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Boundary values of analytic functions on the disc

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"Boundary values of analytic functions on the disc"

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Abstract

In this bachelor's thesis we will solve the Dirichlet problem with an $L^p(\mathbb{T})$ boundary function. First, we will focus on the holomorphic version of the Dirichlet problem and introduce Hardy space theory, from which will follow a sufficient condition on the Fourier coefficients of the boundary function. Then we will prove the Marcinkiewicz interpolation theorem. After that we introduce the conjugate function $\tilde{f}$, which equals the Hilbert transform of f, and use functional analysis to prove an important duality argument of the Hilbert transform. Finally, we will give several different proofs for the boundedness of the map $f \mapsto \tilde{f}$ using the Marcinkiewicz interpolation theorem and the duality argument: the last proof will be done rigorously from scratch, i.e. without relying on (unproved) arguments from other literature.

Preface

This project builds on the lecture notes of the course Fourier Analysis given by Alex Amenta in the third quarter of 2018, an elective course as part of the studies applied mathematics at the Delft University of Technology. Therefore the reader is expected to know about basic Fourier analysis. The project will extend towards harmonic analysis, a huge branch of mathematics.

The Fourier transform on the one-dimensional torus is scaled by 2π, and in different books, this scaling might be in some definitions, while not in others (e.g. the Poisson kernel), so an appendix is added with some basic definitions in Fourier analysis to prevent any ambiguity in these definitions (these will be the same as in *An Introduction to Harmonic Analysis* by Katznelson [1]). The appendix will also contain the theorems that are used throughout this paper and their proofs, which are either not central within Fourier or harmonic analysis, or have not been proven in the lecture notes. The reader is encouraged to read the first paragraph of the appendix before reading this paper, since the notation of some concepts explained in there might be different than usual.

The proofs given in this paper will be rigorous by default. Heuristic proofs will be announced beforehand and usually these kind of proofs will refer to details in other books, which are out of scope of this project, so references can be found at the end of this paper (these heuristic proofs are actually only in Subsection 3.4). Furthermore, the proofs will be detailed, so the reader doesn't have to spend a lot of time fact checking logical conclusions theirself, filling in huge gaps theirself or just assume parts of a proof to be true. For the sake of efficiency, we will keep the conclusion compact and get straight to the point.

I would like to thank Dorothee Frey for guiding me in this project, and for the well-written lecture notes for the course Fourier Analysis. Special thanks to Alex Amenta for being a ripper of a bloke to have had the lectures from.

“Mathematical analysis is as extensive as nature herself.”

— Joseph Fourier

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Introduction

The *Dirichlet problem* on the (complex) unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ (whose boundary $\partial\mathbb{D}$ may be identified with $\mathbb{T}$: see appendix for details), as one may have seen in the course Fourier analysis as mentioned before, or in more general domains in PDE courses, is stated as follows:

Dirichlet problem. *Given a function $f : \partial\mathbb{D} \to \mathbb{C}$, is there an analytic function $u : \overline{\mathbb{D}} \to \mathbb{C}$ such that*

(i) u is harmonic in $\mathbb{D}$, i.e. $\Delta u = 0$ in $\mathbb{D}$, where $\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ (identifying $\mathbb{C}$ with $\mathbb{R}^2$),

(ii) f extends (analytically) to u, i.e. $u = f$ on $\partial\mathbb{D}$ and $\lim\limits_{r \uparrow 1} u(re^{it}) = f(e^{it})$ for each $t \in [0, 2\pi)$?

In the case that f is in the space $C(\mathbb{T})$ of 2π-periodic continuous functions, we have seen in the course Fourier analysis that we can take the convolution of $f_{bd}(t) := f(e^{it})$ with the Poisson kernel P_r to obtain the solution

$$u(re^{it}) = \begin{cases} \frac{1}{2\pi} \int\limits_0^{2\pi} f_{bd}(t-\tau) P_r(\tau)\, d\tau, & r < 1, \\ f_{bd}(t), & r = 1, \end{cases}$$

where $u(re^{it})$ converges uniformly on $\mathbb{T}$ to $f_{bd}(t)$ as $r \to 1$.
For the converse of this problem, we see that we can find f by simply taking the pointwise limit $f(e^{it}) := \lim\limits_{r \uparrow 1} u(re^{it})$.

We would like to generalize this idea to other spaces of functions. For example, if $f_{bd} \in L^p(\mathbb{T})$, would we have the same solution, but then with convergence in $L^p(\mathbb{T})$-norm? What would happen if we replaced *harmonic* by *holomorphic*, which is a stronger statement and more well-suited for complex-valued functions of a complex variable? In this project we will thoroughly study the Hardy space H^p, a space which is very suitable for analysis of this problem. We will see that a function in H^p can be related to a boundary function in $L^p(\mathbb{T})$, their respective norms having the same value (Theorem 1.11) and furthermore that the function is Poisson integral of the boundary function, where the Fourier coefficients of this boundary function must be equal to zero for $n < 0$ (Theorem 1.15).

We will also have a rigorous look at the convergence of Fourier series in $L^p(\mathbb{T})$, which will allow us to find the boundary value of the solution of the Dirichlet problem if we only know the Fourier coefficients of the boundary function, without having to rely on the interior domain (the Poisson integral part) of the solution. For $1 < p < \infty$ we will see that the Fourier series of a function in $L^p(\mathbb{T})$ converges in $L^p(\mathbb{T})$, and in that case it converges back to this function (Theorem 4.4).

1 Hardy spaces

In this section we will look at the *Hardy space* H^p and derive some useful properties, including one that allows us to identify a function in H^p with a boundary function in $L^p(\mathbb{T})$ for $p \geq 1$. For $p = 2$, this will be an easy task, but for $p \neq 2$, we will need to introduce the *Blaschke product* in order to use the *canonical factorization* of functions in H^p for the bigger proofs.

