Optimization of the rotation angle of a quantum particle in a simple quantum system

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Contents

1 Table of Symbols

this section contains a table of all symbols used in this text.

symbol	description	mathematical notation
$\|\psi(t)\rangle$	State vector representing a pure quantum state	
$\hat{H}$	Hamiltonian represents the total energy of the corresponding system	
$\hat{\rho}$	Density matrix representing the quantum state of a quantum system	$\hat{\rho} = \sum_j p_j \|\psi_j\rangle \langle\psi_j\|$
$Tr\left(\hat{A}\right)$	trace sum of the elements on the main diagonal of a matrix	$\mathrm{Tr}\left(\hat{A}\right) = \sum_{i=1}^{n} a_{ii}$
$Tr_A\left(\hat{A} \otimes \hat{B}\right)$	Partial trace a generalization of the trace operator	$Tr_A\left(\sum_i \hat{A}_i \otimes \hat{B}_i\right) = \sum_i Tr\left(\hat{A}_i\right)\hat{B}_i$
$< \hat{A} >$	Notation used for the expectation value of an operator $\hat{A}$	
$\hat{\Sigma}_z$	Pauli-z matrix	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
χ	Counting field	
v	value of the output of a measurement	
T_f	time interval when data is collected	
T_c	time interval during which data needs to be collected to obtain significant results	
$\hat{\Sigma}_x$	Pauli-x matrix	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
$U(\alpha)$	rotation matrix over an angle of α	$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$
α	Angle of rotation of the qubit after a measurement to return it to an equal weight superposition	
γ	Value describing the outside influences on our quantum system	
$\hat{M}$	Observable of the quantum system	
$\hat{Q}$	Operator acting on one of the qubits of the quantum system	
$\|x\rangle$	Eigenstate of the Pauli-x matrix with eigenvalue 1	$\frac{\|+\rangle + \|-\rangle}{\sqrt{2}}$
$\hat{\Sigma}_y$	Pauli-y matrix	$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

2 Introduction

The standard description of quantum mechanics say that a projective measurement is an instantaneous non-unitary process. After such a measurement the quantum system is projected onto an eigenstate of the measured observable. In reality these measurements are never instantaneous but happen over a timescale that is required to obtain a reliable measurement result(1). Continuous weak linear measurement (CWLM)(2) provides a more general and accurate description of the measurement process on a quantum system. Due to recent technological advances it is possible to study CWLM in detail for a set of quantum device setups. In this text, we study CWLM using an iterative simulation procedure. With this procedure we can simulate single quantum trajectories as well as the average trajectory of many quantum trajectories.

The quantum system used to describe a simulation works as follows. The detector is represented by a qubit. At each step of the simulation, which all have a duration of Δt, the qubit is initialized to an equal weight superposition of two states. Then, for the duration of the step, it is coupled to the system being measured. We evaluate the unitary evolution of the system and the qubit during this time. After that, the qubit is measured projectively, the measurement result counts for the detector output at this time interval, and the density matrix gets updated according to this measurement result. The quantum system that will be used for the simulation is a qubit, which initially is in the equal weight superposition $\frac{|+\rangle+|-\rangle}{\sqrt{2}}$. The qubit is measured in the basis of these two states, where $|+\rangle$ stands for spin-up and $|-\rangle$ stands for spin down. We define the direction of the spin to be along the axis of rotation of the quantum particle. As a result of the decoherence the superposition is destroyed at a certain time scale, and the density matrix becomes diagonal. The qubit is now in one of the two states that make up the superposition, and the value of the measurement result freezes at the corresponding value. The resulting signal is normalized such that the final output value of the measurement is equal to ± 1. After a measurement is complete the qubit is no longer in a superposition. The qubit will receive a rotation over an angle α so it can return to the equal weight superposition. We do this so we can continue to collect data from this system. This is interesting because the average of many data sets is less susceptible to chance than a single data set.

In subsection 2.1 we explain what a state vector and a density matrix are used for in a quantum system. Next in Section 3, we show what is theoretically expected of the spin in the x-direction. Then we optimize the rotation angle which returns the qubit in the superposition state.In subsection 3.1, we do this for a system without outside influences. In Subsection 3.2, we do the same for a system that has outside influences that have an effect on the system. In Subsection 3.3 we investigate a relation between the amount of outside influences and the behavior of the system. Then in Section 4 we discuss the simulation and the results generated by the simulation. In Subsection 4.1 we discuss the formulas needed to make the simulation. In Subsection 4.2 we discuss the quantum

trajectories generated by the simulation. In Section 5 the conclusion.

2.1 Density Matrix

In quantum mechanics we work with particles that are too small to be seen with the naked eye. This means we do not always know the exact spin (or even momentum, position, etc.) of a particle. It turns out that a particle does not have set spin direction, rather it can have spin up, spin down and it can be at both spin up and spin down at the same time in a superposition, albeit not with the same likelihood for every spin direction. We call such a particle, or a group of interacting particles, a quantum system. A quantum system can always be described by a state vector. This state vector contains a probability distribution for any observable of the system. An observable is a physical quantity, like position or momentum, that can be measured.

We recognize two situations for quantum states. These are pure states and mixed states. A pure quantum state can be described by a state vector $|\psi\rangle$ and by a density matrix $\hat{\rho}$. A mixed quantum state can only be described by the density matrix. The density matrix of a quantum system can be defined by the state vector, according to equation (1).

$$\hat{\rho} = \sum_j p_j \left|\psi_j\right\rangle \left\langle\psi_j\right| \tag{1}$$

In this equation the p_j represent the probability of the particle of the quantum system to be in the pure state $|\psi_j\rangle$. Now we can give a more mathematical description of a pure state and a mixed state. If a system is in a pure state, one p_j equals 1 and the others are all 0 in equation (1). If a system is in a mixed state, then all $p_j < 1$.

A quantum state does not have to be a static probability distribution. With other words a quantum state can change over time. If a quantum state is dependent on time there are two equations that can be solved to describe the time evolution of the quantum system. One equation can be used if the system is described by a state vector, this equation is equation (2), which is known as the Schrodinger equation. We have set $\hbar$ to be equal to 1 throughout this text.

$$i\hbar \frac{\mathrm{d}}{\mathrm{dt}} \left|\psi(t)\right\rangle = \hat{H} \left|\psi(t)\right\rangle \tag{2}$$

Another equation that can be solved to describe the time evolution of the quantum system is the Liouville–von Neumann equation, given in equation (3).

$$i\hbar \frac{\mathrm{d}\hat{\rho}}{\mathrm{dt}} = [\hat{H}, \hat{\rho}] \tag{3}$$

This equation can be used to solve for the density matrix of the system. In the system that is used in this text we start with a quantum system in a pure state. We will solve the Schrodinger equation and then we use equation (1) to find the density matrix of the system. In equation (3) the brackets denote the commutator of $\hat{H}$ and $\hat{\rho}$.

Now we will give the properties of the density matrix. The first property is that a density matrix is positive semi-definite. This means that equation (4) holds.

$$\langle k| \hat{M} |k\rangle \geq 0 \tag{4}$$

In equation (4) $\hat{M}$ is an $n \times n$ matrix and $|k\rangle$ is an n-dimensional real valued vector. The second property is that a density matrix is Hermitian. This means for a matrix $\hat{A}$ equation (5) holds.

$$a_{ij} = a^*_{ji} \tag{5}$$

The a_{ij} are the elements of the matrix $\hat{A}$. Before we give the last property we introduce the trace of a matrix. The trace of an $n \times n$ matrix $\hat{A}$ is defined by equation (6).

$$Tr\left(\hat{A}\right) = \sum_{i=1}^{n} a_{ii} \tag{6}$$

Now we know what the trace is, the third property of a density matrix is that the trace of a density matrix is always 1.

We have defined the trace of a matrix now. but the trace in this way is not always convenient to use. If we have a system of two particles, we can't use the Hilbert space of only one particle. We create a new space of the quantum system by taking the Kronecker product between the Hilbert space of the first particle and the Hilbert space of the second particle. Any operator working on this quantum system can then be written as the operator in equation (7).

$$\hat{C} = \sum_i \hat{O}_{1,i} \otimes \hat{O}_{2,i} \tag{7}$$

Here $\hat{C}$ is an operator acting on the total quantum system, $\hat{O}_{1,i}$ is an operator acting on particle 1 and $\hat{O}_{2,i}$ is an operator acting on particle 2. In a system of two or more particles the particles can become entangled. This means that measurement outcomes are dependent on each other. If two particles are entangled and one of them has a spin up, the other particle will be in a spin down state the majority of the time. We can now introduce the partial trace. This is similar to the trace that was just introduced, but it can be used on a system of two or more particles. The partial trace works as a trace operator over one matrix of the matrices in the Kronecker product. In the form of an equation the partial trace is the unique operator such that equation (8) holds.

$$Tr_A\left(\sum_i \hat{A}_i \otimes \hat{B}_i\right) = \sum_i Tr\left(\hat{A}_i\right)\hat{B}_i \tag{8}$$

We can also take the partial trace over the second matrix instead of the first, then it looks like equation (9).

$$Tr_B\left(\sum_i \hat{A}_i \otimes \hat{B}_i\right) = \sum_i \hat{A}_i Tr\left(\hat{B}_i\right) \tag{9}$$

It is useful to know if the quantum system describes a pure state or a mixed state. This can be checked. If the density matrix describes a pure state equation (10) holds.

$$\hat{\rho} = \hat{\rho}^2 \tag{10}$$

This means that for a pure state the trace of $\hat{\rho}^2$ is always equal to 1. If we can show that the trace of $\hat{\rho}^2$ is not equal to 1 we know that we do not have a system of a pure state, and thus it must be a mixed state. This will be used later in this text to show that we do have need of the density matrix over a state over to describe our quantum system.

Now we are going to define the expectation value of a measurement using the density matrix. We first recall the expectation value of a measurement for a pure state which we see in equation (11).

$$< \hat{A} >= \langle\psi| \hat{A} |\psi\rangle \tag{11}$$

We use this on each of the pure states that make up a density matrix and we find.

$$\begin{aligned} < \hat{A} >= \sum_j p_j \langle\psi_j| \hat{A} |\psi_j\rangle = \sum_j p_j Tr\left(|\psi_j\rangle \langle\psi_j| \hat{A}\right) = \\ Tr\left(\sum_j p_j \langle\psi_j| \hat{A} |\psi_j\rangle\right) = Tr\left(\hat{\rho}\hat{A}\right) \end{aligned} \tag{12}$$

If we do this, we run a measurement on the system. But a measurement changes the quantum state of the system. To see how it changes first we assume that we can write the observable $\hat{A}$ in the following way.

$$\hat{A} = \sum_i a_i |k_i\rangle \langle k_i| \tag{13}$$

After a measurement on $\hat{A}$ the density matrix changes and equation (14) holds. Note that this is the density matrix for the specific outcome of a measurement.

$$\hat{\rho}_{new} = \frac{|k_i\rangle \langle k_i| \hat{\rho} |k_i\rangle \langle k_i|}{Tr\left(\hat{\rho} |k_i\rangle \langle k_i|\right)} \tag{14}$$

Now we know what a density matrix is and how a measurement on the density matrix works. We will use this in a simulation. But for this simulation we need to apply a rotation after one measurement to make sure the qubit is in an equal weight superposition so we can perform another measurement on the quantum system.

3 Analytical Results

3.1 Ideal System

We want to use our system to perform measurements on it. We make sure our system starts out in an equal weight superposition. After a measurement is done, it will no longer be in a quantum position. If we want to use this same system for another measurement we must rotate it back to this position. However the qubit does not have to be in a spin-up or a spin-down state. It can be in a position that is only a partial spin-up or a partial spin-down. Thus the angle we need to apply a rotation over depends upon the state that is reached at the end of a measurement. In this section we will obtain an expression of the optimal angle. Then we will calculate any constant values that are contained in this expression. We will also look at what the theory predicts about the spin in the x-direction.

To calculate the optimal rotation angle first we shall use a standard differential equation for the density matrix and try to solve it. We will start with a standard differential equation for a density matrix in equation (15).

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = (\hat{\rho} - \hat{\Sigma}_z\hat{\rho}\hat{\Sigma}_z) \tag{15}$$

We will try to evaluate this equation using the counting fields method(2) (3). Before we do this we will give a short explanation on how the counting fields method works. We do not know the behavior of the quantum system. If we run a measurement it can have multiple output values. This means that the output values of the system follow a probability distribution. We now introduce a counting field $\chi(t)$ that is coupled to the output of a measurement. This counting field is the variable of the generating function of the probability distribution of the output. We now treat the evolution of the bra and the ket as slightly different, by giving them different Hamiltonians. This is done by adding/subtracting a term linear in χ to the Hamiltonian.

When we have applied this method we get equation (16), where we now solve the augmented density matrix.

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\frac{\chi^2}{8}\hat{\rho} + i\frac{\chi}{2}(\hat{\Sigma}_z\hat{\rho} + \hat{\rho}\hat{\Sigma}_z) - (\hat{\rho} - \hat{\Sigma}_z\hat{\rho}\hat{\Sigma}_z) \tag{16}$$

To solve equation (16) we will write this out in matrix form and solve for all matrix elements separately. To do this we write the density matrix as in equation (17).

$$\hat{\rho} = \begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix} \tag{17}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\rho_{++} & \rho_{+-}\\ \rho_{-+} & \rho_{--}\end{bmatrix} = -\frac{\chi^2}{8}\begin{bmatrix}\rho_{++} & \rho_{+-}\\ \rho_{-+} & \rho_{--}\end{bmatrix}$$
$$+i\frac{\chi}{2}\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\rho_{++} & \rho_{+-}\\ \rho_{-+} & \rho_{--}\end{bmatrix} + \begin{bmatrix}\rho_{++} & \rho_{+-}\\ \rho_{-+} & \rho_{--}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) \tag{18}$$
$$-\left(\begin{bmatrix}\rho_{++} & \rho_{+-}\\ \rho_{-+} & \rho_{--}\end{bmatrix} - \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\rho_{++} & \rho_{+-}\\ \rho_{-+} & \rho_{--}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right)$$

Now we have four ordinary differential equations and thus we can find the solution for all elements of the density matrix by integrating with respect to time.

$$\rho_{++}(t) = \rho_{++}(0)e^{(-\frac{\chi^2}{8}+i\chi)t} \tag{19}$$

$$\rho_{+-}(t) = \rho_{+-}(0)e^{(-\frac{\chi^2}{8}-2)t} \tag{20}$$

$$\rho_{-+}(t) = \rho_{-+}(0)e^{(-\frac{\chi^2}{8}-2)t} \tag{21}$$

$$\rho_{--}(t) = \rho_{--}(0)e^{(-\frac{\chi^2}{8}-i\chi)t} \tag{22}$$

Note that this is the augmented density matrix, and it is not Hermitian. Now we have the density matrix in terms of time and χ. Our goal was to determine the optimal rotation angle to rotate our system back to its equal weight superposition. To achieve this goal we want to express the augmented density matrix in terms of the value of a measurement. For this we use a Fourier transform. to switch from the variable χ to the value of a measurement, for which we use the variable v:

$$\hat{\rho}(v) = \frac{T}{2\pi}\int_{-\infty}^{\infty}\hat{\rho}(\chi)e^{-i\chi Tv}d\chi \tag{23}$$

In equation (23), T is the time interval data is collected from the system. Then our density matrix becomes, in terms of v:

$$\rho_{++}(v) = \sqrt{\frac{2T}{\pi}}e^{-2T(v-1)^2} \tag{24}$$

$$\rho_{+-}(v) = \sqrt{\frac{2T}{\pi}}e^{-2Tv^2}e^{-2T} \tag{25}$$

$$\rho_{-+}(v) = \sqrt{\frac{2T}{\pi}}e^{-2Tv^2}e^{-2T} \tag{26}$$

$$\rho_{--}(v) = \sqrt{\frac{2T}{\pi}}e^{-2T(v+1)^2} \tag{27}$$

Now we have the density matrix of our system in terms of the value of the measurement.This density matrix is no longer the augmented density matrix, and all the properties discussed for a density matrix hold for this matrix.

We set up a feedback cycle where we want to collect data during a measurement and return the system to the equal-weight superposition after the measurement. The data is collected during a time interval of fixed duration. The time interval in which we collect data we call T_f. During this time interval we need to collect data and we want to get significant results. We will call the time interval we need to get significant results T_c, the characteristic time scale. After a rotation we repeat the cycle. Now we have a problem. If $T_f >> T_c$ the superposition of our system will be destroyed. If $T_f << T_c$ the superposition is not destroyed, but the collected data can show large fluctuations and it might not be an accurate reflection of the final position of the system. To get a criterion for good data we take the average value of Σ_x over an entire cycle, and we define it by equation (28):

$$\bar{\Sigma}_x = \frac{1}{T_f}\int_0^{T_f} < \Sigma_x(t) > dt \tag{28}$$

In equation (28) we take $< \Sigma_x >$ to be the expected value of Σ_x. By running many simulations we we see that this value is approximately equal to e^{-2T}. After we have measured the value of v, we need to apply a rotation to get back to our initial position. This rotation is given by equation (29):

$$\hat{\rho}_a = \int_{-\infty}^{\infty} \hat{U}(\alpha(v))\hat{\rho}(v)\hat{U}^{-1}(\alpha(v))dv \tag{29}$$

When we start a new measurement we want the system to be in the initial position. If we only apply the rotation for the angle at the exact position of the last measurement, the next measurement depends on the result of the previous measurement. To make every measurement independent of other measurements we integrate the applied rotation over every possible angle. With the initial condition being $\hat{\rho}_a = (1+\rho_x\Sigma_x)/2$, we look for a solution that fits both equation (29) and the initial condition, and we find:

$$\hat{\rho}(v) = G_+\frac{1+\hat{\Sigma}_z}{4} + G_-\frac{1-\hat{\Sigma}_z}{4} + \frac{e^{-2T}G\rho_x\hat{\Sigma}_x}{2} \tag{30}$$

Here we have that $G(v) = (2T/\pi)^{1/2}e^{-2Tv^2}$,$G_+ = G(v-1)$,$G_- = G(v+1)$. We see now that a solution exists to this equation.

$$\rho_x = A + \rho_x B \tag{31}$$

$$\rho_x = \frac{A}{1-B} \tag{32}$$

With A and B given by:

$$A = \int_{-\infty}^{\infty} \frac{G_+ - G_-}{2}\sin(2\alpha(v))dv \tag{33}$$

$$B = e^{-2T} \int_{-\infty}^{\infty} G \cos(2\alpha(v)) dv \tag{34}$$

Substitution in equation (28) gives.

$$\bar{\Sigma}_x = \rho_x \frac{1 - e^{-2T}}{2T} = \frac{1 - e^{-2T}}{2T} \frac{A}{1 - B} \tag{35}$$

In equation (35) we have used equation (28) to calculate $\bar{\Sigma}_x$, where we have taken T_f to be 1 in units of T_c. In order to obtain the optimal value of the rotation angle we try to find the optimal value of $\bar{\Sigma}_x$. This means we have an optimal feedback cycle in terms of collected data.

