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Modulus consensus in discrete-time signed networks and properties of special recurrent inequalities

Anton V. Proskurnikov and Ming Cao

***Abstract*— Recently the dynamics of signed networks, where the ties among the agents can be both positive (attractive) or negative (repulsive) have attracted substantial attention of the research community. Examples of such networks are models of opinion dynamics over signed graphs. It has been shown that under mild connectivity assumptions these protocols provide the convergence of opinions in absolute value, whereas their signs may differ. This "modulus consensus" may correspond to the *bipartite consensus* (the opinions split into two clusters, converging to two opposite values) or the asymptotic stability of the system (the opinions always converge to zero).**

In this paper, we demonstrate that the phenomenon of modulus consensus in a signed network is a manifestation of a more general, regarding the solutions of special *recurrent inequalities*, associated to conventional first-order consensus algorithms. Although such a recurrent inequality does not provide the uniqueness of a solution, it can be shown that, under some natural assumptions, each of its bounded solutions has a limit and, moreover, converges to consensus. A similar property has previously been established for special continuous-time differential inequalities in [1]. Besides analysis of signed networks, we link the consensus properties of recurrent inequalities to the convergence properties of distributed optimization algorithms and stability properties of substochastic matrices.

I. Introduction

In the recent years protocols for consensus and synchronization in multi-agent networks have been thoroughly studied [2]–[5]. Much less studied are "irregular" behaviors, exhibited by many real-world networks, such as e.g. cluster synchronization, partial synchronization, desynchronization and chaos [6]–[8]. An important step in understanding these complex behaviors is to elaborate mathematical models for "partial" or cluster synchronization, or simply *clustering* [6], [9], [10]. In social influence theory, this problem is known as the *community cleavage* problem or Abelson's *diversity puzzle* [11], [12]: to disclose mechanisms that hinder reaching consensus among social actors and lead to splitting of their opinions into several clusters.

One reason for clustering in multi-agent networks is the presence of "negative" (repulsive, antagonistic) interactions among the agents [9]. Models of *signed* (or "coopetition") networks with positive and negative couplings among the nodes describe a broad class of real-world systems, from molecular ensembles [13] to continental supply chains [14]. Positive and negative relations among social actors can express, respectively, trust (friendship) or distrust (hostility). Negative ties among the individuals may also result from the *reactance* or *boomerang* effects, first described in [15]: an individual may not only resist the persuasion process, but even adopt an attitude that is contrary to the persuader's one.

A simple yet instructive model of continuous-time opinion dynamics over signed networks has been proposed by Altafini [16], [17] and extended to the discrete-time case in [18]–[20]. In the recent years, Altafini-type coordination protocols over static and time-varying signed graphs have been extensively studied, see e.g. [18]–[26]. It has been shown that under mild connectivity assumptions these models exhibit consensus in absolute value, or *modulus consensus*: the agents' opinions agree in modulus yet may differ in signs. In the recent works [1], [27] it has been shown that the effect of modulus consensus in the continuous-time Altafini model is in fact a manifestation of a more profound result, concerned with the special class of *differential inequalities*

$$\dot{x}(t) \leq -L(t)x(t), \tag{1}$$

where $L(t)$ stands for the Laplacian matrix of a time-varying weighted graph. Although the inequality (1) is a seemingly "loose" constraint, any of its *bounded* solutions (under natural connectivity assumptions) converges to a consensus equilibrium (this property is called *consensus dichotomy*). This implies, in particular, the modulus consensus in the Altafini model [1], [27] since the vector of the opinions' absolute values obeys the inequality (1). In this paper, we extend the theory of differential inequalities to the discrete-time case, where (1) is replaced by the *recurrent* inequality $x(k+1) \leq W(k)x(k)$ with $\{W(k)\}_{k\geq 0}$ being a sequence of stochastic matrices. We establish the consensus dichotomy criteria for these inequalities, which imply the recent results on modulus consensus in the discrete-time Altafini model [19]. We also apply the recurrent inequalities to some problems of matrix theory and the analysis of distributed algorithms for optimization and linear equations solving.

II. Problem Setup

We start with preliminaries and introducing some notation.

