TU Delft

Bachelor Thesis Frank van der Top

On the Theorem of De Rham

05-02-2024/24-06-2024

On the Theorem of De Rham

by

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to obtain the degree of Bachelor of Science
at the Delft University of Technology,
to be defended publicly on Monday June 24, 2024 at 14:15

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LAYMEN'S SUMMARY

In this thesis we look at the theorem of De Rham, which intuitively states that the number of holes in a surface corresponds to the failure of the fundamental theorem of calculus[1] on that surface. This theorem allows us to make analytical conclusions based on the geometry of a surface. In Chapter 1 this correspondence is explored through integration on manifolds.

The number of holes in a surface is measured using singular cohomology, while the failure of this theorem is captured using the De Rham cohomology. Both cohomologies are developed in Chapter 2. Locally these cohomologies coincide, thus showing the theorem of De Rham is locally true. This local isomorphism is extended globally using sheaves.

Sheaves, introduced in Chapter 3, allow for local information to be glued together into global information. However, some information, namely exactness, is lost in this process. This loss is quantified using sheaf cohomology as discussed in Chapter 4.

Finally, in Chapter 5 both the number of holes in a surface and the failure of the fundamental theorem of calculus are shown to coincide with the loss of information of sheaves, thus proving the theorem of De Rham.

[1] This theorem allows for easy evaluation of integrals on surfaces.

CONTENTS

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SUMMARY

As the title of this thesis suggests, in this thesis the theorem of De Rham is scrutinised. This foundational theorem provides a bridge between differential geometry and algebraic topology. In this thesis this bridge is built, providing a link between the geometry of a manifold and the failure of the fundamental theorem of calculus for closed differential forms.

Intuitively, differential forms are smooth volume functions for tangent parallelepipeds on a manifold. These are the volume functions used in integration over smooth manifolds. The famous Stokes' theorem from calculus can be generalised using differential forms. For certain differential forms this leads to a fundamental theorem of calculus, these forms are called exact.

Checking for exactness is often troublesome. As such, we want to check for other characteristics equivalent to exactness. As the exact forms have a vanishing derivative, we want to determine whether this is a sufficient characterisation of exactness. The differential forms with vanishing derivative are called closed.

Closed and not exact forms appear to only exist if the smooth manifold on which they are defined has geometric irregularities such as holes. As such, the number of holes of a smooth manifold has become a question of interest. Additionally, the existence of closed and not exact differential forms appears to have implications regarding the geometry of the smooth manifold on which they are defined. The differential forms and integration on smooth manifolds is studied in Chapter 1.

To determine how many holes a surface has and how many closed and not exact forms exist, we develop the theory of homological algebra in Chapter 2. This theory focuses on assigning abelian groups to smooth manifolds[1] in a topologically invariant way. The singular homology and singular cohomology measure the number of holes a smooth manifold has, whilst the De Rham cohomology measures the number of closed and not exact forms or the extent to which the fundamental theorem of calculus fails for closed differential forms.

For simple smooth manifolds the De Rham cohomology and singular cohomology are isomorphic. Furthermore, any point on a smooth manifold has a neighborhood on which the De Rham cohomology is isomorphic to the singular cohomology. The theorem of De Rham states that indeed these cohomologies are isomorphic globally.

To obtain a global isomorphism between these cohomologies, the local cohomology groups are to be lifted to global cohomology groups. This requires local information to be glued together into global information.

This much is done using sheaves. Intuitively, sheaves assign an abelian group to the open subsets of a smooth manifold such that the elements of the abelian groups can be glued together uniquely. Sheaves are introduced and studied in Chapter 3. As one might expect, both aforementioned cohomologies correspond to sheaves. Moreover, both sheaves are in a class of sheaves called soft sheaves. Under certain conditions soft sheaves allow for elements of local abelian groups to be extended to elements of the global abelian group.

[1] More generally topological Spaces.

Just like for abelian groups, exact sequences of sheaves can be defined. When taking global sections of this sequence exactness need not be maintained. As such, we wish to measure to which extent we lose this exactness when taking global sections. This leads to the notion of cohomology for sheaves which is studied in Chapter 4.

Sheaves have their own form of homological algebra and (co)homology groups aptly called sheaf (co)homology. Three forms of cohomology have been considered; De Rham, singular, and sheaf cohomology. Once more one might wonder whether all cohomologies coincide. And indeed they do. This isomorphism between cohomologies provides the long coveted theorem of De Rham in Chapter 5.

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INTEGRATION ON MANIFOLDS

What is the surface area of the unit sphere? Of course it is 4π, but how was this solution obtained? One might be tempted to claim the solution comes from the surface integral over the unit sphere, but this hardly clarifies the process. Let us pause for a moment and begin by scrutinising integrals over manifolds.

Calculus provides us with a very elementary notion of integration. A surface is approximated using parallelepipeds and by summing the surface of these parallelepipeds the area of this surface is obtained. Intuitively this notion is understandable. However, mathematically, quite a few gaps are left to fill.

As the theorem of De Rham is a theorem about differential forms, we begin with by studying differential forms. These differential forms are volume functions for tangent parallelepipeds of smooth manifolds. The existence of certain differential forms has geometric implications. To prove the existence of these differential forms, the De Rham cohomology is developed in Chapter 2.

Additionally, Stokes' theorem is proven in this chapter. Using this theorem the De Rham cohomology is related to another form of cohomology, this is done in Chapter 4. This relation is later lifted to an isomorphism in Chapter 5 which proves the theorem of De Rham.

During thesis thesis we will use the basics of differential geometry. This theory is not provided in this text and can be found in the first four chapters of (Lee, [9]). We will use the same notation as this book, namely for smooth manifold M; TM is the tangent bundle, T^*M the cotangent bundle, and $\Lambda^k T^*M$ the k-tensor bundle on the tangent space.

1.1 DIFFERENTIAL FORMS

Let us assume there is a smooth manifold, M, on which we want to integrate. Before getting into the integrating itself, we need to establish exactly what it is we are integrating. Let us begin with what the objects are which we integrate and exploring how to interpret them geometrically, along with understanding why these objects are so integrable.

Definition 1.1.1 (Differential k-Form).
A differential k-form is a smooth function $\omega : M \to \Lambda^k T^*M$ such that

$$\omega_p \in \Lambda^k T_p M^k$$

for all $p \in M$. The collection of differential forms of degree k on M is denoted by $\Omega^k(M)$.

When integrating our aim is to approximate the domain of integration using small parallelepipeds. By summing the volume of these parallelepipeds the total volume is obtained. Vector fields assign a tangent parallelepiped to every $p \in M$. This achieves our goal of locally approximating the manifold. Now the task is to sum their volumes. As such, a volume function is needed.

This volume function is provided by ω. Where ω_p measures the volumes of tangent parallelepipeds at $p \in M$. As such, ω can also been interpreted as a function taking vector fields as arguments and returning real numbers. In other words, the differential forms are dual to vector fields. To understand how ω measures these volumes we draw upon concepts from linear algebra.

1.1.1 *Elementary Alternating Tensors*

Suppose that $k = n$. Then we want to assign a real number to a collection of n vectors in $\mathbb{R}^n$. This is what the familiar determinant does. Geometrically, the determinant eats a set of vectors and spits out the volume of the parallelepiped spanned by these vectors. This is exactly what we wanted for integration. But what if we only have $k < n$ tangent vectors? It seems we need to generalise the determinant.

In order to properly generalise the determinant we need to keep track of multiple indices. This is done using a multi-index which is defined as follows.

Definition 1.1.2 (Multi-Index)**.**
An ordered k-tuple containing k indices is called a multi-index. A multi-index is denoted as follows

$$I = (i_i, \dots, i_k).$$

Using this broader notion of indices, the following generalisation of the determinant can be defined.

Definition 1.1.3 (Elementary Alternating Tensors)**.**
The mapping $dx^i : T_pM \to \mathbb{R}$ is defined as $v \mapsto v^i$, the ith coordinate of v. Then define the mapping $dx^I : (T_pM)^k \to \mathbb{R}$ as follows

$$dx^I(v_i, \dots, v_k) = det \begin{pmatrix} dx^{i_1}(v_1) & \dots & dx^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ dx^{i_k}(v_1) & \dots & dx^{i_k}(v_k) \end{pmatrix} = det \begin{pmatrix} v_1^{i_1} & \dots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \dots & v_k^{i_k} \end{pmatrix}.$$

Remark 1.1.1. *From Definition 1.1.1 it is clear that if I contains a repeated index, then $dx^I = 0$.*

Definition 1.1.3 provides us with a function which eats k-vectors, puts these into a matrix, and then takes the determinant of some $(k \times k)$-submatrix. The algebra of these tensors is understood, but the geometric action remains clouded.

Geometrically, the elementary alternating tensors eats a collection of tangent vectors, then looks at the shape formed by these vectors. A parallelepiped is obtained by disregarding some directions of this shape. For example, from a cube spanned by the three base vectors in $\mathbb{R}^3$, a square can be obtained by disregarding the z-coordinates. The volume of this newly formed parallelepiped is then returned.

Addition and scalar multiplication of elementary alternating tensors is defined intuitively. The vector space at $p \in M$ spanned by these elementary alternating forms is called the vector space of volume forms and is denoted by $\Lambda^k(T_p^*M)$.

Every volume function at $p \in M$ is a linear combination of elementary alternating tensors. Definition 1.1.1 stated that the map ω eats a point $p \in M$ and returns a linear combination of elementary alternating tensors $\sum \omega^I dx^I$ on $(T_pM)^k$, with real coefficients ω^I.

As ω depends smoothly on p, we can multiply ω with a smooth real valued function on M. As such, we find $\Omega^k(M)$ is a $C^\infty(M)$-module for any k.

1.1.2 *The Wedge product and Exterior Derivative*

Differential forms can be scaled and added together, but there are other methods by which new differential forms can be obtained. One method is to taking a product of two differential forms. This results in the following definition.

Definition 1.1.4 (Wedge Product)**.**
Let I and J be two multi-indices of length k and l respectively. Then the wedge product is defined as

$$dx^I \wedge dx^J = dx^{IJ},$$

where $IJ = (i_1, \ldots, i_k, j_1, \ldots, j_l)$ is the concatenation of I and J.

Definition 1.1.4 can be interpreted geometrically. For example $dx^{1,2}$ measures volume along the x and y-axis and dx^3 along the z-axis in $\mathbb{R}^3$. The wedge product of the two, $dx^{1,2} \wedge dx^3 = dx^1 \wedge dx^2 \wedge dx^3$, measures the volume along the x, y and z-axis. So, the wedge product combines along which directions volume is measured.

Definition 1.1.4 is often calculated pointwise. As such, the wedge product $\omega \wedge \gamma : M \to \Lambda^k T_p^* M$ of differential forms ω and γ is defined as $p \mapsto \omega_p \wedge \gamma_p$. This pointwise definition of the wedge product is extended linearly to global differential forms. This wedge product has many useful properties. These properties will be stated without proof. The interested reader is advised to review Proposition 14.11 in (Lee, [9]).

Proposition 1.1.1.
The wedge product is associative, left- and right- distributive, and has the following anti-commutative property: For $\alpha \in \Lambda^k(T_p^* M)$ and $\beta \in \Lambda^l(T_p^* M)$

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha.$$

We can now create differential forms from other differential forms. We can even go one step further and lift the degree of a differential form upward. That is obtain a $(k+1)$-differential form from a k-differential form. This is accomplished through the use of the *exterior derivative*.

Definition 1.1.5 (Exterior Derivative)**.**
For a smooth manifold M, the operators $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for $k \geq 0$ satisfying the following axioms are called the exterior derivatives.

1. *For all $a \in \mathbb{R}$ and $\omega \in \Omega^k(M)$, $d(a\omega) = ad(\omega)$.*
2. *For $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then the following holds*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

3. *$d \circ d = 0$*
4. *for $g \in \Omega^0(M)$, df is the differential of f.*

Definition 1.1.5 gives a big list of axioms for the exterior derivative, but it is not defined explicitly. Geometrically, the exterior derivative is a rather unintuitive object. As such, it is often defined using axioms. Luckily, we can actually prove that these mappings exist and that they are unique. This proof is omitted, but is provided in (Lee, [9]) on page 365.

As previously mentioned we can write any k-differential form at p as $\sum \omega^I(p) dx^I$. Thus, if we understand what d does when applied to this sum, then we can understand the exterior derivative.

Example 1.1.1. *Let $\omega = f(x_p, y_p)dx$ on $\mathbb{R}^2$ at $p = (x_p, y_p)$. Then we obtain the following for the exterior derivative.*

$$d(\omega) = d(f(x_p, y_p)dx) =$$
$$\left(\frac{\partial f(x_p, y_p)}{\partial x}dx + \frac{\partial f(x_p, y_p)}{\partial y}dy\right) \wedge dx =$$
$$\frac{\partial f(x_p, y_p)}{\partial x}dx \wedge dx + \frac{\partial f(x_p, y_p)}{\partial y}dy \wedge dx =$$
$$0 - \frac{\partial f(x_p, y_p)}{\partial y}dx \wedge dy = -\frac{\partial f(x_p, y_p)}{\partial y}dx \wedge dy.$$

What can we see from the example above? We can think of ω as measuring the x-component of vectors at $p \in \mathbb{R}^2$ scaled by a factor $f(x_p, y_p)$. To obtain a 2-volume form from a 1-volume form we need to wedge it with another 1-volume form.

The differential df provides a 1-volume form and alters the factor $f(x_p, y_p)$. Taking the wedge product of the differential with dx results in a 2-form. We have successfully upgraded a 1-volume form to a 2-volume form.

Two methods for creating new differential forms from existing ones are in our possession, the exterior derivative and the wedge product. The exterior derivative will play a crucial roll later on as it allows for the construction of the following diagram for a smooth m-manifold M.

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^m(M) \xrightarrow{d} 0$$

This diagram leads to the following two definitions.

Definition 1.1.6 (Closed and Exact Forms)**.**
Any differential form in the kernel of d is called closed. A differential k-form ω is called exact if there exists some differential $(k-1)$-form ν, such that $d\nu = \omega$.

1.1.3 *Pullback of a Differential Form Along a Function*

There is a third important manipulation of differential forms: the pullback along a function. Suppose we have a smooth function $f : M \to N$ between manifolds and $\omega \in \Omega^k(M)$. Then ω can be lifted to a differential form on N using F.

Definition 1.1.7 (Pullback of a Differential Form)**.**
Let $f : M \to N$ be a smooth function and suppose $\omega \in \Omega^k(N)$. Then define the pullback $f^* : \Omega^k(N) \to \Omega^k(M)$ as

$$(f^*\omega)_p(-) := \omega_{f(p)}(Df(-)).$$

The function f in Definition 1.1.7 pulls the domain of the differential form back to M from N by precomposing with Df. One might wonder whether the pullback of a closed/exact form along a smooth map is also closed/exact. This is indeed the case, as stated in the following proposition.

Proposition 1.1.2 (Naturality of the Exterior Derivative)**.**
Let $F : M \to N$ be a smooth map between smooth manifolds, then for every $k \geq 0$ the pullback map $F^* : \Omega^k(N) \to \Omega^k(M)$ commutes with d. In other words, for all $\omega \in \Omega^k(N)$

$$F^*(d\omega) = d(F^*\omega).$$

Suppose that $\omega = d\eta$ is true for $\omega \in \Omega^k(N)$, then we obtain $F^*(\omega) = F^*(d\eta) = d(F^*(\eta))$ per proposition 1.1.2. As such, the pullback along F of ω is also exact. Additionally, $d\omega = 0$ implies $0 = F^*(0) = F^*(d\omega) = d(F^*(\omega))$. Thus, closed forms pull back to closed forms along smooth maps.

For the sake of brevity Proposition 1.1.2 is not proven in this text. If the reader is interested a proof can be found on page 366 of (Lee, [9]).

1.2 ORIENTATION OF MANIFOLDS

After much work we finally understand what the objects are which we integrate, namely differential forms. Before we can start actually integrating these objects, we need to add one additional structure on our manifolds. We need to orient them! This is done by assigning an orientation to the tangent space at each point. First, one might wonder when a smooth manifold even is orientable.

Definition 1.2.1 (Orientable Manifold)**.**
Let M be a smooth m-manifold. If there exists a $\mu = \mu(x)dx^I \in \Omega^m(M)$ such that for all $p \in M$ $\mu(p) \neq 0$, then M is orientable.

Definition 1.2.1 states the orientability of M comes from the existence of a nonvanishing differential form on M. As such, we would expect a nonvanishing differential form to induce an orientation of the tangent space at $p \in M$. This is done in the following fashion

Let $p \in M$ and let ω be a nonvanishing m-form. Suppose that $(E_1, \dots, E_m)$ is a basis for T_pM. The differential form ω induces an orientation by calling this basis positively oriented if $\omega(E_1, \dots, E_m) > 0$. As ω depends smoothly on $p \in M$, the orientation of tangent spaces cannot differ abruptly over M.

It seems the differential m-forms can be categorised into camps, with each camp inducing the same orientation. As such, given two m-forms ω, ω' on M, we say they induce the same orientation the tangent spaces if there is some strictly positive function f such that $\omega = f\omega'$. We denote this by $\omega \sim \omega'$. It is not hard to see that $\sim$ is an equivalence relation. This equivalence relation leads to a proper notion of orientation of a manifold.

Definition 1.2.2 (Orientation)**.**
Let M be a smooth m-manifold. An orientation of M is an equivalence class $[\mu] \in (\Omega^m(M) - \{0\})/\sim$.

Finally, we have a notion of orientation for a smooth manifold. But do there even exist manifolds which satisfy this definition? The answer is yes, and the collection of examples is very large. Some examples include:

Example 1.2.1.

- *S^1 is orientable. A nonvanishing differential form on S^1 is $\omega = -ydx + xdy$. As such, $[\omega]$ is an orientation of S^1.*
- *Furthermore, for $T = S^1 \times S^1$ we can define an orientation. Let $\pi_i : T \to S^1$ be the projection onto the i'th coordinate and let ω be as in the previous example, then the differential form $\mu = \pi_1^*(\omega) \wedge \pi_2^*(\omega)$ is nonvanishing. As such, $[\mu]$ is an orientation of T. This same construction provides an orientation for any finite product of orientable manifolds.*
- *Famously, the Möbius strip, $\mathcal{M}$, is not orientable. Geometrically this is evident. If we fix a 2-form $\omega = fdx^I$ with positive f on $\mathcal{M}$, then at $p \in \mathcal{M}$ $f(p) > 0$. As we then move one turn around $\mathcal{M}$, we apply a reflection which switches the sign of f. Resulting in the existence of some $p' \in \mathcal{M}$ such that $f(p') < 0$. According to the intermediate value theorem provides there must be some point $q \in \mathcal{M}$ such that $f(q) = 0$. As such, every 2-form vanishes on $\mathcal{M}$. Resulting in $\mathcal{M}$ not being orientable.*

When integrating, one often encounters a boundary of the domain of integration[1]. This seems like an issue, as previous discussions only focused on orientation for manifolds without boundaries.

[1] Differential forms can be restricted to a manifold with boundary in the intuitive sense.

Luckily, there is an intuitive fashion in which an orientation can be induced onto a boundary of a manifold.

Definition 1.2.3 (Induced Boundary Orientation)**.**
Let M be an oriented smooth m-manifold and let N be a vector field which is nowhere tangent to ∂M. Suppose $[\mu]$ is an orientation form for M, then

$$[\iota_{\partial M}^*(\mu(N,-))] \in (\Omega^{m-1}(\partial M) - \{0\})/\sim$$

is an orientation form for ∂M, where $\iota_{\partial M} : \partial M \hookrightarrow M$ is the inclusion.

Remark 1.2.1. *The existence of a nowhere tangent vector field $N : \partial M \to TM$ remains to be proven. Furthermore, it can be proven that every outward-pointing vector field along ∂M determines the same orientation for ∂M. This much is proven in Proposition 15.24 of (Lee, [9]).*

Geometrically, the induced boundary orientation does the following: Let N be a nowhere tangent vector field to ∂M. Then at every $p \in \partial M$ we can find a basis for T_pM containing N_p as first basis element. The $(m-1)$-form in Definition 1.2.3 checks whether a basis of $T_p\partial M$ is a positive basis of T_pM when N_p is added as first basis element. If this augmented basis is positive, then the original basis is positive for $T_p\partial M$. This induces an orientation on $T_p\partial M$ in a fashion which is smoothly dependent on $p \in \partial M$.

Example 1.2.2. *S^1 is orientable since it is the boundary of $B_1(0) \subset \mathbb{R}^2$ and the induced boundary orientation coincides with the standard orientation.*

1.3 INTEGRATION ON MANIFOLDS

Finally, the stage is set. We have the objects which we integrate, we have orientations so it is time to integrate! We begin with integration of differential forms which are compactly supported, this means the following.

Definition 1.3.1 (Compactly Supported)**.**
A differential n-form ω om n-manifold M is compactly supported if

$$supp(\omega) = \overline{\{x \in \mathbb{R}^n : (\forall I)(\omega_I(x) = 0)\}}$$

is compact in M.

Compactly supported differential forms are the differential forms which are non-zero only on some compact subset of M. Since compact subsets are very useful to work with, this is a very desirable trait. For the compactly supported differential forms with support in one chart integration can be defined straightforward.

Definition 1.3.2 (Integral of Differential Form I)**.**
Let M be a smooth n-manifold and let ω be an differential n-form with compact support in one chart (U, ϕ) that is either positively or negatively oriented. The integral of ω over M is defined to be

$$\int_M \omega = \pm \int_{\phi(U)} (\phi^{-1})^* \omega = \int_{\phi(U)} \sum_I (\omega_I \circ \phi^{-1})(x) d(\phi^{-1}(x))^I.$$

Where a positively oriented chart induces a plus and a negatively oriented chart a minus sign.

Remark 1.3.1. *The value of the integral in Definition 1.3.2 does not depend on the chosen chart. Any reparametrisation is an orientation preserving diffeomorphism. As such, Proposition 1.3.1 provides that the pullback along the transition function of the charts does not alter the value of the integral.*

As one would expect, not every differential form is compactly supported in a single chart. This, however, can be remedied using a sneaky trick: the smooth partition of unity. The existence of the partitions of unity subordinate to any open cover of a smooth manifold is not proven in this text. If the reader is interested this proof is provided on page forty-three in (Lee, [9]).

Let ω be a compactly supported n-form on M. Let $\{U_i\}$ be a finite open cover of $supp(\omega)$ by oriented smooth charts. Let ψ_i be a smooth partition of unity subordinate to $\{U_i\}$. Then the integral over ω is defined as follows.

Definition 1.3.3 (Integral on Differential Form II).
The integral over ω is defined as

$$\int_M \omega = \sum_i \int_M \psi_i \omega$$

As one would hope, integration is linear. Furthermore, the following property can easily be verified.

Proposition 1.3.1.
If $F : M \to N$ is a diffeomorphism and $dF : TM \to TN$ has positive determinant on M, then

$$\int_N \omega = \int_M F^* \omega$$

Indeed, it seems these integrals function as one would expect. We have abstractly defined what these integrals are and proposed certain properties of these integrals, but how does one actually integrate over a manifold in practise? The following example will hopefully enlighten the reader.

Example 1.3.1 (Integration On S^1).
Let $\omega = x^2 dx$ on $M = S^1$, the goal is to determine

$$\int_M \omega.$$

To achieve this, first a smooth structure on M is required. To this end, we choose the following chart to cover M

$$(\{(cos(\theta), sin(\theta) : \theta \in (0, 2\pi)\}, (cos(\theta), sin(\theta)) \mapsto \theta).$$

Wait, it seems we forgot to cover $\{(1,0)\}$! Indeed we have. However, note that the measure of this singleton is zero. As such, if this singleton is left out while integrating the result is the same.

From the above chart it becomes clear that $\phi^{-1}(\theta) = (cos(\theta), sin(\theta))$. We can integrate to obtain

$$\begin{aligned}
\int_M \omega &= \int_{(0,2\pi)} ((\phi^{-1})^* \omega) \\
&= \int_{(0,2\pi)} cos^2(\theta) d(cos(\theta)) \\
&= -\int_{(0,2\pi)} cos^2(\theta) sin(\theta) d\theta \\
&= [\frac{1}{3} cos^3(\theta)]_0^{2\pi} \\
&= \frac{1}{3} - \frac{1}{3} = 0.
\end{aligned}$$

Indeed this is the final solution.

Integration on manifolds seems to be doable. However, in the above example we still needed to do calculus to compute the integral. If there is any justice in this world, the geometry of manifolds could be harnessed to find the values of integrals in a far easier way. Luckily, this is indeed the case. This leads to the following famous theorem.

1.4 STOKES' THEOREM

Now that we can integrate over manifolds, we are able to state and prove Stokes' Theorem. This is a strong theorem which enables one to integrate far easier. The theorem is first stated, then interpreted, and subsequently proven.

Theorem 1.4.1 (Stokes' Theorem)**.**
Let M be an oriented smooth m-manifold with boundary. Suppose that ω is a compactly supported differential $(m-1)$-form on M. Then

$$\int_M d\omega = \int_{\partial M} \omega.$$

What does Theorem 1.4.1 actually state geometrically? As usual a picture can be enlightening. To this end Figure 1 can help. Figure 1 shows a small part of a smooth 3-manifold.

Remember what ω does? It measures the volume of parallelepipeds in the tangent spaces. For example, in Figure 1 at p a tangent parallelogram in $T_p\partial M$ is drawn in green. ω measures the surface of the green parallelogram and the integral of ω over ∂M then consists of the sum of the surfaces of these parallelograms.

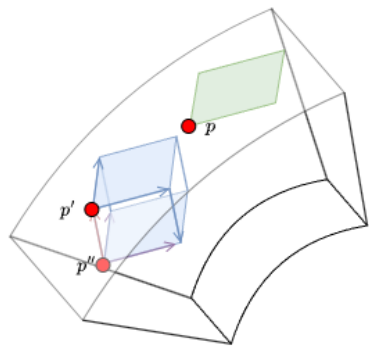


Figure 1: The geometry behind Stokes' theorem.

$d\omega$ gives the volumes of parallelepipeds in T_pM. For example, $d\omega$ would return the volume of the blue parallelepiped in Figure 1. As such, the integral of $d\omega$ over M is the sum of the volumes of each tangent parallelepiped at points in M.

Now geometrically, the integrals are understood. However, the question remains why these are equal?

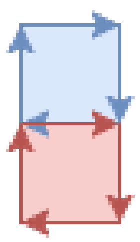


Figure 2: Orientation of integration.