1.1 Conformal mappings and Jensen's inequality

For fixed $0 < |\zeta| < 1$, we define the function

$$b(z, \zeta) := \frac{\bar{\zeta}(\zeta - z)}{|\zeta|(1 - z\bar{\zeta})}.$$

As seen in the course Complex Analysis, this is a conformal mapping from $\mathbb{D}$ to $\mathbb{D}$, which means that b is holomorphic and $b' \neq 0$ in $\mathbb{D}$. It is also clear that $b(\zeta, \zeta) = 0$ and $b(0, \zeta) = |\zeta|$. Furthermore, on $|z| = 1$ we have $|b(z, \zeta)| = 1$ and for $|z| < 1$ we have $|b(z, \zeta)| < 1$, which we have already proved in Section I.1 Exercise 11 in [2]. For $\zeta = 0$ we'll simply define $b(z, 0) := z$.
For some $0 < r < 1$, let f be a function that is holomorphic on $|z| \leq r$ with zeros $\zeta_1, \ldots, \zeta_k$ in $|z| \leq r$ (zeros with a multiplicity greater than 1 are counted again), and let us define the function

$$f_1(z) := f(z) \left(\prod_{n=1}^{k} b\left(\frac{z}{r}, \frac{\zeta_n}{r}\right) \right)^{-1}. \tag{1}$$

Note that with f_1 we actually mean the analytic continuation of the same function to the zeros of f. Now this function is clearly holomorphic, has no zeros and $|f_1(z)| = |f(z)|$ for $|z| = r$ since $|b(1, \zeta)| = 1$. With the help of this function we can now show *Jensen's inequality.*

Lemma 1.1 (Jensen's inequality)**.** *Let $0 < r < 1$, and let f be holomorphic in $|z| \leq r$ for $z \in \mathbb{C}$ with zeros as above. If $f(0) \neq 0$, we have*

$$\log|f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{it})| dt.$$

If f has a zero of order s at $z = 0$, then

$$\log\left|\lim_{z \to 0} \frac{f(z)}{z^s}\right| + \log(r^s) \leq \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{it})| dt.$$

Proof. Since f_1 is holomorphic and has no zeros, we have that (the principal branch) $\mathrm{Log}(f_1(z)) = \log|f_1(z)| + i \,\mathrm{Arg}\,(f_1(z))$ is holomorphic too[1] and thus $\log|f_1(z)|$ is harmonic as the real part of a holomorphic function. The mean value property (see Theorem 11 in the appendix) then gives us

$$\log|f_1(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f_1(re^{it})\right| dt.$$

[1]Since f_1 is holomorphic, we have that f_1' and $\frac{1}{f_1}$ and thus $\frac{f_1'}{f_1}$ are holomorphic as $f_1 \neq 0$, and so its primitive $\mathrm{Log}(f_1(z))$ is holomorphic in the simply connected (closed) domain $|z| \leq r$.

Suppose that f does not vanish on both 0 and $|z| = r$. Writing out f_1 as in Equation (1) and noting that $b\left(0, \frac{\zeta_n}{r}\right) = \frac{|\zeta_n|}{r}$ and $\left|b\left(\frac{re^{it}}{r}, \frac{\zeta_n}{r}\right)\right| = 1$, we obtain

$$\log|f(0)| - \log\left(\prod_{n=1}^{k} \frac{|\zeta_n|}{r}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{it})\right| dt,$$

which is equal to

$$\log|f(0)| + \log\left(\prod_{n=1}^{k} \frac{r}{|\zeta_n|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{it})\right| dt. \tag{2}$$

By continuity in r in the above equality, we may substitute r in Equation (2) by r' (where $|\zeta_n| < r' < r$ for each n) and then let r' approach r and thus we see that it also holds if f has zeros on $|z| = r$. Since $r \geq |\zeta_n|$ for each n, the logarithm yields a non-negative number, which we can delete, and thus we obtain

$$\log|f(0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{it})\right| dt.$$

If f has a zero of order s at $z = 0$, then we split $f_1(z)$ into the product of $g(z) := \frac{f(z)}{z^s}$ and $z^s\left(\prod_{n=1}^{k} b\left(\frac{z}{r}, \frac{\zeta_n}{r}\right)\right)^{-1}$. Now $f_1(0)$ is the product of the limits for $z \to 0$ of both, and noting that $|z|^s\left|\prod_{n=1}^{k} b\left(\frac{z}{r}, \frac{\zeta_n}{r}\right)\right|^{-1} = r^s$ on $|z| = r$ since $b\left(\frac{z}{r}, 0\right) = \frac{z}{r}$, we repeat the proof above to obtain the other inequality. $\square$

1.2 Blaschke products

Let $p > 0$ and f be holomorphic in $\mathbb{D}$. Define $h_p(f, r) = \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^p dt$. We will later use this function to define the Hardy space.
If $p \geq 1$, $0 < r, \rho < 1$, then we see that

$$h_p(f, r\rho) = \|f(r\rho e^{it})\|^p_{L^p(\mathbb{T})} = \|f(re^{it}) * P_\rho(t)\|^p_{L^p(\mathbb{T})},$$

since $f(r\rho e^{it}) = f(re^{it}) * P_\rho(t)$ (see Lemma 8 in the appendix). Now by Minkowski's inequality[2] we get

$$\|f(re^{it}) * P_\rho(t)\|^p_{L^p(\mathbb{T})} \leq \|P_\rho(t)\|^p_{L^1(\mathbb{T})}\|f(re^{it})\|^p_{L^p(\mathbb{T})} = h_p(f, r),$$

since $\|P_\rho(t)\|_{L^1(\mathbb{T})} = 1$ as a non-negative summability kernel. So we have $h_p(f, r\rho) \leq h_p(f, r)$, which means that $h_p(f, r)$ is a non-decreasing function in r. In particular, this holds for $p = 2$, and with this case we will now prove it for all $p > 0$.

Lemma 1.2. *Let $p > 0$ and f be holomorphic in $\mathbb{D}$. Then $h_p(f, r)$ is a non-decreasing function in r.*

[2] It's a special case of Young's convolution inequality, however, we will give the proof in the appendix in Lemma 3, since we will use this inequality more than once.