A. Preliminaries

First we introduce some notation. A vector $x \in \mathbb{R}^n$ is non-negative ($x \geq 0$) if $x_i \geq 0\,\forall i$. Given two vectors $x, y \in$

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$\mathbb{R}^n$, we write $x \geq y$ (respectively, $x \leq y$) if $x - y \geq 0$ (respectively, $y - x \geq 0$). The vector of ones is denoted by $\mathbf{1}_n = (1, \dots, 1)^\top \in \mathbb{R}^n$. Given a matrix $A = (a_{ij})$, we use $|A| = (|a_{ij}|)$ to denote the matrix of element-wise absolute values (the same rule applies to vectors). A matrix $A = (a_{ij})$ is *stochastic* if its entries are non-negative and all rows sum to 1, i.e. $\sum_j a_{ij} = 1\, \forall i$. We use $\rho(A)$ to denote the spectral radius of a square matrix A. The standard Euclidean norm of a vector x is denoted by $\|x\| = \sqrt{x^\top x}$.

A non-negative matrix $A = (a_{ij})_{i,j \in V}$ can be associated to a (directed) weighted graph[1] $\mathcal{G}[A] = (V, E[A], A)$, whose set of arcs is $E[A] = \{(i,j) : a_{ij} \neq 0\}$.

B. Recurrent inequalities and consensus dichotomy.

In this paper, we are interested in the solutions of the following discrete-time, or *recurrent*, inequality

$$x(k+1) \leq W(k)x(k), \quad k = 0, 1, \dots \tag{2}$$

where $x(k) \in \mathbb{R}^n$ is a sequence of vectors and $W(k) \in \mathbb{R}^{n \times n}$ stands for a sequence of *stochastic* matrices.

Replacing the inequality in (2) by the equality, one obtains the well-known averaging, or *consensus* protocol [30]–[32]

$$x(k+1) = W(k)x(k), \tag{3}$$

dating back to the early works on social influence [33], [34], rational decision making [35] and distributed optimization [36]. The algorithm (3) may be interpreted as the dynamics of opinions[2] formation in a network of n agents. At each step of the opinion iteration k agent i calculates the weighted average of its own opinion $x_i(k)$ and the others' opinions; this average is used as the new opinion of the ith agent $x_i(k+1) = \sum_j w_{ij}(k)x_j(k)$. The graph $\mathcal{G}[W(k)]$ naturally represents the interaction topology of the network at step k. Agent i is influenced by agent j if $w_{ij}(k) > 0$, otherwise the jth agent's opinion $x_j(k)$ plays no role in the formation of the new agent i's opinion $x_i(k+1)$.

A similar interpretation can be given to the inequality (2). Unlike the algorithm (3), the opinion of agent i at each step of opinion formation is not uniquely determined by the opinions from the previous step, but is only *constrained* by them $x_i(k+1) \leq \sum_j w_{ij}(k)x_j(k)$. The weight $w_{ij}(k)$ stands for the contribution of agent j's opinion $x_j(k)$ to this constraint, and in this sense it can also be treated as the "influence" weight. The inequality (2) does not provide the solution's uniqueness for a given $x(0)$, but only guarantees the existence of an *upper bound* for the solutions.

Proposition 1: Any solution of (2) obeys the inequality

$$x(k) \leq M\mathbf{1}_n, \quad M \overset{\Delta}{=} \max_i x_i(0).$$

Proof: Proposition 1 is proved via straightforward induction on k. By definition, $x(0) \leq M\mathbf{1}_n$; if $x(k) \leq M\mathbf{1}_n$ then $x(k+1) \leq W(k)x(k) \leq MW(k)\mathbf{1}_n = M\mathbf{1}_n$. ■

[1] We assume that the reader is familiar with the standard concepts of graph theory, regarding directed graphs and their connectivity properties, e.g. walks (or paths), cycles and strongly connected components [28], [29].

[2] In the broad sense, "opinion" is just a scalar quantity of interest; it can stand for e.g. a physical parameters or an attitude to some event or issue.

Although many solutions of (2) are unbounded from below, under certain assumptions any its *bounded* solution converges to a consensus equilibrium $c\mathbf{1}_n$, where $c \in \mathbb{R}$. A similar property, called *consensus dichotomy*[3] has been established in [1], [27] for the differential inequalities (1).

Definition 1: The inequality (2) is said to be *dichotomic* if any of its bounded (from below) solutions has a limit $x_* = \lim_{k \to \infty} x(k)$. It is called *consensus dichotomic* if these limits are consensus equilibria $x_* = c_*\mathbf{1}_n$, where $c_* \in \mathbb{R}$.