Intuitively, integration consists of summing the volumes of tangent parallelepipeds at all points of a small tiling of the manifold with squares. In Figure 2, an example of two tiles on the plane is shown. The lengths of the tangent vectors to each point in the boundaries is summed to calculate the integral. When integrating a 1-form on these tiles the orientation given by the arrows is used.

First, the blue square is integrated over, then the red square. Since the orientations on the shared boundary of the tiles are opposite, their contributions cancel out. As such, only the boundary of the union of the two tiles contributes to the integral.

Now that we understand the theorem, we can proceed to the proof. The proof is split up into two parts. First a proof for Stokes' theorem on $\mathbb{R}^n$ and $\mathbb{H}^n$ is provided. This proof leans quite heavily on calculus, and can be skipped without any loss of continuity. Next, a proof for Stokes' theorem for general manifolds is provided using Stokes' theorem for $\mathbb{R}^n$ and $\mathbb{H}^n$.

Proof for $\mathbb{R}^n$ and $\mathbb{H}^n$.
Suppose that $M = \mathbb{R}^n$. We assumed ω to be a compactly supported $(n-1)$-differential form on M. As such, the support of ω must be contained in some compact subset of $\mathbb{R}^n$. Remember that every compact subset of $\mathbb{R}^n$ is closed and bounded. Thus there exists a box of the form $B = (-R, R)^n$ fully enveloping the support of ω.

As M admits a global chart we can write ω in coordinates as

$$\omega = \sum_{i=1}^{n} \omega_i dx^1 \wedge dx^2 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n.$$

We can now turn our attention to $d\omega$. This becomes

$$d\omega = d\left(\sum_{i=1}^{n} \omega_i dx^1 \wedge dx^2 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n\right) =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge \hat{dx^i} \wedge \cdots \wedge dx^n$$

Note that the double indices return zero in the above expression. Only when $i = j$ does something non-zero occur. By swapping the dx^j to the spot of dx^i we obtain the following

$$d\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Now we start following our noses and compute the following

$$\int_{\mathbb{R}^n} d\omega = \int_{\mathbb{R}^n} \sum_{i=1}^{n} (-1)^{i-1} \frac{\partial \omega_i}{\partial x^i} \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n =$$
$$\sum_{i=1}^{n} (-1)^{i-1} \int_B \frac{\partial \omega_i}{\partial x^i} \wedge dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n =$$
$$\sum_{i=1}^{n} (-1)^{i-1} \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial \omega_i}{\partial x^i} dx^1 dx^2 \dots dx^n.$$

We can now check what happens for when integrating the ith integral. The following holds by the fundamental theorem of calculus.

$$\int_{-R}^{R} \frac{\partial \omega_1}{\partial x^i} dx^i = \Big[\omega_i(x)\Big]_{x^i=-R}^{x^i=R} = \omega_i(R) - \omega_i(-R).$$

Remember how we chose R? R was chosen so big that $\omega_i(\pm R) = 0$ holds. Therefore this integral is 0. This implies that the integral

$$\sum_{i=1}^{n} (-1)^{i-1} \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial \omega_i}{\partial x^i} dx^1 dx^2 \dots dx^n =$$
$$\sum_{i=1}^{n} (-1)^{i-1} \int_{-R}^{R} \cdots \int_{-R}^{R} \frac{\partial \omega_i}{\partial x^i} dx^i dx^1 dx^2 \dots \hat{dx^i} \dots dx^n =$$
$$\sum_{i=1}^{n} (-1)^{i-1} \int_{-R}^{R} \cdots \int_{-R}^{R} 0 \, dx^1 dx^2 \dots \hat{dx^i} \dots dx^n = 0$$

Furthermore, as $M = \mathbb{R}^n$ has no boundary, we obtain that

$$\int_{\partial \mathbb{R}^n} \omega = 0.$$

As such, these two integrals do coincide.

As our manifold is allowed to have a boundary we also have to deal with the case $M = \mathbb{H}^n$. Once again let $R \in \mathbb{R}$ be sufficiently large such that $supp(\omega)$ is contained in the set $B = (-R, R) \times \cdots \times (-R, R) \times [0, R) \subset M$.

When integrating we obtain the same integrals as for the case of $\mathbb{R}^n$. However, now the boundary term introduces a nonvanishing integral. This results in

$$\int_M d\omega = (-1)^n \int_{-R}^{R} \cdots \int_{-R}^{R} \int_0^R w_n(x^1, \dots, x^{n-1}, 0)\, dx^1 dx^2 \dots dx^{n-1}.$$

Now we compare this integral to the integral of ω over ∂M. Note that on the boundary of M, $x^n = 0$ is true per definition. This leads to the integral

$$\int_{\partial M} \omega = \sum_{i=1}^{n} \int_{B \cap \partial \mathbb{H}^n} \omega_n(x^1, \dots, x^{n-1}, 0) dx^1 \dots \hat{dx^i} \dots dx^n.$$

Since x^n is identically zero on the boundary of M, dx^n also becomes zero when restricted to the boundary of M. As such, only when $i = n$ does this integral evaluate to zero. Therefore, only

$$\int_{\partial M} \omega = \int_{B \cap \partial \mathbb{H}^n} \omega_n(x^1, \dots, x^{n-1}, 0) dx^1 \dots dx^{n-1},$$

is nonzero.

It seems we are close to equality, but a pesky minus sign seems to be a potential hazard. Fear not, the orientation of M saves the day.

Fact 1.4.1. *For even n the space $\mathbb{H}^n$ is positively oriented, and for odd n this space is negatively oriented.*

This fact introduces the required minus sign and leads to equality! □

Having proven Stokes' theorem for two special cases, we can focus on arbitrary smooth manifolds.

Stokes' Theorem For Arbitrary manifolds.
Now we can focus on arbitrary smooth manifolds with boundary, let M be such a manifold. We may assume ω to be an $(n-1)$-form which is compactly supported in a single positively oriented chart (U, ϕ).

Suppose that U is an interior chart. That is, U is homeomorphic to and open subset of $\mathbb{R}^n$ under ϕ. Then we obtain

$$\int_M d\omega = \int_{\mathbb{R}^n} (\phi^{-1})^* d\omega = \int_{\mathbb{R}^n} d(\omega \circ \phi^{-1}).$$

As $supp(\omega)$ is compact in M and contained in U, $\phi(supp(\omega)) \subset \phi(U) \subset \mathbb{R}^n$ is also compact. As such, $\omega \circ \phi^{-1}$ is a compactly supported smooth $(n-1)$-form on $\mathbb{R}^n$. As shown before Stokes' theorem is true in $\mathbb{R}^n$

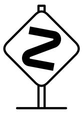
$$\int_{\mathbb{R}^n} d(\omega \circ \phi^{-1}) = \int_{\partial \mathbb{R}^n} \omega \circ \phi^{-1} = 0.$$

Suppose that U is a boundary chart, that is U is homeomorphic to some open subset of $\mathbb{H}^n$ which intersects $\partial \mathbb{H}^n$ under ϕ. If we endow $\partial \mathbb{H}^n$ with the induced boundary orientation, we obtain

$$\int_M d\omega = \int_{\mathbb{H}^n} d(\omega \circ \phi^{-1}).$$

As Stokes' theorem is true in $\mathbb{H}^n$ we obtain this is equal to

$$\int_{\partial \mathbb{H}^n} \omega \circ \phi^{-1}.$$



Geometrically $d\phi$ maps outward pointing vectors on ∂M to outward pointing vectors in $\mathbb{H}^n$. As such, ϕ respects the induced boundary orientation on ∂M and $\partial \mathbb{H}^n$. This implies $d\phi$ has a positive determinant. Proposition 1.3.1 now provides

$$\int_M d\omega = \int_{\partial \mathbb{H}^n} \omega \circ \phi^{-1} = \int_{\partial M} \omega.$$

As such, Stokes' theorem is true.

Of course not every differential form on M will have a compact support fully contained in one chart. This issue is alleviated by the use of a partition of unity. We choose an open cover of $supp(\omega)$ using finitely many positive or negatively oriented smooth charts denoted by $\{U_i\}$. This open cover has a subordinate smooth partition of unity $\{\psi_i\}$.

We use these ψ's to chop our ω up into nice pieces $\psi_i \omega$ for which we have already proven the theorem. Then by gluing these $\psi_i \omega$ together, we obtain ω back.

$$\begin{aligned} \int_{\partial M} \omega = \sum_i \int_{\partial M} \psi_i \omega = \sum_i \int_M d(\psi_i \omega) = \sum_i \int_M d\psi_i \wedge \omega + \psi d\omega = \\ \int_M d\left(\sum_i \psi_i\right) \wedge \omega + \int_M \left(\sum_i \psi_i\right) d\omega = \int_M d(1) \wedge \omega + \int_M 1 \cdot d\omega = \\ 0 + \int_M d\omega. \end{aligned}$$

Indeed, Stokes' Theorem holds in the general case as well. □

Having proven stokes, we notice it provides a generalisation of the fundamental theorem of calculus for exact differential forms. Indeed if $\omega = d\eta$ then we obtain

$$\int_D \omega = \int_{\partial D} \eta,$$

for some domain of integration D. However, for non-exact form this generalisation does not work. As finding η corresponding to ω can often prove to be difficult, we want to look for other sufficient characterisations of exactness for differential forms.

Notice that for any exact form $\omega = d\eta$, we obtain $d\omega = d(d\eta) = 0$. As such, every exact form is closed. It is precisely this property which allows for the development of the theory of homological algebra for the sequence of $\Omega^k(M)$'s. In Chapter 2 this theory is developed resulting in the De Rham cohomology groups.

However, the question remains: Is this a sufficient characterisation for exactness?

1.5 poincaré lemma

To answer this question we begin by looking at a simple manifold, for example $\mathbb{R}^n$. Suppose that $\omega = \sum_{i=1}^n \omega_i dx^i$ is a 1-form on $\mathbb{R}^n$. If this differential form is exact then we obtain

$$\omega(x) = df(x) = \sum_{i=1}^n \frac{\partial f(x)}{\partial x^i} \cdot dx^i$$

for some smooth function $f : M \to \mathbb{R}$. Notice that also $f - f(0) : M \to \mathbb{R}$ has the property that $\omega(x) = d(f(x) - f(0))$. As such, we can assume that $f(0) = 0$ is true. Our goal is to obtain this f from a given exact ω. This leads to a general strategy for determining the potential function of a differential form.

As $\mathbb{R}^n$ is star shaped the following trick from calculus is applied

$$\begin{aligned} f(x) &= \int_0^1 \frac{\partial f(tx)}{\partial t} dt \\ &= \int_0^1 \sum_{i=1}^n \frac{\partial f(tx)}{\partial x^i} x^i dt \\ &= \int_0^1 \sum_{i=1}^n \omega_i(tx) x^i dt. \end{aligned}$$

So, in order to obtain f from ω, the function I_ω defined as

$$I_\omega(x) = \int_0^1 \sum_{i=1}^n \omega_i(tx) x^i dt$$

is to be considered.

It may seem uncertain whether this method truly provides an f for every ω and whether this trick even works for higher order forms. Intuitively it may seem true, but is it?

Luckily, one of the fathers of topology, Henri Poincaré, has already addressed this question for us. This resulted in the following Lemma bearing his name. As described on page 94 in (Spivak, [14]) the following holds true.

Lemma 1.5.1 (Poincaré Lemma)**.**
Every closed form $\omega \in \Omega^k(A)$ on $A \subseteq \mathbb{R}^n$ a star shaped subset with $0 \in A$ is exact for $n \geq 0$ and $k > 0$.

Proof. The goal of this proof is to show that indeed $\omega = d(I_\omega)$ for any closed ω on A. To this end we will prove the following identity

$$\omega = d(I_\omega) + I_{d\omega},$$

as when $d\omega = 0$ this implies $\omega = d(I_\omega)$.

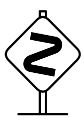
As I_ω will be a linear map in ω it suffices to prove the lemma for any form on A

$$\omega = \omega^I dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \omega^I dx^I.$$

Since A is convex and contains 0, the calculus trick can be applied in the opposite direction. This results in the following definition

$$I_\omega(x) = \sum_{j=1}^k (-1)^{j-1} \left(\int_0^1 t^{k-1} \omega^I(tx) dt \right) x^{i_j} dx^{i_1} \wedge \cdots \wedge \widehat{dx^{i_j}} \wedge \cdots \wedge dx^{i_k},$$

where the hat indicates the dx^{i_j} is omitted. In the sequel we will write I_j for the jth index being omitted from the multi-index.



Note that $\omega(tx)$ is a continuous function in the t-variable. As such $\omega(tx)$ is continuous on $[0,1]$, which implies $\omega(tx)$ is integrable on $[0,1]$. For $k = 0$ we obtain

$$\int_0^1 \omega(tx) dt \cdot x^{i_1},$$

which is convergent. As such, the I_ω map is well defined.

Now our attention is turned to proving $\omega = I_{d\omega} + d(I_\omega)$. There are multiple ways in which this can be accomplished. Here, a calculus oriented proof is provided. As is common in calculus proofs, multiple steps are done at once, making the proof uninsightful, but accessible for most readers.

In Section I.4 of (Bott & Tu, [2]) a proof using homotopy theory is provided. This proof is more insightful, but requires material not covered in this text. The interested reader is advised to review this proof.

First, $d(I_\omega)$ is determined. We obtain

$$d(I_\omega) = \sum_{m=1}^{n} \frac{\partial}{\partial x^m} \Big(\sum_{j=1}^{k} (-1)^{j-1} \big(\int_0^1 t^{k-1} \omega^I(tx) dt) x^{i_j} \Big) \wedge dx^{I_j}$$

Applying the product rule and using the Leibniz rule to interchange integration and differentiation results in the following.

$$\begin{aligned} d(I_\omega) &= k \cdot \big(\int_0^1 t^{k-1} \omega^I(tx) dt) dx^I \\ &+ \sum_{m=1}^{n} \sum_{j=1}^{k} (-1)^{j-1} \big(\int_0^1 t^k \frac{\partial}{\partial x^m} \omega^I(tx) dt) \cdot x^{i_j} dx^{i_m} \wedge dx^{I_j}. \end{aligned}$$

This monster equation has become too much of a hassle to effectively deal with. As such, we turn our attention to $I_{d\omega}$. Per definition the following is true

$$d\omega = \sum_{m=1}^{n} \frac{\partial}{\partial x^m} \omega^I dx^m \wedge dx^I.$$

This results in the following map

$$\begin{aligned} I_{d\omega} &= \sum_{m=1}^{n} \int_0^1 t^k \frac{\partial}{\partial x^m} \omega^I(tx) dt x^{i_m} dx^I \\ &- \sum_{m=1}^{n} \sum_{j=1}^{k} (-1)^{j-1} \big(\int_0^1 t^k \frac{\partial}{\partial x^m} \omega^I(tx) dt) \cdot x^{i_j} dx^m \wedge dx^{I_j}. \end{aligned}$$

Another humongous expression. Luckily, when these expressions are added, the double sums cancel out. Which leaves us with the expression

$$\begin{aligned} I_{d\omega} + d(I_\omega) &= \sum_{m=1}^{n} \int_0^1 t^k \frac{\partial}{\partial x^m} \omega^I(tx) dt \cdot x^{i_m} dx^I + k \cdot \big(\int_0^1 t^{k-1} \omega^I(tx) dt) dx^I \\ &= \int_0^1 \left(\sum_{m=1}^{n} t^k \frac{\partial}{\partial x^m} \omega^I(tx) + k \cdot t^{k-1} \omega^I(tx) \right) dt \cdot dx^I. \end{aligned}$$

The eagle eyed reader might have spotted that this expression is the result of the product rule in the following integral

$$\begin{aligned} &\int_0^1 \frac{\partial}{\partial t} [t^k \omega^I(tx) dt] dx^I \\ &= \big(\omega^I(x) - \omega^I(0) \big) dx^I = \omega^I dx^I = \omega \end{aligned}$$

Thus we obtain $\omega = d(I_\omega) + I_{d\omega}$ and the theorem is true. □

For $\mathbb{R}^n$ it seems every closed form is exact. In order to prove this we needed to ensure the domain of the differential forms contained the path from 0 to x. The domain needed to be contractible.

Furthermore, since $\mathbb{R}^n$ is diffeomorphic to the n-unit ball $\mathbb{B}^n$, the unit ball also has this property. Hmm, but does this hold for every smooth manifold?

Corollary 1.5.1 (Every Closed Form Is Locally Exact)**.** *Let M be a smooth manifold and $p \in M$. Then there is an open neighborhood U of p such that every closed differential form ω on U is exact.*

Proof. Let $p \in M$, we know p is contained in some chart (U, ϕ). Under this chart $\phi(p)$ is contained in some open ball B which per Theorem 1.5.1 has only closed forms which are exact.

Every closed form on $\phi^{-1}(B)$ can be lifted to B via the pullback ϕ^*. Furthermore, every closed form in B can be pulled back to $\phi^{-1}(B)$ along $(\phi^{-1})^*$. Quite clearly these pullbacks are each others inverses. As such, there is an isomorphism between the closed forms on $\phi^{-1}(B)$ and those on B.

Per Proposition 1.1.2 the exterior derivative commutes with pullbacks. As such, every closed form on $\phi^{-1}(B)$ is exact. Thus every $p \in M$ has some open neighborhood on which every closed form is exact. □

This is close to the desired result, however, we need that every closed form is globally exact. Maybe we need to look at $\mathbb{R}^2 - \{0\}$.

Example 1.5.1 (The Punctured Plane)**.** *On* $\mathbb{R}^n - \{0\}$ *define the differential form*

$$\omega = \frac{-ydx + xdy}{x^2 + y^2}.$$

A quick check results in $d\omega = 0$*, as such* ω *is indeed closed. One might recognise* ω *as the derivative of the argument function* θ*, indeed* $\omega = d\theta$*. So, this closed form also seems exact!*

However, remember that on $\mathbb{R}^2 - \{0\}$ θ *is not properly defined. No matter which branch of the argument you take, there is some point at which this function is not continuous. As such, there is no smooth global function of which* ω *can be the derivative. It turns out* ω *is not exact.*

Example 1.5.1 shows not every closed form is exact on every smooth manifold. Being closed is not a sufficient characterisation of exactness for differential forms. As such, there is no fundamental theorem of calculus for all closed differential forms on every smooth manifold.

This raises the question how many closed but not exact form are we separated from a fundamental theorem of calculus for all closed forms on a manifold? To answer this question we need to determine the number of closed and not exact differential forms on a smooth manifold.

Furthermore, the closed but not exact form in this example seems to arise due to the hole in the plane. More precisely, the punctured plane is not contractible due to the presence of this hole. It seems this exactly implies the existence of closed forms which are not exact. In other words, the existence of the closed and not exact forms appears to be closely linked with the geometry of a manifold, particularly the existence of holes.

Also, via study of closed and not exact forms insight into the geometry of a manifold can be obtained. The number of closed and not exact forms a manifold permits, is tied with the irregularity of the geometry of the manifold. As such, the number of holes a manifold has has become an interesting question.

Both these questions lead us into the world of homological algebra. In Chapter 2 the theory of homological algebra is developed providing an answer to both posed questions.

Furthermore, Corollary 1.5.1 showed that locally every closed form is exact. Thus, to obtain globally closed and not exact forms, we need to glue the local information together to obtain global information. A new mathematical structure is used to glue the local differential forms together into global differential forms; sheaves. The theory of sheaves is developed in Chapters 3 and 4.

2

HOMOLOGICAL ALGEBRA: THE BASICS

In Chapter 1 the question was posed how many closed but not exact forms a smooth manifold permits. This question had both geometric and analytic merit. To answer this question we need to look at the world of homological algebra.

Just like the theory of manifolds, homological algebra goes back a long way. Initially homological algebra focused on assigning numbers to topological spaces and manifolds which are topologically invariant. It was only after 1925 that due to Emmy Noether the concept of cohomology groups took centre stage as detailed in (Weibel, [15]).

These homology groups are used to measure the inexactness of a sequence of objects and are topologically invariant. As such, these homological groups have implications regrading the topology, and thus also geometry, of topological space. The homology groups especially are useful for categorising topological spaces.

Many different flavours of homology groups have been developed, each using a different exact sequence associated with a topological space. In this chapter we focus on a dual concept to homological groups, the cohomology groups. Just like homology groups, there are many flavours of cohomology groups, we focus on the De Rham and singular cohomology groups.

Intuitively, the De Rham cohomology groups measure how many closed but not exact differential forms exist on a smooth manifold. The singular cohomology groups are related to measuring the holes in a smooth manifold. These groups provide an answer to questions posed in Chapter 1.

Locally both cohomology groups are trivial and appear isomorphic. The theorem of De Rham states these cohomology groups are globally isomorphic as well. To lift this isomorphism to a global level sheaves are used. Chapters 3 and 4 develop the required sheaf theory to prove this isomorphism. The actual proof is done in Chapter 5.

2.1 INTRODUCTORY THEOREM AND DEFINITIONS

The apparatus of homological algebra is vast and contains many constructions. Some of these concepts can be constructed in a very general manner and lead to a general concept of (co)homology groups. In this section, we will focus on the general construction of (co)homology groups and derive their universal properties. These properties are essential for the study of homological algebra.

2.1.1 *Complexes*

No calculus can be done without any functions, graph theory requires vertices and edges, and probability theorist would die without measures, in the same vain homological algebra relies fundamentally on complexes. The complex is a cornerstone of the homological algebra. But what is a complex?

Definition 2.1.1 (Complex of Abelian Groups).
A complex of abelian groups is a sequence of abelian groups

$$M^\bullet : \qquad \dots \longrightarrow M^{i-1} \longrightarrow M^i \longrightarrow M^{i+1} \longrightarrow \dots$$

such that the composition $M^{i-1} \to M^{i+1}$ *is the zero map. For a complex* $M^\bullet$ *cohomology groups are defined as follows for* $p \in \mathbb{Z}$

$$H^p(M^\bullet) = \frac{Ker(M^p \to M^{p+1})}{Im(M^{p-1} \to M^p)}.$$

In one big swoop complexes and their subsequent cohomology groups are defined! Additionally, the use of cohomology groups has become clear from Definition 2.1.1. Note that if the cohomology groups are zero, then the complex is exact. So, the cohomology groups measure to which extent the complex is not exact.

The most important example of a complex in this thesis is the De Rham complex. This complex has corresponding De Rham cohomology groups as detailed in Definition 2.4.1. Previously we claimed the De Rham cohomology groups measure how many closed and not exact forms exist. Definition 2.1.1 provides us with a more algebraic interpretation. The cohomology groups measure the extent to which the De Rham complex is not exact.

A mathematical object gains significance through its connection to other objects. Two complexes can be related using morphisms or arrows of complexes. These arrows adhere to the differential structure of a complex. This is formalised in the following definition.

Definition 2.1.2 ((Iso)Morphism of Complexes).
A(n) (iso)morphism of complexes $f : A^\bullet \to B^\bullet$ *consists of a collection of (iso)morphisms* $f^i : A^i \to B^i$ *with* $i \in \mathbb{Z}$ *such that the following square Commutes.*

$$\begin{array}{ccc} A^i & \longrightarrow & A^{i+1} \\ \downarrow f^i & & \downarrow f^{i+1} \\ B^i & \longrightarrow & B^{i+1} \end{array}$$

Where the horizontal arrows are the arrows in the complexes $A^\bullet$ *and* $B^\bullet$ *respectively.*

2.1.2 *The Long Exact Sequence of Cohomology*

Now that we can speak of morphisms of complexes, another sequence can be dreamt up. As before we had sequences of abelian groups, why not make a sequences of complexes? This might seem innocent as we just generalise what we did for abelian groups to complexes. However, this leads to a very important theorem of homological algebra.

Theorem 2.1.1 (The Long Exact Sequence of Cohomology).
Let the following be a short exact sequence (s.e.s.) of complexes.

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

Then there is a long exact sequence (l.e.s.) of cohomology groups with natural group homomorphisms.

$$\dots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \xrightarrow{\partial^i} H^{i+1}(A^\bullet) \to \dots$$

Proof.
The proof of this theorem consists of multiple diagram chases. The diagram chases are done in the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A^{i-1} & \xrightarrow{f^{i-1}} & B^{i-1} & \xrightarrow{g^{i-1}} & C^{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a^{i-1}} & & \downarrow{\scriptstyle b^{i-1}} & & \downarrow{\scriptstyle c^{i-1}} & & \\
0 & \longrightarrow & A^{i} & \xrightarrow{f^{i}} & B^{i} & \xrightarrow{g^{i}} & C^{i} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a^{i}} & & \downarrow{\scriptstyle b^{i}} & & \downarrow{\scriptstyle c^{i}} & & \\
0 & \longrightarrow & A^{i+1} & \xrightarrow{f^{i+1}} & B^{i+1} & \xrightarrow{g^{i+1}} & C^{i+1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle a^{i+1}} & & \downarrow{\scriptstyle b^{i+1}} & & \downarrow{\scriptstyle c^{i+1}} & & \\
0 & \longrightarrow & A^{i+2} & \xrightarrow{f^{i+2}} & B^{i+2} & \xrightarrow{g^{i+2}} & C^{i+2} & \longrightarrow & 0
\end{array} \tag{2.1}$$

There are three parts in this proof. First, we will show $\overline{g}^i : H^i(B^\bullet) \to H^i(C^\bullet)$ is well defined and that $Ker(\overline{g}^i) = Im(\overline{f}^i)$. To this end, let $\overline{\sigma} = \overline{\gamma} \in H^i(B^\bullet)$. Per definition $\sigma - \gamma \in Im(b^{i-1})$. As such, there is a $\beta \in B^{i-1}$ such that

$$c^{i-1}(g^{i-1}(\beta)) = g^i(b^{i-1}(\beta)) = g^i(\sigma - \gamma),$$

using the commutativity of the above diagram. We conclude that $g^i(\sigma - \gamma) \in Im(c^{i-1})$. This implies

$$\overline{g}^i(\overline{\sigma}) - \overline{g}^i(\overline{\gamma}) = \overline{g}^i(\overline{\sigma} - \overline{\gamma}) = \overline{g^i(\sigma - \gamma)} = 0.$$

We obtain $\overline{g^i}(\overline{\sigma}) = \overline{g^i}(\overline{\gamma})$. Thus, $\overline{g}^i$ is well defined. The same argument applied to $\overline{f}^i$ shows this arrow is well defined.

