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Abstract In this paper, we present a new approach on how the multiple time-scales perturbation method can be applied to differential-delay equations such that approximations of the solutions can be obtained which are accurate on long time-scales. It will be shown how approximations can be constructed which branch off from solutions of differential-delay equations at the unperturbed level (and not from solutions of ordinary differential equations at the unperturbed level as in the classical approach in the literature). This implies that infinitely many roots of the characteristic equation for the unperturbed differential-delay equation are taken into account and that the approximations satisfy initial conditions which are given on a time-interval (determined by the delay). Simple and more advanced examples are treated in detail to show how the method based on differential and difference operators can be applied.

Keywords Perturbation methods · Delay differential equations · Multiple time-scales · Asymptotic validity

Mathematics Subject Classification 74H10 · 41A60

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1 Introduction

Perturbation theory for differential equations finds its origin in the 19-th century when Poincaré approximately solved systems of ordinary differential equations (ODEs) originating from problems in Celestial Mechanics [28]. The idea is to approximate the solution of a problem in a power series of ε, where ε is a small parameter in the problem. First, the unperturbed (that is, $\varepsilon = 0$) problem is solved, and then small corrections to this solution are added and one finally obtains an approximation of the solution in the form of a (truncated) asymptotic series [13,19,22,35]. The approximations are usually not accurate on long time-scales. To obtain approximations which are valid on long time-scales, a multiple time-scales perturbation method was developed in the period 1935–1970 by Krylov and Bogoliubov, Kuzmak, Kevorkian and Cole, Cochran and Mahony, and Nayfeh. The reader is referred to [5,13,19,21,22,24–27]) for further details on the historical development of the multiple time-scales perturbation method (MTS) for ODEs.

In various fields of science, technology, and engineering, one encounters time delays. Mathematical modelling of problems involving time delays often leads to problems for differential delay equations (DDEs). Examples of these DDEs can be found in mechanical engineering [9,37,38], in biology [10], and in for instance predator–prey problems [1,2,6,12,32, 40]. Stability properties, resonances, bifurcations, and chaotic behaviour are important issues to be studied for these DDEs. These DDEs can also contain a small parameter, and one can try to construct an approximation of the solution by using the MTS-method. In the literature, one can find many papers in which the MTS-method is used for DDEs (see for instance [3,8,11,18,26,30,39]). In all these papers the authors have as the unperturbed equations (that is, for $\varepsilon = 0$) an ODE. So, in the so-called $\mathcal{O}(1)$-order problem, one does not have a DDE. This implies that the usually infinitely many roots present in the characteristic equation for a DDE are "truncated" to only a few ones which are present in the characteristic equation for an ODE. Moreover, for an initial value problem for a DDE, one has to satisfy initial values on an interval, and not at a point as in the case for an initial value problem for an ODE.

A different approach was taken in [7,9,15,23,31,36, 37]. The unperturbed equations are in the form of linear DDEs. The analysis is carried out on systems close to the Hopf bifurcation point. Due to the dominance of purely imaginary characteristic roots, the authors exclude other characteristic roots with a negative real part. In this paper, we want to complete this method by considering all its characteristic roots. The purpose of this approach is to investigate the possibility of the occurrence of secular terms and to obtain approximations which satisfy given initial conditions and are valid on long time-scales. We will propose a new approach on how to apply the MTS method to DDEs such that all roots of the characteristic equation for the DDE are taken into account and such that the initial values (in a specified interval depending on the delay) can be satisfied. The new approach is partly based on the classical MTS method for ODEs and is partly based on the MTS method for ordinary difference equations ($O\Delta Es$) as given by van Horssen and ter Brake in [34].

This paper is organized as follows. In Sect. 2 of this paper, we described shortly the MTS method for ODEs and the MTS method for $O\Delta Es$ (see also [34]). In Sect. 3 of this paper, the MTS method for DDEs will be introduced by using differential and difference operators, and by applying the MTS method to some simple DDEs. The asymptotic validity of the approximations of the solutions of DDEs on long time-scales will be discussed and will be proved in Sect. 4 of this paper. In Sect. 5 more advanced examples for weakly nonlinear DDEs will be treated, and in Sect. 6 analytically obtained approximations are compared with approximation which are obtained by direct numerical integration of the problem. Finally, in Sect. 7 of this paper, some conclusions will be drawn and some remarks on future research will be given.

2 Preliminary: the multiple scales perturbation method for ODEs and for $O\Delta Es$

In this section, we shortly describe the essential features of the method of multiple scales for ODEs (see also [13,19,22,25]), and for $O\Delta Es$ (see [34]). These essential properties are necessary to describe the method of multiple time-scales for DDEs in the next section of this paper.

2.1 ODEs

Let us consider an oscillator problem with weak damping:

$$\ddot{x} + \varepsilon\dot{x} + x = 0, \quad t > 0, x = x(t),$$
$$x(0) = 0, \quad \text{and} \quad \dot{x}(0) = 1, \tag{1}$$

and where ε is a perturbation parameter, $0 < \varepsilon \ll 1$. Of course the exact solution of problem (1) can readily be obtained, and is given by

$$x(t) = \left(1 - \frac{\varepsilon^2}{4}\right)^{-1/2} e^{-\frac{1}{2}\varepsilon t} \sin\left(\left(1 - \frac{\varepsilon^2}{4}\right)^{1/2} t\right). \tag{2}$$

Now, let us assume that we do not know how to construct the exact solution and that we want to expand the solution in a formal expansion given by

$$x(t) = x_0(t) + \varepsilon x_1(t) + \mathcal{O}(\varepsilon^2), \tag{3}$$

where $x_i(t) = \mathcal{O}(1)$ for times t under consideration. By substituting the expansion (3) into problem (1), and by solving the $\mathcal{O}(1)$- problem, one finds

$$x(t) \approx \sin(t) + \frac{1}{2}\varepsilon t \sin(t). \tag{4}$$

Since $x_i(t)$ should be $\mathcal{O}(1)$, it is obvious that the formal approximation is only valid for $t = \mathcal{O}(1)$ and breaks down for larger values of t. By looking at the exact solution (2), it can be seen that two time-scales, that is, $T_0 = t$ and $T_1 = \varepsilon t$, are describing the solution. The idea of the two time-scales perturbation method is now to seek an approximation of the solution of problem (1) in the following form (5)

$$x(t) = \bar{x}(T_0, T_1) = X_0(T_0, T_1) + \varepsilon X_1(T_0, T_1) + \mathcal{O}(\varepsilon^2), \quad (5)$$

where $X_i(T_0, T_1) = \mathcal{O}(1)$ for times t under consideration. By substituting the expansion (5) into problem (1) one obtains as $\mathcal{O}(1)$-problem and as $\mathcal{O}(\varepsilon)$-problem

$$\mathcal{O}(1), \quad \frac{\partial^2 X_0}{\partial T_0^2} + X_0 = 0, \quad (6)$$

$$X_0(0,0) = 0, \quad \text{and} \quad \frac{\partial X_0}{\partial T_0}(0,0) = 1. \quad (7)$$

$$\mathcal{O}(\varepsilon), \quad \frac{\partial^2 X_1}{\partial T_0^2} + X_1 = -2\frac{\partial^2 X_0}{\partial T_0 \partial T_1} - \frac{\partial X_0}{\partial T_0}, \quad (8)$$

$$X_1(0,0) = 0, \quad \text{and} \quad \frac{\partial X_1}{\partial T_0}(0,0) = -\frac{\partial X_0}{\partial T_1}(0,0). \quad (9)$$

respectively. The $\mathcal{O}(1)$-problem (6)–(7) can readily be solved, yielding

$$X_0(T_0, T_1) = A(T_1)\cos(T_0) + B(T_1)\sin(T_0), \quad (10)$$

where $A(T_1)$ and $B(T_1)$ are still arbitrary functions satisfying $A(0) = 0$ and $B(0) = 1$. The arbitrariness of $A(T_1)$ and $B(T_1)$ can be used to solve the $\mathcal{O}(\varepsilon)$-problem (8)–(9) for $X_1(T_0, T_1)$ in such a way that $X_1(T_0, T_1)$ is $\mathcal{O}(1)$ on a sufficiently long time-scale, which is usually a time-scale t of $\mathcal{O}(\varepsilon^{-1})$. In fact $A(T_1)$ and $B(T_1)$ will be chosen in such a way that the coefficients of the resonant terms in the right-hand side of (8) are set equal to zero (that is, the coefficients of $\cos(T_0)$ and of $\sin(T_0)$ in the right-hand side of (8) will be set equal to zero). In this way, one obtains a so-called secular free (in T_0) $X_1(T_0, T_1)$. From (6)–(10) it then follows that $X_0(T_0, T_1) = e^{-\frac{1}{2}T_1}\sin(T_0)$, and

$$x(t) \approx e^{-\frac{1}{2}T_1}\sin(T_0) + \mathcal{O}(\varepsilon), \quad (11)$$

for $t = \mathcal{O}(\varepsilon^{-1})$, and $T_1 = \varepsilon t$, and $T_0 = t$. As illustration the exact solution (2), the formal approximation (4), and the two-time scales perturbation approximation (11) are given in Fig 1 for $\varepsilon = 0.1$. In fact, it can be shown that

$$|x(t) - X_0(T_0, T_1)| = \mathcal{O}(\varepsilon), \quad \text{for } t = \mathcal{O}(\varepsilon^{-1}). \quad (12)$$

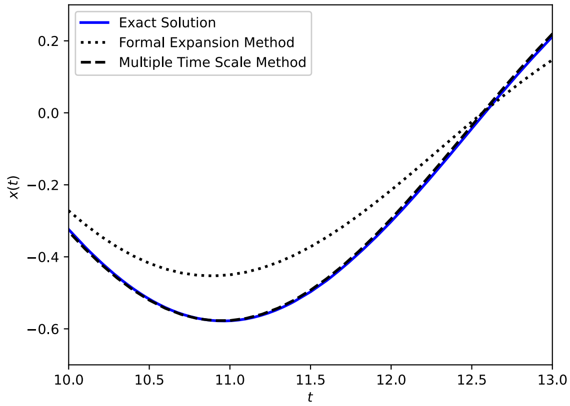

Fig. 1 Plots of the exact solution (2), solid line, the approximation using the formal expansion (4), dashed line, and the approximation (11) using the Multiple time-scales method, dotted line, for $\varepsilon = 0.1$

2.2 $O\Delta Es$

The multiple time-scales perturbation method for $O\Delta Es$ was first introduced by Hoppensteadt and Miranker in [14] by transforming the "differences" into derivatives. In 2009, Van Horssen and Ter Brake introduced in [34] a formulation of the multiple time-scales perturbation method for $O\Delta Es$ completely in terms of diffe-rence operators maintaining in such a way the discrete character of the problem. The classical difference operators are defined in the following way:

$$Ex_n = x_{n+1}, \quad \Delta x_n = x_{n+1} - x_n, \quad \text{and} \quad Ix_n = x_n, \quad (13)$$

where E is the shift operator, Δ the difference operator, and I the identity operator, respectively. When a two-time-scales perturbation method is applied, it is assumed that $x_n = x(n, \varepsilon n)$. By introducing the (forward) partial difference operators

$$\begin{aligned}
\tilde{E}_1 x(n, \varepsilon n) &= x(n+1, \varepsilon n), \\
\tilde{E}_\varepsilon x(n, \varepsilon n) &= x(n, \varepsilon(n+1)), \; I(n, \varepsilon n) = x(n, \varepsilon n), \\
\tilde{\Delta}_1 x(n, \varepsilon n) &= x(n+1, \varepsilon n) - x(n, \varepsilon n) \\
&= (\tilde{E}_1 - I)x(n, \varepsilon n), \\
\tilde{\Delta}_\varepsilon x(n, \varepsilon n) &= x(n, \varepsilon(n+1)) - x(n, \varepsilon n) \\
&= (\tilde{E}_\varepsilon - I)x(n, \varepsilon n) \quad (14)
\end{aligned}$$

and by expanding x_n as

$$x_n = x(n, \varepsilon n) \approx X_0(n, \varepsilon n) + \varepsilon X_1(n, \varepsilon n) + \mathcal{O}(\varepsilon^2), \quad (15)$$

one can set up a perturbation method (see [34]) which leads to accurate approximations of the solutions on time- or iteration scales of order ε^{-1}. In the following example, the method is shortly explained. Let us consider the following problem for x_n (with $n = 0, 1, 2, \ldots$ and $0 < \varepsilon \ll 1$):

$$x_{n+2} + \varepsilon x_{n+1} + x_n = 0, \tag{16}$$

$$x_0 = 0, \quad \text{and} \quad \Delta x_0 = 1. \tag{17}$$

Of course, the exact solution can readily be obtained and is given by

$$x_n = \frac{\sin(n\theta(\varepsilon))}{\sin(\theta(\varepsilon))}, \tag{18}$$

where $\theta(\varepsilon)$ is given by $\cos(\theta(\varepsilon)) = -\varepsilon/2$ and $\sin(\theta(\varepsilon)) = \sqrt{(1-\varepsilon^2/4)}$. When an exact solution (18) is not available, one can try to use the expansion (15). By assuming that $\Delta_\varepsilon = \mathcal{O}(\varepsilon)$ and by using the difference operators (14), one then obtains as $\mathcal{O}(1)-$problem, and as $\mathcal{O}(\varepsilon)-$ problem:

$$\mathcal{O}(1), \quad \left(\Delta_1^2 + 2E_1\right) X_0 = 0, \tag{19}$$

$$X_0(0,0) = 0, \quad \text{and} \quad \Delta_1 X_0(0,0) = 1. \tag{20}$$

$$\mathcal{O}(\varepsilon), \quad (\Delta_1^2 + 2E_1)X_1 = -(2E_1^2\Delta_\varepsilon/\varepsilon + E_1)X_0. \tag{21}$$

$$X_1(0,0) = 0, \quad \text{and} \quad \Delta_1 X_1(0,0) = 0 \tag{22}$$

respectively. By solving (19)–(22) such that $X_1(n, \varepsilon n)$ does not contain secular terms, one obtains as $\mathcal{O}(\varepsilon)$ accurate approximation of $X(n, \varepsilon n)$ for $n = \mathcal{O}(\varepsilon^{-1})$

$$X_0(n, \varepsilon n) = \left(1 + \frac{\varepsilon^2}{4}\right)^{\frac{n}{2}} \sin\left(\frac{1}{2}n\pi + n\mu(\varepsilon)\right) \tag{23}$$

with

$$\cos(\mu(\varepsilon)) = \frac{1}{\sqrt{1+\frac{\varepsilon^2}{4}}},$$

$$\text{and} \quad \sin(\mu(\varepsilon)) = \frac{\varepsilon}{2\sqrt{1+\frac{\varepsilon^2}{4}}}. \tag{24}$$

In Fig. 2, the exact solution (18) of problem (16)–(17), and its approximation $X_0(n, \varepsilon n)$ are given for $\varepsilon = 0.05$ and n up to 200. For more details and other examples for $O\Delta E$s the reader is referred to [34].