Now we will optimize the value of $\bar{\Sigma}_x$ by differentiating to α and calculating when this derivative is zero.

$$\frac{\partial}{\partial \alpha} \bar{\Sigma}_x \propto \frac{(1 - B) \frac{\partial}{\partial \alpha} A - A \frac{\partial}{\partial \alpha} (1 - B)}{(1 - B)^2} \tag{36}$$

In equation (36) the proportional sign is used because the terms of T are independent of alpha. For simplicity these terms have been left out of the equation that we will solve. What we need to solve now is equation (37).

$$0 = \frac{\partial}{\partial \alpha} A(1 - B) - A \frac{\partial}{\partial \alpha} (1 - B) \tag{37}$$

$$\frac{\partial}{\partial \alpha} A = \frac{G_+ - G_-}{2} \cos(2\alpha(v)) 2\alpha' \tag{38}$$

$$\frac{\partial}{\partial \alpha} B = -2e^{-2T} G \sin(2\alpha(v)) \alpha' \tag{39}$$

Now by substituting equation (38) and (39) into equation (37) we get:

$$0 = \frac{G_+ - G_-}{2} \cos(2\alpha(v)) 2\alpha' (1 - B) - A 2e^{-2T} G \sin(2\alpha(v)) \alpha' \tag{40}$$

If α' is unequal to zero we can divide by this term. If it is equal to zero we do not have an equation to solve. Thus we assume that this term is nonzero and we begin to solve this equation. Now we have one equation with two unknowns. This means that there is no unique solution. We must either try to obtain a second equation or we must eliminate one unknown from this one equation. We will do the latter.

We can create another variable, z. This variable is dependent on A and B. A and B are the outcome of an integral and are thus constant. This means that z is also a constant. Now we have one unknown and one equation. This is solvable.

$$\sqrt{z} = \frac{A}{1 - B} \tag{41}$$

$$0 = \frac{G_+ - G_-}{2} \cos(2\alpha(v)) 2 - \sqrt{z} 2e^{-2T} G \sin(2\alpha(v)) \tag{42}$$

We note that we can rewrite this equation in a different way. We introduce a vector F, with $F_x = F\cos(2\alpha)$ and $F_y = F\sin(2\alpha)$. If we do this, we can see that:

$$0 = F_y \cos(2\alpha(v)) - F_x \sin(2\alpha(v)) \tag{43}$$

Combining equation (42) and (43) we find functions for F_x and F_y.

$$F_x = 2e^{-2T} G\sqrt{z} \tag{44}$$

$$F_y = G_+ - G_- \tag{45}$$

Now we can finally get an expression for our rotation angle. We simply rewrite equation (43) until we find equation (46).

$$\tan(2\alpha) = \frac{\sin(2\alpha)}{\cos(2\alpha)} = \frac{F_y}{F_x} \tag{46}$$

Rewriting this we find equation (47).

$$\alpha = \frac{1}{2}\tan^{-1}(\frac{F_y}{F_x}) \tag{47}$$

To find the optimal angle of rotation we only need to calculate the value of z. We will do this by calculating values for A and B and then solve equation (41) for z and A and B, where A and B are now dependent on z. By our definition of F, we can replace the sine and cosine in A and B.

$$A = \int_{-\infty}^{\infty} \frac{(G_+ - G_-)^2}{2F} dv \tag{48}$$

$$B = (e^{-2T})^2 \int_{-\infty}^{\infty} \frac{G^2 2\sqrt{z}}{F} dv \tag{49}$$

Where F is given by:

$$F = 2e^{-2T} G\sqrt{z + \sinh^2(4Tv)} \tag{50}$$

Our domain runs from $-\infty$ to ∞. Since A and B are even integrals , we get:

$$A = 2\int_{0}^{\infty} \frac{(G_+ - G_-)^2}{2F} dv \tag{51}$$

$$B = 2(e^{-2T})^2 \int_{0}^{\infty} \frac{G^2 2\sqrt{z}}{F} dv \tag{52}$$

Now to solve for z and thus find values for A and B, a script has been made to solve these equations numerically. This has been done using MATLAB.

We see from figure 1 that z has a value of 1, independent of time. There are small deviations which are negligible because these deviations are small compared to the value of z itself. In all figures the time will be in units of T_c.

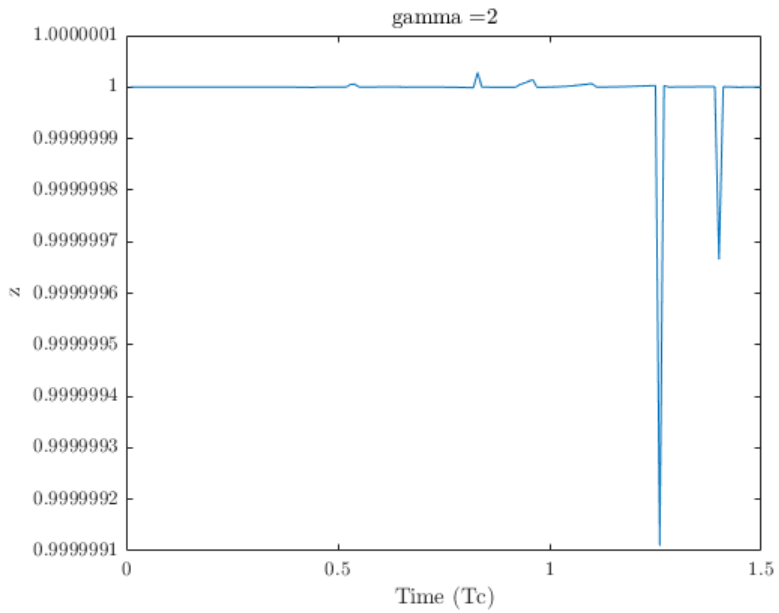


Figure 1: Value of z over time in an ideal system. z has a value of 1. the deviations are caused by numerical calculation errors in MATLAB.

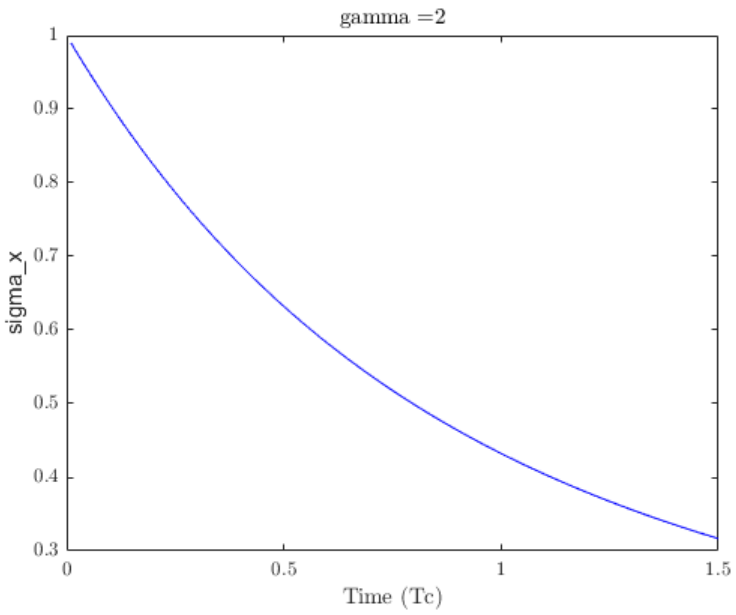


Figure 2: spin of an ideal system in the x-direction over time. The curve follows the expected shape of e^{-2T}.

When we look at figure 2 we see that indeed Σ_x has the shape that was expected, which was e^{-2T}.

3.2 Non Ideal System

We have now found the optimal angle of rotation for an ideal system. In reality a system does not have to be perfect. There could be influences from outside the system that have an effect. It is useful to also know the optimal angle of rotation when such outside influences are present. These influences enter the equation in the form of γ. γ is larger than or equal to 2. If $\gamma = 2$, we have an ideal system. If $\gamma > 2$ we have a non ideal system. In this section our goal is again to determine the optimal angle of rotation, but this time with these outside influences in mind.

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = \frac{\gamma}{2}(\hat{\rho} - \hat{\Sigma_z}\hat{\rho}\hat{\Sigma_z}) \tag{53}$$

We will try to evaluate this equation using the counting fields method(2) (3). Before we do this we will give a short explanation on how the counting fields method works.

We do not know the behavior of the quantum system. If we run a measurement it can have multiple possible output values. This means that the function of output values of the system follows a probability distribution. We introduce now a counting field $\chi(t)$ that is coupled to the output of a measurement. This counting field is the variable of the generating function of the probability distribution of the output. We now treat the evolution of the bra and the ket as slightly different, by giving them different Hamiltonians. This is done by adding/subtracting a term linear in χ to the Hamiltonian. When we have applied this method to the equation we get equation (54):

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\frac{\chi^2}{8}\hat{\rho} + i\frac{\chi}{2}(\hat{\Sigma_z}\hat{\rho} + \hat{\rho}\hat{\Sigma_z}) - \frac{\gamma}{2}(\hat{\rho} - \hat{\Sigma_z}\hat{\rho}\hat{\Sigma_z}) \tag{54}$$

To solve equation (54) we will write this out in matrix form and solve for all matrix elements separately. To do this we write the density matrix as in equation (55).

$$\hat{\rho} = \begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix} \tag{55}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix} &= -\frac{\chi^2}{8}\begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix} \\ +i\frac{\chi}{2}\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix} + \begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \\ -\frac{\gamma}{2}\left(\begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right) \end{aligned} \tag{56}$$

Now we have four ordinary differential equations and thus we can find the solution for all elements of the density matrix by integrating with respect to time.

$$\rho_{++}(t) = \rho_{++}(0)e^{(-\frac{\chi^2}{8}+i\chi)t} \tag{57}$$

$$\rho_{+-}(t) = \rho_{+-}(0)e^{(-\frac{\chi^2}{8}-\gamma)t} \tag{58}$$

$$\rho_{-+}(t) = \rho_{-+}(0)e^{(-\frac{\chi^2}{8}-\gamma)t} \tag{59}$$

$$\rho_{--}(t) = \rho_{--}(0)e^{(-\frac{\chi^2}{8}-i\chi)t} \tag{60}$$

Note that this is the augmented density matrix, and it is not Hermitian. Now we have the density matrix in terms of time and χ. Our goal was to determine the optimal rotation angle to rotate our system back to its equal weight superposition. To achieve this goal we want to express the augmented density matrix in terms of the value of a measurement. For this we use a Fourier transform. to switch from the variable χ to the value of a measurement, for which we use the variable v:

$$\hat{\rho}(v) = \frac{T}{2\pi}\int_{-\infty}^{\infty}\hat{\rho}(\chi)e^{-i\chi Tv}d\chi \tag{61}$$

In equation (61), T is the time interval data is collected from the system. Then our density matrix becomes, in terms of v:

$$\rho_{++}(v) = \sqrt{\frac{2T}{\pi}}e^{-2T(v-1)^2} \tag{62}$$

$$\rho_{+-}(v) = \sqrt{\frac{2T}{\pi}}e^{-2Tv^2}e^{-\gamma T} \tag{63}$$

$$\rho_{-+}(v) = \sqrt{\frac{2T}{\pi}}e^{-2Tv^2}e^{-\gamma T} \tag{64}$$

$$\rho_{--}(v) = \sqrt{\frac{2T}{\pi}}e^{-2T(v+1)^2} \tag{65}$$

Now we have the density matrix of our system in terms of the value of the measurement. This density matrix is no longer the augmented density matrix, and all the properties discussed for a density matrix hold for this matrix.

We set up a feedback cycle where we want to collect data during a measurement and return the system to the equal-weight superposition after the measurement. The data is collected during a time interval. The time interval where we collect data we call T_f. During this time interval we need to collect data and we want to get significant results. We will call the time interval we need to get significant results T_c, the characteristic time scale. After a rotation we repeat the cycle. Now we have a problem. If $T_f >> T_c$ the superposition of our system will be destroyed. If $T_f << T_c$ the superposition is not destroyed,

but the collected data can show large fluctuations and it might not be an accurate reflection of the final position of the system. To get a criterion for good data we take the average value of Σ_x over an entire cycle, and we define it by equation (66):

$$\bar{\Sigma}_x = \frac{1}{T_f}\int_0^{T_f} < \Sigma_x(t) > dt \tag{66}$$

In equation (66) we take $< \Sigma_x >$ to be the expected value of Σ_x. By running many simulations we we see that this value is approximately equal to $e^{-\gamma T}$. After we have measured the value of v, we need to apply a rotation to get back to our initial position. This rotation is given by equation (67):

$$\hat{\rho}_a = \int_{-\infty}^{\infty} \hat{U}(\alpha(v))\hat{\rho}(v)\hat{U}^{-1}(\alpha(v))dv \tag{67}$$

When we start a new measurement we want the system to be in the initial position. If we only apply the rotation for the angle at the exact position of the last measurement, the next measurement depends on the result of the previous measurement. To make every measurement independent of other measurements we integrate the applied rotation over every possible angle. With the initial condition being $\hat{\rho}_a = (1+\rho_x\Sigma_x)/2$, we look for a solution that fits both equation (67) and the initial condition, and we find:

$$\hat{\rho}(v) = G_+\frac{1+\hat{\Sigma}_z}{4} + G_-\frac{1-\hat{\Sigma}_z}{4} + \frac{e^{-\gamma T}G\rho_x\hat{\Sigma}_x}{2} \tag{68}$$

Here we have that $G(v) = (2T/\pi)^{1/2}e^{-2Tv^2}$,$G_+ = exp(4Tv)G(v)exp(-2T)$, $G_- = G(v)exp(-2T)exp(-4Tv)$. We see now that a solution exists to this equation.

$$\rho_x = A + \rho_x B \tag{69}$$

$$\rho_x = \frac{A}{1-B} \tag{70}$$

With A and B given by:

$$A = \int_{-\infty}^{\infty} \frac{G_+ - G_-}{2}\sin(2\alpha(v))dv \tag{71}$$

$$B = e^{-\gamma T}\int_{-\infty}^{\infty} G\cos(2\alpha(v))dv \tag{72}$$

Substitution in equation (66) gives.

$$\bar{\Sigma}_x = \rho_x\frac{1-e^{-\gamma T}}{\gamma T} = \frac{1-e^{-\gamma T}}{\gamma T}\frac{A}{1-B} \tag{73}$$

In equation (73) we have used equation (66) to calculate $\bar{\Sigma}_x$, where we have taken T_f to be 1 in units of T_c. In order to obtain the optimal value of the rotation angle we try to find the optimal value of $\bar{\Sigma}_x$. This means we have an optimal feedback cycle in terms of collected data.

Now we will optimize the value of $\bar{\Sigma}_x$ by differentiating to α and calculating when this derivative is zero.

$$\frac{\partial}{\partial\alpha}\bar{\Sigma}_x \propto \frac{(1-B)\frac{\partial}{\partial\alpha}A - A\frac{\partial}{\partial\alpha}(1-B)}{(1-B)^2} \tag{74}$$

In equation (74) the proportional sign is used because the terms of T are independent of alpha. For simplicity these terms have been left out of the equation that we will solve. What we need to solve now is equation (75).

$$0 = \frac{(1-B)\partial}{\partial\alpha}A - A\frac{\partial}{\partial\alpha}(1-B) \tag{75}$$

$$\frac{\partial}{\partial\alpha}A = \frac{G_+ - G_-}{2}\cos(2\alpha(v))2\alpha' \tag{76}$$

$$\frac{\partial}{\partial\alpha}B = -2e^{-\gamma T}G\sin(2\alpha(v))\alpha' \tag{77}$$

Now by substituting equation (76) and (77) into equation (75) we get:

$$0 = \frac{G_+ - G_-}{2}\cos(2\alpha(v))2\alpha'(1-B) - A2e^{-\gamma T}G\sin(2\alpha(v))\alpha' \tag{78}$$

If α' is unequal to zero we can divide by this term. If it is equal to zero we do not have an equation to solve. Thus we assume that this term is nonzero and we begin to solve this equation. Now we have one equation with two unknowns. This means that there is no unique solution. We must either try to obtain a second equation or we must eliminate one unknown from this one equation. We will do the latter.

We can create another variable, z. This variable is dependent on A and B. A and B are the outcome of an integral and are thus constant. This means that z is also a constant. Now we have one unknown and one equation. This is solvable.

$$\sqrt{z} = \frac{A}{1-B} \tag{79}$$

$$0 = \frac{G_+ - G_-}{2}\cos(2\alpha(v))2 - \sqrt{z}2e^{-\gamma T}G\sin(2\alpha(v)) \tag{80}$$

We note that we can rewrite this equation in a different way. We introduce a vector F, with $F_x = F\cos(2\alpha)$ and $F_y = F\sin(2\alpha)$. If we do this, we can see that:

$$0 = F_y\cos(2\alpha(v)) - F_x\sin(2\alpha(v)) \tag{81}$$

Combining equation (80) and (81) we find functions for F_x and F_y.

$$F_x = 2e^{-\gamma T} G\sqrt{z} \tag{82}$$

$$F_y = G_+ - G_- \tag{83}$$

Now we can finally get an expression for our rotation angle. We simply rewrite equation (81) until we find equation (84).

$$\tan(2\alpha) = \frac{\sin(2\alpha)}{\cos(2\alpha)} = \frac{F_y}{F_x} \tag{84}$$

Rewriting this we find equation (85).

$$\alpha = \frac{1}{2}\tan^{-1}(\frac{F_y}{F_x}) \tag{85}$$

To find the optimal angle of rotation we only need to calculate the value of z. We will do this by calculating values for A and B and then solve equation (79) for z and A and B, where A and B are now dependent on z. By our definition of F, we can replace the sine and cosine in A and B.

$$A = \int_{-\infty}^{\infty} \frac{(G_+ - G_-)^2}{2F} dv \tag{86}$$

$$B = (e^{-2\gamma T}) \int_{-\infty}^{\infty} \frac{G^2 2\sqrt{z}}{F} dv \tag{87}$$

Where F is given by:

$$F = 2G\sqrt{e^{-2\gamma T} z + e^{-4T}\sinh^2(4Tv)} \tag{88}$$

Our domain runs from $-\infty$ to ∞. Since A and B are even integrals , we get:

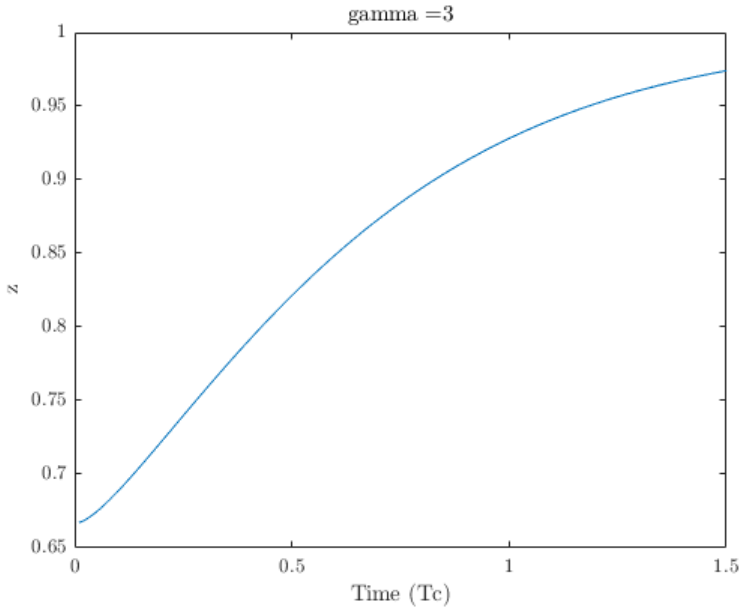
$$A = 2\int_{0}^{\infty} \frac{(G_+ - G_-)^2}{2F} dv \tag{89}$$

$$B = 2(e^{-2\gamma T}) \int_{0}^{\infty} \frac{G^2 2\sqrt{z}}{F} dv \tag{90}$$

Now to solve for z and thus find values for A and B, a script has been made to solve these equations numerically. This has been done using MATLAB.

