \otimes V_2 \cong V_1 \otimes V_2^* \cong \mathrm{Hom}(V_2, V_1),$$

where the isomorphism is given on simple tensors by

$$\begin{aligned} \Phi : V_1 \otimes V_2 &\to \mathrm{Hom}(V_2, V_1), \\ u \otimes v &\mapsto \eta(\cdot, v)u. \end{aligned} \tag{8.10}$$

Under these isomorphisms and using the invariance of η, the G action on $\mathrm{Hom}(V_2, V_1)$ becomes

$$\pi(x)T = \pi_1(x_1) \circ T \circ \pi_2(x_2)^{-1}, \qquad T \in \mathrm{Hom}(V_2, V_1). \tag{8.11}$$

Therefore, $x \in G$ stabilizes $T \in \mathrm{Hom}(V_2, V_1)$ if and only if

$$T \circ \pi_2(x_2) = \pi_1(x_1) \circ T \tag{8.12}$$

That is, if we define the representations of G

$$\begin{aligned} \rho_1 : G &\to GL(V_1), & \rho_2 : G &\to GL(V_2) \\ \rho_1(x_1, x_2) &= \pi_1(x_1) & \rho_2(x_1, x_2) &= \pi_2(x_2), \end{aligned}$$

then (8.12) states precisely that the stabilizer G_T of T in G is the largest subgroup H of G for which T intertwines the representations $\rho_1|_H$ and $\rho_2|_H$ of H.

In case of unitary actions

Consider the special case in which the bilinear form η on V_2 is actually a positive definite inner product. Because κ and η are G_1- and G_2-invariant, respectively, this means that G_1 and G_2 act by unitary transformations. In this case, the action of G on $\mathrm{Hom}(V_2, V_1)$ is in particular compatible with the singular value decomposition, which means that a lot of information of the G action is contained in the separate actions of G_1 and G_2 on $\mathrm{End}(V_1)$ and $\mathrm{End}(V_2)$.

Explicitly, let $T \in \mathrm{Hom}(V_2, V_1)$ be an arbitrary linear map and let $T = U\Sigma V^*$ be its singular value decomposition. Recall that the non-zero elements of Σ are precisely the positive square roots of the eigenvalues of both T^*T and TT^*. Let $S = \{\sigma_k\}$ denote the *distinct non-zero* singular values and let U_k and V_k be the subspaces spanned by the corresponding left- and right-singular vectors. The singular value decomposition tells us that T is a linear isomorphism between the subspaces V_k and U_k for every $\sigma_k \in S$.

Now, since G_1 and G_2 act on V_1 and V_2 by unitary transformations, the action of G on $\mathrm{Hom}(V_2, V_1)$ simply changes the unitary matrices U and V in the singular value decomposition, thus leaving the singular values invariant. Recall that the singular values are the eigenvalues of T^*T and TT^*. We can make the assignments $T \mapsto T^*T$ and $T \mapsto TT^*$ equivariant if we endow $\mathrm{End}(V_1)_{sa}$ and $\mathrm{End}(V_2)_{sa}$ with the G_1 and G_2 actions, respectively, given by:

$$\begin{aligned} x_1 \cdot A &= \pi_1(x_1)A\pi_1(x_1)^{-1}, & A &\in \mathrm{End}(V_1), \quad x_1 \in G_1, \\ x_2 \cdot B &= \pi_2(x_2)B\pi_2(x_2)^{-1}, & B &\in \mathrm{End}(V_2), \quad x_2 \in G_2. \end{aligned}$$

Notice in particular that

$$\begin{aligned} x_1 \cdot A = A &\iff \pi_1(x_1)A = A\pi_1(x_1), && A \in \mathrm{End}(V_1), \quad x_1 \in G_1, \\ x_2 \cdot A = A &\iff \pi_2(x_2)A = A\pi_2(x_2), && B \in \mathrm{End}(V_2), \quad x_2 \in G_2. \end{aligned}$$

