Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft Institute of Applied Mathematics

Elaboration on Kwapień's theorem:
Representing bounded mean zero functions f as coboundary $f = g \circ T - g$

by

Matthijs J. Borst

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Supervisor

prof. dr. ir. M.C Veraar

Other members of thesis committee

dr. B. van den Dries

dr. C. Kraaikamp

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Abstract

In [8] Kwapień proved that every mean zero function $f \in L_\infty[0,1]$ we can write as $f = g \circ T - g$ for some $g \in L_\infty[0,1]$ and some measure preserving transformation T of $[0,1]$. However, as was discovered in [4] there is a gap in the proof for the case that f is not continuous. The aim of this bachelor thesis is filling in that gap in the proof.

We first extend Kwapień's proof for continuous functions to certain other measure spaces. Thereafter, we use the method of proof suggested by Kwapien, to proof the theorem for mean zero function $f \in L_\infty[0,1]$ for which $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. Using this result we then proof that every mean zero function $f \in L_\infty[0,1]$ can be written as a sum $f = (g_1 \circ T_1 - g_1) + (g_2 \circ T_2 - g_2)$ where $g_1, g_2 \in L_\infty[0,1]$ and where T_1, T_2 are measure preserving transformations of $[0,1]$. We finish this thesis with an application of Kwapien's theorem in the study to singular traces.

Preface

When I had to choose my Bachelor Project, this project appealed to me as it seemed an interesting problem in analysis. Once I started the project, and started reading Kwapień's proof, I was quite amazed that the proof used a construction by means of the Cantor set, as I had not expected something like that to appear in the proof. I liked this a lot, as I find the Cantor set a very interesting object. I enjoyed working on the project and I hope you will enjoy reading it. I now want to thank some people. First of all, I want to thank my supervisor Mark Veraar for giving me a lot of helpful feedback and insights. I also want to thank professor F.A. Sukochev and professor A.F. Ber for their helpful feedback and comments. Further I want to thank professor T. Adams, for helping me better understand his line of proof. I want to thank my family for reading and commenting this thesis. Finally, I want to thank the thesis committee.

Contents

1. Introduction

In this Bachelor Thesis we focus on a result of Kwapień [8] that says that we can write every mean-zero function $f \in L_\infty(\Omega, \Sigma, \mu)$, where (Ω, Σ, μ) is a standard measure space, as $f = g \circ T - g$ with $g \in L_\infty(\Omega, \Sigma, \mu)$ and with $T : \Omega \to \Omega$ being a measure preserving map. Kwapień first proved the theorem only for the mean-zero functions $f \in L_\infty([0,1], \lambda)$ that are continuous, and thereafter extended his proof to general mean-zero functions $f \in L_\infty([0,1], \lambda)$ and he noted that the proof for standard measure spaces follows from the proof for the interval $[0,1]$ with the Lebesgue measure λ. However, in [4] was discovered that there is a problem in the proof for general mean-zero functions $f \in L_\infty([0,1], \lambda)$. They also proved an alternative result that says that we can write every mean-zero function $f \in L_\infty([0,1], \lambda)$ as a sum $f = \sum_{i=1}^{k} g_i \circ T_i - g_i$ with k at most 20, and where $g_i \in L_\infty([0,1], \lambda)$ and where T_i is a measure preserving transformation of $[0,1]$. However, at time of this writing, the proof is not available yet.

The aim of this thesis is to investigate the proof of Kwapień and explore how much further we can push his method of proof, in order to obtain a result for general mean-zero functions. We prove the following theorems.

Theorem 1.1. *Let $f \in L_\infty([0,1], \mathcal{B}([0,1]), \lambda)$ be continuous and mean zero. Choose $\epsilon > 0$, then we can a find $g \in L_\infty([0,1], \mathcal{B}([0,1]), \lambda)$ with $||g||_\infty < 4||f||_\infty + \epsilon$ and a measure preserving transformation T of $[0,1]$ such that $f = g \circ T - g$.*

Theorem 1.2. *Let $f \in L_\infty([0,1], \mathcal{B}([0,1]), \lambda)$ be mean-zero and nowhere essentially constant, that is, $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. Choose $\epsilon > 0$, then we can a find $g \in L_\infty([0,1], \mathcal{B}([0,1]), \lambda)$ with $||g||_\infty < 4||f||_\infty + \epsilon$ and a measure preserving transformation T of $[0,1]$ such $f = g \circ T - g$*

Theorem 1.3. *Let $f \in L_\infty([0,1], \mathcal{B}([0,1]), \lambda)$ be mean zero. Choose $\epsilon > 0$, then we can find $g_1, g_2 \in L_\infty([0,1], \mathcal{B}([0,1]), \lambda)$ with $||g_1||_\infty < 4||f||_\infty + \epsilon$ and $||g_2||_\infty < \epsilon$ and measure preserving transformations T_1,T_2 of $[0,1]$ such that $f = (g_1 \circ T_1 - g_1) + (g_2 \circ T_2 - g_2)$.*

We now give an overview of the structure of this thesis. In the Preliminaries, section 2, we will first state our conventions on some notation and further state some definition and theorems from measure theory and topology that we will use throughout this thesis. In section 3 we elaborate on the context around Kwapień's theorem. In section 4 we work out Kwapień's proof for continuous, mean-zero functions in more detail and in a slightly more general setting such that the theorem holds for a larger variety of measure spaces. Theorem 1.1 then follows from a special case of this. In section 5 we consider the proof for general, not necessarily continuous, mean-zero functions and try to fill in the missing gap. We succeeded to prove Kwapień's theorem under the extra assumption that f is nowhere essentially constant, see Theorem 1.2. In section 6 we use the theorems 1.1 and 1.2 to obtain that we can write a general mean-zero functions f as sum $f = (g_1 \circ T_1 - g_1) + (g_2 \circ T_2 - g_2)$, see Theorem 1.3. In section 7, we conclude with an application of last result in the field of singular traces.

We further note that in [2], the full result of Kwapień seems to be proven. The proof uses a different line than Kwapień. In [1], they also obtained a similar result that for $1 \leq p < \infty$, we can write a mean-zero function $f \in L_p([0,1],\lambda)$ as $f = g \circ T - g$ with $g \in L_{p-1}([0,1],\lambda)$ and with T a measure preserving transformation of $[0,1]$.

2. Preliminaries

Throughout this bachelor thesis we will use some terminology and theorems from measure theory as well as from topology. We will state these theorems and definitions here for reference and to introduce our conventions on the notation. We will first introduce our conventions on some general notation, thereafter we will state some definitions and classical theorems from measure theory and subsequently do this as well for definitions and theorems for topology.

2.1. General notation.

We give some quick notes on general notation we use. We use $\mathbb{N}$ to denote $\{1, 2, ..\}$ and $\mathbb{N}_0$ for $\{0, 1, 2, ...\}$. Further, we will use the notation $||\cdot||_\infty$ exclusively for the essential supremum. Further, we will use the notation λ for the Lebesgue measure.

We will further use this extension of the notion of a cyclic permutation, which we will use in Section 4.

Definition 2.1. *We will call a function $T : A \to A$ a cyclic permutation of the set $J = \{I_1, ...I_n\}$ where $I_i \subseteq A$, if for all $I_i \in J$ we have for the image $T(I_i)$ that $T(I_i) \in J$ and if for the image of I_i after k times applying T we have $T^k(I_i) \neq I_i$ for $1 \leq k \leq n-1$ and $T^n(I_i) = I_i$. Note that we do not require the sets in J to be disjoint neither do we require A to be the union of the sets of J. Further note that the definition is equivalent of saying that $T(I_i) = I_{\sigma(i)}$ where σ is a cyclic permutation of $\{1, 2, 3, ..., n\}$ in the usual sense.*

2.2. Topics from measure theory.

Throughout this thesis, we will constantly be working in the space L_∞ on some measure space. Hence we include the definition.

Definition 2.2. *(Essentially bounded functions) Let (Ω, Σ, μ) be a measure space. We let $\mathcal{L}$ denote the set of all functions $f : \Omega \to \mathbb{R}$ and we define an equivalence relation $\sim$ by $f \sim g$ if and only if $\mu(\{f \neq g\}) = 0$. For functions $f \in \mathcal{L}/\sim$ we set $||f||_\infty = \inf\{x \geq 0 : \mu(\{|f| > x\}) = 0\}$ which can be shown to be invariant under the equivalence relation. Furthermore we now let $L_\infty(\Omega, \Sigma, \mu) := \{f \in \mathcal{L}/\sim : ||f||_\infty < \infty\}$ called the space of essentially bounded functions, which we may sometimes abbreviate to $L_\infty(\Omega)$ or just L_∞ when the context is clear. The space L_∞ equipped with the norm $||\cdot||_\infty$ becomes a Banach space.*

We will further be using the following definition of a measure preserving transformation, where we demand invertibility.

Definition 2.3. *(Measure preserving transformation) Let $\mathcal{M} = (\Omega, \Sigma, \mu)$, $\mathcal{M}' = (\Omega', \Sigma', \mu')$ be measure spaces and let $T : \Omega \to \Omega'$ be a bijection. We* *say that T is measure preserving transformation if T and T^{-1} are measurable and if $\mu'(T(A)) = \mu(A)$ for all measurable $A \in \Sigma$. We will also call T an isomorphism. Further, in case such T exists, we call $\mathcal{M}$ and $\mathcal{M}'$ isomorphic.*

We will extend certain results for functions $f \in L_\infty([0,1], \lambda)$ to standard measure spaces, which are the following.

Definition 2.4. *(Standard measure space) We call a measure space $\mathcal{M} = (\Omega, \Sigma, \mu)$ a standard measure space if it is isomorphic to a finite interval with the Lebesgue measurable sets and the Lebesgue measure.*

In order to show that two measures are equal, the following on π-systems will be helpful. Theorem 2.6 can be be found in [6, Corollary 1.6.2]

Definition 2.5. *(π-system) We call a set $\mathcal{C} \subseteq \mathcal{P}(X)$ a π-system on X if for $A, B \in \mathcal{C}$ we have $A \cap B \in \mathcal{C}$, that is, $\mathcal{C}$ is closed under finite intersections.*

Theorem 2.6. *(Equality on π-systems) Let $(X, \mathcal{A})$ be a measurable space, and let $\mathcal{C}$ be a π-system on X such that $\mathcal{A} = \sigma(\mathcal{C})$. If μ and ν are finite measures on X that satisfy $\mu(X) = \nu(X)$ and that satisfy $\mu(C) = \nu(C)$ for each C in $\mathcal{C}$, then $\mu = \nu$.*

In Section 4 we will use the conditional expectation $\mathbb{E}(f|\mathcal{F})$, of f given a σ-algebra $\mathcal{F}$ to approximate the function f. Furthermore, in section 7 we will also be using the conditional expectation. We give the definition.

Definition 2.7. *(Conditional expectation) Let (Ω, Σ, μ) be a measure space and let f be a Σ-measurable function. Further, let $\mathcal{F} \subseteq \Sigma$ be a σ algebra on Ω. The conditional expectation of X given $\mathcal{F}$ is a $\mathcal{F}$-measurable function $\mathbb{E}(f|\mathcal{F})$ for which $\int_F \mathbb{E}(f|\mathcal{F}) d\lambda = \int_F f d\lambda$ for all $F \in \mathcal{F}$.*

Furthermore, in all our cases the σ-algebra will be generated by a partition $\mathcal{A}$ of Ω, in which case it is clear that $\mathbb{E}(f|\mathcal{F})$ exists and is unique, up to a null set. Namely it is the function that is constant on each $A \in \mathcal{A}$ and equal to the average of f over A.

In section 5 we will use the Lusin's theorem which will allow us to reduce the case for general mean-zero functions to continuous mean-zero functions. We state the theorem and a corresponding results that we will use. Theorem 2.9 can be found in [6, Proposition 1.4.1] and Theorem 2.10 in [6, Theorem 7.4.3].

Definition 2.8. *(Regularity) A measure μ on a topological space Ω is called regular if for all measurable sets A we have*

$$\begin{aligned}\mu(A) &= \inf\{\mu(B) : A \subseteq B \textit{and } B \textit{ open and measurable}\} \\ &= \sup\{\mu(B) : B \subseteq A \textit{and } B \textit{ compact and measurable}\}\end{aligned}$$

Theorem 2.9. *(Lebesgue measure regular) The Lebesgue measure λ on the real line is regular.*

Theorem 2.10. *(Lusin's theorem) Let Ω be a locally compact Hausdorff space, let $\mathcal{A}$ be a σ-algebra on Ω that includes $\mathcal{B}(\Omega)$, let μ be a regular measure on $(\Omega, \mathcal{A})$, and let $f : A \to \mathbb{R}$ be $\mathcal{A}$-measurable. If A belongs to $\mathcal{A}$ and satisfies $\mu(A) < \infty$, and if ϵ is a positive number, then there is a* *compact subset K of A such that $\mu(A \setminus K) < \epsilon$ and such that the restriction of f to K is continuous. Moreover, there is a continuous function g with compact support, that agrees with f at each point in K.*

We can equip a Cantor space with a σ-algebra and a product measure. In this way we obtain the result for continuous functions also for Cantor spaces. We give a definition of the product of measure spaces.

Definition 2.11. *(Product of measure spaces) Let $\mathcal{I}$ be some index set and for $i \in \mathcal{I}$ let $\mathcal{M}_i = (\Omega_i, \Sigma_i)$ be a measurable space. We naturally equip the Cartesian product $\Omega := \bigotimes_{i\in\mathcal{I}} \Omega_i$ with the σ-algebra $\Sigma := \sigma(\{(B_i)_{i\in\mathcal{I}} \in \bigotimes_{i\in\mathcal{I}} \Sigma_i : \Omega_i = \Sigma_i$ for all but finitely many $i \in \mathcal{I}\})$, that is, Σ is the smallest σ-algebra such that all projections $p_j : \Omega \to \Omega_j$ that sends $(a)_{i\in\mathcal{I}}$ to a_j, are measurable.*

We need following result, which says that we can define a product measure μ on the product of measure spaces. This theorem can be found in [5, Theorem 3.5.1].

Theorem 2.12. *(Kolmogorov's extension theorem) Let $\mathcal{I}$ be countable and for $i \in \mathcal{I}$ let $(\Omega_i, \Sigma_i, \mu_i)$ be a probability space with Ω_i being a compact metric space. Then there exists a unique probability measure $\mu = \bigotimes_{i\in\mathcal{I}} \mu_i$ on $(\bigotimes_{i\in\mathcal{I}} \Omega_i, \bigotimes_{i\in\mathcal{I}} \Sigma_i)$ with $\mu(A) = \prod_{i=1}^{\infty} \mu_i(A_i)$ for $A \in \bigotimes_{i\in I} \Sigma_i$ with $A_i = \Omega_i$ for all but finitely many $i \in \mathcal{I}$. We call the measure μ, the product measure.*

2.3. **Topics from topology.** We will state some definitions and theorems from topology. We will use these mainly in the proof of Theorem 5.2 where we will construct a homeomorphism between a certain subset of $[0, 1]$ and the Cantor set. We will first introduce the product topology, which we will use to construct the standard Cantor space $\{0, 1\}^{\mathbb{N}}$.

Definition 2.13. *(Product of topological spaces) Let $\mathcal{I}$ be some index set and for $i \in \mathcal{I}$ let Ω_i be a topological space with topology $\mathcal{T}_i$. We will naturally equip the Cartesian product $\Omega := \bigotimes_{i\in\mathcal{I}} \Omega_i$ with the topology $\mathcal{T} := \{(B_i)_{i\in\mathcal{I}} \in \bigotimes_{i\in I} \mathcal{T}_i : B_i = \Omega_i$ for all but finitely many $i \in \mathcal{I}\}$ called the product topology.*

We will use Tychonoff's theorem to show that $\{1, 2\}^{\mathbb{N}}$ is compact. This theorem can be found in [7, Theorem 5.13].

Theorem 2.14. *(Tychonoff's theorem) Every product $\bigotimes_{i\in\mathcal{I}} \Omega_i$ of compact spaces Ω_i is compact. (This theorem for general index sets $\mathcal{I}$ is equivalent with the axiom of choice.)*

Definition 2.15. *A Cantor space is a topological space homeomorphic to the standard Cantor set $\{1, 2\}^{\mathbb{N}}$.*

In Lusin's theorem, theorem 2.10, the space Ω was required to be Hausdorff and locally compact. We state these definitions here.

Definition 2.16. *A topological space Ω is called locally compact if every $x \in \Omega$ has a compact neighbourhood, that is, for $x \in \Omega$ there is an open set U that contains x, and a compact set K with $U \subseteq K$.*

Definition 2.17. *A topological space Ω is called Hausdorff if for every two points $x, y \in \Omega$ with $x \neq y$, there exists disjoint open sets $U, V \subseteq \Omega$ with $x \in U$ and $y \in V$.*

3. Context of Kwapień's theorem

In this section we will introduce some terminology on cochains and coboundaries, to place Kwapień's theorem in a bigger context. We won't be needing this section in the rest of the thesis, hence the reader may skip this. The notation we use in this section is taken as in [10].

Let $(G, \cdot)$ be a group that acts on a set X. For $n \in \mathbb{N}_0$, an n-chain is defined to be a formal linear combination of $n+1$ tuples $(g_1, g_2, ..., g_n, x)$ with $x \in X$ and $g_i \in G$. A tuple $(g_1, ..., g_n, x)$ can be thought of as an oriented simplex with the $n+1$ vertexes $x, g_n x, g_{n-1}g_n x, ..., g_1 g_2 ... g_n x$. An n-chain is then a formal linear combination of oriented simplexes. The space of all n-chains is denoted by $C_n(X, G)$. We can now define the so called boundary maps between $\delta_n : C_n(X, G) \to C_{n-1}(X, G)$ defined on $n+1$-tuples as

$$\begin{aligned}\delta_n(g_1, ..., g_n, x) &= (g_1, ..., g_{n-1}, g_n x) \\ &\quad + \sum_{i=1}^{n-1} (-1)^{n-i} (g_1, ., g_i, g_{i+1} g_{i+2}, g_{i+3}, ..., g_n, x) \\ &\quad + (-1)^n (g_2, ..., g_n, x)\end{aligned}$$

and this definition can be linearly extended to $C_n(X, G)$. Now, if we think of n-chains as oriented simplexes, then we see that this boundary map returns the formal linear combination of the faces of the simplex. The signs ± 1 says what the orientation is of those faces.
Now, for example we have

$$\delta_1(g, x) = gx - x$$

$$\delta_2(g_1, g_2, x) = (g_1, g_2 x) - (g_1 g_2, x) + (g_2, x)$$

Thus, the boundary of the line between x and gx is the formal sum of the endpoints x and gx. The minus-sign is to indicate the orientation.

Now, in the algebra, a chain complex is defined to be a sequence $... \leftarrow A_0 \leftarrow A_1 \leftarrow A_2 \leftarrow ...$ of abelian groups A_i, with given homomorphisms $\tau_i : A_i \to A_{i-1}$ such that the image $\operatorname{Im} \tau_{i+1}$ equals the kernel $\operatorname{Ker} \tau_i$. This is to say that $\tau_i \circ \tau_{i+1} = 0$, where 0 denotes the zero-map. Further, a chain complex has a dual, called a co-chain complex. This is a sequence $... \to B_0 \to B_1 \to B_2 \to ...$, with B_i being an abelian group, with given homomorphisms $\tau^i : B_i \to B_{i+1}$ such that $\tau^{i+1} \circ \tau^i = 0$ is satisfied.
In our case, one can check by writing out that $\delta_n \circ \delta_{n+1} = 0$, where 0 denotes the zero map. Hence, this makes the sequence $.. \leftarrow 0 \leftarrow 0 \leftarrow C_0(X, G) \overset{\delta_1}{\leftarrow} C_1(X, G) \overset{\delta_2}{\leftarrow} C_2(X, G) \overset{\delta_3}{\leftarrow}$ a chain complex, where we denote 0 for the trivial group.

Now, if we let $(U, +)$ be some fixed abelian group, then for $n \in \mathbb{N}_0$ we can define n-cochains as homomorphisms in $C^n(X, G, U) := \operatorname{Hom}(C_n(X, G), U)$. These are the dual of the n-chains. We can also define the coboundary map

$\delta^n : C^n(X,G,U) \to C^{n+1}(X,G,U)$ as $\delta^n F = F \circ \delta_n$. Thus for example:

$$\delta^1 F(g,x) = F(gx) - F(x)$$

$$\delta^2 F(g_1,g_2,x) = F(g_1,g_2x) - F(g_1g_2,x) + F(g_2,x)$$

We have that $\delta^{n+1} \circ \delta^n = 0$ so that $\to 0 \to 0 \to C^0(X,G,U) \xrightarrow{\delta^1} C^1(X,G,U) \xrightarrow{\delta^2} C^2(X,G,U) \xrightarrow{\delta^3} ...$ is a co-chain complex.

Now, in our case the group G is the set of all measure preserving transformations of $[0,1]$ with the group operation being the convolution. This is well-defined, namely, if T_1 and T_2 are measure preserving transformations of $[0,1]$, then so is $T_1 \circ T_2$. We also have that the inverse T_1^{-1} is in G. Further the identity $I_{[0,1]}$ is the identity in G.

Now let $X = U = L_\infty([0,1], \mathcal{B}([0,1]), \lambda)$. The group G acts on X by $Tg \to g \circ T^{-1}$. Now, if we take $F \in C^n(X,G,U)$ the identity map, then we obtain the so called coboundary equation $\delta^1 F(T^{-1}, g) = F(T^{-1}g) - F(g) = T^{-1}g - g = g \circ T - g$. Now the problem that we investigate is to find for which $f \in U$ the coboundary equation can be solved, that is to say, for which $f \in L_\infty([0,1],\lambda)$ we can write f as coboundary $f = g \circ T - g$ for some $g \in L_\infty([0,1],\lambda)$ and measure preserving transformation T of $[0,1]$. Since the map T must be measure preserving, it can be shown that $\int_{[0,1]} g \circ T d\lambda = \int_{[0,1]} g d\lambda$ for all $g \in L_\infty([0,1],\lambda)$, and hence that any function f for which the equation can be solved, must be mean-zero. Now, Kwapień's theorem says that the reverse statement is also true, that is, for every mean zero function $f \in L_\infty([0,1],\lambda)$ the coboundary equation can be solved. This theorem we are going to investigate.