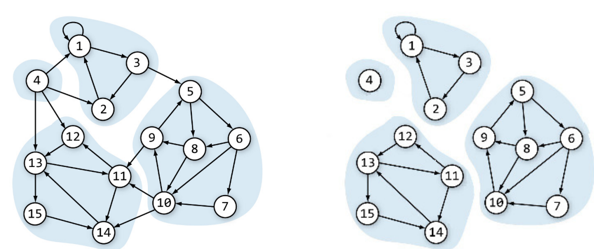
The main goal of this paper is to disclose criteria of consensus dichotomy in the recurrent inequalities (2). In Section IV we discuss applications of these criteria to models of opinion dynamics and algorithms of distributed optimization.

III. Main Results

The first step is to examine *time-invariant* inequalities (2).

A. A dichotomy criterion for the time-invariant case

In this subsection, we assume that $W(k) \equiv W$ is a constant matrix, whose graph $\mathcal{G} \overset{\Delta}{=} \mathcal{G}[W]$ has $s \geq 1$ strongly connected (or *strong*) components $\mathcal{G}_1, \dots, \mathcal{G}_s$; in general, arcs between different components may exist (Fig. 1a). A strong component is *isolated* if no arc enters or leaves it. All strong components are isolated (Fig. 1b) if and only if every arc of the graph belongs to a cycle [28, Theorem 3.2].



(a) (b)

Fig. 1: Non-isolated (a) vs. isolated (b) strong components

Theorem 1: The inequality (2) with the static matrix $W(k) \equiv W$ is dichotomic if and only if all the strong components $\mathcal{G}_1, \dots, \mathcal{G}_s$ of its graph $\mathcal{G}$ are isolated and aperiodic[4]. The inequality is consensus dichotomic if and only if $\mathcal{G}$ is strongly connected ($s = 1$) and aperiodic, or, equivalently, the matrix W is *primitive* [29], [38].

The proof of Theorem 1, as well as the remaining results of this section, is omitted due to the page limit and is available in the extended version of this paper [39].

Remark 1: Let V_j stand for the set of nodes of $\mathcal{G}_j$. Theorem 1 shows that the time-invariant dichotomic inequality (2) reduces to s independent inequalities of lower dimensions

$$x^{(m)}(k+1) \leq W^{(m)}x^{(m)}(k), \quad m = 1, \dots, s, \tag{4}$$

[3] The term *dichotomy* originates from ODE theory. A system is dichotomic if any of its solutions either grows unbounded or has a finite limit [37].

[4] Recall that a graph is *aperiodic* if the greatest common divisor of its cycles' lengths (that is also referred to as the graph's *period*) is equal to 1.

where $x^{(m)}(k) = (x_i(k))_{i\in V_m}$, $W^{(m)} = (w_{ij})_{i,j\in V_m}$ and each inequality (4) is consensus dichotomic.

Remark 2: The matrix is primitive if and only if [12], [29], [38] its powers W^k are strictly positive for large k.

B. Consensus dichotomy in the time-varying case

In this subsection, we extend the result of Theorem 1 to the case of general time-varying inequality (2). Given $\varepsilon > 0$, let S_ε denote the class of all stochastic matrices $W = (w_{ij})_{i,j\in V}$, satisfying the two conditions:

1) $w_{ii} \geq \varepsilon$ for any $i \in V$;
2) the graph $\mathcal{G}_\varepsilon[W] = (V, E_\varepsilon[W])$ is strongly connected, where $E_\varepsilon[W] \overset{\Delta}{=} \{(i,j) \in V \times V : w_{ij} \geq \varepsilon\}$.

In other words, removing from the graph $\mathcal{G}[W]$ all "light" arcs weighted by less than ε, the remaining subgraph $\mathcal{G}_\varepsilon[W]$ is strongly connected and has self-loops at each node.

For any integers $k \geq 0$ and $m > k$ let $\Phi(m,k) = (\varphi_{ij}(m,k))_{i,j=1}^n \overset{\Delta}{=} W(m-1)\dots W(k)$ stand for the evolutionary matrix of the equation (3); for convenience, we denote $\Phi(k,k) = I_n$. It is obvious that any solution of (2) satisfies also the family of inequalities

$$x(m) \leq \Phi(m,k)x(k) \quad \forall m \geq k \geq 0.$$

The following theorem provides a consensus dichotomy criterion for the case of the time-varying matrix $W(k)$.