3 The multiple scales perturbation method for DDEs

In this section of the paper, we will introduce the MTS method for DDEs. Use will be made of differential, shift, and difference operators. We will restrict ourselves to a two-time-scales perturbation method, but more than two-time-scales can be introduced similarly. In the two time-scales perturbation method, one usually encounters the fast time $T_0 = t$ and the slow time $T_1 = \varepsilon t$, and the solution $x(t)$ of the differential equation is usually approximated by

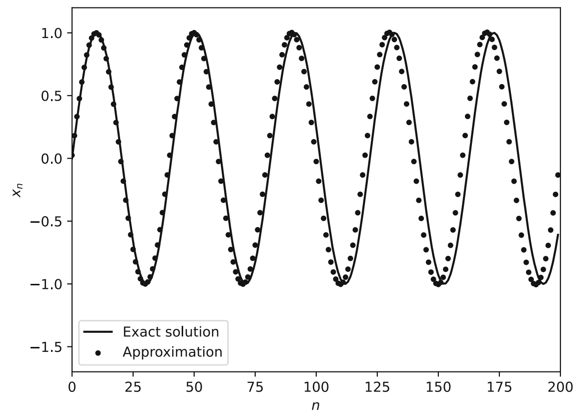

Fig. 2 Plots of the exact solution for Eq. (16) and (17), −, and the approximation $X_0(n, \varepsilon n)$, ·, for $\varepsilon = 0.05$

$$x(t) = \tilde{x}(T_0, T_1) = X_0(T_0, T_1) + \varepsilon X_1(T_0, T_1) + \mathcal{O}(\varepsilon^2), \tag{25}$$

where $X_i(T_0, T_1)$ are (usually) bounded functions on time-scales of order ε^{-1}, with ε a small parameter satisfying $0 < \varepsilon \ll 1$. In DDEs one encounters derivatives of $x(t)$ like $\dot{x}(t) = \frac{d}{dt}x(t), \ddot{x}(t) = \frac{d^2}{dt^2}x(t), \ldots,$ and delayed terms like $x(t-1), \dot{x}(t-1), \ddot{x}(t-1), \ldots$. The ordinary differential operators transform (in the well-known way) into partial differential operators

$$\frac{d}{dt} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1}, \quad \frac{d^2}{dt^2} = \frac{\partial^2}{\partial T_0^2} + 2\varepsilon\frac{\partial^2}{\partial T_0 \partial T_1} + \varepsilon^2\frac{\partial^2}{\partial T_1^2}, \quad \ldots. \tag{26}$$

By introducing the following (backward) shift operators and partial difference operators

$$E_1\tilde{x}(T_0, T_1) = \tilde{x}(T_0 - 1, T_1), \tag{27}$$

$$E_\varepsilon\tilde{x}(T_0, T_1) = \tilde{x}(T_0, T_1 - \varepsilon), \tag{28}$$

$$\Delta_1\tilde{x}(T_0, T_1) = \tilde{x}(T_0, T_1) - \tilde{x}(T_0 - 1, T_1), \tag{29}$$

$$\Delta_\varepsilon\tilde{x}(T_0, T_1) = \tilde{x}(T_0, T_1) - \tilde{x}(T_0, T_1 - \varepsilon), \tag{30}$$

by observing that

$$\Delta_1 = I - E_1, \qquad \Delta_\varepsilon = I - E_\varepsilon, \tag{31}$$

where I is the identity operator, and by observing that $\Delta_1\tilde{x}(T_0,T_1) = \mathcal{O}(1)$ and $\Delta_\varepsilon\tilde{x}(T_0,T_1) = \mathcal{O}(\varepsilon)$ for bounded functions $x(t)$, it follows that the delayed terms $x(t-1)$, $\dot{x}(t-1)$, and $\ddot{x}(t-1)$ can be rewritten in (by using (25) and (26)):

$$\begin{aligned} x(t-1) &= E_1E_\varepsilon\,(X_0+\varepsilon X_1)+\mathcal{O}(\varepsilon^2),\\ &= E_1\,[1-\Delta_\varepsilon]\,(X_0+\varepsilon X_1)+\mathcal{O}(\varepsilon^2), \end{aligned} \tag{32}$$

$$\dot{x}(t-1) = E_1\frac{\partial X_0}{\partial T_0}+\varepsilon E_1\left[\left(\frac{\partial X_0}{\partial T_1}+\frac{\partial X_1}{\partial T_0}\right)-\frac{\Delta_\varepsilon}{\varepsilon}\frac{\partial X_0}{\partial T_0}\right] +\mathcal{O}(\varepsilon^2), \tag{33}$$

$$\ddot{x}(t-1) = E_1\frac{\partial^2 X_0}{\partial T_0^2}+\varepsilon E_1\left[2\frac{\partial^2 X_0}{\partial T_0\partial T_1}+\frac{\partial^2 X_1}{\partial T_0^2}-\frac{\Delta_\varepsilon}{\varepsilon}\frac{\partial X_0^2}{\partial T_0^2}\right] +\mathcal{O}(\varepsilon^2). \tag{34}$$

In the next subparagraphs, we will apply the MTS method to a first-order linear DDE and to second-order linear DDEs.

3.1 First order delay differential equations

Let us apply the perturbation method to the following initial value problem:

$$\dot{x}(t)+ax(t)+bx(t-1)=\varepsilon f(t,x(t)), \tag{35}$$

with

$$x(t)=\phi(t),\quad \text{for}\quad t\in[-1,0],$$

where a and b are constants. Suppose that $f(t,x(t)) = f_0(T_0,T_1,X_0)+\varepsilon f_1(T_0,T_1,X_0,X_1)+\mathcal{O}(\varepsilon^2)$. Substituting the expansion (25) into the initial value problem (35) yields as $O(1)$ problem and as $O(\varepsilon)$ problem:

$$\mathcal{O}(1),\quad \frac{\partial X_0}{\partial T_0}+aX_0+bE_1X_0=0. \tag{36}$$

$$X_0(T_0,T_1)=\phi(T_0),\quad T_0\in[-1,0],\ T_1 =\varepsilon T_0, \tag{37}$$

$$\mathcal{O}(\varepsilon),\quad \frac{\partial X_1}{\partial T_0}+aX_1+bE_1X_1=-\frac{\partial X_0}{\partial T_1} +\frac{b}{\varepsilon}E_1\Delta_\varepsilon X_0+f_0(T_0,T_1,X_0) \tag{38}$$

$$X_1(T_0,T_1)=0,\quad T_0\in[-1,0],\quad T_1=\varepsilon T_0, \tag{39}$$

respectively. For the $\mathcal{O}(1)$ equation, the corresponding characteristic equation is given by

$$h_1(\mu)\equiv\mu+a+be^{-\mu}=0. \tag{40}$$

The set of the characteristic roots of the function h_1 is called the spectrum of h_1 and is denoted as $\chi(h_1)$. For the characteristic Eq. (40), there exists a characteristic root with multiplicity two if and only if $be^a = e^{-1}$, (see also [33]). Now, let us restrict our discussion to the case $be^a \neq e^{-1}$. Hence, all characteristic roots, $\hat{\mu}\in\chi(h_1)$, have multiplicity one. This implies that the general solution of Eq. (36) can be written as

$$X_0(T_0,T_1)=\sum_{\hat{\mu}\in\chi(h_1)}\beta_{\hat{\mu}}(T_1)e^{\hat{\mu}T_0}, \tag{41}$$

where $\beta_{\hat{\mu}}(T_1)$ is still an arbitrary function in T_1. Note that X_0 should also satisfy the initial condition (37), and that X_1 should satisfy the condition $X_1=\mathcal{O}(X_0)$. The arbitrary functions $\beta_{\hat{\mu}}(T_1)$ can be used to avoid secular terms in $X_1(T_0,T_1)$. To avoid these secular (and unbounded) terms in $X_1(T_0,T_1)$, it is well known that the right-hand side of (38) should not contain resonant terms, that is, in this case, terms $e^{\hat{\mu}T_0}$. Obviously, in the right-hand side of (38), the terms $-\frac{\partial X_0}{\partial T_1}$ and $\frac{b}{\varepsilon}E_1\Delta_\varepsilon X_0$ contain such terms. Fistly, let us assume that $f_0(T_0,T_1,X_0)$ in (38) does not contain resonant terms (or equivalently does not contain terms which are solutions of the homogeneous equation related to Eq. (38)). Then, in order to avoid secular terms in $X_1(T_0,T_1)$ it follows from (38) and (41) that $\beta_{\hat{\mu}}(T_1)$ has to satisfy

$$-\frac{d\beta_{\hat{\mu}}(T_1)}{dT_1}+\frac{b}{\varepsilon}e^{-\hat{\mu}}\Delta_\varepsilon\beta_{\hat{\mu}}(T_1)=0 \tag{42}$$

for all $\hat{\mu}\in\chi(h_1)$. Now it should be observed that for $0<\varepsilon\ll 1$

$$\frac{\Delta_\varepsilon\beta_{\hat{\mu}}(T_1)}{\varepsilon}=\frac{d\beta_{\hat{\mu}}(T_1)}{dT_1}+\mathcal{O}(\varepsilon),$$

and so, (42) can be rewritten into (up to $\mathcal{O}(\varepsilon)$):

$$\left(1-be^{-\hat{\mu}}\right)\frac{d\beta_{\hat{\mu}}(T_1)}{dT_1}=0. \tag{43}$$

By assuming that $be^a\neq e^{-1}$ all roots of the characteristic Eq. (40) have multiplicity one, and so, $1-be^{-\hat{\mu}}\neq 0$. From (43) it then implies that $\beta_{\hat{\mu}}(T_1)$ is constant for all $\hat{\mu}\in\chi(h_1)$. By using the initial condition (37), (41), and the Laplace transform method for problem (36)–(37), it follows that:

$$\beta_{\hat{\mu}}(T_1)=\Phi_{\hat{\mu}},\quad \text{for } T_1\in[-\varepsilon,0] \tag{44}$$

with

$$\Phi_{\hat{\mu}}=\frac{1}{1-be^{-\hat{\mu}}}\left(\phi(0)-be^{-\hat{\mu}}\int_{-1}^{0}e^{-\hat{\mu}\theta}\phi(\theta)\,d\theta\right), \tag{45}$$

and $X_0(T_0, T_1)$ is given by (41), and $x(t) = X_0(T_0, T_1) + \mathcal{O}(\varepsilon)$. From (38), that is, from

$$\frac{\partial X_1}{\partial T_0} + aX_1 + bE_1X_1 = f_0(T_0, T_1, X_0), \tag{46}$$

$X_1(T_0, T_1)$ can be computed, and $x(t)$ can be written as

$$x(t) = \sum_{\hat{\mu}\in\chi(h_1)} \Phi_{\hat{\mu}} e^{\hat{\mu}t} + \mathcal{O}(\varepsilon). \tag{47}$$

Now, let us assume that $f_0(T_0, T_1, X_0)$ in (38) contains resonant terms. For that reason, we will consider the following example.

Example 1 Let $x(t)$ satisfy

$$\dot{x}(t) + ax(t) + bx(t-1) = \varepsilon x(t), \tag{48}$$

subject to the initial condition as given in Eq. (35). Following the analysis as given in the beginning of this subparagraph 3.1, it follows that $f_0(T_0, T_1, X_0) = X_0$ and all terms in the right-hand side of (38) are resonant terms. To avoid secular terms in $X_1(T_0, T_1)$ it now follows from (38) and (41) that $\beta_{\hat{\mu}}(T_1)$ has to satisfy

$$-\frac{d\beta_{\hat{\mu}}(T_1)}{dT_1} + \frac{b}{\varepsilon}e^{-\hat{\mu}}\Delta_\varepsilon\beta_{\hat{\mu}}(T_1) + \beta_{\hat{\mu}}(T_1) = 0, \tag{49}$$

or equivalently

$$-(1 - be^{-\hat{\mu}})\frac{d\beta_{\hat{\mu}}(T_1)}{dT_1} + \beta_{\hat{\mu}}(T_1) = 0, \tag{50}$$

for all $\hat{\mu} \in \chi(h_1)$. The ODE (50) can readily be solved, and by using the initial values (44) and (45), one obtains

$$\beta_{\hat{\mu}}(T_1) = \Phi_{\hat{\mu}} e^{\nu_{\hat{\mu}} T_1}. \tag{51}$$

where $\nu_{\hat{\mu}} = (1 - be^{-\hat{\mu}})^{-1}$. And so, the solution of the initial value problem for (48) can be written as

$$x(t) = \sum_{\hat{\mu}\in\chi(h_1)} \Phi_{\hat{\mu}} e^{\nu_{\hat{\mu}}\varepsilon t} e^{\hat{\mu}t} + \mathcal{O}(\varepsilon). \tag{52}$$

This simple example already shows how the MTS method can be applied to a first-order DDE taking into account all (infinitely many) roots of the characteristic equation and taking into account the initial values which are given on the time-interval $[-1, 0]$. In the next subparagraph we will see how the MTS method can be applied to some simple second-order DDEs.