4. Coboundary for continuous, mean zero functions

In [8], Kwapień proved that for continuous mean-zero functions $f \in L_\infty([0,1], \mathcal{A}, \lambda)$, where $\mathcal{A}$ denotes the Lebesgue measurable sets, we have that $f = g \circ T - g$ for some $g \in L_\infty([0,1], \mathcal{A}, \lambda)$ and some measure preserving T of $[0,1]$. He also noted that his proof works evenly well if we replaced the measure space by the Cantor space $\{0,1\}^{\mathbb{N}}$ with the Cantor measure μ, which we will discuss in Example 4.2(2). In this section we will prove Kwapień's theorem for continuous functions in a slightly more general setting such that the case for the measure space $([0,1], \mathcal{A}, \lambda)$ and $(\{0,1\}^{\mathbb{N}}, \mu)$ follow directly from the theorem. Our proof follows the lines of the proof of Kwapień, though we adapted the proof to work for our case and we have worked out the proof in more detail in order to make it better readable and easier verifiable.

We give a small overview of changes with respect to [8]. Lemma 4.3, Lemma 4.4 and Lemma 4.5 are very similar as what Kwapień observed, though we worked out the proofs and adapted them to our case. This made that in Lemma 4.3 we can just obtain convergence in L_∞. Further in Lemma 4.4 we proved an extra bound. In Lemma 4.6 we worked out the observations of the author. Further, Proposition 4.8 already fitted our case, hence we just stated the proposition with the proof and a few extra details such that the proof gets easier to read. Further we included Lemma 4.9 with a proof, which the author left out. Finally, the proof of Lemma 4.6 we worked out in more detail and we obtained the better bound $||g_k||_\infty \leq 4||h_k||_\infty$ instead of $||g_k||_\infty \leq 6||h_k||_\infty$.

Theorem 4.1. *Let $(\Omega, \mathcal{B}(\Omega), \mu)$ be a measure space and let $(m_i)_{i\geq 1}$ be a sequence of natural numbers. For $n \in \mathbb{N}$ we set $\mathcal{E}_n = \bigotimes_{i=1}^n \{1, 2, ..., m_i\}$. Now suppose that for $a \in \mathcal{E}_n$ there exists a measurable subset $\Omega_a \subseteq \Omega$ and for $a, b \in \mathcal{E}_n$ there exists a function $\phi_a^b : \Omega_a \to \Omega_b$ such that if we set $\Omega' = \bigcap_{n=1}^\infty \bigcup_{a\in\mathcal{E}_n} \Omega_a$ the following properties are fulfilled.*

1. *Ω is a compact metric space.*
2. *Ω is of positive finite measure.*
3. *μ is a non-atomic measure.*
4. *For fixed $n \geq 1$ and $a, b \in \mathcal{E}_n$ with $a \neq b$ we have that Ω_a, Ω_b are disjoint.*
5. *For fixed $n \geq 1$ and $a, b \in \mathcal{E}_n$ the given map ϕ_a^b is an isomorphism.*
6. *For fixed $n \geq 1$ and $a, b \in \mathcal{E}_n$ the given map ϕ_a^b is a homeomorphism.*
7. *For $n \geq 1$, if $a \in \mathcal{E}_{n+1}$ and $b \in \mathcal{E}_n$ are such that $a_i = b_i$ for $1 \leq i \leq n$, then $\Omega_a \subseteq \Omega_b$.*
8. *For every chain $\Omega_{a_1} \supset \Omega_{a_2} \supset ...$ with $a_j \in \mathcal{E}_j$ we have $diam(\Omega_{a_j}) \to 0$ as $j \to \infty$*
9. *For $x \in \Omega$ there are at most countably many chains $\Omega_{a_1} \supset \Omega_{a_2} \supset ....$ such that $x \in \bigcap_{i=1}^\infty \overline{\Omega_{a_i}}$*
10. *$\Omega \setminus \Omega'$ is countable.*
11. *For $a \in \mathcal{E}_n$ the set $\overline{\Omega_a} \setminus \Omega_a$ is countable*

Then for every continuous mean zero function $f \in L_\infty(\Omega)$ and $\epsilon_{Theorem} > 0$, we can find $g \in L_\infty(\Omega)$ and measure preserving T of Ω such that $f = g \circ T - g$, and such that moreover $||g||_\infty < 4||f||_\infty + \epsilon_{Theorem}$.

Intuitively, what we demand in property (4),(5),(6),(7) and (10) is that Ω can, up to a countable set, be partitioned into disjoint sets Ω_a for $a \in \mathcal{E}_1$ that are isomorphic as measure spaces and are also homeomorphic. Furthermore, for $n \geq 1$ and $a \in \mathcal{E}_n$ the same we demand for each Ω_a, that is, Ω_a can, up to a countable set, be partitioned in disjoint sets $\Omega_b \subseteq \Omega_a$ with $b \in \mathcal{E}_{n+1}$ such that all Ω_b are isomorphic as measure spaces and are also homeomorphic. This is visualized in Figure 4. Furthermore, what property (8) says is that the diameter of sets Ω_a goes to 0 as we get further in a chain. Further, property (10) says that there are at most countably many point $x \in \Omega$ such that for some $N \geq 1$ we have $x \notin \Omega_a$ for all $a \in \mathcal{E}_N$. Hence, by property (3) these point can be neglected.

Before we continue with the proof of the theorem, we give some examples of measure spaces that satisfy the conditions.

Example 4.2. *Some examples of the measure spaces that satisfy the conditions of Theorem 4.1.*

1. *Any finite interval $[a, b]$ with the Lebesgue measure satisfies the theorem. Namely take $m_1 = 1$ and further $m_i = 2$. Then, we can*

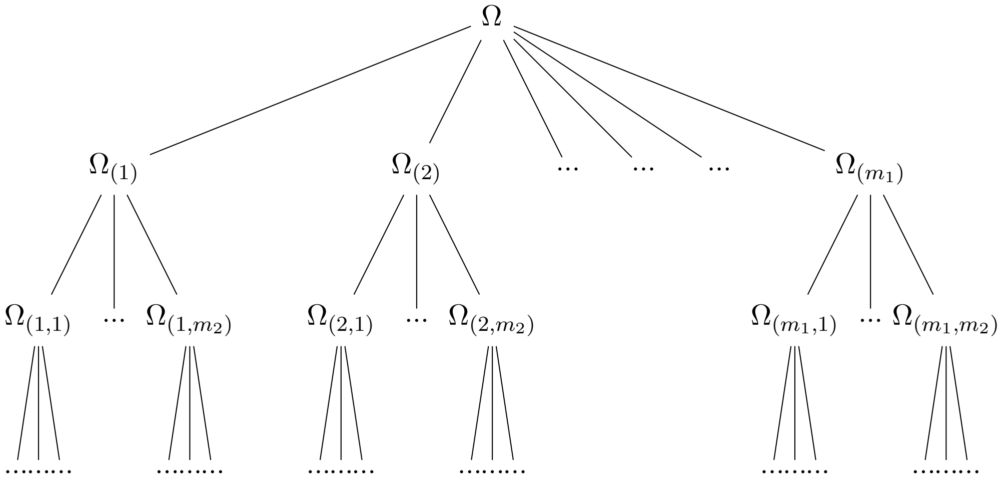

FIGURE 1. Visualisation of the subdivision of Ω in measure spaces. The lines mean that the lower set is included in the upper set. The set $\Omega_{(a_1,...,a_n)}$ is, up to a countable set, the union of the subsets $\Omega_{(a_1,...,a_n,i)}$ with $1 \leq i \leq m_{n+1}$. Further, sets in the same layer in the tree are disjoint, are isomorphic as measure spaces and are homeomorphic.

set $\Omega_{(1)} = [a, b)$ *and for* $n \geq 1$ *and* $a \in \mathcal{E}_{n+1}$ *set* $\Omega_{(a_1,a_2,...,a_n,1)} = [\inf \Omega_a, \frac{\inf \Omega_a + \sup \Omega_a}{2})$ *and* $\Omega_{(a_1,a_2,...,a_n,2)} = [\frac{\inf \Omega_a + \sup \Omega_a}{2}, \sup \Omega_a)$.

(2) *We equip a discrete space* $\{1, 2, ..., n\}$ *with the measure* $\mu_n(A) = \frac{|A|}{n}$. *Now for a sequence of natural number* $(n_i)_{i\geq 1}$ *with* $n_i \geq 2$ *consider the set* $\Omega := \bigotimes_{i=1}^{\infty}\{1, ..., n_i\}$ *with the product measure* $\mu := \bigotimes_{i=1}^{\infty} \mu_{n_i}$ *and the product topology. Then* Ω *satisfies the conditions of the the theorem with the sequence* $(m_i)_{i\geq 1}$ *given by* $m_i = n_i$. *Namely, we can define a metric* d *on* Ω *as* $d(a, b) = \sum_{i=1}^{\infty} \frac{|a_i - b_i|}{1+|a_i - b_i|} \cdot 2^{-i}$ *which induces the topology of* Ω, *hence we can consider* Ω *as a metric space. Further, as a product of compact spaces,* Ω *is also a compact space by Tychonoff's theorem, Theorem 2.14. Hence property (1) is satisfied. Further, property (2) and (3) are also satisfied. Now, if for* $a \in \mathcal{E}_n$ *we take* $\Omega_a = \{x \in \Omega : x_1 = a_1, ..., x_n = a_n\}$, *then property (4) is clearly satisfied. Furthermore for* $a, b \in \mathcal{E}_n$ *the isomorphisms* $\phi_a^b : \Omega_a \to \Omega_b$ *can be taken as* $\phi_a^b(c) = (b_1, b_2, ..., b_n, c_{n+1}, c_{n+2}, ...)$ *which is also a homeomorphism. This shows property (5) and (6). Further we have that the diameter of chains goes to* 0, *which is property (8). Further, property (7) and (10) are also satisfied, and further, property (9) and (11) are satisfied since all sets* Ω_a *are closed. Thus, indeed Theorem 4.1 holds for the measure space* $(\Omega, \mathcal{B}(\Omega), \mu)$. *Note that* Ω *is a Cantor space and that when* $n_i = 2$ *for all* $i \geq 1$, *we have that* $\Omega = \{1, 2\}^{\mathbb{N}}$ *is the standard Cantor set. The measure* μ *we will call the Cantor measure.*

4.1. **Proof of Theorem 4.1.** We now turn our attention to the proof of the theorem. Let (Ω, Σ, μ), the sequence (m_i), the sets Ω_a and the maps

ϕ_a^b be such that the given properties are fulfilled. For convenience we set $\mathcal{E}_0 = \{\varepsilon\}$ where ε denotes the empty string, and we set $\Omega_\varepsilon = \Omega$. We first note some direct consequences of the given properties. First of all, since for chains $\Omega_{a_1} \supset \Omega_{a_2} \supset ...$ with $a_j \in \mathcal{E}_j$ we have $diam(\Omega_{a_j}) \to 0$, we have that $\bigcap_{i=1}^\infty \Omega_{a_j}$ contains at most one element. Since furthermore different sets Ω_a, Ω_b with $a, b \in \mathcal{E}_j$ are disjoint, and since, using property (7), for every set Ω_a with $a \in \mathcal{E}_{j+1}$ there is a $b \in \mathcal{E}_j$ such that $\Omega_a \subseteq \Omega_b$ we have that every element in Ω' corresponds to a unique chain $\Omega \supset \Omega_{a_1} \supset \Omega_{a_2} \supset ..$ of measure spaces. Further we note that by property (7) we have $\bigcup_{a\in\mathcal{E}_n} \Omega_a \subseteq \bigcup_{a\in\mathcal{E}_m} \Omega_a$ for $n \geq m$ and that by property (10) and by definition of Ω' we have $\Omega \setminus \bigcup_{a\in\mathcal{E}_n} \Omega_a \subseteq \Omega \setminus \Omega'$ is countable. Therefore by property (3) we have $\mu(\Omega \setminus \bigcup_{a\in\mathcal{E}_n} \Omega_a) = 0$ for all $n \in \mathbb{N}$ and furthermore by property (4) for every $n \geq 1$ we can partition Ω, up to a countable set, in the subsets Ω_a with $a \in \mathcal{E}_n$. This extends also to sets Ω_b with $b \in \mathcal{E}_k$, that is, for $n \geq k$ we can partition Ω_b, up to a countable set, into its subsets $\Omega_a \subseteq \Omega_b$ with $a \in \mathcal{E}_n$. Namely, using property (7) and the 'partition' for Ω we get $\Omega_b = \Omega_b \cap \Omega = \Omega_b \cap \bigcup_{a\in\mathcal{E}_n} \Omega_a = \bigcup_{a\in\mathcal{E}_n} \Omega_a \cap \Omega_b = \bigcup_{a\in\mathcal{E}_n:\Omega_a\subseteq\Omega_b} \Omega_a$ up to a countable set. Another consequence of the properties is that, for fixed $n \geq 1$ and $a, b \in \mathcal{E}_n$ we have $\mu(\Omega_a) = \mu(\Omega_b)$ which follows from the fact that Ω_a and Ω_b are isomorphic, property (5). Now, using property (4), for any $c \in \mathcal{E}_n$ we have $\mu(\Omega) = \mu(\Omega \setminus \bigcup_{a\in\mathcal{E}_n} \Omega_a) + \sum_{a\in\mathcal{E}_n} \mu(\Omega_a) = |\mathcal{E}_n|\mu(\Omega_c)$. Thus $\mu(\Omega_c) = \frac{1}{|\mathcal{E}_n|}\mu(\Omega)$ for all $c \in \mathcal{E}_n$ and this measure is positive by property (2).

Now let $f \in L_\infty((\Omega, \Sigma, \mu))$ be continuous and mean zero. For $n \in \mathbb{N}_0$ let f_n be the conditional expectation $f_n = \mathbb{E}(f|\sigma(\{\Omega_a : a \in \mathcal{E}_n\}))$, that is, f_n denotes the function which on Ω_a, for $a \in \mathcal{E}_n$, is constant and equal to the average value of f on this set e.g. $\frac{1}{\mu(\Omega_a)} \int_{\Omega_a} f d\lambda$. We have the following Lemmas:

Lemma 4.3. *The sequence (f_n) converges to f in $L_\infty(\Omega)$.*

Proof. Since f is continuous on the compact space Ω it is uniform continuous. Now choose $\epsilon > 0$. Then there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in \Omega$ with $|x - y| < \delta$. Choose $N \in \mathbb{N}$ such that $diam(\Omega_a) < \delta$ for all $a \in \mathcal{E}_N$. This can be done, since otherwise we can find a chain $\Omega_{a_1} \supset \Omega_{a_2} \supset ....$ with $diam(\Omega_{a_j}) \geq \delta$ for all $j \geq 1$, which contradicts property (8). Now for $n \geq N$ choose $b \in \mathcal{E}_n$ then for $x, y \in \Omega_b$ we have $|x - y| < \delta$ and hence $|f(x) - f(y)| < \epsilon$. Hence for $y \in \Omega_b$

$$\begin{aligned} |f_n(y) - f(y)| &= \left| \frac{1}{\mu(\Omega_a)} \int_{\Omega_a} f(x) d\mu(x) - f(y) \right| \\ &= \left| \frac{1}{\mu(\Omega_a)} \int_{\Omega_a} f(x) - f(y) d\mu(x) \right| \\ &= \frac{1}{\mu(\Omega_a)} \int_{\Omega_a} |f(x) - f(y)| d\mu(x) \\ &< \epsilon \end{aligned}$$

Now, for $n \geq N$ this holds for all $y \in \bigcup_{a\in\mathcal{E}_n} \Omega_a$ and thus for all $y \in \Omega'$. Now, since $\lambda(\Omega \setminus \Omega') = 0$ by property (3),(10) and since N did not depend on y, this means that $||f_n - f||_\infty \to 0$ as $n \to \infty$, hence $f_n \to f$ in L_∞. ☐

Lemma 4.4. *For $\epsilon > 0$, we can find a subsequence (n_k) of the non negative integers such that $n_0 = 0$ and $\sum_{k=1}^{\infty} ||f_{n_k} - f_{n_{k-1}}||_\infty < ||f||_\infty + \epsilon$*

Proof. If $f = 0$ then $f_n = 0$ for all $n \in \mathbb{N}_0$ so that any sequence (n_k) suffices. Thus, suppose $f \neq 0$, we will choose a specific sequence (n_k) that satisfies the condition. Choose $\hat{\epsilon} > 0$ such that $\frac{2\hat{\epsilon}}{1-\hat{\epsilon}} < \epsilon||f||_\infty$

Choose $n_0 = 0$ and for $k \geq 1$ choose n_k such that we have $||f_{n_k} - f||_\infty \leq \hat{\epsilon}^k||f||_\infty$ (This can be done since $||f_n - f||_\infty \to 0$ by Lemma 4.3). Note also that, since we assumed f to be mean zero, we have $||f_{n_0} - f||_\infty = ||f||_\infty \leq \hat{\epsilon}^0||f||_\infty$ Hence

$$\begin{aligned}\sum_{k=1}^{\infty} ||f_{n_k} - f_{n_{k-1}}||_\infty &\leq \sum_{k=1}^{\infty} ||f_{n_k} - f||_\infty + ||f - f_{n_{k-1}}||_\infty \\ &\leq ||f||_\infty \sum_{k=1}^{\infty} \hat{\epsilon}^k + \hat{\epsilon}^{k-1} \\ &= ||f||_\infty (\frac{\hat{\epsilon}}{1-\hat{\epsilon}} + \frac{1}{1-\hat{\epsilon}}) \\ &= ||f||_\infty (1 + \frac{2\hat{\epsilon}}{1-\hat{\epsilon}}) < ||f||_\infty + \epsilon\end{aligned}$$

□

Let (n_k) be a sequence as in Lemma 4.4 such that $\sum_{k=1}^{\infty} ||f_{n_k} - f_{n_{k-1}}||_\infty < ||f||_\infty + \frac{1}{4}\epsilon_{Theorem}$ and for $k \in \mathbb{N}$ let us put

$$h_k = f_{n_k} - f_{n_{k-1}}$$

and for $k \in \mathbb{N}_0$ let

$$J_k := \{\Omega_a : a \in \mathcal{E}_{n_k}\}$$

We will prove some properties of the functions h_k in the following Lemma.

Lemma 4.5. *We have the following properties*

(1) $\sum_{k=1}^{\infty} ||h_k||_\infty < \infty$.
(2) $\sum_{k=1}^{\infty} h_k = f$ *in* L_∞
(3) $\int_I h_{k+1} d\lambda = 0$ *for each* $I \in J_k$, *for* $k = 1, 2, ...$

Proof. The first claim follows directly from the choice of the sequence (n_k) (see Lemma 4.4). For the second claim we note that $\sum_{k=1}^{N} h_k = f_{n_N} - f_{n_0} = f_{n_N}$ since f is mean zero. Now from Lemma 4.3 we obtain $\sum_{k=1}^{N} h_k \to f$ in L_∞. For the third claim, let $I \in J_k$, and let $I_1, ..., I_{\frac{|J_{k+1}|}{|J_k|}} \in J_{k+1}$ be the subsets of I in J_{k+1}. As we noted before, each set Ω_b with $b \in \mathcal{E}_j$ can, up to a countable set, be partitioned in its subsets $\Omega_a \subset \Omega_b$ with $a \in \mathcal{E}_n$ for some fixed $n \geq j$. Hence, since countable sets have zero measure by property (3) we have

$$\begin{aligned}\int_I h_{k+1} d\lambda &= \int_I f_{n_{k+1}} - f_{n_k} d\lambda \\ &= \int_I f_{n_{k+1}} d\lambda - \int_I f d\lambda \\ &= \left(\sum_{j=1}^{\frac{|J_{k+1}|}{|J_k|}} \int_{I_j} f_{n_{k+1}} d\lambda \right) - \int_I f d\lambda \\ &= \left(\sum_{j=1}^{\frac{|J_{k+1}|}{|J_k|}} \int_{I_j} f d\lambda \right) - \int_I f d\lambda \\ &= \int_I f d\lambda - \int_I f d\lambda = 0\end{aligned}$$

which proves the claim. □

We now state a Lemma that we will use to construct the function g and the measure preserving T as in Theorem 4.1.

Lemma 4.6. *Given functions $(h_k)_{k=1}^{\infty}$ with the properties from Lemma 4.5, we can construct a sequence $(T_k)_{k=1}^{\infty}$ of measure preserving transformations of Ω and a sequence of functions $(g_k)_{k=1}^{\infty}$ in $L_\infty(\Omega)$ such that for $k \geq 1$ the following are fullfilled.*

(1) *T_k is a cyclic permutation of J_k.*
(2) *T_k is the identiy on $\Omega \setminus \bigcup_{a \in \mathcal{E}_{n_k}} \Omega_a$.*
(3) *T_{k+1} is an extension of T_k in the sense that if $I \in J_k$, $I' \in J_{k+1}$ and $I' \subseteq I$ then $T_{k+1}(I') \subseteq T_k(I)$*
(4) *$||g_k||_\infty \leq 4||h_k||_\infty$*
(5) *g_k is contant on each set $I \in J_k$*
(6) *$h_k = g_k \circ T_k - g_k$*

If we look at this Lemma in perspective to Figure 4, we see that property (1) says that the transformation T_k is a cyclic permutation of the sets from the n_k-th layer of the tree. Further, property (3) says that this is done in a way such that the relation of being an ancestor is preserved, that is, if Ω_a, Ω_b with $a \in \mathcal{E}_{n_k}$ and $b \in \mathcal{E}_{n_{k+1}}$ are such that Ω_a is an ancestor of Ω_b then also $T_k(\Omega_a)$ is an ancestor of $T_{k+1}(\Omega_b)$.

Before we will prove this lemma, we show how this Lemma allows us to find the requested g and T as in Theorem 4.1.

Lemma 4.7. *Having a sequence (T_k) of measure preserving transformations of Ω and a sequence (g_k) in $L_\infty(\Omega)$ that fulfil the properties from Lemma 4.6, we can set T as the pointwise limit $T := \lim_{k\to\infty} T_k$ and take $g = \sum_{k=1}^{\infty} g_k$ where we consider the convergence in L_∞. Then the following hold.*

(1) *The pointwise limit $T := \lim_{k\to\infty} T_k$ exists everywhere.*

(2) *There is a set* $D \subseteq \Omega$ *whose complement is countable, such that* $T|_D$ *is injective.*
(3) *The complement in* Ω *of the image* $T(D)$ *is countable.*
(4) T *is a bijection, after modification on a zero-measure set.*
(5) T *is a measure preserving transformation.*
(6) $g \in L_\infty(\Omega)$ *with* $||g||_\infty \leq 4||f||_\infty + \epsilon_{Theorem}$.
(7) $f = g \circ T - g$

hence this proves Theorem 4.1

Proof. (1) We first prove that the limit $\lim_k T_k$ exists everywhere. Choose $\epsilon > 0$ and choose N such that $diam(I) < \epsilon$ for all $I \in J_N$ which can be done as we saw in the proof of Lemma 4.3. Now choose $x \in \Omega'$ and for $k \in \mathbb{N}$ let $I_k^x \in J_k$ be the set with $x \in I_k^x$. We obtain a chain $I_1^x \supset I_2^x \supset I_3^x$..... Now, by property (2) of Lemma 4.6 we have $T_1(I_1^x) \supset T_2(I_2^x) \supset T_3(I_3^x)$..... Thus, choose $n, m > N$, then we have $T_n(x) \in T_n(I_n^x)$ and $T_m(x) \in T_m(I_m^x)$ so that $T_n(x), T_m(x) \in T_N(I_N^x)$. Now, since T_k permutes the sets of J_k, we have $diam(T_N(I_N^x)) < \epsilon$. Hence, we have $|T_n(x) - T_m(x)| < \epsilon$. Hence $(T_k(x))_{k\geq 1}$ is a Cauchy sequence in Ω, which is a complete metric space. Therefore the limit $T(x) := \lim_{k\to\infty} T_k(x)$ exists for all $x \in \Omega'$.
Now, for $x \notin \Omega'$ we can find a N such that $x \notin \bigcup_{I\in J_N} I$ and hence by property (7) of Theorem 4.1 we have $x \notin \bigcup_{I\in J_k} I$ for $k \geq N$. Hence for $k \geq N$ we have $T_k(x) = x$ so that $T(x) = \lim_{k\to\infty} T_k(x) = x$. Thus the limit exists everywhere.