As the rows in the above diagram are exact, it is not hard to see that $Im(\overline{f}^i) \subseteq Ker(\overline{g}^i)$. Let $\overline{\sigma} \in Ker(\overline{g}^i)$, then per definition $g^i(\sigma) \in Im(c^{i-1})$. Since g^{i-1} is surjective, $g^i(\sigma)$ can be lifted to some $\beta \in B^{i-1}$. Notice that $\sigma - b^{i-1}(\beta) \in Ker(g^i)$. The exactness of the rows of Diagram 2.1 implies the existence of some $\alpha \in A^i$ such that

$$f^i(\alpha) = \sigma - b^{i-1}(\beta).$$

Taking cohomology groups we obtain

$$\overline{f}^i(\overline{\alpha}) = \overline{f^i(\alpha)} = \overline{\sigma - b^{i-1}(\beta)} = \overline{\sigma},$$

per definition of the cohomology groups. As such, $\overline{f}^i(\overline{\alpha}) = \overline{\sigma}$ and $\overline{\sigma} \in Im(\overline{f}^i)$. This implies $Ker(\overline{g}^i) = Im(\overline{f}^i)$. We obtain that the sequence is exact at $H^i(B^\bullet)$ for all i.

Now the connecting morphism $\partial^i : H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ is shown to be well defined with kernel equal to $Im(\overline{g}^i)$. To this end, first the connecting morphism is defined. Let $\overline{\sigma} \in H^i(C^\bullet)$, per definition $\sigma \in Ker(c^i)$. As g^i is surjective we can lift σ to $\beta \in B^i$. Due to the commutativity we obtain

$$g^{i+1}(b^i(\beta)) = c^i(\sigma) = 0$$

so $b^i(\beta) \in Ker(g^{i+1})$. By the exactness of the rows of Diagram 2.1 we obtain the existence of $a \in A^{i+1}$ such that $f^{i+1}(a) = b^i(\beta)$. Additionally, the commutativity provides

$$f^{i+2}(a^{i+1}(a)) = b^{i+1}(b^i(\beta)) = 0.$$

As f^{i+2} is injective we obtain $a^{i+1}(a) = 0$. Now define the map $\partial : H^i(C^\bullet) \to H^{i+1}(A^\bullet)$ as

$$\overline{\sigma} \mapsto \overline{a}.$$

By applying another diagram chase to the object $\overline{\sigma} + \overline{\gamma}$ for $\overline{\sigma}, \overline{\gamma} \in H^i(C^\bullet)$ we obtain that $\partial(\overline{\sigma} + \overline{\gamma}) = \partial(\overline{\sigma}) + \partial(\overline{\gamma})$ Thus proving that ∂ is a homomorphism. Furthermore, ∂ is well defined. Let $\overline{\sigma} = \overline{\gamma} \in H^i(C^\bullet)$, then $\sigma - \gamma \in Im(c^{i-1})$. Since $g^{i-1}$is surjective we can lift to $\beta \in B^{i-1}$. Notice that

$$c^{i-1}(g^{i-1}(\beta)) = g^i(b^{i-1}(\beta)) = \sigma - \gamma,$$

so $\sigma - \gamma$ lifts to $b^{i-1}(\beta)$ under g^i. This results in $b^i(b^{i-1}(\beta)) = 0$. This implies $\partial(\overline{\sigma} - \overline{\gamma}) = 0$ and $\partial(\overline{\sigma}) = \partial(\overline{\gamma})$. We conclude ∂ is well defined.

Additionally, $Ker(\partial) = Im(\overline{g}^i)$. First, suppose $\overline{\sigma} \in Im(\overline{g}^i)$. Then σ lifts to some $\beta \in Ker(b^i)$ under g^i. Under ∂ the map b^i is applied leading to $b^i(\beta) = 0$. As f^{i+1} is injective we obtain $\partial(\overline{\sigma}) = \overline{0}$. Thus, $Im(\overline{g}^i) \subseteq Ker(\partial)$. Now suppose that $\overline{\sigma} \in Ker(\partial)$. Then σ lifts to an element $\beta \in Ker(b^i)$ under g^i. As such, we obtain $\overline{g}^i(\overline{\beta}) = \overline{\sigma}$ and $\overline{\sigma} \in Im(\overline{g}^i)$. Finally, we conclude $Ker(\partial) = Im(\overline{g}^i)$ and the l.e.s. of cohomology is exact at $H^i(C^\bullet)$ for all i.

Finally, the kernel of $\overline{f}^{i+1}$ is shown to be equal to $Im(\partial)$. To this end, let $\overline{a} \in Ker(\overline{f}^{i+1})$ then $f^{i+1}(a) \in Im(b^i)$. As such, we can lift to $\beta \in B^i$ which leads to $\gamma = g^i(\beta) \in C^i$. If $\gamma \in Ker(c^i)$, then we obtain $\partial(\overline{\gamma}) = \overline{a}$ resulting in $Ker(f^{i+1}) \subseteq Im(\partial)$. Per commutativity of Diagram 2.1

$$c^i(\gamma) = g^{i+1}(b^i(\beta)) = g^{i+1}(f^{i+1}(a)) = 0$$

we conclude $\gamma \in Ker(c^i)$. Additionally, if $\overline{a} \in Im(\partial)$, then there is some $\beta \in B^i$ such that $b^i(\beta) = f^{i+1}(a)$. This results in $\overline{f}^{i+1}(\overline{a}) = 0$ and $\overline{a} \in Ker(\overline{f}^{i+1})$. We conclude that $Im(\partial) = Ker(\overline{f}^{i+1})$.

This results in the l.e.s. of cohomology being exact at every object. □

Remark 2.1.1. *The theory here has been developed for increasing complexes $M^\bullet$. However, the same concepts can be defined for decreasing complexes $M_\bullet$ with decreasing indices and objects M_i. In the theorems for decreasing complexes the directions of the arrows also flip.*

2.2 SINGULAR HOMOLOGY

The homological algebra developed so far, has not required any topology or geometry, it was just algebra. For algebraists this is fun, but for geometers this is pure torture. Luckily we now introduce a whole load of geometry!

As noted in Chapter 1 the number of holes a surface has seems to be linked to the existence of closed and not exact holes. Intuitively the singular cohomology groups find holes in a smooth manifold. As such, our geometrical adventure in homological algebra begins with the study of the singular homology groups.

We return to the punctured plane $M = \mathbb{R}^2 \backslash \{0\}$. As noted before, M has one hole in its centre. How would one find this hole mathematically? One property separating a hole from a *non*-hole, is the fact that a path in a smooth manifold can move through a *non*-hole but not through a hole. This property is used when detecting holes.

Begin by taking a closed path, called a loop, around the missing point in M. If we contract our path to a smaller and smaller loop, we always keep the hole enveloped by the loop. In other words, we can never continuously contract this loop into a point. Thus we have detected the hole in M.

By contrast, if the loop were not to envelop the hole, it can be contracted continuously to a point. Thus, resulting in no hole being present in the interior of the loop.

2.2.1 *Simplices on a Smooth Manifold*

As described above, loops play an essential role in smooth singular homology. However, it seems loops cannot find all holes in a smooth manifold. For example, a loop cannot find the hole in

$\mathbb{R}^3 - \{0\}$ as the loop can be contracted to a point by first moving away from the origin. Therefore there is a need to generalise to a broader notion of a loop.

If we view loops as deformed boundary of a triangles. Then a loop which is contractible to a point, actually describes a triangle. The contraction to a point requires the inside of the triangle to be hole-less. As such, a higher dimensional loop would be a tetrahedron. The general notion at play here is the simplex.

Definition 2.2.1 (Simplex in $\mathbb{R}^n$).
An n-simplex in $\mathbb{R}^n$ is the set of points

$$\Delta_n := \left\{ x \in \mathbb{R}^n : \left(\sum_{i=1}^{n} x_i \leq 1 \right) \wedge (\forall i)(x_i \geq 0) \right\}.$$

In other words, a n-simplex is the convex hull of the n standard unit vectors in $\mathbb{R}^n$.

Example 2.2.1. *Simplices can take on many shapes depending on their dimension, some common simplices are*

- 0*-simplices are points,*
- 1*-simplices are lines,*
- 2*-simplices are triangles,*
- 3*-simplices are tetrahedrons.*

A simplex in $\mathbb{R}^n$ is nice, but not very useful for analysis of general smooth manifolds. Luckily, the notion of a simplex can be generalised to a general smooth manifold M[1].

Definition 2.2.2 (Simplex in M).
A simplex in a smooth manifold M is defined as a continuous map $\sigma : \Delta_p \to M$.

Definition 2.2.2 states that a simplex in M is a continuous mapping σ projecting a simplex onto M. In Figure 3 this notion is pictured. A 2-simplex in $\mathbb{R}^2$ is mapped under σ to a subset of M.

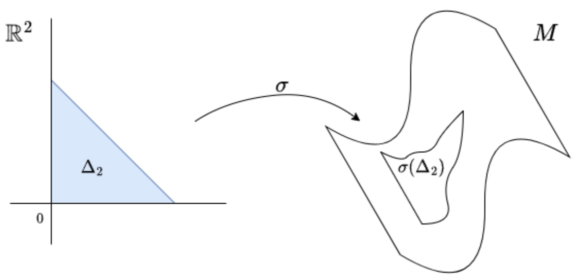


Figure 3: A 2-simplex on the topological space M

2.2.2 *The Chain of Simplices*

Suppose σ and γ are two 1-simplices on M with coinciding endpoints, so σ and γ together form a loop. In order to obtain the loop formed by σ and γ, we need to add these two 1-simplices together. However, there is no additive structure on the collection of 1-simplices of M. This issue is resolved by simply defining an additive structure onto the collection of simplices of M.

[1] The definition of a simplex can even be generalised for any topological space X.

Definition 2.2.3 (Chain Group)**.**
Let M be a smooth manifold and $p \in \mathbb{Z}_{\geq 0}$. The chain group $C_p(M)$ is defined to be free abelian group[2] *with basis the set of p-simplices in M. In other words*

$$C_p(M) = \bigoplus_{\sigma:\Delta_p \to M} \mathbb{Z} \cdot \sigma.$$

For those unfamiliar with free abelian groups, this may seem like a fairly abstract definition. Therefore, it might be helpful to think of $C_p(M)$ as the collection of finite sums of simplices in M.

Example 2.2.2. *Let $3\sigma,\ 5\gamma.\ -2024\delta \in C_p(M)$. Then we define the sum to be $3\sigma + 5\gamma - 2024\delta$.*

This provides us with an additive structure on simplices. Beyond adding simplices together, there is another way to obtain simplices from other simplices.

Looking at the 2-simplex in Figure 3, the boundary of this simplex appears to be a sum of three separate lines, or 1-simplices. By taking the boundary of a k-simplex, we seem to obtain a sum of $(k-1)$-simplices. This leads to the boundary operator.

Definition 2.2.4 (Boundary Operator)**.**
The boundary operator is a map

$$d_p : C_p(X) \to C_{p-1}(X)$$

defined as

$$\sigma \mapsto \sum_{i=0}^{p} (-1)^i (\sigma \circ (e_0, \dots, \hat{e}_i, \dots, e_p)).$$

Where the map $(e_0, \dots, \hat{e}_i, \dots, e_p) = (e_0, \dots, e_{i-1}, e_{i+1}, \dots, \dots, e_p) : \Delta_{p-1} \to \Delta_p \subset \mathbb{R}^p$ is defined as

$$(x_1, \dots, x_p) \mapsto (1 - \sum_{i=1}^{p} x_i)e_1 + x_1 e_i + \dots + x_p e_p.$$

This map sends the endpoint x_i of Δ_{p-1} to the $e_i \in \mathbb{R}^p$.

Definition 2.2.4 provides us with a mapping, which seems to describe a boundary operation. However, this mapping might not immediately intuitive. An example will clarify its meaning.

Example 2.2.3. *We take a look at the standard simplex $\Delta_2 \subset \mathbb{R}^2$. Δ_p has three endpoints $e_0,\ e_1,\ e_2$ as detailed in Figure 4. We will compute the boundary of Δ_2.*

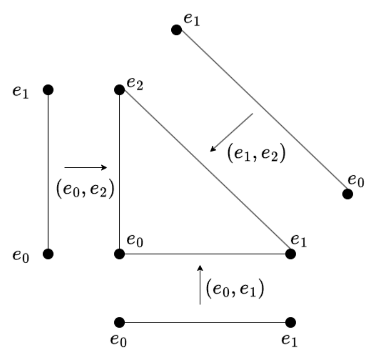


Figure 4: The boundary of Δ_2. Figure adapted from the Figure on page 56 in (Jong & Lugt, [7]).

To this end, we begin by simply applying Definition 2.2.4.

$$d_2\Delta_2 = \sum_{i=0}^{2} (-1)^i (Id_{\mathbb{R}^2} \circ (e_0, \hat{e}_i, e_2)) = (e_1, e_2) - (e_0, e_2) + (e_0, e_1).$$

[2] For a definition and proper treatment of free groups the reader is advised to review I.§12 of (Lang, [8]).

As shown in Figure 4 the mapping (e_1, e_2) coincides with the diagonal edge which start at the e_1 vertex and goes to e_2. The mapping (e_0, e_1) coincides with the bottom edge.

The (e_0, e_2) starts at e_0 and goes to e_2. However, the direction of this map does not coincide with the other edges. As such, we need to flip this map, which is achieved by the (-1) term. Thus we have obtained the boundary of Δ_2

As the reader might have surmised, for any simplex Δ_p we obtain that $(d_{p-1} \circ d_p)(\Delta_p) = 0$. This is due to the chosen $(-1)^i$ term. This much is also readily seen for the example of Δ_2.

Example 2.2.4.

$$\begin{aligned}(d_1 \circ d_2)(\Delta_2) = d_1((e_1, e_2) - (e_0, e_2) + (e_0, e_1)) = \\ d_1((e_1, e_2)) - d_1((e_0, e_2)) + d_1((e_0, e_1)) = \\ e_2 - e_1 - (e_2 - e_0) + e_1 - e_0 = 0\end{aligned}$$

2.2.3 *Singular Homology Groups*

We now have a collection of objects, namely the chain groups, and a collection of boundary maps between these objects. Furthermore, the boundary maps have the property that $d \circ d = 0$. As such, we obtain a complex of chain groups of the following form.

$$C_\bullet(M): \qquad \dots \longrightarrow C_p(M) \xrightarrow{d_p} C_{p-1}(M) \xrightarrow{d_{p-1}} \dots$$

As is common for homological algebra, we wish to find out how exact this sequence is. Now the exactness might seem like a fairly abstract notion, what does this geometrically provide us? To understand this we should first look at the homology groups of this complex.

Definition 2.2.5 (Singular Homology Groups)**.**
The singular cohomology groups are defined as

$$H_p(M) = \frac{Ker(d_p : C_p(M) \to C_{p-1}(M))}{Im(d_{p+1} : C_{p+1}(M) \to C_p(M))}.$$

Definition 2.2.5 sure makes you think. What can this geometrically actually mean? We begin by analysing what the components of this definition entail.

$Ker(d_p : C_p(M) \to C_{p-1}(M))$ can be seen as the collection of chains of simplices which enclose a region. For example, a loop γ in $\mathbb{R}^2$ encloses some region, and has $\gamma(0) = \gamma(1)$, thus the boundary is zero. These simplices are called cycles

$Im(d_{p+1} : C_{p+1}(M) \to C_p(M))$ can be seen as the collection of chains of simplices which are the boundary of a $(p+1)$-simplex. In Example 2.2.3 we saw that $(e_1, e_2) - (e_0, e_2) + (e_0, e_1)$ is the boundary of Δ_2. These simplices are called boundaries.

Thus, the smooth singular homology groups measure how many simplices enclose some area, but are not the boundary of some higher dimensional simplex. More intuitively, it measures how many simplices which enclose an area which cannot be filled in.

Remember that a simplex could not be filled in if there was some hole in its interior. As such, a simplex which encloses some area is not a boundary of some simplex if it encloses some hole! So, we find holes using simplices.

Example 2.2.5. *Let $M = \mathbb{R}^2$. Then any p-cycle is the boundary of a $(p-1)$-simplex. This much we know, as M has no holes. By Contrast, we know $N = \mathbb{R}^2 - \{0\}$ has one hole, and indeed N has cycles which are not boundaries. Namely every loop around 0.*

2.2.4 *Homology of the Klein Bottle*

These definitions are nice and hopefully geometrically intuitive, but we have not actually calculated anything with this theory so far. To test our understanding of the concept of smooth singular homology, we might want to calculate the homology groups of some smooth manifold; The Klein bottle.

Definition 2.2.6 (Klein Bottle)**.**
Define the Klein bottle K as the quotient of $\mathbb{R}^2$ *under the action of the group* $G = \langle \alpha, \beta \rangle = \langle (x+1, y), (-x, y+1) \rangle$[3]*. Thus*

$$K = \mathbb{R}^2 / G.$$

Definition 2.2.6 is a mess. It might describe some object, but the definition does not give us much to work with. Or does it? If we look at α and β, K becomes quite clear!

Under α vertical lines in the plane are identified with each other modulo 1. Under β we also identify horizontal lines with each other modulo 1. However, the $-x$ term of β implies that as we identify horizontal lines with each other, the x coordinates are mirrored in the $y-$axis. As such, the horizontal lines are flipped. This leads to the following unit square

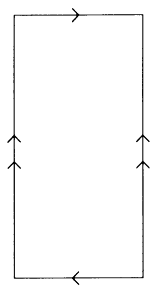


Figure 5: The Klein bottle (Fulton, [4] Section 8b).

The vertical lines are identified with each other, as shown using the double arrow notation. The horizontal lines are identified with the mirrored lines modulo 1, this is shown using the flipped single arrows.

Remark 2.2.1. *K is a smooth* 2*-manifold.*

Using Figure 5, we can also work backwards and construct the Klein bottle. This is done in the following manner. First glue the lengthwise sides together to obtain a cylinder as in Figure 6. Now one end of this cylinder is to be twisted first, and then glued together with the other end. This leads to the familiar Klein bottle as in Figure 6.

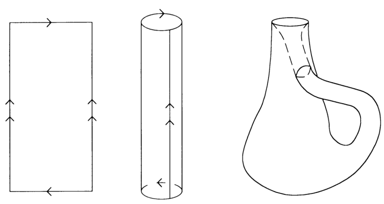


Figure 6: A construction of the Klein bottle (Fulton [4] Section 8b).

[3] Where we identify $(x+1, y)$ with the map $(x, y) \mapsto (x+1, y)$ and the same for $(-x, y+1)$.

We will now calculate the first two singular homology groups of K. To this end, we attack the problem head on. As the kernel of $d_0 : C_0(K) \to 0$ is precisely $C_0(X)$. As the 0-simplices are exactly points, we can identify each $\sigma : \Delta_0 \to K$ with some $x \in K$.

Let $x, y \in K$ be some 0-simplices in K. Then as K is path connected, there is a path $\gamma : \Delta_1 \to K$ with beginning point x and endpoint y. Then $x - y = d_1(\gamma)$. Thus when we take the homology we find $\overline{x} - \overline{y} = \overline{d_1(\gamma)} = 0$. Thus $\overline{x} = \overline{y}$. We obtain x, y differ by one boundary. As such, $H_0(K)$ only contains the class of x scaled by $\mathbb{Z}$. Thus $H_0(K) \cong \mathbb{Z}$.

The first homology group is left to be determined. Any closed 1-simplex contained in the interior of the unit square in Figure 5, is a boundary. This much becomes clear when noting the interior is homeomorphic to $\mathbb{R}^2$. Indeed in $\mathbb{R}^2$ every closed 1-simplex is the boundary of a 2-simplex.

We only need to look at the 1-simplices touching both sides of the square in Figure 7. In Figure 7 three 1-simplices and two 2-simplices are present.

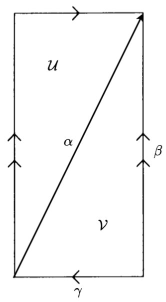


Figure 7: Simplices in the Klein bottle.

The 1-simplices are the diagonal edge α, the vertical edge β -which under the action of G is identified with the other vertical edge-, and the horizontal edge γ -which under the action of G is identified with the mirrored other horizontal edge-. The 2-simplices are the upper triangle denoted by $\mathcal{U}$ and the lower triangle denoted by $\mathcal{V}$.

$H_1(K)$ consists of the cycles which are not the boundary of some simplex. We need to determine the cycles. As in Figure 6 we see the square first is deformed into a cylinder by gluing the β-edges together. This transforms the γ-edge into a cycle, as the begin and endpoint of γ are identified with each other.

Subsequently, as we also glue the γ-edges together, the endpoints of β are identified with each other, thus making it into a cycle. This also transforms α into a cycle as the endpoints are also identified with each other. Thus our cycles are the free group generated by these cycles. $\langle \alpha, \beta, \gamma \rangle$.

By looking at the boundaries of $\mathcal{U}$ and $\mathcal{V}$ the boundary cycles are determined. We see that $d_2(\mathcal{U}) = \beta + \gamma - \alpha$ and $d_2(\mathcal{V}) = \alpha + \beta - \gamma$. Thus the boundaries are generated by $\langle \beta + \gamma - \alpha, \alpha + \beta - \gamma \rangle$. This leads to the following calculation:

$$H_1(K) = \frac{\langle \alpha, \beta, \gamma \rangle}{\langle \beta + \gamma - \alpha, \alpha + \beta - \gamma \rangle} = \frac{\langle \alpha, \alpha + \beta - \gamma, \gamma \rangle}{\langle 2\gamma - 2\alpha, \alpha + \beta - \gamma \rangle} =$$

$$\frac{\langle \alpha, \gamma \rangle}{\langle 2\gamma - 2\alpha \rangle} = \frac{\langle \alpha, \gamma - \alpha \rangle}{\langle 2\gamma - 2\alpha \rangle} = \frac{\langle \alpha, A \rangle}{\langle 2A \rangle} = \frac{\langle A \rangle}{\langle 2A \rangle} \oplus \langle \alpha \rangle \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}.$$

Where we defined $A = \gamma - \alpha$. As such, we obtain that $H_1(K) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$. This leads to the following total list of smooth singular homology groups.

$$H_i(K) = \begin{cases} \mathbb{Z}, \text{ for } i = 0, \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, \text{ for } i = 1. \end{cases} \tag{2.2}$$

Remark 2.2.2. *It should be noted that the calculation above is not an actual proof, but more so an application of the definition of the singular homology groups. A proper proof of this result can be done using Theorem 2.3.2 and mirrors the proof given in Section 2.3.1.*

2.3 SINGULAR COHOMOLOGY

The singular homology allows us to measure the number of holes in a smooth manifold using simplices. However, the theorem of De Rham is a statement about singular *co*homology. The singular cohomology is a dual concept to the singular homology. To make the jump from homology to cohomology, we begin back at the chain complex.

$$C_\bullet(M): \qquad \dots \longrightarrow C_p(M) \xrightarrow{d_p} C_{p-1}(M) \xrightarrow{d_{p-1}} \dots$$

In order to obtain cohomology groups, this downward chain complex has to be transformed into an upward complex. This much is achieved by taking the dual of each abelian group with respect to $\mathbb{R}$. In other words, we take as objects $Hom(C_p(M), \mathbb{R})$ the abelian group of all group homomorphisms from $C_p(M)$ to $\mathbb{R}$. This group will be denoted by $C^p(M)$.

The differential operator $d_p : C_p(M) \to C_{p-1}(M)$ induces a differential operator on the dual group via $\partial^p = - \circ d_p : C^p(M) \to C^{p+1}(M)$, the pre-composition map of d_p. Indeed we see that $\partial^{p+1} \circ \partial^p = - \circ [d_p \circ d_{p+1}] = - \circ 0 = 0$.

Combining this differential operator and the objects described above the following complex is obtained.

$$C^\bullet(M): \qquad \dots \xleftarrow[\partial^p]{} C^p(M) \xleftarrow[\partial^{p-1}]{} C^{p-1}(M) \xleftarrow[\partial^{p-2}]{} \dots$$

This complex is called the cochain complex. Now that we have a complex we can take the cohomology groups of this complex. This is exactly what the singular cohomology entails. Thus we obtain

Definition 2.3.1 (Singular Cohomology)**.**
Let M be a smooth manifold. The cohomology groups of the complex $C^\bullet(M)$, defined as

$$H^p(M) = \frac{Ker(\partial^p : C^p(M) \to C^{p+1}(M))}{Im(\partial^{p-1} : C^{p-1}(M) \to C^p(M))},$$

are the singular cohomology groups of M.

In Definition 2.3.1 the dual operation was first applied before cohomology groups were defined. In general this is not the same as taking the dual of the homology groups. The two concepts can be isomorphic under certain circumstances. Only if $H_{p-1}(M)$ is a free group, then $H^p(M) \cong Hom(H_p(M), \mathbb{R})$ holds true. This theorem is proven in section 3.1 of (Hatcher, [6]).

Many concepts of singular homology generalise to singular cohomology. For example the notion of cochains, coboundaries and cocycles is clear. However, unlike homology the geometric intuition is more clouded.

The central concepts can be interpreted as follows. Cochains assign real numbers to simplices on a smooth manifold. Using the boundary operator a $p-1$ cochain can be lifted to a p cochain. If given a p cochain we wish to know whether it is a lifted cochain, $\partial\omega \equiv 0$ has to hold. The cochains with this property assign zero to every boundary of simplices and are called the cocycles.

The cochains which are lifted cochains are called coboundaries, in other words $\omega = \partial\nu$ for $\omega \in C^p(M)$ and $\nu \in C^{p-1}(M)$ implies ω is a coboundary. As such, the cohomology groups measure to which extent every cocycle is actually a lifted cochain.

Geometrically, one might think of these objects as volume functions on simplices. These functions assign to the simplices a real number which is their volume. There are, however, few restrictions on the assignment of these *volumes*.

Thinking back to Chapter 1, we see a parallel with the differential forms. Both assigned a volume to geometric shapes which locally approximate the surface of a manifold. However, the cochains do not act on the tangent spaces.

With our newfound intuition, the singular cohomology groups of some smooth manifolds can be calculated. We begin with the simplest of manifolds, a single point $M = \bullet$.

The Singular Cohomology of a Point

We turn to the definition of singular cohomology. This provides us with

$$H^p(\bullet) = \frac{Ker(\partial^p : C^p(\bullet) \to C^{p+1}(\bullet))}{Im(\partial^{p-1} : C^{p-1}(\bullet) \to C^p(\bullet))}.$$

We begin simple and look at $p = 0$. As $\partial^{-1} \equiv 0$ we obtain that $Im(\partial^{-1} : C^{-1}(\bullet) \to C^0(\bullet)) = 0$. Now only $Ker(\partial^0 : C^0(\bullet) \to C^1(\bullet))$ has to be determined.