3.2 Second order delay differential equations

Consider the following second order delay differential equation

$$\ddot{x}(t) + a\dot{x}(t-1) + bx(t) = \varepsilon f(t, x), \tag{53}$$

subject to the initial condition

$$x(t) = \phi(t), \quad t \in [-1, 0], \tag{54}$$

where a and b are constants, and where $\phi(t)$ is an $\mathcal{O}(1)-$function independent of $T_1 = \varepsilon t$, and $\phi(t) = \phi(T_0)$. Firstly, we will approximate the solution of (53) by using the expansion (25). Moreover, we will assume that $f(t, x)$ in (53) can be written as $f(t, x) = f_0(T_0, T_1, X_0) + \varepsilon f_1(T_0, T_1, X_0, X_1) + \ldots$. By substituting the expansion (25) for $x(t)$ and by substituting the expansion for $f(t, x)$ into (53), we obtain the following $\mathcal{O}(1)-$problem and $\mathcal{O}(\varepsilon)-$problem,

$$\mathcal{O}(1), \quad \frac{\partial^2 X_0}{\partial T_0^2} + aE_1\frac{\partial X_0}{\partial T_0} + bX_0 = 0,$$
$$T_0, T_1 > 0, \tag{55}$$

$$X_0(T_0, T_1) = \phi(T_0), \quad T_0 \in [-1, 0],$$
$$T_1 = \varepsilon T_0, \tag{56}$$

$$\mathcal{O}(\varepsilon), \quad \frac{\partial^2 X_1}{\partial T_0^2} + aE_1\frac{\partial X_1}{\partial T_0} + bX_1 = -2\frac{\partial^2 X_0}{\partial T_0 \partial T_1}$$
$$+aE_1\left(-\frac{\partial X_0}{\partial T_1} + \frac{1}{\varepsilon}\Delta_\varepsilon\frac{\partial X_0}{\partial T_0}\right)$$
$$+ f_0(T_0, T_1, X_0), \; T_0, T_1 > 0, \tag{57}$$

$$X_1(T_0, T_1) = 0, \quad T_0 \in [-1, 0], T_1 = \varepsilon T_0, \tag{58}$$

respectively. By substituting $e^{\mu T_0}$ into (55) one obtains the characteristic equation

$$h_2(\mu) \equiv \mu^2 + a\mu e^{-\mu} + b = 0. \tag{59}$$

The set of all roots of h_2 is denoted by $\chi(h_2)$. To simplify the computation, it will be assumed that all roots of (59) have multiplicity one; that is, it will be assumed that the constants a and b are such that $2\mu + ae^{-\mu} - a\mu e^{-\mu} \neq 0$ for all roots $\mu \in \chi(h_2)$. Then, the general solution of (53) is given by

$$X_0(T_0, T_1) = \sum_{\hat{\mu}\in\chi(h_2)} \beta_{\hat{\mu}}(T_1)e^{\hat{\mu}T_0}, \tag{60}$$

where $\beta_{\hat{\mu}}(T_1)$ is still an arbitrary function, which will be used to avoid resonant terms in the right-hand side of (57). By using the Laplace-transform method to (55)–(56) it can be shown that $\beta_{\hat{\mu}}(T_1)$ satisfies the following initial condition:

$$\beta_{\hat{\mu}}(T_1) = \frac{N(\hat{\mu})}{2\hat{\mu} + ae^{-\hat{\mu}} - a\hat{\mu}e^{-\hat{\mu}}} = \Phi_k, \quad T_1 \in [-\varepsilon, 0] \tag{61}$$

with

$$N(\hat{\mu}) = \hat{\mu}\phi(0) + \dot{\phi}(0) + a\phi(-1) - a\hat{\mu}e^{-\hat{\mu}}$$
$$\int_{-1}^{0} \phi(T_0)e^{-\hat{\mu}T_0}\, dT_0. \tag{62}$$

If $f_0(T_0, T_1, X_0)$ does not contain resonant terms, then in order to eliminate secular terms, the following condition should be satisfied for $\beta_{\hat{\mu}}$

$$-2\hat{\mu}\beta'_{\hat{\mu}}(T_1) + ae^{-\hat{\mu}}(-\beta'_{\hat{\mu}}(T_1) + \frac{\hat{\mu}}{\varepsilon}\Delta_\varepsilon\beta_{\hat{\mu}}(T_1)) = 0, \tag{63}$$

for all $\hat{\mu} \in \chi(h_2)$. Similar to the first order delay differential equation in Sect. 3.1, by rewriting the definition for the partial difference operator and expanding the delay form by using the Taylor expansion, we obtain that $\beta'_{\hat{\mu}}(T_1) = 0$. Hence, it can be concluded that the solution of Eq. (63) is constant for all $\hat{\mu} \in \chi(h_2)$. As a result, the value for $\beta_{\hat{\mu}}$ is given by (61). However, if f_0 contains resonant terms, then the conditions for $\beta_{\hat{\mu}}$ will be different as well. To determine the conditions for such cases, let us give an illustration for different cases of f_0 in the following examples.

Example 2 Let $x(t)$ satisfy

$$\ddot{x}(t) + a\dot{x}(t-1) + bx(t) = \varepsilon x(t), \tag{64}$$

subject to the initial condition as given in Eq. (54).

Following the analysis as given in the beginning of this subsection, it follows that $f_0(T_0, T_1, X_0) = X_0$ and that the right-hand side of (57) is given by

$$-2\frac{\partial^2 X_0}{\partial T_0 \partial T_1} + aE_1\left[-\frac{\partial X_0}{\partial T_1} + \frac{1}{\varepsilon}\Delta_\varepsilon\frac{\partial X_0}{\partial T_0}\right] + X_0. \tag{65}$$

Therefore, by substituting the solution of the $\mathcal{O}(1)$-problem as given in (60) into formula (65), the condition to remove the resonant terms is

$$-2\hat{\mu}\beta'_{\hat{\mu}}(T_1) + ae^{-\hat{\mu}}(-\beta'_{\hat{\mu}}(T_1) + \frac{\hat{\mu}}{\varepsilon}\Delta_\varepsilon\beta_{\hat{\mu}}(T_1)) + \beta_{\hat{\mu}}(T_1) = 0, \tag{66}$$

for all $\hat{\mu} \in \chi(h_2)$, or equivalently by using the definition of Δ_ε, and the fact that $0 < \varepsilon \ll 1$:

$$(-2\hat{\mu} - ae^{-\hat{\mu}} + a\hat{\mu}e^{-\hat{\mu}})\beta'_{\hat{\mu}}(T_1) + \beta_{\hat{\mu}}(T_1) = 0. \tag{67}$$

By solving (67) for $\beta_{\hat{\mu}}(T_1)$, and by using the initial condition (61) for $\beta_{\hat{\mu}}(T_1)$ one finds

$$\beta_{\hat{\mu}}(T_1) = \Phi_{\hat{\mu}}e^{\nu_{\hat{\mu}}T_1} \tag{68}$$

with

$$\nu_{\hat{\mu}} = [2\hat{\mu} + ae^{-\hat{\mu}} - a\hat{\mu}e^{-\hat{\mu}}]^{-1}. \tag{69}$$

And so, the solution of (64) can be written as

$$x(t) = \sum_{\hat{\mu}\in\chi(h_2)} \Phi_{\hat{\mu}}e^{\nu_{\hat{\mu}}T_1}e^{\hat{\mu}T_0} + \mathcal{O}(\varepsilon). \tag{70}$$

In the next two examples, we will show how the method can be applied when $f(t, x)$ depends explicitly on t.

Example 3 Let $x(t)$ satisfy

$$\ddot{x}(t) + a\dot{x}(t-1) + bx(t) = \varepsilon e^{\varepsilon t}x(t), \tag{71}$$

subject to the initial condition as given in (54). Again following the analysis as given in the beginning of this subsection it follows that $f_0(T_0, T_1, X_0) = e^{T_1}X_0$, and that in order to avoid resonant terms in the right-hand side of (57) that $\beta_{\hat{\mu}}(T_1)$ has to satisfy

$$(-2\hat{\mu} - ae^{-\hat{\mu}} + ae^{-\hat{\mu}})\beta'_{\hat{\mu}}(T_1) + e^{T_1}\beta_{\hat{\mu}}(T_1) = 0. \tag{72}$$

By solving (72) for $\beta_{\hat{\mu}}(T_1)$, and by using the initial condition (61) for $\beta_{\hat{\mu}}(T_1)$, one finally finds that the solution $x(t)$ of (71) can be written as

$$x(t) = \sum_{\hat{\mu}\in\chi(h_2)} \Phi_{\hat{\mu}}e^{\nu_{\hat{\mu}}e^{T_1}}e^{\hat{\mu}T_0} + \mathcal{O}(\varepsilon). \tag{73}$$

where $\Phi_{\hat{\mu}}$ and $\nu_{\hat{\mu}}$ are given in (61) and (69), respectively.

Example 4 Let $x(t)$ satisfy

$$\ddot{x}(t) + 0.5\dot{x}(t-1) + x(t) = \varepsilon\cos(\omega t)x(t), \tag{74}$$

subject to the initial condition as given in Eq. (54), and where ω is a nonzero constant independent of ε, that is, ω is a strict order one constant. The characteristic equation in this case is a particular case of (59) with $a = 0.5$ and $b = 1$. The analysis as given in the beginning of this subsection can be followed again, and $f_0(T_0, T_1, X_0) = \cos(\omega T_0)X_0$. By rewriting $\cos(\omega T_0)$ as $(e^{i\omega T_0} + e^{-i\omega T_0})/2$, and by writing the characteristic roots μ as $v + iw$ with v, and $w \in \mathbb{R}$, it follows that

$$f_0(T_0, T_1, X_0) = \cos(\omega T_0)X_0 = \frac{1}{2}\sum_{v+iw\in\chi(h_2)} \beta_{v+iw}(T_1) \left(e^{[v+i(w+\omega)]T_0} + e^{[v+i(w-\omega)]T_0}\right). \tag{75}$$

As we can see, $v + i(w \pm \omega)$ will contribute to the secular terms if it is again a characteristic root. Next, we will show that it is only possible for $\omega = \pm 2\hat{w}$, where $\hat{w}$ is an imaginary part of a characteristic root, $\hat{v} \pm i\hat{w}$.

Note that Pontryagin's theorem gives necessary and sufficient stability conditions for an exponential polynomial with a principle term. The theorem can be found in [29]. Using this theorem, we obtain that the zero solution of (74) for $a = 0.5$, $b = 1$, and $\varepsilon = 0$ is asymptotically stable since all the real parts of the characteristic roots in problem (74) are negative, that is, $v < 0$.

First, we consider real-valued characteristic roots. For $v \in \mathbb{R}^-$: if $v \in \chi(h_2)$ then it must satify $v^2 + 0.5ve^{-v} + 1 = 0$. Note that $h_2(v) = v^2 + 0.5ve^{-v} + 1$ is monotonically increasing for all $v < 0$, $h_2(-2) < 0$ and $h_2(-1) > 0$. Hence, using the Intermediate value theorem, there exists a unique $v_r \in \mathbb{R}^-$ such that $h_2(v_r) = 0$. Moreover, there is no w_r such that $v_r \pm iw_r \in \chi(h_2)$. This implies that $v_r \pm i\omega$ is again a characteristic root only for $\omega = 0$. This contradicts the assumption that $\omega > 0$. Hence, $v_r \pm i\omega$ will be not contributing to secular terms.

Next, we consider complex-valued characteristic roots. For a given $v + iw \in \chi(h_2)$, v and w satisfy

$$v^2 - w^2 + 0.5ve^{-v}\cos(w) + 0.5we^{-v}\sin(w) + 1 = 0, \tag{76}$$

$$2vw - 0.5v\sin(w)e^{-v} + 0.5we^{-v}\cos(w) = 0. \tag{77}$$

It is easy to see that also $v - iw \in \chi(h_2)$. Next, we will show that for problem (74), if $\hat{v} + i\hat{w} \in \chi(h_2)$, then there is no $\bar{w} \neq \pm\hat{w}$ such that $\hat{v} + i\bar{w} \in \chi(h_2)$. First, rewrite (76) and (77) as

$$v^2 - w^2 + 1 = -0.5ve^{-v}\cos(w) - 0.5we^{-v}\sin(w), \tag{78}$$

$$2vw = 0.5v\sin(w)e^{-v} - 0.5we^{-v}\cos(w). \tag{79}$$

Squaring both equations and adding the so-obtained equations yields

$$w^4 + f(v)w^2 + g(v) = 0, \tag{80}$$

where $f(v) = 2v^2 - 2 - (0.5e^{-v})^2$ and $g(v) = (v^2+1)^2 - (0.5ve^{-v})^2$. Let us define $k(v) = f^2(v) - 4\,g(v) = -16v^2 + e^{-2v} + 0.5^4e^{-4v}$. By observing that (80) is a quadratic equation in w^2, the solutions of (80) can be written as

$$w^2 = \frac{-f(v) + \sqrt{k(v)}}{2} \quad \text{or} \quad w^2 = \frac{-f(v) - \sqrt{k(v)}}{2}. \tag{81}$$

Now, we can show that there are four possibilities for w. Those are

$$w_{1,2} = \pm\sqrt{\frac{-f(v) + \sqrt{k(v)}}{2}}, \quad w_{3,4} = \pm\sqrt{\frac{-f(v) - \sqrt{k(v)}}{2}}. \tag{82}$$

Note that the squared Eq. (80) always has all the original solutions of (78)–(79) but may also have additional solutions because of squaring (78) and (79). So we need to recheck the solutions (82).