Proof. Let $0 < r' < r < 1$. First suppose that f has no zeros on $|z| \leq r$. Now choose the principal branch of $g(z) = (f(z))^{\frac{p}{2}} = \exp(\frac{p}{2}\operatorname{Log}(f(z)))$. This function is holomorphic, since we saw earlier that $\operatorname{Log}(f(z))$ is holomorphic. Now, by the $p = 2$ case, we have that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \left|f(r'e^{it})\right|^p dt &= \frac{1}{2\pi}\int_0^{2\pi} \left|g(r'e^{it})\right|^2 dt \\ &\leq \frac{1}{2\pi}\int_0^{2\pi} \left|g(re^{it})\right|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^p dt.\end{aligned}$$

Now suppose that f has zeros in $|z| < r$, but not on $|z| = r$, then we define $f_1(z)$ again as in Equation (1) with the same r, and note that $|f(z)| = |f_1(z)|$ on $|z| = r$ and that $|f(z)| < |f_1(z)|$ in $|z| < r$, since we had that $\left|b\left(\frac{z}{r}, \frac{\zeta_n}{r}\right)\right| < 1$ in $|z| < r$. Now we get

$$h_p(f, r') < h_p(f_1, r') \leq h_p(f_1, r) = h_p(f, r)$$

by the previous case.
Now suppose that f has zeros on $|z| = r$. As $h_p(f, r)$ is a continuous function in r, we can use the same continuity argument as in the proof of Lemma 1.1. □

We will now set up the definition of the Blaschke product.

Proposition 1.3. *Let $(\zeta_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{C}$ with $|\zeta_n| < 1$ and $\sum_{n=1}^{\infty}(1 - |\zeta_n|) < \infty$. Then we have that*

$$B(z) := \prod_{n=1}^{\infty} b(z, \zeta_n)$$

is absolutely and uniformly convergent in every closed disc with radius $0 < r < 1$.

Proof. Let $0 < r < 1$. As shown in Lemma 15 in the appendix, we only need to show that $\sum_{n=1}^{\infty}|1 - b(z, \zeta_n)|$ converges uniformly on $|z| \leq r$, since $0 < |b(z, \zeta_n)| < 1$ on $|z| \leq r$ with $z \neq \zeta_n$: in the case that $z = \zeta_n$ this infinite product is simply equal to zero. For $0 < |\zeta_n| < 1$, we have that $0 < |z\zeta_n| < r$ and so we see that

$$\begin{aligned}\left|1 - \frac{\bar{\zeta_n}(\zeta_n - z)}{|\zeta_n|(1 - z\bar{\zeta_n})}\right| &= \left|\frac{|\zeta_n| - z\bar{\zeta_n}|\zeta_n| - |\zeta_n|^2 + z\bar{\zeta_n}}{|\zeta_n|(1 - z\bar{\zeta_n})}\right| \\ &= \left|\frac{(|\zeta_n| + z\bar{\zeta_n})(1 - |\zeta_n|)}{|\zeta_n|(1 - z\bar{\zeta_n})}\right| \\ &\leq \frac{|\zeta_n| + |z||\zeta_n|}{|\zeta_n|(1 - |z||\zeta_n|)}(1 - |\zeta_n|) \\ &= \frac{1 + |z|}{1 - |z\zeta_n|}(1 - |\zeta_n|) \\ &\leq \frac{1 + r}{1 - r}(1 - |\zeta_n|).\end{aligned}$$

Due to the assumption that $\sum_{n=1}^{\infty}(1 - |\zeta_n|) < \infty$, we can conclude that the series will be uniformly and absolutely convergent by the Weierstraß M-test, and by the same assumption we also know that there are at most finitely many $\zeta_n = 0$, so the same is true for the infinite product. □

Definition 1.4. *The function $B(z)$ as defined above is called the* Blaschke product *corresponding to the sequence $(\zeta_n)_{n=1}^{\infty}$ in $\mathbb{D}$, assuming that $(\zeta_n)_{n=1}^{\infty}$ satisfies the* Blaschke condition *$\sum_{n=1}^{\infty}(1 - |\zeta_n|) < \infty$.*

The Blaschke product is holomorphic, by noting that $B(z) = \exp(\sum_{n=1}^{\infty} \operatorname{Log}(b(z,\zeta_n)))$ contains a uniformly convergent sum of holomorphic functions in $\mathbb{D}$ wherever $b(z,\zeta_n) \neq 0$ for all n (again, the infinite product would be equal to zero else). $B(z)$ vanishes only in the points ζ_n: all these points have a finite multiplicity, else the Blaschke condition would not hold. Since $|b(z,\zeta_n)| < 1$ in $\mathbb{D}$, we also have $|B(z)| < 1$ in $\mathbb{D}$.

1.3 Hardy spaces

We now have the tools to define the Hardy space H^p and the Nevanlinna class $\mathcal{N}$.

Definition 1.5. *Let $p > 0$. The* Hardy space[3] *H^p is the space of all functions holomorphic in $\mathbb{D}$ such that*

$$\|f\|_{H^p}^p := \sup_{0<r<1} h_p(f,r) < \infty.$$

The Nevanlinna class *$\mathcal{N}$ is the space of all functions holomorphic in $\mathbb{D}$ such that*

$$\|f\|_{\mathcal{N}} := \sup_{0<r<1} \int_0^{2\pi} \log^+|f(re^{it})|\, dt < \infty,$$

where $\log^+$ *is the positive part of the function* $\log$, *i.e.* $\log^+(x) = \sup(0, \log(x))$ *for each* $x \geq 0$.

By Lemma 1.2 we have that $\sup_{0<r<1} h_p(f,r) = \lim_{r\to 1} h_p(f,r)$, which makes it a lot easier to calculate $\|f\|_{H^p}$. For $p \geq 1$, $\|f\|_{H^p}$ is a norm on H^p. We will later see that a function $f \in H^p$ can be identified with a function $f_{bd} \in L^p(\mathbb{T})$. If $0 < p' < p$, then we have $H^p \subseteq H^{p'} \subseteq \mathcal{N}$: the first inclusion follows from Theorem 1.11 later on, the second inclusion is obvious for $p \geq 1$ from the fact that $\log^+ x \leq x^p$ for $x \geq 0$, and so we have $\|f\|_{\mathcal{N}} \leq \|f\|_{H^p}^p$.

We will now prove a useful equivalence in the Hardy space H^2.