When we look at figures 3, 4, 5 and 6 we see that for short times the value of z gets lower as the value of γ gets higher. When the time of a measurement becomes larger z reaches the value of 1. This is why we will look for a relation between γ and z in the next subsection. But first we will look at the value of the spin in the x-direction at some values of γ.

When we look at figures 7, 8, 9 and 10 we see that the spin in the x-direction still takes approximately the same amount of time to reduce in value. One thing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0.85
0.8
0.75
0.7
0.65
z
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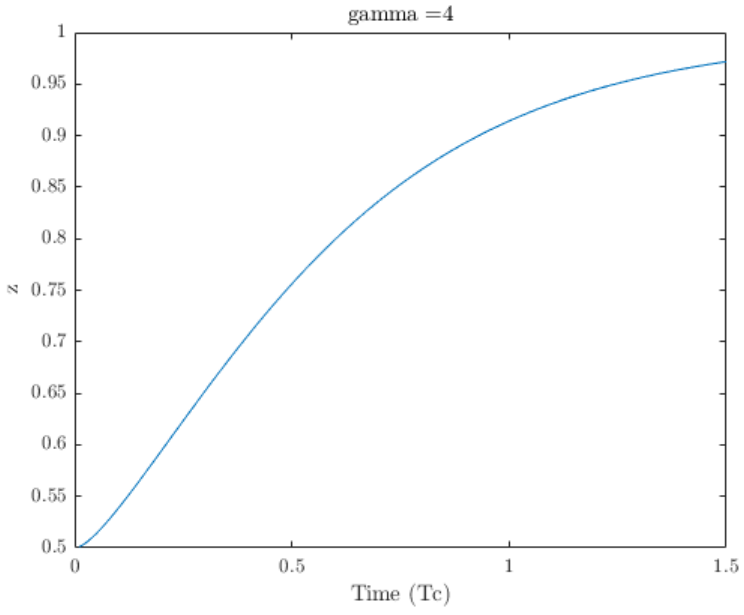
0.5
1
1.5
Time (Tc)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z
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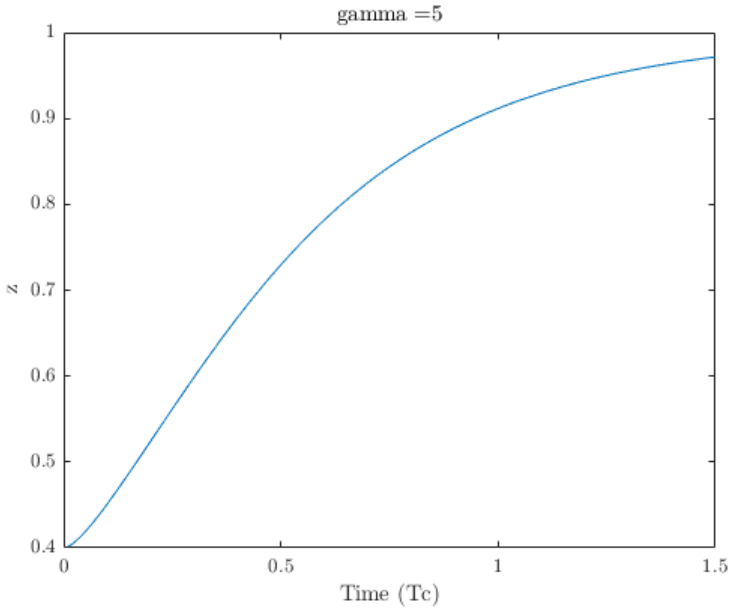
1
1.5
Time (Tc)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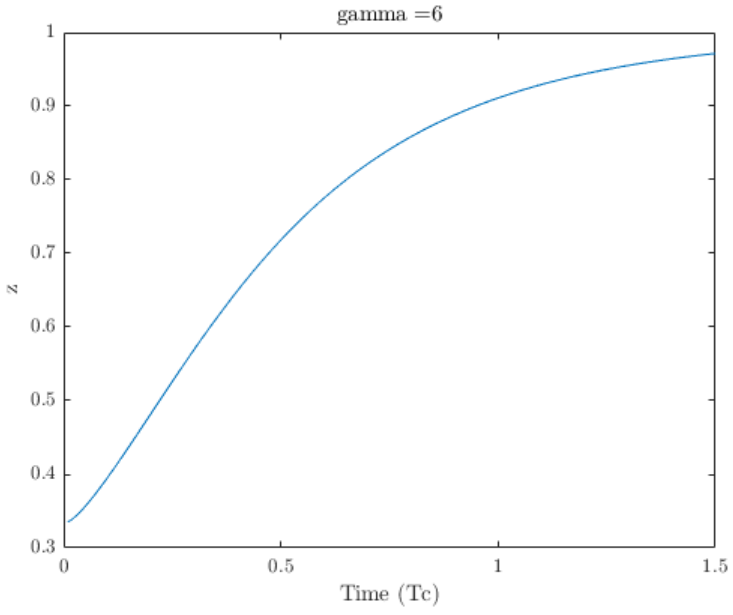
1
1.5
Time (Tc)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0.7
0.6
0.5
0.4
0.3
z
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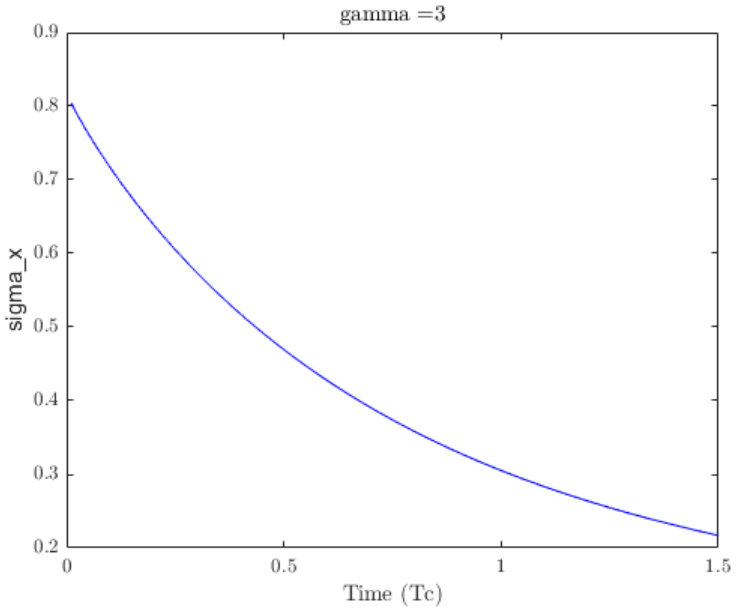
0.5
1
1.5
Time (Tc)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sigma_x
Time (Tc)
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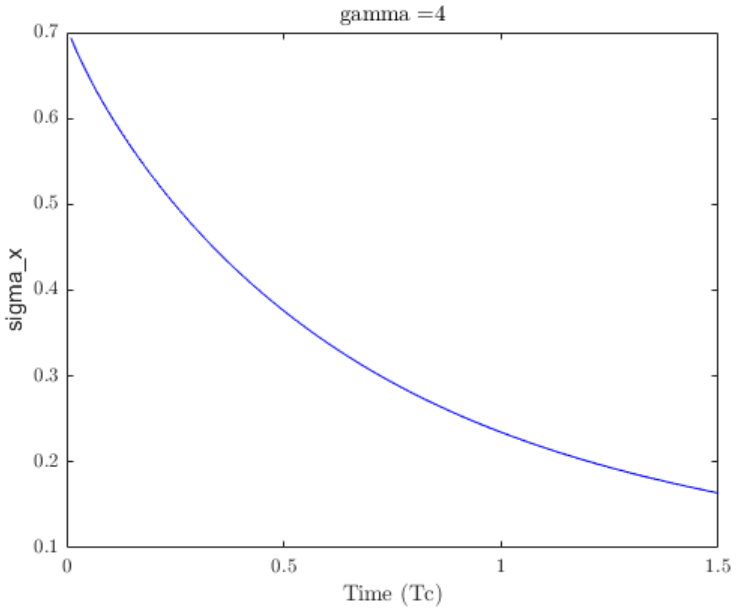
0.6
0.5
0.4
0.3
0.2
0
0.5
1
1.5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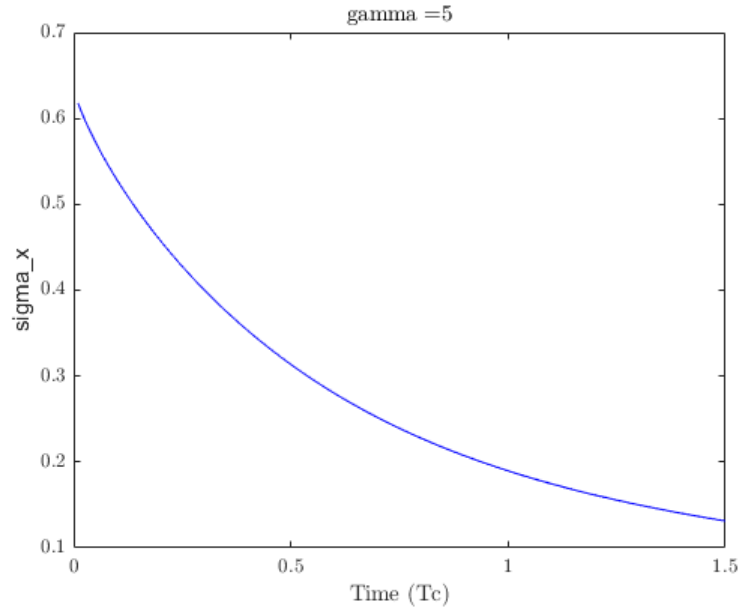
1.5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sigma_x
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0.2
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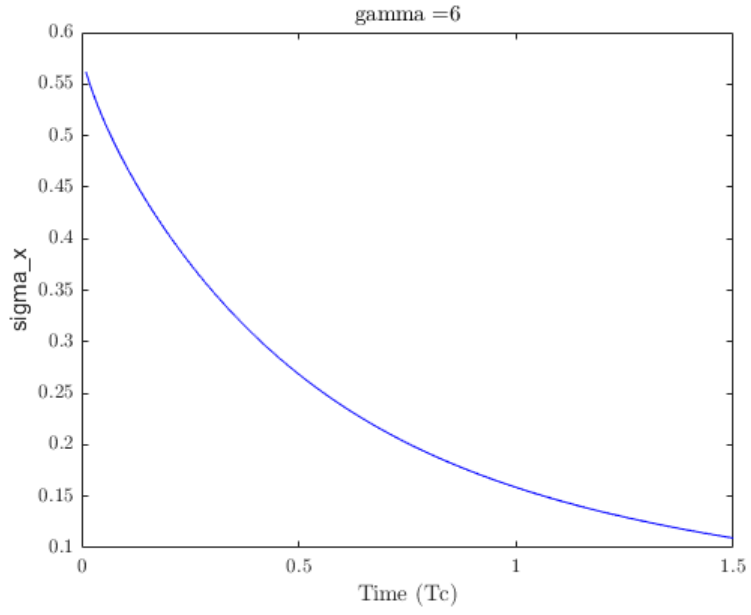
0
0.5
1
1.5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sigma_x
Time (Tc)
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0.55
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0
0.5
1
1.5

that is different is that the starting value of this spin appears to be dependent on γ. This is not in accordance to what was expected. We can still recognize a shape of $e^{-\gamma T}$, but this expected shape had a starting value of 1 at the start of a measurement, which does not seem to happen in figures 7, 8, 9 and 10.

3.3 Small Time Intervals

In the previous subsection we looked at the values of z at different times. We saw that z starts at a lower value if we increase the value of γ. In this section we will see if there is a relation between z and γ at small times.

We will try to simplify A and B in equations (86) and (87).We will use Taylor expansions to simplify our equations. We do this by first simplifying $\frac{1}{F}$ with F. We will only look at terms which are of order one or lower in T. Any higher order terms are negligible, because we are looking at values of T very close to 0.

$$e^x = 1 + x + O(x^2) \tag{91}$$

Now we get:

$$\frac{2G}{F} = \frac{1}{\sqrt{z(1-2\gamma T) + 16T^2v^2}} \tag{92}$$

Now we use:

$$\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2) \tag{93}$$

We find:

$$\frac{2G}{F} = \frac{1}{\sqrt{z}(1 - \gamma T + \frac{8T^2v^2}{z})} \tag{94}$$

Now we use:

$$\frac{1}{1+x} = 1 - x + O(x^2) \tag{95}$$

We find:

$$\frac{2G}{F} = \frac{1}{\sqrt{z}}\left(1 + \gamma T - \frac{8T^2v^2}{z}\right) \tag{96}$$

Now we have approximated $\frac{1}{F}$, We continue expanding A and B. For that we will use:

$$\sinh x = x + O(x^3) \tag{97}$$

By combining equations (86) and (96) we get for A:

$$A = (1 - 4T)\int_{-\infty}^{\infty} \frac{16T^2v^2G}{\sqrt{z}}(1 + \gamma T - \frac{8T^2v^2}{z})dv \tag{98}$$

$$A = \frac{16T^2 < v^2 >}{\sqrt{z}} \tag{99}$$

If we do the same for B we get:

$$B = (1 - 2\gamma T) \int_{-\infty}^{\infty} G(1 + \gamma T - \frac{8T^2 v^2}{z}) dv \tag{100}$$

$$B = 1 - \gamma T - \frac{8T^2 < v^2 >}{z} \tag{101}$$

Now we got A and B again, we can try to simplify z using equation (41).

$$\sqrt{z} = \frac{A}{1 - B} = \frac{\left(\frac{16T^2 < v^2 >}{\sqrt{z}}\right)}{\gamma T + \frac{8T^2 < v^2 >}{z}} \tag{102}$$

A short calculation shows that:

$$< v^2 >= \frac{1}{4T} \tag{103}$$

And we get:

$$z = \frac{4T}{\gamma T + \frac{2T}{z}} \tag{104}$$

This gives us equation (105)

$$z\gamma T + 2T = 4T \tag{105}$$

And from this we get:

$$z = \frac{2}{\gamma} \tag{106}$$

Thus we find that there is a relation between z and γ, and it is the relation from equation (106).

4 Results

4.1 Simulation

In this section we will make a simulation of our system. First we will derive formulas that are necessary to update the density matrix of a general system. Then we will derive these formulas for a more specific system. Then we will show graphs of the quantum trajectories of the spin in the x-direction and in the z-direction.

We will now state our assumptions on the system. After this we will start solving for our density matrix. We first say that there are two particles, which we will call A and B. There is an interaction between these particles, which means that the Hamiltonian is given by equation (107).

$$\hat{H} = \hat{H}_A + \hat{H}_B + \hat{H}_C \tag{107}$$

Here $\hat{H}_C$ is the coupling Hamiltonian which governs the interaction between particles A and B. We use the interaction picture for the simulation, hence at every step we will apply a unitary transformation to switch to the interaction picture. This means that we can ignore the evolution of $\hat{H}_A$ and $\hat{H}_B$. Now we only need a definition of the coupling term of the Hamiltonian. We give this in equation (108).

$$\hat{H}_C = \hat{M} \otimes \hat{Q} \tag{108}$$

We say the system we have is only sensitive to a single observable $\hat{M}$. We also say that $\hat{Q}$ is an operator acting on particle B. Now we introduce our initial density matrix. This is the state before any time has passed and particle A and B have not yet had any interaction. This initial state is given by equation (109).

$$\hat{\rho}(0) = \hat{\rho}^A(0) \otimes \hat{\rho}^B(0) \tag{109}$$

Our last assumption will be that the state vector of a particle does not change in length. It only undergoes a unitary rotation. This means that the time dependence of the state vector can be written as in equation (110).

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle \tag{110}$$

We use this expression and combine it with equation (2). If we do this we get a simple differential equation and we see that the solution is given by equation (111):

$$U(t) = e^{-i\hat{H}t} \tag{111}$$

This is only the time dependence of the state vector of a particle. We fill in this time dependence in equation (1) to get the time dependence of the density matrix of our system. This time dependence is then given by equation (112).

$$\hat{\rho}(\Delta t) = U(\Delta t)\rho_i U^*(\Delta t) \tag{112}$$

We now have a time dependent expression for the density matrix. We want to write this density matrix in terms of the eigenbasis for the observable $\hat{M}$. we note that we can write a matrix $\hat{K}$ in the following way:

$$\hat{K} = \sum_{n,m} |n\rangle \hat{K}_{n,m} \langle m| \tag{113}$$

In equation (113) $K_{n,m} = \langle n| \hat{K} |m\rangle$. To write the density matrix in terms of this eigenbasis we will use this method.

$$\hat{\rho}(\Delta t) = \langle n| U(\Delta t) \sum_{n,m} \rho^A_{n,m} |n\rangle \langle m| \otimes \rho^B(0) U^*(\Delta t) |m\rangle \tag{114}$$

To get to equation (118) from equation (114) we we will use a Taylor expansion for an exponential that was also used in equation (91).

$$U^*(\Delta t) |m\rangle = e^{i\Delta t \hat{M} \otimes \hat{Q}} |m\rangle \tag{115}$$

With the Taylor expansion this becomes.

$$\sum_{l=0}^{\infty} \frac{\left(i\Delta t \hat{M} \otimes \hat{Q}\right)^l}{l!} |m\rangle \tag{116}$$

We now use that the eigenbasis we use are eigenvectors of the matrix $\hat{M}$ with eigenvalues n for an eigenvector $|n\rangle$.

$$\left(1 + i\Delta t \hat{M} \otimes \hat{Q} + ...\right) |m\rangle = \left(1 + i\Delta t m \hat{Q} + ...\right) |m\rangle \tag{117}$$

We can do this since the vector does not change the $\hat{Q}$ matrix in the vector basis of our system. If we apply this we arrive at equation (118).

$$\hat{\rho}(\Delta t) = \sum_{n,m} \rho^A_{n,m} |n\rangle \langle m| \otimes U_n(\Delta t) \rho^B(0) U^*_m(\Delta t) \tag{118}$$

Now the probability of a density matrix is given by its trace. Since here we have a tensor product, we will use the partial trace, which gives us by definition:

$$P(i) = Tr_A(\langle i| \hat{\rho}(\Delta t) |i\rangle) = \sum_n \rho^A_{n,n}(0) \langle i| U_n(\Delta t) \rho^B(0) U^*_n(\Delta t) |i\rangle \tag{119}$$

After measuring the system it is in the state i. The updated density matrix becomes, according to equation (14):

$$\rho^A_{new} = \frac{\sum_{n,m} \rho^A_{n,m}(0) |n\rangle \langle m| \langle i| U_n(\Delta t) \rho^B(0) U^*_m(\Delta t) |i\rangle}{\sum_n \rho^A_{n,n}(0) \langle i| U_n(\Delta t) \rho^B(0) U^*_n(\Delta t) |i\rangle} \tag{120}$$

Now we have derived the equations we need for a general system. Now we are going to define a more specific system that we will use and derive the equations we need for this system. In this system the particle that acts as the detector will initially be in the state shown in equation 121.

