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# APPROXIMATIONS OF STOCHASTIC HYBRID SYSTEMS: A COMPOSITIONAL APPROACH

MAJID ZAMANI<sup>1</sup>, MATTHIAS RUNGGER<sup>1</sup>, AND PEYMAN MOHAJERIN ESFAHANI<sup>2</sup>

ABSTRACT. In this paper we propose a compositional framework for the construction of approximations of the interconnection of a class of stochastic hybrid systems. As special cases, this class of systems includes both jump linear stochastic systems and linear stochastic hybrid automata. In the proposed framework, an approximation is itself a stochastic hybrid system, which can be used as a replacement of the original stochastic hybrid system in a controller design process. We employ a notion of so-called stochastic simulation function to quantify the error between the approximation and the original system. In the first part of the paper, we derive sufficient conditions which facilitate the compositional quantification of the error between the interconnection of stochastic hybrid subsystems and that of their approximations using the quantified error between the stochastic hybrid subsystems and their corresponding approximations. In particular, we show how to construct stochastic simulation functions for approximations of interconnected stochastic hybrid systems using the stochastic simulation function for the approximation of each component. In the second part of the paper, we focus on a specific class of stochastic hybrid systems, namely, jump linear stochastic systems, and propose a constructive scheme to determine approximations together with their stochastic simulation functions for this class of systems. Finally, we illustrate the effectiveness of the proposed results by constructing an approximation of the interconnection of four jump linear stochastic subsystems in a compositional way.

## 1. INTRODUCTION

Stochastic hybrid systems are a general class of dynamical systems consisting of continuous and discrete dynamics subject to probabilistic noise and events. In the past few years, this class of systems has become ubiquitous in many different fields due to the need for a rigorous modeling framework for many safety-critical applications. Examples of those applications include air traffic control [GL04], biochemistry [SH10], communication networks [Hes04], and systems biology [HWS04]. The design of controllers to enforce certain given complex specifications, e.g. those expressed via formulae in linear temporal logic (LTL) [BK08], in a reliable and cost effective way is a grand challenge in the study of many of those safety-critical applications. One promising direction to achieve those objectives is the use of simpler (in)finite approximations of the given systems as a replacement in the controller design process. Those approximations allow us to design controllers for them and then refine the controllers to the ones for the concrete complex systems, while provide us with the quantified errors in this detour controller synthesis scheme.

In the past few years there have been several results on the (in)finite approximations of *continuous-time* stochastic (hybrid) systems. Existing results include the construction of finite approximations for stochastic dynamical systems under contractivity assumptions [Aba09], restricted to models with no control inputs, a finite Markov decision process approximation of a linear stochastic control system [LAB09], however without a quantitative relationship between approximation and concrete model, and the construction of finite bisimilar abstractions for stochastic control systems [ZMM<sup>+</sup>14, ZTA14], for stochastic switched systems [ZAG15], for randomly switched stochastic systems [ZA14], and the construction of sound finite abstractions for stochastic control systems without any stability property [ZEAL13]. Further, the results in [JP09] check the relationship between infinite approximations and a given class of stochastic hybrid systems via a notion of stochastic (bi)simulation functions. However, the results in [JP09] do not provide any approximations and moreover appear to be computationally intractable in the case of systems with inputs because one requires to solve a game in order to quantify the approximation error. Note that all the proposed results in [Aba09, LAB09,

ZMM<sup>+</sup>14, ZTA14, ZAG15, ZA14, ZEAL13, JP09] take a monolithic view of continuous-time stochastic (hybrid) systems, where the entire system is approximated. This monolithic view interacts badly with the construction of approximations, whose complexity grows (possibly exponentially) in the number of continuous state variables in the model.

In this paper, we provide a compositional framework for the construction of *infinite* approximations of the interconnection of a class of stochastic hybrid systems, in which the continuous dynamics are modeled by stochastic differential equations and the switches are modeled as Poisson processes. As special cases, this class of systems includes both jump linear stochastic systems (JLSS) and linear stochastic hybrid automata [JP09]. Our approximation framework is based on a new notion of stochastic simulation functions. In this framework, an approximation, which is itself a stochastic hybrid system (potentially with lower dimension and simpler interconnection topology), acts as a substitute in the controller design process. The stochastic simulation function is used to quantify the error in this detour controller synthesis scheme. Although an approximation in our framework might not be directly amenable to algorithmic synthesis methods based on automata-theoretic concepts [MPS95] which require finite approximations, our approach facilitates the construction of potentially lower-dimensional less-interconnected stochastic hybrid systems as approximations and, hence, can be interpreted as the first pre-processing step in the construction of a finite approximation.

In the first part of the paper, we derive sufficient small-gain type conditions, similar to the ones in [DIW11], under which one can quantify the error between the interconnection of stochastic hybrid subsystems and that of their approximations in a compositional way by using the errors between stochastic hybrid subsystems and their approximations. In the second part of the paper, we focus on JLSS and propose a computational scheme to construct infinite approximations of this class of systems, together with the corresponding stochastic simulation functions. To show the effectiveness of the proposed results, we construct an approximation (two disjoint 3 dimensional JLSS) of the interconnection of four JLSS (overall 10 dimensions) in a compositional way and then use the approximation in order to design a safety controller for the original interconnected system. Note that the controller synthesis would not have been possible without the use of the approximation.

The recent work in [RZ15] provides a compositional scheme for the construction of infinite approximations of interconnected deterministic control systems without any hybrid dynamic. The results in this paper are complementary to the ones in [RZ15] as we extend our focus to the class of stochastic hybrid systems. A preliminary investigation of our results on the compositional construction of infinite approximations of interconnected stochastic hybrid systems appeared in [Zam14]. In this paper we present a detailed and mature description of the results announced in [Zam14], including proposing a new notion of stochastic simulation functions which is computationally more tractable in the case of systems with inputs and providing constructive means to compute approximations of JLSS.

## 2. STOCHASTIC HYBRID SYSTEMS

**2.1. Notation.** We denote by  $\mathbb{N}$  the set of nonnegative integer numbers and by  $\mathbb{R}$  the set of real numbers. We annotate those symbols with subscripts to restrict them in the obvious way, e.g.  $\mathbb{R}_{>0}$  denotes the positive real numbers. The symbols  $I_n$ ,  $0_n$ , and  $0_{n \times m}$  denote the identity matrix, zero vector, and zero matrix in  $\mathbb{R}^{n \times n}$ ,  $\mathbb{R}^n$ , and  $\mathbb{R}^{n \times m}$ , respectively. For  $a, b \in \mathbb{R}$  with  $a \leq b$ , we denote the closed, open, and half-open intervals in  $\mathbb{R}$  by  $[a, b]$ ,  $]a, b[$ ,  $[a, b[$ , and  $]a, b]$ , respectively. For  $a, b \in \mathbb{N}$  and  $a \leq b$ , we use  $[a; b]$ ,  $]a; b[$ ,  $[a; b[$ , and  $]a; b]$  to denote the corresponding intervals in  $\mathbb{N}$ . Given  $N \in \mathbb{N}_{\geq 1}$ , vectors  $x_i \in \mathbb{R}^{n_i}$ ,  $n_i \in \mathbb{N}_{\geq 1}$  and  $i \in [1; N]$ , we use  $x = [x_1; \dots; x_N]$  to denote the vector in  $\mathbb{R}^n$  with  $n = \sum_{i=1}^N n_i$ . Similarly, we use  $X = [X_1; \dots; X_N]$  to denote the matrix in  $\mathbb{R}^{n \times m}$  with  $n = \sum_{i=1}^N n_i$ , given  $N \in \mathbb{N}_{\geq 1}$ , matrices  $X_i \in \mathbb{R}^{n_i \times m}$ ,  $n_i \in \mathbb{N}_{\geq 1}$ , and  $i \in [1; N]$ . Given a vector  $x \in \mathbb{R}^n$ , we denote by  $\|x\|$  the Euclidean norm of  $x$ . The distance of a point  $x \in \mathbb{R}^n$  to a set  $D \subseteq \mathbb{R}^n$  is defined as  $\|x\|_D = \inf_{d \in D} \|x - d\|$ . Given a matrix  $P = \{p_{ij}\} \in \mathbb{R}^{n \times n}$ , we denote by  $\text{Tr}(P) = \sum_{i=1}^n p_{ii}$  the trace of  $P$ .

Given a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$  and  $\bar{x} \in \mathbb{R}^m$ , we use  $f \equiv \bar{x}$  to denote that  $f(x) = \bar{x}$  for all  $x \in \mathbb{R}^n$ . If  $x$  is the zero vector, we simply write  $f \equiv 0$ . Given a function  $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$ , the (essential) supremum of  $f$  is denoted

by  $\|f\|_\infty := (\text{ess})\sup\{\|f(t)\|, t \geq 0\}$ . Measurability throughout this paper refers to Borel measurability. A continuous function  $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ , is said to belong to class  $\mathcal{K}$  if it is strictly increasing and  $\gamma(0) = 0$ ;  $\gamma$  is said to belong to class  $\mathcal{K}_\infty$  if  $\gamma \in \mathcal{K}$  and  $\gamma(r) \rightarrow \infty$  as  $r \rightarrow \infty$ . A continuous function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is said to belong to class  $\mathcal{KL}$  if, for each fixed  $t$ , the map  $\beta(r, t)$  belongs to class  $\mathcal{K}$  with respect to  $r$  and, for each fixed nonzero  $r$ , the map  $\beta(r, t)$  is decreasing with respect to  $t$  and  $\beta(r, t) \rightarrow 0$  as  $t \rightarrow \infty$ .

**2.2. Stochastic hybrid systems.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space endowed with a filtration  $\mathbb{F} = (\mathcal{F}_s)_{s \geq 0}$  satisfying the usual conditions of completeness and right continuity [KS91, p. 48]. Let  $(W_s)_{s \geq 0}$  be a  $\tilde{p}$ -dimensional  $\mathbb{F}$ -Brownian motion and  $(P_s)_{s \geq 0}$  be a  $\tilde{q}$ -dimensional  $\mathbb{F}$ -Poisson process. We assume that the Poisson process and the Brownian motion are independent of each other. The Poisson process  $P_s := [P_s^1; \dots; P_s^{\tilde{q}}]$  model  $\tilde{q}$  kinds of events whose occurrences are assumed to be independent of each other.

**Definition 2.1.** *The class of stochastic hybrid systems with which we deal in this paper is the tuple  $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, r, \mathbb{R}^q, h)$ , where  $\mathbb{R}^n$ ,  $\mathbb{R}^m$ ,  $\mathbb{R}^p$ , and  $\mathbb{R}^q$  are the state, external input, internal input, and output spaces, respectively, and*

- $\mathcal{U}$  is a subset of the set of all  $\mathbb{F}$ -progressively measurable processes with values in  $\mathbb{R}^m$ ; see [KS91, Def. 1.11];
- $\mathcal{W}$  is a subset of the set of all  $\mathbb{F}$ -progressively measurable processes with values in  $\mathbb{R}^p$ ;
- $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n$  is the drift term which is globally Lipschitz continuous: there exist constants  $L_x, L_u, L_w \in \mathbb{R}_{\geq 0}$  such that:  $\|f(x, u, w) - f(x', u', w')\| \leq L_x \|x - x'\| + L_u \|u - u'\| + L_w \|w - w'\|$  for all  $x, x' \in \mathbb{R}^n$ , all  $u, u' \in \mathbb{R}^m$ , and all  $w, w' \in \mathbb{R}^p$ ;
- $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times \tilde{p}}$  is the diffusion term which is globally Lipschitz continuous;
- $r : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times \tilde{q}}$  is the reset function which is globally Lipschitz continuous;
- $h : \mathbb{R}^n \rightarrow \mathbb{R}^q$  is the output map.

A stochastic hybrid system  $\Sigma$  satisfies

$$\Sigma : \begin{cases} d\xi(t) = f(\xi(t), \nu(t), \omega(t)) dt + \sigma(\xi(t)) dW_t + r(\xi(t)) dP_t, \\ \zeta(t) = h(\xi(t)), \end{cases} \quad (2.1)$$

$\mathbb{P}$ -almost surely ( $\mathbb{P}$ -a.s.) for any  $\nu \in \mathcal{U}$  and any  $\omega \in \mathcal{W}$ , where stochastic process  $\xi : \Omega \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^n$  is called a *solution process* of  $\Sigma$  and stochastic process  $\zeta : \Omega \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^q$  is called an *output trajectory* of  $\Sigma$ . We call the tuple  $(\xi, \zeta, \nu, \omega)$  a *trajectory* of  $\Sigma$ , consisting of a solution process  $\xi$ , an output trajectory  $\zeta$ , and input trajectories  $\nu$  and  $\omega$ , that satisfies (2.1)  $\mathbb{P}$ -a.s.. We also write  $\xi_{a\nu\omega}(t)$  to denote the value of the solution process at time  $t \in \mathbb{R}_{\geq 0}$  under the input trajectories  $\nu$  and  $\omega$  from initial condition  $\xi_{a\nu\omega}(0) = a$   $\mathbb{P}$ -a.s., in which  $a$  is a random variable that is  $\mathcal{F}_0$ -measurable. We denote by  $\zeta_{a\nu\omega}$  the output trajectory corresponding to the solution process  $\xi_{a\nu\omega}$ . Here, we assume that the Poisson processes  $P_s^i$ , for any  $i \in [1; \tilde{q}]$ , have the rates of  $\lambda_i$ . We emphasize that the postulated assumptions on  $f$ ,  $\sigma$ , and  $r$  ensure existence, uniqueness, and strong Markov property of the solution processes [Bor89].

**Remark 2.2.** *We refer the interested readers to Section IV in [JP09] showing how one can cast linear stochastic hybrid automata (LSHA) as jump linear stochastic systems (JLSS) (c.f. Section 5) which are a specific class of the ones introduced in Definition 2.1.*

### 3. STOCHASTIC SIMULATION FUNCTION

Before introducing the notion of stochastic simulation functions, we first need to define the infinitesimal generator of a stochastic process.

**Definition 3.1.** *Let  $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, r, \mathbb{R}^q, h)$  and  $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\sigma}, \hat{r}, \mathbb{R}^{\hat{q}}, \hat{h})$  be two stochastic hybrid systems with solution processes  $\xi$  and  $\hat{\xi}$ , respectively. Consider a twice continuously differentiable function  $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \rightarrow \mathbb{R}_{\geq 0}$ . The infinitesimal generator of the stochastic process  $\Xi = [\xi; \hat{\xi}]$ , denoted*

by  $\mathcal{L}$ , acting on function  $V$  is defined in [ØS05, Section 1.3] as:

$$\begin{aligned} \mathcal{L}V(x, \hat{x}) := & [\partial_x V \ \partial_{\hat{x}} V] \begin{bmatrix} f(x, u, w) \\ \hat{f}(\hat{x}, \hat{u}, \hat{w}) \end{bmatrix} + \frac{1}{2} \text{Tr} \left( \begin{bmatrix} \sigma(x) \\ \hat{\sigma}(\hat{x}) \end{bmatrix} [\sigma^T(x) \ \hat{\sigma}^T(\hat{x})] \begin{bmatrix} \partial_{x,x} V & \partial_{x,\hat{x}} V \\ \partial_{\hat{x},x} V & \partial_{\hat{x},\hat{x}} V \end{bmatrix} \right) \\ & + \sum_{i=1}^{\tilde{q}} \lambda_i (V(x + r(x)\mathbf{e}_i, \hat{x} + \hat{r}(\hat{x})\mathbf{e}_i) - V(x, \hat{x})), \end{aligned} \quad (3.1)$$

for every  $x \in \mathbb{R}^n$ ,  $\hat{x} \in \mathbb{R}^{\hat{n}}$ ,  $u \in \mathbb{R}^m$ ,  $\hat{u} \in \mathbb{R}^{\hat{m}}$ ,  $w \in \mathbb{R}^p$ , and  $\hat{w} \in \mathbb{R}^{\hat{p}}$ .