1. For $v < v_r$, since $g(v_r) = 0$, $g'(v_r) > 0$ and g has only one real root, it satisfies $g(v) < 0$. This implies that $k(v) = f^2(v) - 4g(v) > f^2(v)$ for all $v < v_r$. Using this condition, we obtain that $-f(v) - \sqrt{k(v)} < 0$. This implies that $w_{3,4}$ are not real-valued. This contradicts that $w_{3,4}$ are real-valued. In conclusion, $v + iw_{3,4} \notin \chi(h_2)$.
2. For $v_r < v < 0$, there is only one couple of characteristic roots in this interval, and $v + iw_{3,4} \notin \chi(h_2)$ for $v < v_r$. The roots can be seen in the Fig. 3.

This analysis proves that each real part of complex-valued characteristic roots, v, only corresponds with two imaginary parts, $\pm w$. So, $v \pm i(w + \omega)$ is again a characteristic root if $\omega = \mp 2w$.

Now, let us suppose that $\omega = 2\hat{w}$, where $\hat{w}$ is a nonzero imaginary part of an eigenvalue, i.e, there exist a $\hat{v}$ such that $(\hat{v} + i\hat{w}) \in \chi(h_2)$, $\hat{w} \neq 0$. We have to consider three cases.

Case 1. For $\mu \neq \hat{v} \pm i\hat{w}$. In this case, $f_0(T_0, T_1, X_0)$ has no contribution to the secular terms. Therefore, the condition for β_μ to eliminate secular terms is

$$(-2\hat{\mu} - ae^{-\hat{\mu}} + a\hat{\mu}e^{-\hat{\mu}})\beta'_{\hat{\mu}}(T_1) = 0.$$

Hence $\beta_\mu(T_1)$ is a constant function. Using the initial conditions for $\beta_\mu(T_1)$ in (61), we obtain that $\beta_\mu(T_1) = \Phi_\mu$, for $\mu \neq \hat{v} \pm i\hat{w}$.

Case 2. For $\mu = \hat{v} + i\hat{w}$. In this case, $f_0(T_0, T_1, X_0)$ has a contribution to the secular terms. Therefore,

$$-h_2'(\hat{v} + i\hat{w})\beta'_{\hat{v}+i\hat{w}}(T_1) + \frac{1}{2}\beta_{\hat{v}-i\hat{w}}(T_1) = 0. \tag{83}$$

Case 3. For $\mu = \hat{v} - i\hat{w}$. Similar to the previous case, we obtain that

$$-h_2'(\hat{v} - i\hat{w})\beta'_{\hat{v}-i\hat{w}}(T_1) + \frac{1}{2}\beta_{\hat{v}+i\hat{w}}(T_1) = 0. \tag{84}$$

Hence, (83) and (84) are coupled linear first-order differential equations. It is easy to show that for $m = \frac{1}{2}\left(h_2'(\hat{v} + i\hat{w})h_2'(\hat{v} - i\hat{w})\right)^{-1/2}$, the general solution

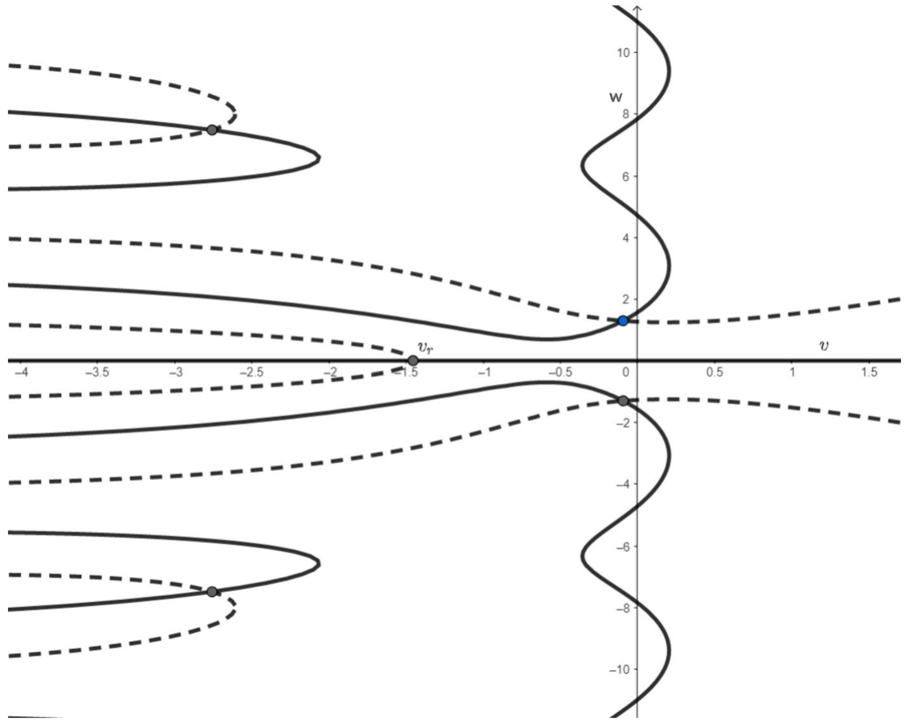

Fig. 3 The real part (Eq. (76)) and the imaginary part (Eq. (77)) of the characteristic equation are plotted by the solid lines and by the dashed lines, respectively. Roots of the characteristic equation are the intersection points of these lines

of these equations is

$$\beta_{\hat{v}+i\hat{w}}(T_1) = c_1 e^{mT_1} + c_2 e^{-mT_1},$$
$$\beta_{\hat{v}-i\hat{w}}(T_1) = \frac{1}{2mh_2'(\hat{v}-i\hat{w})}\left[c_1 e^{mT_1} - c_2 e^{-mT_1}\right]. \tag{85}$$

By using the initial conditions (61) we obtain two linear equations in c_1 and c_2. By solving it, we obtain

$$c_1 = \frac{1}{2}\left[\Phi_{\hat{v}+i\hat{w}} + 2mh_2'(\hat{v}-i\hat{w})\Phi_{\hat{v}-i\hat{w}}\right],$$
$$c_2 = \frac{1}{2}\left[\Phi_{\hat{v}+i\hat{w}} - 2mh_2'(\hat{v}-i\hat{w})\Phi_{\hat{v}-i\hat{w}}\right]. \tag{86}$$

So, the approximation of the solution is

$$X_0(T_0, T_1) = \beta_{\hat{v}+i\hat{w}}(T_1)e^{(\hat{v}+i\hat{w})T_0} + \beta_{\hat{v}-i\hat{w}}(T_1)e^{(\hat{v}-i\hat{w})T_0} + \sum_{\substack{\mu\in\chi(h_2)\\ \mu\neq\hat{v}\pm\hat{w}}} \Phi_\mu e^{\mu T_0}. \tag{87}$$

In the next section, we will prove that the constructed approximations in Example 1 to Example 4 are $\mathcal{O}(\varepsilon)$ accurate for $t = \mathcal{O}(\varepsilon^{-1})$. The presented proof can also be used for the more advanced and weakly nonlinear problems that will be studied in Sect. 5 of this paper.

4 Accuracy of the approximations on a long time-scale

In this section, we are going to show how accurate the constructed approximations of the solutions are on long time-scales for the following initial value problem:

$$\frac{d^n x(t)}{dt^n} + \sum_{m=0}^{n-1} a_m \frac{d^m x(t)}{dt^m} + \sum_{m=0}^{n} b_m \frac{d^m x(t-1)}{dt^m} = \varepsilon f(t, x(t)), \quad t > 0, \tag{88}$$
$$x(t) = \phi(t), \quad \text{for } t \in [-1, 0], \tag{89}$$

where n is a positive integer, and where ε is a small parameter with $0 < \varepsilon \ll 1$. The function $\phi \in C^{n-1}([-1, 0])$, and the function f satisfies the Lipschitz condition, i.e, there exists a constant L such that

$$|f(t, x(t)) - f(t, \tilde{x}(t))| \leq L|x(t) - \tilde{x}(t)|, \tag{90}$$

in a closed interval $0 \leq t \leq t_1$. Let $h(s)$ be the characteristic function related to Eq. (88) with $\varepsilon = 0$. For the characteristic equation $h(s) = 0$, it will be assumed for simplicity that all roots have multiplicity one, and that the roots have a finite maximal real part $\hat{d} \in \mathbb{R}$. Hence, $h(s)$ is analytic in s for all s with $\mathrm{Re}(s) \leq \hat{d}$. According to [4], $h(s)$ has an inverse Laplace transform which is usually called a fundamental solution. Let us denote it as $H(t)$. From the definition of $\hat{d}$, there exists a positive constant M_1 such that

$$|H(t)| \leq M_1 e^{\hat{d}t}. \tag{91}$$

Suppose that the approximation of the solution $x(t)$ is denoted as $\tilde{x}(t)$ and satisfies

$$\frac{d^n\tilde{x}(t)}{dt^n} + \sum_{m=0}^{n-1} a_m \frac{d^m\tilde{x}(t)}{dt^m} + \sum_{m=0}^{n} b_m \frac{d^m\tilde{x}(t-1)}{dt^m} = \varepsilon f(t,\tilde{x}) + R(t,\varepsilon), \tag{92}$$

where $R(t,\varepsilon)$ is called the residual. For the approximation as constructed by using the perturbation method as given in Sect. 3, the residual satisfies

$$|R(t,\varepsilon)| \leq \varepsilon^2 M_2 e^{\hat{d}t}, \tag{93}$$

where M_2 is a constant. It is also assumed that $\phi(t) = \mathcal{O}(1)$ such that $\tilde{x}(t)$ satisfies $\tilde{x}(t) = \phi(t)$ for $t \in [-1, 0]$. In this research, only the first term in Eq. (25) has been determined completely. The $O(\varepsilon)$ equation is considered only to obtain the condition for X_0 such that X_1 does not contain secular terms. Hence, we will show that there exist positive constants c, ε_0, K, and D such that the approximation of the solution $x(t)$ satisfies

$$|x(t) - \tilde{x}(t)| \leq c\varepsilon e^{\hat{d}t} \tag{94}$$

for $0 \leq t \leq K/\varepsilon$, and $0 \leq \varepsilon < \varepsilon_0$. The idea how to prove the validity is motivated by the validity proof of the MTS method for ODE for large times in [21] and for higher order averaging in [27]. First, we rewrite the DDE into an integral equation and then show that uniform validity holds. Using the Laplace transformation method, we obtain the corresponding integral equation for the initial value problem (88), yielding

$$x(t) = \sum_k \frac{1}{h'(\mu_k)} N(\mu_k) e^{\mu_k t} + \varepsilon \int_0^t H(t-s) f(s, x(s))\, ds. \tag{95}$$

with

$$N(\mu_k) = \sum_{j=1}^{n} \mu^{j-1}\phi_{n-j}^R + \sum_{m=1}^{n-1} a_m \sum_{j=1}^{m} \mu_k^{j-1}\phi_{m-j}^R - \sum_{m=0}^{n} b_m \left[\mu_k^m e^{-\mu_k} \int_{-1}^{0} e^{-\mu_k t}\phi(t)\, dt\right] + \sum_{m=1}^{n} b_m \left[\sum_{j=1}^{m} \mu_k^{j-1}\phi_{m-j}^L\right],$$

where $\phi_k^R = \phi^{(k)}(0)$, and $\phi_k^L = \phi^{(k)}(-1)$. The corresponding integral equation for the approximation (see (92)) is given by

$$\tilde{x}(t) = \sum_k \frac{1}{h'(\mu_k)} N(\mu_k) e^{\mu_k t} + \varepsilon \int_0^t H(t-s) f(s, \tilde{x}(s))\, ds + \int_0^t H(t-s) R(s,\varepsilon)\, ds. \tag{96}$$

By subtracting (96) from (95), and by taking the absolute value, one obtains

$$|x(t) - \tilde{x}(t)| \leq \varepsilon \int_0^t |H(t-s)||f(s,x(s)) - f(s,\tilde{x}(s))|\, ds + \int_0^t |H(t-s)||R(s,\varepsilon)|\, ds. \tag{97}$$

To simplify the estimate (97) further it should be observed that f satisfies a Lipschitz condition. Hence there exists a positive constant L such that

$$|f(t,x(t)) - f(t,\tilde{x}(t))| \leq L|x(t) - \tilde{x}(t)|. \tag{98}$$

From Eqs. (91) to (98), we then obtain that

$$\int_0^t |H(t-s)||f(s,x(s)) - f(s,\tilde{x}(s))|\, ds \leq \int_0^t M_1 e^{\hat{d}(t-s)} L|x(s) - \tilde{x}(s)|\, ds, = M_1 L \int_0^t e^{\hat{d}(t-s)} |x(s) - \tilde{x}(s)|\, ds. \tag{99}$$

By using (91) and (93), we further obtain that

$$\int_0^t |H(t-s)||R(s,\varepsilon)|\, ds \leq \int_0^t M_1 e^{\hat{d}(t-s)} \left[\varepsilon^2 M_2 e^{\hat{d}s}\right] ds, = \varepsilon^2 M_1 M_2 t e^{\hat{d}t}, \leq \varepsilon M_1 M_2 K e^{\hat{d}t}. \tag{100}$$

Using the inequalities (99) and (100), the inequality in Eq. (97) becomes

$$|x(t) - \tilde{x}(t)| \leq \varepsilon M_1 L \int_0^t e^{\hat{d}(t-s)} |x(s) - \tilde{x}(s)|\, ds + \varepsilon M_1 M_2 K e^{\hat{d}t}, \tag{101}$$

which is equivalent to

$$e^{-\hat{d}t}|x(t)-\tilde{x}(t)| \leq \varepsilon M_1 L \int_0^t e^{-\hat{d}s}|x(s)-\tilde{x}(s)|\, ds + \varepsilon M_1 M_2 K. \quad (102)$$

By putting $u(t) = e^{-\hat{d}t}|x(t)-\tilde{x}(t)|$, we can rewrite this inequality (102) into

$$u(t) \leq \varepsilon M_1 L \int_0^t u(s)\, ds + \varepsilon[M_1 M_3 K]. \quad (103)$$

Using the Gronwall inequality for (103), we obtain that

$$e^{-\hat{d}t}|x(t)-\tilde{x}(t)| \leq \varepsilon\,[M_1 M_2 K] \exp\left(\int_0^t \varepsilon M_1 L\, ds\right), = \varepsilon\,[M_1 M_2 K]\, e^{\varepsilon M_1 L t} \quad (104)$$

For $t = \mathcal{O}(\varepsilon^{-1})$ the right-hand side is $\mathcal{O}(\varepsilon)$, the absolute error is

$$|x(t)-\tilde{x}(t)| = \mathcal{O}\left(\varepsilon e^{\hat{d}t}\right), \quad (105)$$

and the relative error (compared to the largest possible solution) is:

$$\frac{|x(t)-\tilde{x}(t)|}{e^{\hat{d}t}} = \mathcal{O}(\varepsilon), \quad (106)$$

where $\hat{d}$ is the maximal real part of the roots of the characteristic equation $h(s) = 0$. In conclusion, the absolute error is $\mathcal{O}(\varepsilon e^{\hat{d}t})$, and the relative error is $\mathcal{O}(\varepsilon)$ for all $t = \mathcal{O}(\varepsilon^{-1})$. The main result of this section can be formulated as follows.