Lemma 157. *Suppose that $X, Y \in End(V)$ for some finite dimensional vector space V. Assume further that X is diagonalizable and Y is bijective. Then $[X, Y] = 0$ if and only if Y leaves every eigenspace V_λ of X invariant.*

Proof. If $[X, Y] = 0$ and $Xv = \lambda v$, then $XYv = \lambda Yv$, so $Y : V_\lambda \to V_\lambda$. Conversely, if T leaves every eigenspace V_λ of X invariant, then X and Y commute on all eigenspaces V_λ of X. These span V since X is diagonalizable so X and Y commute everywhere. □

Definition 158. Let $\mathcal{H}$ be a Hilbert space, and let $A \in \mathcal{L}(\mathcal{H})$ be a bounded linear operator on $\mathcal{H}$. A subspace M is called **reducing** for A if both M and $M^\perp$ are A-invariant subspaces.

Remark.

- If M is a reducing subspace for $A \in \mathcal{L}(\mathcal{H})$ as in the definition above, then $A = A|_M \oplus A|_{M^\perp}$.
- A subspace M is reducing for $A \in \mathcal{L}(\mathcal{H})$ if and only if it is invariant for both A and A^*, where A^* denotes the Hermitian adjoint of A. In particular, if A is invertible and M is A-invariant, then we have $AM = M$ and therefore $A^{-1}M = A^{-1}AM = M$ so that M is also A^{-1}-invariant. Thus, if A is unitary then M is reducing for A if and only if M is A-invariant.
- Lemma lemma 157 above implies that

  $$\begin{aligned} x_1 \in (G_1)_{TT^*} &\iff \pi_1(x_1) \text{ leaves all eigenspaces of } TT^* \text{ invariant}, \\ x_2 \in (G_2)_{T^*T} &\iff \pi_2(x_2) \text{ leaves all eigenspaces of } T^*T \text{ invariant}. \end{aligned}$$

  Seeing as both π_1 and π_2 act by unitary transformations, by the above observation this is equivalent to

  $$\begin{aligned} x_1 \in (G_1)_{TT^*} &\iff \text{all eigenspaces of } TT^* \text{ are reducing for } \pi_1(x_1), \\ x_2 \in (G_2)_{T^*T} &\iff \text{all eigenspaces of } T^*T \text{ are reducing for } \pi_2(x_2). \end{aligned}$$

Lemma 159. *Let T and $\{V_k\}_{\sigma_k \in S}$ be as above. Then*

$$G_T = \left\{ (x_1, x_2) \in (G_1)_{TT^*} \times (G_2)_{T^*T} \,:\, \pi_1(x_1) \circ T|_{V_k} = T \circ \pi_2(x_2)|_{V_k} \quad \forall \sigma_k \in S \right\}. \tag{8.13}$$

Moreover, for every $x = (x_1, x_2) \in G_T$, the subspaces U_k and V_k are reducing for $\pi_1(x_2)$ and $\pi_2(x_2)$, respectively.

Proof. Notice first that $G_T \subset (G_1)_{TT^*} \times (G_2)_{T^*T}$ because the assignments $T \mapsto TT^*$ and $T \mapsto T^*T$ are G_1 and G_2-equivariant, respectively. The last statement is clear from the observations above and implies in particular that $\ker T = \left(\bigoplus_{\sigma_k \in S} V_k\right)^\perp$ is $\pi_2|_{G_T}$-invariant. The equation (8.13) is now just a restatement of (8.12), noting that the latter is always satisfied for elements in $\ker T$. □

Remark. Notice that in the lemma above, only the eigenspaces V_k are considered that correspond to *non-zero* singular values.