Now, we introduce a notion of stochastic simulation functions, inspired by the notion of simulation function in [RZ15], for deterministic control systems distinguishing the role of internal and external inputs.

**Definition 3.2.** Let  $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, r, \mathbb{R}^q, h)$  and  $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\sigma}, \hat{r}, \mathbb{R}^q, \hat{h})$  be two stochastic hybrid systems with the same internal input and output space dimension. A twice continuously differentiable function  $V : \mathbb{R}^n \times \mathbb{R}^{\hat{n}} \rightarrow \mathbb{R}_{\geq 0}$  is called a stochastic simulation function from  $\hat{\Sigma}$  to  $\Sigma$  in the  $k$ th moment (SSF- $M_k$ ), where  $k \geq 1$ , if it has polynomial growth rate and for any  $x \in \mathbb{R}^n$  and  $\hat{x} \in \mathbb{R}^{\hat{n}}$  one has

$$\alpha(\|h(x) - \hat{h}(\hat{x})\|^k) \leq V(x, \hat{x}), \quad (3.2)$$

and  $\forall \hat{u} \in \mathbb{R}^{\hat{m}} \ \forall \hat{w} \in \mathbb{R}^{\hat{p}} \ \exists u \in \mathbb{R}^m \ \forall w \in \mathbb{R}^p$  one obtains

$$\mathcal{L}V(x, \hat{x}) \leq -\eta(V(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{u}\|^k) + \rho_{\text{int}}(\|w - \hat{w}\|^k), \quad (3.3)$$

for some  $\mathcal{K}_\infty$  functions  $\alpha, \eta, \rho_{\text{ext}}, \rho_{\text{int}}$ , where  $\mathbf{e}_i \in \mathbb{R}^{\tilde{q}}$  denotes the vector with 1 in the  $i$ th coordinate and 0's elsewhere,  $\alpha, \eta$  are convex functions, and  $\rho_{\text{ext}}, \rho_{\text{int}}$  are concave ones.

In the above definition, the symbols  $\partial_x$  and  $\partial_{x,\hat{x}}$  denote the first and the second order partial derivatives with respect to  $x$  and  $x$  and  $\hat{x}$ , respectively.

We say that a stochastic hybrid system  $\hat{\Sigma}$  is *approximately alternatingly simulated in the  $k$ th moment* by a stochastic hybrid system  $\Sigma$  or  $\Sigma$  *approximately alternatingly simulates in the  $k$ th moment*  $\hat{\Sigma}$ , denoted by  $\hat{\Sigma} \preceq_{\text{AS}}^k \Sigma$ , if there exists a SSF- $M_k$  function  $V$  from  $\hat{\Sigma}$  to  $\Sigma$ . We call  $\hat{\Sigma}$  an *abstraction* of  $\Sigma$ .

**Remark 3.3.** Note that the notion of SSF- $M_k$  here is different from the notion of stochastic simulation function in [JP09, Definition 2] requiring the existences of a supermartingale function [Øks02, Appendix C] whose construction is computationally intractable in the case of (even linear) systems with inputs because one requires to solve a game to compute this function. On the other hand, the notion of stochastic (bi)simulation function in [JP09] is stronger than the notion of SSF- $M_k$  as it provides a lower bound on the probability of satisfaction of specifications for which satisfiability can be obtained at all time instances rather than for a bounded time horizon (cf. Proposition 3.7) or at single time instances (cf. Proposition 3.8). We refer the interested readers to Subsection V.B in [ZMM<sup>+</sup>14] for more detailed information about those differences in satisfiability.

**Remark 3.4.** If the drift, diffusion, and reset terms in  $\Sigma$  and  $\hat{\Sigma}$  in Definition 3.2 are polynomial, one can use some sum of squares based semidefinite programming tools, such as SOSTOOLS [PAV<sup>+</sup>13, Subsection 4.2], in order to efficiently search for a (sum of squares) SSF- $M_k$  function from  $\hat{\Sigma}$  to  $\Sigma$  which may not exist in general.

The following theorem shows the importance of the existence of a SSF- $M_k$  function by quantifying the error between the behaviors of  $\Sigma$  and the ones of its abstraction  $\hat{\Sigma}$ .

**Theorem 3.5.** Let  $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathbb{R}^p, \mathcal{U}, \mathcal{W}, f, \sigma, r, \mathbb{R}^q, h)$  and  $\hat{\Sigma} = (\mathbb{R}^{\hat{n}}, \mathbb{R}^{\hat{m}}, \mathbb{R}^{\hat{p}}, \hat{\mathcal{U}}, \hat{\mathcal{W}}, \hat{f}, \hat{\sigma}, \hat{r}, \mathbb{R}^q, \hat{h})$ . Suppose  $V$  is an SSF- $M_k$  function from  $\hat{\Sigma}$  to  $\Sigma$ . Then, there exist a  $\mathcal{KL}$  function  $\beta$  and  $\mathcal{K}_\infty$  functions  $\gamma_{\text{ext}}, \gamma_{\text{int}}$  such that for any  $\hat{\nu} \in \hat{\mathcal{U}}$ , any  $\hat{w} \in \hat{\mathcal{W}}$ , and any random variable  $a$  and  $\hat{a}$  that are  $\mathcal{F}_0$ -measurable<sup>1</sup>, there exists  $\nu \in \mathcal{U}$

<sup>1</sup>Note that  $\mathcal{F}_0$  may be the trivial sigma-algebra, i.e.,  $a$  and  $\hat{a}$  are deterministic initial conditions.

such that for all  $\omega \in \mathcal{W}$  the following inequality holds:

$$\mathbb{E}[\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\|^k] \leq \beta(\mathbb{E}[V(a, \hat{a})], t) + \gamma_{\text{ext}}(\mathbb{E}[\|\hat{\nu}\|_{\infty}^k]) + \gamma_{\text{int}}(\mathbb{E}[\|\omega - \hat{\omega}\|_{\infty}^k]). \quad (3.4)$$

The proof of Theorem 3.5 requires the following preparatory lemma and is provided in the Appendix.

**Lemma 3.6.** *Let  $g$  be a non-negative constant and  $\eta$  be a  $\mathcal{K}_{\infty}$  function. Suppose that the function  $y : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is continuous and we have  $y(t) \leq y(t_0) + \int_{t_0}^t [-\eta(y(\tau)) + g] d\tau$  for all  $t \geq t_0 \geq 0$ . Then, there exists a  $\mathcal{KL}$  function  $\vartheta$  such that*

$$y(t) \leq \max\left\{\vartheta(y(0), t), \eta^{-1}(2g)\right\}, \quad \forall t \geq 0.$$

The proof of Lemma 3.6 is provided in the Appendix.

Note that the importance of the result provided in Theorem 3.5 is that one can synthesize a controller for the abstraction  $\hat{\Sigma}$ , which is potentially easier (e.g., lower dimension and simpler interconnection topology) to enforce some complex specification, for example given in LTL. Then there exists a controller for the concrete stochastic hybrid system  $\Sigma$  satisfying the same complex specification. The error, introduced in the design process by taking the detour through the abstraction, is quantified by inequality (3.4). In Section 5, we show how one can actually *refine* a controller designed for the abstract JLSS to a controller for the original JLSS via a so-called *interface* function.

The notion of stochastic simulation function in this work can also be used to lower bound the probability that the Euclidean distance between any output trajectory of the abstract model and the corresponding one of the concrete model remains close.

We make the above statement more precise with the following results.

**Proposition 3.7.** *Let  $\Sigma$  and  $\hat{\Sigma}$  be two stochastic hybrid systems with the same internal input and output space dimension. Suppose  $V$  is an SSF- $M_k$  function from  $\hat{\Sigma}$  to  $\Sigma$  and the  $\mathcal{K}_{\infty}$  function  $\eta$  in (3.3) satisfies  $\eta(r) \geq \theta r$  for some  $\theta \in \mathbb{R}_{>0}$  and any  $r \in \mathbb{R}_{\geq 0}$ . For any  $\hat{\nu} \in \hat{\mathcal{U}}$ , any  $\hat{\omega} \in \mathcal{W}$ , and any random variable  $a$  and  $\hat{a}$  that are  $\mathcal{F}_0$ -measurable, there exists  $\nu \in \mathcal{U}$  such that for all  $\omega \in \mathcal{W}$  the following inequalities (3.5) and (3.6) hold provided that there exists a constant  $\epsilon \geq 0$  satisfying  $\epsilon \geq \rho_{\text{ext}}(\|\hat{\nu}\|_{\infty}^k) + \rho_{\text{int}}(\|\omega - \hat{\omega}\|_{\infty}^k)$ :*

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\| \geq \epsilon \mid [a; \hat{a}]\right\} \leq 1 - \left(1 - \frac{V(a, \hat{a})}{\alpha(\epsilon^k)}\right) e^{-\frac{\epsilon T}{\alpha(\epsilon^k)}}, \quad \text{if } \alpha(\epsilon^k) \geq \frac{\epsilon}{\theta}, \quad (3.5)$$

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} \|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\| \geq \epsilon \mid [a; \hat{a}]\right\} \leq \frac{\theta V(a, \hat{a}) + (e^{T\theta} - 1)\epsilon}{\theta \alpha(\epsilon^k) e^{T\theta}}, \quad \text{if } \alpha(\epsilon^k) \leq \frac{\epsilon}{\theta}. \quad (3.6)$$

The proof of Proposition 3.7 is provided in the Appendix.

As an alternative to the previous result, we now use the notion of stochastic simulation function to lower bound the probability of the Euclidean distance between any output trajectory of the abstract model and the corresponding one of the concrete model point-wise in time: this error bound is sufficient to work with those specifications for which satisfiability can be achieved at single time instances, such as next ( $\circ$ ) and eventually ( $\diamond$ ) in LTL. Please look at the explanation after the proof of Proposition 5.11 in [ZMM<sup>+</sup>14] for more details.

**Proposition 3.8.** *Let  $\Sigma$  and  $\hat{\Sigma}$  be two stochastic hybrid systems with the same internal input and output space dimension. Suppose  $V$  is an SSF- $M_k$  function from  $\hat{\Sigma}$  to  $\Sigma$ . For any  $\hat{\nu} \in \hat{\mathcal{U}}$ , any  $\hat{\omega} \in \mathcal{W}$ , and any random variable  $a$  and  $\hat{a}$  that are  $\mathcal{F}_0$ -measurable, there exists  $\nu \in \mathcal{U}$  such that for all  $\omega \in \mathcal{W}$  the following inequality holds for all  $t \in \mathbb{R}_{\geq 0}$ :*

$$\mathbb{P}\left\{\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\| \geq \epsilon\right\} \leq \frac{(\beta(\mathbb{E}[V(a, \hat{a})], t) + \gamma_{\text{ext}}(\mathbb{E}[\|\hat{\nu}\|_{\infty}^k]) + \gamma_{\text{int}}(\mathbb{E}[\|\omega - \hat{\omega}\|_{\infty}^k]))^{\frac{1}{k}}}{\epsilon}, \quad (3.7)$$

where  $\beta$ ,  $\gamma_{\text{ext}}$ , and  $\gamma_{\text{int}}$  are the functions appearing in (3.4).

The proof of Proposition 3.8 is provided in the Appendix.

In the next section, we work with interconnected stochastic hybrid systems *without* internal inputs, resulting from the interconnection of stochastic hybrid subsystems having both internal and external signals. In this case, the interconnected stochastic hybrid systems reduce to the tuple  $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, r, \mathbb{R}^q, h)$  and the drift term becomes  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ . In this view, inequality (3.3) is not quantified over  $w, \hat{w} \in \mathbb{R}^p$ , and, hence, the term  $\rho_{\text{int}}(\|w - \hat{w}\|^k)$  is omitted as well. Similarly, the results in Theorem 3.5 and Propositions 3.7 and 3.8 are modified accordingly, i.e., for systems without internal inputs the inequalities (3.4), (3.5), (3.6), and (3.7) are not quantified over  $\omega, \hat{\omega} \in \mathcal{W}$  and, hence, the term  $\gamma_{\text{int}}(\mathbb{E}[\|\omega - \hat{\omega}\|_{\infty}^k])$  is omitted in inequalities (3.4) and (3.7) and  $\epsilon$  is lower bounded as  $\epsilon \geq \rho_{\text{ext}}(\|\hat{v}\|_{\infty}^k)$  in Proposition 3.7 as well.

The next corollary provides a similar result as the one of Proposition 3.7 but by considering an infinite time horizon and interconnected stochastic hybrid systems and assuming  $\hat{v} \equiv 0$ , resulting in  $\epsilon = 0$ . The relation proposed in this corollary recovers the one proposed in [JP09, Theorem 3].

**Corollary 3.9.** *Let  $\Sigma$  and  $\hat{\Sigma}$  be two interconnected stochastic hybrid systems with the same output space dimension. Suppose  $V$  is an SSF- $M_k$  function from  $\hat{\Sigma}$  to  $\Sigma$ . For  $\hat{v} \equiv 0$  and any random variable  $a$  and  $\hat{a}$  that are  $\mathcal{F}_0$ -measurable, there exists  $\nu \in \mathcal{U}$  such that the following inequality holds:*

$$\mathbb{P} \left\{ \sup_{0 \leq t < \infty} \|\zeta_{a\nu}(t) - \hat{\zeta}_{\hat{a}0}(t)\| > \varepsilon \mid [a; \hat{a}] \right\} \leq \frac{V(a, \hat{a})}{\alpha(\varepsilon^k)}.$$

The proof of Corollary 3.9 is provided in the Appendix.

Note that under the assumptions of Corollary 3.9 any SSF- $M_k$  function is also a stochastic simulation function as in [JP09].

#### 4. COMPOSITIONALITY RESULT

In this section, we analyze interconnected stochastic hybrid systems and show how to construct an abstraction of an interconnected stochastic hybrid system together with the corresponding stochastic simulation function. The definition of the interconnected stochastic hybrid system is based on the notion of interconnected systems introduced in [TI08].

**4.1. Interconnected stochastic hybrid systems.** We consider  $N \in \mathbb{N}_{\geq 1}$  stochastic hybrid subsystems

$$\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, r_i, \mathbb{R}^{q_i}, h_i), \quad i \in [1; N]$$

with partitioned internal inputs and outputs

$$\begin{aligned} w_i &= [w_{i1}; \dots; w_{i(i-1)}; w_{i(i+1)}; \dots; w_{iN}], \quad w_{ij} \in \mathbb{R}^{p_{ij}} \\ y_i &= [y_{i1}; \dots; y_{iN}], \quad y_{ij} \in \mathbb{R}^{q_{ij}} \end{aligned} \quad (4.1)$$

and output function

$$h_i(x_i) = [h_{i1}(x_i); \dots; h_{iN}(x_i)], \quad (4.2)$$

as depicted schematically in Figure 1.

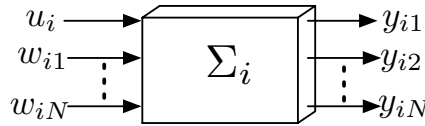


FIGURE 1. Input/output configuration of stochastic hybrid subsystem  $\Sigma_i$ .

We interpret the outputs  $y_{ii}$  as *external* ones, whereas the outputs  $y_{ij}$  with  $i \neq j$  are *internal* ones which are used to define the interconnected stochastic hybrid systems. In particular, we assume that the dimension of  $w_{ij}$  is equal to the dimension of  $y_{ji}$ , i.e., the following *interconnection constraints* hold:

$$p_{ij} = q_{ji}, \quad \forall i, j \in [1; N], \quad i \neq j. \quad (4.3)$$

If there is no connection from stochastic hybrid subsystem  $\Sigma_i$  to  $\Sigma_j$ , then we assume that the connecting output function is identically zero for all arguments, i.e.,  $h_{ij} \equiv 0$ . We define the *interconnected stochastic hybrid system* as the following.