Theorem 2: The inequality (2) is consensus dichotomic if $\varepsilon > 0$ exists that satisfies the following condition: for any $k \geq 0$ there exists $m > k$ such that $\Phi(m,k) \in S_\varepsilon$.

Notice that for the static matrix $W(k) \equiv W$ one has $\Phi(m,k) = W^{m-k}$, so the condition from Theorem 2 means that $W^s \in S_\varepsilon$ for some s. It can be easily shown that in this case $W^{s(n-1)}$ is a strictly positive matrix. On the other hand, if W^d is strictly positive for some d, then $W^d \in S_\varepsilon$ for sufficiently small $\varepsilon > 0$. In view of Remark 2 and Theorem 1, in the static case $W(k) \equiv W$ the sufficient condition of consensus dichotomy from Theorem 2 is in fact also *necessary*, boiling down to the primitivity of W.

The condition from Theorem 2 is implied by the two standard assumptions on the sequence $\{W(k)\}_{k\geq 0}$.

Assumption 1: There exists $\delta > 0$ such that for any $k \geq 0$

1) $w_{ii}(k) \geq \delta$ for any $i = 1,\dots,n$;
2) for any i, j such that $i \neq j$ one has $w_{ij}(k) \in \{0\} \cup [\delta; 1]$.

Assumption 2: (Repeated joint strong connectivity) There exists an integer $B \geq 1$ such that the graph $\mathcal{G}[W(k) + \dots + W(k+B-1)]$ is strongly connected for any k.

Corollary 1: Let Assumptions 1 and 2 hold. Then the inequality (2) is consensus dichotomic.

Proof: We are going to show that the condition from Theorem 2 holds for $\varepsilon = \delta^B$ and $m = k + B$, i.e. $\Phi(k+B,k) \in S_{\delta^B}$ for any k. Indeed, $\varphi_{ii}(m,k) \geq w_{ii}(m-1)\dots w_{ii}(k) \geq \delta^{m-k}\,\forall i$ whenever $m \geq k$ due to Assumption 1. Supposing that $(i,j) \in \mathcal{G}[W(l)]$, where $k \leq l < m$, one has $\Phi(m,k) = \Phi(m,l+1)W(l)\Phi(l,k)$, and therefore $\varphi_{ij}(m,k) \geq \varphi_{ii}(m,l+1)w_{ij}(l)\varphi_{jj}(l,k) \geq \delta^{m-l-1}\delta\delta^{l-k} = \delta^{m-k}$. Applying this to $m = k+B$, one easily notices that i is connected to j in the graph $\mathcal{G}_{\delta^B}[\Phi(k+B,k)]$ whenever $w_{ij}(l) > 0$ for some $l = k,\dots,k+B-1$. Assumption 2 implies now that $\Phi(k+B,k) \in S_{\delta^B}$ for any k. ■

It should be noticed however that the condition of Theorem 2 may hold in many situations where Assumptions 1 and 2 fail. Even in the static case $W(k) \equiv W$, the matrix W can be primitive yet have zero diagonal entries. The following corollary illustrates another situation where both Assumptions 1 and 2 may fail, whereas Theorem 2 guarantees consensus dichotomy.

Corollary 2: Suppose that for any k one has $W(k) \in \{W_0\} \cup \mathcal{W}$, where W_0 stands for the *primitive* matrix and $\mathcal{W}$ is a set of stochastic matrices, commuting with W_0: $W_0 W = W W_0\,\forall W \in \mathcal{W}$. Let the set $K_0 = \{k : W(k) = W_0\}$ be infinite. Then the inequality (2) is consensus dichotomic.