The observations above give a more detailed understanding of the stabilizer of T in G. Indeed, it is the largest subgroup H of G for which the representation $\rho_1|_H$ of H on $\mathrm{Im}(T)$ decomposes as $\mathrm{Im}(T) \cong \bigoplus_{\sigma_k \in S} U_k$, the representation $\rho_2|_H$ of H on $\ker(T)^\perp$ decomposes as $\ker(T)^\perp \cong \bigoplus_{\sigma_K \in S} V_k$ and T defines an equivalence between these two H-representations.

The general case

Now, η is generally not a positive definite inner product and in general there is no way to reconstruct $T \in \mathrm{Hom}(V_2, V_1)$ from an eigen-decomposition of T^*T and TT^*. Nonetheless, we pursue an analysis inspired by previous setting. In the following, we will be using η and κ to identify V_1 and V_2 with their algebraic dual spaces, so let us first convince ourselves of the various basic properties of such identifications.

Lemma 160. *Let V, W be two arbitrary vector spaces. Suppose that κ and η are non-degenerate bilinear forms on V and W, respectively. Denote by $\Psi_\kappa : V^* \to V$ and $\Psi_\eta : W^* \to W$ the isomorphisms they induce. Define*

$$\phi_{\eta,\kappa} \;:\; Hom(W,V) \to Hom(V,W), \quad A \mapsto \Psi_\eta \circ A^* \circ \Psi_\kappa^{-1}$$

Then $\phi_{\eta,\kappa}(A)$ is the unique linear map that satisfies $\kappa(Aw, v) = \eta(w, \phi_{\eta,\kappa}(A)v)$ for all $v \in V$ and $W \in W$.

Proof. Fixing $v \in V$, it is immediate from the non-degeneracy of η that there can be at most one linear map satisfying the above equation. Finally, $\phi(A)$ satisfies this equation by the following quick computation

$$\begin{aligned}\eta(w, \phi_{\eta,\kappa}(A)v) &= (\Psi_\eta^{-1} \circ \phi_{\eta,\kappa}(A)v)(w) \\ &= (A^* \circ \Psi_\kappa^{-1} v)(w) \\ &= \kappa(Aw, v)\end{aligned}$$

□

Now, let us return to the original setting. That is, consider the action of $G = G_1 \times G_2$ on $\mathrm{Hom}(V_2, V_1)$ defined by 8.11 and assume that we are given a G-invariant inner product κ on V_1 and a G-invariant non-degenerate symmetric symmetric bilinear form η on V_2.

We identify $V_2^* \cong V_2$ using η and $V_1^* \cong V_1$ using κ. For $A \in \mathrm{Hom}(V_2, V_1)$, let us write $A^\star = \phi_{\eta,\kappa}(A) \in \mathrm{Hom}(V_1, V_2)$ for its dual under these identifications. Similarly, using the same notation as in lemma 160, write

$$\begin{aligned} R^\star &= \phi_{\kappa,\kappa}(R), & R &\in \mathrm{End}(V_1), \\ S^\star &= \phi_{\eta,\eta}(S), & S &\in \mathrm{End}(V_2). \end{aligned}$$

Remark.

— Notice that if A, R, S are as above, then

$$\phi_{\kappa,\eta}(R \circ A \circ S) = \phi_{\eta,\eta}(S) \circ \phi_{\kappa,\eta}(A) \circ \phi_{\kappa,\kappa}(R),$$

which justifies the notation $(RAS)^\star = S^\star A^\star R^\star$. The situation is depicted in the commutative diagram below.

$$\begin{array}{ccccccc} V_1^* & \xrightarrow{R^*} & V_1^* & \xrightarrow{A^*} & V_2^* & \xrightarrow{S^*} & V_2 \\ \Psi_\kappa \big\downarrow & & \big\downarrow \Psi_\kappa & & \big\downarrow \Psi_\eta & & \big\downarrow \Psi_\eta \\ V_1 & \xrightarrow{R^\star} & V_1 & \xrightarrow{A^\star} & V_2 & \xrightarrow{S^\star} & V_2 \end{array} .$$

In the following it is shown that the orbits of $T \in \mathrm{Hom}(V_2, V_1)$ can be categorized by the eigenvalues of $T^\star T$ and $TT^\star$.