(2) We let $\mathcal{H} = \bigcup_{n=1}^\infty \bigcup_{a\in\mathcal{E}_n} \overline{\Omega_a} \setminus \Omega_a$, which is countable by property (11) of Theorem 4.1, and we set $D = \Omega' \cap T^{-1}(\Omega \setminus \mathcal{H})$. We will show that the complement of D is countable and that T restricted to D is injective. Starting with injectivity, let $x, y \in D$ with $x \neq y$. Let I_k^x and I_k^y be the sets in J_k that contains x respectively y, these exists since $x, y \in \Omega'$. Now, since $x \neq y$ and the diameter of chains goes to 0, there is an N such that I_N^x and I_N^y are different, hence disjoint. Now $T(x) \in \overline{T_N(I_N^x)} \subseteq T_N(I_N^x) \cup \mathcal{H}$ and likewise $T(y) \in \overline{T_N(I_N^y)} \subseteq T_N(I_N^y) \cup \mathcal{H}$. Now since by assumption $T(x), T(y) \notin \mathcal{H}$ we must have $T(x) \in T_N(I_N^x)$ and $T(y) \in T_N(I_N^y)$. Hence, since these sets are disjoint we have $T(x) \neq T(y)$. Hence T restricted to D is injective.

We now show that the complement of D is countable, that is $(\Omega \setminus \Omega') \cup T^{-1}(\mathcal{H})$ is countable. First of all $\Omega \setminus \Omega'$ is countable by assumption, so we only have to show that $T^{-1}(\mathcal{H})$ is countable. Let $y \in \mathcal{H}$ and suppose $T(x) = y$ for some $x \in \Omega$. We can assume that $x \in \Omega'$ since $\Omega \setminus \Omega'$ is countable. Now we can for $k \geq 1$ let I_k^x be the set in J_k that contains x. We then have $T_1(I_1^x) \supset T_2(I_2^x) \supset$ .... by property (3) of Lemma 4.6 so that we must have $y = T(x) = \lim_{k\to\infty} T_k(x) \in \bigcap_{k=1}^\infty \overline{T_k(I_k^x)}$. Now, since by assumption (9) of Theorem 4.1 there are only countably many chains that fulfil this property and since different elements $x_1, x_2 \in \Omega$ correspond to different chains, we have that $T^{-1}(\{y\})$ is countable. Therefore, since $\mathcal{H}$ is also countable by assumption (11) of Theorem 4.1 , we have that $T^{-1}(\mathcal{H})$ is countable.

(3) We will show that the complement of the image $T(D)$ is countable. Let $y \in \Omega'$ so that for $k \geq 1$ there exists a unique set $I_k^y \in J_k$ that contains y. We thus have the chain $I_1^y \supset I_2^y \supset ....$ Now it follows that $T_1^{-1}(I_1^y) \supset T_2^{-1}(I_2^y) \supset ....$ Namely if we let $A_k^y \in J_k$ be the set that contains the set $T_{k+1}^{-1}(I_{k+1}^y)$ then by property (3) from Lemma 4.6 we get $I_{k+1}^y = T_{k+1}(T_{k+1}^{-1}(I_{k+1}^y)) \subseteq T_k(A_k^y)$. Hence $y \in T_k(A_k^y)$ so that $T_k(A_k^y) = I_k^y$ and thus $T_{k+1}^{-1}(I_{k+1}^y) \subseteq A_k^y = T_k^{-1}(I_k^y)$. Thus, indeed we get the inclusions $T_1^{-1}(I_1^y) \supset T_2^{-1}(I_2^y) \supset ...$ We will now show that $T_k^{-1}(y)$ converges as $k \to \infty$. Namely for N and $l, m > N$ we have $T_l^{-1}(y), T_m^{-1}(y) \in T_N^{-1}(I_N^y)$. Now since $|T_l^{-1}(y) - T_m^{-1}(y)| \leq diam(T_N^{-1}(I_N^y))$ gets very small for large N, the sequence $(T_k^{-1}(y))$ is Cauchy and thus converges to some $x \in \Omega$. We have $x \in \overline{T_k^{-1}(I_k^y)}$ for $k \in \mathbb{N}$ and hence $x \in \bigcap_{k=1}^{\infty} \overline{T_k^{-1}(I_k^y)} \subseteq \bigcap_{k=1}^{\infty} T_k^{-1}(I_k^y) \cup \mathcal{H}$.

Now, suppose $x \notin \mathcal{H}$ and $x \in D$ then $x \in \bigcap_{k=1}^{\infty} T_k^{-1}(I_k^y)$, hence $T_k(x) \in I_k^y$ for $k \geq 1$. Further, we get $T_k(x) \in I_k^y \subseteq I_{k-1}^y \subseteq ...$ so that $T_k(x) \in \bigcap_{i=1}^{k} I_i^y$. Hence $T(x) = \lim_{k \to \infty} T_k(x) \in \bigcap_{i=1}^{\infty} \overline{I_i^y}$. Now since furthermore $\overline{I_i^y} \subseteq I_i^y \cup \mathcal{H}$ and since $T(x) \notin \mathcal{H}$ since $x \in D$, we have $T(x) \in \bigcap_{i=1}^{\infty} I_i^y = \{y\}$ so that we get $T(x) = y$ and hence $y \in T(D)$.

We now show that $x \in \mathcal{H}$ or $x \notin D$ for only countably many $y \in \Omega'$. Namely, let $y \in \Omega'$ and suppose that $T_k^{-1}(y)$ converges to some $x \in \mathcal{H}$. Then we have $x \in \bigcap_{k=1}^{\infty} \overline{T_k^{-1}(I_k^y)}$. By assumption (9) of Theorem 4.1, for fixed x there are only countably many chains with this property, hence, since $\mathcal{H}$ as well as D^c is countable, there are only countably many $y \in \Omega'$ such that $T_k^{-1}(y)$ converges to some $x \in \mathcal{H} \cup D^c$ as $k \to \infty$. Therefore, together with what we just showed, all but possibly countably many elements $y \in \Omega'$ have an inverse in D. Now, since furthermore $\Omega \setminus \Omega'$ is also countable, we have that the complement of $T(D)$ is countable.

(4) We have that $T|_D$ is a bijection between D and $T(D)$ and that $\Omega \setminus D$ and $\Omega \setminus T(D)$ are both countable. Now choose a countable infinite set $S \subset D$, so that $T|_{D \setminus S}$ is a bijection between $D \setminus S$ and $T(D \setminus S)$. Then since $\Omega \setminus (D \setminus S)$ and $\Omega \setminus T(D \setminus S)$ are both countably infinite, we can find a bijection τ between them. We can then define the modified transformation T' as $T'(x) = T(x)$ for $x \in D \setminus S$ and further, for $x \in \Omega \setminus (D \setminus S)$ we define T' as the chosen bijection $T'(x) = \tau(x)$. In this way T becomes a bijection after modification on a countable set. Further, since μ is non-atomic, we changed T only on a zero measure set.

(5) We now prove that T is measure preserving. Choose $\epsilon > 0$ and let $A \in J_k$. Then since $T(A) \setminus T_k(A) \subseteq \overline{T_k(A)} \setminus T_k(A)$ is of zero measure we have $\mu(T(A)) \leq \mu(T_k(A)) = \mu(A)$. Now, since this also holds for the complement $\Omega \setminus A$ we have that $\mu(\Omega) = \mu(T(\Omega)) = \mu(T(A)) + \mu(T(A^c)) \leq \mu(A) + \mu(A^c) = \mu(\Omega)$ so that we must have equality, that is, $\mu(T(A)) = \mu(A)$. Now, since $\overline{A} \setminus A$ is countable and μ non-atomic, we also have $\mu(T(\overline{A})) = \mu(\overline{A})$ and therefore also $\mu(T(A^\circ)) = \mu(A^\circ)$.

Now since $\mathcal{I} = \bigcup_{k=1}^{\infty}\{\Omega' \cap \bigcup_{A\in\mathcal{U}} A^\circ : \mathcal{U} \subset J_k\}$ is a π-system, and since $\mu(T(I)) = \mu(I)$ for all $I \in \mathcal{I}$ and $\mu(T(\Omega')) = \mu(\Omega')$ is finite, and since $\sigma(\mathcal{I}) = \mathcal{B}(\Omega')$ we have by Theorem 2.6, that $\lambda \circ T = \lambda$ for all Borel measurable subsets of Ω'. Now, since $\Omega \setminus \Omega'$ is countable, this also follows for Borel measurable subsets of Ω.

(6) We show that g is well defined, and that $g \in L_\infty(\Omega)$. Because g_k has norm $||g_k||_\infty \leq 4||h_k||_\infty$ by construction as in Lemma 4.6, and since $\sum_{k=1}^{\infty} ||h_k||_\infty < ||f||_\infty + \frac{1}{4}\epsilon_{Theorem}$ by our choice of the sequence (n_k), we get that $\sum_{k=1}^{\infty} ||g_k||_\infty \leq 4\sum_{k=1}^{\infty} ||h_k||_\infty < \infty$, hence the sum $\sum_{k=1}^{\infty} g_k$ converges almost everywhere, hence g is well-defined. Furthermore $||g||_\infty \leq \sum_{k=1}^{\infty} ||g_k||_\infty < 4||f||_\infty + \epsilon_{Theorem}$ and $g \in L_\infty(\Omega)$.

(7) We prove the last claim. Let $x \in \Omega'$. Let $I_k^x \in J_k$ be the set that contains x. Then $T(x) \in T(I_k^x) \subseteq \overline{T_k(I_k^x)}$. Now, only countably many elements are sent to $\overline{T_k(I_k^x)} \setminus T_k(I_k^x) \subseteq \mathcal{H}$, so we can assume $T(x) \in T_k(I_k^x)$ Then since g_k is constant on the intervals of J_k, we have $g_k \circ T - g_k = g_k \circ T_k - g_k = h_k$. Now we have $f = \sum_{k=1}^{\infty} h_k = \sum_{k=1}^{\infty} g_k \circ T - g_k = g \circ T - g$. This finishes the proof.

□

To construct the sequences (T_k) and (g_k), the following proposition and lemma are needed.

Proposition 4.8. *Let $(a_{i,j})_{n\times m}$ be a matrix of real numbers such that $|a_{i,j}| \leq C$ for $i = 1, ..., n$ and $j = 1, ..., m$ and such that $\sum_{j=1}^{m} a_{i,j} = 0$ for $i = 1, ..., n$ then there are permutations $\sigma_1, ..., \sigma_n$ of the integers $\{1, ..., m\}$ such that*

$$\left|\sum_{i=1}^{k} a_{i,\sigma_i(j)}\right| \leq 2C$$

for $k = 1, 2, ..., n$ and $j = 1, 2..., m$

Proof. If $b = (\beta_1, ..., \beta_m)$ is a m-dimensional vector. Let us put $||b|| = \max_{1\leq i,j\leq m} |\beta_i - \beta_j|$, however, note that this is not a norm on $\mathbb{R}^m$. Let $b \circ \sigma$, where σ is a permutation of the set $\{1, 2, ...m\}$, denote the vector with the coordinates given by $b\circ\sigma_i = \beta_{\sigma(i)}$. Since in $||b\circ\sigma|| = \max_{1\leq i,j\leq m} |\beta_{\sigma(i)} - \beta_{\sigma(j)}|$ the maximum is taken over all pairs (i, j), the permutation has no influence on the value. Hence $||b \circ \sigma|| = ||b||$ for all permutations. Further, if a, b are two vectors such that the coordinates of a are non-decreasing and the coordinates of b are non-increasing, then for $i \leq j$ we have $a_i - a_j \leq 0 \leq b_i - b_j$ so that $-|a_i - a_j| = (a_i - a_j) \leq (a_i - a_j) + (b_i - b_j) \leq b_i - b_j = |b_i - b_j|$ and thus $|a_i - a_j + b_i - b_j| \leq \max\{|a_i - a_j|, |b_i - b_j\}$ We thus get

$$\begin{aligned} ||a + b|| &= \max_{1\leq i,j\leq m} |(a_i + b_i) - (a_j + b_j)| \\ &\leq \max_{1\leq i,j\leq m} \max\{|a_i - a_j|, |b_i - b_j|\} \\ &= \max\{\max_{1\leq i,j\leq m} |a_i - a_j|, \max_{1\leq i,j\leq m} |b_i - b_j|\} \quad = \max\{||a||, ||b||\} \end{aligned}$$

Now, if $a, b \in \mathbb{R}^m$ are arbitrary, we can find permutations τ, ϕ such that $a \circ \tau$ is non-decreasing and $b \circ \phi$ is non-increasing. Now putting $\sigma = \phi \circ \tau^{-1}$ gives $||a + b \circ \sigma|| = ||(a + b \circ \sigma) \circ \tau|| = ||a \circ \tau + b \circ \phi|| \leq \max\{||a||, ||b||\}$. Hence, we can always find a permutation σ such that $||a + b \circ \sigma|| \leq \max\{||a||, ||b||\}$

We can now define the permutations $\sigma_1, ..., \sigma_n$ from the proposition inductively. For simplicity let us denote $a_i = (a_{i,1}, a_{i,2}, ..., a_{i,m})$ for the i-th row of the matrix. Define $\sigma_1 = id$. Further, assume $\sigma_1, \sigma_2, ...\sigma_k$ are all chosen such that for $j \leq k$ we have

$$||a_1 \circ \sigma_1 + ... + a_j \circ \sigma_j|| \leq \max\{||a_1||, ..., ||a_j||\}$$

Now, as we just saw, there exists a permutation σ_{k+1} such that

$$||a_1 \circ \sigma_1 + ... + a_k \circ \sigma_k + a_{k+1} \circ \sigma_{k+1}|| \leq \max\{||a_1 \circ \sigma_1 + ... + a_k \circ \sigma_k||, ||a_{k+1}||\}$$

Hence, by the assumption on the permutations $\sigma_1, ..., \sigma_k$ we have

$$\begin{aligned} ||a_1 \circ \sigma_1 + ... + a_k \circ \sigma_k + a_{k+1} \circ \sigma_{k+1}|| &\leq \max\{||a_1 \circ \sigma_1 + ... + a_k \circ \sigma_k||, ||a_{k+1}||\} \\ &\leq \max\{||a_1||, ..., ||a_k||, ||a_{k+1}||\} \end{aligned}$$

Thus, inductively we can define the permutations $\sigma_1, ..., \sigma_n$ such that for $1 \leq k \leq n$ we have $||a_1 \circ \sigma_1 + ... + a_k \circ \sigma_k|| \leq \max\{||a_1||, ..., ||a_k||\}$. Further, for $l = 1, ..., n$ and $j = 1, ...m$ holds $|a_{l,j}| \leq C$ so that we have $||a_l|| = \max_{1 \leq i,j \leq m} |a_{l,i} - a_{l,j}| \leq 2 \max_{1 \leq j \leq m} |a_{l,j}| \leq 2C$

Furthermore, since the sum of the coordinates of the vectors a_i equals 0, also the sum of the coordinates of the vector $a_1 \circ \sigma_1 + ... + a_k \circ \sigma_k$ equals 0. Hence, if the value of some coordinate is greater then 0 then there is also a coordinate with value less then 0 and also the other way around. All together this gives us

$$\begin{aligned} \left| \sum_{i=1}^{k} a_{i,\sigma_i(j)} \right| &= |a_1 \circ \sigma_1(j) + ... + a_k \circ \sigma_k(j)| \\ &\leq \max_{1 \leq l \leq m} |(a_1 \circ \sigma_1(j) + ... + a_k \circ \sigma_k(j)) \\ &- (a_1 \circ \sigma_1(l) + ... + a_k \circ \sigma_k(l))| \\ &\leq ||a_1 \circ \sigma_1 + ... + a_k \circ \sigma_k|| \\ &\leq \max\{||a_1||, ..., ||a_k||\} \leq 2C \end{aligned}$$

for $k = 1, ..., n$ and $j = 1, ..., m$ which proves the proposition.

□

Lemma 4.9. *Given a vector $b = (b_1, ..., b_m) \in \mathbb{R}^m$ and a constant C so that $|b_j| \leq C$ and $b_1 + .. + b_m = 0$. Then we can find a permutation σ of $\{1, ..., m\}$ such that for $l = 1, ..., m$ we have*

$$\left| \sum_{j=1}^{l} b_{\sigma(j)} \right| \leq C$$

Proof. We will construct σ_0 inductively. Define $\sigma_0(1) = 1$. Then $|\sum_{j=1}^{1} b_{\sigma_0(j)}| = |b_1| \leq C$. Now suppose that $\sigma_0(1), ..., \sigma_0(u)$ are defined such that $\left|\sum_{j=1}^{l} b_{\sigma_0(j)}\right| \leq C$ for $l = 1, ..., u$. If $u < m$ we define $\sigma_0(u+1)$ in the following way. If $\sum_{j=1}^{u} b_{\sigma_0(j)} \geq 0$ then since $\sum_{j=1}^{m} b_{\sigma_0(j)} = 0$, there must be some $s \in$

$\{1,2,...m\} \setminus \sigma_0(\{1,..,k\})$ with $b_s \leq 0$. Hence we can define $\sigma(u+1) = s$ so that $\sum_{j=1}^{u+1} b_{\sigma_0(j)} = b_{\sigma_0(u+1)} + \sum_{j=1}^{u} b_{\sigma_0(j)} = b_s + \sum_{j=1}^{u} b_{\sigma_0(j)} \leq \sum_{j=1}^{u} b_{\sigma_0(j)} \leq C$ Similar we have $\sum_{j=1}^{u+1} b_{\sigma_0(j)} = b_{\sigma_0(u+1)} + \sum_{j=1}^{u} b_{\sigma_0(j)} \geq b_s \geq -C$ so that we get $\left|\sum_{j=1}^{u+1} b_{\sigma_0(j)}\right| \leq C$. The same can be obtained when $\sum_{j=1}^{u} b_{\sigma_0(j)} \leq 0$ by choosing $b_{\sigma_0(u+1)} \geq 0$. Now, by induction all values $\sigma_0(j)$ for $j = 1,...m$ can be defined so that the property holds. □

We will now finish this section with the proof of Lemma 4.6.

4.2. **Proof of Lemma 4.6.**

Proof. We will inductively define the transformations $T_k : \Omega \to \Omega$ and the functions g_k. We will first define T_0 as $T_0(x) = x$ so that T_0 is a cyclic permutation of $J_0 = \{\Omega\}$. Now, fix $k \geq 0$ and assume that g_k and T_k are defined and that T_k is a cyclic permutation of J_k. We will define T_{k+1} and g_{k+1}. Let $I_1, ..., I_{|J_k|}$ be the sets of J_k enumerated in such a way that the image $T_k(I_i) = I_{i+1}$ for $i = 1,2,...,|J_k|$, here $I_{|J_k|+1} = I_1$. This is possible since T_k is cyclic. Further, set $N = |J_k|, M = \frac{|J_{k+1}|}{|J_k|}$ and for $i = 1,2,...,|J_k|$ let $(I_{i,j})_{j=1}^{M}$ be an enumeration of the sets from J_{k+1} which are contained in I_i. Finally let $a_{i,j}$ denote the value of h_{k+1} on the $I_{i,j}$. Using Lemma 4.5 we have for $i = 1,..,N$ that $0 = \int_{I_i} h_{k+1} d\mu = \sum_{j=1}^{M} \int_{I_{i,j}} h_{k+1} d\mu = \sum_{j=1}^{M} a_{i,j}\mu(I_{i,j}) = \frac{\mu(\Omega)}{|J_{k+1}|} \sum_{j=1}^{M} a_{i,j}$ Hence we have $\sum_{j=1}^{M} a_{i,j} = 0$. Further for $i = 1,...,N$ and $j = 1,...,M$ we have $|a_{i,j}| \leq ||h_{k+1}||_\infty$ because, by definition, the value $a_{i,j}$ is attained by h_{k+1} on a set of positive measure. Now, this means that the conditions of Proposition 4.8 are satisfied with $C = ||h_{k+1}||_\infty$. Let $\sigma_1, ..., \sigma_N$ be the permutations as in the conclusion of Proposition 4.8. Now let us define T_{k+1} by the conditions $T_{k+1}(I_{l,\sigma_l(j)}) = I_{l+1,\sigma_{l+1}(j)}$ for $l = 1,2,...,N-1$ and $j = 1,...,M$. More strictly speaking, writing $\Omega_a := I_{l,\sigma_l(j)}$ and $\Omega_b := I_{l+1,\sigma_{l+1}(j)}$, then for $x \in \Omega_a$ we define $T_{k+1}(x) = \phi_a^b(x)$.

Now, to define T_{k+1} on $I_{N,\sigma_N(j)}$ we will use yet another permutation. Let us put $b_j = \sum_{l=1}^{N} a_{l,\sigma_l(j)}$ for $j = 1,2,...,M$. By writing out we get $\sum_{j=1}^{M} b_j = \sum_{i=1}^{N} \sum_{j=1}^{M} a_{i,j} = 0$. Further, by the conclusion of Proposition 4.8 we have $|b_j| \leq 2||h_{k+1}||_\infty$. Now, applying Lemma 4.9 we obtain a permutation σ_0 of $\{1,..,M\}$ such that for $l = 1,....,M$ we have $|\sum_{j=1}^{l} b_{\sigma_0(j)}| \leq 2||h_{k+1}||_\infty$. We can now define T_{k+1} on $I_{N,\sigma_N(\sigma_0(j))}$ in the following way. Define $T_{k+1}(I_{N,\sigma_N(\sigma_0(j))}) = I_{1,\sigma_1(\sigma_0(j+1))}$ for $j = 1,....,M-1$ and further $T_{k+1}(I_{N,\sigma_N(\sigma_0(M))}) = I_{1,\sigma_1(\sigma_0(1))}$. Again, the strict definition is taken like the definition of T_{k+1} on the other sets, which uses the maps ϕ_a^b. As last, we can define $T_{k+1}(x) = x$ for $x \notin \bigcup_{I \in J_{k+1}} I$ so that we have defined T_{k+1} on the whole of Ω.