Proof: Let d be so large that W_0^d is a positive matrix, whose minimal entry equals $\varepsilon > 0$. For any k, we can find such $m > k$ that the sequence $k, k+1, \dots, m-1$ contains d elements from the set K_0. Since any $W(j)$ commutes with W_0, $\Phi(m,k) = T_k W_0^d$, where T_k is some stochastic matrix, and thus all entries of $\Phi(m,k)$ are not less than ε. ■

Many sequences $\{W(k)\}$, satisfying the conditions of Corollary 2, fail to satisfy Assumptions 1 and 2. For instance, if $\mathcal{W} \ni I_n$ then the sequence $\{W(k)\}$ can contain an arbitrary long subsequence of consecutive identity matrices, which violates Assumption 2. Both the matrix W_0 and matrices from $\mathcal{W}$ may have zero diagonal entries, which also violates Assumption 1. The set $\mathcal{W}$ can also be non-compact, containing matrices with arbitrary small yet non-zero entries.

C. The case of bidirectional interaction

It is known that in the case of bidirectional graphs $w_{ij} > 0 \Leftrightarrow w_{ji} > 0$ the conditions for consensus in the network (3) is reached under very modest connectivity assumptions. Under Assumption 1, consensus is reached if and only if the following relaxed version of Assumption 2 holds [31].

Assumption 3: (Infinite joint strong connectivity) The graph $\mathcal{G}_\infty = (V, E_\infty)$ is strongly connected, where

$$E_\infty = \left\{(i,j) : \sum_{k=1}^{\infty} w_{ij}(k) = \infty\right\}.$$

The following result extends this consensus criterion to the condition of consensus dichotomy in the inequality (2).

Theorem 3: Suppose that Assumption 1 and 3 hold and for any k one has $w_{ij}(k) > 0 \Leftrightarrow w_{ji}(k) > 0$. Then the inequality (2) is consensus dichotomic.

The relaxation of Assumption 1 in Theorem 3 remains a non-trivial open problem. To the best of the authors' knowledge, the same applies to usual consensus algorithms (3): most of the existing results for consensus in discrete-time switching networks [3], [30]–[32] rely on Assumption 1 or at least require uniformly positive diagonal entries $w_{ii}(k)$.

IV. Examples and Applications

In this section we apply the criteria from Section III to the analysis of several multi-agent coordination protocols.

A. Modulus consensus in the discrete-time Altafini model

We first consider the discrete-time Altafini model [18]–[20] of opinion formation in a signed network. This model is similar to the consensus protocol (3) and is given by

$$\begin{gathered}\xi(k+1) = A(k)\xi(k) \in \mathbb{R}^n, \quad \text{or, equivalently} \\ \xi_i(k+1) = \sum_{j=1}^{n} a_{ij}(k) x_j(k).\end{gathered} \tag{5}$$

Here the matrix $(a_{ij}(k))$ satisfies the following assumption.

Assumption 4: For any $k = 0, 1, \ldots$, the matrix $A(k) = (a_{ij}(k))$ has non-negative diagonal entries $a_{ii}(k) \geq 0$, and the modulus matrix $|A(k)| = (|a_{ij}(k)|)$ is stochastic.

The non-diagonal entries $a_{ij}(k)$ in (5) may be both positive and negative. Considering the elements $\xi_i(k)$ as "opinions" of n agents, the positive value $a_{ij}(k) > 0$ can be treated as trust or attraction among agents i and j. In this case, agent i shifts its opinion towards the opinion of agent j. Similarly, the negative value $a_{ij}(k) < 0$ stands for distrust or repulsion among the agents: the ith agent's opinion is shifted away from the opinion of agent j. The central question concerned with the model (5) is reaching consensus in absolute values [19], [26], or *modulus consensus*.

Definition 2: We say that modulus consensus is established by the protocol (5) if the coincident limits exist

$$\lim_{k\to\infty} |\xi_1(k)| = \ldots = \lim_{k\to\infty} |\xi_n(k)| \quad \text{for any } \xi(0) \in \mathbb{R}^n.$$

The absolute values $x_i(k) = |\xi_i(k)|$ obey the inequalities

$$x_i(k+1) \leq \sum_{j=1}^{n} |a_{ij}(k)| x_j(k) \quad \forall i, \tag{6}$$

and hence the vector $x(k) = (x_1(k), \ldots, x_n(k))^\top$ obeys (2) with $W(k) = |A(k)|$. If this recurrent inequality is consensus dichotomic, then modulus consensus in (5) is established. Theorems 2 and 3 yield in in the following criterion.

Theorem 4: Modulus consensus in (5) is established, if the sequence of matrices $W(k) = |A(k)|$ satisfies the conditions of Theorem 2 or Theorem 3.