Lemma 1.6. *Let $f(z)$ be of the form $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f \in H^2$ if and only if $\sum_{n=0}^{\infty} |a_n|^2 < \infty$.*

Proof. In the lecture notes of the course Fourier Analysis we have seen that the sequence $\left(\frac{1}{\sqrt{2\pi}} e^{in(\cdot)}\right)_{n=-\infty}^{\infty}$ is an orthonormal system in $L^2(\mathbb{T})$, and by Lemma 4.28 in [3] we know that

$$(2\pi)^2 \left\| \sum_{n=0}^{\infty} a_n r^n \frac{1}{\sqrt{2\pi}} e^{int} \right\|_{L^2(\mathbb{T})}^2 = \sum_{n=0}^{\infty} |a_n|^2 r^{2n},$$

by taking the coefficients $a_n r^n$ with $a_n = 0$ for $n < 0$ and $0 < r < 1$, where the left part has a fixed finite value if and only if the left sum converges in $L^2(\mathbb{T})$ if and only if the right part is finite. Now we note that

$$h_2(f,r) = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{n=0}^{\infty} a_n r^n e^{int} \right|^2 dt = (2\pi)^2 \left\| \sum_{n=0}^{\infty} a_n r^n \frac{1}{\sqrt{2\pi}} e^{int} \right\|_{L^2(\mathbb{T})}^2 .$$

[3] We may also define H^∞ as the space of all functions holomorphic in $\mathbb{D}$ such that $\|f\|_{H^\infty} := \sup_{0<|z|<1} |f(z)| < \infty$, but we will not use this space. Note that H is for Hardy and $\mathcal{N}$ is for Nevanlinna.

By the monotone convergence theorem[4] applied on the function $|a_n|^2 r^{2n}$, which is non-decreasing in r, we obtain $\lim_{r\to 1} \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} |a_n|^2$ and thus we have

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2,$$

from which the desired conclusion follows. □

By Corollary 4.32 in [3] we can find a unique $f_{bd} \in L^2(\mathbb{T})$ with $\widehat{f_{bd}} = a_n$, and we see that $f_{bd}(t) = \sum_{n=0}^{\infty} a_n e^{int}$ (in $L^2(\mathbb{T})$). The convolution of $f_{bd}(t)$ with the Poisson kernel $P_r(t)$ is equal to $f(re^{it})$, which means that f is the Poisson integral of f_{bd}. Since the Poisson kernel is a summability kernel, we have for $1 \le p < \infty$ that the Poisson integral of some function in $L^p(\mathbb{T})$ converges in the same space to this function by Corollary 4.23 in [3], so in particular we have that $\lim_{r\to 1} f(re^{it}) = f_{bd}(t)$ in $L^2(\mathbb{T})$. By Theorem 3.11 (a theorem that we will see later in a subsection that is independent of this section) we have convergence almost everywhere, since $L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$.

1.4 Canonical factorization

Proposition 1.7. *Let $f \in \mathcal{N}$, then its zeros (counted as many times as their multiplicities) satisfy the Blaschke condition.*

Proof. Note that f is holomorphic in $\mathbb{D}$, so f is holomorphic in $|z| \le r$ for all $0 < r < 1$. Without loss of generality we assume that $f(0) \neq 0$ (the proof goes analogously if we iterate over these s zeros first, using the second Jensen's inequality of Lemma 1.1). From Equation (2) in the proof of Lemma 1.1 we obtain

$$\log|f(0)| - \sum_{n=1}^{k} \log|\zeta_n| + k\log(r) \le \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{it})\right| dt,$$

where we only consider the first k zeros of f and $0 < \zeta_n < r$ for each $1 \le n \le k$. Note that we now have an inequality sign, because we took the other zeros out of the left side of the equality. Now we let $r \to 1$ and see that

$$\log|f(0)| - \|f\|_{\mathcal{N}} \le \sum_{n=1}^{k} \log|\zeta_n|.$$

This inequality holds for all k and so it holds also as $k \to \infty$, and so $\sum_{n=1}^{\infty} \log|\zeta_n|$ is bounded from below and thus converges to some negative real number by the monotone convergence theorem for sequences of real numbers, which we combine with the observation in the proof of Lemma 15 to see that the infinite product $\prod_{n=1}^{\infty} |\zeta_n|$ converges, and by the same lemma we obtain $\sum_{n=1}^{\infty} (1 - |\zeta_n|) < \infty$. □

[4]This works as explained in Footnote 7, noting that we have convergence almost everywhere, as $L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$.

By Proposition 1.7 and the fact that $H^p \subseteq \mathcal{N}$ we may define the Blaschke product B on $\mathbb{D}$ for every $f \in H^p$ (corresponding to the sequence of zeros of f), and both functions share exactly the same zeros. Since this Blaschke product is holomorphic in $\mathbb{D}$, we have that the analytic continuation of $F(z) := f(z)(B(z))^{-1}$ (we will just call this $F(z)$) to the zeros of f is holomorphic, has no zeros and satisfies $|F(z)| = |f(z)||B(z)|^{-1} > |f(z)|$ in $\mathbb{D}$.

Definition 1.8. *Let $f \in \mathcal{N}$. We call $f = BF$ the* canonical factorization of f, *where B is the Blaschke product corresponding to the sequence of zeros of f.*

We note that a Blaschke product corresponding to the sequence of zeros of f can be found for every function f in $\mathcal{N}$, since a holomorphic function in $\mathbb{D}$ either has no zeros, countably many zeros[5], or is the zero function[6]: in the first case $B = 1$ (that's by definition the product of no elements), and in the last case it makes sense to define $B = 0$ in $\overline{\mathbb{D}}$, since $\zeta = 0$ has a "multiplicity of infinity", and so B will "consist of infinitely many products of z", which means that $B = 0$ in $\mathbb{D}$ and we extend this analytically: however, F will not be defined, but since $f = 0$ is a trivial case, there is no information to be obtained from this f.

As we may expect, there is a useful relation between f and F.

Theorem 1.9. *Let $f \in H^p$ for $p > 0$, and let $f = BF$ be its canonical factorization. Then we have that $F \in H^p$ and $\|F\|_{H^p} = \|f\|_{H^p}$.*

Proof. The Blaschke product B is of the form

$$B(z) = \lim_{k\to\infty} z^s \prod_{n=1}^{k} b(z, \zeta_n)$$

and so we define $F_k(z) := f(z)\left(z^s \prod_{n=1}^{k} b(z,\zeta_n)\right)^{-1}$, which is clearly holomorphic and converges uniformly to F in every closed disc with radius $0 < r < 1$. Also, for each k we have that $\left|z^s \prod_{n=1}^{k} b(z,\zeta_n)\right|$ converges uniformly to 1 as $|z| \to 1$ (and so does its $-p$-th power), which means that for any $\varepsilon > 0$ we have

$$\begin{aligned} &\left|\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^p dt - \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^p \left|z^s \prod_{n=1}^{k} b(z,\zeta_n)\right|^{-p} dt\right| \\ \leq\ & \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^p \left|\left|z^s \prod_{n=1}^{k} b(z,\zeta_n)\right|^{-p} - 1\right| dt \\ <\ & \varepsilon \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^p dt = \varepsilon h_p(f,r) \end{aligned}$$

[5]Each zero will have finite multiplicity as reasoned underneath Definition 1.4: else the Blaschke condition would clearly not hold.