Lemma 161. *Let $T \in Hom(V_2, V_1)$, The eigenvalues of the linear maps $(x \cdot T)^\star (x \cdot T)$ and $(x \cdot T)(x \cdot T)^\star$ are the same for all $x \in G$, counting multiplicities. That is, they are constant on G-orbits.*

Proof. By the invariance of η, we have $\eta(\pi_2(x_2)v, w) = \eta(v, \pi_2(x_2)^{-1} w)$ for $v, w \in V_2$ and therefore $\pi_2(x_2)^\star = \pi_2(x_2)^{-1}$. Using this fact we find that

$$\begin{aligned}(x \cdot T)^\star (x \cdot T) &= \pi_2(x_2) T^\star T \pi_2(x_2)^{-1} \\ (x \cdot T)(x \cdot T)^\star &= \pi_1(x_1) T T^\star \pi_1(x_1)^{-1}\end{aligned}$$

Since conjugation by invertible linear maps preserves the eigenvalues, the result follows. □

Lemma 162. *The non-zero eigenvalues of $T^\star T$ and $TT^\star$ coincide, counting multiplicities.*

Proof. Observe first that if v is an eigenvector of $T^\star T$ with eigenvalue $\lambda \neq 0$, then $Tv \neq 0$ is an eigenvector of $TT^\star$ with eigenvalue λ. Interchanging the roles of T and $T^\star$ shows that T maps the eigenspace V_λ of $T^\star T$ into the eigenspace W_λ of $TT^\star$ and $T^\star$ maps in the converse direction. It remains to show that the restrictions of T and $T^\star$ to these eigenspaces V_λ and W_λ are injective, so that the eigenspaces have the same dimension. This follows from the definition of the eigenspaces. If $T^\star T v = \lambda v$ with $v \neq 0$, then in particular $Tv \neq 0$, so T is injective on V_λ. Similarly, $T^\star$ is injective on W_λ. It follows that T and $T^\star$ are bijections between the eigenspaces V_λ and W_λ. □

Notice that $V_1 \otimes V_2$ is equipped naturally with the bilinear form $\eta \otimes \kappa$. The following lemma determines the corresponding bilinear form β on $\mathrm{Hom}(V_2, V_1)$ under the isomorphism $\Phi : V_1 \otimes V_2 \to \mathrm{Hom}(V_2, V_1)$ defined in (8.10). Denote for a bilinear map α on $V_1 \otimes V_2$ the corresponding bilinear map on $\mathrm{Hom}(V_2, V_1)$ by

$$\Phi_*(\alpha)(A, B) = \alpha(\Phi^{-1}(A), \Phi^{-1}(B)), \qquad A, B \in \mathrm{Hom}(V_2, V_1)$$

Lemma 163. *Define* $\beta \stackrel{d}{=} \Phi_*(\eta \otimes \kappa)$. *Then* $\beta(A, B) = \mathrm{tr}(A^\star B)$.

Proof. We show that an identity holds between the two on simple tensors, which by linearity shows the claim. Suppose $t = u \otimes x$. Then $T := \Phi(t) = \eta(\cdot, x)u$ and $T^\star = \kappa(u, \cdot)x$. Thus, $T^\star T = \eta(\cdot, x)\kappa(u, u)x$ and $TT^\star = \eta(x, x)\kappa(u, \cdot)u$. Choosing any orthogonal basis of V_1 containing u, we find immediately that

$$\mathrm{tr}(TT^\star) = \mathrm{tr}(T^\star T) = \eta(x, x)\kappa(u, u) = (\eta \otimes \kappa)(x \otimes u, x \otimes u).$$

The identity now follows on simple tensors by an application of the polarization identity. □

8.5 Some relevant facts on rotations

In the proof of theorem 117, some facts regarding rotations are needed. These are collected in this section. Write $R_x(\theta) \in SO(3)$ for the counter clock-wise rotation about $x \neq 0$ of angle θ.