$$\hat{\rho}^B = |x\rangle \langle x| \tag{121}$$

In equation (121) the x denotes an eigenstate of the Pauli-x-matrix $\hat{\sigma}_x$ with eigenvalue 1. By taking the Kronecker product between the measured system and the detector, the joint density matrix takes the form of equation (122).

$$\hat{\rho}(0) = \hat{\rho}^A \otimes |x\rangle \langle x| \tag{122}$$

We are only interested in the evolution of the measured system and not of the detector system. Thus at each time step the detector system will be initialized to the system in equation (121). At each time step the Hamiltonian will be turned on. In the time evolution of the entire system the Hamiltonian will be used. In the simulation the evolution of the measured system ($\hat{H}_a$) and of the detector ($\hat{H}_b$) are small and can be neglected. Thus for the time evolution only the coupling Hamiltonian will be used. This coupling Hamiltonian is shown in equation (123).

$$\hat{H}_c = \hat{M} \otimes \hat{\sigma}_y \tag{123}$$

Here $\hat{\sigma}_y$ is the Pauli-y-matrix. At every step the density matrix is adjusted by using equation (112). To simplify the calculations that must be done in the simulation, we will use equation (118) to rewrite the density matrix in terms of the eigenbasis of the operator $\hat{M}$. The resulting density matrix is shown in equation (124).

$$\hat{\rho}_{n,m}(\Delta t) = \hat{\rho}_{n,m}(0) \otimes e^{-iM_n\hat{\sigma}_y\Delta t} |x\rangle \langle x| e^{iM_m\hat{\sigma}_y\Delta t} \tag{124}$$

Now we have the density matrix after a time step. Before we go on we will simplify the density matrix in equation (124). We will do this by using an identity for a Pauli matrix in an exponent. This identity is shown in equation (125).

$$e^{ia\hat{\sigma}} = I\cos(a) + i\hat{\sigma}\sin(a) \tag{125}$$

In equation (125) a is a constant and I is the identity matrix. Now we will substitute equation (125) in equation (124) and we find:

$$\begin{aligned}\hat{\rho}_{n,m}(\Delta t) = \hat{\rho}_{n,m}(0) \otimes (I\cos(M_n\Delta t) - i\hat{\sigma}_y\sin(M_n\Delta t))\,|x\rangle \\ \langle x|\,(I\cos(M_m\Delta t) + i\hat{\sigma}_y\sin(M_m\Delta t))\end{aligned} \tag{126}$$

We can still simplify equation (126). To do this we rewrite the equation until we get equation (127). Now we have a density matrix in terms of another density matrix and one other term. We will calculate the value of this other term, which is called V in equation (127). Note that V in equation (127) is given by equation (128).

$$\hat{\rho}_{n,m}(\Delta t) = \hat{\rho}_{n,m}(0) V V^* \tag{127}$$

$$V = \left(I \cos(M_n \Delta t) + i\hat{\sigma}_y \sin(M_n \Delta t)\right) |x\rangle \tag{128}$$

In equation (127) V^* is the conjugate transpose of V. Now V will be calculated. If we have this value we can also calculate the value of the conjugate transpose of V. A quick calculation shows that with the value of V the new density matrix can be written as in equation (129).

$$\begin{aligned} \hat{\rho}_{n,m}(\Delta t) = \frac{1}{2}\hat{\rho}_{n,m}(0) \left(|1,1\rangle \cos M_n \Delta t - |-1,1\rangle \sin M_n \Delta t\right) \\ \left(\langle 1,1| \cos M_m \Delta t - \langle -1,1| \sin M_m \Delta t\right) \end{aligned} \tag{129}$$

Now we know the density matrix after a time step. We will use this density matrix in combination with equation (119) to calculate the probability to measure a $+1$ or a -1. Initially our measured qubit is aligned with the x-axis. At every step this qubit is projected onto the z-axis. A result of $+1$ means the qubit is aligned more with the positive z-axis compared to the previous time step and a result of a -1 means the qubit is more aligned with the negative z-axis compared to the previous time step. After a result has been recorded the new density matrix is calculated using equation (120). To calculate the probability of getting a result we now apply equation (119) to our density matrix. This probability is recorded in equation (130).

$$P(\pm) = \frac{1}{2} \sum_n \rho_{n,n} \left(1 \pm \sin 2M_n \Delta t\right) \tag{130}$$

With the probability calculated the density matrix can be updated. To do this we apply equation (120). If we do this we find that the elements of the updated density matrix become.

$$\rho^A_{n,m,\pm} = \frac{1}{4P(\pm)} (\cos n\Delta t \pm \sin n\Delta t)(\cos m\Delta t \pm \sin m\Delta t) \tag{131}$$

Now we have calculated all the formulas needed for a simulation. We will now look at the code of the simulation program. This code was made using the programming language Python. The code is shown in appendix B.

We will now run through the code and an explanation will be given how it works. First a function 'step' is defined. In this function the probability of getting a $+1$ or a -1 is calculated. Then a random number generator chooses a number between 0 and 1. The density matrix is updated according to this result. This function returns the updated components of the density matrix and the result that was measured. After this a function 'loop' is defined. This function applies the function 'step' a set number of times. This number of times is set so that a significant total result can be obtained. This function returns the weighted result, the density matrix elements and the entire quantum trajectory. Third a function 'rotate' is defined. This function rotates the density matrix

to its position before any measurement had started. The function returns the rotated density matrix. Last a function 'total' is defined. This function completes the function 'loop' and then the function 'rotate' on the density matrix that 'loop' returns. the function 'total' does this n times, where n is an integer that can be chosen.

4.2 Simulation Results

In this section we will first show that we need to use the density matrix to describe our system. we will do this by showing that equation (10) does not hold. Then we will explain the value of the constants of our simulation, mainly the value of the characteristic time, then we will show and discuss the quantum trajectories in the z- and x- directions. Now we will show a plot of the trace of $\hat{\rho}^2$ for $\gamma = 3$. In figure 11 we see the trace of the matrix $\hat{\rho}^2$. It is clearly visible that this trace does not always have the value of 1. This means that $\hat{\rho}^2 \neq \hat{\rho}$, and thus we do not have a pure state in our quantum system. Now we will move on to the values of the constants that we used in our simulation program.

To simulate a continuous weak linear measurement we require the strength of a measurement to be small. This means we require:

$$M\Delta t < 1 \tag{132}$$

There are no other requirements for M and Δt so these can be chosen. For this report these values have been chosen such that $\Delta t = 0.01$ and $M = 1$. Now we use the fact that for a decisive measurement, we require the standard deviation of the average output to be smaller than the separation between the discrete values for $< \Sigma_z >= \pm 1$. This gives us:

$$\sqrt{\frac{\Delta t}{T_c}} < M\Delta t \tag{133}$$

We solve this equation by taking the square and isolating T_c. We find.

$$T_c = \frac{1}{M^2 \Delta t} \tag{134}$$

Now the values of the constants used are explained we will move on to the plots. We will discuss quantum trajectories in the x-direction and in the z-direction. We will do this for integer values of γ from 2 to 6.

First we look at the quantum trajectory in the z-direction. Our qubit always begins in an equal weight quantum position. Therefore the expectation is that the average spin in the z-direction of a large number of quantum trajectories will go to zero.

In figure 12 we can see a quantum trajectory. This trajectory goes to the value of +1. If we compare this to figures 13,14,15 and 16 we see no big differences. But these are just single trajectories. To see if our expectations are correct we will look at averages of multiple quantum trajectories. We first look at the average over 20 trajectories.

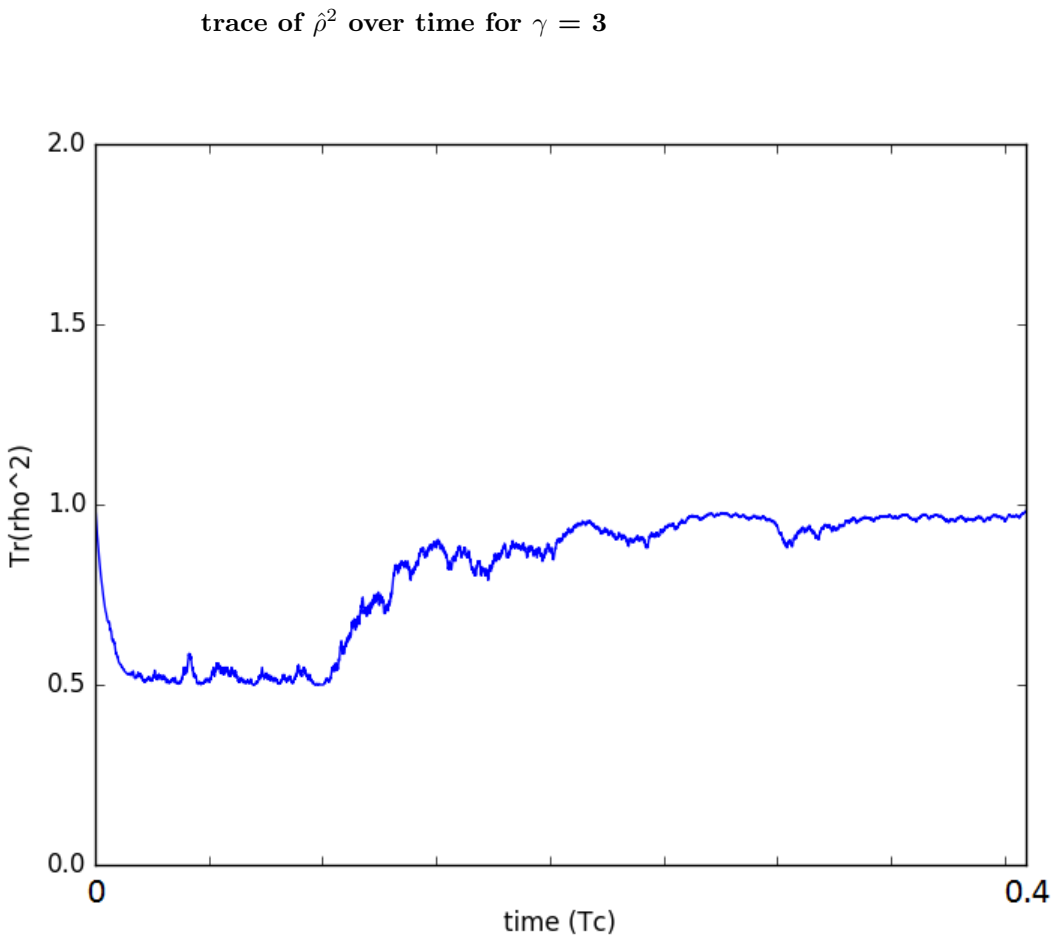

Figure 11: trace of $\hat{\rho}^2$ over time for $\gamma = 3$. If we have a non ideal system we see that the trace of $\hat{\rho}^2$ is not equal to 1. This means that $\hat{\rho}^2 \neq \hat{\rho}$, and thus we do not have a pure state, but rather a mixed state and it is necessary to use the density matrix.

spin in the z-direction over time for $\gamma = 2$

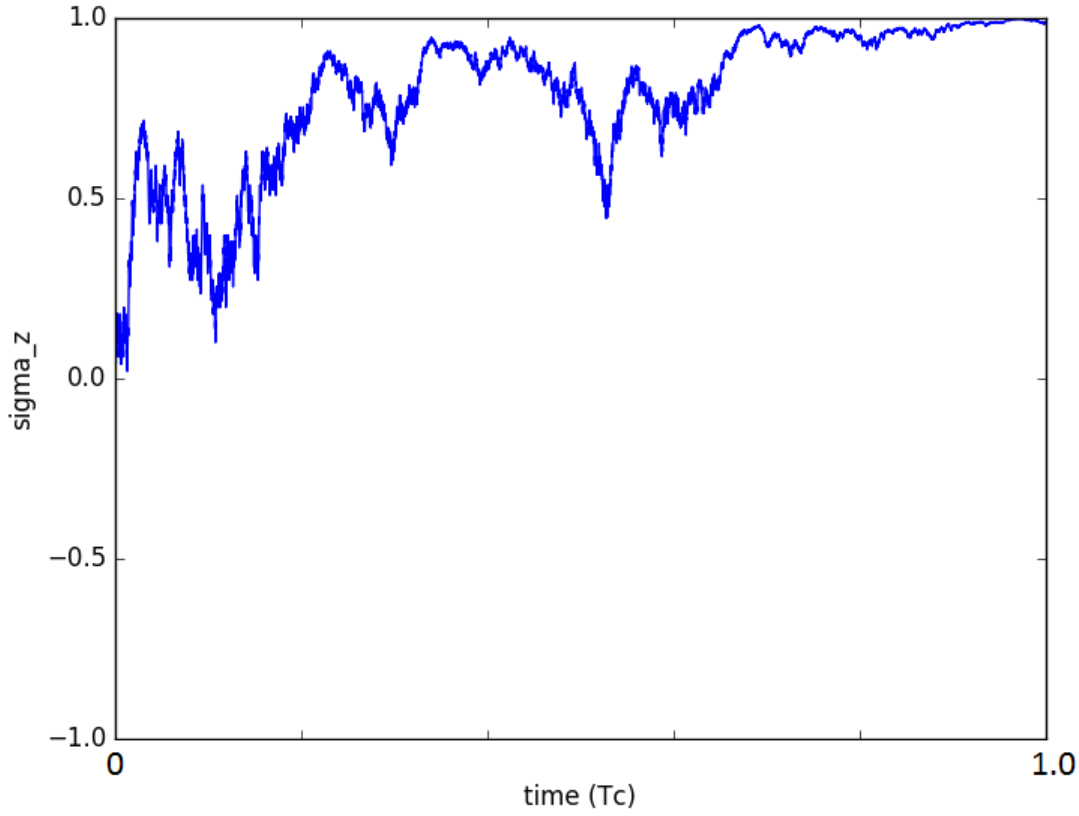

Figure 12: a quantum trajectory for $\gamma = 2$. This shows the spin in the z-direction. In this case the qubit ends in a spin up state.

spin in the z-direction over time for $\gamma = 3$

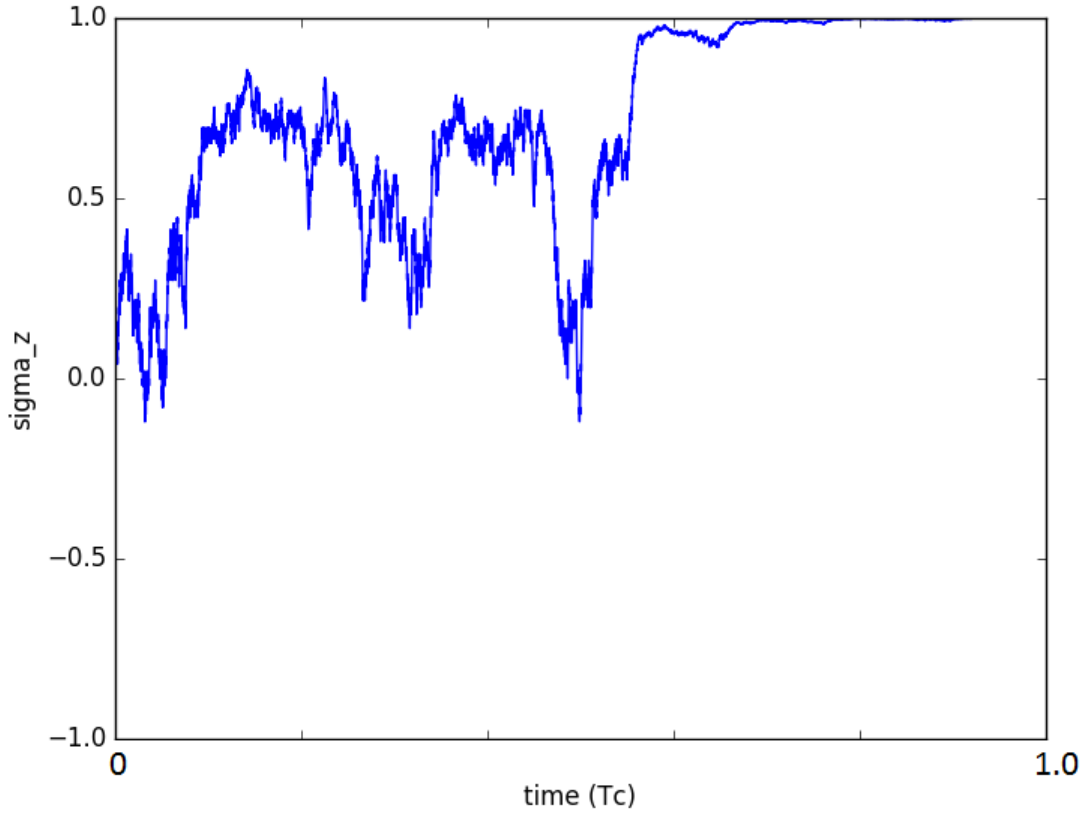

Figure 13: a quantum trajectory for $\gamma = 3$. This shows the spin in the z-direction. In this case the qubit ends in a spin up state.

spin in the z-direction over time for $\gamma = 4$

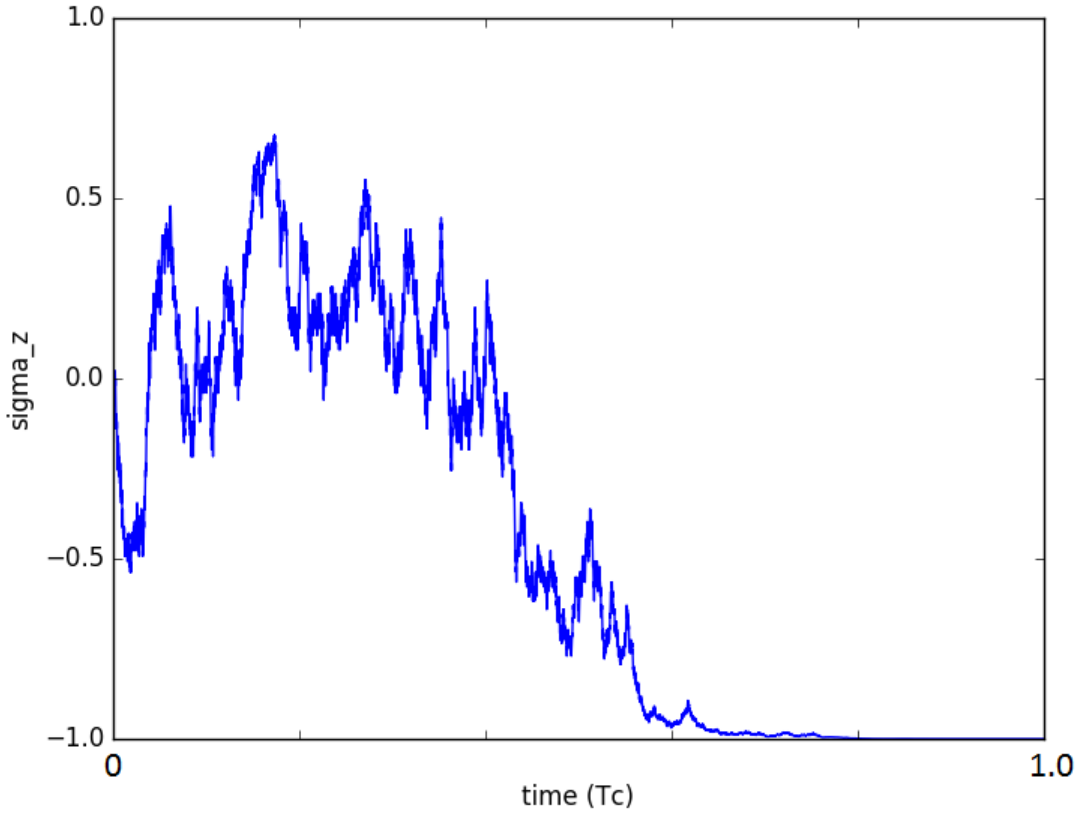

Figure 14: a quantum trajectory for $\gamma = 4$. This shows the spin in the z-direction. In this case the qubit ends in a spin down state.