[6]If $f \in \mathcal{N}$ has uncountably many zeros, then we can find a limit point of these zeros in $\mathbb{D}$, which implies that $f = 0$ in $\mathbb{D}$ by the identity theorem. The limit point exists in $\mathbb{D}$: suppose there are only countably many zeros in $\{z \in \mathbb{C} : |z| < 1 - \frac{1}{n}\}$ for each $n \in \mathbb{N}$. Taking the countable union of these infinitely many n, being equal to $\mathbb{D}$, yields a contradiction with $\mathbb{D}$ having uncountably many zeros. Now we can find a compact subset of $\mathbb{D}$ (by taking the closure of some subset of the form as above) which contains infinitely many zeros and will thus have a limit point within this compact subset.

for r close enough to 1. Now $\lim_{r\to 1} h_p(f,r)$ is finite by assumption, so we have that both terms are equal as $r \to 1$, and thus we get

$$\begin{aligned}\|F_k\|_{H^p}^p &= \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\left|F_k(re^{it})\right|^p dt = \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{it})\right|^p \left|z^s\prod_{n=1}^{k} b(z,\zeta_n)\right|^{-p} dt \\ &= \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{it})\right|^p dt = \|f\|_{H^p}^p,\end{aligned}$$

which means that $\lim_{k\to\infty}\|F_k\|_{H^p}^p = \|f\|_{H^p}^p$. Now let $r<1$. Since F_k converges uniformly to F on $|z|\le r$, we may swap the limit $k\to\infty$ with the integral in the definition of $h_p(F_k,r)$, and $h_p(F_k,r)$ is non-decreasing as $r\to 1$ by Lemma 1.2, so we see that

$$h_p(F,r) \le \lim_{k\to\infty} h_p(F_k,r) \le \lim_{k\to\infty}\lim_{r\to 1} h_p(F_k,r) = \lim_{k\to\infty}\|F_k\|_{H^p} = \|f\|_{H^p}^p,$$

which gives $\|F\|_{H^p} \le \|f\|_{H^p}$ when letting r approach 1, and since $\|f\|_{H^p}<\infty$, we have that $F\in H^p$. For the reverse inequality, we simply recall that $|F(z)| > |f(z)|$ in $\mathbb{D}$ to obtain

$$\|f\|_{H^p}^p = \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{it})\right|^p dt \le \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\left|F(re^{it})\right|^p dt = \|F\|_{H^p}^p$$

and now we can also conclude that $\|F\|_{H^p} = \|f\|_{H^p}$. □

This relation is useful in the theory of Hardy spaces, because we may work with non-vanishing functions, and thus can be raised to any power in order to move to other Hardy spaces that may be more convenient.

With the theorem we can prove an interestic fact about the continuous extension of the Blaschke product to the boundary of $\mathbb{D}$.

Corollary 1.10. *Let B be a Blaschke product in $\mathbb{D}$, then we have that $|B(e^{it})| = 1$ for almost every t.*

Proof. We recall that $|B(z)|<1$ and that B is harmonic (by Proposition 10 in the appendix) in $\mathbb{D}$, so by Lemma 9 in the appendix, we know that B is the Poisson integral of some bounded function (a function in $L^\infty(\mathbb{T}) \subseteq L^2(\mathbb{T}) \subseteq L^1(\mathbb{T})$), which means that $B_{bd}(t) := B(e^{it})$ exists as a radial limit (i.e. $B(e^{it}) := \lim_{r\to 1} B(re^{it})$ exists) for almost every t by Theorem 3.11. Now if we take $f=B$, we see that $F=1$ everywhere in the canonical factorization and since $|B(re^{it})|$ is non-decreasing in r, we may apply the monotone convergence theorem[7] to obtain

$$\|f\|_{H^2}^2 = \lim_{r\to 1}\frac{1}{2\pi}\int_0^{2\pi}\left|B(re^{it})\right|^2 dt = \frac{1}{2\pi}\int_0^{2\pi}\left|B(e^{it})\right|^2 dt = \|B_{bd}\|_{L^2(\mathbb{T})}^2 < \infty,$$

so $B\in H^2$ and by the previous theorem we have $\|f\|_{H^2} = \|F\|_{H^2} = 1$, and combining the equalities we obtain $\frac{1}{2\pi}\int_0^{2\pi}\left|B(e^{it})\right|^2 dt = 1$, which can only hold if $\left|B(e^{it})\right| = 1$ almost everywhere, since $\left|B(e^{it})\right| \le 1$ on $\overline{\mathbb{D}}$ by continuity. □

[7] As the monotone convergence theorem holds for the sequence of continuous functions $|B(r_n e^{it})|^2$ with limit $|B(e^{it})|^2$ almost everywhere for all sequences $r_n\to 1$ as $n\to\infty$, this simply holds too for the limit $r\to 1$.

1.5 Boundary values of a function in H^p

Now we arrive at an important theorem that gives us information about the boundary function of functions in Hardy spaces.

Theorem 1.11. *Let $p > 0$ and suppose $f \in H^p$. Then the limit $f_{bd}(t) = f(e^{it}) := \lim_{r\to 1} f(re^{it})$ exists for almost all $t \in \mathbb{T}$, $f_{bd} \in L^p(\mathbb{T})$ and we have*

$$\|f\|_{H^p} = \|f_{bd}\|_{L^p(\mathbb{T})}.$$

Proof. If $p = 2$, then the first two statements are indeed true, as we have already discussed after Lemma 1.6. We recall from the lemma that