**Definition 4.1.** Consider  $N \in \mathbb{N}_{\geq 1}$  stochastic hybrid subsystems  $\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, r_i, \mathbb{R}^{q_i}, h_i)$ ,  $i \in [1; N]$ , with the input-output configuration given by (4.1)-(4.3). The interconnected stochastic hybrid system  $\Sigma = (\mathbb{R}^n, \mathbb{R}^m, \mathcal{U}, f, \sigma, r, \mathbb{R}^q, h)$ , denoted by  $\mathcal{I}(\Sigma_1, \dots, \Sigma_N)$ , follows by  $n = \sum_{i=1}^N n_i$ ,  $m = \sum_{i=1}^N m_i$ ,  $q = \sum_{i=1}^N q_{ii}$ , and functions

$$\begin{aligned} f(x, u) &:= [f_1(x_1, u_1, w_1); \dots; f_N(x_N, u_N, w_N)], \\ \sigma(x) &:= [\sigma_1(x_1); \dots; \sigma_N(x_n)], \\ r(x) &:= [r_1(x_1); \dots; r_N(x_n)], \\ h(x) &:= [h_{11}(x_1); \dots; h_{NN}(x_N)], \end{aligned}$$

where  $u = [u_1; \dots; u_N]$  and  $x = [x_1; \dots; x_N]$  and with the interconnection variables constrained by  $w_{ij} = y_{ji}$  for all  $i, j \in [1; N], i \neq j$ .

The interconnection of two stochastic hybrid subsystems  $\Sigma_i$  and  $\Sigma_j$  from a group of  $N$  subsystems is illustrated in Figure 2.

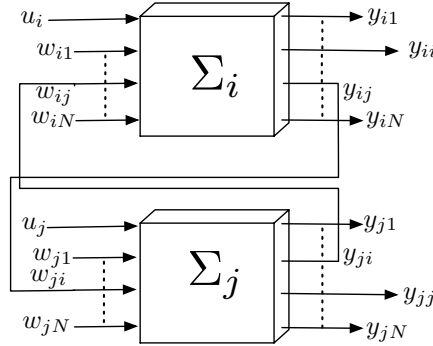


FIGURE 2. Interconnection of two stochastic hybrid subsystems  $\Sigma_i$  and  $\Sigma_j$ .

**4.2. Compositional construction of abstractions and simulation functions.** We assume that we are given  $N$  stochastic hybrid subsystems  $\Sigma_i = (\mathbb{R}^{n_i}, \mathbb{R}^{m_i}, \mathbb{R}^{p_i}, \mathcal{U}_i, \mathcal{W}_i, f_i, \sigma_i, r_i, \mathbb{R}^{q_i}, h_i)$ , together with their corresponding abstractions  $\hat{\Sigma}_i = (\mathbb{R}^{\hat{n}_i}, \mathbb{R}^{\hat{m}_i}, \mathbb{R}^{\hat{p}_i}, \hat{\mathcal{U}}_i, \hat{\mathcal{W}}_i, \hat{f}_i, \hat{\sigma}_i, \hat{r}_i, \mathbb{R}^{\hat{q}_i}, \hat{h}_i)$  and with SSF- $M_k$  functions  $V_i$  from  $\hat{\Sigma}_i$  to  $\Sigma_i$ . In order to provide the main compositionality result, we require the following assumption:

**Assumption 1.** For any  $i, j \in [1; N], i \neq j$ , there exist  $\mathcal{K}_\infty$  convex functions  $\gamma_i$  and constants  $\tilde{\lambda}_i \in \mathbb{R}_{>0}$  and  $\delta_{ij} \in \mathbb{R}_{\geq 0}$  such that for any  $s \in \mathbb{R}_{\geq 0}$

$$\eta_i(s) \geq \tilde{\lambda}_i \gamma_i(s) \quad (4.4a)$$

$$h_{ji} \equiv 0 \implies \delta_{ij} = 0 \text{ and} \quad (4.4b)$$

$$h_{ji} \neq 0 \implies \rho_{\text{int}}((N-1)^{\max\{\frac{k}{2}, 1\}} \alpha_j^{-1}(s)) \leq \delta_{ij} \gamma_j(s), \quad (4.4c)$$

where  $\eta_i$ ,  $\alpha_i$ , and  $\rho_{\text{int}}$  represent the corresponding  $\mathcal{K}_\infty$  functions of subsystems  $\Sigma_i$  appearing in Definition 3.2.



For notational simplicity in the rest of the paper, we define matrices  $\Lambda$  and  $\Delta$  in  $\mathbb{R}^{N \times N}$  with their components given by  $\Lambda_{ii} = \tilde{\lambda}_i$ ,  $\Delta_{ii} = 0$  for  $i \in [1; N]$  and  $\Lambda_{ij} = 0$ ,  $\Delta_{ij} = \delta_{ij}$  for  $i, j \in [1; N], i \neq j$ . Moreover, we define  $\Gamma(\vec{s}) := [\gamma_1(s_1); \dots; \gamma_N(s_N)]$ , where  $\vec{s} = [s_1; \dots; s_N]$ .

The next theorem provides a compositional approach on the construction of abstractions of interconnected stochastic hybrid systems and that of the corresponding SSF- $M_k$  functions.

**Theorem 4.2.** *Consider the interconnected stochastic hybrid system  $\Sigma = \mathcal{I}(\Sigma_1, \dots, \Sigma_N)$  induced by  $N \in \mathbb{N}_{\geq 1}$  stochastic hybrid subsystems  $\Sigma_i$ . Suppose that each stochastic hybrid subsystem  $\Sigma_i$  approximately alternately simulates a stochastic hybrid subsystem  $\hat{\Sigma}_i$  with the corresponding SSF- $M_k$  function  $V_i$ . If Assumption 1 holds and there exists a vector  $\mu \in \mathbb{R}_{>0}^N$  such that the inequality*

$$\mu^T(-\Lambda + \Delta) < 0 \quad (4.5)$$

is satisfied<sup>2</sup>, then

$$V(x, \hat{x}) := \sum_{i=1}^N \mu_i V_i(x_i, \hat{x}_i)$$

is an SSF- $M_k$  function from  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \dots, \hat{\Sigma}_N)$  to  $\Sigma$ .

*Proof.* Note that for any  $x = [x_1; \dots; x_N]$ , where  $x_i \in \mathbb{R}^{n_i}$  and  $i \in [1; N]$ , one obtains:

$$\|x\|^k \leq \sum_{i=1}^N \|x_i\|^k,$$

for any  $k \in [1, 2]$  due to triangle inequality and appropriate equivalency between different norms and

$$\|x\|^k = (\|x\|^2)^{\frac{k}{2}} = \left( \sum_{i=1}^N \|x_i\|^2 \right)^{\frac{k}{2}} \leq N^{\frac{k}{2}-1} \sum_{i=1}^N \|x_i\|^k,$$

for any  $k > 2$  due to Jensen's inequality [BV09] for convex functions. By combining the previous inequalities, one gets

$$\|x\|^k \leq N^{\max\{\frac{k}{2}, 1\}-1} \sum_{i=1}^N \|x_i\|^k, \quad (4.6)$$

for any  $k \geq 1$  and any  $x = [x_1; \dots; x_N]$ , where  $x_i \in \mathbb{R}^{n_i}$  and  $i \in [1; N]$ .

First we show that inequality (3.2) holds for some convex  $\mathcal{K}_\infty$  function  $\alpha$ . Using (4.6), for any  $x = [x_1; \dots; x_N] \in \mathbb{R}^n$  and  $\hat{x} = [\hat{x}_1; \dots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$ , one gets:

$$\begin{aligned} \|\hat{h}(\hat{x}) - h(x)\|^k &\leq N^{\max\{\frac{k}{2}, 1\}-1} \sum_{i=1}^N \|\hat{h}_{ii}(\hat{x}_i) - h_{ii}(x_i)\|^k \leq N^{\max\{\frac{k}{2}, 1\}-1} \sum_{i=1}^N \|\hat{h}_i(\hat{x}_i) - h_i(x_i)\|^k \\ &\leq N^{\max\{\frac{k}{2}, 1\}-1} \sum_{i=1}^N \alpha_i^{-1}(V_i(x_i, \hat{x}_i)) \leq \bar{\alpha}(V(x, \hat{x})), \end{aligned}$$

where  $\bar{\alpha}$  is a  $\mathcal{K}_\infty$  function defined as

$$\bar{\alpha}(s) := \begin{cases} \max_{\vec{s} \geq 0} & N^{\max\{\frac{k}{2}, 1\}-1} \sum_{i=1}^N \alpha_i^{-1}(s_i) \\ \text{s.t.} & \mu^T \vec{s} = s. \end{cases}$$

Now we show that  $\bar{\alpha}$  is a concave function. Let us recall that by assumptions  $\alpha_i$  are convex functions and, hence,  $\alpha_i^{-1}$  are concave<sup>3</sup>. Thus, from an optimization point of view, the function  $\bar{\alpha}$  is a *perturbation* function

<sup>2</sup>We interpret the inequality component-wise, i.e., for  $x \in \mathbb{R}^N$  we have  $x < 0$  iff every entry  $x_i < 0$ ,  $i \in \{1, \dots, N\}$ .

<sup>3</sup>Note that the inverse of a strictly increasing convex (resp. concave) function is a strictly increasing concave (resp. convex) one.

$$\begin{aligned}
 \mathcal{L}V(x, \hat{x}) &= \sum_{i=1}^N \mu_i \mathcal{L}V_i(x_i, \hat{x}_i) \leq \sum_{i=1}^N \mu_i \left( -\eta_i(V_i(x_i, \hat{x}_i)) + \rho_{i\text{int}}(\|w_i - \hat{w}_i\|^k) + \rho_{i\text{ext}}(\|\hat{u}_i\|^k) \right) \\
 &\leq \sum_{i=1}^N \mu_i \left( -\eta_i(V_i(x_i, \hat{x}_i)) + \sum_{j=1, j \neq i}^N \rho_{i\text{int}}((N-1)^{\max\{\frac{k}{2}, 1\}} \|w_{ij} - \hat{w}_{ij}\|^k) + \rho_{i\text{ext}}(\|\hat{u}_i\|^k) \right) \\
 &\leq \sum_{i=1}^N \mu_i \left( -\eta_i(V_i(x_i, \hat{x}_i)) + \sum_{j=1, j \neq i}^N \rho_{i\text{int}}((N-1)^{\max\{\frac{k}{2}, 1\}} \|y_{ji} - \hat{y}_{ji}\|^k) + \rho_{i\text{ext}}(\|\hat{u}_i\|^k) \right) \\
 &\leq \sum_{i=1}^N \mu_i \left( -\eta_i(V_i(x_i, \hat{x}_i)) + \sum_{j=1, j \neq i}^N \rho_{i\text{int}}((N-1)^{\max\{\frac{k}{2}, 1\}} \|h_j(x_j) - \hat{h}_j(\hat{x}_j)\|^k) + \rho_{i\text{ext}}(\|\hat{u}_i\|^k) \right) \\
 &\leq \sum_{i=1}^N \mu_i \left( -\eta_i(V_i(x_i, \hat{x}_i)) + \sum_{j=1, j \neq i}^N \rho_{i\text{int}}\left( (N-1)^{\max\{\frac{k}{2}, 1\}} \alpha_j^{-1}(V_j(x_j, \hat{x}_j)) \right) + \rho_{i\text{ext}}(\|\hat{u}_i\|^k) \right) \\
 &\leq \sum_{i=1}^N \mu_i \left( -\tilde{\lambda}_i \gamma_i(V_i(x_i, \hat{x}_i)) + \sum_{i \neq j, j=1}^N \delta_{ij} \gamma_j(V_j(x_j, \hat{x}_j)) + \rho_{i\text{ext}}(\|\hat{u}_i\|^k) \right) \\
 &= \mu^\top (-\Lambda + \Delta) \Gamma([V_1(x_1, \hat{x}_1); \dots; V_N(x_N, \hat{x}_N)]) + \sum_{i=1}^N \mu_i \rho_{i\text{ext}}(\|\hat{u}_i\|^k). \tag{4.7}
 \end{aligned}$$

which is known to be a concave function; see [BV09, Section 5.6.1, p. 249] for further details. By defining the convex<sup>3</sup>  $\mathcal{K}_\infty$  function  $\alpha(s) = \bar{\alpha}^{-1}(s)$ ,  $\forall s \in \mathbb{R}_{\geq 0}$ , one obtains

$$\alpha(\|\hat{h}(\hat{x}) - h(x)\|^k) \leq V(x, \hat{x}),$$

satisfying inequality (3.2). Now we show that inequality (3.3) holds as well. Consider any  $x = [x_1; \dots; x_N] \in \mathbb{R}^n$ ,  $\hat{x} = [\hat{x}_1; \dots; \hat{x}_N] \in \mathbb{R}^{\hat{n}}$ , and  $\hat{u} = [\hat{u}_1; \dots; \hat{u}_N] \in \mathbb{R}^{\hat{m}}$ . For any  $i \in [1; N]$ , there exists  $u_i \in \mathbb{R}^{m_i}$ , consequently, a vector  $u = [u_1; \dots; u_N] \in \mathbb{R}^m$ , satisfying (3.3) for each pair of subsystems  $\Sigma_i$  and  $\hat{\Sigma}_i$  with the internal inputs given by  $w_{ij} = h_{ji}(x_j)$  and  $\hat{w}_{ij} = \hat{h}_{ji}(\hat{x}_j)$ . We derive the chain of inequalities in (4.7), where we use the inequalities (4.6) and:

$$\rho_{i\text{int}}(r_1 + \dots + r_{N-1}) \leq \sum_{i=1}^{N-1} \rho_{i\text{int}}((N-1)r_i),$$

which are valid for any  $k \geq 1$ ,  $\rho_{i\text{int}} \in \mathcal{K}_\infty$ ,  $x_i \in \mathbb{R}^{n_i}$ , and any  $r_i \in \mathbb{R}_{\geq 0}$ ,  $i \in [1; N[$ . Note that if  $\rho_{i\text{int}}$  satisfies the triangle inequality, one gets the less conservative inequality

$$\rho_{i\text{int}}(r_1 + \dots + r_{N-1}) \leq \sum_{i=1}^{N-1} \rho_{i\text{int}}(r_i),$$

and it suffices that (4.9) holds instead of (4.4c). Define the functions

$$\eta(s) := \begin{cases} \min_{\vec{s} \geq 0} & -\mu^\top (-\Lambda + \Delta) \Gamma(\vec{s}) \\ \text{s.t.} & \mu^\top \vec{s} = s, \end{cases} \tag{4.8a}$$

$$\rho_{\text{ext}}(s) := \begin{cases} \max_{\vec{s} \geq 0} & \sum_{i=1}^N \mu_i \rho_{i\text{ext}}(s_i) \\ \text{s.t.} & \|\vec{s}\| \leq s. \end{cases} \tag{4.8b}$$

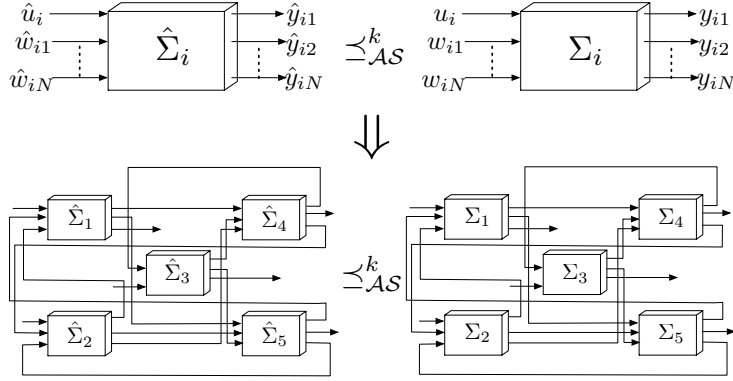


FIGURE 3. Compositionality results.