Theorem 1 *Suppose that $x(t)$ is the exact solution of the initial value problem (88)–(89), and $\tilde{x}(t)$ is an approximation of $x(t)$ and satisfies Eq. (92). Assuming that the initial function $\phi(t)$ is $\mathcal{O}(1)$ for $t \in [-1, 0]$, it then follows that the absolute error between $x(t)$ and $\tilde{x}(t)$ is $\mathcal{O}(\varepsilon e^{\hat{d}t})$, and the relative absolute error between $x(t)$ and $\tilde{x}(t)$ is $\mathcal{O}(\varepsilon)$, on a timescale $t = \mathcal{O}\left(\varepsilon^{-1}\right)$, where $\hat{d}$ is the maximal real part of the characteristic equation related to (88) with $\varepsilon = 0$.*

Next, in Sect. 5 of this paper, it will be shown how the MTS method can be applied to more advanced problems, i.e, to weakly nonlinear initial value problems.

5 Advanced problems

In Sect. 3 of this paper, the MTS method for DDEs has been introduced and has been applied to some simple, linear problems. In this section, it will be shown how the MTS method for DDEs can be applied to more complicated problems, that is, to weakly nonlinear problems.

5.1 A first Order DDE with a weak quadratic nonlinearity

Example 5 Consider the following DDE for $x = x(t)$:

$$\dot{x}(t) + x(t) - x(t-1) = \varepsilon x^2(t), t > 0, \quad (107)$$

subject to the initial condition as given in the initial value problem (35).

In fact, this is a particular example of problem (35) with $a = 1, b = -1$, and $f(t, x) = x^2$. Substituting the expansion (25) into the equation of problem (107) and into the initial value yields as $O(1)$-problem and as $O(\varepsilon)$-problem

$$\mathcal{O}(1), \quad \frac{\partial X_0}{\partial T_0} + X_0 - E_1 X_0 = 0, \quad (108)$$

$$X_0(T_0, T_1) = \phi(T_0), \quad T_0 \in [-1, 0], T_1 = \varepsilon T_0, \quad (109)$$

$$\mathcal{O}(\varepsilon), \quad \frac{\partial X_1}{\partial T_0} + X_1 - E_1 X_1 = -\frac{\partial X_0}{\partial T_1} - \frac{1}{\varepsilon} E_1 \Delta_\varepsilon X_0 + X_0^2, \quad (110)$$

$$X_1(T_0, T_1) = 0, \quad T_0 \in [-1, 0], T_1 = \varepsilon T_0. \quad (111)$$

Now, let $\hat{\mu}$ be a root of the characteristic equation corresponding to (108), that is,

$$h_3(\mu) \equiv \mu + 1 - e^{-\mu} = 0. \quad (112)$$

For all $\mu = \hat{\mu}$ with $h_3(\hat{\mu}) = 0$, we have $h_3'(\hat{\mu}) \neq 0$, and so, all roots of (112) have multiplicity one. Hence, the general solution of Eq. (108) can be written as

$$X_0(t) = \sum_{\hat{\mu} \in \chi(h_3)} \beta_{\hat{\mu}}(T_1) e^{\hat{\mu} T_0}. \quad (113)$$

It is easy to show that zero is one of the characteristic roots of (112). Using the Laplace transform method and the initial condition, we obtain that $\beta_k(T_1)$ should satisfy the initial condition

$$\beta_{\hat{\mu}}(T_1) = \Phi_{\hat{\mu}}, \quad T_1 \in [-\varepsilon, 0] \quad (114)$$

with

$$\Phi_{\hat{\mu}} = \frac{1}{1+e^{-\hat{\mu}}}\left[\phi(0) + e^{-\hat{\mu}} \int_{-1}^0 e^{-\hat{\mu}s} \phi(s)\, ds\right]. \quad (115)$$

By substituting the solution (113) into the right-hand side of the $\mathcal{O}(\varepsilon)$ Eq. (110), it follows that this right-hand side can be written as:

$$\sum_{\hat{\mu} \in \chi(h_3)} \left(-\frac{d\beta_{\hat{\mu}}(T_1)}{dT_1} - e^{-\hat{\mu}} \frac{\Delta_\varepsilon}{\varepsilon} \beta_{\hat{\mu}}(T_1)\right) e^{\hat{\mu} T_0} + \sum_{\hat{\mu}_1 \in \chi(h_3)} \sum_{\hat{\mu}_2 \in \chi(h_3)} \beta_{\hat{\mu}_1}(T_1) \beta_{\hat{\mu}_2}(T_1) e^{(\hat{\mu}_1 + \hat{\mu}_2) T_0}. \quad (116)$$

Resonant terms in this right-hand side due to the quadratic nonlinearity can occur whenr $\hat{\mu}_1 + \hat{\mu}_2 \in \chi(h_3)$, where $\hat{\mu}_1$ and $\hat{\mu}_2$ satisfy (112), that is,

$$\hat{\mu}_1 + 1 - e^{-\hat{\mu}_1} = 0, \tag{117}$$

$$\hat{\mu}_2 + 1 - e^{-\hat{\mu}_2} = 0, \text{ and} \tag{118}$$

$$(\hat{\mu}_1 + \hat{\mu}_2) + 1 - e^{(\hat{\mu}_1+\hat{\mu}_2)} = 0. \tag{119}$$

By substituting the exponential form from (117) and (118) into (119), we obtain that

$$\hat{\mu}_1\hat{\mu}_2 = 0. \tag{120}$$

Hence, the sum of two roots will be a root of the characteristic Eq. (112) if and only if at least one of these roots is identically equal to zero. The right-hand side (116) can now be rewritten as

$$\left(-\frac{d\beta_0(T_1)}{dT_1} - \frac{\Delta_\varepsilon}{\varepsilon}\beta_0(T_1) + \beta_0^2(T_1)\right) e^{0\cdot T_0}$$
$$+ \sum_{\substack{\hat{\mu}\in\chi(h_3)\\ \hat{\mu}\neq 0}} \left(-\frac{d\beta_{\hat{\mu}}(T_1)}{dT_1} - e^{-\hat{\mu}}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat{\mu}}(T_1)\right.$$
$$\left. +2\beta_0(T_1)\beta_{\hat{\mu}}(T_1)\right) e^{\hat{\mu}T_0}$$
$$+ \sum_{\substack{\hat{\mu}_1\in\chi(h_3),\\ \hat{\mu}_1\neq 0}} \sum_{\substack{\hat{\mu}_2\in\chi(h_3),\\ \hat{\mu}_2\neq 0}} \beta_{\hat{\mu}_1}(T_1)\beta_{\hat{\mu}_2}(T_1)e^{(\hat{\mu}_1+\hat{\mu}_2)T_0}. \tag{121}$$

Obviously, the first two terms in (121) are resonant terms, and the last term is not resonant. To avoid secular terms in $X_1(T_0, T_1)$ it now follows from (110) and (121) that $\beta_0(T_1)$ has to satisfy

$$-\frac{d\beta_0(T_1)}{dT_1} - \frac{\Delta_\varepsilon}{\varepsilon}\beta_0(T_1) + \beta_0^2(T_1) = 0. \tag{122}$$

and that $\beta_{\hat{\mu}}(T_1)$ for $\hat{\mu} \neq 0$ has to satisfy

$$-\frac{d\beta_{\hat{\mu}}(T_1)}{dT_1} - e^{-\hat{\mu}}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat{\mu}}(T_1) + 2\beta_0(T_1)\beta_{\hat{\mu}}(T_1) = 0. \tag{123}$$

Since $\Delta_\varepsilon\beta_0(T_1) = \beta_0(T_1) - \beta_0(T_1 - \varepsilon) = \varepsilon\beta_0'(T_1) + \mathcal{O}(\varepsilon^2)$, it follows from our perturbation procedure (that is, (122) only contains $\mathcal{O}(1)$ terms) and from (122) that $\beta_0(T_1)$ satisfies:

$$-2\beta_0'(T_1) + \beta_0^2(T_1) = 0. \tag{124}$$

The Eq. (124) for $\beta_0(T_1)$ can readily be solved, and by using the initial condition (114) for $\hat{\mu} = 0$, it follows that $\beta_0(T_1)$ is given by

$$\beta_0(T_1) = \left[-\frac{T_1}{2} + \Phi_0\right]^{-1}, \tag{125}$$

where $\Phi_0 = 2\left[\phi(0) + \int_{-1}^{0}\phi(s)\,ds\right]^{-1}$. Similarly, the Eq. (123) for $\beta_{\hat{\mu}}(T_1)$ with $\hat{\mu} \neq 0$ can be written as

$$-(1 + e^{-\hat{\mu}})\beta_{\hat{\mu}}'(T_1) + 2\beta_0(T_1)\beta_{\hat{\mu}}(T_1) = 0, \tag{126}$$

and can be solved accordingly, yielding

$$\beta_{\hat{\mu}}(T_1) = \Phi_{\hat{\mu}}(-2\Phi_0)^{4/h_3'(\hat{\mu})}(T_1 - 2\Phi_0)^{-4/h_3'(\hat{\mu})} \tag{127}$$

So far, $X_0(T_0, T_1)$ has been determined completely, and $X_1(T_0, T_1)$ does not contain secular terms (that is, $X_1 = \mathcal{O}(X_0)$ for $t = \mathcal{O}(\varepsilon^{-1})$). And so, we can conclude that

$$X_0(T_0, T_1) = \sum_{\hat{\mu}\in\chi(h_3)} \beta_{\hat{\mu}}(T_1)e^{\hat{\mu}T_0}, \tag{128}$$

where $\beta_0(T_1)$ and $\beta_{\hat{\mu}}(T_1)$ with $\hat{\mu} \neq 0$ are given by (125) and (127), respectively, and that

$$x(t) = X_0(T_0, T_1) + \mathcal{O}(\varepsilon X_0(T_0, T_1)) \tag{129}$$

for $t = \mathcal{O}\left(\varepsilon^{-1}\right)$.

5.2 A second order DDE with a weak quadratic nonlinearity.

Example 6 Consider the following initial value problem for $x = x(t)$:

$$\ddot{x}(t) + x(t) - x(t-1) = \varepsilon x^2(t), t > 0,$$
$$x(t) = \phi(t), t \in [-1, 0], \tag{130}$$

where it is assumed that ϕ is a smooth function of order 1, and where ε is a small parameter with $0 < \varepsilon \ll 1$. By substituting the expansion (25) for $x(t)$ into (130), and by collecting terms of order 1, and of order ε, we obtain the following $\mathcal{O}(1)$-, and $\mathcal{O}(\varepsilon)-$ problems:

$$\mathcal{O}(1), \quad \frac{\partial^2 X_0}{\partial T_0^2} + X_0 - E_1X_0 = 0, \tag{131}$$

$$X_0(T_0, T_1) = \phi(T_0), \quad T_0 \in [-1, 0],$$
$$T_1 = \varepsilon T_0, \tag{132}$$

$$\mathcal{O}(\varepsilon), \quad \frac{\partial^2 X_1}{\partial T_0^2} + X_1 - E_1X_1$$
$$= -2\frac{\partial^2 X_0}{\partial T_0\partial T_1} - E_1\frac{\Delta_\varepsilon}{\varepsilon}X_0 + X_0^2, \tag{133}$$

$$X_1(T_0, T_1) = 0, \quad T_0 \in [-1, 0], T_1 = \varepsilon T_0. \tag{134}$$

First, we will investigate the multiplicity of the characteristic roots of Eq. (131). Let us define the characteristic function h_4 as

$$h_4(\mu) \equiv \mu^2 + 1 - e^{-\mu}. \tag{135}$$

Suppose that there exists a characteristic root, $\hat{\mu} \in \chi(h_4)$, with multiplicity greater than one. Hence, it satisfies

$$\hat{\mu}^2 + 1 - e^{-\hat{\mu}} = 0, \quad \text{and} \quad 2\hat{\mu} + e^{-\hat{\mu}} = 0. \tag{136}$$

By adding those two equations in (136), we obtain that

$$\hat{\mu}^2 + 2\hat{\mu} + 1 = 0. \tag{137}$$

Equation (137) can only be satisfied when $\hat{\mu} = -1$. However, $-1 \notin \chi(h_4)$. This contradicts the assumption that $\hat{\mu}$ is a characteristic root with multiplicity greater than one. Therefore, we can conclude that there are no characteristic roots with a multiplicity greater than one. The general solution of the $\mathcal{O}(1)$ problem (131)–(132) can now be written as

$$X_0(T_0, T_1) = \sum_{\hat{\mu} \in \chi(h_4)} \beta_{\hat{\mu}}(T_1) e^{\hat{\mu} T_0}. \tag{138}$$