In particular, if Assumption 1 holds, then modulus consensus is ensured by the repeated strong connectivity (Assumption 2), which can be relaxed to the infinite strong connectivity (Assumption 3) if the network is bidirectional $w_{ij}(k) > 0 \Leftrightarrow w_{ji}(k) > 0$. Theorem 4 includes thus the results of Theorems 2.1 and 2.2 in [19]. As discussed in Section III, the condition from Theorem 2 holds in many situations where Assumption 1 fails, e.g. $W(k) \equiv W$ may be a constant primitive matrix with zero diagonal entries. Unlike consensus algorithms (3), where the gains $w_{ij}(k)$ are design parameters, the social influence (or "social power") of an individual over another one depends on many uncertain factors [40], and the uniform positivity of the non-zero gains $|a_{ij}(k)|$ may become a restrictive assumption.

The most interesting case of modulus consensus is *bipartite consensus*, or "bimodal polarization": the agents split into two groups, whose opinions converge to two opposite (non-zero) values. Modulus consensus is also established, however, if the system is asymptotically stable, i.e., all opinions converge to 0. We do not consider here conditions criteria for bipartite consensus and stability, which can be found e.g. in the recent works [26] (see also Theorem 2.3 in [19]). Notice, however, that the criteria from [26] primarily deal with the case of *exponentially* convergent Altafini's model, whereas the general criterion from Theorem 4, in general, does not guarantee exponential convergence.

B. Substochastic matrices and the Friedkin-Johnsen model

A non-negative matrix $A = (a_{ij})$ is called *substochastic* if $\sum_{j=1}^{n} a_{ij} = 1\,\forall i$. We say that the ith row of A is a *deficiency* row of A if the latter inequality is strict $\sum_j a_{ij} < 1$. Unlike a stochastic matrix, always having an eigenvalue at 1, a substochastic square matrix is usually Schur stable $\rho(A) < 1$. Theorem 1 allows to give an elegant proof of the Schur stability criterion for substochastic matrices [41], [42].

Lemma 1: Let $\mathcal{G} = \mathcal{G}[A]$ be the graph of a substochastic square matrix A and $I_d = \{i : \sum_j a_{ij} < 1\}$ is the subset of its nodes, corresponding to the deficiency rows of A. If any node j either belongs to the set I_d, or I_d is reachable from it in $\mathcal{G}$ via some walk, then $\rho(A) < 1$.

Proof: Consider the matrix $W = (w_{ij})$, defined by

$$w_{ij} \triangleq a_{ij} + \frac{1}{n}\left(1 - \sum_l a_{il}\right) \geq a_{ij}.$$

Obviously, $W = (w_{ij})$ is stochastic and $w_{ij} > a_{ij} \geq 0\,\forall j$ when $i \in I_d$. Hence in the graph $\mathcal{G}[W]$ each node $i \in I_d$ is connected to any other node and to itself, and hence $\mathcal{G}[W]$ is aperiodic. The condition of Lemma 1 implies that $\mathcal{G}[W]$ is also strongly connected. Choosing an arbitrary non-negative vector $x_0 \geq 0$, the vectors $x(k) = A^k x_0$ are non-negative for any $k \geq 0$ and satisfy the inequality (2) with $W(k) \equiv W$. Thanks to Theorem 1, $x(k) \to c\mathbf{1}$, where $c \geq 0$. It remains to notice that $\mathbf{1}$ is not an eigenvector of A since $I_d(A) \neq \emptyset$, and hence $c = 0$. Thus $A^k x_0 \to 0$ as $k \to \infty$ for any $x_0 \geq 0$, which implies the Schur stability of A since any vector x_0 is a difference of two non-negative vectors. ∎

Notice that Lemma 1 implies the following well-known property of substochastic irreducible matrices [38]: if $\mathcal{G}$ is strongly connected then A is either stochastic or Schur stable. The condition from Lemma 1 is not only sufficient but also necessary for the Schur stability [42]. Lemma 1 implies the condition of opinion convergence in the *Friedkin-Johnsen* model of opinion formation [11], [42], [43]

$$x(k) = \Lambda W x(k) + (I - \Lambda)u, \quad u = x(0). \tag{7}$$

Here W is a stochastic matrix of influence weights, and Λ is a *diagonal* matrix of the agents' *susceptibilities* to the social influence [43], $0 \leq \lambda_{ii} \leq 1$. Without loss of generality, one may suppose that $\lambda_{ii} = 0 \Leftrightarrow w_{ii} = 1$; in this case agent i is *stubborn* $x_i(k) \equiv x_i(0)$ (often it is assumed [43] that $\lambda_{ii} = 1 - w_{ii}$). Another extremal case is $\lambda_{ii} = 1$, which means that agent i "forgets" its initial opinion $u_i = x_i(0)$ and iterates the usual procedure of opinion averaging $x_i(k+1) = \sum_j w_{ij} x_j(k)$. If $0 < \lambda_{ii} < 1$, then agent i is "partially