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2,$$

and by Corollary 4.32 in [3] we also have that

$$\sum_{n=0}^{\infty} |a_n|^2 = \sum_{n=-\infty}^{\infty} |\widehat{f_{bd}}(n)|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f_{bd}(t)|^2 dt = \|f_{bd}\|_{L^2(\mathbb{T})}^2$$

and so it's proven for $p = 2$.
If $p > 0$, then we consider the canonical factorization $f = BF$. Define $G(z) := (F(z))^{\frac{p}{2}}$ and note that $G \in H^2$, since $F \in H^p$ by Theorem 1.9, and subsequently we have that $G(e^{it}) := \lim_{r\to 1} G(re^{it})$ exists for almost every $t \in \mathbb{T}$ by the case $p = 2$, and thus the same holds for $F(e^{it}) := \lim_{r\to 1} F(re^{it})$, where $\left(F(e^{it})\right)^{\frac{p}{2}} = G(e^{it})$. As $B(e^{it})$ exists a.e. and $|B(e^{it})| = 1$ a.e. by the previous corollary, we obtain that $f(e^{it}) := \lim_{r\to 1} f(re^{it}) = \lim_{r\to 1} B(re^{it})F(re^{it})$ exists a.e. and $|f(e^{it})|^{\frac{p}{2}} = |G(e^{it})|$ a.e. (also after taking squares), from which we may conclude that

$$\|f\|_{H^p}^p = \|F\|_{H^p}^p = \|G\|_{H^2}^2 = \frac{1}{2\pi}\int_0^{2\pi} |G(e^{it})|^2 dt = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{it})|^p dt = \|f_{bd}\|_{L^p(\mathbb{T})}^p,$$

where the first equality holds due to Theorem 1.9. □

Before proving the next theorem, we recall that $f \in H^2$ is the Poisson integral of some function $f_{bd} \in L^2(\mathbb{T})$ and that this Poisson integral converges to f_{bd} in $L^2(\mathbb{T})$, as we stated after Lemma 1.6. The same is true if we replace 2 by $p \geq 2$, since $f_{bd}(t) := \lim_{r\to 1} f(re^{it})$ exists a.e. and $f_{bd} \in L^p(\mathbb{T})$ by the previous theorem, and $L^p(\mathbb{T}) \subseteq L^2(\mathbb{T})$ for $p \geq 2$, so $f_{bd} \in L^2(\mathbb{T})$ is the same limit a.e. as stated after the Lemma 1.6, so f is the Poisson integral of f_{bd}. Convergence of this Poisson integral in $L^p(\mathbb{T})$ is secured by said discussion after the lemma.

We now want to prove the same for H^1, and by the same reasoning as above this will then follow for H^p with $p \geq 1$. But first we prove a theorem that allows us to "upgrade" to Hardy spaces of greater p.

Theorem 1.12. *Let $0 < p < p'$ and suppose that $f \in H^p$. If we have $f(e^{it}) \in L^{p'}(\mathbb{T})$, then $f \in H^{p'}$.*

Proof. Let $f = BF$ be the canonical factorization of f, then define $G(z) = (F(z))^{\frac{p}{2}}$. We have that $G \in H^2$, hence G is the Poisson integral of $G(e^{it}) \in L^2(\mathbb{T})$, but also $G(e^{it}) \in L^{\frac{2p'}{p}}(\mathbb{T})$, using the assumption that $f(e^{it}) \in L^{p'}(\mathbb{T})$ and the fact that $|f(e^{it})|^{\frac{p}{2}} = |G(e^{it})|$ a.e. as seen in the proof of the previous theorem. We note that $\frac{2p'}{p} \geq 2$, and so by the discussion above we have that $G(re^{it})$ converges to $G(e^{it})$ in $L^{\frac{2p'}{p}}(\mathbb{T})$ as a Poisson integral. Since

$$\left| \|G(re^{it})\|_{L^{\frac{2p'}{p}}(\mathbb{T})} - \|G(e^{it})\|_{L^{\frac{2p'}{p}}(\mathbb{T})} \right| \leq \|G(re^{it}) - G(e^{it})\|_{L^{\frac{2p'}{p}}(\mathbb{T})} \to 0$$

as $r \to 1$ by the reverse triangle inequality, we obtain

$$\lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} |G(re^{it})|^{\frac{2p'}{p}} dt = \lim_{r\to 1} \|G(re^{it})\|_{L^{\frac{2p'}{p}}(\mathbb{T})}^{\frac{2p'}{p}} = \|G(e^{it})\|_{L^{\frac{2p'}{p}}(\mathbb{T})}^{\frac{2p'}{p}} < \infty$$

and so $G \in H^{\frac{2p'}{p}}$. This means that $F \in H^{p'}$, and repeating last part of the proof of Theorem 1.9 (restating the theorem by swapping f with F while keeping the canonical factorization $f = BF$ the same), we conclude that $f \in H^{p'}$. □

Now we will prove the boundary value case for H^1. For this we need the following lemma.

Lemma 1.13. *Any function in H^1 can be decomposed into a product of two functions in H^2.*

Proof. Suppose $f \in H^1$ and let $f = BF$ be its canonical factorization. Take $f_1 := F^{\frac{1}{2}}$ and $f_2 := BF^{\frac{1}{2}}$. Now clearly $f_1 \in H^2$ and the same is true for f_2, since $\left|B(e^{it})\right| \leq 1$ on $\overline{\mathbb{D}}$ as noted in Corollary 1.10. □

Theorem 1.14. *Let $f \in H^1$ and suppose that f_{bd} is its boundary value. Then f is the Poisson integral of f_{bd}.*

Proof. We write $f = f_1 f_2$ with $f_j \in H^2$ with $j = 1, 2$ as in the previous lemma and note that

$$f(re^{it}) - f(e^{it}) = f_1(re^{it})f_2(re^{it}) - f_1(e^{it})f_2(re^{it}) + f_1(e^{it})f_2(re^{it}) - f_1(e^{it})f_2(e^{it}).$$

Using the Cauchy-Schwarz inequality twice in the last line, we obtain

$$\begin{aligned}
& \|f(re^{it}) - f(e^{it})\|_{L^1(\mathbb{T})} \\
= \quad & \|f_2(re^{it})\left(f_1(re^{it}) - f_1(e^{it})\right) + f_1(e^{it})\left(f_2(re^{it}) - f_2(e^{it})\right)\|_{L^1(\mathbb{T})} \\
\leq \quad & \|f_2(re^{it})\left(f_1(re^{it}) - f_1(e^{it})\right)\|_{L^1(\mathbb{T})} + \|f_1(e^{it})\left(f_2(re^{it}) - f_2(e^{it})\right)\|_{L^1(\mathbb{T})} \\
\leq \quad & \|f_2(re^{it})\|_{L^2(\mathbb{T})}\|f_1(re^{it}) - f_1(e^{it})\|_{L^2(\mathbb{T})} + \|f_1(e^{it})\|_{L^2(\mathbb{T})}\|f_2(re^{it}) - f_2(e^{it})\|_{L^2(\mathbb{T})} \\
\to \quad & 0
\end{aligned}$$