By construction, we readily have

$$\dot{V}(x, \hat{x}) \leq -\eta(V(x, \hat{x})) + \rho_{\text{ext}}(\|\hat{u}\|^k),$$

where the functions  $\eta$  and  $\rho_{\text{ext}}$  are  $\mathcal{K}_\infty$  functions. It remains to show that  $\eta$  is a convex function and  $\rho_{\text{ext}}$  is a concave one. Let us recall that by assumptions  $\mu^T(-\Lambda + \Delta) < 0$  and  $\gamma_i$ , the  $i$ -th element of  $\Gamma$ , is convex. Thus, the function  $\eta$  in (4.8a) is a perturbation function which is a convex one. Note that by assumption each function  $\rho_{i\text{ext}}$  is concave, and for the same reason as above, the function (4.8b) is also concave. Hence, we conclude that  $V$  is an SSF- $M_k$  function from  $\hat{\Sigma}$  to  $\Sigma$ .  $\square$

**Remark 4.3.** As shown in [DIW11, Lemma 3.1], a vector  $\mu \in \mathbb{R}_{>0}^N$  satisfying  $\mu^T(-\Lambda + \Delta) < 0$  exists if and only if the spectral radius of  $\Lambda^{-1}\Delta$  is strictly less than one.

**Remark 4.4.** If the functions  $\rho_{i\text{int}}$ ,  $i \in [1; N]$ , satisfy the triangle inequality,  $\rho_{i\text{int}}(a + b) \leq \rho_{i\text{int}}(a) + \rho_{i\text{int}}(b)$  for all non-negative values of  $a$  and  $b$ , then the condition (4.4c) reduces to

$$h_{ji} \neq 0 \implies \rho_{i\text{int}}((N-1)^{\max\{\frac{k}{2}, 1\}-1} \alpha_j^{-1}(s)) \leq \delta_{ij} \gamma_j(s). \quad (4.9)$$

Figure 3 illustrates schematically the result of Theorem 4.2.

## 5. JUMP LINEAR STOCHASTIC SYSTEMS

In this section, we focus on a specific class of stochastic hybrid systems, namely, jump linear stochastic systems (JLSS) [JP09] and *quadratic* SSF- $M_2$  functions  $V$ . In the first part, we assume that we are given an abstraction  $\hat{\Sigma}$  and provide conditions under which  $V$  is an SSF- $M_2$  function. In the second part we show how to construct the abstraction  $\hat{\Sigma}$  together with the SSF- $M_2$  function  $V$ .

A JLSS is defined as a stochastic hybrid system with the drift, diffusion, reset, and output functions given by

$$\begin{aligned} d\xi(t) &= (A\xi(t) + B\nu(t) + D\omega(t)) dt + E\xi(t) dW_t + \sum_{i=1}^{\tilde{q}} R_i \xi(t) dP_t^i, \\ \zeta(t) &= C\xi(t), \end{aligned} \quad (5.1)$$

where

$$A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, D \in \mathbb{R}^{n \times p}, E \in \mathbb{R}^{n \times n}, R_i \in \mathbb{R}^{n \times n}, \forall i \in [1; \tilde{q}], C \in \mathbb{R}^{q \times n}.$$

The matrices  $R_i$ ,  $\forall i \in [1; \tilde{q}]$ , parametrize the jump associated with event  $i$ . We use the tuple

$$\Sigma = (A, B, C, D, E, R),$$

where  $\mathbf{R} = \{R_1, \dots, R_{\tilde{q}}\}$ , to refer to a JLSS of the form (5.1). Note that in this section we consider JLSS driven by a scalar Brownian motion for the sake of simple presentation, though the proposed results can be readily generalized for the systems driven by multi-dimensional Brownian motions as well.

**5.1. Quadratic SSF-M<sub>2</sub> functions.** In this section, we assume that for some constant  $\hat{\kappa} \in \mathbb{R}_{>0}$  there exist a positive definite matrix  $M \in \mathbb{R}^{n \times n}$  and matrix  $K \in \mathbb{R}^{m \times n}$  such that the matrix inequalities

$$C^T C \preceq M, \quad (5.2)$$

$$\left( A + BK + \sum_{i=1}^{\tilde{q}} \lambda_i R_i \right)^T M + M \left( A + BK + \sum_{i=1}^{\tilde{q}} \lambda_i R_i \right) + E^T M E + \sum_{i=1}^{\tilde{q}} \lambda_i R_i^T M R_i \preceq -\hat{\kappa} M, \quad (5.3)$$

hold.

Note that condition (5.3) is sufficient and necessary for the asymptotic stability of  $\Sigma = (A, B, C, 0_{n \times p}, E, \mathbf{R})$  equipped with a linear feedback control law  $u = Kx$  in the mean square sense (second moment)<sup>4</sup> as showed in the next lemma. Condition (5.2) is always satisfied for any positive definite matrix  $M$  up to multiplication by some positive scalar which does not violate the satisfaction of (5.3).

**Lemma 5.1.** *A JLSS  $\Sigma = (A, B, C, 0_{n \times p}, E, \mathbf{R})$  equipped with a linear feedback control law  $u = Kx$  is asymptotically stable in the mean square sense if and only if there exists a positive definite matrix  $M \in \mathbb{R}^{n \times n}$  such that the matrix inequality (5.3) is satisfied for given feedback gain  $K$  and some positive constant  $\hat{\kappa}$ .*

The proof of Lemma 5.1 is provided in the Appendix.

The matrices  $K$  and  $M$  in (5.2) and (5.3) can be computed jointly using semidefinite programming as explained in the following lemma.

**Lemma 5.2.** *Denoting  $\bar{K} = KM^{-1}$  and  $\bar{M} = M^{-1}$ , matrix inequalities (5.2) and (5.3) are equivalent to the following linear matrix inequalities:*

$$\begin{bmatrix} \bar{M} & \bar{M}C^T \\ C\bar{M} & I_q \end{bmatrix} \succeq 0 \quad (5.4)$$

$$\begin{bmatrix} \bar{M} & 0 & \cdots & 0 & E\bar{M} \\ 0 & \bar{M} & \ddots & \vdots & \lambda_{\tilde{q}}^{\frac{1}{2}} R_{\tilde{q}} \bar{M} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \bar{M} & \lambda_1^{\frac{1}{2}} R_1 \bar{M} \\ \bar{M}E^T & \lambda_{\tilde{q}}^{\frac{1}{2}} \bar{M}R_{\tilde{q}}^T & \cdots & \lambda_1^{\frac{1}{2}} \bar{M}R_1^T & \bar{Q} \end{bmatrix} \succeq 0, \quad (5.5)$$

where 0's denote zero matrices of appropriate dimensions and

$$\bar{Q} := -\hat{\kappa}\bar{M} - \bar{M} \left( A + \sum_{i=1}^{\tilde{q}} \lambda_i R_i \right)^T - \left( A + \sum_{i=1}^{\tilde{q}} \lambda_i R_i \right) \bar{M} - \bar{K}^T B^T - B\bar{K}.$$

The proof is a simple consequence of using Schur complements [BV09] and is omitted here for the sake of brevity.

Here, we consider a quadratic SSF-M<sub>2</sub> function of the following form

$$V(x, \hat{x}) = (x - P\hat{x})^T M (x - P\hat{x}), \quad (5.6)$$

<sup>4</sup>A stochastic hybrid system  $\Sigma$  is said to be asymptotically stable in the mean square sense if all  $\mathcal{F}_0$ -measurable initial states  $a$  yield  $\lim_{t \rightarrow \infty} \mathbb{E}[\|\xi(t)\|^2] = 0$ .

where  $P$  is a matrix of appropriate dimension. Assume that the equalities

$$AP = P\hat{A} - BQ \quad (5.7a)$$

$$D = P\hat{D} - BS \quad (5.7b)$$

$$CP = \hat{C} \quad (5.7c)$$

$$EP = P\hat{E} \quad (5.7d)$$

$$R_i P = P\hat{R}_i, \quad \forall i \in [1; \tilde{q}], \quad (5.7e)$$

hold for some matrices  $Q$  and  $S$  of appropriate dimensions. In the following theorem, we show that those conditions imply that (5.6) is an SSF- $M_2$  function from  $\tilde{\Sigma}$  to  $\Sigma$ .

**Theorem 5.3.** *Consider two JLSS  $\Sigma = (A, B, C, D, E, R)$  and  $\tilde{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{R})$  with  $p = \hat{p}$  and  $q = \hat{q}$ . Suppose that there exist matrices  $M, K, P, Q,$  and  $S$  satisfying (5.2), (5.3), and (5.7), for some constant  $\hat{\kappa} \in \mathbb{R}_{>0}$ . Then,  $V$  defined in (5.6) is an SSF- $M_2$  function from  $\tilde{\Sigma}$  to  $\Sigma$ .*

*Proof.* Note that  $V$  is twice continuously differentiable. We show that for every  $x \in \mathbb{R}^n$ ,  $\hat{x} \in \mathbb{R}^{\hat{n}}$ ,  $\hat{u} \in \mathbb{R}^{\hat{m}}$ ,  $\hat{w} \in \mathbb{R}^{\hat{p}}$ , there exists  $u \in \mathbb{R}^m$  such that for all  $w \in \mathbb{R}^p$ ,  $V$  satisfies  $\|Cx - \hat{C}\hat{x}\|^2 \leq V(x, \hat{x})$  and

$$\begin{aligned} \mathcal{L}V(x, \hat{x}) &:= \frac{\partial V(x, \hat{x})}{\partial x} (Ax + Bu + Dw) + \frac{\partial V(x, \hat{x})}{\partial \hat{x}} (\hat{A}\hat{x} + \hat{B}\hat{u} + \hat{D}\hat{w}) \\ &\quad + \frac{1}{2} \text{Tr} \left( \begin{bmatrix} Ex \\ \hat{E}\hat{x} \end{bmatrix} \begin{bmatrix} x^T E^T & \hat{x}^T \hat{E}^T \end{bmatrix} \begin{bmatrix} \partial_{x,x} V & \partial_{x,\hat{x}} V \\ \partial_{\hat{x},x} V & \partial_{\hat{x},\hat{x}} V \end{bmatrix} \right) + \sum_{i=1}^{\tilde{q}} \lambda_i (V(x + R_i x, \hat{x} + \hat{R}_i \hat{x}) - V(x, \hat{x})) \\ &\leq -(\hat{\kappa} - \pi)V(x, \hat{x}) + \frac{2\|\sqrt{M}D\|^2}{\pi} \|w - \hat{w}\|^2 + \frac{2\|\sqrt{M}(B\tilde{R} - P\hat{B})\|^2}{\pi} \|\hat{u}\|^2, \end{aligned} \quad (5.8)$$

for any positive constant  $\pi < \hat{\kappa}$  and some matrix  $\tilde{R}$  of appropriate dimension.

From (5.7c), we have  $\|Cx - \hat{C}\hat{x}\|^2 = (x - P\hat{x})^T C^T C (x - P\hat{x})$  and using  $M \succeq C^T C$ , it can be readily verified that  $\|Cx - \hat{C}\hat{x}\|^2 \leq V(x, \hat{x})$  holds for all  $x \in \mathbb{R}^n$ ,  $\hat{x} \in \mathbb{R}^{\hat{n}}$ . We proceed with showing the inequality in (5.8). Note that

$$\begin{aligned} \partial_x V(x, \hat{x}) &= 2(x - P\hat{x})^T M, \quad \partial_{\hat{x}} V(x, \hat{x}) = -2(x - P\hat{x})^T MP, \quad \partial_{x,x} V(x, \hat{x}) = 2M, \quad \text{and} \\ \partial_{\hat{x},\hat{x}} V(x, \hat{x}) &= P^T \partial_{x,x} V(x, \hat{x}) P, \quad \partial_{x,\hat{x}} V(x, \hat{x}) = (\partial_{\hat{x},x} V(x, \hat{x}))^T = -\partial_{x,x} V(x, \hat{x}) P \end{aligned}$$

holds. Given any  $x \in \mathbb{R}^n$ ,  $\hat{x} \in \mathbb{R}^{\hat{n}}$ ,  $\hat{u} \in \mathbb{R}^{\hat{m}}$ , and  $\hat{w} \in \mathbb{R}^{\hat{p}}$ , we choose  $u \in \mathbb{R}^m$  via the following linear *interface* function:

$$u = \nu_{\hat{\nu}}(x, \hat{x}, \hat{u}, \hat{w}) := K(x - P\hat{x}) + Q\hat{x} + \tilde{R}\hat{u} + S\hat{w}, \quad (5.9)$$

for some matrix  $\tilde{R}$  of appropriate dimension.

By using the equations (5.7a) and (5.7b) and the definition of the interface function in (5.9), we simplify

$$Ax + B\nu_{\hat{\nu}}(x, \hat{x}, \hat{u}, \hat{w}) + Dw - P(\hat{A}\hat{x} + \hat{B}\hat{u} + \hat{D}\hat{w})$$

to  $(A + BK)(x - P\hat{x}) + D(w - \hat{w}) + (B\tilde{R} - P\hat{B})\hat{u}$  and obtain the following expression for  $\mathcal{L}V(x, \hat{x})$ :

$$\begin{aligned} \mathcal{L}V(x, \hat{x}) &= 2(x - P\hat{x})^T M [(A + BK)(x - P\hat{x}) + D(w - \hat{w}) + (B\tilde{R} - P\hat{B})\hat{u}] \\ &\quad + \begin{bmatrix} x \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} E^T & 0 \\ 0 & \hat{E}^T \end{bmatrix} \begin{bmatrix} M & -MP \\ -P^T M & P^T MP \end{bmatrix} \begin{bmatrix} E & 0 \\ 0 & \hat{E} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} x \\ \hat{x} \end{bmatrix}^T \sum_{i=1}^{\tilde{q}} \lambda_i \begin{bmatrix} R_i^T & 0 \\ 0 & \hat{R}_i^T \end{bmatrix} \begin{bmatrix} M & -MP \\ -P^T M & P^T MP \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \\ &\quad + \begin{bmatrix} x \\ \hat{x} \end{bmatrix}^T \begin{bmatrix} M & -MP \\ -P^T M & P^T MP \end{bmatrix} \sum_{i=1}^{\tilde{q}} \lambda_i \begin{bmatrix} R_i & 0 \\ 0 & \hat{R}_i \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \\ &\quad + \begin{bmatrix} x \\ \hat{x} \end{bmatrix}^T \sum_{i=1}^{\tilde{q}} \lambda_i \begin{bmatrix} R_i^T & 0 \\ 0 & \hat{R}_i^T \end{bmatrix} \begin{bmatrix} M & -MP \\ -P^T M & P^T MP \end{bmatrix} \begin{bmatrix} R_i & 0 \\ 0 & \hat{R}_i \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}, \end{aligned}$$

where 0's denote zero matrices of appropriate dimensions. We use (5.7d) and (5.7e) to obtain the following expression for  $\mathcal{L}V(x, \hat{x})$ :

$$\begin{aligned} \mathcal{L}V(x, \hat{x}) = & (x - P\hat{x})^T \left[ \left( A + BK + \sum_{i=1}^{\tilde{q}} \lambda_i R_i \right)^T M + M \left( A + BK + \sum_{i=1}^{\tilde{q}} \lambda_i R_i \right) \right. \\ & \left. + E^T M E + \sum_{i=1}^{\tilde{q}} \lambda_i R_i^T M R_i \right] (x - P\hat{x}) + 2(x - P\hat{x})^T M [D(w - \hat{w}) + (B\tilde{R} - P\hat{B})\hat{u}]. \end{aligned}$$

Using Young's inequality [You12] as

$$ab \leq \frac{\epsilon}{2} a^2 + \frac{1}{2\epsilon} b^2,$$

for any  $a, b \geq 0$  and any  $\epsilon > 0$ , and with the help of Cauchy-Schwarz inequality and (5.3) one gets the following upper bound for  $\mathcal{L}V(x, \hat{x})$ :

$$\mathcal{L}V(x, \hat{x}) \leq -\hat{\kappa}V(x, \hat{x}) + \pi V(x, \hat{x}) + \frac{2\|\sqrt{M}D\|^2}{\pi} \|w - \hat{w}\|^2 + \frac{2\|\sqrt{M}(B\tilde{R} - P\hat{B})\|^2}{\pi} \|\hat{u}\|^2,$$

for any positive constant  $\pi < \hat{\kappa}$ .