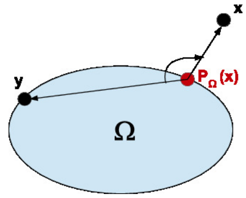


Fig. 2: The projection onto a closed convex set

stubborn" or *prejudiced* [12], [44]: such an agent adopts the others' opinions, however it is "attached" to its initial opinion $x_i(0)$ and factors it into every opinion iteration.

If the substochastic matrix ΛW is Schur stable, then the opinion vector $x(k)$ in (7) converges to the equilibrium

$$x(k) \xrightarrow[k\to\infty]{} (I-\Lambda W)^{-1}(I-\Lambda)u. \tag{8}$$

By noticing that the graphs $\mathcal{G}[\Lambda W]$ and $\mathcal{G}[W]$ differ only by the structure of self-loops (recall that $\lambda_{ii}>0$ unless $w_{ii}=1$ and $w_{ij}=0\,\forall j\neq i$), Lemma 1 implies the following.

Corollary 3: [42] The opinions (8) converge if from each agent i with $\lambda_{ii}=1$ there exists a walk in $\mathcal{G}[W]$ to some agent j with $\lambda_{jj}<1$, that is, each agent is either prejudiced or influenced (directly or indirectly) by a prejudiced agent.

Using Theorems 2 and 3, some stability criteria for the time-varying extension [44] of the Friedkin-Johnsen model can be obtained that are beyond the scope of this paper.

C. Constrained consensus

In this subsection, we consider another application of the recurrent inequalities case, related to the problem of *constrained* or "optimal" consensus that is closely related to distributed convex optimization [45]–[47] and distributed algorithms, solving linear equations [48]–[50].

For any closed convex set $\Omega\subset\mathbb{R}^d$ and $x\in\mathbb{R}^d$ the *projection* operator $P_\Omega : x\in\mathbb{R}^d\mapsto P_\Omega(x)\in\Omega$ can be defined, mapping a point to the closest element of Ω, i.e. $\|x-P_\Omega(x)\| = \min_{y\in\Omega}\|x-y\|$. This implies that $\measuredangle(y-P_\Omega(x),x-P_\Omega(x))\geq\pi/2$ (Fig. 2) and

$$\|x-y\|^2\geq\|x-P_\Omega(x)\|^2+\|y-P_\Omega(x)\|^2\quad\forall y\in\Omega. \tag{9}$$

The distance $d_\Omega(x)\triangleq\|x-P_\Omega(x)\|$ is a convex function.

Consider a group of n discrete-time agents with the state vectors $\xi_i(k)\in\mathbb{R}^d$. Each agent is associated with a closed convex set $\Xi_i\subseteq\mathbb{R}^d$ (e.g., the set of minima of some convex function). The agents' cooperative goal is to find some point $\xi_*\in\Xi\triangleq\Xi_1\cap\ldots\cap\Xi_n$. To solve this problem, various modifications of the protocol (3) have been proposed. We consider the following three algorithms

$$\xi_i(k+1)=P_{\Xi_i}\left[\sum\nolimits_{j=1}^n w_{ij}(k)\xi_j(k)\right], \tag{10}$$

$$\xi_i(k+1)=P_{\Xi_i}\left[\sum\nolimits_{j=1}^n w_{ij}(k)P_{\Xi_j}(\xi_j(k))\right], \tag{11}$$

$$\xi_i(k+1)=w_{ii}(k)P_{\Xi_i}(\xi_i(k))+\sum_{j\neq i}w_{ij}(k)\xi_j(k). \tag{12}$$

Here $W(k)=(w_{ij}(k))$ stands for the sequence of stochastic matrices. The protocol (10) has been proposed in the influential paper [45] (see also [47]), dealing with distributed optimization problems. The special cases of protocols (11) and (12) naturally arise in distributed algorithms, solving linear equations, see respectively [48], [49] and [50]; a randomized version of (12) has been also examined in [46].