Lemma 164. *Let $v, w \in S^2$. Suppose that $R \in SO(3)$ is such that $Rv = w$. Then*

$$R_w(\theta) = R \circ R_v(\theta) \circ R^{-1}$$

Proof. Choosing an orthogonal basis (x, y, v) of $\mathbb{R}^3$ of positive orientation so that $R_v(\theta)$ is given by

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A direct computation shows that this is also the matrix representation of $R \circ R_v(\theta) \circ R^{-1}$ with respect to the basis (Rx, Ry, w). Since (Rx, Ry, w) is again of positive orientation, this is also the matrix of $R_w(\theta)$ with respect to this basis and so we are done. □

Corollary 165. *Let $R \in SO(3)$ be arbitrary. Then there exists a rotation $R_0 \in SO(3)$ such that $R = R_0 \circ R_{e_3}(\theta) \circ R_0^{-1}$ for some angle $\theta \in [0, 2\pi)$.*

Proof. $SO(3)$ acts transitively on S^2. Now apply lemma 164. □

Remark. By lemma 47, this corollary implies that the adjoint representation of $SU(2)$ on its Lie algebra $\mathfrak{su}(2)$ is surjective onto $SO(\mathfrak{su}(2), \kappa)$.

Lemma 166. *Let $R = R_{e_3}(\theta) \in SO(3)$ be a rotation about e_3 with angle θ and let $z = e^{i\theta}$. Then its complexification of R has eigenvalues z^k for $k = -1, 0, 1$. The corresponding eigenvectors are*

$$q_0 = e_3, \qquad q_1 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix} \qquad q_{-1} = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

Proof. One checks directly that $Re_3 = e_3$ and $Rq_{\pm 1} = e^{\pm i\theta} q_{\pm 1}$. □

Combining the previous lemma with corollary 165 reveals some relevant facts:

Corollary 167.

1. *the eigenvalues of $R_v(\theta)$ are 1 and $e^{\pm i\theta}$ for any $v \in S^2$.*
2. *For a rotation R the two (possibly complex) eigenvectors v_1, v_{-1} other than its axis of rotation are related by $\overline{v_1} = v_{-1}$.*
3. *The only invariant elements in $\mathbb{R}^3$ of a non-trivial rotation are those in its axis of rotation.*

Proof.

1. This is immediate from lemma 166 and corollary 165.
2. Notice that $\overline{q_1} = q_{-1}$. Any rotation is conjugate to $R_{e_3}(\theta)$ by corollary 165 so that its eigenvalues are of the form Rq_k for some rotation R. Then $\overline{Rq_1} = R\overline{q_1} = Rq_{-1}$.
3. By the above lemma, there is a single one-dimensional real eigenspace with eigenvalue 1 unless R is trivial.

□

Corollary 168. *Let $R = R_a(\theta) \in SO(3)$ be a non-trivial rotation about a of angle $\theta \in [0, 2\pi)$. The following hold:*

1. *The subspaces Span$\{a\}$ and Span$\{a\}^\perp$ of $\mathbb{R}^3$ are R-invariant.*
2. *If $\theta \in \{0, \pi\}$ then every one-dimensional subspace of $\mathbb{R}^3$ contained in Span$\{a\}^\perp$ is R-invariant. Every two-dimensional subspace of $\mathbb{R}^3$ containing Span$\{a\}$ is R-invariant.*

Moreover, these exhaust all invariant subspaces.