Using this computed upper bound, we obtain (5.8) which completes the proof. Note that the  $\mathcal{K}_\infty$  functions  $\alpha$ ,  $\eta$ ,  $\rho_{\text{ext}}$ , and  $\rho_{\text{int}}$ , in Definition 3.2 associated with the SSF-M<sub>2</sub> function in (5.6) are given by  $\alpha(s) := s$ ,  $\eta(s) := (\hat{\kappa} - \pi)s$ ,  $\rho_{\text{ext}}(s) := \frac{2\|\sqrt{M}(B\tilde{R} - P\hat{B})\|^2}{\pi} s$  and  $\rho_{\text{int}}(s) := \frac{2\|\sqrt{M}D\|^2}{\pi} s$ ,  $\forall s \in \mathbb{R}_{\geq 0}$ .

□

**Remark 5.4.** Using the linear functions  $\alpha$  and  $\eta$ , as computed in Theorem 5.3, the functions  $\beta$ ,  $\gamma_{\text{ext}}$ , and  $\gamma_{\text{int}}$ , appearing in Theorem 3.5, are simplified as the following:  $\beta(r, t) := re^{-(\hat{\kappa} - \pi)t}$ ,  $\gamma_{\text{ext}}(r) := \frac{1}{\hat{\kappa} - \pi} \rho_{\text{ext}}(r)$ , and  $\gamma_{\text{int}}(r) := \frac{1}{\hat{\kappa} - \pi} \rho_{\text{int}}(r)$  for any  $r, t \in \mathbb{R}_{\geq 0}$ .

**Remark 5.5.** Note that Theorem 5.3 does not impose any condition on matrix  $\tilde{R}$ . Similar to the results in [GP09, Proposition 1] for the deterministic case, we propose a choice of  $\tilde{R}$  which minimize function  $\rho_{\text{ext}}$  for  $V$ . The choice of  $\tilde{R}$  minimizing  $\rho_{\text{ext}}$  is given by

$$\tilde{R} = (B^T M B)^{-1} B^T M P \hat{B}. \quad (5.10)$$

**Remark 5.6.** Consider  $\Sigma_i = (A_i, B_i, C_i, D_i, E_i, R_i)$  and its abstraction  $\hat{\Sigma}_i = (\hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{D}_i, \hat{E}_i, \hat{R}_i)$ . Assume  $D_i = [d_i^1 \cdots d_i^p]$  and  $\hat{D}_i = [\hat{d}_i^1 \cdots \hat{d}_i^p]$ . Using equation (5.7b), one can readily conclude that if  $d_i^j \in \text{im } B$ , for some  $j \in [1; p]$ , then the corresponding  $\hat{d}_i^j$  can be chosen as  $\hat{d}_i^j = 0_{\hat{n}}$ . This choice for columns of  $\hat{D}$  makes the interconnection topology of abstract subsystems potentially simpler and, hence, their analysis easier. We refer the interested readers to Section 6 for an example of such choice for  $\hat{D}$ .

As of now, we derived various conditions on the original system  $\Sigma$ , the abstraction  $\hat{\Sigma}$ , and the matrices appearing in (5.6) and (5.9), to ensure that (5.6) is an SSF-M<sub>2</sub> function from  $\hat{\Sigma}$  to  $\Sigma$  with the corresponding interface function in (5.9) lifting any control policy designed for  $\hat{\Sigma}$  to the one for  $\Sigma$ . However, those conditions do not impose any requirements on the abstract external input matrix  $\hat{B}$ . As an example, one can choose  $\hat{B} = I_{\hat{n}}$  which makes the abstract system  $\hat{\Sigma}$  fully actuated and, hence, the synthesis problem over  $\hat{\Sigma}$  much easier. Similar to [GP09, Subsection 4.1] in the context of deterministic control systems, one can also choose an external input matrix  $\hat{B}$  which *preserves* all the behaviors of the original JLSS  $\Sigma$  on the abstraction  $\hat{\Sigma}$ : for every trajectory  $(\xi, \zeta, \nu, \omega)$  of  $\Sigma$  there exists a trajectory  $(\hat{\xi}, \hat{\zeta}, \hat{\nu}, \hat{\omega})$  of  $\hat{\Sigma}$  such that  $\hat{\zeta} = \zeta$  P-a.s..

Note that using the following choice of external input matrix  $\hat{B}$ , the results in [RZ15] for the linear deterministic control system are fully recovered by the corresponding ones here providing that the JLSS is not affected by any noise, implying that  $E$ ,  $\hat{E}$ ,  $R_i$ , and  $\hat{R}_i$ ,  $\forall i \in [1; \tilde{q}]$ , are identically zero.

**Theorem 5.7.** Consider two JLSS  $\Sigma = (A, B, C, D, E, R)$  and  $\hat{\Sigma} = (\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{E}, \hat{R})$  with  $p = \hat{p}$  and  $q = \hat{q}$ . Suppose that there exist matrices  $P, Q,$  and  $S$  satisfying (5.7) and that the abstract external input matrix  $\hat{B}$  is given by

$$\hat{B} = [\hat{P}B \ \hat{P}AG], \quad (5.11)$$

where  $\hat{P}$  and  $G$  are assumed to satisfy

$$C = \hat{C}\hat{P} \quad (5.12a)$$

$$I_n = P\hat{P} + GF \quad (5.12b)$$

$$I_{\hat{n}} = \hat{P}P \quad (5.12c)$$

$$0_{\hat{n} \times n} = \hat{P}EGF \quad (5.12d)$$

$$0_{\hat{n} \times n} = \hat{P}R_iGF, \quad \forall i \in [1; \tilde{q}], \quad (5.12e)$$

for some matrix  $F$ . Then, for every trajectory  $(\xi, \zeta, \nu, \omega)$  of  $\Sigma$  there exists a trajectory  $(\hat{\xi}, \hat{\zeta}, \hat{\nu}, \hat{\omega})$  of  $\hat{\Sigma}$  so that  $\zeta = \hat{\zeta}$  holds  $\mathbb{P}$ -a.s..

*Proof.* Let  $(\xi, \zeta, \nu, \omega)$  be a trajectory of  $\Sigma$ . We are going to show that  $(\hat{\xi}, \hat{\zeta}, \hat{\nu}, \omega)$  with

$$\hat{\zeta} = \zeta, \quad \hat{\xi} = \hat{P}\xi, \quad \text{and} \quad \hat{\nu} = \begin{bmatrix} \nu - Q\hat{P}\xi - S\omega \\ F\xi \end{bmatrix},$$

$\mathbb{P}$ -a.s. is a trajectory of  $\hat{\Sigma}$ . We use (5.7d), (5.7e), (5.12b), (5.12c), (5.12d), and (5.12e) and derive

$$\begin{aligned} d\hat{P}\xi &= (\hat{P}A\xi + \hat{P}B\nu + \hat{P}D\omega) dt + \hat{P}E\xi dW_t + \sum_{i=1}^{\tilde{q}} \hat{P}R_i\xi dP_t^i \\ &= (\hat{P}A\hat{P}\xi + \hat{P}A(I_n - P\hat{P})\xi + \hat{P}B\nu + \hat{P}D\omega) dt + \hat{P}E(P\hat{P} + GF)\xi dW_t + \sum_{i=1}^{\tilde{q}} \hat{P}R_i(P\hat{P} + GF)\xi dP_t^i \\ &= (\hat{P}A\hat{P}\xi + \hat{P}AGF\xi + \hat{P}B\nu + \hat{P}D\omega) dt + \hat{P}E\hat{P}\xi dW_t + \sum_{i=1}^{\tilde{q}} \hat{P}R_i\hat{P}\xi dP_t^i \\ &= (\hat{P}A\hat{P}\xi + \hat{P}AGF\xi + \hat{P}B\nu + \hat{P}D\omega) dt + \hat{E}\hat{P}\xi dW_t + \sum_{i=1}^{\tilde{q}} \hat{R}_i\hat{P}\xi dP_t^i. \end{aligned}$$

Now we use the equations (5.7a) and (5.7b) and the definition of  $\hat{B}$  and  $\hat{\nu}$  to derive

$$\begin{aligned} d\hat{P}\xi &= (\hat{P}(P\hat{A} - BQ)\hat{P}\xi + \hat{P}AGF\xi + \hat{P}B\nu + \hat{P}(P\hat{D} - BS)\omega) dt + \hat{E}\hat{P}\xi dW_t + \sum_{i=1}^{\tilde{q}} \hat{R}_i\hat{P}\xi dP_t^i \\ &= (\hat{A}\hat{P}\xi + [\hat{P}B \ \hat{P}AG]\hat{\nu} + \hat{D}\omega) dt + \hat{E}\hat{P}\xi dW_t + \sum_{i=1}^{\tilde{q}} \hat{R}_i\hat{P}\xi dP_t^i \\ &= (\hat{A}\hat{P}\xi + \hat{B}\hat{\nu} + \hat{D}\omega) dt + \hat{E}\hat{P}\xi dW_t + \sum_{i=1}^{\tilde{q}} \hat{R}_i\hat{P}\xi dP_t^i \end{aligned}$$

showing that  $(\hat{P}\xi, \hat{\zeta}, \hat{\nu}, \omega)$  is a trajectory of  $\hat{\Sigma}$ . From  $C = \hat{C}\hat{P}$  in (5.12a), it follows that  $\hat{\zeta} = \zeta$   $\mathbb{P}$ -a.s. which concludes the proof.  $\square$

**5.2. Construction of abstractions.** In this subsection, we provide constructive methods to compute the abstraction  $\hat{\Sigma}$  along with the various matrices involved in the definition of the stochastic simulation function and its corresponding interface function.

First, let us recall Lemma 2 in [GP09], showing that there exist matrices  $\hat{A}$  and  $Q$  satisfying (5.7a) if and only if columns of  $P$  span an  $(A, B)$ -controlled invariant subspace, see e.g. [BM92, Definition 4.1.1].

**Lemma 5.8.** *Consider matrices  $A$ ,  $B$ , and  $P$ . There exist matrices  $\hat{A}$  and  $Q$  satisfying (5.7a) if and only if*

$$\text{im } AP \subseteq \text{im } P + \text{im } B. \quad (5.13)$$

Given that  $P$  satisfies (5.13), it is straightforward to compute  $\hat{A}$  and  $Q$  such that (5.7a) holds, by solving  $\hat{n}$  linear equations.

Similar to Lemma 5.8, we give necessary and sufficient conditions for the existence of matrices  $\hat{D}$  and  $S$  appearing in condition (5.7b).

**Lemma 5.9.** *Given  $P$  and  $B$ , there exist matrices  $\hat{D}$  and  $S$  satisfying (5.7b) if and only if*

$$\text{im } D \subseteq \text{im } P + \text{im } B. \quad (5.14)$$

The proof of Lemma 5.9 is provided in the Appendix.

Now we provide necessary and sufficient conditions for the existence of matrices  $\hat{E}$  and  $\hat{R}_i, \forall i \in [1; \tilde{q}]$ , appearing in conditions (5.7d) and (5.7e).

**Lemma 5.10.** *Given  $P$  and  $E$ , there exists a matrix  $\hat{E}$  satisfying (5.7d) if and only if*

$$\text{im } EP \subseteq \text{im } P. \quad (5.15)$$

The proof is recovered from the one of Lemma 5.8 by substituting  $A$ ,  $\hat{A}$ , and  $B$  with  $E$ ,  $\hat{E}$ , and  $0_{n \times m}$ , respectively.

**Lemma 5.11.** *Given  $P$  and  $R_i, \forall i \in [1; \tilde{q}]$ , there exist matrices  $\hat{R}_i, \forall i \in [1; \tilde{q}]$ , satisfying (5.7e) if and only if*

$$\text{im } R_i P \subseteq \text{im } P, \quad (5.16)$$

for any  $i \in [1; \tilde{q}]$ .

The proof is recovered from the one of Lemma 5.8 by substituting  $A$ ,  $\hat{A}$ , and  $B$  with  $R_i, \hat{R}_i, \forall i \in [1; \tilde{q}]$ , and  $0_{n \times m}$ , respectively.

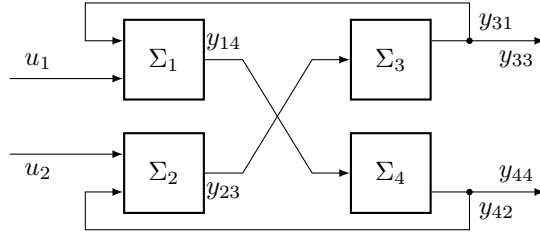
Lemmas 5.8, 5.9, 5.10, and 5.11 provide necessary and sufficient conditions on  $P$  which lead to the construction of matrices  $\hat{A}$ ,  $\hat{D}$ ,  $\hat{E}$ , and  $\hat{R}_i, \forall i \in [1; \tilde{q}]$ , together with the matrices  $Q$ ,  $S$  appearing in the definition of the interface function in (5.9). The output matrix  $\hat{C}$  simply follows by  $\hat{C} = CP$ . As we already discussed, the abstract external input matrix can be chosen arbitrarily. For example one can choose  $\hat{B} = I_{\hat{n}}$  making the abstract system  $\hat{\Sigma}$  fully actuated and, hence, the synthesis problem over it much simpler. One can also choose  $\hat{B}$  as in (5.11) guaranteeing preservation of all behaviors of  $\Sigma$  on  $\hat{\Sigma}$  under extra conditions in (5.12). Lemma 3 in [GP09], as recalled next, provides necessary and sufficient conditions on  $P$  and  $C$  for the existence of  $\hat{P}$ ,  $G$ , and  $F$  satisfying (5.12a), (5.12b), and (5.12c).

**Lemma 5.12.** *Consider matrices  $C$  and  $P$  with  $P$  being injective and let  $\hat{C} = CP$ . There exists matrix  $\hat{P}$  satisfying (5.12a), (5.12b), and (5.12c), for some matrices  $G$  and  $F$  of appropriate dimensions, if and only if*

$$\text{im } P + \ker C = \mathbb{R}^n. \quad (5.17)$$

The conditions (5.13)-(5.16) (resp. (5.13)-(5.17)) complete the characterization of matrix  $P$ , together with the system matrices  $\{A, B, C, D\}$  leading to the abstract matrices  $\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\}$ , where  $\hat{B}$  can be chosen arbitrarily (resp.  $\hat{B}$  is computed as in (5.11) for the sake of preservation of all behaviors of  $\Sigma$  on  $\hat{\Sigma}$  as long as conditions (5.12d) and (5.12e) are also satisfied). Note that there always exists an injective matrix  $P \in \mathbb{R}^{n \times \hat{n}}$  that satisfies the conditions (5.13)-(5.17). In the worst-case scenario, we can pick the identity matrix with  $\hat{n} = n$ .