Of course, $\beta_{\hat{\mu}}(T_1)$ has to satisfy the initial condition given by (61)–(62). Besides that the arbitrary functions $\beta_{\hat{\mu}}(T_1)$ have to be chosen such that no resonant terms occur in the right-hand side of (133) (or equivalently, $\beta_{\hat{\mu}}(T_1)$ has to be chosen such that no secular terms occur in $X_1(T_0, T_1)$ for all $\hat{\mu} \in \chi(h_4)$). The right-hand side of (133) can be rewritten into

$$\sum_{\hat{\mu} \in \chi(h_4)} \left[-2\beta'_{\hat{\mu}}(T_1)\hat{\mu} - e^{-\hat{\mu}} \frac{\Delta_\varepsilon}{\varepsilon} \beta_{\hat{\mu}}(T_1) \right] e^{\hat{\mu} T_0} + \sum_{\hat{\mu}_1 \in \chi(h_4)} \sum_{\hat{\mu}_2 \in \chi(h_4)} \beta_{\hat{\mu}_1}(T_1) \beta_{\hat{\mu}_2}(T_1) e^{(\hat{\mu}_1 + \hat{\mu}_2) T_0}, \tag{139}$$

and so, we need to investigate whether $\hat{\mu}_1 + \hat{\mu}_2$ can be a root of the characteristic equation $h_4(\mu) = 0$, or equivalently

$$\hat{\mu}_1^2 + 1 - e^{-\hat{\mu}_1} = 0, \tag{140}$$

$$\hat{\mu}_2^2 + 1 - e^{-\hat{\mu}_2} = 0, \tag{141}$$

$$(\hat{\mu}_1 + \hat{\mu}_2)^2 + 1 - e^{-(\hat{\mu}_1 + \hat{\mu}_2)} = 0. \tag{142}$$

Substituting the Eqs. (140) and (141) into (142) yields

$$(\hat{\mu}_1 + \hat{\mu}_2)^2 + 1 = \left(\hat{\mu}_1^2 + 1\right)\left(\hat{\mu}_2^2 + 1\right). \tag{143}$$

By expanding both sides in (143), we obtain that $\hat{\mu}_1^2 + 2\hat{\mu}_1\hat{\mu}_2 + \hat{\mu}_2^2 + 1 = \hat{\mu}_1^2\hat{\mu}_2^2 + \hat{\mu}_1^2 + \hat{\mu}_2^2 + 1$, which implies that $\hat{\mu}_1^2\hat{\mu}_2^2 = 2\hat{\mu}_1\hat{\mu}_2$. And so, it follows that

$$\hat{\mu}_1 = 0, \text{ or } \hat{\mu}_2 = 0, \text{ or } \hat{\mu}_1\hat{\mu}_2 = 2. \tag{144}$$

It is clear that the sum of two characteristic roots is again a characteristic root if at least one of them is zero. So, we still have to investigate the case $\hat{\mu}_1\hat{\mu}_2 = 2$. First, note that the only real-valued characteristic root is zero. Hence, we need to check whether there is a possibility that the multiplication of two complex-valued characteristic roots is equal to 2. This can only occur if there exists a complex-valued characteristic root with a distance less than or equal to $\sqrt{2}$ to the origin in the complex-plane. Let us write $\hat{\mu} = v + iw$, with $v, w \in \mathbb{R}$. Then,

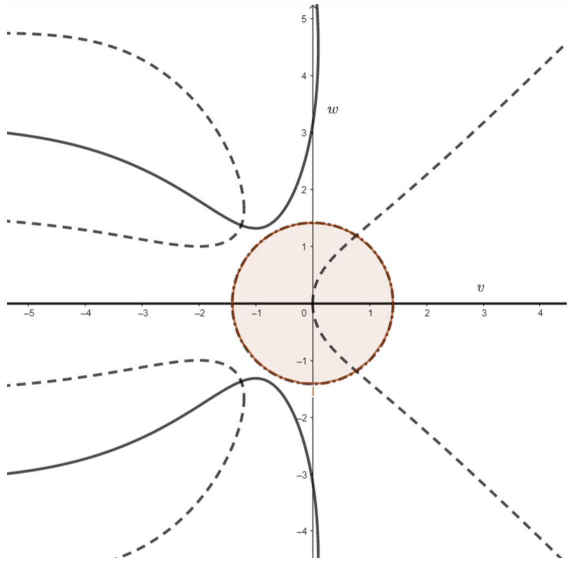

Fig. 4 The real part (Eq. (146)) is depicted by solid lines, the imaginary part (Eq. (147)) is given by dashed lines, and the inequality (Eq. (148)) is given by the shaded area

$$[v + iw]^2 + 1 - e^{-(v+iw)} = 0. \tag{145}$$

By separating the real and the imaginary parts in (145), we obtain that v and w should satisfy

$$v^2 - w^2 + 1 - e^{-v}\cos(w) = 0, \tag{146}$$

$$2vw + e^{-v}\sin(w) = 0, \tag{147}$$

$$v^2 + w^2 \leq 2. \tag{148}$$

In Fig. 4, the two equalities (146) and (147), and the inequality (148) are plotted in the (v, w)-plane.

As can be seen from Fig. 4, no intersection of the equality (146) lies inside the grey area of the inequality (148), and so, there are no characteristic roots which have modulus less than or equal to $\sqrt{2}$. Hence, the sum of the characteristic roots can only be a characteristic root if and only if at least one of those roots is equal to zero. The right-hand side of (133), or equivalently (139), can now be rewritten into

$$-\frac{\Delta_\varepsilon}{\varepsilon}\beta_0(T_1)+\sum_{\substack{\hat\mu\in\chi(h_4)\\ \hat\mu\neq 0}}\left[-2\beta'_{\hat\mu}(T_1)\hat\mu-e^{-\hat\mu}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat\mu}(T_1)\right]e^{\hat\mu T_0}$$
$$+\beta_0^2(T_1)+2\sum_{\substack{\hat\mu\in\chi(h_4)\\ \hat\mu\neq 0}}\beta_0(T_1)\beta_{\hat\mu}(T_1)e^{\hat\mu T_0}$$
$$+\sum_{\substack{\hat\mu_1\in\chi(h_4)\\ \hat\mu_1\neq 0}}\sum_{\substack{\hat\mu_2\in\chi(h_4)\\ \hat\mu_2\neq 0}}\beta_{\hat\mu_1}(T_1)\beta_{\hat\mu_2}(T_1)e^{(\hat\mu_1+\hat\mu_2)T_0}. \tag{149}$$

From (149), it follows that $\beta_{\hat\mu}(T_1)$ has to satisfy for $\hat\mu=0$

$$-\frac{\Delta_\varepsilon}{\varepsilon}\beta_0(T_1)+\beta_0^2(T_1)=0, \tag{150}$$

and for all $\hat\mu\in\chi(h_4)$ with $\hat\mu\neq 0$

$$-2\beta'_{\hat\mu}(T_1)\hat\mu-e^{-\hat\mu}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat\mu}(T_1)+2\beta_0(T_1)\beta_{\hat\mu}(T_1)=0 \tag{151}$$

in order to avoid secular terms in $X_1(T_0,T_1)$. Since (150) and (151) should be strict order 1 equations, it simply follows that (150) and (151) can be rewritten into

$$-\beta'_0(T_1)+\beta_0^2(T_1)=0,\text{ and} \tag{152}$$
$$-h'_4(\hat\mu)\beta'_{\hat\mu}(T_1)+2\beta_0(T_1)\beta_{\hat\mu}(T_1)=0, \tag{153}$$

respectively. The ODEs (152) and (153) can easily be solved, and by using the initial conditions (61)–(62) one finds

$$\beta_0(T_1)=\frac{-1}{T_1+C_0},\ \text{ and }\ \beta_{\hat\mu}(T_1)=C_{\hat\mu}\left[T_1+C_0\right]^{\frac{-2}{h'_4(\hat\mu)}}, \tag{154}$$

for $\hat\mu\in\chi(h_4)$ with $\hat\mu\neq 0$. Using the initial conditions of $\beta_{\hat\mu}$, we have $C_0=-\Phi_0^{-1}$ and $C_{\hat\mu}=\Phi_{\hat\mu}(-\Phi_0)^{-\frac{2}{h'_4(\hat\mu)}}$. And so, $X_0(T_0,T_1)$ has been completely determined and is given by

$$X_0(T_0,T_1)=\frac{-1}{T_1+C_0}+\sum_{\hat\mu,\hat\mu\neq 0}C_{\hat\mu}\left[T_1+C_0\right]^{\frac{-2}{h'_4(\hat\mu)}}e^{\hat\mu T_0}. \tag{155}$$

And so, by using the theorem from Sect. 4 it follows that $x(t)$ can be written as

$$x(t)=X_0(T_0,T_1)+\mathcal{O}(\varepsilon), \tag{156}$$

for $t=\mathcal{O}(\varepsilon^{-1})$.

5.3 A first order DDE with a weak cubic nonlinearity.

Example 7 Consider the following initial value problem for $x=x(t)$:

$$2\dot x(t)+\dot x(t-1)+2x(t)+x(t-1)=\varepsilon x^3(t),\ t>0, \tag{157}$$
$$x(t)=\phi(t),\ \text{for all}\quad t\in[-1,0], \tag{158}$$

where ϕ is a smooth function of order 1, and where ε is a small parameter with $0<\varepsilon\ll 1$. By substituting the expansion (25) for $x(t)$ into (157)–(158), and by taking together terms of order 1, and of order ε, we obtain the following $\mathcal{O}(1)-$, and $\mathcal{O}(\varepsilon)-$problems:

$$\mathcal{O}(1),\quad 2\frac{\partial X_0}{\partial T_0}+E_1\frac{\partial X_0}{\partial T_0}+2X_0+E_1X_0=0,$$
$$X_0(T_0,T_1)=\phi(T_0),\ T_0\in[-1,0],\ T_1=\varepsilon T_0, \tag{159}$$
$$\mathcal{O}(\varepsilon),\quad 2\frac{\partial X_1}{\partial T_0}+E_1\frac{\partial X_1}{\partial T_0}+2X_1+E_1X_1$$
$$=-2\frac{\partial X_0}{\partial T_1}-E_1\frac{\partial X_0}{\partial T_1}+E_1\frac{\Delta_\varepsilon}{\varepsilon}\frac{\partial X_0}{\partial T_0}$$
$$+E_1\frac{\Delta_\varepsilon}{\varepsilon}X_0+X_0^3,$$
$$X_1(T_0,T_1)=0,\ T_0\in[-1,0],\ T_1\in[-\varepsilon,0], \tag{160}$$

respectively. To determine the general solution of the $\mathcal{O}(1)$-problem (159), we first have to study the characteristic roots of the DDE in (159). The characteristic equation is given by

$$h_5(\mu)\equiv 2\mu+\mu e^{-\mu}+2+e^{-\mu}$$
$$=(\mu+1)(2+e^{-\mu})=0. \tag{161}$$

The roots $\mu=\hat\mu$ of this equation are given by

$$\hat\mu=-1,\quad\text{or}\quad \hat\mu=-\ln(2)+i(2n+1)\pi,$$

for all integers n. Now we will show that all roots have multiplicity one. If a root $\hat\mu$ has multiplicity two, then $\hat\mu$ should satisfy (161) and $h'_5(\hat\mu)=2-\hat\mu e^{-\hat\mu}=0$. For $\hat\mu=-1$, we obtain that $h'_5(-1)=2+e\neq 0$, and also for $\hat\mu=-\ln(2)+i\,(2n+1)\,\pi$, we obtain that $h'_5(\hat\mu)=2-\hat\mu e^{-\hat\mu}=2+2\left[-\ln(2)+i(2n+1)\pi\right]\neq 0,\ \forall n\in\mathbb{Z}$. Hence, we can conclude that all characteristic roots of Eq. (161) have multiplicity one. This implies that the general solution of Eq. (159) can be written as

$$X_0(T_0,T_1)=\sum_{\hat\mu\in\chi(h_5)}\beta_{\hat\mu}(T_1)e^{\hat\mu T_0}, \tag{162}$$

where the still arbitrary functions $\beta_{\hat\mu}(T_1)$ have to satisfy the initial conditions in (159), and have to be chosen in such a way that no secular terms occur in $X_1(T_0, T_1)$. By substituting (162) into (160), the right hand side of the equation for $X_1(T_0, T_1)$ becomes:

$$\sum_{\hat{\mu}\in\chi(h_5)} \left[-2\beta'_{\hat{\mu}}(T_1) + \hat{\mu}e^{-\hat{\mu}}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat{\mu}}(T_1)\right.$$
$$\left. -e^{-\hat{\mu}}\beta'_{\hat{\mu}}(T_1) + e^{-\hat{\mu}}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat{\mu}}(T_1)\right] e^{\hat{\mu}T_0}$$
$$+ \sum_{\hat{\mu}_1\in\chi(h_5)} \sum_{\hat{\mu}_2\in\chi(h_5)} \sum_{\hat{\mu}_3\in\chi(h_5)} \beta_{\hat{\mu}_1}(T_1)\beta_{\hat{\mu}_2}(T_1)$$
$$\beta_{\hat{\mu}_3}(T_1)e^{(\hat{\mu}_1+\hat{\mu}_2+\hat{\mu}_3)T_0}. \tag{163}$$

As we can see from (163) the sum of three characteristic roots will contribute to secular terms if it is again a characteristic root. Therefore, we will investigate the possibility whether the sum of three characteristic roots is again a characteristic root or not, that is, whether $h_5(\hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3) = 0$ or not. There are four different cases we have to consider to see whether $\hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3$ is a root. These cases are the following ones.