as $r \to 1$, since $\|f_j(re^{it}) - f_j(e^{it})\|_{L^2(\mathbb{T})} \to 0$ as a Poisson integral. Now let $\sum_{n=-\infty}^{\infty} \widehat{f_{bd}}(n) r^{|n|} e^{int}$ be the Poisson integral of f_{bd} and write $f(z) = \sum_{n=0}^{\infty} a_n z^n$. This means that

$$\| \sum_{n=-\infty}^{\infty} (a_n - \widehat{f_{bd}}(n)) r^{|n|} e^{int} \|_{L^1(\mathbb{T})} = \|f(re^{it}) - f(e^{it})\|_{L^1(\mathbb{T})} \to 0$$

as $r \to 1$ with $a_n = 0$ for $n < 0$, and since for each $0 \leq r < 1$ the terms $r^{|n|}e^{int}$ are linearly independent, we see that this can only hold if $a_n = \widehat{f_{bd}}(n)$ for all n. We can now conclude that f is the Poisson integral of f_{bd}. □

We arrive at the final theorem of this section, which states the relationship between Hardy spaces and the Fourier coefficients of the function on the boundary.

Theorem 1.15. *Let $p \geq 1$. Then we have that $f \in H^p$ if and only if f is the Poisson integral of $f_{bd} \in L^p(\mathbb{T})$ with $\widehat{f_{bd}}(n) = 0$ for all $n < 0$.*

Proof. Suppose $f \in H^1$, then by the previous theorem, we have indeed that f is the Poisson integral of f_{bd} with $\widehat{f_{bd}}(n) = 0$ for all $n < 0$, and $f_{bd} \in L^1(\mathbb{T})$ by Theorem 1.11. As explained above Theorem 1.12, we have that the same now holds for $f \in H^p$ with $p \geq 1$. Now suppose that f is the Poisson integral of $f_{bd} \in L^p(\mathbb{T})$ with $\widehat{f_{bd}}(n) = 0$ for each $n < 0$, then $f(re^{it}) = \sum_{n=-\infty}^{\infty} \widehat{f_{bd}}(n) r^{|n|} e^{int} = \sum_{n=0}^{\infty} \widehat{f_{bd}}(n)(re^{it})^n$, so f is holomorphic in $\mathbb{D}$, and by Minkowski's inequality we obtain

$$\|f(re^{it})\|_{L^p(\mathbb{T})} = \|(f_{bd} * P_r)(t)\|_{L^p(\mathbb{T})} \leq \|P_r(t)\|_{L^1(\mathbb{T})} \|f_{bd}(t)\|_{L^p(\mathbb{T})} = \|f_{bd}(t)\|_{L^p(\mathbb{T})} < \infty,$$

and by the monotone convergence theorem for sequences of real numbers we have that $h_p(f, r) = \|f(re^{it})\|^p_{L^p(\mathbb{T})}$ converges, which means that

$$\lim_{r \to 1} h_p(f, r) = \lim_{r \to 1} \|f(re^{it})\|^p_{L^p(\mathbb{T})} < \infty,$$

and so $f \in H^p$. □

Theorem 1.15 gives us a sufficient condition for a boundary function in $L^p(\mathbb{T})$ in the holomorphic case of the Dirichlet problem: for $p \geq 1$, given a function $f_{bd} \in L^p(\mathbb{T})$, we may calculate the Fourier coefficients of f_{bd}, and if we have $\widehat{f_{bd}}(n) = 0$ for each $n < 0$, then we can use the same expression as for a continuous boundary function as seen in the introduction. We may take the Poisson integral of f_{bd}, which is in that case a holomorphic function, and the Poisson integral converges back to f_{bd} in $L^p(\mathbb{T})$. Note that if $f_{bd} \in C(\mathbb{T}) \subseteq L^1(\mathbb{T})$ with the assumption as above, then by the weaker (harmonic) case, we have uniform convergence on $\mathbb{T}$ of the Poisson integral as $r \to 1$. Conversely, if we are given a function $f \in H^p$ for some $p \geq 1$, then we can solve the converse of this problem by letting $r \to 1$, which then converges in $L^p(\mathbb{T})$ to some boundary function $f_{bd} \in L^p(\mathbb{T})$.

For the harmonic case of the Dirichlet problem we just combine two arguments of which one we have already used extensively throughout this section: for $f_{bd} \in L^p(\mathbb{T})$ (with $p \geq 1$), the Poisson integral of f_{bd} converges back to f_{bd} in $L^p(\mathbb{T})$, as discussed at the end of Section 1.3, and the Poisson integral is uniformly convergent[8] on every closed disc with center 0 in $\mathbb{D}$. This means that we may swap derivatives and series in the interior of this disc, so we only need to check that the terms $\widehat{f_{bd}}(n) r^{|n|} e^{int}$ are harmonic for each $n \in \mathbb{Z}$. Using the polar coordinates expression of the Laplace operator Δ we have

$$\begin{aligned}
\Delta \widehat{f_{bd}}(n) r^{|n|} e^{int} &= \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial t^2} \right) \widehat{f_{bd}}(n) r^{|n|} e^{int} \\
&= |n|(|n|-1)\widehat{f_{bd}}(n) r^{|n|-2} + |n|\widehat{f_{bd}}(n) r^{|n|-2} - n^2 \widehat{f_{bd}}(n) r^{|n|-2} \\
&= (n^2 - |n| + |n| - n^2)\widehat{f_{bd}}(n) r^{|n|-2} \\
&= 0
\end{aligned}$$

for $|n| \neq 0, 1$. For $n = 0$ it's trivial, since any derivative of a constant is equal to zero, and for $|n| = 1$ the first term evaluates to zero, while the other two terms cancel each other

[8] Using the fact that $|\widehat{f_{bd}}(n)| \leq 1$ for all $|n|$ great enough by the Riemann-Lebesgue lemma (Theorem 3.9 in [3]) and the Weierstraß M-test with the geometric series.

out. So the Poisson integral of f is harmonic in every open disc with center 0 within $\mathbb{D}$, and thus harmonic in $\mathbb{D}$ itself.