FIGURE 4. The interconnected system  $\Sigma = \mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ .

Of course, we would like to have the abstraction  $\hat{\Sigma}$  as simple as possible and, therefore, we should aim at a  $P$  with  $\hat{n}$  as small as possible.

We summarize the construction of the abstraction  $\hat{\Sigma}$  in Table 1.

- |   |
|---|
| <ol style="list-style-type: none"> <li>1. Compute matrices <math>M</math> and <math>K</math> satisfying (5.2) and (5.3).</li> <li>2. Pick an injective <math>P</math> satisfying (5.13)-(5.16) (resp. (5.13)-(5.17) only if the computed matrices <math>\hat{P}</math>, <math>G</math>, and <math>F</math> satisfy (5.12d) and (5.12e));</li> <li>3. Compute <math>\hat{A}</math> and <math>Q</math> from (5.7a);</li> <li>4. Compute <math>\hat{D}</math> and <math>S</math> from (5.7b);</li> <li>5. Compute <math>\hat{C} = CP</math>;</li> <li>6. Choose <math>\hat{B}</math> arbitrarily (resp. <math>\hat{B} = [\hat{P}B \ \hat{P}AG]</math>);</li> <li>7. Compute <math>\tilde{R}</math>, appearing in (5.9), from (5.10);</li> <li>8. Compute <math>\hat{E}</math> from (5.7d) (resp. <math>\hat{E} = \hat{P}EP</math>);</li> <li>9. For any <math>i \in [1; \hat{q}]</math>, compute <math>\hat{R}_i</math> from (5.7e) (resp. <math>\hat{R}_i = \hat{P}R_iP</math>).</li> </ol> |
|---|

TABLE 1. Construction of an abstract JLSS  $\hat{\Sigma}$  for a given JLSS  $\Sigma$ .

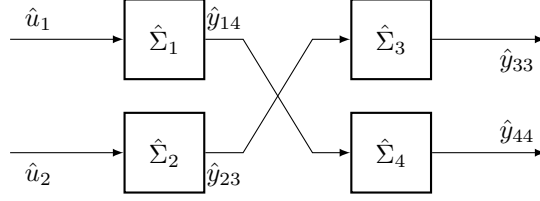
## 6. AN EXAMPLE

Let us demonstrate the effectiveness of the proposed results by synthesizing a controller for an interconnected system consisting of four JLSS  $\Sigma = \mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ . The interconnection scheme of  $\Sigma$  is illustrated in Figure 4. The system has two outputs and we synthesize a controller to enforce them to stay approximately (in the 2nd moment metric) within the safety constraint

$$S = [0 \ 5] \times [0 \ 5].$$

We refer the interested readers to the explanation provided before [ZMM<sup>+</sup>14, Remark 5.5] or to [ZTA14, Subsection 5.1] concerning the interpretation of the satisfaction of a safety constraint in the moment over the concrete stochastic systems.

In designing a controller for  $\Sigma$  we proceed as follows. In the first step, we compute abstractions  $\hat{\Sigma}_i$  of the individual subsystems to obtain an abstraction  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3, \hat{\Sigma}_4)$  of the interconnected system  $\Sigma$ . The interconnection scheme changes for  $\hat{\Sigma}$  (see Remark 5.6) and the abstract system is given by two identical independent interconnected systems  $\hat{\Sigma}_{14} = \mathcal{I}(\hat{\Sigma}_1, \hat{\Sigma}_4)$  and  $\hat{\Sigma}_{23} = \mathcal{I}(\hat{\Sigma}_2, \hat{\Sigma}_3)$ . The abstract system  $\hat{\Sigma}$  is illustrated in Figure 5. In the second step, we determinize the stochastic systems  $\hat{\Sigma}_{14}$  and  $\hat{\Sigma}_{23}$  by neglecting the diffusion and reset terms. We obtain two identical deterministic control systems  $\tilde{\Sigma}_{14}$  and  $\tilde{\Sigma}_{23}$ . We show that  $\tilde{\Sigma}_i$  is an abstraction of  $\hat{\Sigma}_i$ ,  $i \in \{14, 23\}$  by computing an SSF-M<sub>2</sub> function from  $\tilde{\Sigma}_i$  to  $\hat{\Sigma}_i$ . In the third step, we fix a sampling time  $\tau > 0$  and use the MATLAB Toolbox MPT [HKJM13] to synthesize a safety controller that enforces the safety constraints on  $\tilde{\Sigma} = \mathcal{I}(\tilde{\Sigma}_{14}, \tilde{\Sigma}_{23})$  at all sampling times  $k\tau$ ,  $k \in \mathbb{N}$ . In the final step, we refine

FIGURE 5. The abstract interconnected system  $\hat{\Sigma} = \mathcal{I}(\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3, \hat{\Sigma}_4)$ .

the computed controller for  $\tilde{\Sigma}$  to a controller for  $\Sigma$ . We use Theorem 3.5 to establish a bound on the distance in the 2nd moment metric between the output trajectories of  $\Sigma$  and the safe set  $S$ .

**6.1. The interconnected system.** Let us consider the system illustrated in Figure 4. The subsystems  $\Sigma_1$  and  $\Sigma_2$  are double integrators and  $\Sigma_3$  and  $\Sigma_4$  are autonomous triple integrators. All systems are affected by a scalar Brownian motion and a Poisson process. For  $j \in \{1, 2\}$  the system matrices are given by

$$A_j = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, B_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_j^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, E_j = 0.4I_2, R_j = 0.1I_2,$$

and for  $i \in \{3, 4\}$  by

$$A_i = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix}, B_i = 0, C_i^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, E_i = 0.4I_3, R_i = 0.1I_3.$$

The rate of the Poisson process  $P_t$  is  $\lambda = 4.2$ . The output of  $\Sigma_1$  (resp.  $\Sigma_2$ ) is connected to the internal input of  $\Sigma_4$  (resp.  $\Sigma_3$ ) and the output of  $\Sigma_3$  ( $\Sigma_4$ ) connects to the internal input of  $\Sigma_1$  (resp.  $\Sigma_2$ ). The output functions  $h_{ij}(x_i) = C_{ij}x_i$  are determined by  $C_{ii} = C_{i(i-2)} = [1 \ 0 \ 0]$  for  $i \in \{3, 4\}$ ,  $C_{23} = C_{14} = [1 \ 0]$  and  $h_{ij} \equiv 0$  for the remaining  $i, j \in [1; 4]$ . Correspondingly, the internal input matrices are given by

$$D_{41} = D_{32} = \begin{bmatrix} 0 \\ -d \\ 5d \end{bmatrix}, D_{j(j+2)} = \begin{bmatrix} 0 \\ d \end{bmatrix}, d \neq 0, j \in \{1, 2\}.$$

Subsequently, we use  $C_1 = C_{14}$ ,  $C_2 = C_{23}$ ,  $C_i = C_{ii}$ ,  $i \in \{3, 4\}$ ,  $D_1 = D_{13}$ ,  $D_2 = D_{24}$ ,  $D_3 = D_{32}$ ,  $D_4 = D_{41}$ , and denote the JLSS by  $\Sigma_i = (A_i, B_i, C_i, D_i, E_i, R_i)$ .

**6.2. The abstract subsystems.** In order to construct an abstraction for  $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$  we construct an abstraction  $\hat{\Sigma}_i$  of each individual subsystem  $\Sigma_i$ ,  $i \in \{1, 2, 3, 4\}$ . We begin with  $i \in \{1, 2\}$  and follow the steps outlined in Table 1. First, we fix  $\hat{\kappa} = 3$  and solve an appropriate LMI (see Lemma 5.2) to determine the matrices  $M_i$  and  $K_i$  so that (5.2) and (5.3) hold. We obtain

$$M_i = \begin{bmatrix} 1.68 & 0.4 \\ 0.4 & 0.23 \end{bmatrix}, K_i^T = \begin{bmatrix} -9 \\ -4 \end{bmatrix}.$$

We continue with step 2. and determine

$$P_i^T = [1 \ -2],$$

so that (5.13)-(5.17) hold. The matrices  $\hat{P}_i$ ,  $F_i$ , and  $G_i$  that (5.12b)-(5.12e) hold, follow by  $\hat{P}_i = [1 \ 0]$ ,  $G_i^T = [0 \ 2]$ , and  $F_i = [1 \ 0]$ . We continue with steps 3.-8. and get the scalar abstract JLSS subsystems  $\hat{\Sigma}_i$ ,  $i \in \{1, 2\}$  with

$$\hat{A}_i = -2, \hat{B}_i = 1, \hat{D}_i = 0, \hat{C}_i = 1, \hat{E}_i = 0.4, \hat{R}_i = 0.1.$$

Simultaneously, we compute  $Q_i = 2$  and  $S_i = -d$ . As already discussed in Remark 5.6,  $D_i \in \text{im } B_i$  and we can choose  $\hat{D}_i = 0$ . It follows that the subsystems  $\hat{\Sigma}_i$ ,  $i \in \{1, 2\}$ , are not affected by internal inputs, which implies

that the interconnection between  $\Sigma_3$  (resp.  $\Sigma_4$ ) and  $\Sigma_1$  (resp.  $\Sigma_2$ ) is absent on the abstract interconnected system  $\hat{\Sigma}$ ; compare also Figure 4 and Figure 5.

We continue with the construction of  $\hat{\Sigma}_i$  for  $i \in \{3, 4\}$ . We repeat the procedure and obtain

$$M_i = \begin{bmatrix} 6.924 & 3.871 & 0.468 \\ 3.871 & 2.534 & 0.315 \\ 0.468 & 0.315 & 0.054 \end{bmatrix}, \quad K_i = 0.$$

In step 2., we compute

$$P_i^T = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -3 & 9 \end{bmatrix},$$

so that (5.13)-(5.17) hold. The equations (5.12b)-(5.12e) are satisfied by

$$\hat{P}_i = \frac{1}{6} \begin{bmatrix} 0 & -9 & -3 \\ 0 & 4 & 2 \end{bmatrix},$$

$G_i^T = [1 \ 0 \ 0]$ , and  $F_i = \frac{1}{6} [6 \ 5 \ 1]$ . We follow steps 3.-8. and get the 2D abstract JLSS subsystems  $\hat{\Sigma}_i$ ,  $i \in \{3, 4\}$ , where

$$\hat{A}_i = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \hat{B}_i = \begin{bmatrix} 12 \\ -8 \end{bmatrix}, \hat{D}_i = d \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \hat{C}_i = [1 \ 1],$$

with the diffusion and reset terms again given by  $\hat{E}_i = 0.4I_2$  and  $\hat{R}_i = 0.1I_2$ . Moreover,  $Q_i = 0$  and  $S_i = 0$ .

For all  $i \in \{1, 2, 3, 4\}$ , equations (5.2), (5.3), and (5.7) hold. Hence, Theorem 5.3 applies and we see that  $V_i(x_i, \hat{x}_i) = (x_i - P_i \hat{x}_i)^T M_i (x_i - P_i \hat{x}_i)$  is an SSF-M<sub>2</sub> function from  $\hat{\Sigma}_i$  to  $\Sigma_i$  for all  $i \in [1; 4]$ . Moreover, (5.12) holds and Theorem 5.7 implies that all the behaviors of  $\Sigma_i$  are preserved on  $\hat{\Sigma}_i$ . Following the proof of Theorem 5.3, we see that the interface function for  $i \in \{1, 2\}$  follows by (5.9) as

$$\nu_{i\hat{\nu}_i}(x_i, \hat{x}_i, \hat{u}_i, \hat{w}_i) = K_i(x_i - P_i \hat{x}_i) - 2\hat{x}_i - 2.5\hat{u}_i - d\hat{w}_i, \quad (6.1)$$

and  $\nu_{i\hat{\nu}_i} \equiv 0$  for  $i \in \{3, 4\}$ . Here we used (5.10) to compute  $\tilde{R}_i = -2.5$  for  $i \in \{1, 2\}$ . Although the internal input matrices for  $\Sigma_1$  and  $\Sigma_2$  are zero, the internal inputs  $\hat{w}_1 = \hat{y}_3$  and  $\hat{w}_2 = \hat{y}_4$  still appear in the interface function. As provided in the proof of Theorem 5.3 and by fixing  $\pi = 1$ , the  $\mathcal{K}_\infty$  functions for  $i \in \{1, 2\}$  and  $j \in \{3, 4\}$  are given by

$$\begin{aligned} \alpha_i(s) &= s, \quad \eta_i(s) = 2s, \quad \rho_{i\text{ext}}(s) = 0.16s, \quad \rho_{i\text{int}}(s) = 1.3d^2s, \\ \alpha_j(s) &= s, \quad \eta_j(s) = 2s, \quad \rho_{j\text{ext}}(s) = 150s, \quad \rho_{j\text{int}}(s) = 7.9d^2s, \end{aligned}$$

for any  $s \in \mathbb{R}_{\geq 0}$ .

**6.3. The interconnected abstraction.** We now proceed with Theorem 4.2 to construct a stochastic simulation function from  $\hat{\Sigma}$  to  $\Sigma$ . We start by checking the Assumption 1. Note that  $\rho_{i\text{int}}$  satisfies the triangle inequality and we use Remark 4.4 to see that Assumption 1 holds for  $\gamma_i(s) = s$ ,  $\tilde{\lambda}_i = 2$ , and  $\delta_{ij}$  are given by

$$\Delta = d^2 \begin{bmatrix} 0 & 0 & 1.3 & 0 \\ 0 & 0 & 0 & 1.3 \\ 0 & 7.9 & 0 & 0 \\ 7.9 & 0 & 0 & 0 \end{bmatrix}.$$

Additionally, we require the existence of a vector  $\mu \in \mathbb{R}_{\geq 0}^4$  satisfying (4.5), which is the case if and only if the spectral radius of  $\Delta$  is strictly less than one, i.e.,  $1/2\sqrt{1.3 \times 7.9}d^2 < 1$ , which holds for example for  $d = 1/2$ . One can choose the vector  $\mu$  as  $\mu = [2 \ 2 \ 1 \ 1]$  and, hence, it follows that

$$V(x, \hat{x}) = \sum_{i=1}^2 2V_i(x_i, \hat{x}_i) + \sum_{i=3}^4 V_i(x_i, \hat{x}_i),$$

is an SSF-M<sub>2</sub> from  $\mathcal{I}(\hat{\Sigma}_1, \hat{\Sigma}_2, \hat{\Sigma}_3, \hat{\Sigma}_4)$  to  $\mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$  where the interface function follows from (6.1). Following the proof of Theorem 4.2, we see that  $V$  satisfies (3.2) with  $\alpha(s) = s$  and (3.3) with  $\eta(s) = 1.35s$ ,  $\rho_{\text{ext}}(s) = 150s$ , and  $\rho_{\text{int}} \equiv 0$ . Here, we computed  $\eta$  and  $\rho_{\text{ext}}$  according to (4.8a) and (4.8b). Subsequently, we design a controller for  $\Sigma$  via the abstraction  $\hat{\Sigma}$ . We restrict external inputs for  $\hat{\Sigma}_3$  and  $\hat{\Sigma}_4$  to zero, so that we can set  $\rho_{j\text{ext}} \equiv 0$ ,  $j \in \{3, 4\}$ . As a result  $\rho_{\text{ext}}$  reduces to  $\rho_{\text{ext}}(s) = 0.16s$ ,  $\forall s \in \mathbb{R}_{\geq 0}$ , and we use Theorem 3.5 in combination with Remark 5.4 to derive the inequality

$$\mathbb{E}[\|\zeta_{av}(t) - \hat{\zeta}_{\hat{a}\hat{v}}(t)\|^2] \leq e^{-1.35t} \mathbb{E}[V(a, \hat{a})] + 0.12 \mathbb{E}[\|\tilde{\nu}\|_{\infty}^2]. \quad (6.2)$$