Notice that there is only one 0-simplex, namely $\sigma \equiv \bullet$. As such, any 0-cochain can be identified with a constant function on $\bullet$. Additionally, notice that

$$d_0(\sigma) = \sigma(1) - \sigma(0) = \bullet - \bullet = 0.$$

Thus we obtain that any 0-cochain is in $Ker(\partial^0)$. As there are precisely $\mathbb{R}$ constant functions from $\bullet$ to $\mathbb{R}$, we obtain $Ker(\partial^0 : C^0(\bullet) \to C^1(\bullet)) \cong \mathbb{R}$. We immediately see $H^0(\bullet) \cong \mathbb{R}$.

Now we can calculate higher ordered cohomology groups. We split this procedure into two cases. First we assume $p > 0$ to be odd. We begin once more by reviewing $Ker(\partial^p : C^p(\bullet) \to C^{p+1}(\bullet))$.

Let $\gamma \in Ker(\partial^p : C^p(\bullet) \to C^{p+1}(\bullet))$ and let σ be some $(p+1)$-simplex in $\bullet$. Then per assumption

$$\partial^p(\gamma)(\sigma) = (\gamma \circ d_p)(\sigma) = \gamma(\sum_{i=0}^{p+1}(-1)^i(\sigma \circ (e_i, \ldots, \hat{e}_i, \ldots, e_{p+1}))).$$

Note that in $\bullet$ the function $\sigma \circ (e_i, \ldots, \hat{e}_i, \ldots, e_{p+1}) : \mathbb{R}^{p-1} \to \bullet$ is constant. Thus we obtain the above sum is equal to

$$\gamma(\sum_{i=0}^{p+1}(-1)^i(\sigma \circ (e_i, \ldots, \hat{e}_i, \ldots, e_{p+1}))) = \sum_{i=0}^{p+1}(-1)^i\gamma(\bullet).$$

Per assumption p is odd and there are $p+2$ elements in the above sum, thus there are an odd number of elements in the sum. This implies all but one element of the sum cancel out due to the $(-1)^i$ term. We find

$$\gamma(\bullet) = 0$$

has to hold. Thus γ is the zero map, so ∂^p is injective! We obtain $H^p(\bullet) = 0$ per definition.

But what if $p > 0$ is even? If we look at $Ker(\partial^p : C^p(\bullet) \to C^{p+1}(\bullet))$, we obtain the same as before

$$\sum_{i=0}^{p+1}(-1)^i\gamma(\bullet) = 0.$$

However, now p is even, so all terms cancel out! ∂^p is the zero map. We are left with

$$Ker(\partial^p : C^p(\bullet) \to C^{p+1}(\bullet)) = C^p(\bullet) = Hom(C_p(\bullet), \mathbb{R}).$$

As the p-simplices on $\bullet$ are once more constant, this is equal to $Hom(\bullet, \mathbb{R}) \cong \mathbb{R}$.

Now we turn our attention to $Im(\partial^{p-1} : C^{p-1}(\bullet) \to C^p(\bullet))$. As p is even, $p-1$ is odd. As shown before we find ∂^{p-1} is injective.

Let $(\gamma \circ d_{p-1}) \in Im(\partial^{p-1} : C^{p-1}(\bullet) \to C^p(\bullet))$. Then for any p-simplex σ we find

$$(\gamma \circ d_{p-1})(\sigma) = \sum_{i=0}^{p} (-1)^i \gamma(\bullet).$$

As p is even, $p+1$ is odd. Thus we obtain this sum is equal to $\gamma(\bullet)$. Since γ is constant, we can identify $Im(\partial^{p-1} : C^{p-1}(\bullet) \to C^p(\bullet))$ with the constant function to $\mathbb{R}$. As noted before, there are precisely $\mathbb{R}$ of these functions.

Thus the smooth singular cohomology groups for even p becomes

$$H^p(\bullet) \cong \frac{\mathbb{R}}{\mathbb{R}} = 0.$$

This leads to the following total overview of smooth singular cohomology groups.

$$H^p(\bullet) \cong \begin{cases} \mathbb{R}, & \text{for } p = 0, \\ 0, & \text{for } p \geq 1. \end{cases}$$

This was hard work. If the singular cohomology of this simple smooth manifold is so hard to compute, is there any hope to calculate this for more complex manifolds? Luckily there is! Many theorems have been dreamt up which allow for easier calculation of the cohomology groups.

One way to obtain the cohomology groups of a smooth manifold is by relating it to another smooth manifold using a smooth mapping. As the smooth mapping relates the geometry of the two manifolds, it also relates their singular cohomology groups. This much is captured in the following theorem.

Theorem 2.3.1. *Homeomorphisms and contractions induce isomorphisms in cohomology groups. As such, the following isomorphisms are true*

$$H^p(\mathbb{B}^n) \cong H^p(\mathbb{R}^n) \cong H^p(\bullet).$$

Theorem 2.3.1 allows us to relate the cohomology groups of smooth manifolds using only certain mappings between them. A proof of Theorem 2.3.1 is omitted in this text, a proof of this theorem can be found on page 201 in (Hatcher, [6]).

Another important theorem is the Mayer-Vietoris theorem. Intuitively this theorem allows us to cut up our manifold M into two open sets U, V. If we know the cohomology groups of $U, V, U \cap V$, we can use this to calculate the cohomology groups of M.

Theorem 2.3.2 (Mayer-Vietoris)**.**
Let M be a smooth manifold and let U, V be two open subsets of M such that $M = U \cup V$. Then there is a sequence of inclusions

$$M \xleftarrow{k} U \sqcup V \underset{j}{\overset{i}{\leftleftarrows}} U \cap V$$

By taking the singular chain complexes of the objects in this sequence, a s.e.s. of the following form is obtained.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^\bullet(M) & \longrightarrow & C^\bullet(U) \bigoplus C^\bullet(V) & \xrightarrow{i^* - j^*} & C^\bullet(U \cap V) & \longrightarrow & 0 \\ & & & & (\omega, \nu) & \longmapsto & \nu - \omega & & \end{array}$$

This induces a l.e.s. of cohomology of the following form

$$\dots \longrightarrow H^p(M) \longrightarrow H^p(U) \oplus H^p(V) \longrightarrow H^p(U \cap V) \longrightarrow$$
$$\hookrightarrow H^{p+1}(M) \longrightarrow H^{p+1}(U) \oplus H^{p+1}(V) \longrightarrow H^{p+1}(U \cap V) \longrightarrow \dots$$

The proof of this theorem requires some more advanced notions not treated in the text so far. As such, no proof of this theorem is provided here. However, a proof of this theorem can be found in Chapter one of (Bott & Tu, [2]).

From this theorem the following fact[4] can be derived.

Fact 2.3.1. *The singular cohomology groups of the unit circle S^1 are the following*

$$H^p(S^1) \cong \begin{cases} \mathbb{R}, & \text{for } p = 0, 1, \\ 0, & \text{for } p \geq 2. \end{cases}$$

2.3.1 *Singular Cohomology of the Klein Bottle*

Using theorem 2.3.2 we can calculate the cohomology groups of more complex smooth manifolds. Fact 2.3.1 already provides some singular cohomology groups without any proof. Let us now take the time to properly calculate the singular cohomology groups of a more complex smooth manifold. To this end, we return to the Klein bottle.

We will calculate the singular cohomology groups of K using the theorem of Mayer-Vietoris. We begin by choosing an easy open cover of K. We choose an open cover of $\mathcal{U}$ and of $\mathcal{V}$ as shown in Figure 8.

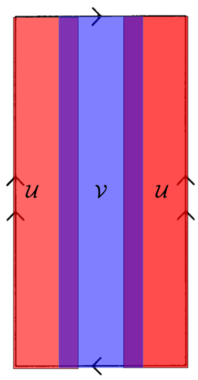


Figure 8: The open cover of the Klein bottle.

$\mathcal{U}$ and $\mathcal{V}$ are strips of which the ends are glued together after one flip has been applied. This might be know to the reader as a Möbius-strip! The intersection $\mathcal{U} \cap \mathcal{V}$ is actually a cylinder. The action of G on the square glues the left top to the bottom right, and the top right to the bottom left. During this process no flips are introduced and we are left with a cylinder.

Fact 2.3.2. *The Möbius-strip and cylinder have the same singular cohomology groups as S^1.*

We now turn to the l.e.s. of Mayer-Vietoris. This results in

4 This fact can be constructed using all the theory developed in the text. If the reader is interested, they are advised to do so.

$$0 \to H^0(K) \to H^0(\mathcal{U}) \bigoplus H^0(\mathcal{V}) \to H^0(\mathcal{U} \cap \mathcal{V}) \to H^1(K) \to H^1(\mathcal{U}) \bigoplus H^1(\mathcal{V}) \to H^1(\mathcal{U} \cap \mathcal{V}) \to H^2(K) \to H^2(\mathcal{U}) \bigoplus H^2(\mathcal{V}) \to H^2(\mathcal{U} \cap \mathcal{V}) \to \dots$$

Due to Fact 2.3.2 we know $\mathcal{U}, \mathcal{V}, \mathcal{U} \cap \mathcal{V}$ and S^1 have the same singular cohomology groups. These can be filled in the above diagram to obtain the following exact sequence.

$$0 \longrightarrow H^0(K) \longrightarrow \mathbb{R} \bigoplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow H^1(K) \longrightarrow \mathbb{R} \bigoplus \mathbb{R} \longrightarrow \mathbb{R} \longrightarrow H^2(K) \longrightarrow 0 \longrightarrow \dots$$

Look at the map $\mathbb{R}^2 \to \mathbb{R}$, what does this map look like? This arrow corresponds with the arrow $H^p(\mathcal{U}) \oplus H^p(\mathcal{V}) \to H^p(\mathcal{U} \cap \mathcal{V})$ for $p = 0, 1$ which was induced by $i^* - j^*$. This arrow is defined as

$$(\overline{\omega}, \overline{\nu}) \mapsto \overline{\nu - \omega}.$$

Under the identification $H^p(\mathcal{U}) \cong \mathbb{R}$ for $p = 0, 1$ we assign to $\overline{\omega} \in H^p(\mathcal{U})$ a number $m \in \mathbb{R}$. The map $H^p(\mathcal{U}) \oplus H^p(\mathcal{V}) \to H^p(\mathcal{U} \cap \mathcal{V})$ then corresponds to $\phi : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$(m, n) \mapsto n - m.$$

Quite clearly $\mathbb{R}^2 \to \mathbb{R}$ is surjective. By the exactness of the above sequence $Ker(\mathbb{R} \to H^1(K)) = Im(\mathbb{R}^2 \to \mathbb{R}) = \mathbb{R}$. As such, the connecting map has its entire domain as its kernel! So, it has to be the zero map. Now the Mayer-Vietoris sequence can be split into two exact parts

$$0 \longrightarrow H^0(K) \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \longrightarrow 0$$

$$0 \longrightarrow H^1(K) \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \longrightarrow H^2(K) \longrightarrow 0$$

Note that

$$Ker(\phi) = \{(m, n) \in \mathbb{R}^2 : m = n\} \cong \mathbb{R}.$$

Per exactness of the sequences, $Im(H^0(K) \to \mathbb{R}^2) = Ker(\phi) \cong \mathbb{R}$. As $H^0(K) \to \mathbb{R}^2$ is injective we obtain $H^0(K) \cong Im(H^0(K) \to \mathbb{R}^2) \cong \mathbb{R}$. Finally, the result is $H^0(K) \cong \mathbb{R}$. By the exact same argument $H^1(K) \cong \mathbb{R}$ is also true.

Per exactness, $Ker(\mathbb{R} \to H^2(K)) = Im(\mathbb{R}^2 \to \mathbb{R})$ holds true. As $\mathbb{R}^2 \to \mathbb{R}$ is surjective, we obtain $Ker(\mathbb{R} \to H^2(K)) = \mathbb{R}$. Thus, $\mathbb{R} \to H^2(K)$ is the zero map. By the first isomorphism theorem we obtain $\mathbb{R}/\mathbb{R} \cong 0 \cong= H^2(K)$, which results in $H^2(K) \cong 0$.

In total we obtain

$$H^p(K) = \begin{cases} \mathbb{R}, & \text{for } p = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

Having calculated the singular cohomology and singular homology of the Klein bottle, it might be instructive to compare these results. First, we notice that cohomology changes the group $\mathbb{Z}$ to the group $\mathbb{R}$. During this process the torsion of the singular homology groups is lost. However, for all non-torsion groups this process has no further effect. As such, intuitively we can see singular cohomology as a measure for the number of holes in a surface up to torsion.

2.4 DE RHAM COHOMOLOGY

After our dive into the world of homological algebra, we find ourselves back in the world of differential forms. Here we encounter an important concept: the De Rham cohomology. The De Rham cohomology will answer our question of how many closed and not exact forms exist on a given manifold which was posed in Section 1.5.

We have already developed all the required machinery for the De Rham cohomology. In fact, the De Rham cohomology is the cohomology of a complex we have already seen way back in Chapter 1, the complex of differential forms:

$$\Omega^\bullet(M): \qquad \dots \xrightarrow{d} \Omega^{m-1}(M) \xrightarrow{d} \Omega^m(M) \xrightarrow{d} \dots$$

Previously, we have not explicitly stated that this is a complex. It is, however, not hard to see that this is indeed so. Each of the objects in this sequence is an abelian group, and as in definition 1.1.5 $d \circ d \equiv 0$. As discussed in Section 2.1 we can take the cohomology groups of this complex. This results in:

Definition 2.4.1 (De Rham Cohomology)**.**
The cohomology groups of the complex $\Omega^\bullet(M)$ are defined as

$$H^p_{DR}(M) = \frac{Ker(d^p : \Omega^p(M) \to \Omega^{p+1}(M))}{Im(d^{p-1} : \Omega^{p-1}(M) \to \Omega^p(M))}.$$

As described in Chapter 1 the existence of closed and not exact differential forms was related to the failure of a fundamental theorem of calculus for differential forms. Definition 2.4.1 measures the number of closed forms which are not exact. As mentioned in Chapter 1 this forms a measure of the failure for the fundamental theorem of calculus for closed differential forms. As such, the De Rham cohomology groups provide a measure for the extent of this failure.

In fact, our work in Chapter 1 already provides us with much knowledge regarding the De Rham cohomology. Remember the Poincaré Lemma 1.5.1? This lemma stated that every closed form on $\mathbb{R}^n$ was also exact. This lemma can be restated in terms of the De Rham cohomology.

Lemma 2.4.1 (Poincaré Lemma)**.**
Let $M \in \mathbf{Man}^\infty$. Suppose that $\phi : M \to \bullet$ is a contraction[5]*, then ϕ induces an isomorphism of De Rham cohomology groups. In other terms*

$$H^p_{DR}(M) \cong H^p_{DR}(\bullet) \cong \begin{cases} \mathbb{R}, & \textit{for } p = 0, \\ 0, & \textit{for } p \geq 1. \end{cases}$$

The renewed Poincaré lemma provides us with the cohomology groups for many smooth manifold. In particular, we get the cohomology groups of $\mathbb{R}^n$ for free, since these manifolds are contractible.

Remark 2.4.1. *The theorem of Mayer-Vietoris is also true for the De Rham cohomology groups.*

The cochains consisted of volume forms on chains of simplices, while the differential forms are volume forms on tangent parallelepipeds. A simplex and a parallelepiped seem not to far removed from each other. As such, it might seem like the two cohomologies are closely related. But are they?

Example 2.4.1. *The De Rham cohomology groups of S^1 can be determined to be*

$$H^p_{DR}(S^1) \cong \begin{cases} \mathbb{R}, & \textit{for } p = 0, 1 \\ 0, & \textit{for } p \geq 1. \end{cases}$$

[5] This lemma can also be generalised in terms of homotopy equivalences. If the reader wishes to know more they are advised to review Corollary 4.1.2 of (Bott & Tu, [2]).

The De Rham cohomology groups seem to be identical to the smooth singular cohomology groups! It makes you wonder, are these cohomologies isomorphic? Maybe S^1 and $\bullet$ were just flukes. After all, a single point and a circle are simple smooth manifolds. Perhaps a more complex smooth manifold will prove to be a counterexample.

2.4.1 *De Rham Cohomology of the Klein Bottle*

In order to test our hypothesis, we begin with calculating the De Rham cohomology groups of K. As Mayer-Vietoris also holds for De Rham cohomology, we just apply this theorem once more. We even choose the same open cover for K. Which leads to the following l.e.s. of Mayer-Vietoris.

$$\begin{array}{l} 0 \longrightarrow H^0(K) \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \\ \hookrightarrow H^1(K) \longrightarrow \mathbb{R}^2 \longrightarrow \mathbb{R} \\ \hookrightarrow H^2(K) \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots \end{array}$$

We have seen this sequence before when calculating the singular cohomology groups! Not surprisingly, this leads to the same groups. We once more obtain

$$H^p_{DR}(K) = \begin{cases} \mathbb{R}, & \text{for } i = 0, 1, \\ 0, & \text{for } i \geq 2. \end{cases}$$

It seems even more complex smooth manifolds, like the Klein bottle, have isomorphic singular and De Rham cohomology. The following preview will show our hypothesis to be truthful.

2.5 PREVIEW OF THE THEOREM OF DE RHAM

The singular cohomology and the De Rham cohomology groups are closely related. In fact, there exists an isomorphism between these two cohomology groups. Sadly, we are far removed from proof of this theorem. However, we can go ahead and take a sneak peak at this wonderful theorem.

Theorem 2.5.1 (De Rham)**.**
There is an isomorphism of groups between the pth singular cohomology and the p'th De Rham cohomology groups.

$$\mathfrak{c} : H^p_{DR}(M) \xrightarrow{\sim} H^p(M)$$

The isomorphism $\mathfrak{c}$ *is defined as*

$$[\omega] \mapsto \left(\int_{(-)} \omega : \sigma \mapsto \int_\sigma \omega \right).$$

Previously, we drew a parallel between cochains and differential forms. In particular, their shared roll in assigning volume to objects which locally approximate the manifold. If we commit the cardinal sin of identifying tangent parallelepipeds with actual parallelepipeds on a manifold, this isomorphism has intuitive merit.

As mentioned before, the singular cohomology measures holes up to torsion. This intuition allows us to completely rephrase the theorem of De Rham. The theorem of De Rham states that the degree to which the fundamental theorem of calculus fails for closed differential forms on a smooth manifold is precisely the number of holes, up to torsion, in this smooth manifold. This provides a direct relation between the geometry and analysis on a smooth manifold. A weak form of this theorem can already be proven using the material covered so far.

Theorem 2.5.2 (Weak Theorem of De Rham)**.**
For any $p \in M$, there is an open neighborhood U for which the following isomorphism is true

$$H^p_{DR}(U) \cong H^p(U),$$

for every $p \in \mathbb{Z}$.

Proof.
Let $p \in M$, then there is an open neighborhood U of p which is homeomorphic to $\mathbb{R}^n$. As $\mathbb{R}^n$ is contractible to a point, so is U. The Poincaré lemma 2.4.1 now provides that

$$H^p_{DR}(U) \cong \begin{cases} \mathbb{R}, & \text{for } p = 0, \\ 0, & \text{for } p \geq 1. \end{cases}$$

As U is homeomorphic to $\mathbb{R}^n$, Theorem 2.3.1 provides that

$$H^p(U) \cong \begin{cases} \mathbb{R}, & \text{for } p = 0, \\ 0, & \text{for } p \geq 1. \end{cases}$$

As such, we obtain

$$H^p_{DR}(U) \cong H^p(U),$$

for every $p \in \mathbb{Z}$. □

At the local level, Theorem 2.5.2 provides the desired isomorphism. However, it seems uncertain whether this local isomorphism can be lifted to a global isomorphism. To solve this issue, we need to lift local information to global information. Enter sheaves, a powerful mathematical tool which will help us achieve this goal. In the upcoming chapter, we will delve into the theory of sheaves and develop how they lift local information to global information.

Subsequently, in Chapter 4 the homological algebra of sheaves will be developed. This theory generalises many concepts developed in this chapter to sheaves. In Chapter 5 the homological algebra of sheaves is used to prove the theorem of De Rham.

3

SHEAF THEORY

Our adventure towards the theorem of De Rham continues with the introduction of sheaves. Sheaves were introduced in 1946 by the french mathematician Jean Leray as described in Dieudonné's *A History of Algebraic and Differential Topology* (Dieudonné, [3]).

During the Nazi occupation of France, Leray was a prisoner of war (POW) in the Edelbach prison. Fearing he would be forced to work for the Nazis due to his expertise in applied mathematics, he turned to pure mathematics specialising in algebraic topology. It was in this prison Leray concocted sheaves and revolutionised the field of algebraic topology. If the reader wishes to learn more about Leray in Edelbach, and the *University in Captivity* he managed in prison, they are invited to review *Leray in Edelbach* (Anna et al, [1]).

After having been introduced by Leray, sheaves have become a powerful tool for gluing local data together into global data. Intuitively, sheaves assign abelian groups to open sets of a topological space in such a way that the elements of the abelian groups can be glued together to obtain a global element and abelian group. This is especially useful for topological spaces which are locally very simple, such as smooth manifolds. The gluing property of sheaves will prove indispensable for the proof of the theorem of De Rham in Chapter 5.

In this chapter we first develop the theory of sheaves. This encompasses what sheaves are, how to create sheaves as well as providing a whole zoo of useful sheaves. Subsequently, in Chapter 4 the theory of sheaf cohomology is developed, which can be seen as generalisation of the notions from Chapter 2.

3.1 CATEGORY THEORY

When talking about sheaves the language of category theory proves useful. As such, this theory is used in this and subsequent chapters. A basic understanding of this theory is assumed from the reader. If the reader wishes to refresh their memory or become acquainted with category theory, they are advised to review the first chapter of (Riehl, [13]).

In the sequel, we will write $A \in \mathcal{C}$ to mean that A is an object in the category $\mathcal{C}$. As such, we will write $M \in \mathbf{Man}^\infty$ to mean that M is an element of the category of smooth manifolds. This category is defined as follows.

Definition 3.1.1 (Category of Smooth Manifolds)**.**
The category $\mathbf{Man}^\infty$ *is the category of smooth manifolds, and has the following objects and arrows.*

- *The objects are the smooth manifolds.*
- *The arrows are the smooth functions between smooth manifolds.*

Another frequently used category is the category of open subsets of a smooth manifold M. This category is defined as follows.

Definition 3.1.2 (Category of Open Subsets)**.**
Let $M \in \mathbf{Man}^\infty$. Then the category of open subsets of M denoted by $\mathbf{Op}(M)$ has the following objects and arrows.

- *The objects are the open subsets of M.*
- *For any pair $U, V \in \mathbf{Op}(M)$ with $V \subseteq U$ the arrow between these objects is inclusion map $i_{VU} : V \hookrightarrow U$.*

3.2 PRESHEAVES

Sheaves can be seen as objects assigning abelian groups to open subsets of a smooth manifold M, such that elements of the abelian groups can be glued together. This gluing property seems to be an extra requirement on a mapping from open sets of manifold to the category of abelian groups. As such, we start our treatise of sheaves with a less powerful notion, the presheaf.

Definition 3.2.1 (Presheaf)**.**
A presheaf F on a smooth manifold M contains the following components

- *For each open $U \subseteq M$ an abelian group $F(U)$,*
- *For every pair of opens $U, V \subseteq M$ such that $V \subseteq U$, there is a restriction map $\rho_{UV} : F(U) \to F(V)$.*

These objects abide by the following

- *$\rho_{UU} = Id_U$ for all open $U \subseteq M$,*
- *For every tower of open sets $A \subseteq B \subseteq C$ in M, the following functoriality holds $\rho_{CA} = \rho_{BA} \circ \rho_{CB}$.*

The elements $s \in F(U)$ are called sections.

Remark 3.2.1. *Definition 3.2.1 can be restated in the language of category theory. A presheaf on $M \in \mathbf{Man}^\infty$ is a contravariant functor $F : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$. In the sequel the functor notation will be used for presheaves.*

Remark 3.2.2. *Presheaves are defined as functors from the category of open subsets to $\mathbf{Ab}$. This definition can be generalised by mapping into other categories than $\mathbf{Ab}$. Some examples are presheaves of rings, vector spaces, fields, modules...*

What does Definition 3.2.1 actually tell us about presheaves? A presheaf F takes an open $U \subseteq M$ and maps this to some abelian group $F(U)$. It seems, we have properly defined a basic notion from which we can build sheaves.

Presheaves are fairly ubiquitous in mathematics, most likely the reader has already become acquainted with a large scale of presheaves prior to reading this text. Examples are the additive group of continuous functions on a topological space, or analytic functions on $\mathbb{C}$. Some other important examples are.

Example 3.2.1. *Let $F = \mathbb{R}$ be the functor assigning the abelian group $\mathbb{R}$ to any open subset of X. The restriction maps are given by the identity maps on $\mathbb{R}$. This functor is indeed a presheaf.*

Given any open $U \in \mathbf{Op}(M)^{op}$ we find that $\rho_{UU} = Id_U$ per definition. As such, the first presheaf axiom is satisfied. Furthermore, for a tower of open sets $A \subseteq B \subseteq C$ in M, we find

$$\rho_{CA} = Id_{\mathbb{R}} = Id_{\mathbb{R}} \circ Id_{\mathbb{R}} = \rho_{BA} \circ \rho_{CB}.$$

Thus, also the second presheaf axiom is satisfied, making $\mathbb{R}$ into a presheaf.

Example 3.2.2. *The functor $C^\infty(-, \mathbb{R}) : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$ is a presheaf which assigns to $U \in \mathbf{Op}(M)^{op}$ the additive group of smooth functions $C^\infty(U, \mathbb{R})$. The restriction maps are given by restricting the domains of the smooth functions.*

Indeed for any $U \in \mathbf{Op}(M)^{op}$ *and* $f \in C^\infty(U, \mathbb{R})$,

$$\rho_{UU}(f) = f|_U = f = Id(f).$$

The first presheaf axiom is true. Furthermore, for any tower of open sets $A \subseteq B \subseteq C$ *in* M, *we obtain*

$$\rho_{CA}(f) = (f)|_A = (f|_B)|_A = \rho_{BA} \circ \rho_{CB}(f)$$

for any $f \in C^\infty(C, \mathbb{R})$. *As such,* $\rho_{CA} = \rho_{BA} \circ \rho_{CB}$ *and the second presheaf axiom is satisfied, making* $C^\infty(-, \mathbb{R})$ *into a presheaf.*

One might wonder when presheaves are the *same*? As presheaves are contravariant functors, we get a notion of morphism of presheaves for free. This leads to (iso)morphisms of sheaves.