First of all we will show that T_{k+1} is a measure preserving transformation. Every $I \in J_{k+1}$ can uniquely be represented as $I = I_{i,\sigma_i(\sigma_0(j))}$ for $i = 1,...,N$ and $j = 1,...,M$. Namely, by property (4),(7) of Theorem 4.1 there is a unique set $I_i \in J_k$ such that $I \subseteq I_i$. Hence $I = I_{i,l}$ for some $1 \leq l \leq M$. Now, since permutations are bijective, there is a unique $j \in \{1, ..., M\}$ such that $\sigma_i(\sigma_0(j)) = l$. Thus, indeed I can uniquely be represented as $I = I_{i,\sigma_i(\sigma_0(j))}$. Now this means that T_{k+1} is well-defined and values taken on different sets of J_{k+1} are different since different sets $I_{i,\sigma_i(j)}$ were send to different sets. Further the values taken on different elements in the same set of J_k are also different, since the functions ϕ_a^b are bijective. Thus, since T_{k+1} is simply the identity on $\Omega \setminus \bigcup_{I \in J_{k+1}} I$, we have that T_{k+1} is injective. Now, again since the representation $I = I_{i,\sigma_i(\sigma_0(j))}$ is unique and since T_{k+1} is the identity on $\Omega \setminus \bigcup_{I \in J_{k+1}} I$, we have that T_{k+1} is also surjective. Now for a measurable set $A \in \mathcal{B}(\Omega)$ we have that A is the union of the disjoint sets $\{A \cap I : I \in J_{k+1}\} \cup \{A \setminus \bigcup_{I \in J_{k+1}} I\}$ which are all measurable, since A and $I \in J_{k+1}$ are measurable. Now T_{k+1} on $A \cap I$ equals ϕ_a^b for some $a, b \in \mathcal{E}_{n_{k+1}}$, so that $T_{k+1}(A \cap I)$ is measurable and $\mu(T_{k+1}(A \cap I)) = \mu(A \cap I)$. Further $T_{k+1}(A \setminus \bigcup_{I \in J_{k+1}} I) \subseteq T_{k+1}(\Omega \setminus \Omega')$ is countable, hence is measurable and has zero measure. Now this means that $T_{k+1}(A)$ is measurable and, since all the images $T_{k+1}(A \cap I)$ are all disjoint, we have $\mu(T_{k+1}(A)) = \mu(T_{k+1}(A \setminus \bigcup_{I \in J_{k+1}} I)) + \sum_{I \in J_{k+1}} \mu(T_{k+1}(A \cap I)) = \sum_{I \in J_{k+1}} \mu(A \cap I) = \mu(A)$. Thus T_k is measure preserving.

(1) We will now show that T_{k+1} is a cyclic permutation of J_{k+1}. By consecutively applying T_{k+1}, we see that $T_{k+1}^l(I_{1,\sigma_1(\sigma_0(j))}) = I_{l+1,\sigma_{l+1}(\sigma_0(j))}$ for $l = 0, ..., N-1$ and $j = 1, ..., M$. Now furthermore $T_{k+1}^N(I_{i,\sigma_i(\sigma_0(j))}) = I_{i,\sigma_i(\sigma_0(j+1))}$ for $j = 1, ..., M-1$ and $i = 1, ..., N$ so that $T_{k+1}^{qN}(I_{i,\sigma_i(\sigma_0(1))}) = I_{i,\sigma_i(\sigma_0(q+1))}$ for $q = 0, .., M-1$. Now for $0 \leq l \leq NM-1$ write $l = qN + r$ with $0 \leq r \leq N-1$ and $0 \leq q \leq M-1$. We thus have

$$\begin{aligned} T_{k+1}^l(I_{1,\sigma_1(\sigma_0(1))}) &= T_{k+1}^r(T_{k+1}^{qN}(I_{1,\sigma_1(\sigma_0(1))})) \\ &= T_{k+1}^r(I_{1,\sigma_1(\sigma_0(q+1))}) \\ &= I_{r+1,\sigma_{r+1}(\sigma_0(q+1))} \end{aligned}$$

Now if $r \neq 0$ then $T_{k+1}^{qN+r}(I_{1,\sigma_1(\sigma_0(1))}) \neq I_{1,\sigma_1(\sigma_0(1))}$. Further, if $r = 0$ but $q \neq 0$ then $\sigma_1(\sigma_0(1)) \neq \sigma_1(\sigma_0(q+1))$ so that again $T_{k+1}^{qN+r}(I_{1,\sigma_1(\sigma_0(1))}) \neq I_{1,\sigma_1(\sigma_0(1))}$. Hence, the only possibility with $T_{k+1}^{qN+r}(I_{1,\sigma_1(\sigma_0(1))}) = I_{1,\sigma_1(\sigma_0(1))}$ and $0 \leq qN + r \leq NM - 1$ is the case that $r = q = 0$. Now $T_{k+1}^l(I_{1,\sigma_1\sigma_0(1)})$ must all be different for $l = 0, ..., NM-1$. Further, we see that $T_{k+1}^{NM}(I_{1,\sigma_1(\sigma_0(1))}) = T_{k+1}(I_{N,\sigma_N(\sigma_0(M))}))) = I_{1,\sigma_1(\sigma_0(1)}$. Now since $|J_{k+1}| = NM$ we must have that T_{k+1} is a cyclic permutation of J_{k+1}.

(2) By definition of T_{k+1} we have that T_{k+1} is the identity on $\Omega \setminus \bigcup_{I \in J_{k+1}} I$.

(3) Now, to see that T_{k+1} is an extension of T_k, note that by definition for any set $I_{i,j} \subseteq I_i$ with $1 \leq i < N$ and $1 \leq j \leq M$ we have $T_{k+1}(I_{i,j}) \subseteq I_{i+1} = T_k(I_i)$. Further for $I_{N,j} \subseteq I_N$ we have $T_{k+1}(I_{N,j}) \subseteq I_1 = T_k(I_N)$. So, T_{k+1} is indeed an extension of T_k.

We will now continue by defining g_{k+1}. For $l = 1, ..., NM$ we define g_{k+1} on the set $T_{k+1}^l(I_{1,1})$ by the constant value of $h_{k+1} + h_{k+1} \circ T_{k+1} + ... + h_{k+1} \circ T_{k+1}^{l-1}$ on $I_{1,1}$. Note that this is well-defined since every set $I \in J_{k+1}$

can uniquely be represented as $T_{k+1}^l(I_{1,1})$ for some $l = 1, ..., NM$ since T_{k+1} is cyclic. Further note that $\Omega \setminus \bigcup_{I \in J_k} I$ is countable, hence a null set.

(4) We will prove that $||g_{k+1}||_\infty \leq 4||h_{k+1}||_\infty$. Choose $1 \leq l \leq NM$ and write $l - 1 = qN + r$ as before. The function

$$\sum_{u=0}^{l-1} h_{k+1} \circ T_{k+1}^u = \left(\sum_{u=0}^{q-1} \sum_{v=0}^{N-1} h_{k+1} \circ T_{k+1}^{uN+v} \right) + \sum_{v=0}^{r} h_{k+1} \circ T_{k+1}^{qN+v}$$

attains on $I_{1,1}$ a value $A(l)$ with

$$\begin{aligned} |A(l)| &= \left| \left(\sum_{u=0}^{q-1} \sum_{v=0}^{N-1} a_{v+1,\sigma_{v+1}(\sigma_0(u+1))} \right) + \sum_{v=0}^{r} a_{v+1,\sigma_{v+1}(\sigma_0(q+1))} \right| \\ &\leq \left| \sum_{u=1}^{q} b_{\sigma_0(u)} \right| + \left| \sum_{v=1}^{r+1} a_{v,\sigma_v(\sigma_0(q+1))} \right| \\ &\leq 2||h_{k+1}||_\infty + 2||h_{k+1}||_\infty = 4||h_{k+1}||_\infty \end{aligned}$$

Where we used Proposition 4.8 and Lemma 4.9 for the two bounds $\left|\sum_{v=1}^{r+1} a_{v,\sigma_v(\sigma_0(q+1))}\right|$ and $|\sum_{u=0}^{q-1} b_{\sigma_0(u)}|$ respectively. Now g_{k+1} attains on $T_{k+1}^l(I_{1,1})$ precisely the value $A(l)$ so that $||g_{k+1}||_\infty \leq 4||h_{k+1}||_\infty$.

(5) By definition the value taken by g_{k+1} on a set $I \in J_{k+1}$ is constant.

(6) We show that $g_{k+1} \circ T_{k+1} - g_{k+1} = h_{k+1}$ which ends the proof. To see that $g_{k+1} \circ T_{k+1} - g_{k+1} = h_{k+1}$ choose a set $T_{k+1}^l(I_{1,1}) \in J_{k+1}$ (every set $I \in J_{k+1}$ can be written in this form since T_{k+1} cyclic). Now on $I = T_{k+1}^l(I_{1,1})$ the function $g_{k+1} \circ T_{k+1} - g_{k+1}$ attains the constant value

$$\begin{aligned} g_{k+1} \circ T_{k+1}(I) - g_{k+1}(I) &= g_{k+1} \circ T_{k+1}^{l+1}(I_{1,1}) - g_{k+1} \circ T_{k+1}^l(I_{1,1}) \\ &= \sum_{j=0}^{l} h_{k+1} \circ T_{k+1}^j(I_{1,1}) - \sum_{j=0}^{l-1} h_{k+1} \circ T_{k+1}^j(I_{1,1}) \\ &= h_{k+1} \circ T_{k+1}^l(I_{1,1}) \\ &= h_{k+1}(I) \end{aligned}$$

Now since this holds for any $I \in J_k$ we have that $g_{k+1} \circ T_{k+1} - g_{k+1} = h_{k+1}$

Now this completes the construction of the functions T_{k+1} and g_{k+1} with the stated properties. □

5. Coboundary for nowhere constant, mean zero function

In [8], Kwapień extended the result for continuous mean zero functions $f \in L_\infty([0, 1])$ to general mean zero functions $f \in L_\infty(\Omega, \Sigma, \mu)$ where (Ω, Σ, μ) is a standard measure space. For $f \in L_\infty([0, 1], \lambda)$ this procedure was described as follows:

"If f is a bounded measurable function on $[0, 1]$ then, using the Lusin Theorem, we can find a sequence $A_n, n = 1, 2, ...,$ of disjoint subsets of $[0, 1]$ such that $\lambda([0, 1] \setminus \bigcup_{n=1}^\infty A_n) = 0$ and such that each A_n fulfils the conditions

(1) A_n is a closed subset, homeomorphic to the Cantor set,
(2) f restricted to A_n is a continuous function,
(3) $\lambda(A_n) > 0$ and $\int_{A_n} f d\lambda = 0$

For each n, A_n equipped with the measure $\frac{\lambda}{\lambda(A_n)}$, there exists $1-1$ map from A_n onto the Cantor set which is a homeomorphism and which maps the measure on the Cantor measure. Therefore by the first case and the above remark fore each n there exists T_n - a measure preserving transformation of A_n and $g_n \in L_\infty(A_n, \lambda)$ such that $f|_{A_n} = g_n \circ T_n - g_n$. Hence, if we denote by T the map which for n coincides with T_n on A_n and by g the function in $L_\infty([0,1], \lambda)$ such that $g|_{A_n} = g_n$ for each n we obtain that $f = g \circ T - g$. Obviously T is a measure preserving transformation and this proves Theorem."

However, as in [4] was discovered, there is a problem in this proof. Namely, in general there does not exist a homeomorphism between a given Cantor set A_n and the standard Cantor set $\{0,1\}^{\mathbb{N}}$ that also maps the scaled Lebesgue measure $\frac{\lambda}{\lambda(A_n)}$ to the Cantor measure μ. We show this using following lemma.

Lemma 5.1. *Let $C \subseteq [0,1]$ be a set of positive measure and suppose we can write $C = C_1 \cup C_2$ with C_1, C_2 being compact, disjoint and such that $\frac{\lambda(C_1)}{\lambda(C_1 \cup C_2)}$ is irrational. Then there is no homeomorphism $\varphi : C \to \{0,1\}^{\mathbb{N}}$ that maps the scaled Lebesgue measure $\frac{\lambda}{\lambda(C)}$ to the Cantor measure μ on $\{0,1\}^{\mathbb{N}}$.*

Proof. Suppose there is an homeomorphism $\varphi : C \to \{0,1\}^{\mathbb{N}}$. We have that C_1 is compact, and also that C_1 is open in C. Therefore $\varphi(C_1)$ is also compact and open. Hence, since $\mathcal{B} := \{\Omega_a : a \in \{0,1\}^N \text{ for some } N \in \mathbb{N}\}$, where $\Omega_a = \{x \in \{0,1\}^{\mathbb{N}} : x_1 = a_1, ..., x_N = a_N\}$, is a basis for the topology for $\{0,1\}^{\mathbb{N}}$, we can write $\varphi(C_1) = \bigcup_{A \in \mathcal{A}} A$ for some $\mathcal{A} \subseteq \mathcal{B}$. Now, since $\mathcal{A}$ is an open cover for $\varphi(C_1)$, and since $\varphi(C_1)$ is compact we can even write $\varphi(C_1) = \bigcup_{i=1}^{N} A_i$ for some $N \in \mathbb{N}$ and some $A_i \in \mathcal{A}$. Now, we can also write $\varphi(C_1) = \bigcup_{i=1}^{N'} B_i$ for some $N' \in \mathbb{N}$ and with $B_i \in \mathcal{B}$ such that the sets B_i are furthermore disjoint. Hence we have $\mu(\varphi(C_1)) = \frac{j}{2^l}$ for some $j, l \in \mathbb{N}$. Now, since $\frac{\lambda(C_1)}{\lambda(C_1 \cup C_2)}$ is irrational, the homeomorphism φ does not map the scaled Lebesgue measure to the Cantor measure μ. Now, since φ was arbitrary, this holds for any homeomorphism. □

Now, we can construct such set C homeomorphic to the Cantor set $\{0,1\}^{\mathbb{N}}$, that also satisfies the conditions of Lemma 5.1. Namely we can construct two disjoint sets $C_1, C_2 \subseteq [0,1]$ both homomorphic to the Cantor set $\{0,1\}^{\mathbb{N}}$ and both of positive measure. These sets are then automatically compact. We can furthermore scale the set C_1 such that the ratio $\frac{\lambda(C_1)}{\lambda(C_1 \cup C_2)}$ is irrational. Now the set $C = C_1 \cup C_2$ is homeomorphic to $\{0,1\}^{\mathbb{N}}$ and satisfies the conditions of the Lemma. Hence, this shows that there does not always exist a homeomorphism between a given Cantor set, and the standard Cantor set, that maps $\frac{\lambda}{\lambda(C)}$ to μ.

In this section we will work out the construction of the sets A_n with given properties in full detail. The proof is quite tedious. For convenience, our homeomorphism will be from A_n to some arbitrary product space $\bigotimes_{i=1}^{\infty}\{1,2,...,m_i\}$ as we saw in Example 4.2(2). Further, we will construct A_n such that we get the addition property that the homeomorphism maps $\frac{\lambda}{\lambda(A_n)}$ to the Cantor measure μ, as this is needed for the proof. We will then obtain the function g and transformation T. However, we could only work out the construction of the sets A_n under the extra assumption that f is nowhere essentially constant, that is, $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. We thus prove the following theorem.

Theorem 5.2. *Let (Ω, Σ, μ) be a standard measure space. Let $f \in L_\infty(\Omega, \Sigma, \mu)$ be mean zero and such that $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. Choose $\epsilon_{Theorem} > 0$, then there is a $g \in L_\infty(\Omega, \Sigma, \mu)$ with $||g||_\infty < 4||f||_\infty + \epsilon_{Theorem}$ and a measure preserving transformation T of Ω such that $f = g \circ T - g$.*

As we will prove in Lemma 5.4 it is enough to consider the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$. The idea behind the proof of the theorem is to first use Lusin's theorem, theorem 2.10, to find a suitable compact subset $A \subseteq [0,1]$ so that f is continuous on A, and so that $\lambda([0,1] \setminus A)$ is small. We will then construct a compact subset $A' \subseteq A$ such that $\lambda([0,1] \setminus A')$ is still small, and so that further $\int_{A'} f d\lambda = 0$ and so that there is an isomorphism ϕ between $(A', \frac{\lambda}{\lambda(A')})$ and some product space $(\bigotimes_{i=1}^{\infty}\{1,2,...,m_i\}, \mu)$ that is also a homeomorphism. We can then apply Theorem 4.1 on $\bigotimes_{i=1}^{\infty}\{1,2,...,m_i\}$ with $f \circ \phi^{-1}$ and from this we can obtain the g and T so that $f|_{A'} = g \circ T - g$. Now, this process can be repeated on $[0,1] \setminus A'$ so that we get disjoint compact sets $A_1, A_2, ...$ and measure preserving transformations T_i of A_i and essentially bounded functions g_i on A_i such that $f|_{A_i} = g_i \circ T_i - g_i$ and such that $[0,1]$ is essentially equal to the union of all A_i. Then the functions g and T from the theorem can be constructed by defining $g|_{A_i} = g_i$ and $T|_{A_i} = T_i$, which then finishes the proof. The hard part in the proof of this theorem lies in the construction of the set A' from A. We do this by first shrinking the set A slightly in a way such that the integral of f over A equals 0 and such that $\lambda([0,1] \setminus A)$ is still small. Once we have done that, we divide A from left to right in m_1 pieces (m_i chosen in a certain way) which have an equal measure and which are disjoint except for their endpoints. We then shrink those created subsets in a way that their measures get only slightly less, that they are disjoint, that they are compact and such that the integral of f over the pieces together is still 0. We repeat this process on the created compact subsets so that we get chains of subsets, analogue to the construction of the Cantor set as subset of $[0,1]$ using closed intervals. This is visualized in Figure 5. We then obtain the compact set A' with $\int_{A'} f d\lambda = 0$ and for which we can create an isomorphism, that is also a homeomorphism, to some Cantor space $(\bigotimes_{i=1}^{\infty}\{1,2,...,m_i\}, \mu)$. Further, f is continuous on A' as f is continuous on A. In this construction, some care had to be taken to ensure that the diameter of chains goes to 0, as well as ensuring continuity of the inverse of the created homeomorphism. This all makes the proof quite tedious. We will further show that we can not do this construction by leaving out the assumption that $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. This is not to say that the theorem in not true in that case.

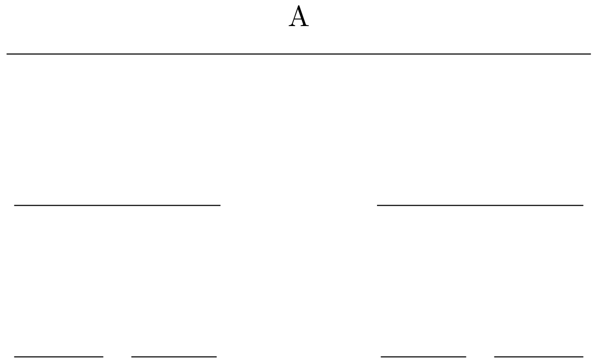

FIGURE 2. This figure gives an intuitive impression of the process to construct A' from A. We start with a set A which we from left to right in multiple pieces. Those pieces are then shrunken. This forms the second layer. This process is repeated. The final set A' is the intersection of all the layers. We note that in general the sets in the layers may be much more complicated, not just unions of intervals. This may depend on the initial set A and the function f.

Example 5.3. *Choose $\alpha \in (0,1)$ irrational and let $f \in L_\infty([0,1],\lambda)$ be given by $f = \mathbb{1}_{[0,\alpha]} - \frac{\alpha}{1-\alpha}\mathbb{1}_{(\alpha,1]}$. We then have $\int_{[0,1]} f d\lambda = 0$. We now show that there is no compact set $A \subseteq [0,1]$ of positive measure on which f is continuous, $\int_A f d\lambda = 0$ and such that there is an isomorphism between $(A, \frac{\lambda}{\lambda(A)})$ and a Cantor space $(\bigotimes_{i=1}^{\infty}\{1,...,m_i\},\mu)$ that is also a homeomorphism.*

Choose any compact subset $A \subseteq [0,1]$ on which f is continuous and for which we have $\int_A f d\lambda = 0$. Then we have $\lambda(A\cap[0,\alpha]) - \frac{\alpha}{1-\alpha}\lambda(A\cap(\alpha,1]) = 0$. Therefore $\lambda(A) = \lambda(A\cap[0,\alpha]) + \lambda(A\cap(\alpha,1]) = (1+\frac{1-\alpha}{\alpha})\lambda(A\cap[0,\alpha])$ and thus $\frac{\lambda(A\cap[0,\alpha])}{\lambda(A)} = \alpha$.

Now, since f is not continuous at α and since A is closed we have either $(\alpha-\epsilon,\alpha)\cap A = \varnothing$ or $(\alpha,\alpha+\epsilon)\cap A = \varnothing$ for some $\epsilon > 0$. In the case $(\alpha,\alpha+\epsilon)\cap A = \varnothing$ we set $B := A\cap[0,\alpha]$, so that B and its complement $A\setminus B = A\cap[\alpha+\epsilon,1]$ are both compact. Furthermore, as we saw, we have $\frac{\lambda(B)}{\lambda(A)} = \alpha$.

In the case $(\alpha-\epsilon,\alpha)\cap A = \emptyset$ we set $B := A\cap[0,\alpha) = A\cap[0,\alpha-\epsilon]$ so that again B and $A\setminus B = A\cap[\alpha,1]$ are both compact. Further, in this case we also have $\frac{\lambda(B)}{\lambda(A)} = \alpha$.

Now, in both cases B and $A\setminus B$ are compact, and furthermore $\frac{\lambda(B)}{\lambda(B\cup A\setminus B)} = \alpha$ is irrational. Hence, applying Lemma 5.1 shows that there exists no homeomorphism from A to $\{0,1\}^{\mathbb{N}}$ that maps the scaled Lebesgue measure $\frac{\lambda}{\lambda(A)}$ to the product measure μ. Further, the proof of Lemma 5.1 works evenly well for a general product space $(\bigotimes_{i=1}^{\infty}\{1,...,m_i\},\mu)$. Hence, this finishes this proof.