Case 1. $\hat{\mu}_1 = -1, \hat{\mu}_2 = -1, \hat{\mu}_3 = -1$. This implies $\hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3 = -3$. Hence,

$$h_5(\hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3) = h_5(-3)$$
$$= [-3 + 1]\left[2 + e^{-(-3)}\right] = -2(2 + e^3) \neq 0.$$

Case 2. $\hat{\mu}_1 = -1, \hat{\mu}_2 = -1$ and $\hat{\mu}_3 = -\ln(2) + i(2m + 1)\pi$. This implies $\hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3 = -2 - \ln(2) + i(2m + 1)\pi$. Hence,

$$h_5(\hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3)$$
$$= [-2 - \ln(2) + i(2m + 1)\pi + 1]$$
$$\left[2 + e^{2+\ln(2)-i(2m+1)\pi}\right]$$
$$= [-1 - \ln(2) + i(2m + 1)\pi]\left[2 - e^{2+\ln(2)}\right]$$
$$= [-1 - \ln(2) + i(2m + 1)\pi]\left[2 - 2e^2\right]$$
$$\neq 0.$$

Case 3. $\hat{\mu}_1 = -1, \hat{\mu}_2 = -\ln(2) + i(2l + 1)\pi$ and $\hat{\mu}_3 = -\ln(2) + i(2m + 1)\pi$. Hence,

$$h_5(\hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3)$$
$$= [-1 - 2\ln(2) + i2(l + m + 1)\pi + 1]$$
$$\left[2 + e^{1+2\ln(2)-i2(l+m+1)\pi}\right]$$
$$= [-2\ln(2) + i2(l+m+1)\pi]\left[2-e^{1+2\ln(2)}\right]$$
$$= [-2\ln(2) + i2(l + m + 1)\pi]\,[2 - 4e]$$
$$\neq 0.$$

Case 4. $\hat{\mu}_1 = -\ln(2) + i(2k + 1)\pi,\ \hat{\mu}_2 = -\ln(2) + i(2l+1)\pi$ and $\hat{\mu}_3 = -\ln(2)+i(2m+1)$. Note that each characteristic root here is satisfying $e^{-\hat{\mu}} = -2$. Hence

$$h_5(\hat{\mu}_1 + \hat{\mu}_2 + \hat{\mu}_3)$$
$$= [-3\ln(2) + i(2(k + l + m) + 3)\pi + 1]$$
$$[2 + (-2)(-2)(-2)]$$
$$= -6\,[1 - 3\ln(2) + i(2(k + l + m) + 3)\pi]$$
$$\neq 0.$$

So, in all cases in (163) the sum of three characteristic roots can not be a characteristic root. Hence, the corresponding condition to remove resonant terms in (163) is:

$$-2\beta'_{\hat{\mu}}(T_1) + \hat{\mu}e^{-\hat{\mu}}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat{\mu}}(T_1) - e^{-\hat{\mu}}\beta'_{\hat{\mu}}(T_1)$$
$$+e^{-\hat{\mu}}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat{\mu}}(T_1) = 0, \tag{164}$$

and the corresponding equation for $X_1(T_0, T_1)$ becomes

$$2\frac{\partial X_1}{\partial T_0} + E_1\frac{\partial X_1}{\partial T_0} + 2X_1 + E_1X_1 = X_0^3. \tag{165}$$

Equation (164) can be reduced (as in the previous examples) to: $\beta'_{\hat{\mu}}(T_1) = 0$ for $T_1 > 0$. By using the initial condition (159) it also follows for $T_1 \in [-\varepsilon, 0]$ that

$$\beta_{\hat{\mu}}(T_1) = \frac{N(\hat{\mu})}{h'_5(\hat{\mu})}, \tag{166}$$

with $N(s) = 2\phi(0) + \phi(-1) - (s + 1)e^{-s}\int_{-1}^{0}\phi(T_0)$ $e^{-sT_0}\,dT_0$. Since $\beta'_{\hat{\mu}}(T_1) = 0$ for $T_1 > 0$ it then follows that (166) also holds for $T_1 > 0$, and so

$$X_0(T_0, T_1) = \frac{N(-1)}{h'_5(-1)}e^{-T_0} + \sum_{\substack{\hat{\mu}\in\chi(h_5)\\ \hat{\mu}\neq-1}} \frac{N(\hat{\mu})}{h'_5(\hat{\mu})}e^{\hat{\mu}T_0}. \tag{167}$$

Again it follows from Sect. 4 that $x(t) = X_0(T_0, T_1) + \mathcal{O}(\varepsilon)$ for $t = \mathcal{O}(\varepsilon^{-1})$. The following example looks similar to this Example 7, but it will turn out that the cubic nonlinearity gives rise to complicated, resonant terms.

5.4 A second order DDE with a weak cubic nonlinearity

Example 8 Consider the following initial value problem for $x = x(t)$:

$$2\ddot{x}(t) + \ddot{x}(t - 1) + 2x(t) + x(t - 1) = \varepsilon x^3(t), \tag{168}$$
$$x(t) = \phi(t), \quad \text{for all } t \in [-1, 0], \tag{169}$$

where $\phi(t)$ is a smooth function, and where ε is a small parameter with $0 < \varepsilon \ll 1$.

By substituting the expansion (25) for $x(t)$ into (168), and by collecting terms of order 1, and of order ε, we obtain the following $\mathcal{O}(1)$-, and $\mathcal{O}(\varepsilon)$- problems:

$$\mathcal{O}(1),\quad 2\frac{\partial^2 X_0}{\partial T_0^2} + E_1\frac{\partial^2 X_0}{\partial T_0^2} + 2X_0 + E_1 X_0 = 0,$$
$$X_0(T_0, T_1) = \phi(T_0),\ T_0 \in [-1, 0], T_1 = \varepsilon T_0, \quad (170)$$
$$\mathcal{O}(\varepsilon),\quad 2\frac{\partial^2 X_1}{\partial T_0^2} + E_1\frac{\partial^2 X_1}{\partial T_0^2} + 2X_1 + E_1 X_1$$
$$= -4\frac{\partial^2 X_0}{\partial T_1 \partial T_0} - 2E_1\frac{\partial^2 X_0}{\partial T_1 \partial T_0} + E_1\frac{\Delta_\varepsilon}{\varepsilon}\frac{\partial^2 X_0}{\partial T_0^2} + E_1\frac{\Delta_\varepsilon}{\varepsilon}X_0 + X_0^3,$$
$$X_1(T_0, T_1) = 0, \text{ for } T_0 \in [-1, 0], \text{ and } T_1 = \varepsilon T_0, \quad (171)$$

respectively. The characteristic equation belonging to the DDE in (170) is given by

$$h_6(\mu) \equiv 2\mu^2 + \mu^2 e^{-\mu} + 2 + e^{-\mu} = (\mu^2 + 1)(e^{-\mu} + 2), \quad (172)$$

and its roots are given by

$$\hat{\mu} = \pm i, \quad \text{or} \quad \hat{\mu} = -\ln(2) + i(2n+1)\pi, \forall n \in \mathbb{Z}. \quad (173)$$

Since $h_6'(\mu) \neq 0$ for all $\mu \in \chi(h_6)$, we can conclude that all characteristic roots have multiplicity one. Therefore, the solution of the $\mathcal{O}(1)$- problem (170) can be written as

$$X_0(T_0, T_1) = \sum_{\hat{\mu} \in \chi(h_6)} \beta_{\hat{\mu}}(T_1) e^{\hat{\mu} T_0}, \quad (174)$$

where for $T_1 \in [-\varepsilon, 0]$, $\beta_{\hat{\mu}}(T_1) = \frac{N(\hat{\mu})}{h_6'(\hat{\mu})} = \Phi_{\hat{\mu}}$ with $N(s) = 2s\phi(0) + 2\dot{\phi}(0) + s\phi(-1) + \dot{\phi}(-1) - (s^2 + 1)\int_{-1}^{0} \phi(T_0) e^{-s(T_0+1)} dT_0$. For $T_1 > 0$, $\beta_{\hat{\mu}}(T_1)$ is still undetermined and can be used to avoid resonant terms in the right-hand side of the DDE in (171). The right-hand side is given by

$$\sum_{\hat{\mu} \in \chi(h_6)} \left[-4\hat{\mu}\beta_{\hat{\mu}}'(T_1) - 2\hat{\mu}\beta_{\hat{\mu}}'(T_1)e^{-\hat{\mu}} + \hat{\mu}^2 e^{-\hat{\mu}}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat{\mu}}(T_1) + e^{-\hat{\mu}}\frac{\Delta_\varepsilon}{\varepsilon}\beta_{\hat{\mu}}(T_1)\right] e^{\hat{\mu} T_0}$$
$$+ \sum_{\mu_1 \in \chi(h_6)} \sum_{\mu_2 \in \chi(h_6)} \sum_{\mu_3 \in \chi(h_6)} \beta_{\mu_1}(T_1)\beta_{\mu_2}(T_1)\beta_{\mu_3}(T_1) e^{(\mu_1+\mu_2+\mu_3)T_0}, \quad (175)$$

and can be simplified (similarly as has been done in the previous examples) to:

$$\sum_{\hat{\mu} \in \chi(h_6)} -h_6'(\hat{\mu})\beta_{\hat{\mu}}'(T_1) e^{\hat{\mu} T_0} + \sum_{\mu_1 \in \chi(h_6)} \sum_{\mu_2 \in \chi(h_6)} \sum_{\mu_3 \in \chi(h_6)} \beta_{\mu_1}(T_1)\beta_{\mu_2}(T_1)\beta_{\mu_3}(T_1) e^{(\mu_1+\mu_2+\mu_3)T_0}. \quad (176)$$

From (176) it is obvious that the cubic nonlinearity in (168) can lead to resonant terms in the right-hand side of (172) when a sum of three characteristic roots μ_1, μ_2 and μ_3 is again a characteristic root, that is, when $h_6(\mu_1 + \mu_2 + \mu_3) = 0$, or equivalently, when a sum of three characteristic roots is equal to i, or $-i$, or $-\ln(2) + i(2n+1)\pi$. So, we have to consider the following three cases.

Case 1. $\mu_1 + \mu_2 + \mu_3 = i$.
This case only holds for $(\mu_1, \mu_2, \mu_3) = (i, i, -i)$, $(\mu_1, \mu_2, \mu_3) = (i, -i, i)$, or $(\mu_1, \mu_2, \mu_3) = (-i, i, i)$. Hence, the coefficient of e^{iT_0} should satisfy the condition (177)

$$-h_6'(i)\beta_i'(T_1) + 3\beta_i^2(T_1)\beta_{-i}(T_1) = 0. \quad (177)$$

Case 2. $\mu_1 + \mu_2 + \mu_3 = -i$.
This case only holds for $(\mu_1, \mu_2, \mu_3) = (i, -i, -i)$, $(\mu_1, \mu_2, \mu_3) = (-i, i, -i)$, or $(\mu_1, \mu_2, \mu_3) = (-i, -i, i)$. Hence, the coefficient of e^{-iT_0} should satisfy the condition (178)

$$-h_6'(-i)\beta_{-i}'(T_1) + 3\beta_{-i}^2(T_1)\beta_i(T_1) = 0. \quad (178)$$

Case 3. $\mu_1 + \mu_2 + \mu_3 = -\ln(2) + i(2n+1)\pi$.
This case only holds for the six permutations of $(i, -i, -\ln(2) + i(2n+1)\pi)$. To simplify, let us define $\hat{\mu}_n = -\ln(2) + i(2n+1)\pi$. Hence the coefficient of $e^{\hat{\mu}_n T_0}$ should satisfy the condition

$$-h_6'(\hat{\mu}_n)\beta_{\hat{\mu}_n}'(T_1) + 6\beta_i(T_1)\beta_{-i}(T_1)\beta_{\hat{\mu}_n}(T_1) = 0. \quad \forall n \in \mathbb{Z}. \quad (179)$$

So, to avoid resonant terms in the right-hand side of (176) it follows that the functions $\beta_{\hat{\mu}}(T_1)$ in (174) have to satisfy (177), (178), and (179). The Eqs. (177) and (178) for $\beta_i(T_1)$ and $\beta_{-i}(T_1)$ can readily be solved, and by using these solutions the Eq. (179) for $\beta_{\hat{\mu}_n}$ can be solved. Finally, by using the initial conditions for the function $\beta_{\hat{\mu}}$, one obtains

$$\beta_i(T_1) = \Phi_i e^{3\Phi_i \Phi_{-i} T_1/h_6'(i)}, \tag{180}$$

$$\beta_{-i}(T_1) = \Phi_{-i} e^{3\Phi_i \Phi_{-i} T_1/h_6'(-i)}, \tag{181}$$

$$\beta_{\hat{\mu}_n}(T_1) = \Phi_{\hat{\mu}_n} e^{6\Phi_i \Phi_{-i} T_1/h_6'(\hat{\mu}_n)}, \tag{182}$$

for all $\hat{\mu}_n \in \chi(h_6) \setminus \{i, -i\}$. As in the previous examples, one simply obtains that

$$x(t) = \sum_{\hat{\mu} \in \chi(h_6)} \beta_{\hat{\mu}}(T_1) e^{\hat{\mu} T_0} + \mathcal{O}(\varepsilon), \tag{183}$$

for $t = \mathcal{O}(\varepsilon^{-1})$.