Definition 3.2.2 ((Iso)Morphisms of Presheaves)**.**
A morphism of presheaves is a natural transformation $\phi : F \to G$ *such that for every* $U \in \mathbf{Op}(M)^{op}$ *and every open* $V \subseteq U$ *the following diagram commutes.*

$$\begin{array}{ccc} F(U) & \xrightarrow{\phi_U} & G(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho_{UV}} \\ F(V) & \xrightarrow{\phi_V} & G(V) \end{array}$$

The morphism ϕ *is called an isomorphism of presheaves if the arrows* ϕ_U, ϕ_V *are isomorphisms for all pairs* U, V.

Definition 3.2.2 allows us to create arrows between presheaves. So, we have objects, namely the presheaves, and arrows, the presheaf morphisms. If there is any justice in this world, then this would form a category.

Definition 3.2.3 (Category of Presheaves)**.**
For any $M \in \mathbf{Man}^\infty$ *the category of presheaves* $\mathbf{Psh}(M)$ *consists of the following objects and arrows.*

- *The objects are the presheaves on* M.
- *The arrows are the morphisms of presheaves as described in Definition 3.2.2.*

One might wonder whether we need to go any further, is this not already strong enough to patch together sections of abelian groups? Since we already have a category full of presheaves it may seem meant to be. Let us check this.

Suppose we take a simple smooth manifold, say $\mathbb{R}$ with the standard smooth structure and as presheaf take the constant presheaf of $\mathbb{Z}$. Let $U = B_{\frac{1}{2}}(0)$ and $V = B_{\frac{1}{2}}(1)$ be opens in $\mathbb{R}$. Note that $U \cap V = \emptyset$.

Let $3 \in \mathbb{Z}(U)$ and $1 \in \mathbb{Z}(V)$ be sections. If $\mathbb{Z}$ has the gluing property, these sections would need to be gluable. Since on $U \cap V = \emptyset$, 1 and 3 do coincide. However, if we glue these two sections together, we do not obtain a section on $U \cup V$, as the glued section is not constant. It has value 3 on U and 1 on V. Therefore, this glued section is not an element of $F(U \cup V) = \mathbb{Z}$. Thus this presheaf assigns abelian groups to open subsets in such a way, that these cannot be glued together.

Now the reader might proclaim we fix this problem by requiring the intersection to be non-empty. However, geometrically this leads to problems. In Figure 9 three open neighborhoods are displayed in the plane. The intersection of A and C is clearly empty. According to this argument, we cannot glue elements of any presheaf applied to A and C together.

However, if we first glue sections of $\mathbb{Z}$ applied to A to those of B, and then those of B to those of C we obtain a section of $\mathbb{Z}$ applied to $A \cup B \cup C$. Restricting this section to $A \cup C$ provides us with a gluing of sections of $\mathbb{Z}$ over A and C. If we want to obtain a global abelian group, we need to glue on empty intersections: vacuous gluing.

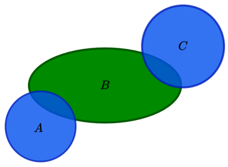


Figure 9: The gluing property required vacuous gluing.

3.3 SHEAVES

This gluing property requires extra attention. The gluing of abelian groups we wish to do has two intuitive requirements. It would be nice if we could obtain global abelian groups and if the gluing of sections was unique. This is exactly what we add to presheaves to obtain sheaves.

Definition 3.3.1 (Sheaves).
$\mathcal{F} \in \mathbf{Psh}(X)$ is called a sheaf if for every $U \in \mathbf{Op}(M)^{op}$ and every open cover $(U_i)_{i \in I}$ of U the following is satisfied

- *if $s, t \in \mathcal{F}(U)$ and $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$,*
- *if $s_i \in \mathcal{F}(U_i)$ and if $U_i \cap U_j \neq \emptyset$ implies*

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j},$$

  for all $i \in I$, then there is an $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

What does this gluing property look like? The reader might adopt the picture detailed in Figure 10. On the left in Figure 10, a presheaf is displayed. This presheaf takes open sets in M and relates abelian groups $\mathcal{F}(U)$ to U and $\mathcal{F}(V)$ to V. These abelian groups need not be gluable, and as such are displayed separately. On the right a sheaf is displayed. The sheaf also assigns $\mathcal{F}(U)$ and $\mathcal{F}(V)$ to U and V, however, this data does overlap and can be thought of as being gluable.

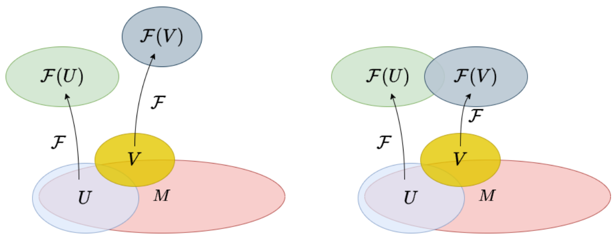


Figure 10: The difference between a sheaf and presheaf.

Just like before, we can create arrows between sheaves using morphisms of functors. As such, a morphisms of sheaves is just a morphism of presheaves. Additionally, this also induces a category structure on the sheaves.

Definition 3.3.2 (Category of Sheaves).
For any $M \in \mathbf{Man}^\infty$, the category of sheaves on M denoted by $\mathbf{Sh}(M)$ contains the following.

- *The objects are the sheaves on M as defined in Definition 3.3.1.*
- *The arrows are the morphisms of presheaves as in Definition 3.2.2.*

Now the reader might wonder, how does one obtain a sheaf? The answer is two pronged, one can find sheaves in *the wild* or one can grow sheaves. We begin with some examples of sheaves found in the wild. How to grow sheaves will be discussed in Section 3.4.

Example 3.3.1. *Denote by $Map(-) : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$ the sheaf of functions. This functor eats an open set $U \in \mathbf{Op}(M)^{op}$ and returns the abelian group of functions from U to $\mathbb{R}$. The restriction mappings are given by restriction of domain.*

A quick check reveals $Map(-) \in \mathbf{Psh}(M)$. Let $U \in \mathbf{Op}(M)^{op}$ and $(U_i)_{i \in I}$ be a open cover of U. Let $s, t \in Map(U)$, and assume $s|_{U_i} = t|_{U_i}$ for all $i \in I$. Then s and t coincide at every point in U, as such $s = t$ must hold. Thus the first sheaf axiom is satisfied.

Suppose that $s_i \in Map(U_i)$ for all $i \in I$ and $U_i \cap U_i \neq \emptyset$ with

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}.$$

Notice that any $x \in U$ is contained in some U_i. Now define $s : U \to \mathbb{R}$ as $x \mapsto s_i(x)$. This mapping is well defined as we assumed s_i and s_j are equal on the intersection of their domains.

Furthermore, $s|_{U_i} = s_i$. The second sheaf axiom is satisfied, making $Map(-)$ into a sheaf.

Example 3.3.2. *For any $G \in \mathbf{Ab}$ a sheaf can be defined. This is the constant sheaf G_M on any smooth manifold M. This sheaf endows G with the discrete topology and assigns to $U \in \mathbf{Op}(M)^{op}$ the abelian group of continuous functions from $U \to G$. This coincides with the abelian group of locally constant functions on M to G.*

Example 3.3.3 (Sheaf of Kernels)**.** *Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then define the functor $Ker(\phi(-)) : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$ as*

$$U \mapsto Ker(\phi(U)).$$

Since $Ker(\phi(U))$ is a subgroup of $\mathcal{F}(U)$ for every $U \in \mathbf{Op}(M)^{op}$, $Ker(\phi(-))$ inherits the restrictions mappings from $\mathcal{F}$ which satisfy the presheaf axioms. As such, it indeed is a presheaf.

Let $s, t \in ker(\phi(U))$ and take an open cover, $(U_i)_{i \in I}$, of U. Suppose that $s|_{U_i} = t|_{U_i}$. As $Ker(\phi(U)) \subseteq \mathcal{F}(U)$ is a subgroup, $s, t \in \mathcal{F}(U)$ is true. Per assumption $\mathcal{F}$ is a sheaf, so $s = t$ has to hold via the first sheaf axiom. Therefore, the first sheaf axiom is satisfied.

Let $s_i \in Ker(\phi(U_i))$ and $U_i \cap U_j \neq \emptyset$. Then once more $s_i \in \mathcal{F}(U_i)$ as well. Since $\mathcal{F}$ is a sheaf there is an $s \in \mathcal{F}(U)$ such that $s|_{u_i} = s_i$. Per definition of a morphism of sheaves, $\phi(s)|_{U_i} = \phi(s|_{u_i}) = \phi(s_i)$. As $s_i \in Ker(\phi(U_i))$ we obtain that $\phi(s)|_{U_i} = \phi(s|_{u_i}) = \phi(s_i) = 0$, thus $\phi(s)|_{U_i} = 0$ for all $i \in I$. This implies $\phi(s) = 0$, and thus $s \in Ker(\phi(U))$.

We conclude that there is an $s \in Ker(\phi(U))$ such that $s|_{U_i} = s_i$. The second sheaf axiom is satisfied making $Ker(-)$ into a sheaf.

Example 3.3.4 (De Rham Sheaf)**.** *The sheaf $\Omega^k : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$ for any $k \geq 0$ is a sheaf on smooth manifolds. This sheaf assigns to every open subset of M the abelian group of differential k-forms. The restriction maps are the restriction maps to the domain of the differential forms. Differential forms are glued together as functions.*

Example 3.3.5 (Singular Cochain Groups)**.** *One might wonder whether the functor $C^k : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$ is a sheaf. One readily checks this functor does have restriction maps, restricting the cochain domains, which satisfy the presheaf axioms.*

However, the gluing property is troublesome. Suppose there are two open subsets U_1 and U_2 in the plane as in Figure 11. Suppose that $\sigma : C_1(U_1) \to \mathbb{R}$ and $\gamma : C_1(U_2) \to \mathbb{R}$ are two cochains which agree on the intersection $U_1 \cap U_2$.

Then $\sigma(v_1)$ and $\gamma(v_2)$ are defined. However, for both cochains $\sigma(v_3)$ and $\gamma(v_3)$ are not defined as v_3 is not an element of either σ or γ's domain.

If we glue σ and γ together to form $\tau : C_1(U_1 \cup U_2) \to \mathbb{R}$, then $v_3 \in C_1(U_1 \cup U_2)$. As such, $\tau(v_3) \in \mathbb{R}$ has to hold. The value of $\tau(v_3)$ can be chosen arbitrarily as when restricting τ to U_i this 1-simplex falls away.

But then σ and γ cannot be uniquely glued together. As $\tau(v_3) = 1$ or $\tau(v_3) = 0$ both can be the glued object of σ and γ. By the same argument C^k is not a sheaf for any $k > 0$. A presheaf which satisfies only the second sheaf axiom is called an epipresheaf.

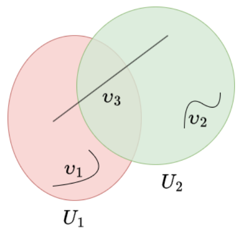


Figure 11: The cochain functor is a epipresheaf.

Oh no! The Functor of singular cochain groups is not a sheaf. This seems troublesome as we wished to obtain global data from the local data of singular cochain groups, and now this cannot be done. Fear not, as this can be solved using a ...

3.4 SHEAFIFICATION

After seeing some sheaves in the wild, one might wonder how to grow sheaves yourself. Growing sheaves can be compared to growing plants, one starts with a seed and with some love and time a beautiful and mesmerising result is obtained; a sheaf. For our seeds we wish to start out with something which is close to a sheaf, we start with a presheaf F. In order to grow our F into a sheaf, we need some additional equipment.

Definition 3.4.1 (Espace Étalé)**.**
*Let $X \in$ **Top** be a topological space. An Espace Étalé is a topological space $Y \in$ **Top** together with a projection map $\pi : Y \to X$ such that π is a local homeomorphism.*

For those familiar with a cover space, an espace étalé is a bit more general as it does not require a fixed cardinality of fibres of π. Just like any projection mapping, sections can be constructed. We denote the set of continuous sections from $U \subseteq X$ to Y by $\Gamma(U, Y)$.

3.4.1 *Germs and Stalks*

The great power of sheaves is glueing local abelian groups into global abelian groups. In order to do this efficiently, we would like to have a proper understanding of these local abelian groups. Previously, we looked at open neighborhoods of a smooth manifold M, but we would like to know what is happening at specific points $p \in M$. This is achieved by zooming in!

Zooming in indefinitely on the point p seems to coincide with taking a limit of local neighborhoods. This provides the *purest* local abelian groups. Zooming in indefinitely is done using stalks.

Definition 3.4.2 (Stalk As a Limit)**.**
Let $F \in \mathbf{Psh}(M)$ and $p \in M$ for $M \in \mathbf{Man}^\infty$. Define the stalk of F at p as follows

$$F_p = \lim_{U \ni p} F(U).$$

Where $U \in \mathbf{Op}(M)^{op}$.

Geometrically, Definition 3.4.2 states that we zoom in as close as possible to $x \in M$ and look at how F behaves in this close neighborhood. As seen in Figure 12 the zooming in coincides with taking the abelian group of smaller and smaller open neighborhoods of $x \in M$. This leads to more local abelian groups around x.

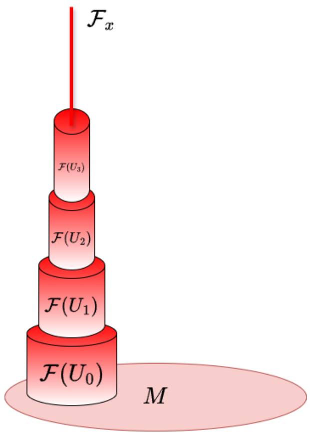


Figure 12: The stalk generation of a stalk $\mathcal{F}_x$ on M.

As we zoom in we say that elements $s, t \in F(U)$ are the same, if they coincide in a neighborhood of x. In other words: if when we zoom in enough we cannot differentiate between s and t, then the limits of s and t should coincide. This leads to the following equivalent definition of stalks.

Definition 3.4.3 (Stalk As Equivalence Classes)**.**
Let $\mathcal{F} \in \mathbf{Psh}(M)$ and $p \in M$ for $M \in \mathbf{Man}^\infty$. The stalk of $\mathcal{F}$ at p is defined as

$$\mathcal{F}_p = \frac{\{(U, s) : U \in \mathbf{Op}(M)^{op} \wedge s \in \mathcal{F}(U)\}}{\sim}.$$

Where $(U, s) \sim (V, t)$ if there is some open neighborhood $W \subset U \cap V$ of p, such that $s|_W = t|_W$.

For every $U \in \mathbf{Op}(M)^{op}$ and $p \in M$ there is a projection map $q_p : \mathcal{F}(U) \to \mathcal{F}_p$ defined as $s \mapsto [(U, s)] \in \mathcal{F}_p$.

Remark 3.4.1. *Let $\mathcal{F} : \mathbf{Op}(M)^{op} \to \mathbf{Ab} \in \mathbf{Psh}(M)$, then the stalk of $\mathcal{F}$ at $p \in M$ is an abelian group $\mathcal{F}_p$.*

Definition 3.4.3 states two elements of the stalk are the same if we cannot distinguish between them in some open neighborhood of $p \in M$. This definition does capture our intuitive notion of zooming in.

Example 3.4.1. *Let $M \in \mathbf{Man}^\infty$ and let A be a finite set. Let $\vartheta \in \mathbf{Sh}(M)$, where $\vartheta(U)$ consists of all locally constant functions from $M \supseteq U$ to $\mathbb{Z}$ which vanish on A. The stalks of ϑ will be calculated using Definition 3.4.3.*

Suppose (ϕ, V) and (ψ, W) are two pairs of sections and open neighborhoods of p. Note that these pairs are equivalent if and only if $\phi(p) = \psi(p)$. If $p \notin A$, we obtain $\phi(p)$ can have any value in $\mathbb{Z}$. Therefore any pair (ϕ, V) is always equivalent to the pair $(\phi(p), M)$, where $\phi(p)$ is the constant function with value $\phi(p) \in \mathbb{Z}$. We conclude that $\vartheta_p \cong \mathbb{Z}$.

However, if $p \in A$, then p must have an open neighborhood on which any section of ϑ is zero. Therefore, any pair (ϕ, V) must be equivalent to $(0, M)$ per definition of ϑ. Thus, we obtain that ϑ_p consists of one equivalence class. We conclude $\vartheta_p \cong 0$.

As seen in Example 3.4.1 the equivalence class of any pair (ϕ, V) details the behaviour of ϕ in a neighborhood of p. Namely, the triviality of ϑ_p encoded that $p \in A$ and that any section of ϑ vanishes on A. Using the map q_p from definition 3.4.3 we can map a section to the equivalence class encoding its local behaviour. This leads to the following definition.

Definition 3.4.4 (Germ)**.**
The projection of $s \in \mathcal{F}(U)$ under $q_p : \mathcal{F}(U) \to \mathcal{F}_p$ is called the germ of s and is denoted by s_p.

Furthermore, any morphism of presheaves $f : \mathcal{A} \to \mathcal{B}$ induces a map on the stalks of the presheaves.

Definition 3.4.5 (Induced Map on Stalks)**.**
Let $f : \mathcal{A} \to \mathcal{B}$ be a morphism of presheaves and let $p \in M$. Then f induces a map $f_p : \mathcal{A}_p \to \mathcal{B}_p$ defined as

$$[(U, s)] \mapsto [(U, f \circ s)].$$

3.4.2 *Growing a Sheaf*

Now we turn to growing a sheaf. Our goal is to grow a sheaf from a presheaf and keep the stalks the same. This way the local information of our presheaf remains in the resulting sheaf. We begin with a seed, namely $\mathcal{F} \in \mathbf{Psh}(M)$. Our goal is to construct an espace étalé over M using the stalks of $\mathcal{F}$. Then by looking at the sections of this espace étalé, we can look for a sheaf.

Our first step in constructing the espace étalé is collecting all the local information of $\mathcal{F}$ we have in one big pile. Since we wish to keep the stalks of $\mathcal{F}$ as the stalks of our final sheaf, we begin by collecting the stalks. Define the following pile of stalks of $\mathcal{F}$.

$$\overline{\mathcal{F}} := \bigsqcup_{p \in M} \mathcal{F}_p$$

This pile of stalks of $\mathcal{F}$ can be paired with a projection map $\pi : \overline{\mathcal{F}} \to M$ and a topology to obtain an espace étalé. We choose the intuitive projection map of $s_x \mapsto x$. For the proper topology a more thorough look at the wanted result is needed. We begin by defining the section of this espace étalé.

Definition 3.4.6 (Section of $(\overline{\mathcal{F}}, \pi)$)**.**
Let $U \in \mathbf{Op}(M)^{op}$. For each $s \in \mathcal{F}(U)$ define the set theoretic function

$$\bar{s} : U \to \overline{\mathcal{F}}, \;\; U \ni x \mapsto \bar{s}(x) = s_x$$

Note that $\pi \circ \bar{s} = Id_U$.

As $\overline{\mathcal{F}}$ has no topology, we don't know which sections are continuous. The sections in Definition 3.4.6 are nice enough sections such that we want these to be continuous. As such, the following basis for the topology of $\overline{\mathcal{F}}$ is chosen,

$$\{\bar{s}(U) : U \in \mathbf{Op}(M)^{op} \wedge s \in \mathcal{F}(U)\}.$$

One might wonder whether this basis ensures our desired sections are continuous. Luckily, this is indeed the case according to the following argument. If $\bar{s} : U \to \overline{\mathcal{F}}$ as in Definition 3.4.6, then $\bar{s}^{-1}(\bar{s}(U)) = U$ which is open in M. Thus the pre-image of the basis element is open.

Now we turn our attention to $\bar{s}^{-1}(\bar{t}(U))$ for some section $\bar{t}$ and $U \in \mathbf{Op}(M)^{op}$. Supposing that $\bar{s} \neq \bar{t}$, we obtain that $\bar{s}^{-1}(\bar{t}(U)) = \bar{s}^{-1}(\bar{s}(U) \cap \bar{t}(U))$. Define the set

$$\Lambda := \{x \in U : s_x = t_x\}.$$

Quite clearly $\Lambda \subseteq \bar{s}^{-1}(\bar{s}(U) \cap \bar{t}(U))$. Furthermore, let $x \in \bar{s}^{-1}(\bar{s}(U) \cap \bar{t}(U))$. Then per definition there is a $y \in U$ such that $\bar{s}(x) = \bar{t}(y)$. However, as $\bar{s}$ and $\bar{t}$ are both section of π, we obtain $\bar{s}(x) = s_x = t_y \bar{t}(y)$ implies $x = \pi(s_x) = \pi(t_y) = y$. As such, $x \in \Lambda$. We obtain the following

$$\bar{s}^{-1}(\bar{s}(U) \cap \bar{t}(U)) = \{x \in U : s_x = t_x\} = \Lambda.$$

Now if Λ is open, then every basis element has an open inverse under $\bar{s}$, making $\bar{s}$ continuous. To this end, let $x \in \Lambda$. Then per definition of Λ, $s_x = t_x$. Per definition of a stalk there is an open neighborhood U_x of x such that $s|_{U_x} = t|_{U_x}$.

So, for all $y \in U_x$, U_x is an open neighborhood enveloped by U in which $s|_{U_x} = t|_{U_x}$. Thus $s_y = t_y$, and thus $y \in \Lambda$ which implies $U_x \subseteq \Lambda$. As such, every $x \in \Lambda$ has an open neighborhood which is contained in Λ. Thus Λ is open.

To summarise, we have now obtained the situation as in Figure 13. There is a large covering of M with the stalks of the presheaf $\mathcal{F}$. This cover is combined with a projection map π and a topology to obtain an espace étalé. For this espace étaléthe continuous sections were found, namely the $\bar{s}$.

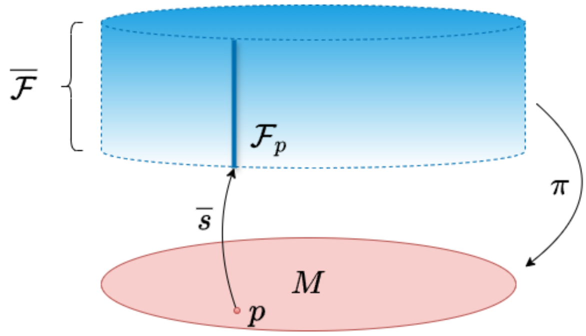


Figure 13: The espace étaléconstructed during the sheafification.

And now we are done! We have grown a sheaf. "Huh?!", you surely proclaim. Where is this elusive sheaf we have defined? Well it is right under our noses, the continuous sections form a sheaf!

Definition 3.4.7 (Sheafifified Presheaf).
Let $\mathcal{F} \in \mathbf{Psh}(M)$. Then the sheafification of $\mathcal{F}$, denoted by $\mathcal{F}^+$, is the functor

$$\mathcal{F}^+(-) = \Gamma(-, \overline{\mathcal{F}}) : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$$

defined as

$$\mathbf{Op}(M)^{op} \ni U \mapsto \Gamma(U, \overline{\mathcal{F}}).$$

Where $\Gamma(U, \overline{\mathcal{F}})$ was the abelian group of continuous sections of $(\overline{\mathcal{F}}, \pi)$. The restriction maps are provided by restriction of the domain of the continuous sections.

Remark 3.4.2. *This construction makes the name of an element of $\mathcal{F}(U)$ a lot clearer. We called this a section, since it really does coincide with a section.*

Example 3.4.2 (Sheaf of Quotients). *Let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on $M \in \mathbf{Man}^\infty$. Suppose that for every $U \in \mathbf{Op}(M)^{op}$ the abelian group $\mathcal{F}(U)$ is a subgroup of $\mathcal{G}(U)$. Then define the presheaf of quotients $\mathcal{Q} : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$ as*

$$U \mapsto \frac{\mathcal{G}(U)}{\mathcal{F}(U)}.$$

This presheaf need not be a sheaf, but using our new sheafification construction we can define $\mathcal{Q}^+$ as the sheaf of quotients.

Suppose $f : \mathcal{A} \to \mathcal{B}$ is a morphism of sheaves. One might wonder whether f induces a map from the sections of $\mathcal{A}^+(U)$ to those of $\mathcal{B}^+(U)$ for any $U \in \mathbf{Op}(M)^{op}$. Indeed this is so.

Definition 3.4.8 (Induced Sheafified Map).
Let $\mathcal{A}$ and $\mathcal{B}$ be presheaves on smooth manifold M and let $U \in \mathbf{Op}(M)^{op}$. The morphism of presheaves $f : \mathcal{A} \to \mathcal{B}$ induces a map $f_U : \mathcal{A}^+ \to \mathcal{B}^+$ defined as

$$(\bar{s} : U \to \mathcal{A}^+) \mapsto (\overline{f \circ s} : U \to \mathcal{B}^+)$$

which form a sheaf morphism.

Theorem 3.4.1 (Sheaf Invariance).
Let $\mathcal{F} \in \mathbf{Sh}(M)$. Then there is a sheaf isomorphism $\mathcal{F} \cong \mathcal{F}^+$.

A proof of this theorem is omitted in this text and can be found on page 45 of (Wells, [17]). This theorem has a very strong implication for sheaves, which is captured in the following observation.

Observation 3.4.1. *The stalks determine the sheaf up to isomorphism.*

When any two sheaves have the same stalks, they induce isomorphic espace étalé's. Since the espace étalé's are isomorphic, they have the same continuous sections. As such, these sheaves are isomorphic.

The principal information a sheaf contains, seems to be contained in the stalks. It makes sense that an exact sequence of stalks respects this. This leads to the following definition of an exact sequence of sheaves.

Definition 3.4.9 (Exact Sequence of Sheaves)**.**
Let $\mathcal{A}_i \in \mathbf{Sh}(M)$ for every $i \in \mathbb{Z}$ and

$$\dots \xrightarrow{f^{i-2}} \mathcal{A}^{i-1} \xrightarrow{f^{i-1}} \mathcal{A}^{i} \xrightarrow{f^{i}} \mathcal{A}^{i+1} \xrightarrow{f^{i+1}} \dots$$

a sequence of sheaf morphisms. This sequence is exact at $\mathcal{A}_i$ if for every $p \in M$ the induces sequence of abelian groups on stalks

$$\dots \xrightarrow{f_p^{i-2}} \mathcal{A}_p^{i-1} \xrightarrow{f_p^{i-1}} \mathcal{A}_p^{i} \xrightarrow{f_p^{i}} \mathcal{A}_p^{i+1} \xrightarrow{f_p^{i+1}} \dots$$

is exact at $\mathcal{A}_p^i$. If the sequence is exact at every $i \in \mathbb{Z}$, then the sequence is called an exact sequence.