Theorem 5: Let the set Ξ_i be closed and convex, and assume that $\Xi=\Xi_1\cap\ldots\cap\Xi_n\neq\emptyset$. Suppose that the matrices $W(k)$ satisfy Assumptions 1 and 2. Then each of the protocols (10)-(12) establishes *constrained* consensus:

$$\lim_{k\to\infty}x_1(k)=\ldots=\lim_{k\to\infty}x_n(k)\in\Xi. \tag{13}$$

Proof: Due to the page limit, we give only an outline of the proof. By assumption, there exists some $\xi_0\in\Xi$. Denote $P_i(\cdot)\triangleq P_{\Xi_i}(\cdot)$, $d_i(\cdot)\triangleq d_{\Xi_i}(\cdot)$ and let $\eta_i(k)\triangleq\sum_j w_{ij}(k)\xi_j(k)$. Under Assumptions 1 and 2, to prove the constrained consensus (13) it suffices to show [47] that

$$e_i(k)\triangleq\xi_i(k+1)-\eta_i(k)\xrightarrow[k\to\infty]{}0,\quad d_i(\xi_i(k))\xrightarrow[k\to\infty]{}0. \tag{14}$$

Applying (9) to $\Omega=\Xi_i$, $x=\xi$, $y=\xi_0\in\Xi_i$, one gets

$$\|\xi-\xi_0\|^2\geq\|P_i(\xi)-\xi_0\|^2+d_i(\xi)^2\quad\forall\xi\in\mathbb{R}^d, \tag{15}$$

and therefore $\|\xi-\xi_0\|\geq\|P_i(\xi)-\xi_0\|$. Each protocol (10)-(12) thus implies the recurrent inequality (2), where $x_i(k)\triangleq\|\xi_i(k)-\xi_0\|\,\forall i$. For instance, the equation (10) entails that

$$0\leq x_i(k+1)\leq\left\|\sum_{j=1}^n w_{ij}(k)\xi_j(k)-\xi_0\right\|\leq\sum_{j=1}^n w_{ij}(k)x_j(k).$$

Corollary 1 implies the existence of the common limit $x_*=\lim_{k\to\infty}x_i(k)\geq 0$. We are now going to prove (14) for the protocol (10). The second statement in (14) is obvious since $d_i(\xi_i(k+1))\equiv 0$. Substituting $\xi=\eta_i(k)$ into (15),

$$\begin{aligned}\|e_i(k)\|^2 &\overset{(10)}{=} d_i(\eta_i(k))^2\overset{(15)}{\leq}\|\eta_i(k)-\xi_0\|^2-x_i(k+1)^2\leq\\ &\leq\sum\nolimits_j w_{ij}(k)x_j(k)-x_i(k+1)\xrightarrow[k\to\infty]{}0.\end{aligned} \tag{16}$$

To prove (14) for the protocol (12), notice that

$$\begin{aligned}x_i(k+1) &\overset{(10)}{\leq} w_{ii}(k)\|P_i(\xi_i(k))-\xi_0\|+\sum_{j\neq i}w_{ij}x_j(k)\\ &\overset{(15)}{\leq} w_{ii}(k)\sqrt{x_i(k)^2-d_i(\xi_i(k))^2}+\sum_{j\neq i}w_{ij}x_j(k).\end{aligned} \tag{17}$$

Recalling that $w_{ii}(k)\geq\delta$ and $x_i(k)\to x_*\forall i$, it can be shown that $d_i(\xi_i(k))\to 0$ and hence $\|e_i(k)\|=w_{ii}(k)d_i(\xi_i(k))\to 0$. The property (14) for the protocol (11) is proved similarly, combining the arguments from (16) and (17). ■

V. Conclusions

In this paper, we have examined a class of recurrent inequalities (2), inspired by the analysis of "modulus consensus" in signed networks. Under natural connectivity assumptions the inequality is shown to be *consensus dichotomic*, that is, any of its solution is either unbounded or converges to consensus. Besides signed networks, we illustrate the applications of this profound property to some problems of matrix theory and distributed optimization algorithms.

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