spin in the z-direction over time for $\gamma = 5$

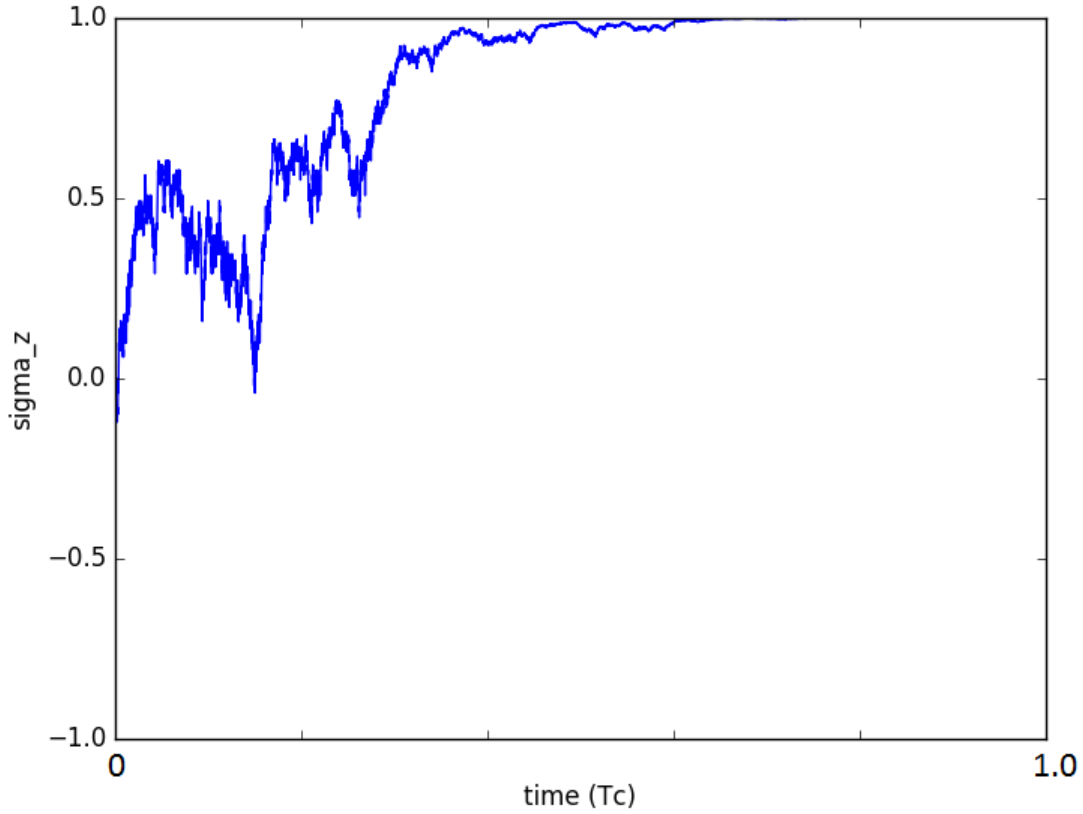

Figure 15: a quantum trajectory for $\gamma = 5$. This shows the spin in the z-direction. In this case the qubit ends in a spin up state.

spin in the z-direction over time for $\gamma = 6$

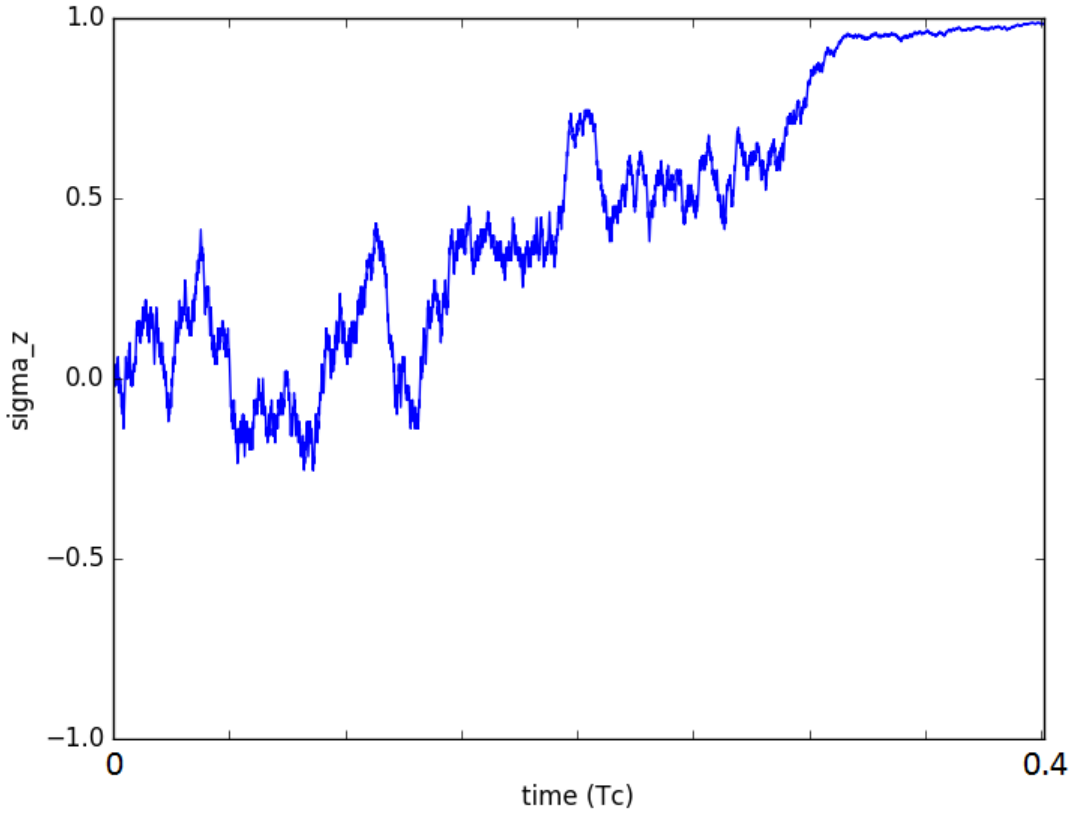

Figure 16: a quantum trajectory for $\gamma = 6$. This shows the spin in the z-direction. In this case the qubit ends in a spin up state.

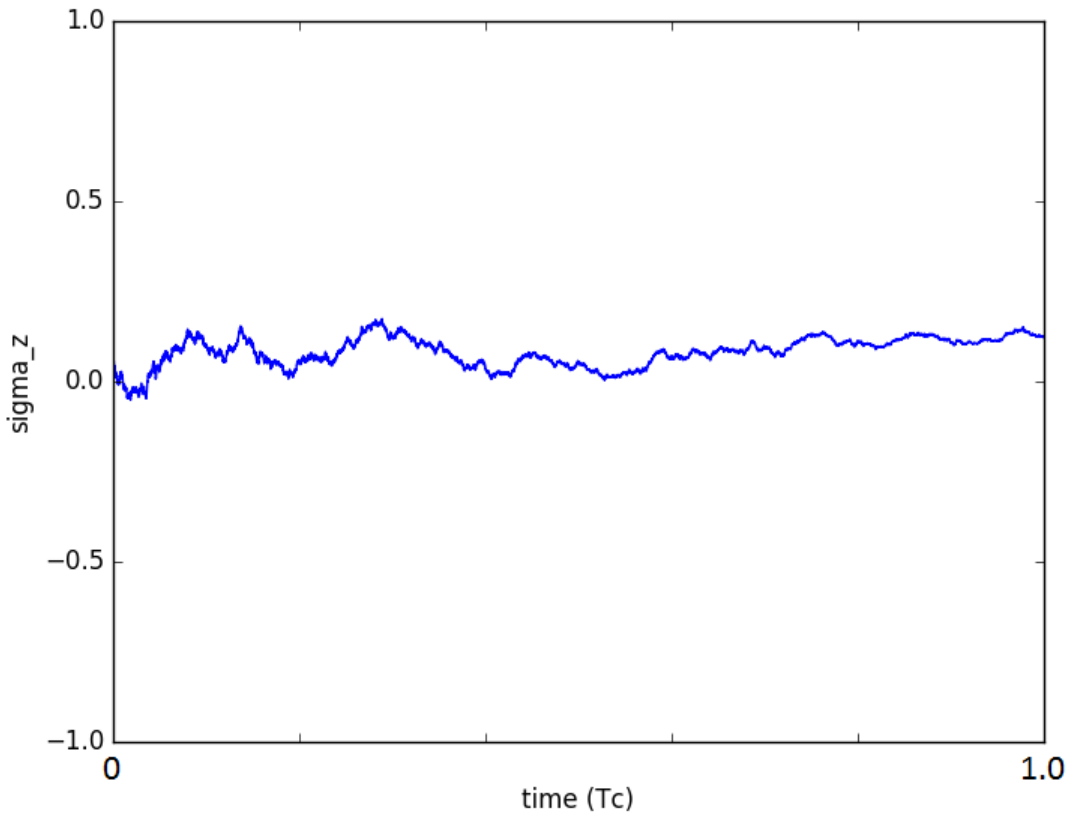

Figure 17: average of 20 quantum trajectories for $\gamma = 2$. The curve goes slightly higher than 0. We expect it to stay near zero. This is only a small deviation, which might be caused by a small amount of trajectories in the average.

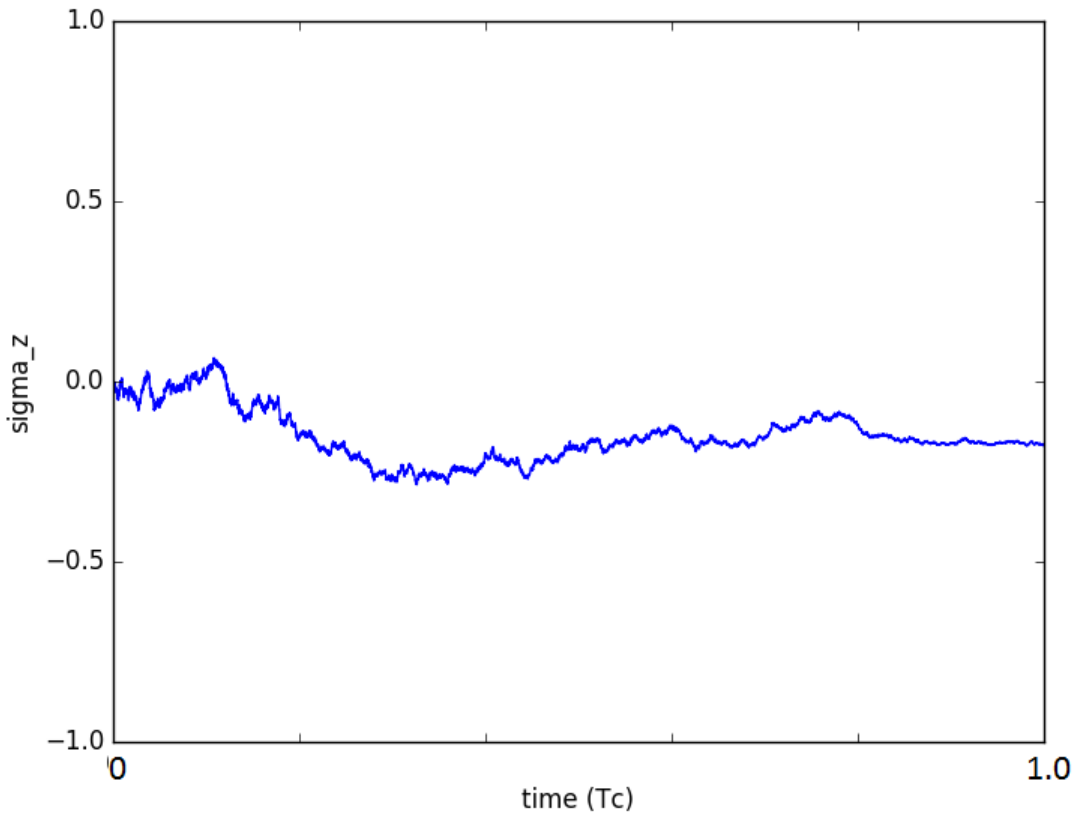

Figure 18: average of 20 quantum trajectories for $\gamma = 3$. The curve goes slightly lower than 0. We expect it to stay near zero. This is only a small deviation, which might be caused by a small amount of trajectories in the average.

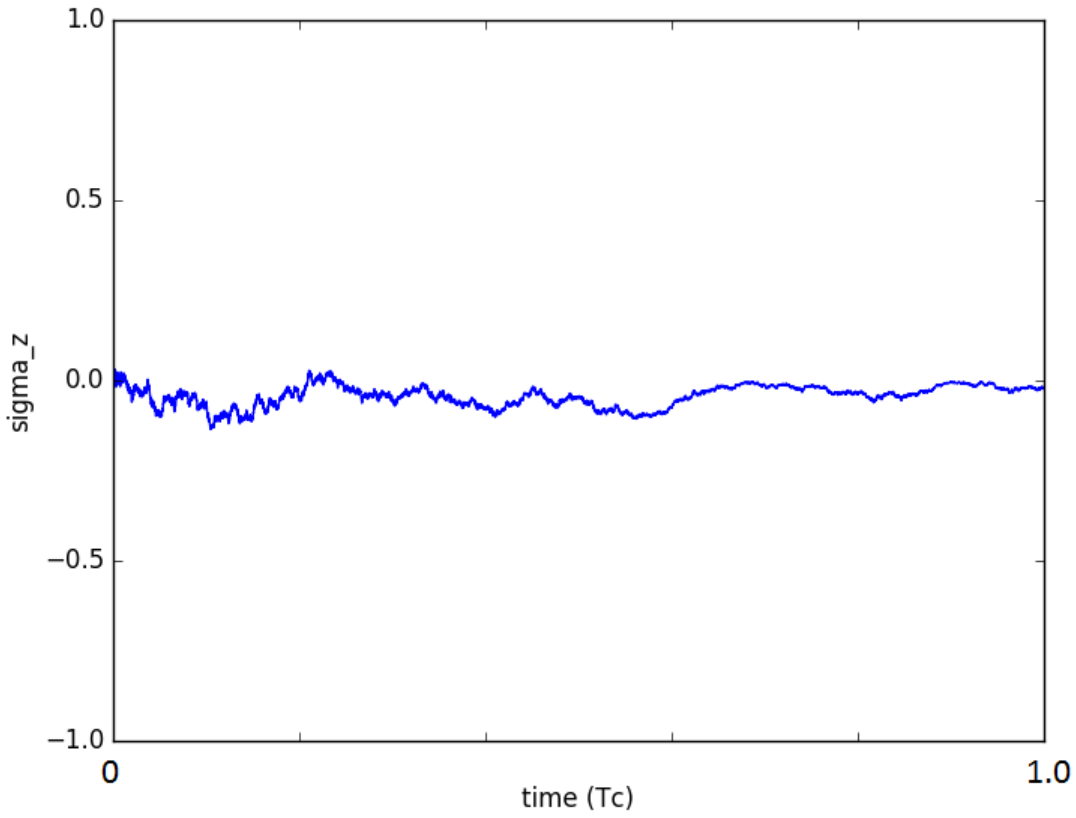

Figure 19: average of 20 quantum trajectories for $\gamma = 4$. The curve stays near 0, which is what we expect of the average.

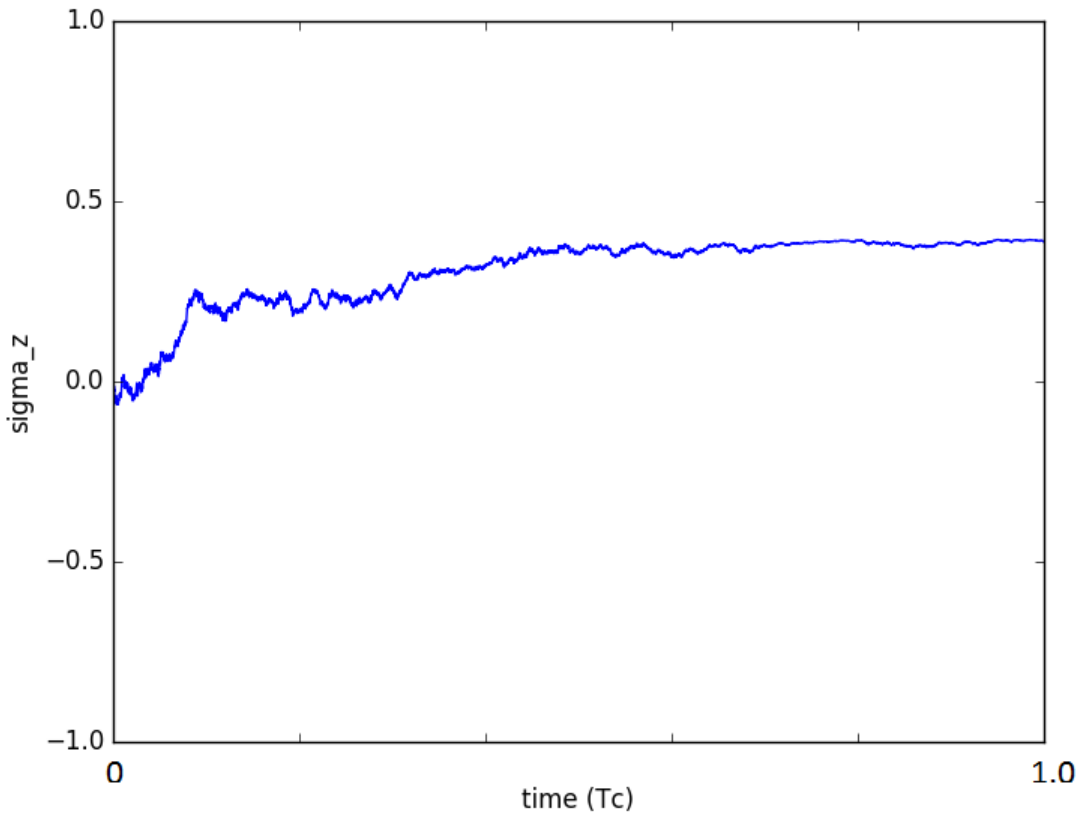

Figure 20: average of 20 quantum trajectories for $\gamma = 5$. The curve goes higher than 0. We expect it to stay near zero. This is not a small deviation, but it might be caused by a small amount of trajectories in the average.