Remark 3.4.3. *The exactness of sheaves are defined on the stalk level, and not on the open set level! This is exactly what makes sheaves so powerful. It allows us to measure to which extent local exactness does not lift to global exactness.*

Remark 3.4.4. *The notion of an exact sequence can also be defined for sequences of presheaves. A sequence of presheaves is exact, if the sequence evaluated at every $U \in \mathbf{Op}(M)^{op}$ is exact.*

However, it should be noted this is not equivalent to the definition of exactness for sequences of sheaves. Exactness of a sequence of presheaves is defined on the level of open sets.

Since a s.e.s. of presheaves can be defined, one might wonder whether sheafififying the presheaves in this s.e.s. results in a s.e.s. of sheaves. A s.e.s. of sheaves was defined at the stalk level for every $p \in M$, the exactness is encoded into the stalks.

A s.e.s. of presheaves is exact for every open neighborhood of every $p \in M$. As such, as we take the limit over these open neighborhoods to the stalks at p, the sequence remains exact. We obtain a s.e.s. of abelian groups, namely the stalks.

As mentioned before the sheafification process does not alter the stalks. The local data remains the same. As such, the sequence of sheaves will also have an exact sequence of stalks for all $p \in M$. As such, by sheafififying a s.e.s. of presheaves one obtains a s.e.s. of sheaves.

In the language of category theory we say the sheafification functor $(-)^+ : \mathbf{Psh}(M) \to \mathbf{Sh}(M)$ is an exact functor.

Now that we know what an exact sequence of sheaves is, they seem to appear all around us. In fact, they often can be related to exact sequences of abelian groups.

Example 3.4.3. *Let $M \in \mathbf{Man}^\infty$. In Example 3.4.1 $\vartheta \in \mathbf{Sh}(M)$ was defined and the stalks were calculated. We have the following sequence of sheaves.*

$$0 \longrightarrow \vartheta \xhookrightarrow{i} \mathbb{Z}_M \xrightarrow{q} \frac{\mathbb{Z}_M}{\vartheta} \longrightarrow 0 \tag{3.3}$$

Using the same argument as in Example 3.4.1 the stalk of $\mathbb{Z}_M$ at $p \in M$ can be determined to be isomorphic to $\mathbb{Z}$. We obtain that if $p \notin A$, then Equation 3.3 induces the following sequence of stalks.

$$0 \longrightarrow \mathbb{Z} \xhookrightarrow{i_p} \mathbb{Z} \xrightarrow{q_p} \frac{\mathbb{Z}}{\mathbb{Z}} = 0 \longrightarrow 0$$

Which is quite clearly exact as $\mathbb{Z} \cong \mathbb{Z}$.

Furthermore, if $p \in A$, then the stalk ϑ_p was determined to be trivial. Sequence 3.3 induces the following sequence of stalks.

$$0 \longrightarrow 0 \xhookrightarrow{i_p} \mathbb{Z} \xrightarrow{q_p} \frac{\mathbb{Z}}{0} = \mathbb{Z} \longrightarrow 0$$

Which is also exact. We conclude Sequence 3.3 is exact at the stalk level for any $p \in M$ and as such is an exact sequence of sheaves.

Notice that by applying the functor $\Gamma(M, -)$ to Sequence 3.3, we retain exactness at ϑ and $\mathbb{Z}_M$. However, when M is connected, any section of $\mathbb{Z}_M(M)$ has a constant value on A, while sections of $(\mathbb{Z}_M/\vartheta)(M)$ can have varying values on A[1]. Under q_M the sections with varying values on A are not covered. As such the map q_M fails to be surjective if M is connected.

Sadly, the global sections functor does not maintain exactness when applied. Following this potential loss of exactness, one might wonder whether the global sections functor does always maintain the exactness on the left? Luckily, this is the case.

Theorem 3.4.2.
The sections functor $\Gamma(U, -) : \mathbf{Sh}(M) \to \mathbf{Ab}$ is left exact for any $U \in \mathbf{Op}(M)^{op}$. That is, for any s.e.s. of sheaves

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

the sequence of abelian groups

$$0 \longrightarrow A(U) \xrightarrow{f_U} B(U) \xrightarrow{g_U} C(U)$$

is exact for any open set $U \in \mathbf{Op}(M)^{op}$.

Proof. First, we will show f_S is injective. Note that per Definition 3.3.3 $Ker(f_U) = Ker(f(U))$. Per assumption f is injective, resulting in $Ker(f(-)) = 0$. As such, $Ker(f_U) = 0$ and f_U is injective.

Now the exactness at $B(U)$ is proven. Let $\phi \in A(U)$ and let $p \in U$. Per definition of a s.e.s. of sheaves $g_U(f_U(\phi)) = 0$ on some open neighborhood V_p of p. This provides us with an open cover of U namely $\{V_p\}_{p \in U}$ for which $g_U(f_U(\phi))|_{V_p} = 0$. By the first sheaf axiom we now obtain that $g_U(f_U(\phi)) = 0$, resulting in $f_U(\phi) \subseteq Ker(g_U)$. We conclude $Im(f_U) \subseteq Ker(g_U)$.

Let $\phi \in Ker(g_U)$. By the definition of exactness of a s.e.s. of sheaves there must exist an open neighborhood V_p of p such that $\phi|_{V_p} = f_{V_p}(\psi^p)$. This provides us with an open cover of U consisting of $\{V_p\}_{p \in U}$. Suppose that $V_p \cap V_q \neq \emptyset$ for $p, q \in U$, then $f_{V_p}(\psi^p)|_{V_p \cap V_q} = \phi|_{V_p \cap V_q} = f_{V_q}(\psi^q)$. As f is injective we obtain that $f_{V_p}(\psi^p)|_{V_p \cap V_q} = f_{V_q}(\psi^q)$. The second sheaf axiom provides us with a section $f_U(\psi) = \phi$. Thus proving $Ker(g_U) = Im(f_U)$. □

As Example 3.4.3 shows us, we cannot do better than Theorem 3.4.2 when taking global sections. We can always lose exactness. As we wish to lift local information to global information for the theorem of De Rham, we wish to know to which extent taking sections preserves exactness. To answer this question we return to the world of homological algebra.

The exact sequence of sheaves we have defined allows for a theory of homological algebra analogous to that of Chapter 2. Intuitively, this homological algebra for sheaves, aptly named sheaf cohomology, measures the extent to which exactness is lost when taking global sections of sequences.

Chapter 4 delves into the theory of sheaf cohomology, providing all relevant notions and definitions and generalising results from Chapter 2 to sheaves. Subsequently, Chapter 5 uses the theory of sheaf cohomology to prove the theorem of De Rham.

[1] This is a direct result of the sheafification process during the construction of the quotient sheaf. The sections of the associated espace étalé map points to the associated germ. Example 3.4.1 shows the germs to be trivial for $x \notin A$ and Definition 3.4.7 allows for sections with varying values on A.

4

HOMOLOGICAL ALGEBRA: SHEAF COHOMOLOGY

Our study of sheaves has led to the formulation of a short exact sequence of sheaves in Chapter 4. Just like for sequences of abelian groups, this allows for the development of homological algebra. In this chapter the apparatus of sheaf cohomology is developed.

Just like sheaves themselves, sheaf cohomology was also developed by Leray whilst a POW. After the war Leray published his papers on sheaves before leaving algebraic topology to return to analysis. The cohomology of sheaves has since become an essential tool in both algebraic topology and algebraic geometry, being popularised by the likes of Cartan and Grothendieck. For the full story of Leray's influential papers in algebraic topology the reader is advised to review (Miller,[10]).

Intuitively, sheaf cohomology measures to which extent lifting local exact sequences to global exact sequences results in exactness. Our development of this theory starts with complexes of sheaves and the notion of cohomology of such a complex. Subsequently, the definition of a soft sheaf is provided.

Soft sheaves can be viewed as sheaves with extendable sections. Once a section is defined on a closed subset of a manifold, it can be extended to a section on the whole manifold. Many useful properties of soft sheaves are derived before taking a deeper dive into sheaf cohomology.

The sheaf cohomology developed in this chapter allows for a relation between the singular cohomology, De Rham cohomology and sheaf cohomology. This very relation is lifted to an isomorphism in Chapter 5 proving the theorem of De Rham.

4.1 RESOLUTIONS AND COMPLEXES OF SHEAVES

Our goal is clear. Just like for abelian groups, we wish to develop a theory of cohomology of sheaves. Cohomology of sheaves can be obtained in two ways. It can be found in the wild, and it can be grown by building. We will begin with the cohomology of sheaves found in the wild. This leads to the following definition.

Definition 4.1.1 (Complex of Sheaves)**.**
Let $M \in \mathbf{Man}^\infty$. A complex of sheaves in $\mathbf{Sh}(M)$ is an sequence of sheaves of the form

$$\mathcal{F}^\bullet : \qquad \dots \xrightarrow{d^{i-2}} \mathcal{F}^{i-1} \xrightarrow{d^{i-1}} \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \xrightarrow{d^{i+1}} \dots$$

such that for all $j \in \mathbb{Z}$ $\mathcal{F}^j \in \mathbf{Sh}(M)$ and the composition $d^{j+1} \circ d^j \equiv 0$.

The cohomology groups of this sequence are defined as

$$H^p(M, \mathcal{F}^\bullet) = \frac{Ker(d^p : \mathcal{F}^p(M) \to \mathcal{F}^{p+1}(M))}{Im(d^{p-1} : \mathcal{F}^{p-1}(M) \to \mathcal{F}^p(M))}.$$

As previously mentioned, sheaf cohomology measures the loss of exactness of taking global sections. This is captured in Definition 4.1.1. First, one applies the global section functor to the complex of sheaves, resulting in a complex of abelian groups. Subsequently, the cohomology groups measure the inexactness of this complex of abelian groups.

Now that we know what a complex of sheaves is and what the associated cohomology groups measure, it might be instructive to look at these complexes as objects themselves. As always, this requires defining the arrows between objects.

Definition 4.1.2 (Morphism of Complexes).
A morphism of complexes of sheaves $f : A^\bullet \to B^\bullet$ is a collection of morphisms of sheaves $f^j : A^j \to B^j$ such that

$$d_B \circ f^j = f^{j-1} \circ d_A.$$

Where d_A and d_B are the differential operators of the complexes $A^\bullet$ and $B^\bullet$.

4.1.1 *Resolutions of Sheaves*

Global sections of sheaves can be rather tough to completely determine. For example, the sheaf $C^\infty(-) : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$ requires us to understand the collection of all smooth functions on M. This is a lot of information we need to understand even for simple smooth manifolds like S^1. As such, we often wish we could approximate a sheaf using other sheaves. This is done using a resolution.

Definition 4.1.3 (Resolution of Sheaves).
Let $\mathcal{F} \in \mathbf{Sh}(M)$. A resolution, $\mathcal{F}^\bullet$, of $\mathcal{F}$ is an exact sequence of sheaves of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \dots$$

This is abbreviated by

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^\bullet$$

Remark 4.1.1. *The notion of a resolution can be generalised to resolutions for sheaves of abelian groups, commutative rings, fields,. . .*

Intuitively, resolutions allow for the approximation of sheaves using other sheaves. If we wish to approximate the sheaf $\mathcal{F}$, then the sheaves $\mathcal{F}^i$ in the resolution are chosen in such a manner that the exactness of each map in Definition 4.1.3 implies something about the original sheaf.

For example, if we were to have the following resolution

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow 0 \longrightarrow \dots$$

Then immediately one can make statements about $\mathcal{F}$ just by knowing what the sequence from $\mathcal{F}^0$ onwards looks like. The structure of sheaf resolutions comes by fairly often in mathematics. Even in Chapter 1 we already saw a complex which was close to a resolution. Namely:

Example 4.1.1 (The De Rham Resolution).
Let $\mathbb{R}_M$ be the constant sheaf on a smooth m-manifold M. Let $\Omega^k : \mathbf{Op}(M)^{op} \to \mathbf{Ab}$ be the sheaf of differential k-forms on M. Then there is a resolution of $\mathbb{R}_M$ given by

$$0 \longrightarrow \mathbb{R}_M \xhookrightarrow{i} \Omega^0_M \xrightarrow{d^0} \Omega^1_M \xrightarrow{d^1} \dots \xrightarrow{d^{m-1}} \Omega^m_M \longrightarrow 0$$

Here i is the inclusion of smooth locally constant $\mathbb{R}$-functions into the smooth functions Ω^0_M. As usual d is the exterior derivative, so $d \circ d \equiv 0$.

First, let $p \in M$ and take an open neighborhood U of p which under some chart (V, ϕ) has $\phi(U) \cong \mathbb{B}^m$. Now take some $f \in \Omega^0(U)$ such that $df = 0$. Under the chart (V, ϕ), $d(f \circ \phi^{-1})$ corresponds to the standard derivative of $f \circ \phi^{-1}$. Since this is zero, $f \circ \phi^{-1}$ is constant on $\phi(U)$. As such, also f is constant on U.

We can conclude f is a locally constant smooth function from M to $\mathbb{R}$. Note that this was exactly the description of elements of $\mathbb{R}_M$! Therefore, $f \in \mathbb{R}_M$ holds true. Resulting in $Im(i) = Ker(d^0)$. Thus, the sequence is exact at Ω^0_M.

As U was chosen homeomorphic to $\mathbb{B}^m$, as such the De Rham cohomology groups of $\mathbb{B}^m$ and U coincide. As shown before

$$H^p_{DR}(\mathbb{B}^m) = \begin{cases} \mathbb{R}, \text{ for } p = 0, \\ 0, \text{ for } p \geq 1. \end{cases}$$

Thus, for $p \geq 1$ per exactness $Im(d^{p-1}) = Ker(d^p)$. Indeed the resolution is exact for the sheaves applied to U.

Thus for every $p \in M$ there is an open neighborhood U for which the sequence of sheaves applied to U is exact. When the sequence is further restricted to the stalks at $p \in M$, it remains exact. As such, this is a resolution of sheaves. Furthermore, the De Rham cohomology coincides with the cohomology of this resolution per definition.

Example 4.1.2 (The Singular Cochain Resolution)**.**
Let $\mathbb{R}_M$ once more be the constant sheaf on some smooth manifold M. Our goal is to create a resolution of this sheaf using another complex than the De Rham complex. In Chapter 2 another complex was discussed, the singular cochain complex $C^\bullet$.

However, as noted in Example 3.3.5 the functor $C^p(-)$ is not a sheaf for any $p > 0$. As such, no resolution of sheaves can be constructed using these presheaves. Or can it?

First, we choose some open neighborhood U of $p \in M$ which is homeomorphic to $\mathbb{B}^m$ under a chart (V, ϕ). We identify this neighborhood of p with $\mathbb{B}^n$. For a unit ball the singular smooth cohomology groups are

$$H^p(\mathbb{B}^m) = \begin{cases} \mathbb{R}, \text{ for } p = 0, \\ 0, \text{ for } p \geq 1. \end{cases}$$

For the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$ we now obtain the following exact sequence.

$$0 \longrightarrow \mathbb{R} \overset{i}{\hookrightarrow} C^0(\mathbb{B}^n) \overset{\partial^0}{\longrightarrow} C^1(\mathbb{B}^n) \overset{\partial^1}{\longrightarrow} \dots \overset{\partial^{m-1}}{\longrightarrow} C^m(\mathbb{B}^n) \overset{\partial^m}{\longrightarrow} \dots$$

Just as for the De Rham resolution, this open neighborhood of p can be contracted to the stalks of p whilst remaining exact. Since this can be done for every $p \in M$ we obtain the exact sequence of presheaves.

$$0 \longrightarrow \mathbb{R} \overset{i}{\hookrightarrow} C^0 \overset{\partial^0}{\longrightarrow} C^1 \overset{\partial^1}{\longrightarrow} \dots \overset{\partial^{m-1}}{\longrightarrow} C^m \overset{\partial^m}{\longrightarrow} \dots$$

Using the sheafification process of Section 3.4.2 a sheaf $\mathcal{C}^p = (C^p)^+$ can be grown for any $C^p \in C^\bullet$. The boundary operator $\partial : C^p \to C^{p+1}$ induces a mapping of sheaves by Definition 3.4.8 defined as

$$\partial : \mathcal{C}^p \to \mathcal{C}^{p+1}.$$

As noted in Section 3.4.2, applying the sheafification functor to an exact sequence of presheaves results in an exact sequence of Sheaves. This results in the following exact sequence of sheaves.

$$0 \longrightarrow \mathbb{R}_M \overset{i}{\hookrightarrow} \mathcal{C}^0 \overset{\partial^0}{\longrightarrow} \mathcal{C}^1 \overset{\partial^1}{\longrightarrow} \dots \overset{\partial^{m-1}}{\longrightarrow} \mathcal{C}^m \overset{\partial^m}{\longrightarrow} \dots \tag{4.4}$$

We have derived a resolution of sheaves using the singular cochain presheaf.

One might wonder whether the cohomology groups of Resolution 4.4 are isomorphic to the singular cohomology groups of M. Intuitively, the cochain presheaf is very close to being a sheaf. If the unique lifting issue from Example 3.3.5 can be fixed, then we obtain a sheaf. Maybe the sheafification process alters the cochain presheaf just enough to make it into a sheaf, but too little to alter the cohomology?

Theorem 4.1.1 (Isomorphism of Sheafified Cochain Presheaf).
There is an isomorphism of abelian groups

$$H^p(M) \cong H^p(M, \mathcal{C}^\bullet).$$

The proof of this fact is beyond the scope of this text, but a sketch of the proof is provided. For a proper proof the reader is advised to review (Mustaţă, [11]). In the sequel these two cohomologies will be identified with one another. As such, $H^p(M)$ will also denote the cohomology groups of the sheafifified cochain presheaf.

Sketch of Proof.
In Example 3.3.5 the lack of a unique gluing property of the singular cochain epipresheaf was discussed. One intuitive solution to the issue posed in this example, would be subdividing the troublesome simplices into a sum of simplices fully contained in U_1 or U_2, as done on the right in Figure 14.

However, this leads to another issue, when further restricting to V_1 and V_2 we obtain that α_2 is not fully contained in V_2 or V_1. As such, we would have to subdivide this simplex as well. A cycle of subdividing and restricting seems to form in which the issue seems to return each time.

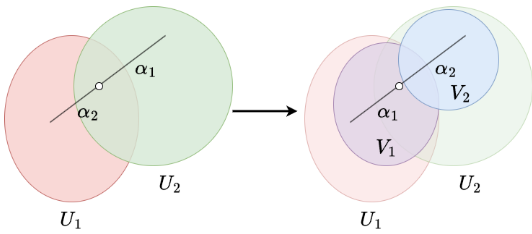


Figure 14: There exists no uniform subdivision of 1-chains.

Luckily, there is a way out of this vicious cycle. Let $\mathcal{U}$ be an open cover of $M \in \mathbf{Man}^\infty$. Then denote by $C_\bullet(M, \mathcal{U})$ the chains of simplices which are fully contained in one element of $\mathcal{U}$.

Intuitively, smaller simplices still measure the same holes in a surface, so the singular homology of subdivided simplices is isomorphic to that of singular homology. In more mathematical terms, the inclusion of $i : C_\bullet(M, \mathcal{U}) \hookrightarrow C_\bullet(M)$ induces an isomorphism on the homology groups of $C_\bullet(M, \mathcal{U})$ and $C_\bullet(M)$. By the same argument the projection mapping $\pi = - \circ i : C^\bullet(M) \to C^\bullet(M, \mathcal{U})$ also induces an isomorphism of cohomology groups.

Having provided the intuitive required isomorphisms, the more technical part of the proof begins. First, notice that $Ker(\pi)$ consists of cochains which vanish on simplices fully contained in elements of $\mathcal{U}$. Additionally, for any epipresheaf there is a surjective natural mapping $\tau : C^\bullet(-) \to \mathcal{C}^\bullet(-)$. This mapping results in a s.e.s. of complexes.

$$0 \longrightarrow V^\bullet \xhookrightarrow{i} C^\bullet(M) \xrightarrow{\tau} \mathcal{C}^\bullet(M) \longrightarrow 0$$

In the induced l.e.s. of cohomology we see that $H^p(V^\bullet) = 0$ for all p implies the required isomorphism. One can prove that $V^\bullet$ is the direct limit of the complexes $V^\bullet(\mathcal{U})$, where $\mathcal{U}$ is an open cover of M and $V^p(\mathcal{U})$ consists of all the p-cochains which vanish on all simplices fully contained in elements of $\mathcal{U}$. If for every $\mathcal{U}$ and p we obtain $H^p(V^\bullet(\mathcal{U})) = 0$, then taking the direct limit over $\mathcal{U}$ also results in $H^p(V^\bullet) = 0$.

Notice that our description of $V^\bullet(\mathcal{U})$ coincides with the definition of $Ker(\pi)$. As such, we can create the following s.e.s. of complexes.

$$0 \longrightarrow V^\bullet(\mathcal{U}) \xhookrightarrow{i} C^\bullet(M) \xrightarrow{\pi} \mathcal{C}^\bullet(M, \mathcal{U}) \longrightarrow 0 \tag{4.5}$$

As mentioned before, π induces an isomorphism on cohomology. As such, by looking at the induced l.e.s. of cohomology of Sequence 4.5 we obtain $H^p(V^\bullet(\mathcal{U})) = 0$. This results in the theorem being proven. □

Just like with complexes of abelian groups, one can define morphisms of complexes for sheaves. This induces a notion of morphisms of resolutions. In fact, a morphism of resolutions is a morphism of complexes.

Example 4.1.3 (Morphism from $\Omega^\bullet$ to $C^\bullet$.).
Let M be a smooth manifold. Before we have seen two resolutions of the constant sheaf $\mathbb{R}_M$ on M. A very useful morphism of resolutions between $\Omega^\bullet$ and $C^\bullet$ will be constructed using integration on manifolds! First, remember we had the following resolutions of $\mathbb{R}_M$.

$$0 \longrightarrow \mathbb{R}_M \longrightarrow \Omega^\bullet$$

$$0 \longrightarrow \mathbb{R}_M \longrightarrow C^\bullet$$

Now define the following map

$$\mathfrak{c}_U : \Omega^\bullet(U) \to C^\bullet(U)$$

for any $U \in \mathbf{Op}(M)^{op}$ as being given by

$$\mathfrak{c}_U(\omega) = \left(\int_{(-)} \omega : \sigma \mapsto \int_\sigma \omega \right)$$

for some piecewise smooth *chain σ in $C_k(U)$ for some $k \geq 0$. By the linearity of the integral this is indeed a homomorphism. For this map to be a morphism of resolutions, it needs to commute with the differential operators of $\Omega^\bullet$ and $C^\bullet$. That is we require*

$$\mathfrak{c} \circ d = \partial \circ \mathfrak{c}.$$

Where d is the exterior derivative and $\partial = - \circ \delta$ the boundary operator for cochains. On the left hand side we obtain

$$\omega \mapsto dw \mapsto \int_{(-)} d\omega.$$

On the right hand side we obtain

$$\omega \mapsto \int_{(-)} \omega$$

By Composting with $\partial = - \circ \delta$ this coincides with first taking the boundary or δ of a chain and then integrating ω over this. In other words

$$\partial \circ \mathfrak{c} = \left[\int_{(-)} \omega : \sigma \mapsto \int_{\partial\sigma} \omega \right].$$

Due to Stokes' theorem 1.4.1 this is equal to

$$\int_{(-)} d\omega : \sigma \mapsto \int_\sigma d\omega = (\mathfrak{c} \circ d)(\omega)(\sigma).$$

So, $\mathfrak{c}$ commutes with the differentials. As such, this is a morphism of resolutions.

Remark 4.1.2. *It should be noted that σ is required to be (piecewise) smooth in Example 4.1.3. Otherwise, Stokes' theorem cannot be applied. However, for any $M \in \mathbf{Man}^\infty$ this is no issue, as any cochain can be made smooth using a smoothing operator. This makes the morphism constructed in Example 4.1.3 into a properly defined morphism. The construction of the smoothing operator is not provided in this text, the interested reader is advised to review Lemma 18.8 of (Lee, [9]).*

4.2 SOFT SHEAVES

All sheaves are useful, but some are more useful than others. One particularly useful type of sheaf will be our focus for this section; soft sheaves. Soft sheaves are soft, in the sense that they are malleable. One can reshape soft sheaves in such a way that local abelian groups can easily be extended to global abelian groups.

Definition 4.2.1 (Soft Sheaves)**.**
A sheaf $\mathcal{A}$ is called soft if for ever closed subset $S \subseteq M \in \mathbf{Man}^\infty$ the restriction map

$$\mathcal{A}(M) \to \mathcal{A}(S)$$

is surjective.

Remark 4.2.1. *The above definition tells us that $\mathcal{A}$ is soft if any section of $\mathcal{A}$ over S can be extended to a global section of $\mathcal{A}$.*

Wait, wait, wait. A closed set in a sheaf, have we lost our minds? Only for open sets have we defined what a sheaf does. Indeed this requires some attention. How could we use open sets to approximate closed sets? Well, just like the stalks we can just approximate the closed set with open sets. This leads to

Definition 4.2.2 (Sheaf Applied To a Closed Subset)**.**
Let $\mathcal{F} \in \mathbf{Sh}(M)$ and let $S \subseteq M$ be closed. Then define

$$\mathcal{F}(S) = \lim_{U \supseteq S} \mathcal{F}(U),$$

for $U \in \mathbf{Op}(M)^{op}$.

Remark 4.2.2. *In the language of espace étalé's the abelian group $\mathcal{F}(S)$ is the set of continuous sections $s : S \to \pi^{-1}(S)$.*

The reason for defining soft sheaves becomes apparent quickly. The following theorem shows explicitly why soft sheaves are so useful.

Theorem 4.2.1 (Soft Property)**.**
Let A be a soft sheaf and

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

a s.e.s. of sheaves, then for any closed $S \subseteq M$ the following s.e.s. is obtained.

$$0 \longrightarrow A(S) \xrightarrow{f_S} B(S) \xrightarrow{g_S} C(S) \longrightarrow 0$$

Proof.
Theorem 3.4.2 already provides us with exactness everywhere except at $C(S)$. Only the surjectivity of g_S remains to be proven.

The goal is to show that there exists a section $b \in B(S)$ such that $g_S(b) = c$. As the sequence of sheaves is exact, the following sequence is exact for $p \in M$.

$$0 \longrightarrow A_p \xrightarrow{f_p} B_p \xrightarrow{g_p} C_p \longrightarrow 0$$

Thus for $c_p = [(U, c)] \in C_p$ there exists some $b_p = [(W, b)] \in B_p$, such that on $V_p \subseteq W \cap U$ $g(b) = c|_{V_p}$. As $p \in M$ was arbitrary and V_p is open, S can be covered by the open sets $\bigcup_{p \in S} V_p$.