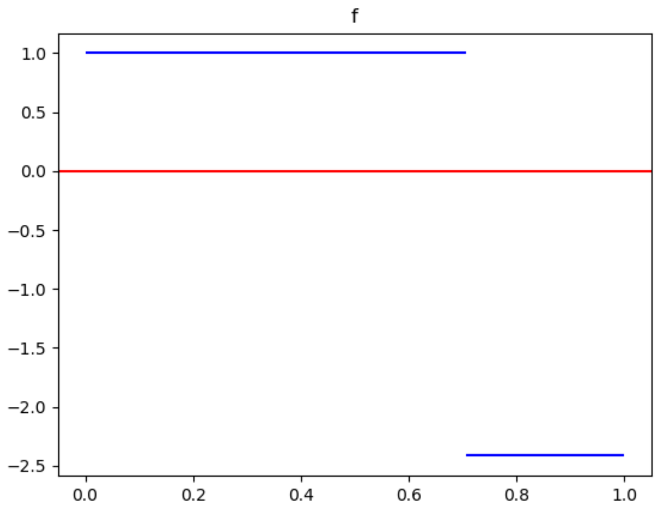

FIGURE 3. Example of the function f where $\alpha = \frac{1}{2}\sqrt{2}$

5.1. **Reduction for standard measure spaces.** We will now proof that it is enough to only consider the case that (Ω, Σ, μ) is the interval $[0,1]$ with the Borel measurable sets and the Lebesgue measure.

Lemma 5.4. *Let $\mathcal{M} = ([0,1], \mathcal{B}([0,1]), \lambda)$ be the measure space on $[0,1]$ with the Lebesgue measure λ. If Theorem 5.2 holds for $\mathcal{M}$, that is, if for every $\epsilon > 0$ and every function in $f \in L_\infty(\mathcal{M})$ with $\int_{[0,1]} f d\lambda = 0$ and $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$ we can find a $g \in L_\infty(\mathcal{M})$ with $||g||_\infty < 4||f||_\infty + \epsilon$ and a measure preserving transformation T of $\mathcal{M}$ such that $f = g \circ T - g$, then the same is true for any standard measure space.*

Proof. Assume the theorem holds for the measure space $([0,1], \mathcal{B}([0,1]), \lambda)$. Then for $f \in L_\infty([0,1]\mathcal{B}([0,1]), \lambda)$ we can find the g and measure preserving T. Now, for any Lebesgue measurable set $A \subseteq [0,1]$, it follows from the regularity of the Lebesgue measure, Theorem 2.9, that we can find Borel measurable sets $B_1 \subseteq A \subseteq B_2$ with $\lambda(B_1) = \lambda(B_2)$. Now $T(B_1) \subseteq T(A) \subseteq T(B_2)$, and $\lambda(T(B_1)) = \lambda(B_1) = \lambda(B_2) = \lambda(T(B_2))$. Thus also $T(A)$ is Lebesgue measurable and $\lambda(T(A)) = \lambda(A)$. The same is true for T^{-1}. Hence T is also a measure preserving transformation of $([0,1], \mathcal{A}, \lambda)$ where $\mathcal{A}$ denote the Lebesgue measurable sets. Thus, the theorem holds also for the measure space $([0,1], \mathcal{A}, \lambda)$.

Now, let $\mathcal{M}_1 = (I, \mathcal{A}_1, \lambda)$ be a measure space on a closed finite interval $I = [a,b]$ with $a < b$. with the Lebesgue measurable sets $\mathcal{A}_1$. Let $\phi : I \to [0,1]$ be a bijection given by $\phi(x) = \frac{x-a}{b-a}$. Since ϕ and ϕ^{-1} are continuous they are measurable functions. Now choose $\epsilon > 0$ and let $f \in L_\infty(I, \mathcal{A}_1, \lambda)$ with

$\int_I f d\lambda = 0$ and $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$, then $f \circ \phi^{-1}$ is a function in $L_\infty([0,1], \mathcal{A}, \lambda)$ with $\int_{[0,1]} f \circ \phi^{-1} d\lambda = \int_I f \circ d\lambda \circ \phi = \frac{1}{b-a} \int_I f d\lambda = 0$ and $\lambda((f \circ \phi^{-1})^{-1}(\{x\}) = \lambda(\phi(f^{-1}(\{x\})) = \frac{1}{(b-a)} \lambda(f^{-1}(\{x\}) = 0$ for all $x \in \mathbb{R}$. Hence, there exists a $g \in L_\infty([0,1], \mathcal{A}, \lambda)$ with $||g||_\infty < 4||f \circ \phi^{-1}||_\infty + \epsilon = 4||f||_\infty + \epsilon$ and a measure preserving T of $([0,1], \mathcal{A}, \lambda)$ such that $f \circ \phi^{-1} = g \circ T - g$. Thus, first applying ϕ to both sides gives, $f = \tilde{g} \circ \tilde{T} - \tilde{g}$ where $\tilde{g} := g \circ \phi \in L_\infty(I, \mathcal{A}_1, \lambda)$ with $||\tilde{g}||_\infty = ||g||_\infty < 4||f||_\infty + \epsilon$ and further $\tilde{T} := \phi^{-1} \circ T \circ \phi$ is a measure preserving transformation of $(I, \mathcal{A}_1, \lambda)$. Namely, $\tilde{T}$ is a bijection of I to itself, since T and ϕ are bijections. Further $\tilde{T}$ and $\tilde{T}^{-1}$ are measurable functions since T, T^{-1}, ϕ and ϕ^{-1} are measurable functions. And, last $\lambda(\tilde{T}(A)) = \lambda(\phi^{-1} \circ T \circ \phi(A)) = (b-a)\lambda(T \circ \phi(A)) = (b-a)\lambda(\phi(A)) = \lambda(A)$ where we used that T is measure preserving and that λ is translation-invariant. Thus we find that the theorem also holds for $(I, \mathcal{A}_1, \lambda)$. Now if I' were an open, or half-open interval (a,b), $(a,b]$ or $[a,b)$, and if $f \in L_\infty(I', \mathcal{A}_1, \lambda)$ with $\int_{I'} f d\lambda = 0$ and $\lambda(f^{-1}(x)) = 0$ for all $x \in \mathbb{R}$ then we can consider f as a function $f \in L_\infty(I, \mathcal{A}_1, \lambda)$ with $\int_I f d\lambda = 0$ and $\lambda(f^{-1}(x)) = 0$ for all $x \in \mathbb{R}$ so that, for $\epsilon > 0$ we get a $g \in L_\infty(I, \mathcal{A}_1, \lambda)$ with $||g||_\infty < 4||f||_\infty + \epsilon$ and measure preserving T of $(I, \mathcal{A}_1, \lambda)$ such that $f = g \circ T - g$. Now redefining $T^{-1}(a)$ and $T^{-1}(b)$ as $T(a)$ and $T(b)$ respectively, makes $T|_{I'}$ a measure preserving transformation of I'. Hence, the theorem holds also for any measure space $(I, \mathcal{A}_1, \lambda)$ where I is an arbitrary finite interval. Now, if $\mathcal{M}_2 = (\Omega, \Sigma, \mu)$ is some standard measure space, then it is isomorphic to some measure space $(I, \mathcal{A}_1, \lambda)$ where I is some finite interval and where $\mathcal{A}_1$ are the Lebesgue measurable sets. Now letting $\phi : \Omega \to I$ be the bijection, we can prove the theorem for $\mathcal{M}_2$ in the same way as we just did for $(I, \mathcal{A}_1, \lambda)$. Thus this finishes the proof. □

5.2. **Compensation of Integral.** For the proof for $([0,1], \mathcal{B}([0,1]), \lambda)$ we will need the following lemma that says that if we have a set D and an essentially bounded, non-zero, function f on D, whose integral over D equals 0. Then, if we choose $\epsilon > 0$ small enough (depending on f how small), and if we have a compact subset $E \subseteq D$ with $\lambda(D \setminus E) < \epsilon$ such that f is continuous on E, then we can find a slightly smaller compact subset $K \subseteq E$ such that the integral of f over K equals 0.

Lemma 5.5. *Let $D \subseteq [0,1]$ and let $f \in L_\infty(D, \mathcal{B}(D), \lambda)$ with $f \neq 0$ and such that $\int_D f d\lambda = 0$. Then $\tau^+ = \lambda(\{f > \frac{1}{2}||f^+||_\infty\}) > 0$ and $\tau^- = \lambda(\{f < -\frac{1}{2}||f^-||_\infty\}) > 0$. Let $0 < \epsilon < \frac{1}{4}\min\{\tau^+ \frac{||f^+||_\infty}{||f||_\infty}, \tau^- \frac{||f^-||_\infty}{||f||_\infty}\}$. Then, for any compact subset $E \subset D$ with $\lambda(E) \geq \lambda(D) - \epsilon$ and for which $f|_E$ is continuous, there is a compact subset $K \subseteq E$ with $\int_K f d\lambda = 0$ and $\lambda(K) \geq \lambda(D) - \left(1 + 2||f||_\infty \max\{\frac{1}{||f^+||_\infty}, \frac{1}{||f^-||_\infty}\}\right)\epsilon$.*

Proof. Let D, E and f and ϵ so that all conditions are fulfilled. Further, set $\tilde{\epsilon} = \int_E f d\lambda$. Since, $f \neq 0$ and $\int_D f d\lambda = 0$ we have that $f^+, f^- \neq 0$, hence $\tau^+, \tau^- > 0$. We have $|\tilde{\epsilon}| = |\int_E f d\lambda| = |\int_{D \setminus E} f d\lambda| \leq \lambda(D \setminus E)||f||_\infty \leq \epsilon||f||_\infty$. We will further suppose that $\tilde{\epsilon} \geq 0$. The set $f|_E^{-1}((\frac{1}{2}||f^+||_\infty, \infty))$ is open in E since $f|_E$ is continuous. Now, for $0 \leq r \leq 1$ define the following measurable set, $R_r := f|_E^{-1}((\frac{1}{2}||f^+||_\infty, \infty)) \cap [0, r) \subseteq E$. Now, let

$F : [0,1] \to \mathbb{R}$ be given by $F(r) = \int_{E \setminus R_r} f d\lambda$. This is a continuous function. Further we have $F(0) = \tilde{\epsilon} \geq 0$ by assumption. Furthermore, since $\lambda(R_1) = \lambda(f^{-1}((\frac{1}{2}||f^+||_\infty, \infty)) \cap E) \geq \tau^+ - \lambda(D \setminus E) \geq \tau^+ - \epsilon \geq \frac{1}{2}\tau^+$ we have $F(1) = \int_E f d\lambda - \int_{R_1} f d\lambda \leq \tilde{\epsilon} - \lambda(R_1)\frac{1}{2}||f^+||_\infty \leq \tilde{\epsilon} - \tau^+\frac{1}{4}||f^+||_\infty \leq (\epsilon - \frac{1}{4}\tau^+\frac{||f^+||_\infty}{||f||_\infty})||f||_\infty < 0$, hence, there must be a r_0 with $F(r_0) = 0$. Now set $K = E \setminus R_{r_0} \subseteq D$. Since R_{r_0} is open in E, we have that K is compact. Further we have $\int_K f d\lambda = F(r_0) = 0$. Finally, we have $\tilde{\epsilon} = \int_E f d\lambda = \int_{R_{r_0}} f d\lambda \geq \lambda(R_{r_0})\frac{1}{2}||f^+||_\infty$ so that $\lambda(R_{r_0}) \leq \frac{2\tilde{\epsilon}}{||f^+||_\infty} \leq \frac{2||f||_\infty}{||f^+||_\infty}\epsilon$. Hence $\lambda(K) = \lambda(E) - \lambda(R_{r_0}) \geq (\lambda(D) - \epsilon) - \frac{2||f||_\infty}{||f^+||_\infty}\epsilon = \lambda(D) - (1 + \frac{2||f||_\infty}{||f^+||_\infty})\epsilon \geq \lambda(D) - \left(1 + 2||f||_\infty \max\{\frac{1}{||f^+||_\infty}, \frac{1}{||f^-||_\infty}\}\right)\epsilon$. The case that $\tilde{\epsilon} < 0$ now follows by replacing f by $-f$. This finishes the proof. □

5.3. **Shrinking Lemma.** What the next Lemmas say is that if we have a compact set K and a continuous, essentially bounded function f on K with integral 0 over K, then for $0 < c < \lambda(K)$ we can find a compact subset E of K such that $\lambda(K \setminus E) = c$, the integral of f over E equals 0, and such that E does not contain the endpoints of K.

Lemma 5.6. *Let $K \subset [0,1]$ be compact and of positive measure and let $f \in L_\infty(K, \lambda)$ be continuous, mean zero and such that $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. Let $c \in (0, \lambda(K))$ then there is a compact subset $E \subseteq K$ of measure $\lambda(E) = \lambda(K) - c$ such that $\int_E f d\lambda = 0$.*

Proof. Let K, f and c be given. Let $\upsilon^\pm = \lambda(\{f^\pm > 0\})$. We have $\upsilon^\pm > 0$ since f is mean zero and not constant. For $0 \leq r \leq \upsilon^+$ let $A_r = f^{-1}((0, a_r))$ where $a_r \geq 0$ is chosen maximally such that $\lambda(A_r) = r$. Furthermore, for $0 \leq r \leq \upsilon^-$ let $B_r = f^{-1}((-\infty, b_r))$ where $b_r \leq 0$ is chosen maximally such that $\lambda(B_r) = r$. Note that we can choose a_r and b_r in this way because of the assumption that $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. We have $A_{r_1} \subset A_{r_2}$ and $B_{r_1} \subset B_{r_2}$ whenever $r_1 < r_2$. Let $F_\pm : [0, \upsilon^\pm] \to \mathbb{R}^+$ be given by $F_+(r) = \int_{A_r} f d\lambda$ and $F_-(r) = -\int_{B_r} f d\lambda$. These are continuous, strictly increasing functions with $F_+(0) = F_-(0) = 0$ and $F_+(\upsilon^+) = F_-(\upsilon^-)$ as f is mean zero on K.

Let $G : [0, \upsilon^+] \times [0, \upsilon^-] \to \mathbb{R}$ be given by $G(t, r) = F_-(r) - F_+(t)$ so that we have $G(t, 0) = F_-(0) - F_+(t) \leq 0$ and $G(t, \upsilon^-) = F_-(\upsilon^-) - F_+(t) = F_+(\upsilon^+) - F_+(t) \geq 0$ for $t \in [0, \upsilon^+]$. Hence, since $G(t, \cdot)$ is continuous, there is a $0 \leq x \leq \upsilon^-$ with $G(t, x) = 0$. Further, since $G(t, \cdot)$ is strictly increasing this x is unique. Now we can define $H : [0, \upsilon^+] \to [0, \upsilon^-]$ by letting $H(t) \in [0, \upsilon^-]$ be the unique number so that $G(t, H(t)) = 0$.

For $t \in [0, \upsilon^+]$ we now have $F_-(H(t)) - F_+(t) = 0$, which means $\int_{A_t} f d\lambda + \int_{B_{H(t)}} f d\lambda = 0$. Hence, set $E_t = K \setminus (A_t \cup B_{H(t)})$, which is compact since $A_t, B_{H(t)}$ are open in K, then $\int_{E_t} f d\lambda = 0$. Now $\lambda(E_t) = \lambda(K) - (H(t) + t)$. Now, since G is continuous, strictly decreasing in t and strictly increasing in r we have that H is continuous and strictly increasing. Now, since $H(0) = 0$ and since $\upsilon^+ + H(\upsilon^+) = \upsilon^+ + \upsilon^- = \lambda(K)$ as f is mean zero and nowhere constant, we have that there exists a $t_0 \in (0, \upsilon^+)$ such that $t_0 + H(t_0) = c$.

Now we can take $E = E_{t_0}$ so that $\lambda(E) = \lambda(K) - c$ and such that f is mean zero on E. This finishes the proof. $\square$

Lemma 5.7. *Let $K \subseteq [0,1]$ be compact and of positive measure and let $f \in L_\infty(K,\lambda)$ be continuous, mean zero and such that $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. Let $c \in (0, \lambda(K))$ then there is a compact subset $E \subseteq K \cap (\inf K, \sup K)$ of measure $\lambda(E) = \lambda(K) - c$ such that $\int_E f d\lambda = 0$.*

Proof. Let K, f and c be given. Let $\tau^\pm = (\{f^\pm > \frac{1}{2}||f^\pm||_\infty\})$ which are positive since f is mean zero and nowhere constant. Choose $\epsilon > 0$ with $\epsilon < \frac{1}{4}\min\{\tau^+ \frac{||f^+||_\infty}{||f||_\infty}, \tau^- \frac{||f^-||_\infty}{||f||_\infty}\}$ and $(1 + 2||f||_\infty \max\{\frac{1}{||f^+||_\infty}, \frac{1}{||f^-||_\infty}\})\epsilon < c$ Now choose $0 < r < \lambda(K)$ such that $\lambda(K \cap [\inf K + r, \sup K - r]) = \lambda(K) - \epsilon$. Now we can apply Lemma 5.5 on $K \cap [\inf K + r, \sup K - r] \subseteq K$ with f and ϵ to get a compact subset $\widetilde{K} \subseteq K \cap [\inf K + r, \sup K - r]$ with $\int_{\widetilde{K}} f d\lambda = 0$ and with $\lambda(\widetilde{K}) \geq \lambda(K) - (1 + 2||f||_\infty \max\{\frac{1}{||f^+||_\infty}, \frac{1}{||f^-||_\infty}\})\epsilon > \lambda(K) - c$. Now let $y = \lambda(K) - \lambda(\widetilde{K})$ so that $0 < c - y < \lambda(\widetilde{K})$. We can now apply Lemma 5.6 on $\widetilde{K}$ with f and $(c-y)$ to get a compact subset $E \subseteq \widetilde{K} \subseteq K \cap (\inf K, \sup K)$ with $\int_E f d\lambda = 0$ and $\lambda(E) = \lambda(\widetilde{K}) - (c - y) = \lambda(K) - c$. Hence this proves the statement. $\square$

5.4. **Splitting Lemma.** The following two lemmas we need to ensure that the diameter of sets in the chains goes to 0. To do this, in case there is an open interval $I \subseteq [\inf K, \sup K]$ disjoint with K, we need the ratio of the measure on the left, compared to the entire measure, to be of the form p/q so that we can partition K in a multiple of q pieces, which then, in essence, lie either entirely at the left or entirely at the right of the interval I.

Lemma 5.8. *Let $K = K_1 \cup K_2$ be of positive measure, with K_1, K_2 being compact subsets of $[0,1]$ with $\lambda(K_1 \cap K_2) = 0$. Further let $f \in L_\infty(K, \mathcal{B}(K), \lambda)$ be a continuous function with $\int_K f d\lambda = 0$ and such that $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. Then, for $\epsilon > 0$ we can find a compact subset $E \subseteq K$ of positive measure such that $\lambda(E) \geq \lambda(K) - \epsilon$ and $\int_E f d\lambda = 0$ and such that $\frac{\lambda(E \cap K_1)}{\lambda(E)} = \frac{p}{q}$ for some $p \in \mathbb{N}_0$ en $q \in \mathbb{N}$.*

Proof. Let K_1, K_2, f and ϵ be given as stated. We can assume that both $\lambda(K_1), \lambda(K_2) > 0$ otherwise we can take $E = K$. Write $f_1 = f|_{K_1}$ and $f_2 = f|_{K_2}$. We have $f_1, f_2 \neq 0$ since f is nowhere essentially constant. Now let $\tau_1^\pm = \lambda(\{f_1^\pm > \frac{1}{2}||f_1^\pm||_\infty\})$ and $\tau_2^\pm = \lambda(\{f_2^\pm > \frac{1}{2}||f_2^\pm||_\infty\})$. We must have $\tau_1^+ > 0$ or $\tau_1^- > 0$ and likewise $\tau_2^+ > 0$ or $\tau_2^- > 0$, since $f_1, f_2 \neq 0$. Now if $\tau_1^+ > 0$ and $\tau_2^+ > 0$ then we must also have $\tau_1^- > 0$ or $\tau_2^- > 0$, since f is mean zero. Hence we can assume that $\tau_1^+, \tau_2^- > 0$, the case that $\tau_1^-, \tau_2^+ > 0$ being similar by considering $-f$ instead of f. For $0 \leq r \leq \tau_1^+$ let $A_r = f_1^{-1}((0, a_r))$ where $a_r \geq 0$ is chosen maximally such that $\lambda(A_r) = r$. Furthermore, for $0 \leq r \leq \tau_2^-$ let $B_r = f_2^{-1}((-\infty, b_r))$ where $b_r \leq 0$ is chosen maximally such that $\lambda(B_r) = r$. Note that we can choose a_r and b_r in this way because of the assumption that $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. We have $A_{r_1} \subset A_{r_2}$ and $B_{r_1} \subset B_{r_2}$ whenever $r_1 < r_2$. Let $F_1, F_2 : [0, \min\{\tau_1^+, \tau_2^-\}] \to \mathbb{R}^+$ be given by $F_1(r) = \int_{A_r} f d\lambda$ and $F_2(r) = -\int_{B_r} f d\lambda$. These are continuous, strictly increasing functions with $F_1(0) = F_2(0) = 0$. We look at their right-derivatives. For $h > 0$ we have $\left|\frac{F_1(r+h)-F_1(r)}{h} - a_r\right| = \left|\frac{\int_{A_{r+h}\setminus A_r} f - a_r d\lambda}{h}\right| \leq \frac{\int_{A_{r+h}\setminus A_r} |f-a_r| d\lambda}{h} \leq |a_{r+h} - a_r|$. Now, since a_r is chosen maximally such that $\lambda(A_r) = r$, we must have for $\epsilon > 0$ that $\lambda(f_1^{-1}(0, a_r + \epsilon)) > \lambda(A_r) = r$. Hence if $r < r + h < \lambda(f_1^{-1}(0, a_r + \epsilon))$ we have $a_r < a_{r+h} < a_r + \epsilon$, so that $|a_{r+h} - a_r| \to 0$ as $h \to 0$. This means that for the right derivative we have $F_1'(r^+) = a_r$, which is strictly increasing.

Now, similarly for F_2. For $h > 0$ we have

$$\begin{aligned}\left|\frac{F_2(r+h) - F_2(r)}{h} + b_r\right| &= \left|\frac{\int_{B_{r+h}\setminus B_r} b_r - f d\lambda}{h}\right| \\ &\leq \frac{\int_{B_{r+h}\setminus B_r} |b_r - f| d\lambda}{h} \leq |b_{r+h} - b_r|\end{aligned}$$

. Now, since b_r is chosen maximally such that $\lambda(B_r) = r$, we must have for $\epsilon > 0$ that $\lambda(f_2^{-1}(-\infty, b_r + \epsilon)) > \lambda(B_r) = r$. Hence if $r < r + h < \lambda(f_2^{-1}(-\infty, b_r + \epsilon))$ we have $b_r < b_{r+h} < b_r + \epsilon$, so that $|b_{r+h} - b_r| \to 0$ as $h \to 0$. This means that for the right derivative we have $F_2'(r^+) = -b_r$, which is strictly decreasing.

Now fix $0 < r_0 < \min\{\tau_1^+, \tau_2^-\}$ and choose $0 < t_0 < \min\{\tau_1^+, \tau_2^-\}$ with $0 < F_1(t_0) < F_2(r_0)$, which can be done since F_1 is continuous and F_1, F_2 are strictly increasing.