Proof. Notice first that by corollary 165, we may assume that $a = e_3$. Let us consider first the one-dimensional invariant subspaces. Consider the basis $\{q_k\}_{k=-1}^{1}$ of $\mathbb{C}^3$ as in lemma 166, and let $v = \sum_k c_k q_k \in \mathbb{R}^3$ be non-zero. Notice that $v \in \mathbb{R}^3 \implies \overline{c_1} = c_{-1}$ and $\overline{c_0} = c_0$. Now, suppose that $Rv = \mu v$ for some $\mu \in \mathbb{C}$. Since $Rq_k = z^k q_k$, where $z = e^{i\theta}$, this implies that $\sum_{k=-1}^{1} z^k c_k v_k = \mu \sum_{k=-1}^{1} c_k v_k$ or equivalently $z^k c_k = \mu$ for all k for which $c_k \neq 0$. Notice that since $\overline{c_1} = c_{-1}$ we have $c_1 \neq 0 \iff c_{-1} \neq 0$. Moreover, since R is non-trivial we can not have $c_k \neq 0$ for all k. This leaves two cases

1. If $c_0 \neq 0$, then $c_{-1} = c_1 = 0$ and thus $v \in \operatorname{Span}\{a\}$.
2. If $c_0 = 0$, then $v \in \operatorname{Span}\{a\}^\perp$ since both c_1 and c_{-1} are non-zero. In this case we require $z^{-1} = z$, i.e., $z^2 = 1$. This implies that $z = e^{ik\pi}$ for some $k \in \mathbb{Z}$ so that $\theta \in \{0, \pi\}$.

This completes the statements regarding one-dimensional subspaces. Those regarding two-dimensional subspaces can be reduced to the first case by the observation that since R is orthogonal, the orthogonal complement of an R-invariant subspace is again R-invariant. □

The following few lemmas are concerned with elements in $SU(2)$ covering certain specific rotations.

Lemma 169. *Let $\phi : SU(2) \to SO(3)$ be the covering homomorphism as in corollary 48. Then*

1. $\phi^{-1}(R_{e_k}(\pi)) = \{\pm\sigma_k\}$,
2. $\phi^{-1}(R_{e_3}(\theta)) = \pm \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$,
3. *If $v = R_{e_3}(\theta)e_1$, then* $\phi^{-1}(R_v(\pi)) = \pm \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$.

Proof. Recall from corollary 48 under the linear isomorphism $\mathbb{R}^3 \to \mathfrak{su}(2)$ given by $x_i e_i \mapsto x_i i\sigma_i$, ϕ becomes the adjoint representation Ad of $SU(2)$ in $\mathfrak{su}(2)$ and moreover that ϕ is a double covering.

1. It suffices to check that both the elements $\pm\sigma_k$ act map to $R_{e_k}(\pi)$. Under the isomorphism $\mathbb{R}^3 \to \mathfrak{su}(2)$ above, this amounts to checking that $\mathrm{Ad}(\sigma_i)\sigma_i = \sigma_i$ and $\mathrm{Ad}(\sigma_i)\sigma_j = -\sigma_j$ for $j \neq i$. This follows immediately from the equations
$$\begin{aligned} \sigma_k^2 &= I, && \forall k = 1,2,3, \\ \sigma_i\sigma_j + \sigma_j\sigma_i &= 0, && i \neq j. \end{aligned}$$
2. This follows from the fact that if $z = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}$, then
$$\begin{aligned} \mathrm{Ad}_z(\sigma_1) &= \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix} &&= \cos(\theta)\sigma_1 - \sin(\theta)\sigma_2, \\ \mathrm{Ad}_z(\sigma_2) &= \begin{pmatrix} 0 & -ie^{i\theta} \\ ie^{-i\theta} & 0 \end{pmatrix} &&= \sin(\theta)\sigma_1 + \cos(\theta)\sigma_2. \end{aligned}$$
3. Notice that by lemma 164, $R_v(\pi) = R_{e_3}(\theta)R_{e_1}(\pi)R_{e_3}(\pi)^{-1}$. By the first point we know that $\{\pm\sigma_1\}$ covers $R_{e_1}(\pi)$ and by the second point we know that z covers $R_{e_3}(\theta)$. It follows that $\pm\mathrm{Ad}_z(\sigma_1)$ covers $R_v(\pi)$, where z is as above. We have already seen that $\mathrm{Ad}_z(\sigma_1) = \begin{pmatrix} 0 & e^{i\theta} \\ e^{-i\theta} & 0 \end{pmatrix}$ and we are done.