If these b_p's can be glued together to a global section, then g is surjective. As M is paracompact per definition, so is S. Therefore, there exists a locally finite refinement $\{F_i\}_{i \in I}$ of $\{V_p\}_p$ such that each F_i is a closed set and the interior of F_i's cover S.

Let Λ denote the set of pairs (b, F_J). Where $J \subseteq I$ such that $F_J = \bigcup_{j \in J} F_j$, and $b \in B(F_J)$ which satisfies $g(b) = c|_{F_J}$. This set can be partially ordered. Define an ordering as

$$(b, F_J) \leq (b', F_{J'})$$

when $J \subseteq J'$ and $b'|_{F_J} = b$.

Our goal is to apply Zorn's Lemma 4.2.1 to obtain a maximal element of Λ.

Lemma 4.2.1 (Zorn)**.** *Let S be a non-empty partially ordered set. If every chain in S has an upper bound, then S contains a maximal element.*

Suppose

$$\cdots \leq (b_i, F_{J_i}) \leq (b_{i+1}, F_{J_{i+1}}) \leq (b_{i+2}, F_{J_{i+2}}) \leq \ldots$$

is a chain in Λ[1]. We will prove this chain has an upper bound.

Define

$$F = \bigcup_i F_{J_i}.$$

Note that the closed sets F_{J_i} form a closed cover of F. Note that per definition of the chain on $F_{ij} := F_{J_i} \cap F_{J_j} \neq \emptyset$ we have

$$b_i|_{F_{ij}} = b_j|_{F_{ij}}$$

for all i, j. As B is a sheaf, the second sheaf axiom now provides a section $\beta \in B(F)$ such that $\beta|_{F_{J_i}} = b_i$ for all i.

We obtain the pair $(\beta, F) \in \Lambda$. Note that indeed

$$(b, F_J) \leq (\beta, F).$$

As such, the chain has an upper bound.

Since every totally ordered chain has an upper bound, Zorn's Lemma 4.2.1 states there is a maximal set F and a maximal section $b \in B(F)$. This section has the following property $g(b) = c|_F$. If $F = S$ then the surjectivity is proven.

Suppose the contrary. Then there should be a set $F_i \in \{F_j\}_j$ so that $F_i \not\subseteq F$ and and element $b_i \in B(F_i)$, such that $g_{F \cap F_i}(b - b_i) = c - c = 0$. We conclude $b - b_i \in Ker(g(F \cap F_i))$. Now we focus on the following sequence of abelian groups, which exact per Theorem 3.4.2.

$$0 \longrightarrow A(F \cap F_i) \xrightarrow{f_{F \cap F_i}} B(F \cap F_i) \xrightarrow{g_{F \cap F_i}} C(F \cap F_i)$$

As this is exact at $B(F \cap F_i)$, there must be a section $a \in A(F \cap F_i)$ such that $f(a) = b - b_i$.

As A is soft, a can be extended to the whole of M, this section can then be restricted to $F \cup F_i$. This result in a section a on the whole of $F \cup F_i$. Define the section $\hat{b} \in B(F \cup F_i)$ as follows

$$\hat{b} = \begin{cases} b, & \text{on } F, \\ b_i + f(a), & \text{on } F_i. \end{cases}$$

It follows that $g(\hat{b}) = c|_{F \cup F_i}$ and $(b, F) \leq (\hat{b}, F \cup F_i)$. This implies that $(\hat{b}, F)$ is not maximal which is a contradiction! As such, g_S is surjective. □

Theorem 4.2.1 has some very powerful corollaries which make the homological algebra of soft sheaves simpler. A direct result of Theorem 4.2.1 is the following corollary.

[1] Here it is understood the chain can also be uncountably infinite

Corollary 4.2.1. *Suppose A is a soft sheaf on M, and*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is an exact sequence of sheaves on M. Then B is soft if and only if C is soft.

Proof.
Let $S \subseteq M$ be closed. The goal of this proof is to show the restriction map $C(M) \to C(S)$ is surjective. By theorem 4.2.1 we obtain that the following commutative diagram has exact rows as A is soft.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A(M) & \xrightarrow{f_M} & B(M) & \xrightarrow{g_M} & C(M) & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow b & & \downarrow c & & \\ 0 & \longrightarrow & A(S) & \xrightarrow{f_S} & B(S) & \xrightarrow{g_S} & C(S) & \longrightarrow & 0 \end{array} \tag{4.6}$$

Suppose that B is soft and let $\phi \in C(S)$. Per assumption g_S is surjective, thus ϕ can be lifted to $\psi \in B(S)$. As B is soft, the map b is surjective. Thus, we can lift ψ to $\beta \in B(M)$. The commutativity of Diagram 4.6 implies that

$$\phi = (g_S \circ b)(\beta) = (c \circ g_M)(\beta).$$

As such, $\phi \in Im(c)$. We conclude that $C(S) = Im(c)$, resulting in C being soft.

Now assume that C is soft, then per definition c is surjective. Let $\phi \in Ker(g_S)$, then per exactness of the rows of Diagram 4.6 ϕ lifts to a $\psi \in A(S)$. The same diagram chase as above provides that $\phi \in Im(b)$.

If $\phi \notin Ker(g_S)$, then under c we can lift $g_S(\phi)$ to $\beta \in B(M)$ using the surjectivity of c and g_M. The commutativity of Diagram 4.6 implies that $(g_S \circ b)(\beta) = g_S(\phi)$. As such, $b(\beta) - \phi \in Ker(g_S)$. The exactness of the rows of Diagram 4.6 implies the existence of $\alpha \in A(S)$ such that $f_S(\alpha) = b(\beta) - \phi$.

By the surjectivity of a we can lift α to $\alpha' \in A(M)$. Now define $\beta - f_M(\alpha') \in B(M)$. Applying b to this element and using the commutativity of Diagram 4.6 results in

$$b(\beta - f_M(\alpha')) = b(\beta) - f_S(\alpha) = b(\beta) - (b(\beta) - \phi) = \phi.$$

We conclude that $\phi \in Im(b)$. We obtain that b is surjective, so B is soft. □

This corollary might not seem like a powerful statement, but it allows for easy confirmation whether a sheaf is soft. Additionally, it leads to the following corollary which extends our homological algebra of soft sheaves.

Corollary 4.2.2. *If the following is a long exact sequence of soft sheaves,*

$$0 \longrightarrow S^0 \longrightarrow S^1 \longrightarrow S^2 \longrightarrow \dots$$

then the sequence of global sections is also exact

$$0 \longrightarrow S^0(M) \longrightarrow S^1(M) \longrightarrow S^2(M) \longrightarrow \dots$$

Proof. The goal of this proof is to build a big pile of s.e.s. of global sections of soft sheaves, and then create a l.e.s. of global sections of soft sheaves by gluing these s.e.s. together. To this end, a big pile of s.e.s. of soft sheaves is needed.

Define $\kappa^i = Ker(S^i \to S^{i+1})$. This induces the following sequences for every $i \geq 0$.

$$0 \longrightarrow \kappa^i \hookrightarrow S^i \longrightarrow \kappa^{i+1} \longrightarrow 0 \tag{4.7}$$

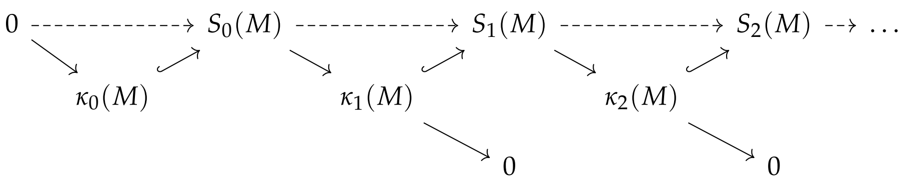
Let $p \in M$, we will show Sequence 4.7 is exact. As the sequence of sheaves in Corollary 4.2.2 is exact, we know that $Im(S_p^i \to S_p^{i+1}) = Ker(S_p^{i+1} \to S_p^{i+2}) = \kappa_p^{i+1}$. As such the left arrow is surjective on the stalk level.
Furthermore, as the right arrow is an inclusion it is injective on the stalk level. Additionally, $Ker(S_p^i \to \kappa_p^{i+1}) = Ker(S_p^i \to S_p^{i+1}) = \kappa_p^i$ is true. Thus Sequence 4.7 is exact at the stalk for any $p \in M$ and any $i \geq 0$. We conclude that Sequence 4.7 of sheaves is exact for every $i \geq 0$.

For $i = 0$, we can note that $\kappa_0 = S_0$ which is soft per assumption. Per Corollary 4.2.1 we obtain κ_1 is soft. Applying this argument to the s.e.s. containing S_1 we obtain once more κ_2 is soft. This in turn allows for an iterative application of this argument, which leads to κ_i being soft for all $i \geq 0$.

The soft property from Theorem 4.2.1 claims the following sequence is exact for every $i \geq 0$.

$$0 \longrightarrow \kappa_i(M) \longrightarrow S_i(M) \longrightarrow \kappa_{i+1}(M) \longrightarrow 0$$

We now have our a big pile of separate s.e.s. of global sections of soft sheaves. How to glue these sequences together? This actually is somewhat easy, we just need to know which commutative diagram to draw.



In the above diagram we see s.e.s. which were created being deformed to allow for arrows between the $S_i(M)$. The dashed arrows $S_i(M) \to S_{i+1}(M)$ are the compositions of the arrows $S_i(M) \to \kappa_i(M)$ and $k_i(M) \hookrightarrow S_{i+1}(M)$. If this composed arrow is exact, then we are done.

Let $i \geq 0$. Then $Ker(S_i(M) \to S_{i+1}(M)) = Ker(S_i(M) \to \kappa_{i+1}(M))$ as $\kappa_{i+1}(M) \hookrightarrow S_{i+1}(M)$ is injective. By the exactness of the created s.e.s. we know $Ker(S_i(M) \to \kappa_{i+1}(M)) = Im(\kappa_i(M) \hookrightarrow S_i(M))$.

Furthermore, as the arrow $S_{i-1}(M) \to \kappa_i(M)$ is surjective per construction, we obtain

$$Im(S_{i-1}(M) \to S_i(M)) = Im(\kappa_i(M) \hookrightarrow S_i(M)).$$

By transitivity we obtain

$$\begin{aligned} Ker(S_i(M) \to S_{i+1}(M)) &= Ker(S_i(M) \to \kappa_{i+1}(M)) \\ &= Im(\kappa_i(M) \hookrightarrow S_i(M)) = Im(S_{i-1}(M) \to S_i(M)) \\ \Rightarrow Ker(S_i(M) \to S_{i+1}(M)) &= Im(S_{i-1}(M) \to S_i(M)). \end{aligned}$$

Thus indeed the sequence

$$0 \longrightarrow S_0(M) \longrightarrow S_1(M) \longrightarrow S_2(M) \longrightarrow \ldots$$

is exact. $\square$

4.2.1 *Fine Sheaves*

Some powerful properties of soft sheaves have been discussed, but no examples have been discussed to far. The next collection of sheaves provides us with a big pile of examples of soft sheaves. These are the fine sheaves.

Definition 4.2.3 (Fine Sheaves)**.**
A sheaf of abelian groups $\mathcal{F}$ over a smooth manifold M is called fine if for any locally finite open cover $(U_i)_i$ of M there is a family of sheaf morphisms

$$\{\psi_i : \mathcal{F} \to \mathcal{F}\}$$

such that

1. $\sum_i \psi_i \equiv 1,$
2. *for all* $p \in supp(\psi_i)^c$, $\psi_i(\mathcal{F}_p) = 0.$

The family of sheaf morphisms is called a partition of unity of $\mathcal{F}$ subordinate to $(U_i)_i$.

Using this definition we obtain a whole pile of examples. Many more can be cooked up by creating partitions of unity for sheaves. One particularly important example is the following.

Example 4.2.1 (Ω^k is Fine)**.** *Let $k \geq 0$, then the sheaf $\Omega^k(-)$ on M is fine. To see this we cover M with some locally finite $(U_i)_i$. Luckily, as proven in (Lee, [9]) on page 43, for any open cover $\{U_i\}_i$ of a smooth manifold M, there exists a smooth partition of unity $\{\psi_i\}_i$.*

This induces a map of differential forms as follows: $\psi_i : \Omega^k \to \Omega^k$ by $\omega \mapsto \psi_i \cdot \omega$. Note that indeed

$$\sum_i \psi_i(\omega) = \sum_i \psi_i \cdot \omega \equiv 1 \cdot \omega = \omega.$$

Furthermore, per definition the support of ψ_i is contained in U_i. For any $p \in supp(\psi_i)^c \subset M$ the following is true for any $\omega_p \in \Omega^k_p(M)$,

$$\psi_i(\omega_p) = \psi_i(p)\omega_p = 0 \cdot \omega_p = 0.$$

Thus, we obtain indeed that $\psi_i(\Omega^k(M)_p) = 0$ for all $p \in supp(\psi_i)^c$. As such the resolution $\Omega^\bullet$ is fine per Definition 4.2.3.

It was claimed that the fine sheaves are in fact soft, and indeed this is so. In differential geometry, the smooth partition of unity is often used to extend local functions to global functions. For example, as was done in the proof of Stokes' theorem 1.4.1.

Acquiring global sections of a sheaf works in the same fashion. Using the partition of unity a local section is extended to a global section. This results in the coveted theorem.

Theorem 4.2.2 (Fine Sheaves Are Soft)**.**
*Let $\mathcal{F}$ be a fine sheaf on $M \in$ **Man**$^\infty$, then $\mathcal{F}$ is soft.*

Proof.
Suppose that $\mathcal{F}$ is a fine sheaf over $M \in \mathbf{Man}^\infty$. Let $S \subseteq M$ be a closed subset and let $s \in \mathcal{F}(S)$. For any open cover $(U_i)_i$ of S in M there are sections $s_i \in \mathcal{F}(U_i)$ such that

$$s|_{U_i \cap S} = s_i|_S.$$

We extend our open cover to cover all of M by adding $U_0 = M - S$ to the open cover and setting $s_0 = 0$. Note that any smooth manifold is locally paracompact, and as such we assume $(U_i)_i$ is locally finite and hence it has a subordinate sheaf partition of unity denoted by $(\phi_i)_i$. Using this partition of unity a global extension of s will be constructed thus proving surjectivity of the restriction mapping.

Note that $\psi_i(s_i)$ is a section of $\mathcal{F}(U_i)$ per definition of ψ_i. Furthermore, $\psi_i(s_i)$ is identically zero on $supp(\psi_i(s_i))^c \subsetneq U_i$. As such, we can extend $\psi_i(s_i)$ to a global section as follows.

$$\psi_i(s_i)(p) = \begin{cases} \psi_i(s_i)(p), \text{ if } p \in U_i, \\ 0, \text{ if } p \in U_i^c. \end{cases}$$

Using the global extensions of $\psi_i(s_i)$ we can define the following section

$$\bar{s} = \sum_i \psi_i(s_i).$$

Indeed, $\bar{s} \in \mathcal{F}(M)$ and

$$\bar{s}|_S = \sum_i \psi_i(s_i)|_S = \sum_i \psi_i(s_i)|_{U_i \cap S} = \sum_i \psi_i(s|_{U_i \cap S}) = s|_{U_i \cap S}.$$

As such the restriction map is indeed surjective and $\mathcal{F}$ is soft. □

4.2.2 *Sheaf of Modules*

In Example 4.2.1 the smooth partition of unity of M induced a partition of unity on Ω^k. If we could induce a soft sheaf structure onto other sheaves, we could obtain even more soft sheaves.

There is a way to obtain soft sheaves from other soft sheaves. This is done by endowing a sheaf with a module structure for another sheaf. A module of a sheaf, what would that be?

Definition 4.2.4 (Sheaf of $\mathcal{R}$-Modules)**.**
Let $\mathcal{R}$ be a sheaf of commutative rings on M. Let $\mathcal{F}$ be a sheaf of abelian groups on M. If for any $V \subseteq U \in \mathbf{Op}(M)^{op}$ the abelian group $\mathcal{F}(U)$ is a $\mathcal{R}(U)$-module such that for $a \in \mathcal{R}(U)$ and $b \in \mathcal{F}(U)$

$$ab|_V = a|_V b|_V.$$

Then $\mathcal{F}$ is a sheaf of $\mathcal{R}$-modules.

The soft property of a sheaf can be transferred to another sheaf via the scalar multiplication of sections under the module structure. This much leads to the following extremely useful lemma.

Lemma 4.2.2.
Let $\mathcal{R}$ be a sheaf of commutative rings and let $\mathcal{F}$ be a sheaf of $\mathcal{R}$-modules. If $\mathcal{R}$ is soft, then so is $\mathcal{F}$.

Proof. First a sketch of the proof. We wish to show that for any closed subset S of M we can extend $s \in \mathcal{F}(S)$ to a global section in $\mathcal{F}(M)$. This is done by scaling with an element of $\mathcal{R}$ in a proper fashion. So the main goal of this proof is to find a proper factor to multiply s by.

Let $S \subset M$ be closed. Let $s \in \mathcal{F}(S)$. Since s is defined in a neighborhood of every point in S, we obtain an open neighborhood of S, call this $U \subset M$, to which s can be extended. The extension of s will also be denoted by s.

Now the proper scaling factor is determined. Let $r \in \mathcal{R}(S \cup U^c)$. These two sets are disjoint and closed. We choose r as follows.

$$r(x) \equiv \begin{cases} 1_x, \text{ if } x \in S, \\ 0_x, \text{ if } x \in U^c. \end{cases}$$

Note that in the definition of r, 1_x and 0_x denote the germs of the constant sections of additive and multiplicative identities and not elements of $\mathbb{R}$. Since $\mathcal{R}$ is soft and $S \cup U^c$ is closed, we obtain a global extension of r denoted by $\bar{r}$. Using this extension we can define the global section $\bar{s} \in \mathcal{F}(M)$ as

$$\bar{s}(p) = \begin{cases} [\bar{r} \cdot s](p), & \text{if } p \in U, \\ 0, & \text{if } p \in U^c. \end{cases}$$

$\bar{s}$ is a global extension of s. As such, any section of $\mathcal{F}(S)$ can be extended to a global section and $\mathcal{F}$ is soft. □

Lemma 4.2.2 gives us another large pile of soft sheaves. Many can be dreamt up, but one important one for us is the following example.

Example 4.2.2 (C^p is Soft)**.** *Let $p \geq 0$ and $M \in \mathbf{Man}^\infty$. Let $U \in \mathbf{Op}(M)^{op}$, $C^0(U)$ is defined as the collection of continuous mappings from the 1-chains on U to $\mathbb{R}$. Similarly, $C^p(U)$ is defined as the continuous mappings from the p-chains on U to $\mathbb{R}$. Look at the following action of $C^0(U)$ on $C^p(U)$. For $\sigma \in C^0(U)$ and $\omega \in C^p(U)$*

$$(\sigma \smile \omega) : C_p(U) \to \mathbb{R}, \quad \Delta_p \mapsto \sigma(\Delta_p \circ (e_0)) \cdot \omega(\Delta_p).$$

Geometrically, the action $\smile$ does the following. $\sigma(\Delta_p \circ (e_0))$ first takes the first vertex of Δ_p, a 0-chain, and applies σ. This is then multiplied by ω applied to Δ_p. This action is called the cup product of cochains.

It is not hard to see that for every $U \in \mathbf{Op}(M)^{op}$ the abelian group $C^p(U)$ is a module of the ring $C^0(U)$ under the cup product. Furthermore, as for an open $V \subseteq U$

$$(\sigma \smile \omega)|_V = \sigma(- \circ (e_0))|_V \cdot \omega(-)|_V$$

the cup product makes $C^p(-)$ into a sheaf of $C^0(-)$-modules per Definition 4.2.4.

As mentioned before C^0 consist of smooth mappings from 0-chains to $\mathbb{R}$. As the 0-simplices coincide with the points of M, every 0-cochain can be identified with a continuous mapping from M to $\mathbb{R}$. Under this identification we obtain $C^0(M) \cong C(M, \mathbb{R})$.

As stated before, every open cover of M induces a smooth partition of unity $(\psi_i)_i$. In particular $\psi_i \in C(M, \mathbb{R})$ for every i. As such, we obtain a partition of unity of $C^0(-)$. We conclude $C^0(-)$ is fine and thus also soft. As shown before $C^p(-)$ is a sheaf of $C^0(-)$-modules. By Lemma 4.2.2 we obtain that since $C^0(-)$ is soft, also $C^p(-)$ is soft.

Soft sheaves seem to be useful. The soft property makes taking global sections intuitive, and ensure the homological algebra of soft sheaves is relatively simple. Furthermore, soft sheaves are quite ubiquitous.

One might wonder whether we can approximate any sheaf using soft sheaves. That is, does any sheaf have a resolution of soft sheaves? This would allow us to understand any sheaf using just soft sheaves.

4.2.3 *Godement Resolution*

The answer surprisingly is a resounding yes! The resolution given here is often called the Canonical Godement resolution, being constructed by Godement in his *Topologie algébrique et théorie des faisceaux* in 1958 (Godement, [5]).

Our construction of a soft resolution begins with a smooth manifold $M \in \mathbf{Man}^\infty$ and a sheaf $\mathcal{F} \in \mathbf{Sh}(M)$ with espace étalé$\pi : \overline{\mathcal{F}} \to M$. We now use a very powerful sheaf, the sheaf of discontinuous sections of $\mathcal{F}$!

This sheaf is defined by

$$\mathcal{S}(\mathcal{F})(U) = \{f : U \to \overline{\mathcal{F}} : \pi \circ f = Id_u\},$$

for $U \in \mathbf{Op}(M)^{op}$. The restriction mappings are restrictions of the domains of the sections. This indeed forms a sheaf.

Our goal is to construct a soft resolution of $\mathcal{F}$, so we begin with an initial attempt:

$$0 \longrightarrow \mathcal{F} \hookrightarrow \mathcal{S}(\mathcal{F}).$$

The arrow $\mathcal{F} \to \mathcal{S}(\mathcal{F})$ is the inclusion of $\mathcal{F}$ into $\mathcal{S}(F)$ where $\mathcal{F}$ is identified with the continuous sections of $\overline{\mathcal{F}}$. Is this sequence exact? Well almost, the inclusion is indeed injective, but $\mathcal{F} \to \mathcal{S}(\mathcal{F})$ is not always surjective. By adding another object to this sequence it can be made exact.

One would expect a cokernel construction to allow for an exact sequence of soft sheaves in this case. As such we set $\mathcal{S}(\mathcal{F})/\mathcal{F}$ as our new final object to obtain

$$0 \longrightarrow \mathcal{F} \hookrightarrow \mathcal{S}(\mathcal{F}) \longrightarrow \frac{\mathcal{S}(\mathcal{F})}{\mathcal{F}} \longrightarrow 0$$

which is exact. However, the quotient sheaf need not be soft. As such, we want to make it soft by applying the sheaf of discontinuous sections to it. This results in the following exact sequence.

$$0 \longrightarrow \mathcal{F} \hookrightarrow \mathcal{S}(\mathcal{F}) \longrightarrow \mathcal{S}\left(\frac{\mathcal{S}(\mathcal{F})}{\mathcal{F}}\right) \longrightarrow \frac{\mathcal{S}\left(\frac{\mathcal{S}(\mathcal{F})}{\mathcal{F}}\right)}{\mathcal{S}(\mathcal{F})} \longrightarrow 0$$

Another cokernel object is added to keep the exactness. Hmm, it seems we have obtained another quotient sheaf. So once more we would need to apply the sheaf of discontinuous sections. However, this would once more result in a quotient sheaf being present in the exact sequence. As such, if this argument is repeated ad infinitum then we obtain a resolution of soft sheaves! Indeed we do just that to obtain

$$0 \longrightarrow \mathcal{F} \hookrightarrow \mathcal{S}(\mathcal{F}) \longrightarrow \mathcal{S}\left(\frac{\mathcal{S}(\mathcal{F})}{\mathcal{F}}\right) \longrightarrow \mathcal{S}\left(\frac{\mathcal{S}\left(\frac{\mathcal{S}(\mathcal{F})}{\mathcal{F}}\right)}{\mathcal{S}(\mathcal{F})}\right) \longrightarrow \dots$$

Which is indeed an exact sequence of soft sheaves. This resolution is abbreviated as

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{S}^\bullet$$

What does this resolution even do? In all honesty, no clue and it is not important. The importance of this resolution is not in its looks but in its existence. Why is this so important? This resolution is essential for...

4.3 ARTIFICIAL SHEAF COHOMOLOGY

After much work with sheaf cohomology in the wild, we can continue with artificial sheaf cohomology. Many pages have been spent working towards this moment. Just like sheaf cohomology in the wild, artificial sheaf cohomology is requires a complex of sheaves.

Often it is not clear which complex one would take to obtain sheaf cohomology as we might only have one single sheaf on M and no resolution. Luckily, the Godement resolution provides us with a nice resolution of soft sheaves for any sheaf $\mathcal{F}$.

Definition 4.3.1 (Sheaf Cohomology of a Sheaf)**.**
Let $M \in \mathbf{Man}^\infty$, $\mathcal{F} \in \mathbf{Sh}(M)$ and let $\mathcal{S}^\bullet$ be the Godement resolution of $\mathcal{F}$. The sheaf cohomology of $\mathcal{F}$ is defined as the cohomology groups of the complex

$$\mathcal{S}^\bullet(M): \qquad 0 \longrightarrow \mathcal{S}(\mathcal{F})(M) \longrightarrow \mathcal{S}\left(\frac{\mathcal{S}(\mathcal{F})}{\mathcal{F}}\right)(M) \longrightarrow \dots$$

The cohomology groups of the sheaf cohomology of the sheaf $\mathcal{F}$ are denoted as

$$H^p(M, \mathcal{F}).$$

4.3.1 *Properties of Sheaf Cohomology*

We now have two constructions of sheaf cohomology. Using complexes from the wild and using the Godement resolution! The sheaf cohomology of a single sheaf may seem unwieldy at first, so it might be instructive to first analyse some properties of this cohomology of a single sheaf.

Proposition 4.3.1 (Properties of Sheaf Cohomology).
For any sheaf $\mathcal{F}$ over a smooth manifold M,

1. $H^0(M, \mathcal{F}) = \mathcal{F}(M)$,
2. *If $\mathcal{F}$ is soft, then $H^p(M, \mathcal{F}) = 0$ for $p > 0$.*

Proof.
(1) By definition of sheaf cohomology of a sheaf

$$H^0(M, \mathcal{F}) = Ker(q : \mathcal{S}(\mathcal{F})(M) \to \mathcal{S}\left(\frac{\mathcal{S}(\mathcal{F})}{\mathcal{F}}\right)(M)).$$

If q is understood, then $H^0(M, \mathcal{F})$ is known. First what does q do? It takes a discontinuous global section from $\mathcal{F}$ and maps this to its quotient class in $(\mathcal{S}(\mathcal{F})/\mathcal{F})(M)$. This quotient class is a section from M to $\overline{\mathcal{S}(\mathcal{F})/\mathcal{F}}$, and thus can be seen as an element of $\mathcal{S}(\mathcal{S}(\mathcal{F})/\mathcal{F})$.