Let $G : [0, t_0] \times [0, r_0] \to \mathbb{R}$ be given by $G(t, r) = F_2(r) - F_1(t)$ so that we have $G(t, 0) = F_2(0) - F_1(t) \leq 0$ and $G(t, r_0) = F_2(r_0) - F_1(t) > 0$ for $t \in [0, t_0]$. Hence, since $G(t, \cdot)$ is continuous, there is a $0 \leq x < r_0$ with $G(t, x) = 0$. Further, since $G(t, \cdot)$ is strictly increasing this x is unique. Now define $H : [0, t_0] \to [0, r_0)$ as the unique number $H(t) \in [0, r_0)$ with $G(t, H(t)) = 0$.

For $r \in [0, t_0]$ we now have $F_2(H(r)) - F_1(r) = 0$, which means $\int_{A_r} f d\lambda + \int_{B_{H(r)}} f d\lambda = 0$. Hence, set $E_r = K \setminus (A_r \cup B_{H(r)})$, which is compact since $A_r, B_{H(r)}$ are open in K_1, K_2 respectively, then $\int_{E_r} f d\lambda = 0$. Now $\lambda(E_r) = \lambda(K) - (H(r) + r)$. Now, since G is continuous, and strictly increasing in t we have that H is continuous and strictly increasing. Hence, since $H(0) = 0$, we can choose $t_1 > 0$ very small such that $\lambda(E_{t_1}) \geq \lambda(K) - \epsilon$, that is $H(t_1) + t_1 \leq \epsilon$. Now let $R(t) = \frac{\lambda(E_t \cap K_1)}{\lambda(E_t)} = \frac{\lambda(K_1) - t}{\lambda(K) - (t + H(t))}$. We have $R(0) = \frac{\lambda(K_1)}{\lambda(K)}$. Now, suppose that R is constant on $[0, t_1]$ then solving for H we get $H(t) = \frac{\lambda(K) - \lambda(K_1)}{\lambda(K_1)} t$ for $t \in [0, t_1]$. Further we have $F_2(H(t)) = F_1(t)$ so that differentiating from the right gives $F_2'(H(t))H'(t) = F_1'(t)$. Now, since $F_2' \circ H$ is strictly decreasing and F_1' strictly increasing and H' is a positive constant, this gives a contradiction. We thus conclude that R is not constant on $[0, t_1]$. Hence, since R is continuous we can find a $t_2 \in [0, t_1]$ such that $R(t_2) = \frac{p}{q}$ for some $p \in \mathbb{N}_0$ and $q \in \mathbb{N}$. Now set $E = E_{t_2}$ so that $\lambda(E) \geq \lambda(K) - \epsilon$ and $\int_E f d\lambda = 0$ and $\frac{\lambda(E \cap K_1)}{\lambda(E)} = R(r_2) = \frac{p}{q}$. This finishes the proof. □

Lemma 5.9. *Let $K \subseteq [0,1]$ be compact of positive measure. Let $k^M = \frac{\inf K + \sup K}{2}$, and set $K^L = K \cap [0, k^M]$ and $K^R = K \cap [k^M, 1]$. Further, let $f \in L_\infty(K, \mathcal{B}(K), \lambda)$ be continuous with $\int_K f d\lambda = 0$ and such that $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. In case $\lambda(K^L) = 0$ or $\lambda(K^R) = 0$ choose $c \in (0, \lambda(K))$, otherwise choose $c \in (0, \min\{\lambda(K^L), \lambda(K^R)\})$. Then there exists a compact subset $E \subset K \cap (\inf K, \sup K)$ with $\frac{\lambda(E \cap K^L)}{\lambda(E)} = \frac{p}{q}$ for some $p \in \mathbb{N}_0$ en $q \in \mathbb{N}$ and such that $\lambda(E) = \lambda(K) - c$ and $\int_E f d\lambda = 0$.*

Proof. If $\lambda(K^L) = 0$ we can apply Lemma 5.7 on K^R to get the compact subset $E \subseteq K^R \cap (\inf K^R, \sup K^R) \subseteq K \cap (\inf K, \sup K)$ with $\int_E f d\lambda = 0$ and $\lambda(E) = \lambda(K^R) - c = \lambda(K) - c$. Moreover $\frac{\lambda(E \cap K^L)}{\lambda(K)} = 0 \in \mathbb{Q}$ so that E satisfies the properties of this Lemma. This goes similar when $\lambda(K^R) = 0$, by interchanging the roles of K^L and K^R. We can thus assume that $\lambda(K^L), \lambda(K^R) > 0$. We use Lemma 5.8 on $K = K^L \cup K^R$ with f and $\frac{c}{2}$ to get a compact subset $\tilde{K} \subseteq K$ with $\int_{\tilde{K}} f d\lambda = 0$ and $\frac{\lambda(\tilde{K} \cap K^L)}{\lambda(\tilde{K})} = \frac{p}{q}$ and $\lambda(\tilde{K}) \geq \lambda(K) - \frac{c}{2}$. Now set $\tilde{K}^L = \tilde{K} \cap K^L$ and $\tilde{K}^R = \tilde{K} \cap K^R$. Further set $y = \lambda(K) - \lambda(\tilde{K})$ so that we have $0 < y < c$. Furthermore we have $\lambda(\tilde{K}^L) = \lambda(K^L) - \lambda(K^L \cap (K \setminus \tilde{K})) \geq \lambda(K^L) - y$ and likewise $\lambda(\tilde{K}^R) \geq \lambda(K^R) - y$. Now $0 < (c - y) < \min\{\lambda(K^L) - y, \lambda(K^R) - y\} \leq \min\{\lambda(\tilde{K}^L), \lambda(\tilde{K}^R)\}$. Now set $h^L = f - \frac{1}{\lambda(\tilde{K}^L)} \int_{\tilde{K}^L} f d\lambda$ and $h^R = f - \frac{1}{\lambda(\tilde{K}^R)} \int_{\tilde{K}^R} f d\lambda$. We can apply Lemma 5.7 on $\tilde{K}^L$ with h^L and $(c - y)\frac{p}{q}$ and on $\tilde{K}^R$ with h^R and $(c - y)(1 - \frac{p}{q})$ to obtain compact subsets $E^L \subseteq \tilde{K}^L \cap (\inf \tilde{K}^L, \sup \tilde{K}^L)$ and $E^R \subseteq \tilde{K}^R \cap (\inf \tilde{K}^R, \sup \tilde{K}^R)$ with $\lambda(E^L) = \lambda(\tilde{K}^L) - (c - y)\frac{p}{q}$ and $\lambda(E^R) = \lambda(\tilde{K}^R) - (c - y)(1 - \frac{p}{q})$. and moreover $\int_{E^L} h^L d\lambda = \int_{E^R} h^R d\lambda = 0$.

The last assertion means that $\int_{E^L} f d\lambda = \frac{\lambda(E^L)}{\lambda(\tilde{K}^L)} \int_{\tilde{K}^L} f d\lambda$ and $\int_{E^R} f d\lambda = \frac{\lambda(E^R)}{\lambda(\tilde{K}^R)} \int_{\tilde{K}^R} f d\lambda$. Now let $E = E^L \cup E^R \subseteq K \cap (\inf K, \sup K)$, which is a compact set. We have that

$$\int_E f d\lambda = \frac{\lambda(E^L)}{\lambda(\tilde{K}^L)} \int_{\tilde{K}^L} f d\lambda + \frac{\lambda(E^R)}{\lambda(\tilde{K}^R)} \int_{\tilde{K}^R} f d\lambda \tag{1}$$

$$= (1 - (c - y)\frac{\frac{p}{q}}{\lambda(\tilde{K}^L)}) \int_{\tilde{K}^L} f d\lambda + (1 - (c - y)\frac{1 - \frac{p}{q}}{\lambda(\tilde{K}^R)}) \int_{\tilde{K}^R} f d\lambda \tag{2}$$

$$= (1 - \frac{(c - y)}{\lambda(\tilde{K})}) \int_{\tilde{K}^L} f d\lambda + (1 - \frac{(c - y)}{\lambda(\tilde{K})}) \int_{\tilde{K}^R} f d\lambda \tag{3}$$

$$= (1 - \frac{(c - y)}{\lambda(\tilde{K})}) \int_{\tilde{K}^L \cup \tilde{K}^R} f d\lambda = 0 \tag{4}$$

Furthermore $\lambda(E) = \lambda(\tilde{K}^L) - (c - y)\frac{p}{q} + \lambda(\tilde{K}^R) - (c - y)(1 - \frac{p}{q}) = \lambda(\tilde{K}) - (c - y) = \lambda(K) - c$.

Finally, as we just saw we have $\frac{\lambda(E^L)}{\lambda(\tilde{K}^L)} = \frac{\lambda(E^R)}{\lambda(\tilde{K}^R)}$ so that

$$\frac{\lambda(E^L)}{\lambda(E)} = \frac{\lambda(E^L)}{\lambda(E^L)+\lambda(E^R)} = \frac{\lambda(\tilde{K}^L)}{\lambda(\tilde{K}^L)+\lambda(\tilde{K}^R)} = \frac{\lambda(\tilde{K}^L)}{\lambda(\tilde{K})} = \frac{p}{q} \tag{5}$$

□

5.5. **Construction of Chains.** We will now prove the following lemma, which is used to construct a set $C \subseteq K$ for which we will in Lemma 5.11 create an homeomorphism to a Cantor set, that is also an isomorphism.

Lemma 5.10. *Let $K \subseteq [0,1]$ compact and of positive measure. Further let $f \in L_\infty(K, \mathcal{B}(K), \lambda)$ continuous and with $\int_K f d\lambda = 0$ and $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. Now, if we let $\lambda(K) > \epsilon > 0$ then there exists a sequence of natural numbers $(m_n)_{n=1}^\infty$ with $m_n \geq 2$, such that, if we set $\mathcal{E}_n = \bigotimes_{j=1}^n \{1, ..., m_j\}$ for $n \geq 1$, there exist families $\mathcal{C}_n = \{K_a : a \in \mathcal{E}_n\}$ for $n \in \mathbb{N}$ such that if we set $C_n = \bigcup_{K_a \in \mathcal{C}_n} K_a$ and $C = \bigcap_{n=1}^\infty C_n$ the following properties hold:*

1. *Every $K_a \in \mathcal{C}_n$ is a compact subset of $[0,1]$*
2. *For fixed $n \geq 1$ we have that $K_a, K_b \in \mathcal{C}_n$ are disjoint if $a \neq b$.*
3. *For fixed $n \geq 1$ we have that any $K_a, K_b \in \mathcal{C}_n$ have equal positive measure $M_n := \lambda(K_a) = \lambda(K_b) > 0$*
4. *If $a \in \mathcal{E}_n$ and $b \in \mathcal{E}_{n-1}$ are such that $a_i = b_i$ for $1 \leq i \leq n-1$, then $K_a \subseteq K_b$.*
5. *For any $n \geq 1$ we have $\int_{C_n} f d\lambda = 0$, and furthermore $\int_C f d\lambda = 0$*
6. *For $K_a \in \mathcal{C}_n$ we have $|\mathcal{E}_n| \cdot \lambda(K_a) \geq \lambda(K) - (1 - 2^{-n})\epsilon > \lambda(K) - \epsilon$*
7. *For $n \geq 0$ every set $K_a \in \mathcal{C}_n$ is open in C_n*
8. *For every chain $K_{c_1} \supset K_{c_2} \supset ....$ with $c_j \in \mathcal{E}_j$ we have $diam(K_{c_j}) \to 0$ as $j \to \infty$.*
9. *The set $\{K_a \cap C : K_a \in \mathcal{C}_n$ for some $n \geq 1\}$ is a basis for the topology of C.*

Proof. For $n \geq 1$ we will choose m_n and define families $\mathcal{C}_n = \{K_a : a \in \mathcal{E}_n\}$ with given properties. We first set $\mathcal{C}_0 = \{K\}$ for convenience and choose $m_1 = 2$. Now we will inductively define m_{n+2} and $\mathcal{C}_{n+1}$ for all $n \geq 0$. Let $n \geq 0$, we can assume that property (1) to (7) hold for $0 \leq j \leq n$. Now choose $b \in \mathcal{E}_n$. We will consider $K_b \in \mathcal{C}_n$.

Choose $x_b^0, x_b^1, ...x_b^{m_{n+1}} \in [\inf K_b, \sup K_b]$ such that for $1 \leq i \leq m_{n+1}$ we have $\lambda(K_b \cap [x_b^{i-1}, x_b^i]) = \frac{1}{m_{n+1}} \lambda(K_b)$, that is, we divide K_b in m_{n+1} pieces of equal measure. For $i = 1, ..., m_{n+1}$, we will let K_b^i denote the set $K_b \cap [x_b^{i-1}, x_b^i]$, let $K_b^{i,L}$ denote the set $K_b \cap [\inf K_b^i, \frac{\inf K_b^i + \sup K_b^i}{2}]$ and let $K_b^{i,R}$ denote the set $K_b \cap [\frac{\inf K_b^i + \sup K_b^i}{2}, \sup K_b^i]$. Furthermore set $h_b^i = f|_{K_b^i} - \frac{1}{\lambda(K_b^i)} \int_{K_b^i} f d\lambda$ so that $\int_{K_b^i} h_b^i d\lambda = 0$. Now choose $\epsilon_n > 0$ with

$$\epsilon_n < \frac{1}{3} \min\{\frac{\epsilon}{2^{n+1}|\mathcal{E}_n| m_{n+1}}, M_n\} \cup \{\lambda(K_c^{i,L}), \lambda(K_c^{i,R}) : 1 \leq i \leq m_{n+1}, c \in \mathcal{E}_n\} \backslash \{0\}$$

Now, since by construction K_b is compact, also the sets K_b^i are compact. Now, further f is continuous on K_b so also on the subsets. Now we apply Lemma 5.9 on K_b^i with h_b^i and ϵ_n so that we get compact subsets $\tilde{K}_b^{i,L} \subset K_b^{i,L} \cap (\inf K_b^i, \sup K_b^i)$ and $\tilde{K}_b^{i,R} \subset K_b^{i,R} \cap (\inf K_b^i, \sup K_b^i)$, so that, if we set $\tilde{K}_b^i = \tilde{K}_b^{i,L} \cup \tilde{K}_b^{i,R}$, we have that $\lambda(\tilde{K}_b^i) = \lambda(K_b^i) - \epsilon_n$ and $\int_{\tilde{K}_b^i} h_b^i d\lambda = 0$ and further $\frac{\lambda(\tilde{K}_b^{i,L})}{\lambda(\tilde{K}_b^i)} = \frac{p_b^i}{q_b^i}$ for some $p_b^i \in \mathbb{N}_0$ and $q_b^i \in \mathbb{N}$.

Now let $(I_j)_{j=1}^\infty$ be an enumeration of all open intervals contained in $[0,1]$ with rational endpoints and let $\mathcal{U}_b^i = \{I_j \cap \tilde{K}_b^i : j \in \mathbb{N}, \lambda(I_j \cap \tilde{K}_b^i) = 0\}$ Now if $a = (b_1, b_2, ..., b_n, i)$ with $1 \leq i \leq m_{n+1}$ then we define $K_a \in \mathcal{C}_{n+1}$ as $K_a = \tilde{K}_b^i \setminus \bigcup_{U \in \mathcal{U}_b^i} U$. Note that we only removed a set of zero measure. Now, to conclude the construction we set $m_{n+2} = 2 \prod_{b \in \mathcal{E}_n} \prod_{i=1}^{m_{n+1}} q_b^i$.

Before we prove that our construction satisfies the stated properties (1)-(9), we will give an intuitive idea of what we are doing.

We have divided the set K_b from the left to the right in sets K_b^i, all of equal measure. Further, these sets are disjoint, except for their endpoints. We then want to define the sets $K_a \in \mathcal{C}_{n+1}$ as subset of the sets K_b^i in such a way that the sets K_a do not contain the endpoints, are of equal measure, are still compact, and such that the integral of f over their union $\bigcup_{a \in \mathcal{C}_{n+1}} \widetilde{K}_a$ remains zero. Further we need to make sure that the diameter of chains goes to zero. This is not trivial since the endpoints x_b^i are not equidistantly distributed, but distributed depending on the set K_b. Hence, if there is an interval (u, v) not contained in K_b, but $\inf K_b \leq u$ and $v \leq \sup K_b$, then we might create a chain $K_{a_1} \supset K_{a_2} \supset ......$ such that $\inf \bigcap_{i=1}^\infty K_{a_i} \leq u$ and $v \leq \sup \bigcap_{i=1}^\infty K_{a_i}$. This we have to avoid.

To do all of this, we defined the sets $K_b^{i,L}$ and $K_b^{i,R}$. These sets have diameter at most half of the diameter of K_b. Hence, if eventually sets in the chains are either a subset of $K_b^{i,L}$ or of $K_b^{i,R}$ for some i, then this ensures that the diameter of chains goes to zero. To do this we needed Lemma 5.9. This lemma gives us compact subsets $\widetilde{K}_b^i \subseteq K_b^i$ that do not contain the endpoints, and such that the functions h_b^i that were mean zero on K_b^i are also mean zero on $\widetilde{K}_b^i$. This last property says that the average of f over K_b^i equals the average of f over $\widetilde{K}_b^i$. Now, since the removed measure $\lambda(K_b^i \setminus \widetilde{K}_b^i) = \epsilon_n$ is equal for all $b \in \mathcal{C}_n$ and $1 \leq i \leq m_{n+1}$, this implies that the integral of f over the union $\bigcup_{b \in \mathcal{C}_n} \bigcup_{i=1}^{m_{n+1}} \widetilde{K}_b^i$ is zero. Now the subsets $K_a \subseteq K_b$ with $a \in \mathcal{E}_{n+1}$ are essentially set equal to some set $\widetilde{K}_b^i$. Now, since the lemma gave us that $\frac{\lambda(\widetilde{K}_b^i \cap K_b^{i,L})}{\lambda(\widetilde{K}_b^i)} \in \mathbb{Q}$ and because of the way we choose m_{n+2} we have that all measure of the sets K_a^j, with $1 \leq j \leq m_{n+2}$, are contained in either $K_b^{i,L}$ or $K_b^{i,R}$ for some i. Further, the construction using the enumerations of the open intervals $(I_j)_{j=1}^\infty$ with rational endpoints is done to ensure that we remove sets of zero measure that may keep the diameter large. After we removed these sets, the subsets $K_c \subseteq K_a$ with $c \in \mathcal{E}_{n+2}$ are really fully contained in either $K_b^{i,L}$ or $K_b^{i,R}$. In this way we ensure that the diameter goes to zero.

We will now show the stated properties hold.

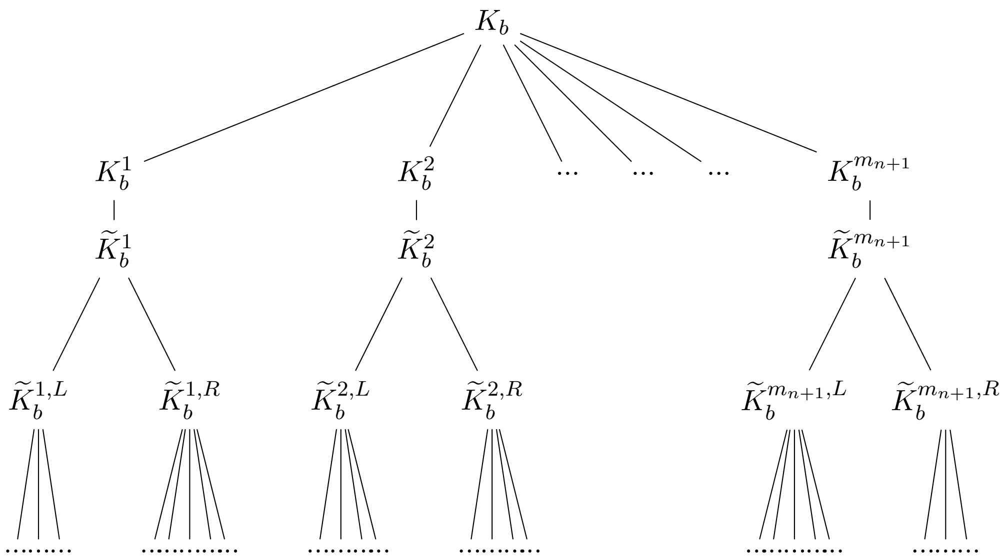

FIGURE 4. Visualisation of the subdivision of some K_b, with $b \in \mathcal{E}_n$, in measure spaces. The lines mean that the lower set is included in the upper set. The subsets $K_a \subseteq K_b$ with $a \in \mathcal{E}_{n+1}$ are set equal, up to a null set, to the sets $\widetilde{K}_b^i$. The number m_{n+2} is now chosen such that all the measure of some set K_a^i, for some $a \in \mathcal{E}_{n+1}$ with $K_a \subseteq K_b$, is contained either in some $K_b^{j,L}$ or $K_b^{j,R}$. This ensures that the diameter of chains $K_b \supset K_a \supset ....$ goes to zero.

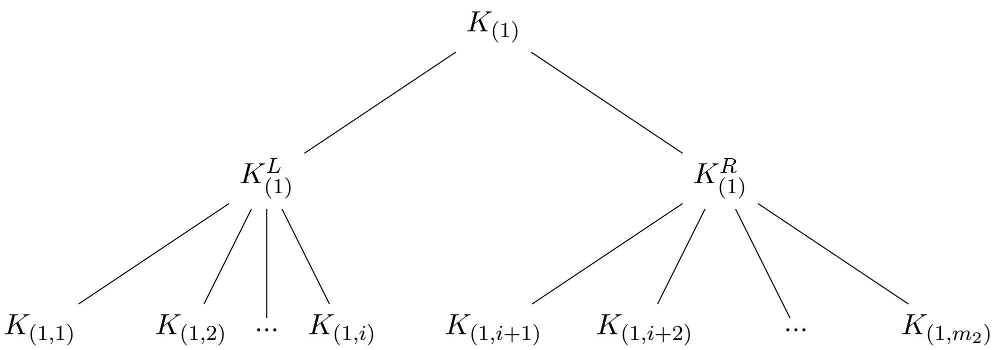

FIGURE 5. Visualisation of the subdivision of some K_b, with $b \in \mathcal{E}_n$, in measure spaces. The lines mean that the lower set is included in the upper set. The subsets $K_a \subseteq K_b$ with $a \in \mathcal{E}_{n+1}$ are either fully contained in $K_{(1)}^L$ or $K_{(1)}^R$. This ensures that the diameter of chains $K_b \supset K_a \supset ....$ goes to zero.