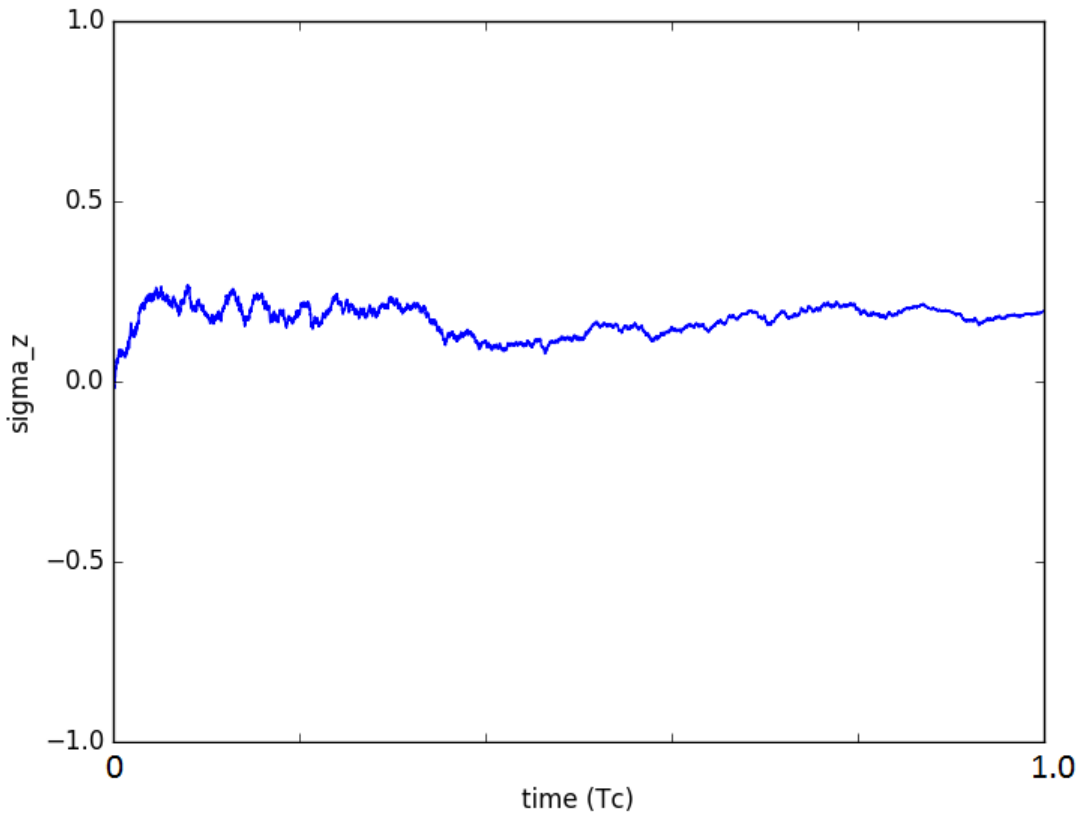

Figure 21: average of 20 quantum trajectories for $\gamma = 6$. The curve goes slightly higher than 0. We expect it to stay near zero. This is only a small deviation, which might be caused by a small amount of trajectories in the average.

Only in figure 19 the average seems to go to 0. In the figures 17,18,20 and 21 the average does not go to 0. In figure 20 the deviation from 0 appears to be bigger than in the other figures. This could be the case because we took the average over a too small number of trajectories. We want to know if we still see a deviation if we increase the amount of trajectories in the average trajectory. Therefore we will now take the average over 100 trajectories. When we look at the averages over 100 trajectories in figure 22 we see that the average goes to zero as expected. The average over 100 trajectories for $\gamma = 4$ is also shown in figure 23. Here we still see a small deviation from 0. This is not a big deviation thus it could be caused by a small amount of trajectories in the average trajectory. We again increase the amount of trajectories in the average trajectory to make sure this is the cause of the deviations. If so, we expect the deviations to decrease. For the other values of γ the plots are not shown here because they are similar to the given figures. We will look now at the average over 500 trajectories. Because the plots for the values of γ other than 2 again show no significant differences from previous plots, we will only show the average of 500 trajectories for $\gamma = 2$. Again the average of all trajectories goes to 0. We draw the conclusion that the average value over a large number of trajectories does indeed go to 0 as was expected. Any deviation from this average we have seen can be accounted to not having enough trajectories for the average to approach the theoretical values. We also conclude that outside influences, which come into play when $\gamma > 2$, do not have a significant influence over the quantum trajectory in the z direction.

Now we will discuss the results of the spin in the x-direction. In the last chapter the evolution of the spin in the x-direction has been theoretically calculated. We will compare these theoretical results with the results the simulation gives us. Now we look at the x-trajectories for integer values of γ from 2 to 6.

In figure 25 we see the evolution of the spin in the x-direction. This curve does not look smooth. It stays at the value of 1 before it decreases in value. This is but a single trajectory. The curve may approach the theoretical value better if we take an average over multiple trajectories. When we look at figures 26, 27, 28 and 29 we note that the curves already appear smoother than in figure 26. Another big difference is the rate at which the curves go to 0. The trajectory for $\gamma = 2$ approaches 0 over the course of the characteristic time scale. For values of γ bigger than 2 the trajectory is already close to 0 when 0.05 or less time has passed in units of the characteristic time scale. If we compare this to the theoretical curves we see that the actual trajectories decrease in value significantly faster then is expected. It was also expected that the starting value would decrease while the actual trajectories in the x-direction always start at the value of 1. We will now start looking at the average over multiple trajectories. Since the curves for $\gamma > 2$ are similar in shape we will only look at the values of $\gamma = 2$ and $\gamma = 3$.

When we take the average over 20 trajectories we see no difference in figure 31 for $\gamma = 3$. When we increase the number of trajectories in the average we do not expect to see that curve change anymore, because the curve is already smooth and any deviation from the average does not appear to be big. If we

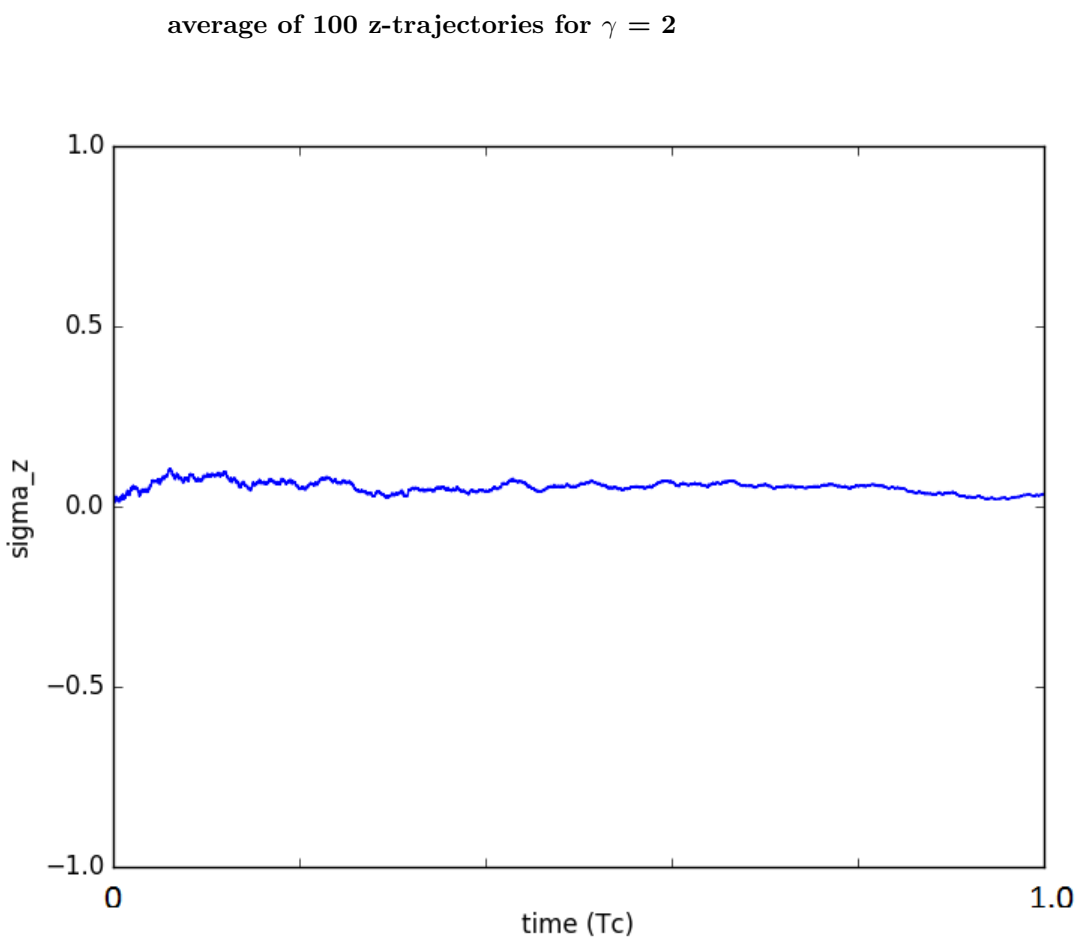

Figure 22: average of 100 quantum trajectories for $\gamma = 2$. The average value does not show significant deviations from 0. This means it behaves as we expect.

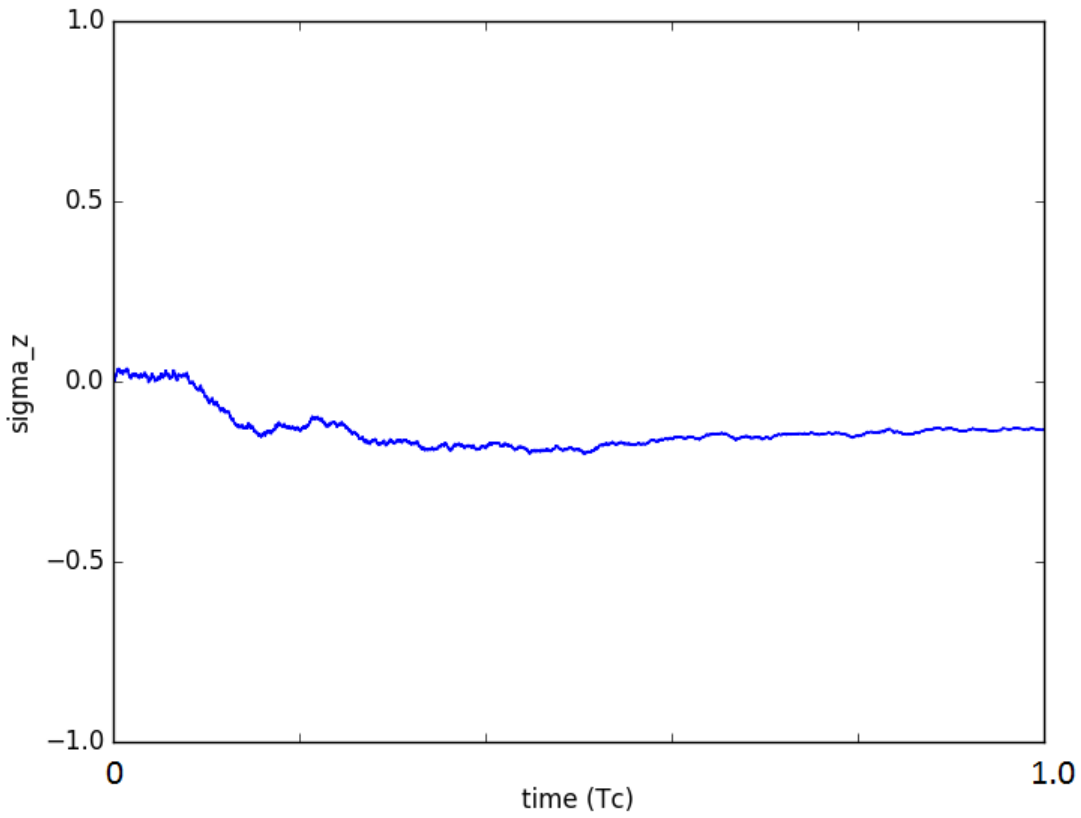

Figure 23: average of 100 quantum trajectories for $\gamma = 4$. The average still shows a small deviation from 0. This is a small deviation, thus it might still be caused by a small amount of trajectories in the average trajectory.

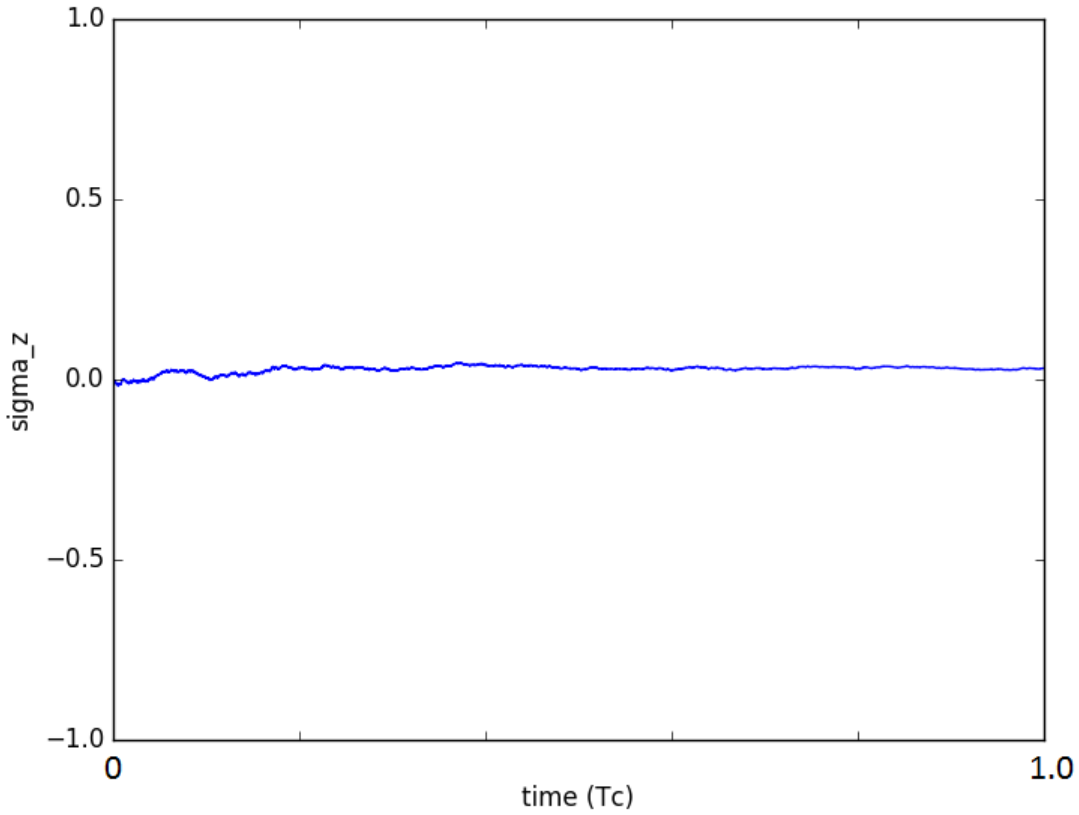

Figure 24: average of 500 quantum trajectories for $\gamma = 2$. We see that the deviations that have occurred are very small. The curve follows the expected value of 0.

spin in the x-direction over time for $\gamma = 2$

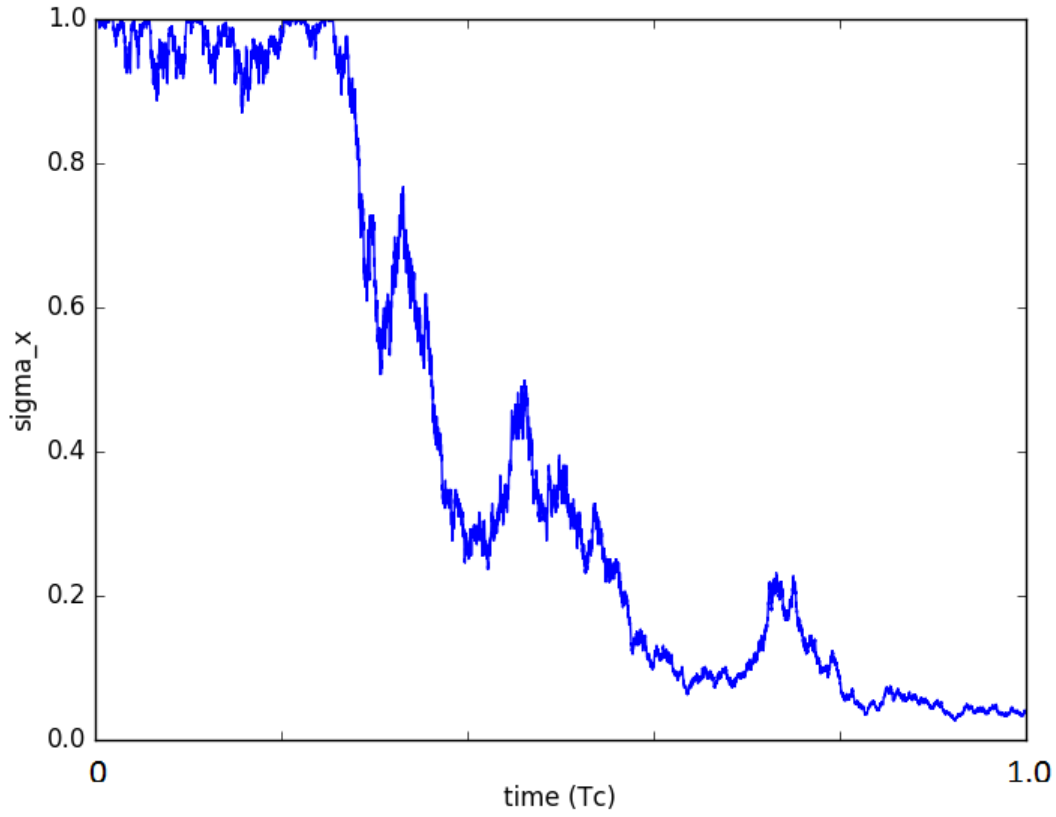

Figure 25: a quantum x-trajectory for $\gamma = 2$. The qubit seems to keep its spin in the positive x-direction initially instead of immediately aligning along the z-direction. After a time of 1 T_c the spin has aligned itself along the z-direction. This is in line with the theoretical expectation

spin in the x-direction over time for $\gamma = 3$

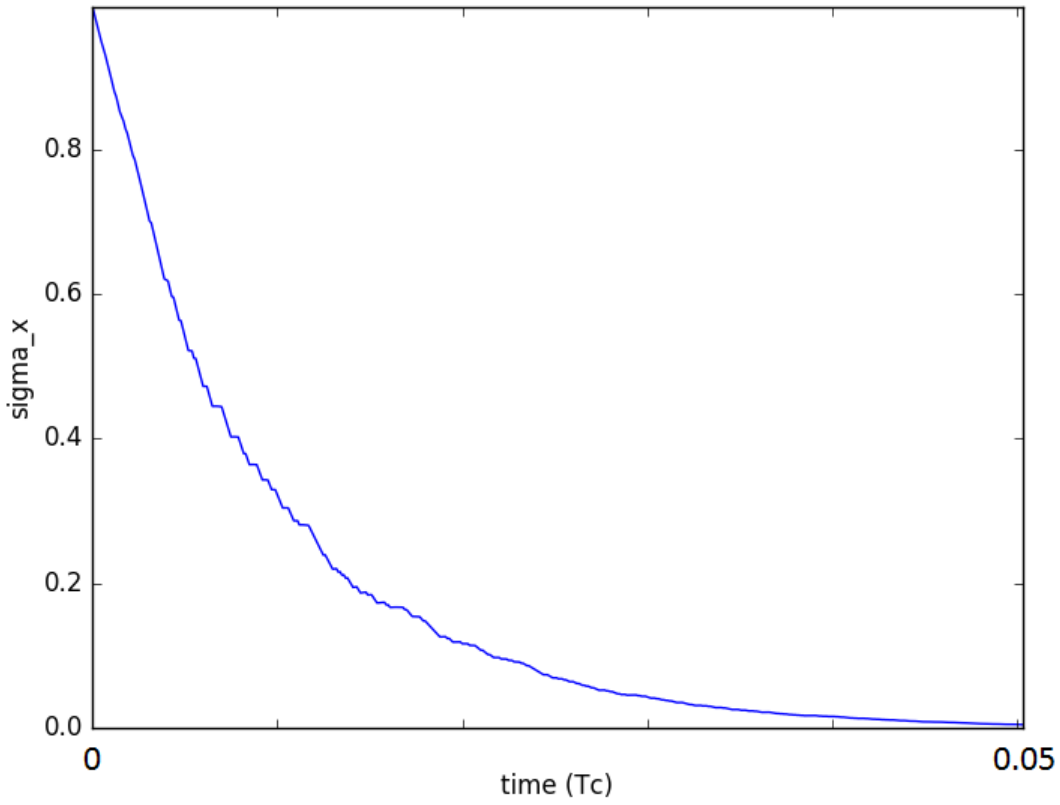

Figure 26: a quantum x-trajectory for $\gamma = 3$. The value of the spin along the x-direction immediately decreases and goes to 0. It goes to 0 in a time of $0.05T_c$, which is faster than the theoretical expectation. It also starts at a value of 1, which is not what the theory predicted.