As such, we can see q as the composition of the quotient map $\mathcal{S}(\mathcal{F})(M) \to (\mathcal{S}(\mathcal{F})/\mathcal{F})(M)$ and the inclusion $(\mathcal{S}(\mathcal{F})/\mathcal{F})(M) \hookrightarrow \mathcal{S}(\mathcal{S}(\mathcal{F})/\mathcal{F})(M)$. As the second of these mappings is injective, the kernel of q is completely determined by the kernel of $\mathcal{S}(\mathcal{F})(M) \to (\mathcal{S}(\mathcal{F})/\mathcal{F})(M)$. It is not hard to see this kernel is precisely $\mathcal{F}(M)$.

This allows for the following conclusion

$$H^0(M, \mathcal{F}) = \mathcal{F}(M).$$

(2) Suppose now that $\mathcal{F}$ is a soft sheaf on M. Per construction the Godement resolution of a sheaf is exact.

$$0 \longrightarrow \mathcal{F} \hookrightarrow \mathcal{S}(\mathcal{F}) \longrightarrow \mathcal{S}(\tfrac{\mathcal{S}(\mathcal{F})}{\mathcal{F}}) \longrightarrow \dots$$

According to Theorem 4.2.2 as these sheaves are soft, the sequence remains exact when taking global sections. This implies

$$0 \longrightarrow \mathcal{F}(M) \hookrightarrow \mathcal{S}(\mathcal{F})(M) \longrightarrow \mathcal{S}(\tfrac{\mathcal{S}(\mathcal{F})}{\mathcal{F}})(M) \longrightarrow \dots$$

is exact. This implies that except at $\mathcal{S}(\mathcal{F})$ the following sequence is exact.

$$0 \longrightarrow \mathcal{S}(\mathcal{F})(M) \longrightarrow \mathcal{S}(\tfrac{\mathcal{S}(\mathcal{F})}{\mathcal{F}})(M) \longrightarrow \dots$$

The exactness implies the following for cohomology groups with $p > 0$

$$H^p(M, \mathcal{F}) = 0.$$

□

Proposition 4.3.1 provides another reason why soft sheaves are so nice. The Godement resolution of a soft sheaf is exact almost everywhere. This is to be expected in view of Theorem 4.2.1 in combination with Corollary 4.2.2.

Property (2) of Proposition 4.3.1 is of note. A resolution of sheaves with this property, maintains exactness when taking global sections. It will turn out that sheaves with this property are particularly useful. This property is called acyclicity.

Definition 4.3.2 (Acyclic Resolution)**.**
A resolution of a sheaf $\mathcal{F}$ on $M \in \mathbf{Man}^{\infty}$,

$$0 \longrightarrow \mathcal{F} \longrightarrow A^{\bullet},$$

is acyclic if $H^q(M, A^p) = 0$ for every $q > 0$ and $p \geq 0$.

The use of acyclicity will become apparent in Section 4.3.2. The remainder of this section will focus on the Godement resolution as a functor. In particular, the Godement resolution can be seen as a covariant functor from $\mathbf{Sh}(M)$ to the category of complexes of sheaves on M.

The category of complexes of sheaves on M is denoted as $\mathbf{Comp}(M)$ and has complexes of sheaves on M as objects and morphisms of complexes as arrows. The Godement functor has one particularly useful property.

Lemma 4.3.1 (The Godement Functor is Exact)**.**
Let $\mathcal{S}^{\bullet}(-) : \mathbf{Sh}(M) \to \mathbf{Comp}(M)$ denote the Godement functor defined as

$$\mathcal{F} \mapsto \mathcal{S}^{\bullet}(\mathcal{F}).$$

The Godement functor is exact.

Proof.
To start this proof we begin with a s.e.s. of sheaves on $M \in \mathbf{Man}^{\infty}$.

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \tag{4.8}$$

Using the Godement resolution this s.e.s. of sheaves can be augmented. This results in the following diagram.

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathcal{S}^{\bullet}(A) & & \mathcal{S}^{\bullet}(B) & & \mathcal{S}^{\bullet}(C) & & \end{array} \tag{4.9}$$

The action of the Godement functor on sheaves is clear, however, its action on the morphism of sheaves remains clouded. The Godement functor applied to f results in the following.

An inductive definition of $h := \mathcal{S}^{\bullet}(f)$ will be provided. Begin by defining $h^0 : \mathcal{S}^0(A) \to \mathcal{S}^0(B)$ as

$$s \mapsto (h \circ s).$$

Now we can continue with the inductive part of the definition. Let $h^p : \mathcal{S}^p(A) \to \mathcal{S}^p(B)$ be a morphism of sheaves. We now turn to the following diagram to obtain the next map.

$$\begin{array}{ccc} \mathcal{S}^p(A) & \xrightarrow{h^p} & \mathcal{S}^p(B) \\ \downarrow q & & \downarrow q' \\ \frac{\mathcal{S}^p(A)}{\mathcal{S}^{p-1}(A)} & \longrightarrow & \frac{\mathcal{S}^p(B)}{\mathcal{S}^{p-1}(B)} \end{array}$$

As $\mathcal{S}^{p-1}(B) \subset \mathcal{S}^p(B)$ is a subsheaf the quotient is indeed well defined and the quotient map q' is the natural mapping. Our goal is to obtain the red arrow.

Via composition we obtain a map $\mathcal{S}^p(A) \to \frac{\mathcal{S}^p(B)}{\mathcal{S}^{p-1}(B)}$. Notice that $\alpha - \beta \in \mathcal{S}^{p-1}(A) \subset \mathcal{S}^p(A)$, implies $h^p(\alpha_x - \beta_x) = (h^{p-1} \circ (\alpha - \beta))_x \in \mathcal{S}^{p-1}(B)$. As such, $q'((h^{p-1} \circ (\alpha - \beta))_x) = 0$ and we obtain $q'(h^{p-1}(\alpha)_x) = q'(h^{p-1}(\beta)_x)$. Now via the universal property of quotient mappings the red arrow exists and the diagram commutes. Denote this red arrow by $\overline{h}^p$.

Then $\overline{h}^p$ induces a map

$$\begin{aligned} h^{p+1} : \mathcal{S}^{p+1}(A) = \mathcal{S}\left(\frac{\mathcal{S}^p(A)}{\mathcal{S}^{p-1}(A)}\right) &\longrightarrow \mathcal{S}\left(\frac{\mathcal{S}^p(B)}{\mathcal{S}^{p-1}(B)}\right) = \mathcal{S}^{p+1}(B), \\ s &\longmapsto (\overline{h}^p \circ s) \end{aligned}$$

We obtain $\mathcal{S}^\bullet(f)$ using this inductive procedure. Now that we know how $\mathcal{S}^\bullet$ acts on arrows, the lemma can be proven.

Applying $\mathcal{S}^\bullet$ to Sequence 4.8 results in Diagram 4.9 with additional arrows between the Godement resolutions. The exactness of the row containing $\mathcal{S}^0(A)$ is a direct result of the definition of the sheaf of discontinuous sections. As such, we will prove the exactness of the remaining rows using strong induction.

Supposing that all rows up to the p'th row are exact in Diagram 4.9, we will prove that the $p+1$'th row is exact. To this end, the following diagram will be used.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{S}^{p-1}(A) & \xrightarrow{f^{p-1}} & \mathcal{S}^{p-1}(B) & \xrightarrow{g^{p-1}} & \mathcal{S}^{p-1}(C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \phi_a} & & \downarrow{\scriptstyle \phi_b} & & \downarrow{\scriptstyle \phi_c} & & \\ 0 & \longrightarrow & \mathcal{S}^{p}(A) & \xrightarrow{f^{p}} & \mathcal{S}^{p}(B) & \xrightarrow{g^{p}} & \mathcal{S}^{p}(C) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle q_a} & & \downarrow{\scriptstyle q_b} & & \downarrow{\scriptstyle q_c} & & \\ 0 & \longrightarrow & \frac{\mathcal{S}^p(A)}{\mathcal{S}^{p-1}(A)} & \xrightarrow{\overline{f}} & \frac{\mathcal{S}^p(B)}{\mathcal{S}^{p-1}(B)} & \xrightarrow{\overline{g}} & \frac{\mathcal{S}^p(C)}{\mathcal{S}^{p-1}(C)} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_A} & & \downarrow{\scriptstyle i_B} & & \downarrow{\scriptstyle i_C} & & \\ 0 & \longrightarrow & \mathcal{S}^{p+1}(A) & \xrightarrow{f^{p+1}} & \mathcal{S}^{p+1}(B) & \xrightarrow{g^{p+1}} & \mathcal{S}^{p+1}(C) & \longrightarrow & 0 \end{array} \quad (4.10)$$

Let $x \in M$, and let $s_x \in Ker(f_x^{p+1})$. Then x has some open neighborhood U such that for all $y \in U$ we have $\overline{f}(s(y)) = 0$[2]. In other words, any representative of this class is contained in $\mathcal{S}^{p-1}(B)$, and as such can be lifted to $a \in \mathcal{S}^{p-1}(B)$.

Via commutativity we obtain the image of this lift under g^{p-1} must be contained in $Ker(\phi_c)$. Using the exactness of the columns, we lift this element to $\mathcal{S}^{p-2}(C)$. This element can be further lifted to $\mathcal{S}^{p-2}(B)$ via the surjectivity of g^{p-2}. This lifted element can then be mapped to $\alpha \in \mathcal{S}^{p-1}(B)$. Via commutativity we obtain $a - \alpha \in Ker(g^{p-1})$ and $\phi_b(a-\alpha) = \phi_b(a)$.

As such, we may assume our lift of $s(y)$ in $\mathcal{S}^{p-1}(B)$ to be contained in $Ker(g^{p-1})$. Using the exactness of the rows, it can be lifted to an element of $\mathcal{S}^{p-1}(A)$. We obtain that any representative of $s(y)$ is contained in $\mathcal{S}^{p-1}(A)$. As such, $s(y) = 0$ for all $y \in U$. As such, f^{p+1} is injective.

Suppose that $s \in \mathcal{S}^{p+1}(C)$ and let $x \in M$ as before. For $y \in U$ we obtain that since $\overline{g}$ is surjective, s can be pointwise lifted to a section $\overline{g}^{-1}(s) \in \mathcal{S}^{p+1}(B)$ defined as mapping y to the lift of $s(y)$ under $\overline{g}$. Note that via commutativity $g^{p+1}(\overline{g}^{-1}(s)) = s$. We obtain that g^{p+1} is surjective.

A direct consequence of the exactness of the p'th row, is $Im(f^{p+1}) \subseteq Ker(g^{p+1})$. As such, only the reversed inclusion remains to be proven. Suppose that $s_x \in Ker(g_x^{p+1})$ for any $x \in M$. Then for any $y \in U$ we obtain $\overline{g}(s(y)) = 0$. As such, we can lift under q_c to an element $\beta \in \mathcal{S}^{p-1}(C)$. As g^{p-1} is surjective, β can be lifted to $\mathcal{S}^{p-1}(B)$ and subsequently mapped to $\alpha \in \mathcal{S}^p(B)$.

[2] Note that s is a section from U to the espace étalé of $\mathcal{S}^p(A)/\mathcal{S}^{p+1}(A)$, and as such $s(y)$ is equal to the germ of a section at y. This allows for a diagram chase through Diagram 4.10 when looking at the stalks at y proving pointwise exactness. As s is defined pointwise, pointwise exactness suffices for a proof of exactness.

Additionally, $s(y)$ can be lifted to a representative $a \in \mathcal{S}^p(B)$. Note that $g^p(a - \alpha) = 0$ per commutativity. As such, $a - \alpha$ can be lifted to $\eta \in \mathcal{S}^p(A)$ using exactness of the rows. Using commutativity we observe $\overline{f}(q_a(\eta)) = q_b(\alpha) = s(y)$. Define the discontinuous section $b \in \mathcal{S}^{p+1}(A)$ as $y \mapsto q_a(\eta)$. This is indeed a discontinuous section defined on U and $f^{p+1}(b) = s$ on U. As such, the reverse inclusion is proven and the p'th row is shown to be exact. □

Remark 4.3.1. *Using the proof of Corollary 4.2.2 we obtain that $\mathcal{S}$ applied to any exact sequence of sheaves results in an exact sequence of complexes of sheaves.*

This might seem like the perfect moment to include an example of the sheaf cohomology of some sheaf on a smooth manifold. However, in practise the definition of sheaf cohomology of a single sheaf is rarely used when computing sheaf cohomology.

The definition of artificial sheaf cohomology is too unwieldy to effectively be used to compute sheaf cohomology. What is often done is the following. One finds another resolution of the same sheaf and computes its cohomology. Then the cohomology groups are shown to be isomorphic to that of the artificial sheaf cohomology. As such, we need to compare wild and artificial sheaf cohomology.

4.3.2 *Abstract De Rham Theorem*

Both wild and artificial sheaf cohomology have implication for a fixed sheaf. Both constructions vary greatly, and one may wonder when these sheaf cohomologies coincide.

Intuitively, that these cohomologies can even coincide may seem unlikely. Example 4.2.1 showed that the sheaf Ω^k is acyclic. Furthermore, the cohomology of this resolution was closely linked to the analysis of the manifold on which it is defined. The artificial sheaf cohomology does not seem to have any immediate analytic implications. As such, an isomorphism in sheaf cohomologies would be unexpected.

In order for these two cohomologies to be compared, they need to talk to each other, this is done via a morphism. The following Lemma, often called the *abstract De Rham theorem*, provides such a morphism. Furthermore, it provides an answer to our original question.

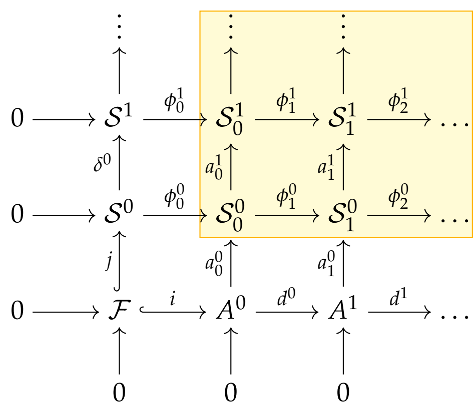
Theorem 4.3.1 (Abstract De Rham Theorem)**.**
*Let $\mathcal{F}$ be a sheaf over a smooth manifold $M \in$ **Man**$^\infty$. Let*

$$0 \longrightarrow \mathcal{F} \longrightarrow A^\bullet \tag{4.11}$$

be an acyclic resolution of $\mathcal{F}$. Then there is a natural isomorphism

$$g^p : H^p(M, A^\bullet) \to H^p(M, \mathcal{F}).$$

Proof.
As with many proofs, we begin with a commutative diagram of sheaves.



(4.12)

Here the sheaves $\mathcal{S}_j^i$ are the i'th Godement resolution sheaf associated to the Godement resolution of A^j. As in the proof of Lemma 4.3.1 the arrows between $\mathcal{F}$ and $A^\bullet$ induce arrows between their Godement resolutions, resulting in the horizontal arrows. Additionally, as Sequence 4.11 is exact, this same proof provides that Diagram 4.7 has both exact rows and columns.

Applying the global sections functor to Diagram 4.12 results in a diagram of abelian groups. Since $A^\bullet$ was assumed to be acyclic, only the most left column and the bottom row of Diagram 4.12 are not exact after applying the global sections functor. As such, the arrows in the yellow square in Diagram 4.12 are exact even after applying the global sections functor. Using this exactness the required isomorphism will be constructed.

Let $\alpha \in Ker(d^p)$, then under a_p^0 we can map to $a_p^0(\alpha) \in S_p^0$. Via commutativity of the diagram we obtain that $a_p^0(\alpha) \in Ker(\phi_{p+1}^0)$. Using the exactness of the rows, we can lift to $\beta \in S_{p-1}^0$. We are now in the same situation as at the start of this diagram chase. By repeating this procedure $p+1$ times, an element $\gamma \in Ker(\delta^p) \subseteq S^p$ is obtained.

What is actually done in this construction is the following. We begin with an element in $Ker(d^p)$ and we lift this element diagonally to S_{p-1}^0. We then apply the same diagram chase to the lift in S_{p-1}^0 to obtain a lift of S_{p-2}^1. This procedure results in a staircase-like lift from α to γ.

Using this map we can define the map $g^p : H^p(M, A^\bullet) \to H^p(M, \mathcal{F})$ as $\overline{\alpha} \mapsto \overline{\gamma}$. Following the same diagram chase as above, the map g^p can be found to be well defined and to be a homomorphism. If the morphism g^p can be shown to be bijective for all p, then the theorem is proven.

First, surjectivity will be shown. Let $\overline{\gamma} \in H^p(M, \mathcal{F})$, then per definition $\gamma \in Ker(\delta^p)$. Then we can map to $\phi_1^p(\gamma) \in S_0^p$. Via commutativity we obtain $\phi_1^p(\gamma) \in Ker(a_0^1)$. Per exactness in the columns this implies we can lift to an element of S_0^{p-1}. Once more we can map to S_1^{p-1} under ϕ_1^{p-1}. Once again the commutativity of the diagram ensures that the image is contained in $Ker(a_1^{p-1})$. We are once again in the situation in which we started this diagram chase. As such, the same procedure can be applied to obtain a pre-image for $\overline{\gamma}$ in $H^p(M, A^\bullet)$. We conclude g^p is surjective.

Now the injectivity will be shown. To this end let $g^p(\overline{\alpha}) = 0$, then $\gamma \in Im(\delta^{p-1})$. This implies γ can be lifted to $s \in S^{p-1}$. Under ϕ_0^{p-1} and ϕ_1^{p-1} this lift is mapped to zero. As such, during our construction of γ, we lift to an element in $Ker(a_1^{p-2})$.

Via the exactness of the columns, this element can be lifted to S_1^{p-3}. By repeating this procedure an element $\beta \in A^{p-1}$ is obtained such that $d^{p-1}(\beta) = \alpha$. Thus, we conclude $Ker(g^p) = 0$ and g^p is injective. This results in g^p being an isomorphism. □

Theorem 4.3.1 states that any acyclic resolution of a sheaf has isomorphic cohomology to the sheaf cohomology of this sheaf. As such, any acyclic resolution can be used to compute the sheaf cohomology of a sheaf. Additionally, any pair of acyclic resolutions have sheaf cohomologies which are both isomorphic to sheaf cohomology.

If we return to the theorem of De Rham, then we wish to construct an isomorphism between the singular cohomology and De Rham cohomology groups. Sheaf cohomology appears to be the perfect middle man to proving this. Our plan of attack is the following.

$$\{\text{De Rham}\} \rightleftarrows \{\text{Sheaf}\} \rightleftarrows \{\text{Singular}\}, \qquad \{\text{De Rham}\} \overset{\sim}{\longleftrightarrow} \{\text{Singular}\}$$

We want to show that the De Rham cohomology and Singular cohomolohy are both isomorphic to the sheaf cohomology of a fixed sheaf. Using these two isomorphism we want to construct an isomorphism between these two cohomology groups. The final attack and completion of the proof is done in the following chapter.

5

THE DE RHAM THEOREM

Our attention can now be focussed on the theorem of De Rham. This theorem was first stated and proven by Georges De Rham himself in 1931 in his doctoral paper *Sur l'analysis situs des variétés à n dimensions* (De Rham, [12]). This theorem was later generalised and proven using sheaf cohomology by a founder of the Bourbaki-group, André Weil, in the paper *Sur les théorèmes de de Rham* (Weil, [16]). The proof we will give is a modification to the proof of Weil, which can be found in (Wells, [17]).

The De Rham theorem states that smooth singular cohomology and De Rham cohomology are both isomorphic for smooth manifolds. Even better, the theorem provides an explicit isomorphism between the two cohomology groups, as was discussed in Section 2.5.

In Chapter 4 a relation between the cohomology of an acyclic resolution and sheaf cohomology was provided. In the proof of the theorem of De Rham this relation is lifted to an isomorphism of two cohomologies of acyclic resolutions. As both the De Rham and Cochain complex are acyclic, this provides our coveted isomorphism.

5.1 REQUIRED LEMMA

The Abstract Theorem of De Rham 4.3.1 provides a connection between wild and grown sheaf cohomology. In fact, it even provides an isomorphism if the resolution of the wild cohomology is acyclic. One might wonder whether every pair of acyclic resolutions have isomorphic cohomology groups?

Lemma 5.1.1 (Morphism of Resolutions).
Let $S, T \in \mathbf{Sh}(M)$ for $M \in \mathbf{Man}^\infty$. Suppose $A^\bullet$ and $B^\bullet$ are acyclic resolutions of S and T respectively and let $f : S \to T$ be an isomorphism of sheaves. If the following diagram of morphism of resolutions of sheaves is commutative,

$$\begin{array}{ccccc} 0 & \longrightarrow & S & \xrightarrow{a} & A^\bullet \\ & & \big\downarrow f & & \big\downarrow g \\ 0 & \longrightarrow & T & \xrightarrow{b} & B^\bullet \end{array}$$

then there is an induced isomorphism

$$g^* : H^p(A^\bullet(M)) \to H^p(B^\bullet(M)).$$

Proof.
We begin by looking at the induced diagram on cohomology groups for an arbitrary $p \geq 0$. This leads to the following commutative diagram.

$$\begin{array}{ccc} H^p(X,S) & \xrightarrow{a^*} & H^p(A^\bullet(X)) \\ \downarrow{f^*} & & \downarrow{g^*} \\ H^p(X,T) & \xrightarrow{b^*} & H^p(B^\bullet(X)) \end{array}$$

In the above diagram the horizontal arrows are courteously provides by Theorem 4.3.1. We now turn our attention to the map g^*.

Remember that g^* was defined as the map induced on the cohomology groups by g. In other words $[s] \mapsto [g \circ s]$. Indeed g^* is a group homomorphism. Thus we have obtained the first main result from this theorem.

If we also assume that f is an isomorphism and $A^\bullet, B^\bullet$ are acyclic, the real power of this theorem becomes apparent. From the commutativity of the above diagram, the following identity becomes clear

$$g^* \circ a^* = b^* \circ f^*.$$

The goal is to show that g^* is an isomorphism. If a^* and b^* are isomorphisms, then we are done. As then we can use the inverse of a^* to obtain the equality

$$g_p^* = b^* \circ f^* \circ (a^*)^{-1}.$$

On the right hand side this is a composition of only isomorphisms. Thus we obtain g_p^* must be an isomorphism.

Now remember that Theorem 4.3.1 stated that if $A^\bullet$ and $B^\bullet$ were cyclic, then a^* and b^* are isomorphisms. Since $A^\bullet$ and $B^\bullet$ are cyclic per assumption, we can conclude that a^* and b^* are isomorphisms. As such the theorem is proven. □

5.2 THE PROOF

Finally we are there, the theorem of De Rham. The theorem was already stated and motivated in Section 2.5. As such, let us jump in and immediately state and proof this beast of a theorem.

Theorem 5.2.1.
Let $M \in \mathbf{Man}^\infty$. Then there is an isomorphism

$$\mathfrak{c}^* : H^p_{DR}(M) = H^p(M, \Omega^\bullet) \xrightarrow{\sim} H^p(M, C^\bullet) = H^p(M).$$

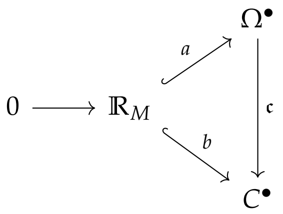
The isomorphism $\mathfrak{c}$ is defined by the natural mapping as

$$[\omega] \mapsto \left[\int_{(-)} \omega : \sigma \mapsto \int_\sigma \omega \right].$$

Proof.
First fix $M \in \mathbf{Man}^\infty$. Remember that $\Omega^\bullet$ and $C^\bullet$ are both resolutions of sheaves of the constant sheaf $\mathbb{R}_M$ as described in Examples 4.1.1 and 4.1.2. Furthermore, there is a map $\mathfrak{c} : \Omega^\bullet \to C^\bullet$ between these two complexes defined as

$$\omega \mapsto \left(\int_{(-)} \omega : \sigma \mapsto \int_\sigma \omega \right),$$

described in Example 4.1.3. This leads to the following commutative diagram of morphisms of sheaf resolutions.



Lemma 5.1.1 states the following about this diagram. There is an induced morphism $\mathfrak{c}_p^* : H^p(\Omega^\bullet(M)) \to H^p(C^\bullet(M))$. Furthermore, if $\Omega^\bullet$ and $C^\bullet$ are acyclic, then this $\mathfrak{c}_p^*$ is an isomorphism.

Remark 5.2.1. *The requirement that f is an isomorphism is trivially satisfied since in the diagram above f coincides with the identity map on the constant sheaf $\mathbb{R}_M$ which is certainly an isomorphism.*

So, our goal is to show $\Omega^\bullet$ and $C^\bullet$ are both acyclic. From Example 4.2.1 we obtain Ω^k is fine for every $k \geq 0$. As such, $\Omega^\bullet$ is comprised of fine sheaves. Each fine sheaf is soft and as such acyclic. Thus $\Omega^\bullet$ is an acyclic resolution. Additionally, recall Example 4.2.2. In this example we already showed that C^p is soft for every $p \in \mathbb{Z}$. As such, C^p is also acyclic. Indeed Lemma 5.1.1 provides that the natural map $\mathfrak{c}^*$ is an isomorphism. □

5.3 conclusion

In conclusion, Theorem 5.2.1 allows us to conclude that the theorem of De Rham indeed holds true, by providing an ismorphism between the singular cohomology and De Rham cohomology. Intuitively, this implies the degree of failure of the fundamental theorem of calculus fails for closed differential forms coincides with the number of holes of a surface, up to torsion, with the integral map, $\mathfrak{c}^*$, providing the required isomorphism explicitly.

Additionally, Theorem 5.2.1 establishes an isomorphism between the sheaf cohomology of $\mathbb{R}_M$ and both the singular cohomology and De Rham cohomology. Showing a correspondence between all covered types of cohomology.

This result directly links the geometry of a surface to both the analysis and sheaves on this surface. Thereby providing a bridge between algebraic geometry and differential geometry as detailed in Chapter 0.

LIST OF FIGURES

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