(1) Fix $n \geq 0$. Assuming that all sets $K_b \in \mathcal{C}_n$ were compact, we constructed the compact sets $\tilde{K}_b^i$. Further, since the sets $U \in \mathcal{U}_b^i$ are open in $\tilde{K}_b^i$, also $\tilde{K}_b^i \setminus \bigcup_{U \in \mathcal{U}_b^i} U$ is compact. Thus also the sets in $\mathcal{C}_{n+1}$ are compact. This proofs the claim by induction.

(2) Fix $n \geq 0$ and assume that the sets in $\mathcal{C}_n$ are all disjoint. Further choose $b \in \mathcal{E}_n$. For $i \neq j$ we have that $K_b^i \cap K_b^j \subseteq \{x_b^1, ..., x_b^{m_{n+1}}\}$ by construction. Further, by the construction we have the endpoint $x_b^{l-1}, x_b^l \notin \tilde{K}_b^l$ for $l = 1, ..., m_{n+1}$ so that $\tilde{K}_b^i$ and $\tilde{K}_b^j$ are disjoint. Hence all sets in $\mathcal{C}_{n+1}$ are disjoint. Hence we have proven the statement by induction.
(3) Fix $n \geq 0$ and assume that the sets in $\mathcal{C}_n$ have all equal measure. Let $K_a, K_c \in \mathcal{C}_{n+1}$ and let $K_b, K_d \in \mathcal{C}_n$ such that $K_a \subset K_b^i$ and $K_c \subset K_d^j$ for some $1 \leq i, j \leq m_{n+1}$. Then, by the construction we have $\lambda(K_a) = \lambda(K_b^i) - \epsilon_n = \frac{\lambda(K_b)}{m_{n+1}} - \epsilon_n = \frac{\lambda(K_d)}{m_{n+1}} - \epsilon_n = \lambda(K_d^j) - \epsilon_n = \lambda(K_c)$. Hence all sets in $\mathcal{C}_{n+1}$ have equal measure.
(4) This property holds directly by the construction of the sets K_a.
(5) By the construction we have $\int_{\tilde{K}_b^i} h_b^i d\lambda = 0$ hence it follows that

$$\begin{aligned}\int_{\tilde{K}_b^i} f d\lambda &= \frac{\lambda(\tilde{K}_b^i)}{\lambda(K_b^i)} \int_{K_b^i} f d\lambda \\ &= (1 - \frac{\epsilon_n}{\lambda(K_b^i)}) \int_{K_b^i} f d\lambda = (1 - \frac{\epsilon_n m_{n+1}}{\lambda(K_b)}) \int_{K_b^i} f d\lambda\end{aligned}$$

Therefore, since $M_n = \lambda(K_b)$ does not depend on $b \in \mathcal{E}_n$ we have

$$\begin{aligned}\int_{C_{n+1}} f d\lambda &= \int_{\bigcup_{b \in \mathcal{E}_n} \bigcup_{i=1}^{m_{n+1}} \tilde{K}_b^i} f d\lambda \\ &= \sum_{b \in \mathcal{E}_n} \int_{\bigcup_{i=1}^{m_{n+1}} \tilde{K}_b^i} f d\lambda \\ &= (1 - \frac{\epsilon_n m_{n+1}}{M_n}) \sum_{b \in \mathcal{E}_n} \int_{\bigcup_{i=1}^{m_{n+1}} K_b^i} f d\lambda \\ &= (1 - \frac{\epsilon_n m_{n+1}}{M_n}) \sum_{b \in \mathcal{E}_n} \int_{K_b} f d\lambda \\ &= (1 - \frac{\epsilon_n m_{n+1}}{M_n}) \int_{C_n} f d\lambda = 0\end{aligned}$$

Now, furthermore, for $n \geq 1$ we have $|\int_C f d\lambda| = |\int_{C_n} f d\lambda - \int_{C_n \setminus C} f d\lambda| \leq \lambda(C_n \setminus C) ||f||_\infty$. Hence, since $\lambda(C \setminus C_n)$ goes to 0 as $n \to \infty$, we have $\int_C f d\lambda = 0$.
(6) We have $\lambda(K_a) = \lambda(K)$ for $K_a \in \mathcal{C}_0$. Further, choose $n \geq 1$ and assume $|\mathcal{E}_n| \lambda(K_b) \geq \lambda(K) - (1 - 2^{-n})\epsilon$ for $b \in \mathcal{E}_n$. Then for $K_a \in \mathcal{C}_{n+1}$ with $K_a \subseteq K_b$ we have $\lambda(K_a) = \lambda(K_b^i) - \epsilon_n \geq \frac{\lambda(K_b)}{m_{n+1}} - \frac{\epsilon}{2^{n+1}|\mathcal{E}_n| m_{n+1}}$. Hence $\lambda(K_a) \geq \frac{1}{|\mathcal{E}_n| m_{n+1}}(\lambda(K) - (1 - 2^{-n})\epsilon) - \frac{\epsilon}{2^{n+1}|\mathcal{E}_n| m_{n+1}}$. Hence $|\mathcal{E}_{n+1}| \lambda(K_a) \geq \lambda(K) - (1 - 2^{-(n+1)})\epsilon$.
(7) By construction in the sets $K_a \in \mathcal{C}_{n+1}$ that were created are contained in $K_a \subseteq \tilde{K}_b^i \subset K_b^i \cap (\inf K_b^i, \sup K_b^i) \subseteq K_b \cap (x_b^{i-1}, x_b^i)$ for some $b \in \mathcal{E}_n$ and $1 \leq i \leq m_{n+1}$. Hence $K_a = C_{n+1} \cap (x_b^{i-1}, x_b^i)$ and $K_a \cap C = C \cap (x_b^{i-1}, x_b^i)$ so that K_a is open in C_{n+1} as well as in C.
(8) Let $K_a \in \mathcal{C}_{n+2}$ with $K_a \subseteq K_b^i \subset K_b \subseteq K_c^j \subset K_c \in \mathcal{C}_n$. We show that we have $diam(K_a) < \frac{1}{2} \cdot diam(K_c)$ so that for every chain $K_{d_1} \supset K_{d_2} \supset ...$ of

subsets with $d_k \in \mathcal{E}_k$ we have $diam(K_{d_k}) \to 0$ as $k \to \infty$.
The set K_b is constructed in such a way that $\frac{\lambda(K_b \cap K_c^{j,L})}{\lambda(K_b)} = \frac{p_c^j}{q_c^j}$. Now m_{n+2} is taken as multiple of q_c^j, so that $\frac{\lambda(K_b \cap K_c^{j,L})}{\lambda(K_b)}$ is a multiple of $\frac{1}{m_{n+2}}$. Now if $K_b^i \subseteq K_c^{j,L}$ or $K_b^i \subseteq K_c^{j,R}$ then $diam(K_a) \leq diam(K_b^i) \leq \frac{1}{2} diam(K_c)$ and we are done. Hence we can assume that $K_b^i \cap K_c^{j,L}$ and $K_b^i \cap K_c^{j,R}$ are non-empty, so that $x_b^{i-1} \in K_b^{j,L}$ and $x_b^i \in K_b^{j,R}$. Now for $1 \leq l \leq m_{n+1}$ we have $\frac{\lambda(K_b \cap [x_b^{l-1}, x_b^l])}{\lambda(K_b)} = \frac{1}{m_{n+1}}$ so that $\frac{\lambda(K_b \cap K_c^{j,L} \cap [x_b^{i-1}, x_b^i])}{\lambda(K_b)}$ is either 0 or $\frac{1}{m_{n+1}}$. Hence, all measure of K_b^i is either contained in $K_c^{j,L}$ or in $K_c^{j,R}$. Now K_a is set equal to a subset of $\tilde{K}_b^i \subseteq K_b^i$ from which either $\tilde{K}_b^i \cap (x_b^{i-1}, 1)$ or $\tilde{K}_b^i \cap (0, x_b^i)$ is removed, since it has zero measure. Hence $K_a \subseteq K_c^{j,L}$ or $K_a \subseteq K_c^{j,R}$ hence $\lambda(K_a) \leq \frac{1}{2}\lambda(K_c)$.
(9) The set $\{[0,u) \cap C : [0,u) \subseteq [0,1]\} \cup \{(v,1] \cap C : (v,1] \subseteq [0,1]\}$ is a subbasis for the topology of C. Now let $\mathcal{B} = \{L \cap C : L \in \mathcal{C}_n,$ for some $n \geq 1\}$. We show that $\mathcal{B}$ is a basis for the topology of C. Let $[0,u) \subset [0,1]$ be an open interval. Let $x \in C$ with $x < u$. Then, since the diameter of sets in the chains goes to 0, we can find an $N \in \mathbb{N}$ and a set $K^x \in \mathcal{C}_N$ such that $x \in K^x$ and $\sup K^x < u$. Now let $U = \bigcup_{x \in C \cap [0,u)} K^x \cap C$, then we have $U = [0,u) \cap C$. Thus $[0,u) \cap C$ is generated by the sets in $\mathcal{B}$. This proof goes identical for sets $(v,1] \subset [0,1]$. Hence $\mathcal{B}$ is a subbasis for the topology of C. Now since finite intersections of sets $A, B \in \mathcal{B}$ are the union of sets in $\mathcal{B}$, we have that $\mathcal{B}$ is actually a basis for the topology of C.

□

5.6. **Construction of Cantor space and homeomorphism.**

Lemma 5.11. *Let $K \subseteq [0,1]$ be compact and of positive measure. Furthermore let $f \in L_\infty(K, \mathcal{B}(K), \lambda)$ such that f is continuous and $\int_K f d\lambda = 0$ and $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$. Now, choose $\epsilon > 0$. Then there is a subset $C \subseteq K$ such that*

1. *C is compact*
2. *$\lambda(C) \geq \lambda(K) - \epsilon$ and $\lambda(C) > 0$*
3. *$\int_C f d\lambda = 0$*
4. *For some sequence (n_i) of natural numbers, there is a homeomorphism ϕ between C and the product space $\bigotimes_{i=1}^{\infty}\{1, ..., n_i\}$ from Example 4.2(2), such that ϕ maps the measure $\frac{\lambda}{\lambda(C)}$ to the product measure μ.*

Proof. Let K, f and ϵ as stated. We apply Lemma 5.10 on K with f and with $\frac{1}{2}\min\{\epsilon, \lambda(K)\}$ to get a sequence $(m_i)_{i=0}^{\infty}$, and the families $\mathcal{C}_n$ and the sets $C_n = \bigcup_{a \in \mathcal{E}_n} K_a$ and $C = \bigcap_{n=1}^{\infty} C_n$.
We will prove the stated properties.
(1) C is compact since all C_n are compact, since all K_a are compact.
(2)We have $\lambda(C) = \lim_{n\to\infty} \lambda(C_n) \geq \lambda(K) - \frac{1}{2}\min\{\epsilon, \lambda(K)\}$. Hence $\lambda(C) > \lambda(K) - \epsilon$ and $\lambda(C) > 0$.
(3) Follows from Lemma 5.10.
(4) Since $\lambda(C) > 0$ we have that $(C, \mathcal{B}(C), \frac{\lambda}{\lambda(C)})$ is a well-defined measure

space. We now construct a homeomorphism ϕ between C and $\bigotimes_{i=1}^{\infty}\{1,...,m_i\}$ that is also an isomorphism between the measure spaces $(C, \mathcal{B}(C), \frac{\lambda}{\lambda(C)})$ and $(\bigotimes_{i=1}^{\infty}\{1,...,m_i\}, \mu)$. Let $x \in C$ then there is a unique sequence $a^x = (a_1^x, a_2^x, ...)$ with $1 \le a_i \le m_i$ such that, for $n \ge 1$ we have $x \in K_{(a_1^x, a_2^x, ..., a_n^x)}$. Furthermore, since for every sequence $a = (a_1, a_2, ..)$ with $1 \le a_i \le m_i$ we have $diam(K_{(a_1,....,a_n)}) \to 0$ as $n \to \infty$ and since for $n \ge 1$ we have that $K_{(a_1,...,a_n)}$ is closed, we have that the intersection $\bigcap_{n=1}^{\infty} K_{(a_1,...,a_n)}$ contains exactly a single point. Therefore every $x \in C$ corresponds to a unique sequence in $\Omega := \bigotimes_{i=1}^{\infty}\{1,...,m_i\}$ and we can define a bijection $\phi : C \to \Omega$ as $\phi(x) = (a_1^x, a_2^x, ...)$. We show that ϕ is a homeomorphism. For $a \in \mathcal{E}_n := \bigotimes_{i=1}^{n}\{1,...,m_i\}$ set $\Omega_a = \{x \in \Omega : x_1 = a_1, ..., x_n = a_n\}$ so that $\bigcup_{n=1}^{\infty}\{\Omega_a : a \in \mathcal{E}_n\}$ is a basis for the topology of Ω. Then, for $a \in \mathcal{E}_n$ we have that $\phi^{-1}(\Omega_a) = K_a \cap C$ which is open in C, hence ϕ is continuous. Now the set $\mathcal{B} = \{K_a \cap C : K_a \in \mathcal{C}_n \text{ for some } n \ge 1\}$ is by construction of Lemma 5.10 a basis for the topology of C. Further, for $K_a \in \mathcal{C}_n$ we have $\phi(K_a \cap C) = \Omega_a$ is open in Ω. Hence ϕ^{-1} is continuous. Thus ϕ is a homeomorphism.

To see that this homeomorphism is also an isomorphism between the measure spaces we note that we have for every $K_a \in \mathcal{C}_n$ that $\frac{\lambda(K_a \cap C)}{\lambda(C)} = \lim_{j\to\infty} \frac{\lambda(K_a \cap C_j)}{\lambda(C_j)} = \lim_{j\to\infty} \frac{1}{|\mathcal{E}_n|} = \frac{1}{|\mathcal{E}_n|} = \mu(\Omega_a) = \mu(\phi(K_a \cap C))$. Now, since $\bigcup_{n=1}^{\infty}\{\bigcup_{a\in\mathcal{I}} K_a \cap C : \mathcal{I} \subseteq \mathcal{E}_n\}$ is a π-system, and since it generates all Borel sets $\mathcal{B}(C)$, we have by Theorem 2.6 that the measure $\frac{\lambda}{\lambda(C)}$ equals $\mu \circ \phi$ for all Borel measurable sets.

□

5.7. **Finishing proof of Theorem 5.2.** We now finish the proof of theorem 5.2 by constructing, for arbitrary mean zero $f \in L_\infty([0,1], \lambda)$ with $\lambda(f^{-1}(\{x\})) = 0$ for all $x \in \mathbb{R}$, a measure preserving transformation T and a function $g \in L_\infty([0,1], \lambda)$ such that $f = g \circ T - g$. We will do this by defining measurable sets $A_n \subseteq [0,1]$ for $n = 1, ...$ such that

(1) Every set A_n is compact.
(2) Different sets A_n, A_m are disjoint.
(3) $\int_{A_n} f d\lambda = 0$
(4) $\lambda([0,1] \setminus \bigcup_{n=1}^{\infty} A_n) = 0$
(5) f is continuous on A_n.
(6) There is a sequence (m_i) corresponding to A_n such that there is a homeomorphism ϕ_n from A_n to $\Omega_n := \bigotimes_{i=1}^{\infty}\{1,...,m_i\}$ that is also an isomorphism between the measure spaces $(A_n, \frac{\lambda}{\lambda(A_n)})$ and (Ω_n, μ_n)

Once we have that we can use the proof of Kwapień's theorem for continuous functions on Ω_n with $f|_{A_n} \circ \phi_n^{-1}$ to get measure preserving T_n of Ω_n and $g_n \in L_\infty(\Omega_n, \mu_n)$ with $||g_n||_\infty < 4||f|_{A_n} \circ \phi_n^{-1}||_\infty + \frac{1}{2}\epsilon_{Theorem} = 4||f_{A_n}|| + \frac{1}{2}\epsilon_{Theorem}$ and so that $f|_{A_n} \circ \phi_n^{-1} = g_n \circ T_n - g_n$. Hence, we can set $\tilde{g}_n = g_n \circ \phi_n$ and $\tilde{T}_n = \phi_n^{-1} \circ T_n \circ \phi_n$, the last being a measure preserving transformation of A_n. Further we can define the transformation $T : [0,1] \to [0,1]$ as $\tilde{T}_n$ on A_n and $T(x) = x$ for $x \notin A_n$ and define g as $\tilde{g}_n$ on A_n. Then we have $f = g \circ T - g$. Further $g \in L_\infty([0,1], \lambda)$ with $||g||_\infty \le \sup\{||\tilde{g}_n||_\infty : n \ge$

$1\} \leq \sup\{4||f|_{A_n}||_\infty + \frac{1}{2}\epsilon_{Theorem} : n \geq 1\} < 4||f|| + \epsilon_{Theorem}$. Furthermore, T is a measure preserving transformation since all transformations $\tilde{T}_n$ are measure preserving and, since further the identity on $[0,1] \setminus \bigcup_{i=1}^\infty A_i$ is also measure preserving.

We thus construct the sets A_n. We define them inductively. For $n \geq 1$ set $D_n = [0,1] \setminus \bigcup_{j=1}^{n-1} A_j$ which is open in $[0,1]$ hence locally compact. Further, as D_n is a metric space it is also Hausdorff. Further, for D_n we have $\int_{D_n} f d\lambda = \int_{[0,1]} f d\lambda - \sum_{j=1}^{n-1} \int_{A_j} f d\lambda = 0$. Set $\tau_n^{\pm} = \lambda(\{f|_{D_n}^{\pm} \geq \frac{1}{2}||f|_{D_n}^{\pm}||\})$ and set $z_n = (1 + 2||f|_{D_n}||_\infty \max\{\frac{1}{||f|_{D_n}^+||_\infty}, \frac{1}{||f|_{D_n}^-||_\infty}\})$. We can now choose $\epsilon_n > 0$ with $\epsilon_n < \frac{1}{4}\min\{\frac{1}{z_n}\lambda(D_n), \tau_n^+ \frac{||f^+|_{D_n}||_\infty}{||f|_{D_n}||_\infty}, \tau_n^- \frac{||f^+|_{D_n}||_\infty}{||f|_{D_n}||_\infty}\}$. Now, since D_n is locally compact, Hausdorff, and since the Lebesgue measure λ is regular by Theorem 2.9, we can apply Lusin's theorem, theorem 2.10, to get a compact set $E_n \subseteq D_n$ of measure $\lambda(E_n) > \lambda(D_n) - \epsilon_n$ and so that f is continuous on E_n. Now $\epsilon_n < \frac{1}{4}\min\{\tau^+ \frac{||f^+|_{D_n}||_\infty}{||f|_{D_n}||_\infty}, \tau^- \frac{||f^+|_{D_n}||_\infty}{||f|_{D_n}||_\infty}\}$ so that we can then apply Lemma 5.5 on $E_n \subseteq D_n$ with $f|_{D_n}$ and ϵ_n to get a compact subset $K_n \subseteq E_n$ of measure $\lambda(K_n) > \lambda(E_n) - z_n\epsilon_n \geq \lambda(E_n) - \frac{1}{4}\lambda(D_n)$ such that $\int_{K_n} f d\lambda = 0$. Now, we can apply Lemma 5.11 on K_n with $f|_{K_n}$ and $\frac{1}{4}\lambda(D_n)$ to get a compact subset $A_n \subseteq K_n$ with $\lambda(A_n) \geq \lambda(K_n) - \frac{1}{4}\lambda(D_n) \geq \lambda(E_n) - \frac{1}{2}\lambda(D_n) \geq \frac{1}{4}\lambda(D_n)$, and $\int_{A_n} f d\lambda = 0$ and such that there is a homeomorphism between A_n and $\bigotimes_{i=1}^\infty \{1, ..., m_i\}$ for some sequence (m_i) with $m_i \geq 2$, that maps the measure $\frac{\lambda}{\lambda(A_n)}$ to the Cantor measure. We now see that the stated properties hold, hence this finishes the proof.

6. Representation of mean zero functions as sum of two coboundaries

We will now use Theorem 4.1 and Theorem 5.2 to show that any mean zero function $f \in L_\infty([0,1], \lambda)$ can be written as the sum of two coboundaries, that is $f = (g_1 \circ T_1 - g_1) + (g_2 \circ T_2 - g_2)$ with $g_1, g_2 \in L_\infty([0,1], \lambda)$ and T_1, T_2 being measure preserving transformations of $[0,1]$. Furthermore we can obtain a bound of $||g_1||_\infty$ using $||f||_\infty$ and we can get $||g_2||_\infty$ arbitrary small. We further note that the proof can, like we did in 5.4, be extended to general standard measure spaces. Further, in Section 7 we will apply this result in the field of singular traces.

Theorem 6.1. *Let $f \in L_\infty([0,1], \mathcal{B}([0,1]), \lambda)$ be mean zero, and choose $\epsilon > 0$. Then there exist $g_1, g_2 \in L_\infty([0,1], \lambda)$ with $||g_1||_\infty < 4||f||_\infty + \epsilon$ and $||g_2||_\infty < \epsilon$ and there exist measure preserving transformations T_1, T_2 of $[0,1]$ such that $f = (g_1 \circ T_1 - g_1) + (g_2 \circ T_2 - g_2)$.*

Proof. In this proof we will construct a continuous mean zero function h such that for $f - h$ we have $\lambda((f-h)^{-1}(\{c\})) = 0$ for all $c \in \mathbb{R}$. In this way we can apply Theorem 5.2 to $f - h$ and Theorem 4.1 to h, such that we we get $f = (f-h)+h = (g_1 \circ T_1 - g_1) + (g_2 \circ T_2 - g_2)$ with the stated properties.

Define the set $A := \{a \in \mathbb{R} : \exists c_a \in \mathbb{R} : \lambda(\{x \in [0,1] : f(x) = ax + c_a\}) > 0\}$. Now for $a \in A$ let us denote the set $G_a := \{x \in [0,1] : f(x) = ax + c_a\}$ for which we thus have $\lambda(G_a) > 0$.

Suppose A is uncountable. Then, since we have $A = \bigcup_{n=1}^{\infty}\{a \in A : \lambda(G_a) > \frac{1}{n}\}$, there must be a $\delta > 0$ such that $\lambda(G_a) > \delta$ for uncountably many $a \in A$. Now choose a sequence $(a_i)_{i\geq 1}$ in A with $a_i \neq a_j$ whenever $i \neq j$ and such that for $i \geq 1$ we have $\lambda(G_{a_i}) > \delta$. Now for $a_i \neq a_j$ we have that $G_{a_i} \cap G_{a_j}$ contains at most one point, since the equation $a_i x + c_{a_i} = a_j x + c_{a_j}$ has only one solution. We thus have $\lambda(\bigcup_{i=1}^{\infty} G_{a_i}) = \lambda(G_{a_1}) + \lambda(\bigcup_{i=2}^{\infty} G_{a_i}) - \lambda(\bigcup_{i=2}^{\infty}(G_{a_i} \cap G_{a_1})) = \lambda(G_{a_1}) + \lambda(\bigcup_{i=2}^{\infty} G_{a_i})$. Repeating this we get $\lambda(\bigcup_{i=1}^{\infty} G_{a_i}) = \sum_{i=1}^{\infty} \lambda(G_{a_i}) \geq \sum_{i=1}^{\infty} \delta = \infty$. However, this is obviously wrong since $\bigcup_{i=1}^{\infty} G_{a_i} \subseteq [0,1]$. Hence we must have that A is countable.