6 Comparisons of analytically obtained approximations and numerically obtained approximations

In this section, the results obtained from the proposed study, which has at least two purely imaginary characteristic roots, are compared. The first example is Example 8, see Eq. (168). For $\phi(t) = 1$, the corresponding analytical approximation is given in Eq. (183), where the coefficients are given by (180), (181), and (182) with

$$\Phi_{\hat{\mu}} = \frac{N(\hat{\mu})}{h_6'(\hat{\mu})}, \; N(\hat{\mu}) = 3\hat{\mu} - (\hat{\mu}^2 + 1) \cdot \frac{1 - e^{-\hat{\mu}}}{\hat{\mu}}, \quad \text{and}$$

$$h_6'(\hat{\mu}) = 4\hat{\mu} - (\hat{\mu} - 1)^2 e^{-\hat{\mu}}, \tag{184}$$

for all $\hat{\mu} \in \chi(h_6)$. To show how accurate the analytical approximation is, we will compare the analytical approximation with a numerical approximation which is obtained by numerically integrating (168)–(169) with a Runge-Kutta 45 method. The results can be seen in Fig. 5. The first one is the approximate solution if we only consider the purely imaginary characteristic roots, see (173). Next, for the proposed method, we consider two purely imaginary roots and six pair of complex conjugates (14 roots in total). Both will be compared with the numerical approximation. As we can see from Fig. 5, both approximate analytical solutions describe the behaviour of the numerical solution very well. Since all the real parts of $\hat{\mu}_n$ are negative (except for the two purely imaginary roots) we obtain that the amplitudes of the almost purely oscillatory part of the solution will slowly increase in time (see (180), (181), and (184)). This slow increase in amplitude can also be clearly seen in Fig. 5. For a constant initial

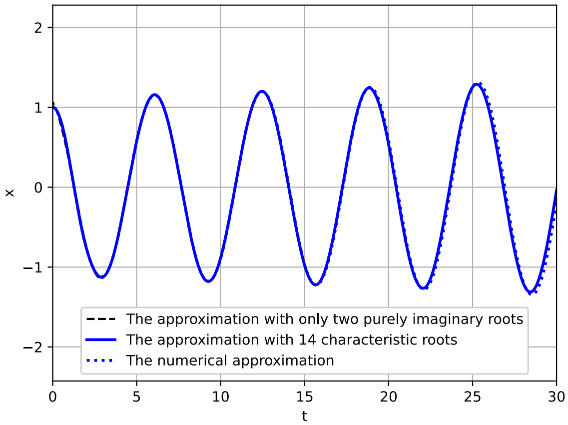

Fig. 5 Plot of the approximate solutions for Example 8, where $\phi(t) = 1$ and $\varepsilon = 0.1$

value $\phi(t) \equiv 1$, the analytical results and the numerical results agree very well. However, when $\phi(t)$ is not constant, a simple truncation to only the first two purely imaginary roots will give rather inaccurate results for small times as can be seen in Fig. 6. Taking into account the first 14 roots gives a much better result on the whole time interval. In Fig. 6, we took $\phi(t) = \cos(2\pi t)$, so that,

$$N(\hat{\mu}) = 3\hat{\mu} - (\hat{\mu}^2 + 1)\hat{\mu} \left[\frac{1 - e^{-\hat{\mu}}}{\hat{\mu}^2 + 4\pi^2} \right], \tag{185}$$

for all $\hat{\mu} \in \chi(h_6)$. Hence, the graphical representation of these results can be seen in Fig. 6. Here, we can see that the existing method has different behaviour in the beginning but it starts to coincide for $t > 4$.

Next, let us consider another example with at least two purely imaginary roots. Here, we consider a second-order delay Mathieu Equation, see also [16,17, 20].

Example 9 Consider the following second-order Mathieu Equation.

$$\ddot{x}(t) + 2\pi^2 x(t) + \pi^2 x(t-1)$$

$$= 2\varepsilon \cos(2\pi t) x(t), \; t > 0, \tag{186}$$

$$x(t) = \sin(2\pi t), \quad \text{for all } t \in [-1, 0]. \tag{187}$$

Using the perturbation procedure as described in this paper, we obtain at the $\mathcal{O}(1)$-level as characteristic function

$$h_7(\mu) \equiv \mu^2 + 2\pi^2 + \pi^2 e^{-\mu}. \tag{188}$$

Once again, the roots of the characteristic function (188) are obtained by computing the intersection points

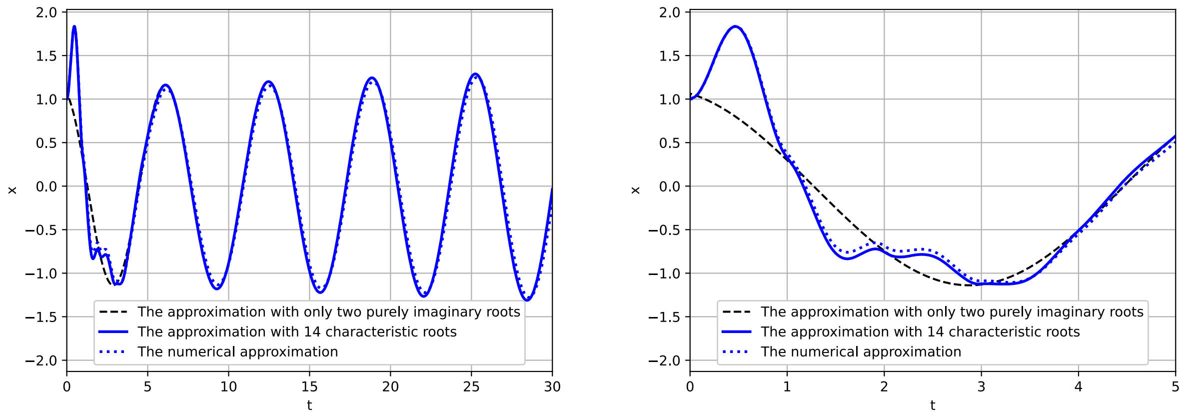

Fig. 6 Plot of the approximate solutions for Example 8, where $\phi(t) = \cos(2\pi t)$ and $\varepsilon = 0.1$

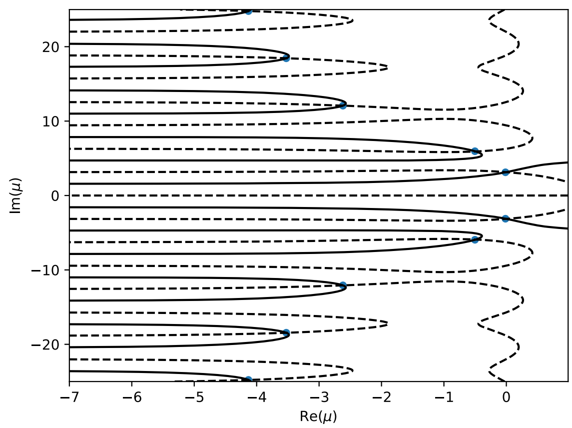

Fig. 7 Plot of some of the roots of h_7

of the real and imaginary curves as given in Fig. 7. Obviously, we have two purely imaginary roots $\pm i\pi$

Suppose that $\hat{\mu}$ is a characteristic root for the function h_7, $\hat{\mu} \in \chi(h_7)$. The solution of the $\mathcal{O}(1)$ equation can be written as

$$X_0(T_0, T_1) = \sum_{\hat{\mu} \in \chi(h_7)} \beta_{\hat{\mu}}(T_1) e^{\hat{\mu} T_0}, \tag{189}$$

where the initial conditions for $\beta_{\hat{\mu}}(T_1)$ are

$$\beta_{\hat{\mu}}(T_1) = \Phi_{\hat{\mu}}, \quad \text{for } T_1 \in [-\varepsilon, 0], \tag{190}$$

with

$$\begin{aligned} \Phi_\mu &= \frac{1}{2\mu - \pi^2 e^{-\mu}} \left(2\pi - \pi^2 e^{-\mu} \int_{-1}^{0} e^{-\mu\theta} \sin(2\pi\theta)\, d\theta \right), \\ &= \frac{1}{2\mu - \pi^2 e^{-\mu}} \left(2\pi - 2\pi^3 \cdot \frac{1 - e^{-\mu}}{\mu^2 + 4\pi^2} \right). \end{aligned} \tag{191}$$

To determine the secular terms in the $\mathcal{O}(\varepsilon)$-problem, we have to study the right-hand side of the $\mathcal{O}(\varepsilon)$ equation. The right-hand side is given by:

$$\sum_{\hat{\mu} \in \chi(h_7)} \left[\left(-2\hat{\mu} \frac{d\beta_{\hat{\mu}}(T_1)}{dT_1} + \frac{\pi^2}{\varepsilon} e^{-\hat{\mu}} \Delta_\varepsilon \beta_{\hat{\mu}}(T_1) \right) e^{\hat{\mu} T_0} + 2\beta_{\hat{\mu}}(T_1) e^{\hat{\mu} T_0} \cos(2\pi T_0) \right]. \tag{192}$$

Since $2\cos(2\pi T_0) = e^{i2\pi T_0} + e^{-i2\pi T_0}$ and $\pm i\pi$ are two of the characteristic roots, $\hat{\mu} \pm i2\pi$ will contribute to the secular terms for $\hat{\mu} = -i\pi$ and $\hat{\mu} = i\pi$. Hence, the conditions to avoid secular terms are

$$\begin{aligned} \frac{d\beta_{i\pi}(T_1)}{dT_1} &= -\frac{\pi}{i2\varepsilon} \Delta_\varepsilon \beta_{i\pi}(T_1) + \frac{1}{i2\pi} \beta_{-i\pi}(T_1), \\ \frac{d\beta_{-i\pi}(T_1)}{dT_1} &= \frac{\pi}{i2\varepsilon} \Delta_\varepsilon \beta_{-i\pi}(T_1) + \frac{1}{-i2\pi} \beta_{i\pi}(T_1). \\ \frac{d\beta_{\hat{\mu}}(T_1)}{dT_1} &= 0, \quad \text{for } \hat{\mu} \neq \pm i\pi. \end{aligned} \tag{193}$$

Solving these Eq. (193) and using the given initial conditions (190), we obtain that

$$\begin{aligned} \beta_{i\pi}(T_1) &= c_1 e^{mT_1} + c_2 e^{-mT_1}, \\ \beta_{-i\pi}(T_1) &= \frac{-m^{-1}}{\pi(2i + \pi)} \left[c_1 e^{mT_1} - c_2 e^{-mT_1} \right], \\ \beta_{\hat{\mu}}(T_1) &= \Phi_{\hat{\mu}}, \quad \text{for } \hat{\mu} \neq \pm i\pi. \end{aligned} \tag{194}$$

where $m = \left[\pi\sqrt{4+\pi^2}\right]^{-1}$, $c_1 = \frac{1}{2}\left[\Phi_{i\pi} + \frac{\pi - 2i}{\sqrt{4+\pi^2}} \Phi_{-i\pi}\right]$, $c_2 = \frac{1}{2}\left[\Phi_{i\pi} - \frac{\pi - 2i}{\sqrt{4+\pi^2}} \Phi_{-i\pi}\right]$.

The approximation based on only the two purely imaginary roots, the approximation based on the first 10 characteristic roots (including the two purely imaginary ones), see also Fig. 7 for the roots, and the numerical approximation of the solution are given in Fig. 8. Again it can be seen that for short times just after $t = 0$, the

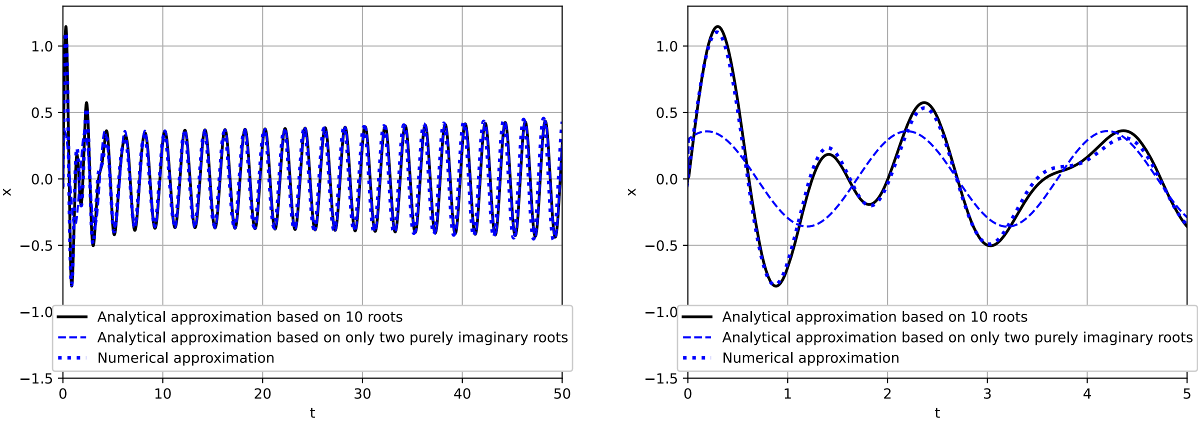

Fig. 8 Plot of the approximations for Example 9 with $\varepsilon = 0.1$

approximation based on only the two purely imaginary roots is rather inaccurate. for larger times, all approximations coincide.

7 Conclusion and remarks

In this paper, it has been shown how the multiple time-scales perturbation method can be applied to differential-delay equations. Approximations of the solutions can be constructed which are accurate on long timescales of order ε^{-1}. By using differential and difference operators, it is shown how approximations can be obtained which branch off from solutions of differential-delay equations at the unperturbed level. In the classical approach in the literature, only approximations of solutions which branch off from solutions of ordinary differential equations were considered. In this classical approach, only a finite number (usually only two) of the infinitely many roots of the characteristic equation of the DDE are considered. Moreover, in the classical approach, the approximations are not satisfying the initial conditions which are given on a time-interval determined by the delay. By using the multiple-time scales perturbation method for DDEs which is based on differential and difference operators (as presented in this paper), one can take into account the infinitely many roots of the characteristic equation of the DDE, and one can satisfy the initial condition which are given on the time-interval determined by the delay. In this paper, some simple and some more complicated examples are treated in detail to show how the method can be applied and to indicate what kind of underlying problems related to internal resonances might occur. As far as we can conclude now, the presented method can be applied to a large class of problems for DDEs, and for future research, one can try to extend the presented approach to problems for partial differential equations with delays.

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Data availability We declare that there is no data used in the study.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest and no data in this manuscript.

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