**6.4. The deterministic system and the controller.** The synthesis of the safety controller is based on a deterministic system  $\tilde{\Sigma}$  which results from  $\hat{\Sigma}$  by omitting the diffusion and reset terms. In particular, we determinize the identical systems  $\hat{\Sigma}_{14} = \mathcal{I}(\hat{\Sigma}_1, \hat{\Sigma}_4)$  and  $\hat{\Sigma}_{23} = \mathcal{I}(\hat{\Sigma}_2, \hat{\Sigma}_3)$  and obtain for  $i \in \{14, 23\}$  the systems

$$\tilde{\Sigma}_i : \begin{cases} \dot{\tilde{\xi}}_i(t) = \begin{bmatrix} -2 & 0 & -d \\ 0 & -3 & d \\ 0 & 0 & -2 \end{bmatrix} \tilde{\xi}_i(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tilde{\nu}_i(t), \\ \tilde{\zeta}_i(t) = [1 \quad 1 \quad 0] \tilde{\xi}_i(t). \end{cases}$$

We compute an SSF-M<sub>2</sub> function  $\hat{V}(\hat{x}, \hat{x}) = [\hat{x}; \hat{x}]^T \hat{M} [\hat{x}; \hat{x}]$  from  $\tilde{\Sigma} = \mathcal{I}(\tilde{\Sigma}_{14}, \tilde{\Sigma}_{23})$  to  $\hat{\Sigma}$ , by solving an appropriate LMI. The matrix  $\hat{M}$  results in

$$\hat{M} = \begin{bmatrix} m_1 & 0 & -m_2 & 0 \\ 0 & m_1 & 0 & -m_2 \\ -m_2^T & 0 & m_3 & 0 \\ 0 & -m_2^T & 0 & m_3 \end{bmatrix}$$

with

$$m_1 = \begin{bmatrix} 1.1400 & 1.3072 & 0.0052 \\ 1.3072 & 1.6968 & 0.0228 \\ 0.0052 & 0.0228 & 0.0104 \end{bmatrix}, \quad m_2 = \begin{bmatrix} 1.1437 & 1.3112 & 0.0060 \\ 1.3365 & 1.7181 & 0.0218 \\ 0.0089 & 0.0230 & 0.0085 \end{bmatrix}, \quad m_3 = \begin{bmatrix} 1.1793 & 1.3649 & 0.0081 \\ 1.3649 & 1.7631 & 0.0224 \\ 0.0081 & 0.0224 & 0.0079 \end{bmatrix}.$$

The associated  $\mathcal{K}_{\infty}$  functions for  $\hat{V}$  are given by  $\alpha(r) = r$ ,  $\eta(r) = 0.82r$ ,  $\rho_{\text{ext}}(r) = 0.32r$ , and  $\rho_{\text{int}} \equiv 0$ . Again we use Theorem 3.5 and Remark 5.4 to establish

$$\mathbb{E}[\|\hat{\zeta}_{\hat{a}\hat{v}}(t) - \tilde{\zeta}_{\hat{a}\hat{v}}(t)\|^2] \leq e^{-0.82t} \mathbb{E}[\hat{V}(\hat{a}, \hat{a})] + 0.4 \|\tilde{\nu}\|_{\infty}^2. \quad (6.3)$$

Next we design a safety controller to restrict the output  $\tilde{y} \in \mathbb{R}$  of  $\tilde{\Sigma}_i$ ,  $i \in \{14, 23\}$  to  $[0 \ 5]$ . Additionally, to control the mismatch between the trajectories of  $\Sigma$  and  $\tilde{\Sigma}$ , we limit the inputs to  $\tilde{u} \in [-1 \ 1]^2$ . We fix the sampling time to  $\tau = 0.1$  secs and use the MATLAB Toolbox MPT [HKJM13] to compute a safety controller  $K : \mathbb{R}^6 \rightarrow 2^{[-1 \ 1]^2}$ , which when applied in a sample-and-hold manner to  $\tilde{\Sigma}$  enforces the constraints at the sampling instances  $t = k\tau$ ,  $k \in \mathbb{N}$ . A part of the domain of the controller, which restricts the initial states of  $\tilde{\Sigma}$  is illustrated in Figure 6. Note that  $K$  is a set-valued map that provides, for each state  $\tilde{x}$  in the domain of  $K$ , possibly a set of admissible inputs  $K(\tilde{x}) \subseteq [-1 \ 1]^2$ .

**6.5. Input trajectory generation and performance guarantees.** We use the closed-loop system consisting of  $\tilde{\Sigma}$  and  $K$  to generate input trajectories for  $\Sigma$ . Let  $(\tilde{\xi}, \tilde{\zeta}, \tilde{\nu})$  be a trajectory of  $\tilde{\Sigma}$  that satisfies  $K$ , i.e.,  $\tilde{\nu}$  is constant on the intervals  $\tau[k, (k+1)[$ ,  $k \in \mathbb{N}$ , and satisfies  $\tilde{\nu}(k\tau) \in K(\tilde{\xi}(k\tau))$  for all  $k \in \mathbb{N}$ . We use the interface (6.1) to compute the input trajectory  $\nu$  for  $\Sigma$ . Using the bounds in (6.2) and (6.3), the overall estimate between output trajectories of  $\tilde{\Sigma}$  and  $\Sigma$  follows to

$$\begin{aligned} \left( \mathbb{E}[\|\zeta_{av}(t) - \tilde{\zeta}_{\hat{a}\hat{v}}(t)\|^2] \right)^{\frac{1}{2}} &\leq \left( \mathbb{E}[\|\zeta_{av}(t) - \hat{\zeta}_{\hat{a}\hat{v}}(t)\|^2] \right)^{\frac{1}{2}} + \left( \mathbb{E}[\|\hat{\zeta}_{\hat{a}\hat{v}}(t) - \tilde{\zeta}_{\hat{a}\hat{v}}(t)\|^2] \right)^{\frac{1}{2}} \\ &\leq e^{-0.67t} \mathbb{E}[V(a, \hat{a})]^{\frac{1}{2}} + e^{-0.41t} \mathbb{E}[\hat{V}(\hat{a}, \hat{a})]^{\frac{1}{2}} + \|\tilde{\nu}\|_{\infty}. \end{aligned} \quad (6.4)$$

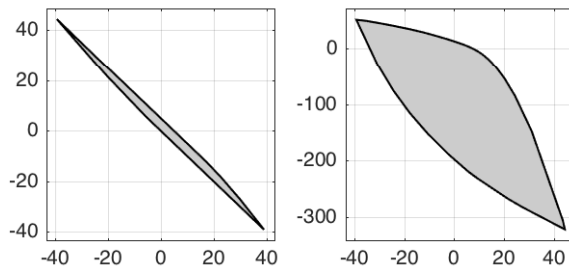


FIGURE 6. Part of the domain of the safety controller. The left figure shows the projection on  $\tilde{x}_1$  and  $\tilde{x}_2$ . The right figure shows the projection on  $\tilde{x}_2$  and  $\tilde{x}_3$ .

We show some simulation results of the controlled system in Figure 7. The initial state of  $\Sigma$  is fixed as  $a = [1; -1; -5; 1; -1; -5; 1; -2; 1; -2]$ . We determine the initial state for  $\hat{\Sigma}$  as well as  $\tilde{\Sigma}$  as the vector  $\tilde{a} \in \mathbb{R}^6$  lying in the domain of the controller and minimizing  $V(a, \tilde{a})$  which is  $\tilde{a} = [1.44; -0.69; 1.44; -0.69; 1; 1]$ . We randomly pick the input  $\tilde{\nu}(k\tau)$  in  $K(\tilde{\xi}(k\tau))$ . In the top two plots of the figure, we see a realization of the observed process  $\zeta_1$  (resp.  $\zeta_2$ ) and  $\hat{\zeta}_1$  (resp.  $\hat{\zeta}_2$ ) of  $\Sigma$  and  $\hat{\Sigma}$ , respectively. On the middle plot, we show the corresponding evolutions of the refined input signals  $\nu_1$  and  $\nu_2$  for  $\Sigma$ . On the 2nd plot from bottom, we show the square root of the average value (over 1000 experiments) of the squared distance in time of the output trajectory of  $\Sigma$  to the one of  $\hat{\Sigma}$ , namely,  $\|\zeta_{av} - \hat{\zeta}_{av}\|^2$ . The solid black curve denotes the error bound given by the right-hand-side of (6.2). On the bottom part, we show the square root of the average value (over 1000 experiments) of the squared distance in time of the output trajectory of  $\Sigma$  to the set  $S$ , namely,  $\|\zeta_{av}(t)\|_S$ . Notice that the square root of this empirical (averaged) squared distances is significantly lower than the computed bound given by the right-hand-side of (6.4), as expected since the stochastic simulation functions can lead to conservative bounds. (One can improve the bounds by seeking optimized stochastic simulation functions.)

## 7. SUMMARY

In this paper we proposed a compositional framework for the construction of infinite approximations of interconnected stochastic hybrid systems by leveraging some small-gain type conditions. We introduced a new notion of stochastic simulation functions to quantify the error between the stochastic hybrid systems and their approximations. In comparison with the similar notion in [JP09], our proposed notion of stochastic simulation functions is computationally more tractable for stochastic hybrid systems with inputs. Moreover, we provided a constructive approach on the construction of those infinite approximations for a class of stochastic hybrid systems, namely, jump linear stochastic systems. Finally, we illustrated the effectiveness of the results by constructing an infinite approximation of an interconnection of four jump linear stochastic systems in a compositional manner. We employed the constructed approximation as a substitute in the controller synthesis scheme to enforce a safety constraint on the concrete interconnected system, would not have been possible to enforce without the use of the approximation.

## APPENDIX

*Proof of Lemma 3.6.* Lemma 3.6 is an extension of Lemma 4.4 in [LSW96] and the proof follows similar ideas. The proof includes two steps. We first show that the set  $[0, s_0]$ ,  $s_0 := \eta^{-1}(2g)$ , is forward invariant, i.e., if  $y(t_0) \in [0, s_0]$ , then  $y(t) \in [0, s_0]$  for all  $t \geq t_0$ . For the sake of contradiction, suppose the trajectory  $y$  visits  $[0, s_0]$  and then later leaves it. Due to the continuity of  $y$ , this implies that there exist a time instance  $t > t_0$  and positive value  $\varepsilon > 0$  such that  $y(t_0) = s_0$  and  $y(t) = s_0 + \varepsilon$ , and  $y(\tau) \geq s_0$  for all  $\tau \in [t_0, t]$ . In view of the

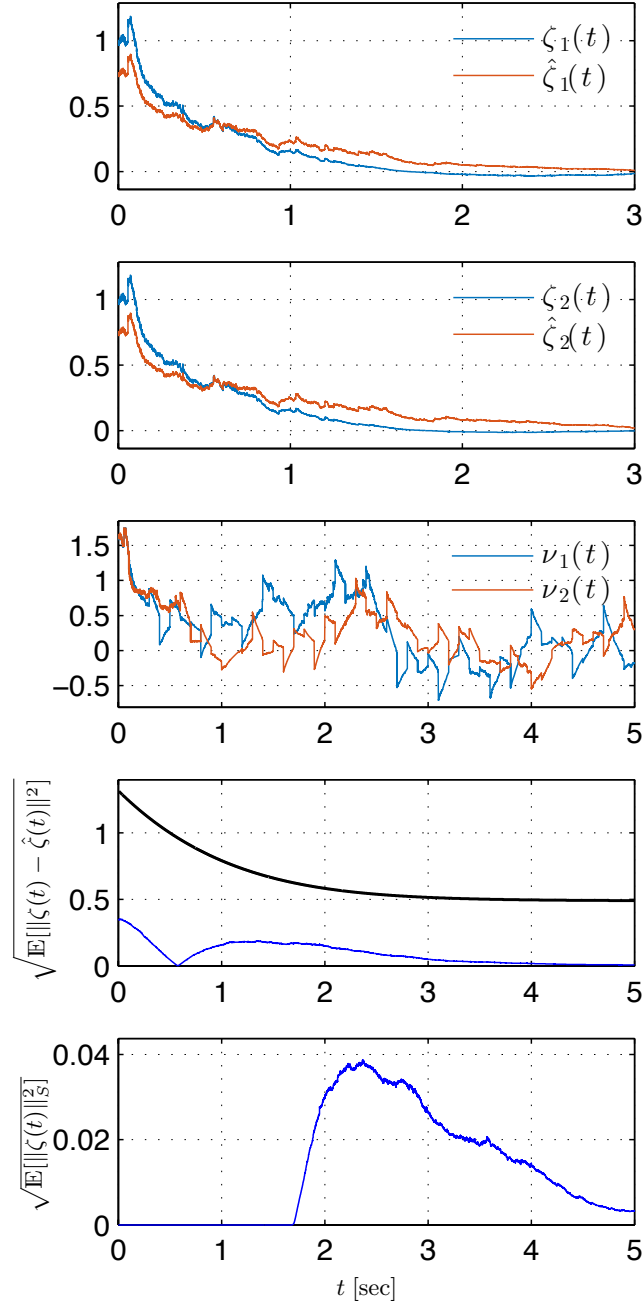


FIGURE 7. Top two plots: One realization of  $\zeta_1$  (resp.  $\zeta_2$ ) (—) and  $\hat{\zeta}_1$  (resp.  $\hat{\zeta}_2$ ) (—). The middle plot: the corresponding realization of external inputs  $\nu_1$  (—) and  $\nu_2$  (—) of  $\Sigma$ . The 2nd plot from bottom: Square root of the average values (over 1000 experiments) of the squared distance of the output trajectory of  $\Sigma$  to the one of  $\hat{\Sigma}$ . The solid black line indicates the error bound given by the right-hand-side of (6.2). Bottom plot: Square root of the average values (over 1000 experiments) of the squared distance of the output trajectory of  $\Sigma$  to the safe set  $S$ .

lemma hypothesis, we then have

$$0 < \varepsilon = y(t) - y(t_0) \leq \int_{t_0}^t [-\eta(y(\tau)) + g] \leq 0,$$

which concludes the first step. In the second step, we assume that  $y(0) > s_0$ . Consider the function  $\kappa : \mathbb{R}_{>0} \rightarrow \mathbb{R}$  defined as

$$\kappa(s) := \int_1^s \frac{-d r}{\min\{\eta(r), r\}}.$$

Let  $t_s$  be the first time that the process  $y$  reaches  $s_0$ , i.e.,  $t_s := \inf\{t \geq 0 : y(t) \leq s_0\}$ .<sup>5</sup> In the following we show that the function

$$\vartheta(r, t) := \begin{cases} \kappa^{-1}(\kappa(r) + t/2), & r > 0 \\ 0 & r = 0, \end{cases} \quad (7.1)$$

is indeed the desired  $\mathcal{KL}$  function for the lemma assertion. Note that for all  $t \in [0, t_s]$ , we have  $\eta(y(t)) \geq 2g$ , and that we have

$$\kappa(y(t)) - \kappa(y(0)) = \int_{y(0)}^{y(t)} \frac{-d(y(\tau))}{\min\{\eta(y(\tau)), y(\tau)\}} \geq \int_0^t \frac{\eta(y(\tau)) - g}{\min\{\eta(y(\tau)), y(\tau)\}} d\tau \geq \int_0^t \frac{\frac{1}{2}\eta(y(\tau))}{\eta(y(\tau))} d\tau \geq \frac{t}{2}.$$