spin in the x-direction over time for $\gamma = 4$

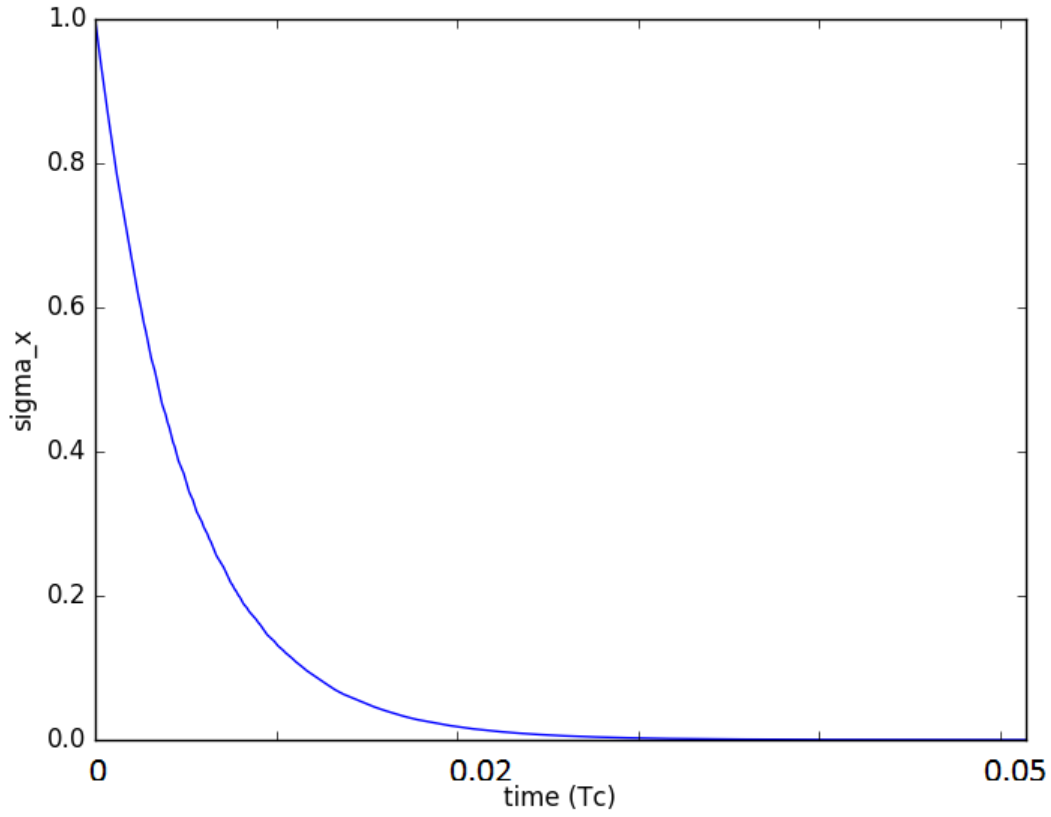

Figure 27: a quantum x-trajectory for $\gamma = 4$. The value of the spin along the x-direction immediately decreases and goes to 0. It goes to 0 in a time of $0.02T_c$, which is faster than the theoretical expectation. It also starts at a value of 1, which is not what the theory predicted.

spin in the x-direction over time for $\gamma = 5$

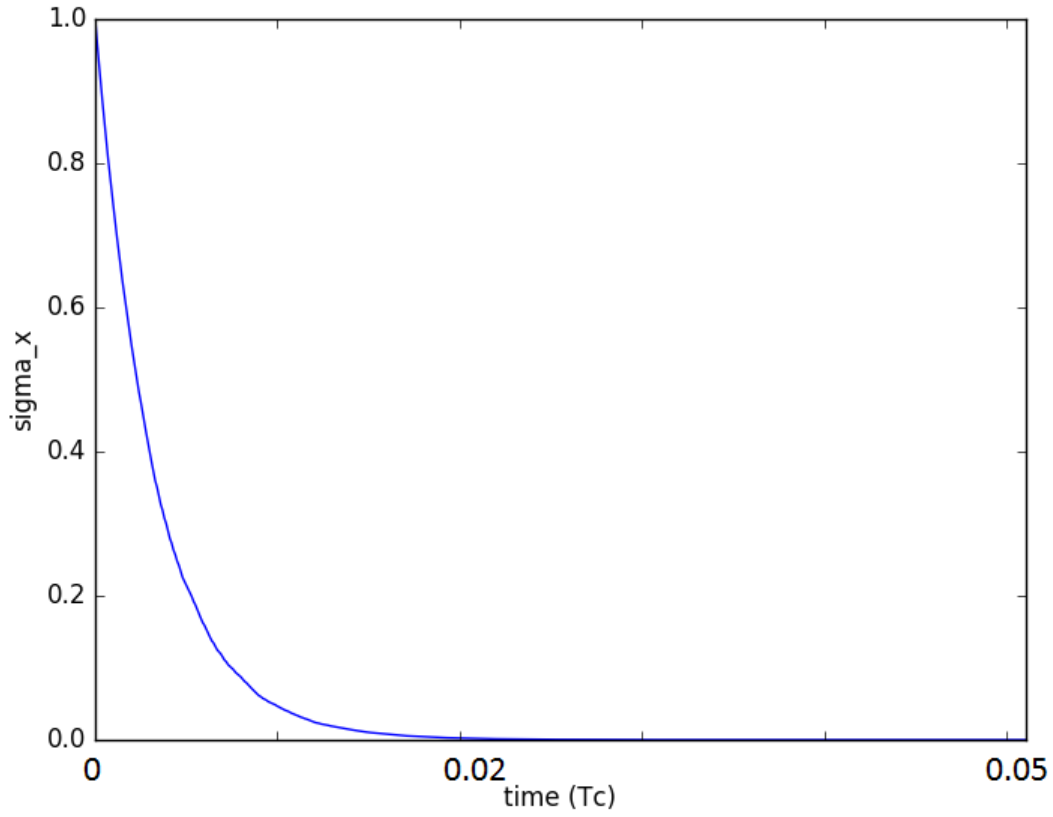

Figure 28: a quantum x-trajectory for $\gamma = 5$. The value of the spin along the x-direction immediately decreases and goes to 0. It goes to 0 in a time of $0.02T_c$, which is faster than the theoretical expectation. It also starts at a value of 1, which is not what the theory predicted.

spin in the x-direction over time for $\gamma = 6$

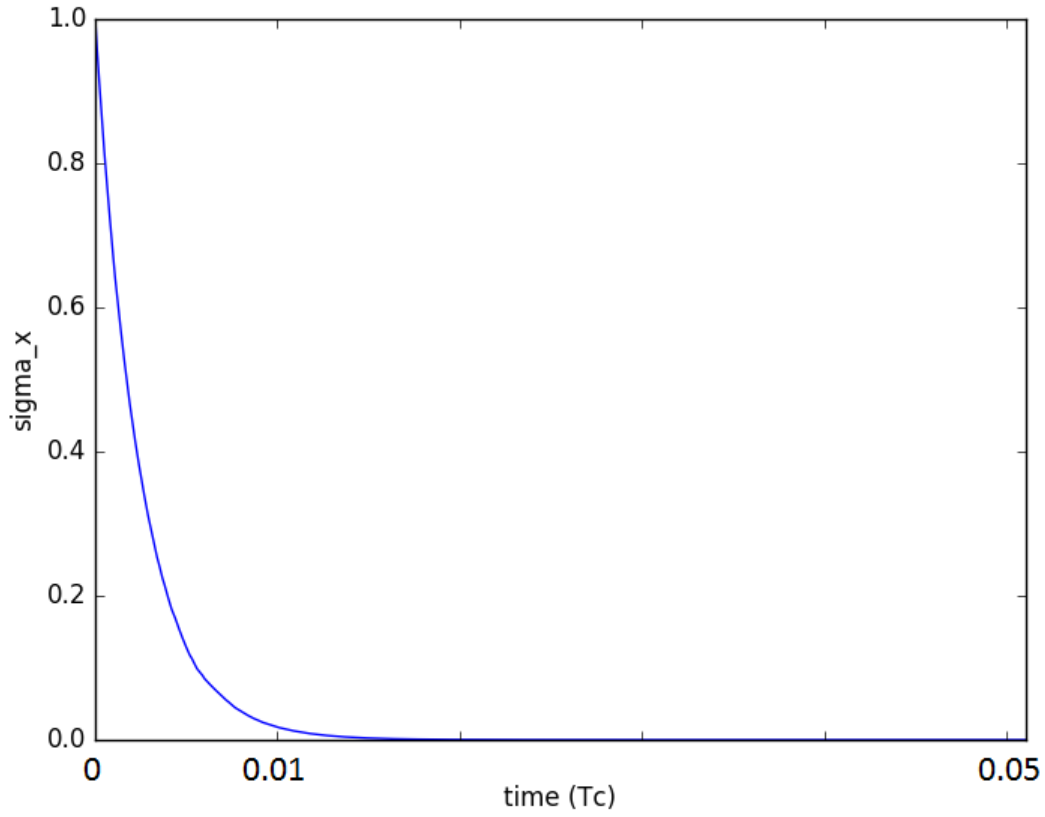

Figure 29: a quantum x-trajectory for $\gamma = 6$. The value of the spin along the x-direction immediately decreases and goes to 0. It goes to 0 in a time of $0.01T_c$, which is faster than the theoretical expectation. It also starts at a value of 1, which is not what the theory predicted.

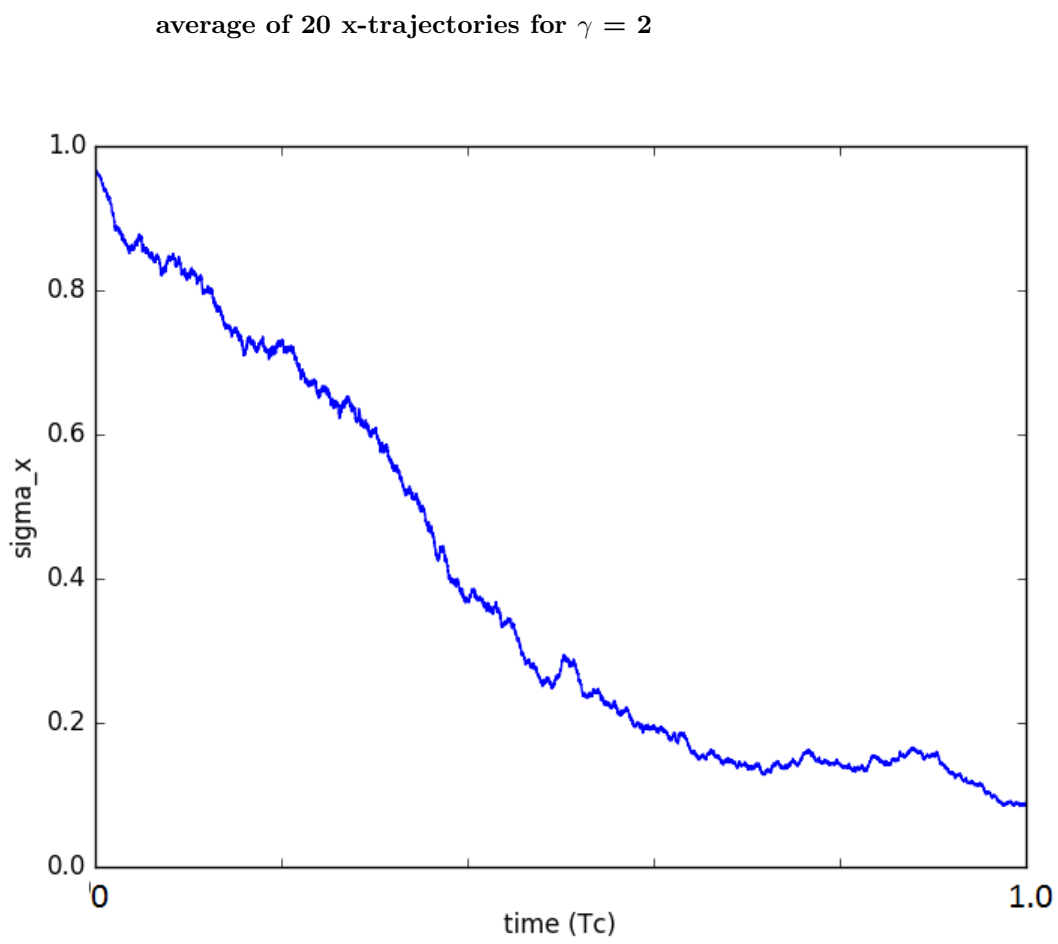

Figure 30: average of 20 quantum trajectories for $\gamma = 2$. The qubit immediately goes to 0 in a time of $1T_c$. This is in line with the prediction of the theory.

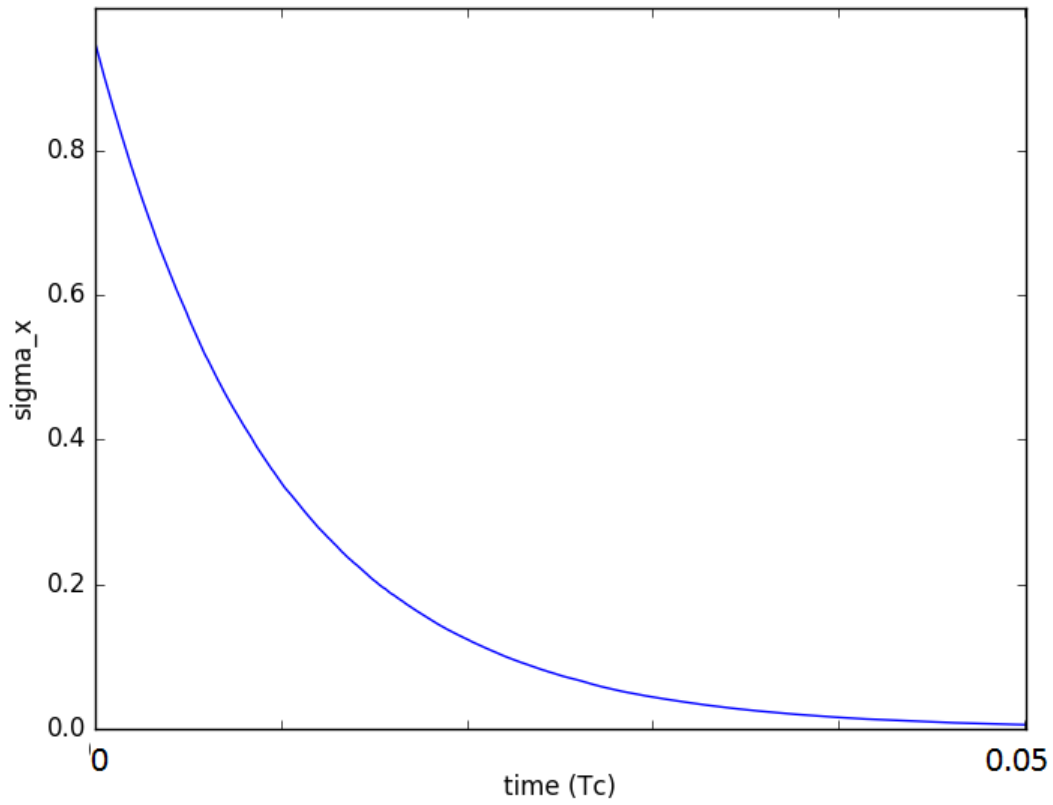

Figure 31: average of 20 quantum trajectories for $\gamma = 3$. The qubit immediately goes to 0 in a time of $0.05T_c$. This is not in line with the prediction of the theory. The starting value is at 1, which is also not in line with the theoretical prediction.

now look at figure 30 we see that the curve appears smoother than in figure 25. The curve does not linger at 1 but its value decreases immediately. We will now increase the number of trajectories in the average to 100 and afterwards to 500.

For $\gamma = 3$ we see that the shape has not changed and the differences from the theoretical curve have not changed either. These differences have already been discussed. For $\gamma = 2$ the curve does not show any significant difference from the theoretical curve. The value starts at 1 and it starts decreasing to 0. It takes the entire time scale of the characteristic time to get close to 0. The conclusion is that for an ideal system with no outside influence, the spin in the x-direction follows the theoretical shape. When outside influences are present, the theoretical curve does not correctly predict the behavior of the spin in the x-direction.

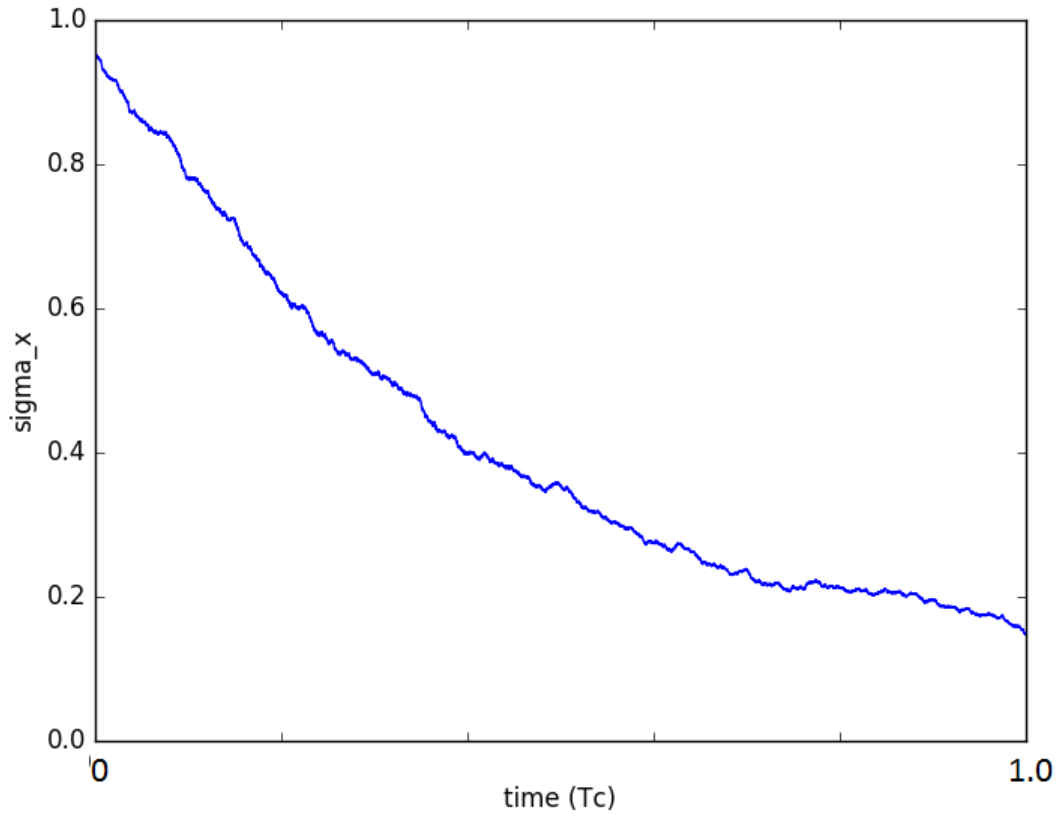

Figure 32: average of 100 quantum trajectories for $\gamma = 2$. The qubit immediately goes to 0 in a time of $1T_c$. This is in line with the prediction of the theory.