□

Remark.

1. If $\phi(u)v = w$, then $\phi^{-1}(R_w(\theta)) = u\phi^{-1}(R_v(\theta))u^{-1}$
2. The rotations about some fixed axis $a \in S^2$ are covered by a subgroup conjugate to $U(1)$. We denote this subgroup by $U_a(1)$ and simply write $U(1)$ if $a = e_3$.
3. In fact, the stabilizer of the action of $SU(2)$ on S^2 via ϕ is $U(1)$, so that $S^2 \cong SU(2)/U(1)$ and by corollary 18, $SU(2) \to S^2$ is a principal $U(1)$-bundle, called the *Hopf-fibration*.

4. We know from lemma 169 that the elements of the form $\begin{pmatrix} 0 & \overline{u} \\ u & 0 \end{pmatrix}$ covers the rotations of the form $R_v(\pi)$ for some $v \in \text{Span}\{e_1\}^\perp$.

Lemma 170. *Let q_1 and q_{-1} be as in lemma 166. Let $q \in \mathbb{C}^3$ be non-zero and such that $q \perp \overline{q}$ and $q \perp e_3$. Then q is a complex multiple of either q_1 or q_{-1}.*

Proof. The condition $q \perp e_3$ implies that $q = (z_1, z_2, 0)$ for some $z_1, z_2 \in \mathbb{C}$. Now, $q \perp \overline{q}$ states that $z_1^2 + z_2^2 = 0$ and therefore $z_2 = \pm i z_1$. Thus $q = (z_1, \pm i z_1, 0) = z_1 \cdot (1, \pm i, 0)$. □

Corollary 171. *Let $q \in \mathbb{C}^3$ be non-zero and such that $q \perp \overline{q}$. Let $a \in Span\{q, \overline{q}\}^\perp$. Then q is an eigenvector of $\phi(u)$ for any $u \in U_a(1)$.*

Proof. There exists some rotation R such that $Re_3 = a$. Notice that the complexification of a rotation is unitary. This means that $R^{-1}q$ satisfies the requirements of lemma 170 so that $R^{-1}q$ is an eigenvector of $\phi(u)$ for any $u \in U_{e_3}(1)$. Then by lemma 164 q is an eigenvector of $\phi(u)$ for any $u \in U_a(1)$. □

Corollary 172. *Let $q \in \mathbb{C}^3$ be non-zero, $u \in SU(2)$ and $s \in U(1)$. The following hold.*

1. *If $q \in \mathbb{R}^3$, then $\phi(u)q = q$ if and only if $u \in U_q(1)$.*
2. *If $q \perp \overline{q}$, then q is an eigenvector of $\phi(u)$ if and only if $u \in U_a(1)$, where $a \in Span\{q, \overline{q}\}^\perp$.*
3. *If $q \notin \mathbb{R}^3$ and $q \not\perp \overline{q}$, then q is an eigenvector of $\phi(u)$ if and only if $\phi(u) = I$.*

Proof.

1. This is immediate from corollary 168.
2. By corollary 171 it remains to show the 'only if' direction. As such, suppose that q is an eigenvector of $\phi(u)$. The condition $q \perp \overline{q}$, implies in particular that $q \notin \mathbb{R}^3$ and so by corollary 168, q and $\overline{q}$ must be eigenvectors corresponding to the conjugate eigenvalues. This implies that $u \in U_a(1)$, where $a \in \text{Span}\{q, \overline{q}\}^\perp \subseteq \mathbb{R}^3$.
3. By corollary 168 Any eigenvector of a non-trivial rotation $\phi(u)$ not in its axis of rotation must be part of a conjugate pair of eigenvectors.

□

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