Now, since A is countable, we can choose $b \in [0, \frac{1}{4}\epsilon) \setminus A$. Thus, since $b \notin A$ we have for all $c \in \mathbb{R}$ that $\lambda(\{x \in [0,1] : f(x) = bx + c\}) = 0$. Now, define $h \in L_\infty([0,1], \lambda)$ as $h(x) := b(x - \frac{1}{2})$, which is continuous and mean zero and has norm $||h||_\infty = \frac{1}{2}b < \frac{1}{8}\epsilon$. Previous statement says exactly that we have $\lambda((f-h)^{-1}(\{c\})) = \lambda(\{x \in [0,1] : f(x) - bx - \frac{1}{2}b = c\}) = 0$ for all $c \in \mathbb{R}$. Further, $f - h$ is mean zero since f and h are mean zero. Hence, we can apply Theorem 5.2 to the functions $f - h$ to get a $g_1 \in L_\infty([0,1], \lambda)$ with $||g_1||_\infty < 4||f-h||_\infty + \frac{1}{2}\epsilon \leq 4(||f||_\infty + ||h||_\infty) + \frac{1}{2}\epsilon < 4||f||_\infty + \epsilon$ and a measure preserving T_1 of $[0,1]$ such that $f - h = g_1 \circ T_1 - g_1$. Further, since h is continuous and mean zero, we can apply Theorem 4.1 to get a function $g_2 \in L_\infty([0,1], \lambda)$ with $||g_2||_\infty < 4||h||_\infty + \frac{1}{2}\epsilon < \epsilon$ and a measure preserving transformation T_2 of $[0,1]$ such that $h = g_2 \circ T_2 - g_2$. Now we have $f = (f-h) + h = (g_1 \circ T_1 - g_1) + (g_2 \circ T_2 - g_2)$. This finishes the proof. □

7. Application of Kwapień's Theorem

To empathise the importance of Kwapien's theorem in its general form, we now conclude with an application in singular traces. We will give a proof of Theorem 7.1, as done in [9], which uses Kwapień's theorem. In [9] this theorem is then used for other applications in the field of singular traces. We will not go deeper into this as it is quite advanced material and outside the scope of this bachelor thesis.
We further note that, in [4] a weaker version of Kwapień's theorem was proved, which says that we can write a mean zero functions f on $[0,1]$ as sum $f = \sum_{i=1}^{k} g_i \circ T_i - g_i$ with k at most 20, and where $g_i \in L_\infty([0,1])$ and where T_i is measure preserving. This weaker version was already sufficient to prove Theorem 7.1.

We will introduce some notation from singular traces. The terminology we use can be found in more detail in [9] and [3]. We consider the set $\mathcal{L}(0,1)$ of measurable functions $f : (0,1) \to \mathbb{R}$ and likewise the set $\mathcal{L}(0,\infty)$ of measurable functions on $(0,\infty)$. We let $\sim$ be the equivalence relation of equality, almost everywhere. For such function $f \in \mathcal{L}(0,1)/\sim$ and $f \in \mathcal{L}(0,1)/\sim$ we let $\mu(f)$ denote the decreasing rearrangement of $|f|$, that is, $\mu(f)(t) = \inf\{s \geq 0 : \lambda(|f| > s) \leq t\}$. Now let $S(0,1)$ be the set of all measurable functions on $(0,1)$ and let $S(0,\infty)$ be the set of all measurable functions f on $(0,\infty)$ such that $\lambda(\{|f|>s\})<\infty$ for large enough s. A symmetric function space E on $(0,1)$ is a Banach space $E\subseteq S(0,1)$ such that $f\in E$, $g\in S(0,1)$ and $\mu(g)\le\mu(f)$ imply $g\in E$ and such that furthermore for $f,g\in E$ with $\mu(g)\le\mu(f)$ we have for the norm $||g||\le||f||$. The definition for a symmetric function space on $(0,\infty)$ is similar, by replacing $(0,1)$ by $(0,\infty)$. For a symmetric function space E on $(0,1)$ respectively $(0,\infty)$ we define the following.

We let

$$D_E=Span\{x\in E:x=\mu(x)\}=\{\mu(a)-\mu(b):a,b\in E\} \tag{6}$$

and further let

$$Z_E=Span\{x_1-x_2:0\le x_1,x_2\in E,\mu(x_1)=\mu(x_2)\} \tag{7}$$

Now, for the case $(0,\infty)$ we will furthermore define a function $C:(L_\infty+L_1)(0,\infty)\to S(0,\infty)$ by

$$C(x)=\frac{1}{t}\int_0^t x(s)ds \tag{8}$$

and for the case $(0,1)$ we take $C:L_1(0,1)\to S(0,1)$ defined in the same way.

We now state a theorem from [9, Theorem 4.5.1] which we will prove.

Theorem 7.1. *We have the following*

1. *Let E be a symmetric function space on $(0,\infty)$ and let $x\in D_E$. We have $x\in Z_E$ if and only if $Cx\in E$.*
2. *Let E be a symmetric function space on $(0,1)$ and let $x\in D_E$. We have $x\in Z_E$ if and only if we have $Cx\in E$ and $\int_0^1 xd\lambda=0$.*

The 'only if' part of the statements does not use Kwapień's theorem, its proof can be found in [9, Theorem 4.5.1]. To prove the 'if' part of Theorem 7.1, we will use Theorem 6.1 and we will additionally need the following result from [3, Theorem 5.11] about the dilation operator on symmetric function spaces.

Theorem 7.2. *Let E be a symmetric function space. Then the dilation operator σ_s defined by $\sigma_s(f)(t)=f(\frac{t}{s})$ maps E to itself.*

We give the proof of the 'if' part of the Theorem 7.1.

Proof of the 'if' part of Theorem 7.1. (1) Let E be a symmetric function space on $(0,\infty)$ and let $x\in D_E$. Assume that $Cx\in E$, we will show that $x\in Z_E$. For $n\in\mathbb{Z}$ let $I_n=(2^n,2^{n+1}]$ and define the partition $\mathcal{A}=\{I_n:n\in\mathbb{Z}\}$ of $(0,\infty)$. Further set $x_1=\mathbb{E}(x|\sigma(\mathcal{A}))$. Since $x\in D_E$ we can write $x=\mu(a)-\mu(b)$ with $a,b\in E$, and hence $x_1=\mathbb{E}(\mu(a)|\sigma(\mathcal{A}))-\mathbb{E}(\mu(b)|\sigma(\mathcal{A}))$. Now, since $\mu(a)$ is decreasing we see that $\mathbb{E}(\mu(a)|\sigma(\mathcal{A}))(t)\le\mu(a)(2^{\lfloor\log_2(t)\rfloor})\le\mu(a)(\frac{t}{2})$ hence $\mathbb{E}(\mu(a)|\sigma(\mathcal{A}))\le\sigma_2(\mu(a))$ where σ_2 is the dilation operator. Likewise we have $\mathbb{E}(\mu(b)|\sigma(\mathcal{A}))\le\sigma_2(\mu(b))$. Now $\sigma_2(\mu(a))$ and $\sigma_2(\mu(b))$ are in E by Theorem 7.2. Now, by the given inequalities and by definition of E, we also have that $\mathbb{E}(\mu(a)|\sigma(\mathcal{A}))$ and $\mathbb{E}(\mu(b)|\sigma(\mathcal{A}))$ are in E. Hence also $x_1\in E$.

Now, since $\mu(a) \geq 0$ and decreasing we have for $t \in I_n$ that $|\mu(a)(t) - \mathbb{E}(\mu(a)|\sigma(\mathcal{A}))| \leq \max\{\mu(a)(t), \mathbb{E}(\mu(a)|\sigma(\mathcal{A}))\} \leq \mu(a)(\frac{t}{2})$ and hence by definition of the conditional expectation we have

$$\begin{aligned}\left|\frac{1}{t}\int_0^t \mu(a) - \mathbb{E}(\mu(a)|\sigma(\mathcal{A}))(t)d\lambda\right| &= \left|\frac{1}{t}\int_n^t \mu(a) - \mathbb{E}(\mu(a)|\sigma(\mathcal{A}))(t)d\lambda\right| \\ &\leq \frac{1}{t}\int_{2^n}^t |\mu(a) - \mathbb{E}(\mu(a)|\sigma(\mathcal{A}))|\, d\lambda \\ &\leq \frac{t-2^n}{t}\mu(a)(\frac{t}{2}) \\ &\leq \frac{1}{2}\sigma_2(\mu(a))(t)\end{aligned}$$

Hence

$$\begin{aligned}|C(x_1) - C(x)| &= |C(x_1 - x)| \\ &\leq \left|\frac{1}{t}\int_0^t \mu(a) - \mathbb{E}(\mu(a)|\sigma(\mathcal{A}))d\lambda\right| \\ &+ \left|\frac{1}{t}\int_0^t \mu(b) - \mathbb{E}(\mu(b)|\sigma(\mathcal{A}))d\lambda\right| \\ &\leq \frac{1}{2}\sigma_2(\mu(a) + \mu(b))\end{aligned}$$

Now we have $\mu(a), \mu(b) \in E$ thus also $\mu(a) + \mu(b) \in E$. Now, again by theorem 7.2 this means that also $\sigma_2(\mu(a) + \mu(b)) \in E$. Now we have $\mu(C(x_1) - C(x)) = \mu(|C(x_1) - C(x)|) \leq \frac{1}{2}\sigma_2(\mu(a) + \mu(b))$. Thus, by definition of E we have that also $C(x_1) - C(x) \in E$. Now, since by assumption $C(x) \in E$, we also have that $C(x_1) \in E$. We now define the function:

$$z(t) = C(x_1)(2^{n+1}) \text{ for } t \in I_n \text{ for some } n \in \mathbb{Z}$$

We have on $(2^n, 2^{n+1})$ that $C(x_1)'(t) = \frac{-1}{t^2}\int_0^t x_1 d\lambda + \frac{1}{t}x_1(t) = \frac{x_1(t) - C(x_1)(t)}{t}$. Now, since x_1 is constant on I_n, it follows that $|C(x_1)'(t)|$ is decreasing on $(2^n, 2^{n+1})$. Now, for $t \in I_n$ we either have $\sqrt{2}t \in I_n$ or $\frac{t}{\sqrt{2}} \in I_n$. Suppose $\sqrt{2}t \in I_n$, then we get

$$\begin{aligned}|C(x_1)(2^{n+1})| &\leq |C(x_1)(t)| + |2^{n+1} - t| \cdot \frac{|C(x_1)(2^{n+1}) - C(x_1)(t)|}{|2^{n+1} - t|}| \\ &\leq |C(x_1)(t)| + |2^{n+1} - t| \cdot \frac{|C(x_1)(\sqrt{2}t) - C(x_1)(t)|}{|\sqrt{2}t - t|} \\ &= |C(x_1)(t)| + \frac{2^{n+1} - t}{(\sqrt{2} - 1)t} \cdot |C(x_1)(\sqrt{2}t) - C(x_1)(t)| \\ &\leq |C(x_1)(t)| + \frac{1}{\sqrt{2} - 1} \cdot |C(x_1)(\sqrt{2}t) - C(x_1)(t)|\end{aligned}$$

Otherwise, if $\frac{t}{\sqrt{2}} \in I_n$, we have

$$\begin{aligned}|C(x_1)(2^{n+1})| &\le |C(x_1)(t)| + |2^{n+1}-t|\cdot\frac{|C(x_1)(2^{n+1})-C(x_1)(t)|}{2^{n+1}-t}\\ &\le |C(x_1)(t)| + |2^{n+1}-t|\cdot\frac{|C(x_1)(\frac{t}{\sqrt{2}})-C(x_1)(t)|}{|\frac{t}{\sqrt{2}}-t|}\\ &= |C(x_1)(t)| + \frac{2^{n+1}-t}{(1-\frac{1}{\sqrt{2}})t}\cdot|C(x_1)(\frac{t}{\sqrt{2}})-C(x_1)(t)|\\ &\le |C(x_1)(t)| + \frac{1}{1-\frac{1}{\sqrt{2}}}\cdot|C(x_1)(\frac{t}{\sqrt{2}})-C(x_1)(t)|\end{aligned}$$

Hence we have $|z| \le |C(x_1)| + \frac{1}{\sqrt{2}-1}|\sigma_{\frac{1}{\sqrt{2}}}C(x_1) - C(x_1)| + \frac{1}{1-\frac{1}{\sqrt{2}}}|\sigma_{\sqrt{2}}C(x_1) - C(x_1)|$ and this shows that $z \in E$.

Further, we have for $n \in \mathbb{N}$ that

$$\begin{aligned}\int_{[2^n,2^{n+1}]} 2z(t)-\sigma_2(z)(t)dt &= 2^n(2C(x_1)(2^{n+1}) - C(x_1)(2^n))\\ &= 2^n(\frac{2}{2^{n+1}}\int_0^{2^{n+1}} x_1(t)dt - \frac{1}{2^n}\int_0^{2^n} x_1(t)dt)\\ &= \int_{[2^n,2^{n+1}]} \mathbb{E}(x|\sigma(\mathcal{A}))(t)dt\\ &= \int_{[2^n,2^{n+1}]} x(t)dt\end{aligned}$$

Hence, for every set $A \in \sigma(\mathcal{A})$ the integral of $2z - \sigma_2 z$ over A equals the integral of x over A. This means that $x_1 = \mathbb{E}(x|\sigma(\mathcal{A})) = 2z - \sigma_2 z$.

Set $z_a(t) = \frac{1}{2^{n+1}}\int_0^{2^{n+1}} \mu(a)d\lambda$ for $t \in I_n$. Now let $D = \bigcup_{n\in\mathbb{Z}}(2^n, 2^n + 2^{n-1})$ and define the transformations $\tau_1 : D \to (0,\infty)$ and $\tau_2 : D^c \to (0,\infty)$ as $\tau_1(t) = t - 2^{n-1}$ for $t \in I_n$ and $\tau_2(t) = t - 2^n$ for $t \in I_n$. Since both these transformations are just translations on the sets I_n, they are one-sided measure preserving transformations. We then have $z_a(\tau_1(t)) = \sigma_2(z_a)(t)$ on D and similarly $z_a(\tau_2(t)) = \sigma_2(z_a)(t)$ on D^c. Hence $\sigma_2(z_a) = z_a \circ \tau_1 \cdot \mathbb{1}_D + z_a \circ \tau_2 \cdot \mathbb{1}_{D^c}$. Now, by [7, Theorem 7.2] we have that $\mu(z_a \circ \tau_1) = \mu(z_a \circ \tau_2) = \mu(z_a)$. Therefore we have $\mu(\sigma_2(z_a)) = 2\mu(z_a)$. Further we have $z_a \ge 0$. Now, we have that $x_1 = 2z - \sigma_2(z) = (2z_a - \sigma_2(z_a)) - (2z_b - \sigma_2(z_b))$. Now this means that $x_1 \in Z_E$.

Consider the function $x_1 - x$ on I_n We have that $x_1 - x$ is mean zero on I_n and further, since we have $||x\mathbb{1}_{I_n}||_\infty \le ||\mu(a)\mathbb{1}_{I_n} - \mu(b)\mathbb{1}_{I_n}||_\infty \le \mu(a)(2^n) + \mu(b)(2^n) < \infty$ we have that $||(x_1 - x)\mathbb{1}_{I_n}||_\infty < 2||x\mathbb{1}_{I_n}||_\infty < \infty$ so that $(x_1 - x)\mathbb{1}_{I_n} \in L_\infty$. Therefore, we can apply theorem 6.1 so that we can write $(x_1 - x)\mathbb{1}_{I_n} = (g_{1,n} \circ T_{1,n} - g_{1,n}) + (g_{2,n} \circ T_{2,n} - g_{2,n})$ with $g_{1,n}, g_{2,n} \in L_\infty(I_n, \lambda)$ with $||g_{1,n}||_\infty < 4||(x_1 - x)\mathbb{1}_{I_n}||_\infty + \epsilon$ and $||g_{2,n}||_\infty < \epsilon$ and with $T_{1,n}, T_{2,n}$ being measure preserving transformations of I_n. If $(x_1 - x)\mathbb{1}_{I_n} = 0$ then we can take $g_1 = g_2 = 0$. Hence, in all cases we can take $\epsilon = ||(x_1 - x)\mathbb{1}_{I_n}||_\infty$.

We now define functions $y_{i,n}$ supported on I_n as $y_{1,n} = g_{1,n} \circ T_{1,n} + ||g_{1,n}||_\infty$, $y_{2,n} = g_{1,n} + ||g_{1,n}||_\infty$, $y_{3,n} = g_{2,n} \circ T_{2,n} + ||g_{2,n}||_\infty$ and $y_{4,n} = g_{2,n} + ||g_{2,n}||_\infty$. We then have $(x_1 - x)\mathbb{1}_{I_n} = (y_{1,n} - y_{2,n}) + (y_{3,n} - y_{4,n})$ and further $0 \leq y_{1,n}, y_{2,n}, y_{3,n}, y_{4,n}$. Further, as $y_{1,n} = y_{2,n} \circ T_{1,n}$ and $y_{3,n} = y_{4,n} \circ T_{2,n}$ and $T_{1,n}, T_{2,n}$ measure preserving, we have by [7, Theorem 7.2] that $\mu(y_{1,n}) = \mu(y_{2,n})$ and $\mu(y_{3,n}) = \mu(y_{4,n})$. Now for $i = 1, 2, 3, 4$ set $y_i = \sum_{n \in \mathbb{Z}} y_{i,n}$. We further have for $t \in I_n$ that $|(x_1 - x)(t)| \leq |\mathbb{E}(\mu(a)|\sigma(\mathcal{A}))(t) - \mu(a)(t)| + |\mathbb{E}(\mu(b)|\sigma(\mathcal{A}))(t) - \mu(b)(t)| \leq 2\mu(a)(\frac{t}{2}) + 2\mu(b)(\frac{t}{2}) = 2\sigma_2(\mu(a) + \mu(b))$. Now we thus have $||(x_1 - x)\mathbb{1}_{I_n}||_\infty \leq ||2\sigma_2(\mu(a) + \mu(b))\mathbb{1}_{I_n}||_\infty \leq 2\sigma_4(\mu(a) + \mu(b))$ This means that on I_n we have $y_i < 10||(x_1 - x)\mathbb{1}_{I_n}||_\infty \leq 20\sigma_4(\mu(a) + \mu(b))$ and hence $y_i \in E$. Now further $0 \leq y_1, y_2, y_3, y_4$. Now since $x_1 - x = (y_1 - y_2) + (y_3 - y_4)$ we have $x_1 - x \in Z_E$. Now, as we already saw that $x_1 \in Z_E$, we have that also $x \in Z_E$. This finishes the proof for this case.

(2) We now do the proof for the case that E is a symmetric function space on $(0, 1)$. Choose $x \in D_E$ with $\int_0^1 x d\lambda = 0$ and such that $C(x) \in E$. We first define for $n \leq -1$ the interval $I_n = [2^n, 2^{n+1})$ and let $\mathcal{A} = \{I_{-n} : n \in \mathbb{N}\}$ be our partition of $(0, 1)$. Since $x \in D_E$ we can write $x = \mu(a) - \mu(b)$ with $a, b \in E$. Now we set $x_1 = \mathbb{E}(x|\sigma(\mathcal{A})) = \mathbb{E}(\mu(a)|\sigma(\mathcal{A})) - \mathbb{E}(\mu(b)|\sigma(\mathcal{A}))$. Again we have $\mathbb{E}(\mu(a)|\sigma(\mathcal{A})) \leq \sigma_2(\mu(a))$ and $\mathbb{E}(\mu(b)|\sigma(\mathcal{A})) \leq \sigma_2(\mu(b))$ which means that $x_1 \in E$.

We can now do the same calculation as before to obtain that $C(x_1) \in E$. We can now define $z : E \to S(0, 1)$ as

$$z(t) = \begin{cases} C(x_1)(2^{n+1}) & t \in I_n \text{ for some } n \leq -2 \\ 0 & t \in I_{-1} \end{cases} \tag{9}$$

We again have $C(x_1)'(t) = \frac{x_1(t) - C(x_1)(t)}{t}$ and since x_1 is constant on each I_n, we have that $|C(x_1)'|$ is decreasing on each interval $(2^n, 2^{n+1})$. Now, for $n \leq -2$ we have, by the same calculation as in the case for $(0, \infty)$, on I_n the bound $|z| \leq |C(x_1)| + \frac{1}{\sqrt{2}-1}|\sigma_{\frac{1}{\sqrt{2}}} C(x_1) - C(x_1)| + \frac{1}{1-\frac{1}{\sqrt{2}}}|\sigma_{\sqrt{2}} C(x_1) - C(x_1)|$. Now, this bound also holds on I_{-1} since $z = 0$ on I_{-1}. This means that $z \in E$.

Now, as in the case on $(0, \infty)$, we get for $n \leq -2$ that $\int_{I_n} 2z(t) - \sigma_2(z)(t)dt - \int_{I_n} x dt$. This means that $x_1 = 2z - \sigma_2(z)$ on $(0, \frac{1}{2})$. Now we also have, in the same manner as done on $(0, \infty)$ that $\mu(2z) = \mu(\sigma_2(z))$. Now, this means that $\int_{(0,1)} 2z - \sigma_2(z) d\lambda = 0$. Hence, since x_1 is also mean zero, and since $x_1 = 2z - \sigma_2(z)$ on $(0, \frac{1}{2})$, we also have $\int_{I_{-1}} x_1 = \int_{I_{-1}} 2z - \sigma_2(z) d\lambda$. Hence, by definition of the conditional expectation, we now have $x_1 = 2z - \sigma_2(z)$ on $(0, 1)$. Now, again we obtain $x_1 \in Z_E$.

We can now again consider the function $x_1 - x$ on I_n for some $n \leq -1$. We have $||x\mathbb{1}_{I_n}||_\infty \leq \mu(a)(2^n) + \mu(b)(2^n) < \infty$ and hence again $||(x_1 - x)\mathbb{1}_{I_n}|| < \infty$ so that $(x_1 - x)\mathbb{1}_{I_n} \in L_\infty(I_n)$. Now, we can continue as in the case for $(0, \infty)$ to obtain the functions y_1, y_2, y_3, y_4 from which follows that

$x_1 - x \in Z_E$. Hence we have $x \in Z_E$. This finishes the proof.

□

8. Notes

At this moment, I am still working on the proof of Kwapień. It seems that combining a result from [4] with the theorem for nowhere constant functions yields the full result of Kwapień.

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