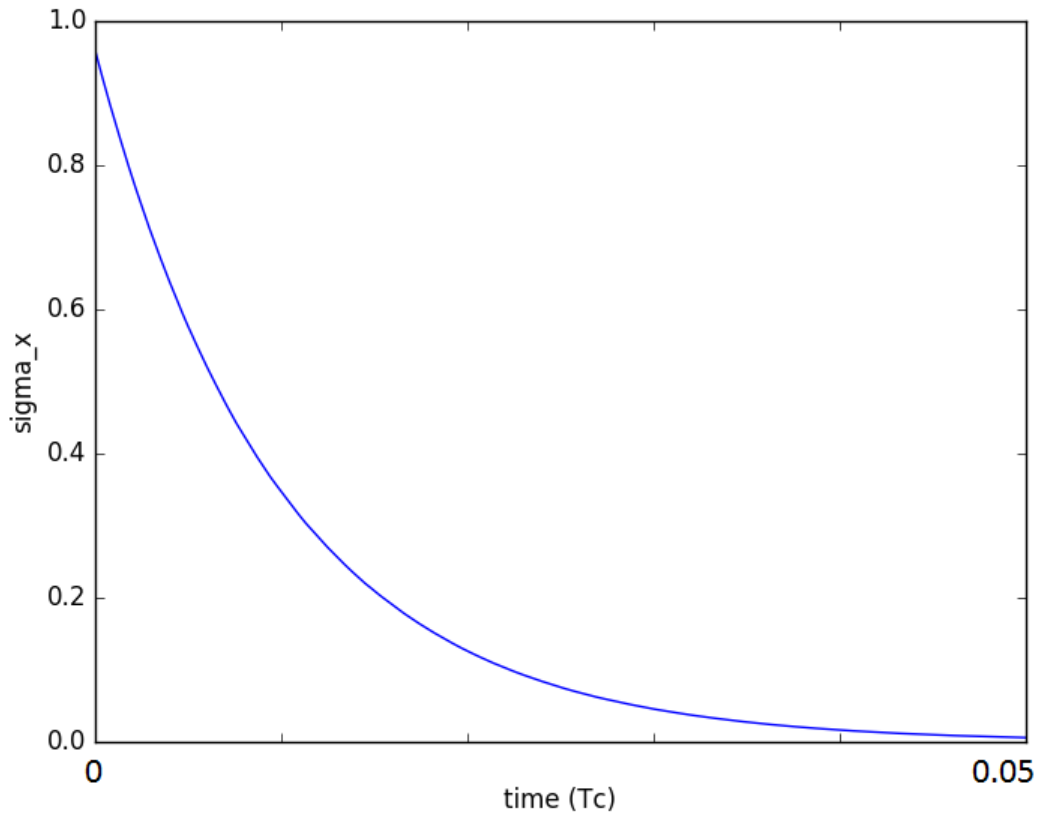

Figure 33: average of 100 quantum trajectories for $\gamma = 3$. The qubit immediately goes to 0 in a time of $0.05T_c$. This is not in line with the prediction of the theory. The starting value is at 1, which is also not in line with the theoretical prediction.

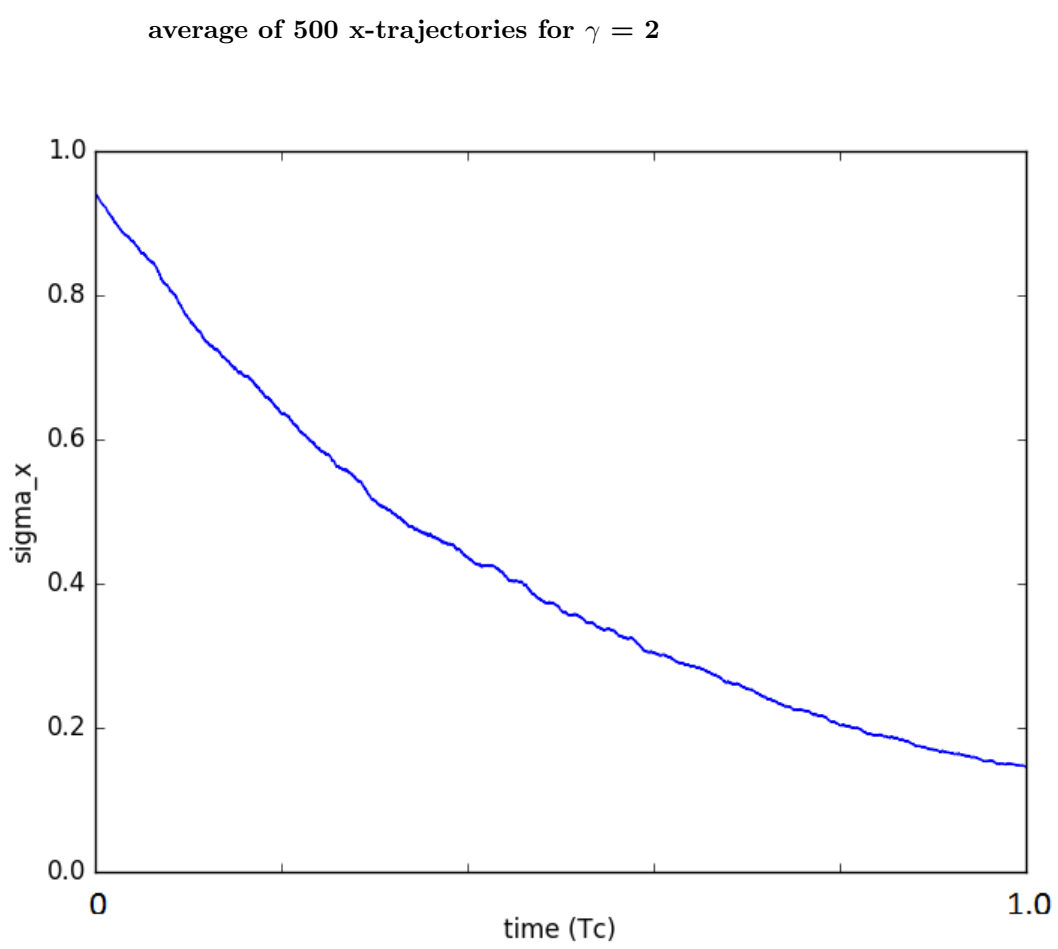

Figure 34: average of 500 quantum trajectories for $\gamma = 2$. The qubit immediately goes to 0 in a time of $1T_c$. This is in line with the prediction of the theory.

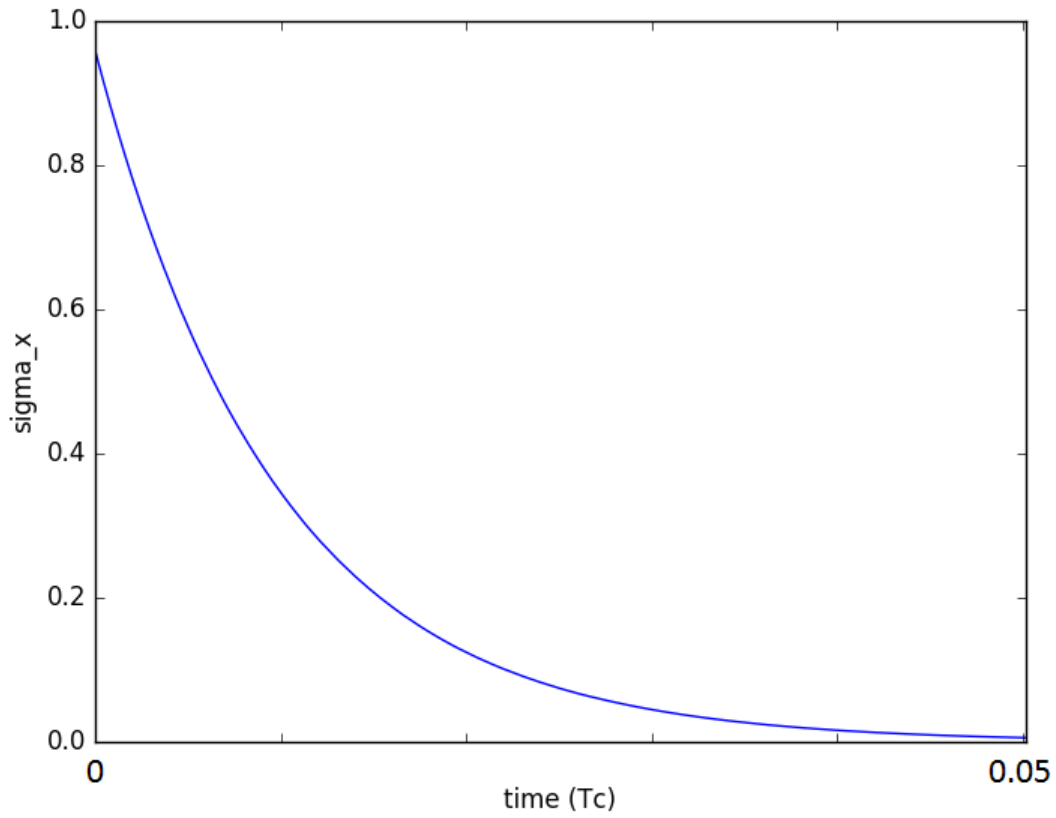

Figure 35: average of 500 quantum trajectories for $\gamma = 3$. The qubit immediately goes to 0 in a time of $0.05T_c$. This is not in line with the prediction of the theory. The starting value is at 1, which is also not in line with the theoretical prediction.

5 Conclusion

In this text we used a quantum system of two quantum particles, specifically two qubits. One qubit acts as a detector in an equal weight superposition of spin up and spin down. The two qubits in the system are entangled. To do multiple measurements on the system, we want to rotate the system back to the equal weight superposition, which is destroyed after a measurement.

We have calculated the time evolution of the density matrix of this quantum system using a standard differential equation. We altered this differential equation using the counting fields method. After this we used the density matrix to calculate the optimal angle of rotation to return the qubit to the equal weight superposition. We did this for an ideal system without outside influences and for a system with outside influences. We derived a relationship between the outside influences on the system, γ, where $\gamma = 2$ represents an ideal system, and a variable inside the system, z. This relation is $\frac{2}{\gamma} = z$.

We also described an algorithm to simulate a random quantum trajectory. We found that the value of the spin in the z-direction does go to the expected value of 0 when the amount of single trajectories in the average trajectory is large. This is true for all values of γ. For the value of the spin in the x-direction we found that the theoretical curve correctly predicts the behavior of an ideal system. We found that for a non ideal system, when $\gamma > 2$, the theoretical curve does not correctly predict the behavior of a trajectory. The theory fails to predict the speed with which the spin in the x-direction goes to 0 in a non ideal system, where the theory is significantly slower at going to 0. Also the theory predicts the starting value to be lower than 1 in a non ideal system, which is not the case in practice.

acknowledgements

This report was part of the final Bachelor Project of the combined bachelors study program Applied Physics and Applied Mathematics at Delft University of Technology. I want to thank two people in particular who have helped me during this project. The first person is my main supervisor, Prof. Dr. Yuli Nazarov. He has helped me during every part of this project. The second person is my second supervisor, Dr. P.M. Visser, who has helped me with the organization around the project.

References

[1] M. Nielsen and I. Chuang. *Quantum computation and Quantum Information* (Cambridge University Press,2000)

[2] H.Wei and Y. V. Nazarov. Phys. Rev. B 78, 045308 (2008)

[3] A. Franquet and Y. V. Nazarov. Phys. Rev. B 95, 085427 (2017)

Appendices

A MATLAB code

```
clear all; close all; clc;
set(0, 'defaultlegendlocation', 'best')
set(0, 'defaultlegendinterpreter', 'Latex')
set(0, 'defaultaxesticklabelinterpreter', 'Latex')
set(0, 'defaulttextinterpreter', 'Latex')

for k = 2:1:6
gamma = k;
j = 1;

fun_1 = @(x,z,T)exp(-4*T).*sqrt(2*T./pi).*x.^(-log(x)./(8*T)).*(1./x-x).^2.*1./((4*T*x).*sqrt(4*z*exp(-2*gamma*T)+exp(-4*T)*(1./x-x).^2));
fun_2 = @(x,z,T)4*exp(-2*gamma*T).*sqrt(z).*sqrt(2*T./pi).*x.^(-log(x)./(8*T)).*1./((4*T*x).*sqrt(4*z*exp(-2*gamma*T)+exp(-4*T).*(1./x-x).^2));

I_1 = @(z,T)integral(@(x)fun_1(x,z,T),0,1);
I_2 = @(z,T)integral(@(x)fun_2(x,z,T),0,1);
CC = @(z,T)I_1(z,T)./(1-I_2(z,T));
Wortel = @(z)sqrt(z);


A = @(z,T)I_1(z,T);
B = @(z,T)I_2(z,T);
sigma_x = @(z,T)((1-exp(-gamma*T))/(gamma*T))*Wortel(z);
%in een loop om te laten werken voor alle waarden van T
i = 1;

a_z = @(v,T)tanh(4*T*v);
a_x = @(v,T)exp((-gamma+2)*T)/cosh(4*T*v);
a = @(v,T)a_z(v,T).^2+a_x(v,T).^2;

while i < 151
fun = @(z)(CC(z,i/100)-Wortel(z));
sol_z(i+1) = fzero(fun,[0,1.1]);

%value_A(i+1) = A(sol_z(i+1),i/500);
%value_B(i+1) = B(sol_z(i+1),i/500);
value_sigma_x(i+1) = sigma_x(sol_z(i+1),i/100);
i = i+1;
end
```

```
    sigma_optT_ind
= max(value_sigma_x);
T_opt = (T_ind − 1)/100;
sigma_mat(j) = sigma_opt;
T_opt;
opt_z(k) = sol_z(2)


y = linspace(0, 1.5, 151);
figure
plot(y(2 : length(y)), value_sigma_x(2 : length(value_sigma_x)),' −b')
xlabel('Time(Tc)')
ylabel('sigma'_x)
title(['gamma =', num2str(k)])
figure
plot(y(2 : length(y)), sol_z(2 : length(sol_z)))
xlabel('Time(Tc)')
ylabel('z')
title(['gamma =', num2str(k)])
save = table(y, value_sigma_x);
filename = ['gammais', num2str(k),' .txt'];

fid = fopen(filename,' wt');
fprintf(fid, ['Time''''Sigma''']);
fprintf(fid,' %f%f', [y'value_sigma'_x]'); fclose(fid); end
```

B python code

```
import numpy
import math

    # constantes een waarde geven
delta_t = 0.01
mn = 1
#mm = −mn
cn = numpy.cos(mn ∗ delta_t)
#cm = numpy.cos(mm ∗ delta_t)
sn = numpy.sin(mn ∗ delta_t)
#sm = numpy.sin(mm ∗ delta_t)
sin2 = numpy.sin(2 ∗ mn ∗ delta_t)
Tc = 1/(mn ∗ ∗2 ∗ delta_t ∗ ∗2)
gamma = 2.

#functions
```

$def step(rhopp, rhopm):$
$treshold = 0.5 * (1 + 2 * rhopp * sin2 - sin2)$
$if numpy.random.rand() < treshold:$
$result = 1.0$
$rhopp_new = 0.5 * rhopp * (cn + sn) * (cn + sn)/(treshold)$
$rhopm_new = numpy.exp(-(gamma - 2) * delta_t) * 0.5 * rhopm * (cn + sn) *$
$(cn - sn)/(treshold)$
$else:$
$result = -1.0$
$rhopp_new = 0.5 * rhopp * (cn - sn) * (cn - sn)/(1.0 - treshold)$
$rhopm_new = numpy.exp(-(gamma - 2) * delta_t) * 0.5 * rhopm * (cn - sn) *$
$(cn + sn)/(1.0 - treshold)$
$return rhopp_new, rhopm_new, result$

$def loop(rhopp, rhopm, k):$
$restot = 0.0$
$rhopptot = [rhopp]$
$rhopmtot = [rhopm]$
$for i in range(0, k):$
$rhopp_new, rhopm_new, result = step(rhopp, rhopm)$
$restot += result$
$rhopp = rhopp_new$
$rhopm = rhopm_new$
$rhopptot.append(rhopp)$
$rhopmtot.append(rhopm)$
$return rhopp, rhopm, restot/(k * sin2), rhopptot, rhopmtot$

$def rotate(rhopp, rhopm, result, k):$
$hoek = -0.5 * math.atan(numpy.sinh(4. * result * delta_t * k * gamma/2.))$
$rhopp_rot = rhopp*(numpy.cos(hoek)**2-numpy.sin(hoek)**2)+numpy.sin(hoek)*$
$*2 + 2 * rhopm * numpy.cos(hoek) * numpy.sin(hoek)$
$rhopm_rot = (1 - 2 * rhopp) * numpy.cos(hoek) * numpy.sin(hoek) + rhopm *$
$(numpy.cos(hoek) * *2 - numpy.sin(hoek) * *2)$
$return rhopp_rot, rhopm_rot$

$def total(n):$
$rhopp = 0.5$
$rhopm = 0.5$
$k = int(Tc)$
$listv = []$
$rhopptotav = [0] * (k + 1)$
$rhopmtotav = [0] * (k + 1)$
$for i in range(0, n):$

$rhopp_new, rhopm_new, v, rhoppav, rhopmav = loop(rhopp, rhopm, k)$
$listv.append(v)$
$rhopptotav = numpy.add(rhoppav, rhopptotav)$
$rhopmtotav = numpy.add(rhopmav, rhopmtotav)$
$rhopp, rhopm = rotate(rhopp_new, rhopm_new, v, k)$
$returnlistv, rhopptotav/n, rhopmtotav/n$

$\#\#\#\#\#\#\#\#\#\#\#\#\#$
$importmatplotlib.pyplotasplt$
$v, rpl, rpm = total(1)$
$T = numpy.linspace(0, int(Tc * delta_t), num = len(rpl))$
$mylist = 2 * rpl$
$sig_z = [x - 1 for x in mylist]$
$sig_x = [2 * x for x in rpm]$
$plt.plot(T, sig_z)$
$plt.ylabel('sigma'_z)$
$plt.xlabel('time(Tc)')$
$plt.axis([0, int(Tc * delta_t), -1, 1])$
$plt.show()$