The above observation together with the fact that the function  $\kappa$  is strictly decreasing on  $(0, \infty)$  imply that

$$y(t) \leq \kappa^{-1}(\kappa(y(0)) + t/2), \quad \forall t \in [0, t_s].$$

Note that  $\lim_{s \downarrow 0} \kappa(s) = \infty$ , and since  $\kappa$  is strictly decreasing on  $(0, \infty)$ , the function  $\vartheta(r, t)$  defined in (7.1) is a  $\mathcal{K}_\infty$  function in the first argument for each  $t$ , and decreasing with respect to the second argument for each nonzero  $r$ . As such, the function  $\vartheta(r, t)$  is a  $\mathcal{KL}$  function. Combining the results of the two steps concerning the intervals  $[0, t_s]$  and  $(t_s, \infty)$  concludes the desired assertion.  $\square$

*Proof of Theorem 3.5.* For any time instances  $t \geq t_0 \geq 0$ , any  $\hat{\nu}(t) \in \mathbb{R}^{\hat{m}}$ , any  $\hat{\omega}(t) \in \mathbb{R}^p$ , and any random variable  $a$  and  $\hat{a}$  that are  $\mathcal{F}_0$ -measurable, there exists  $\nu(t) \in \mathbb{R}^m$  such that for all  $\omega(t) \in \mathbb{R}^p$ , one obtains

$$\begin{aligned} \mathbb{E} \left[ V(\xi_{a\nu\omega}(t), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t)) \right] &= \mathbb{E} \left[ V(\xi_{a\nu\omega}(t_0), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t_0)) + \int_{t_0}^t \mathcal{L}V(\xi_{a\nu\omega}(s), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(s)) ds \right] \\ &\leq \mathbb{E} \left[ V(\xi_{a\nu\omega}(t_0), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t_0)) \right] + \mathbb{E} \left[ \int_{t_0}^t -\eta \left( V(\xi_{a\nu\omega}(s), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(s)) \right) + \rho_{\text{ext}}(\|\hat{\nu}(s)\|^k) + \rho_{\text{int}}(\|\omega(s) - \hat{\omega}(s)\|^k) ds \right] \\ &\leq \mathbb{E} \left[ V(\xi_{a\nu\omega}(t_0), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t_0)) \right] + \int_{t_0}^t -\eta \left( \mathbb{E} \left[ V(\xi_{a\nu\omega}(s), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(s)) \right] \right) + \mathbb{E} \left[ \rho_{\text{ext}}(\|\hat{\nu}\|_\infty^k) + \rho_{\text{int}}(\|\omega - \hat{\omega}\|_\infty^k) \right] ds, \end{aligned}$$

where the first equality is an application of the Itô's formula for jump diffusions thanks to the polynomial rate of the function  $V$  [ØS05, Theorem 1.24], and the last inequality follows from Jensen's inequality due to the convexity assumption on the function  $\eta$  [Oks02, p. 310]. Let us define the process  $y(t) := \mathbb{E} \left[ V(\xi_{a\nu\omega}(t), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t)) \right]$ . Note that in view of the Itô's formula, the process  $y(\cdot)$  is continuous provided that the solution processes  $\xi_{a\nu\omega}$  and  $\hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}$  have finite moments. This is indeed the case under our model setting in Definition 2.1, in particular due to the Lipschitz continuity of functions  $f, \sigma, r, \hat{f}, \hat{\sigma}, \hat{r}$  [ØS05, 1.19]. Therefore, the process  $y(t)$  meets all the required assumptions of Lemma 3.6, implying that there exists a  $\mathcal{KL}$  function  $\vartheta$  such that

$$\mathbb{E} \left[ V(\xi_{a\nu\omega}(t), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t)) \right] \leq \vartheta(\mathbb{E} \left[ V(a, \hat{a}) \right], t) + \eta^{-1} \left( 2\mathbb{E} \left[ \rho_{\text{ext}}(\|\hat{\nu}\|_\infty^k) + \rho_{\text{int}}(\|\omega - \hat{\omega}\|_\infty^k) \right] \right). \quad (7.2)$$

<sup>5</sup>By convention,  $\inf \emptyset = \infty$ .

In view of Jensen's inequity and using equation (3.2), the convexity of  $\alpha$  and the concavity of  $\rho_{\text{ext}}, \rho_{\text{int}}$ , we have

$$\begin{aligned} \alpha\left(\mathbb{E}\left[\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\|^k\right]\right) &\leq \mathbb{E}\left[\alpha\left(\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\|^k\right)\right] \leq \mathbb{E}\left[V(\xi_{a\nu\omega}(t), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t))\right] \\ &\leq \vartheta(\mathbb{E}[V(a, \hat{a})], t) + \eta^{-1}\left(2\rho_{\text{ext}}(\mathbb{E}[\|\hat{\nu}\|_{\infty}^k]) + 2\rho_{\text{int}}(\mathbb{E}[\|\omega - \hat{\omega}\|_{\infty}^k])\right), \end{aligned}$$

which in conjunction with the fact that  $\alpha \in \mathcal{K}_{\infty}$  leads to

$$\begin{aligned} \mathbb{E}\left[\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\|^k\right] &\leq \alpha^{-1}\left(\vartheta(\mathbb{E}[V(a, \hat{a})], t) + \eta^{-1}\left(2\rho_{\text{ext}}(\mathbb{E}[\|\hat{\nu}\|_{\infty}^k]) + 2\rho_{\text{int}}(\mathbb{E}[\|\omega - \hat{\omega}\|_{\infty}^k])\right)\right) \\ &\leq \alpha^{-1}\left(2\vartheta(\mathbb{E}[V(a, \hat{a})], t)\right) + \alpha^{-1}\left(2\eta^{-1}\left(4\rho_{\text{ext}}(\mathbb{E}[\|\hat{\nu}\|_{\infty}^k])\right)\right) + \alpha^{-1}\left(2\eta^{-1}\left(4\rho_{\text{int}}(\mathbb{E}[\|\omega - \hat{\omega}\|_{\infty}^k])\right)\right). \end{aligned}$$

Therefore, by introducing functions  $\beta$ ,  $\gamma_{\text{ext}}$ , and  $\gamma_{\text{int}}$  as

$$\begin{aligned} \beta(r, t) &:= \alpha^{-1}(2\vartheta(r, t)), \\ \gamma_{\text{ext}}(r) &:= \alpha^{-1}\left(2\eta^{-1}\left(4\rho_{\text{ext}}(r)\right)\right), \\ \gamma_{\text{int}}(r) &:= \alpha^{-1}\left(2\eta^{-1}\left(4\rho_{\text{int}}(r)\right)\right), \end{aligned} \tag{7.3}$$

inequality (3.4) is satisfied. Note that if  $\alpha^{-1}$  and  $\eta^{-1}$  satisfies the triangle inequality (i.e.,  $\alpha^{-1}(a+b) \leq \alpha^{-1}(a) + \alpha^{-1}(b)$  and  $\eta^{-1}(a+b) \leq \eta^{-1}(a) + \eta^{-1}(b)$  for all  $a, b \in \mathbb{R}_{\geq 0}$ ), one can divide all the coefficients by factor 2 in the expressions of  $\beta$ ,  $\gamma_{\text{ext}}$ , and  $\gamma_{\text{int}}$  in (7.3) to get a less conservative upper bound in (3.4).  $\square$

*Proof of Proposition 3.7.* Since  $V$  is an SSF- $M_k$  function from  $\hat{\Sigma}$  to  $\Sigma$  and  $\eta(r) \geq \theta r$  for some  $\theta \in \mathbb{R}_{>0}$  and any  $r \in \mathbb{R}_{\geq 0}$ , for any  $\hat{\nu} \in \hat{\mathcal{U}}$ , any  $\hat{\omega} \in \mathcal{W}$ , and any random variable  $a$  and  $\hat{a}$  that are  $\mathcal{F}_0$ -measurable, there exists  $\nu \in \mathcal{U}$  such that for all  $\omega \in \mathcal{W}$  one obtains:

$$\mathcal{L}V\left(\xi_{a\nu\omega}(t), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\right) \leq -\theta V\left(\xi_{a\nu\omega}(t), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\right) + \rho_{\text{ext}}(\|\hat{\nu}\|_{\infty}^k) + \rho_{\text{int}}(\|\omega - \hat{\omega}\|_{\infty}^k).$$

Since there exists a constant  $\epsilon \geq 0$  such that  $\epsilon \geq \rho_{\text{ext}}(\|\hat{\nu}\|_{\infty}^k) + \rho_{\text{int}}(\|\omega - \hat{\omega}\|_{\infty}^k)$ , one obtains:

$$\mathcal{L}V\left(\xi_{a\nu\omega}(t), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\right) \leq -\theta V\left(\xi_{a\nu\omega}(t), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\right) + \epsilon, \tag{7.4}$$

and the following chain of inequalities hold:

$$\begin{aligned} \mathbb{P}\left\{\sup_{0 \leq t \leq T} \left\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\right\| \geq \varepsilon \mid [a; \hat{a}]\right\} &= \mathbb{P}\left\{\sup_{0 \leq t \leq T} \alpha\left(\left\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\right\|^k\right) \geq \alpha(\varepsilon^k) \mid [a; \hat{a}]\right\} \\ &\leq \mathbb{P}\left\{\sup_{0 \leq t \leq T} V\left(\xi_{a\nu\omega}(t), \hat{\xi}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\right) \geq \alpha(\varepsilon^k) \mid [a; \hat{a}]\right\}. \end{aligned} \tag{7.5}$$

Using inequalities (7.4), (7.5), and Theorem 1 in [Kus67, Chapter III], one obtains the inequalities (3.5) and (3.6).  $\square$

*Proof of Proposition 3.8.* The proof is a simple consequence of Theorem 3.5 and Markov inequality [Oks02], used as the following:

$$\begin{aligned} \mathbb{P}\left\{\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\| \geq \varepsilon\right\} &\leq \frac{\mathbb{E}\left[\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\|\right]}{\varepsilon} \leq \frac{\left(\mathbb{E}\left[\|\zeta_{a\nu\omega}(t) - \hat{\zeta}_{\hat{a}\hat{\nu}\hat{\omega}}(t)\|^k\right]\right)^{\frac{1}{k}}}{\varepsilon} \\ &\leq \frac{(\beta(\mathbb{E}[V(a, \hat{a})], t) + \gamma_{\text{ext}}(\mathbb{E}[\|\hat{\nu}\|_{\infty}^k]) + \gamma_{\text{int}}(\mathbb{E}[\|\omega - \hat{\omega}\|_{\infty}^k]))^{\frac{1}{k}}}{\varepsilon}. \end{aligned} \tag{7.6}$$

$\square$



*Proof of Corollary 3.9.* Since  $V$  is an SSF- $M_k$  function from  $\hat{\Sigma}$  to  $\Sigma$ , for  $\hat{\nu} \equiv 0$  and any random variable  $a$  and  $\hat{a}$  that are  $\mathcal{F}_0$ -measurable, there exists  $\nu \in \mathcal{U}$  such that one obtains:

$$\mathcal{L}V\left(\xi_{a\nu}(t), \hat{\xi}_{\hat{a}0}(t)\right) \leq -\eta\left(V\left(\xi_{a\nu}(t), \hat{\xi}_{\hat{a}0}(t)\right)\right),$$

implying that  $V\left(\xi_{a\nu}(t), \hat{\xi}_{\hat{a}0}(t)\right)$  is a nonnegative supermartingale [Oks02, Appendix C]. As a result, we have the following chain of inequalities:

$$\begin{aligned} \mathbb{P}\left\{\sup_{0 \leq t < \infty} \left\|\zeta_{a\nu}(t) - \hat{\zeta}_{\hat{a}0}(t)\right\| > \varepsilon \mid [a; \hat{a}]\right\} &= \mathbb{P}\left\{\sup_{0 \leq t < \infty} \alpha\left(\left\|\zeta_{a\nu}(t) - \hat{\zeta}_{\hat{a}0}(t)\right\|^k\right) > \alpha(\varepsilon^k) \mid [a; \hat{a}]\right\} \\ &\leq \mathbb{P}\left\{\sup_{0 \leq t < \infty} V\left(\xi_{a\nu}(t), \hat{\xi}_{\hat{a}0}(t)\right) > \alpha(\varepsilon^k) \mid [a; \hat{a}]\right\} \leq \frac{V(a, \hat{a})}{\alpha(\varepsilon^k)}, \end{aligned}$$

where the last inequality is implied from  $V(\xi_{a\nu}(t), \hat{\xi}_{\hat{a}0}(t))$  being a nonnegative supermartingale and [Kus67, Lemma1].  $\square$

*Proof of Lemma 5.1.* Consider the jump linear stochastic system  $\Sigma$  with a linear feedback control law  $u = Kx$ , where  $K \in \mathbb{R}^{m \times n}$ , satisfying

$$d\xi(t) = (A + BK)\xi(t) dt + E\xi(t) dW_t + \sum_{i=1}^{\tilde{q}} R_i \xi(t) dP_t^i.$$

Define the matrix-valued deterministic process  $\Phi(t) := \mathbb{E}[\xi(t)\xi^T(t)]$ . Applying the Itô's formula for jump diffusions [ØS05] leads to the following differential equations describing the time-evolution of the deterministic process  $\Phi(t)$ :

$$\dot{\Phi}(t) = (A + BK + \sum_{i=1}^{\tilde{q}} \lambda_i R_i) \Phi(t) + \Phi(t) (A + BK + \sum_{i=1}^{\tilde{q}} \lambda_i R_i)^T + E\Phi(t)E^T + \sum_{i=1}^{\tilde{q}} \lambda_i R_i \Phi(t) R_i^T. \quad (7.7)$$

To see further details on how the above ODE is derived, one can view each element of the matrix  $\Phi(t)$  as an  $\mathbb{R}$ -valued mapping and treat it in the same way as we considered the Lyapunov function in the proof of Theorem 5.3, and consequently arrives at (7.7). From linear system theory, one can readily check that the ODE in (7.7) is asymptotically stable (implying  $\Sigma$  is mean square asymptotically stable) if and only if  $V(\Phi(t)) = \text{Tr}(M\Phi(t)) = \mathbb{E}[\xi(t)^T M \xi(t)]$  is a Lyapunov function for (7.7) for a positive definite matrix  $M$  satisfying condition (5.3), which completes the proof.  $\square$

*Proof of Lemma 5.9.* Suppose that  $\text{im } D \not\subseteq \text{im } P + \text{im } B$ , then there exists  $w \in \mathbb{R}^p$  so that  $Dw \neq P\hat{x} - Bu$  holds for all  $\hat{x} \in \mathbb{R}^{\hat{n}}$ ,  $u \in \mathbb{R}^m$ . Hence (5.7b) cannot hold for any matrix  $\hat{D}$  and  $S$ . Now suppose  $\text{im } D \subseteq \text{im } P + \text{im } B$ . Let  $e_i$  denote the columns of  $I_p$ . Then there exist  $\hat{d}_i \in \mathbb{R}^{\hat{n}}$  and  $s_i \in \mathbb{R}^m$  so that  $De_i = P\hat{d}_i - Bs_i$  holds for all  $i \in \{1, \dots, p\}$  and the matrices  $\hat{D} = [\hat{d}_1 \dots \hat{d}_p]$  and  $S = [s_1 \dots s_p]$  satisfy (5.7b).  $\square$

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<sup>1</sup>DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING, TECHNISCHE UNIVERSITÄT MÜNCHEN, D-80290 MUNICH, GERMANY.

E-mail address: {zamani,matthias.rungger}@tum.de

URL: <http://www.hcs.ei.tum.de>

<sup>2</sup>AUTOMATIC CONTROL LABORATORY AT ETH ZÜRICH AND THE RISK ANALYTICS AND OPTIMIZATION CHAIR AT ETH LAUSANNE, SWITZERLAND.

*E-mail address:* mohajerin@control.ee.ethz.ch

*URL:* <http://control.ee.ethz.ch/~peymanm>

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