We have convergence almost everywhere by Theorem 3.11, and if the function $f_{bd} \in L^p(\mathbb{T})$ has a continuous representative, then this representative has a Poisson integral that converges uniformly as seen in Theorem 6.4 in [3] and in particular everywhere. So the convergence almost everywhere may be "completed" whenever the boundary function in $L^p(\mathbb{T})$ has a continuous representative.

We might not be satisfied yet. Suppose we were only given the Fourier coefficients of the boundary function $f_{bd} \in L^p(\mathbb{T})$ for some $p \geq 1$. Is it possible to find or construct f_{bd} without having to solve the Dirichlet problem (i.e. without constructing the Poisson integral)? Can we do this directly? Of course, building the Fourier series with these Fourier coefficients would be an option. This would be a nice result, since we then only have to take one limit, instead of taking the limit of an infinite series like with the Poisson integral. But what if the Fourier series doesn't converge? We would now like to construct the boundary function f_{bd} while only knowing its Fourier coefficients $\widehat{f_{bd}}(n)$ for $n \in \mathbb{Z}$ by building the Fourier series of f_{bd}: this can be done by proving that the Fourier series of f_{bd} converges back to f_{bd} in $L^p(\mathbb{T})$ for $1 < p < \infty$. The next three sections will be dedicated to proving this fact.

2 Interpolation

In this section we will introduce the *weak L^p space*, a generalization of the L^p space, which will be used to state the *Marcinkiewicz interpolation theorem*. This theorem will be applied abundantly in Section 3 and Section 4.

2.1 Distribution functions

We recall that $L^p(\mathbb{T})$ spaces are defined using the Lebesgue measure λ (as $\mathbb{T}$ corresponds to a subset of $\mathbb{R}$) while looking back at Proposition 1 in the appendix. We will now define the distribution function d_f.

Definition 2.1. *The* distribution function *of a measurable function $f : \mathbb{T} \to \mathbb{C}$ is the function $d_f : [0, \infty) \to [0, 2\pi]$ defined by*

$$d_f(x) := \lambda(\{t \in \mathbb{T} : |f(t)| > x\}).$$

This is a non-increasing function that gives us information about the height of f (not pointwise). Intuitively, for positive simple functions we just rearrange the blocks under the function from high to low and then switch the axes, thus measuring their height. See the example below.

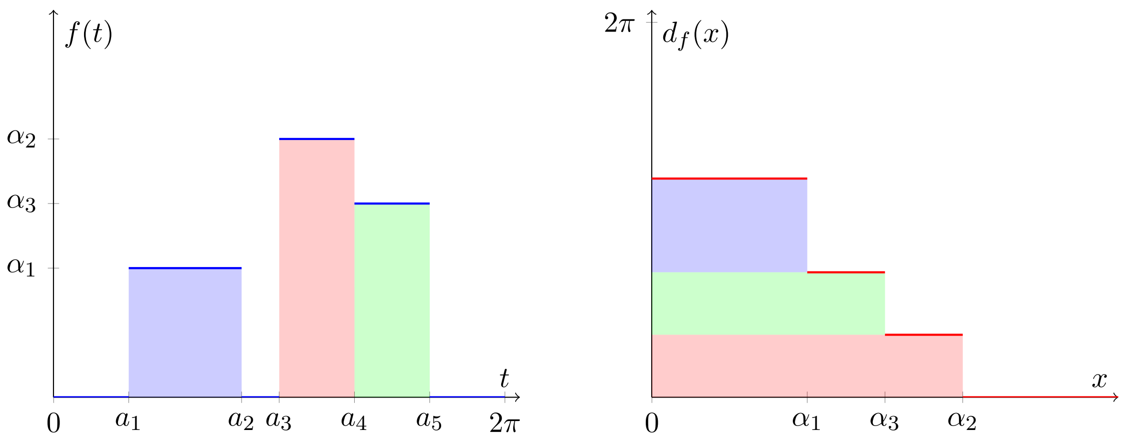

Figure 1: The simple function $f(t) = \alpha_1 \mathbb{1}_{[a_1,a_2)}(t) + \alpha_2 \mathbb{1}_{[a_3,a_4)}(t) + \alpha_3 \mathbb{1}_{[a_4,a_5)}(t)$ (given by the blue line on the left) and its distribution function $d_f(x)$ (given by the red line on the right).

Note that d_f is a measure, since it's equal to $\lambda\left(|f|^{-1}\left((x, \infty]\right)\right)$. Observe that $d_f(x)$ is also equal to $2\pi - \lambda(\{t \in \mathbb{T} : |f(t)| \leq x\})$ and[9] $\int_0^{2\pi} \mathbb{1}_{\{|f(t)|>x\}}(x,t)\, d\lambda(t)$.

The following lemma shows the connection between the $L^p(\mathbb{T})$-norm and the distribution function.

Lemma 2.2. *Let $0 < p < \infty$, and suppose that $f \in L^p(\mathbb{T})$. Then we have*

$$\|f\|^p_{L^p(\mathbb{T})} = \frac{p}{2\pi} \int_0^\infty x^{p-1} d_f(x)\, dx.$$

[9]Even though it's an integral that we could scale by $\frac{1}{2\pi}$, we don't scale it: the sole purpose of a distribution function is to measure a (specific) subset of $\mathbb{T}$.

Proof. Note that

$$\begin{aligned}
\|f\|^p_{L^p(\mathbb{T})} &= \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^p \, d\lambda(t) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{|f(t)|} px^{p-1} \, dx \, d\lambda(t) \\
&= \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} px^{p-1}\mathbb{1}_{\{|f(t)|>x\}}(x,t) \, dx \, d\lambda(t),
\end{aligned}$$

and since this integral is finite, we may swap the integrals due to Fubini's theorem to obtain

$$\begin{aligned}
\|f\|^p_{L^p(\mathbb{T})} &= \frac{1}{2\pi}\int_0^{\infty}\int_0^{2\pi} px^{p-1}\mathbb{1}_{\{|f(t)|>x\}}(x,t) \, d\lambda(t) \, dx \\
&= \frac{p}{2\pi}\int_0^{\infty} x^{p-1}\int_0^{2\pi} \mathbb{1}_{\{|f(t)|>x\}}(x,t) \, d\lambda(t) \, dx \\
&= \frac{p}{2\pi}\int_0^{\infty} x^{p-1} d_f(x) \, dx.
\end{aligned}$$

□

2.2 Weak L^p spaces

Let $0 < p < \infty$, and suppose that $f \in L^p(\mathbb{T})$. For any $x > 0$, we have the estimate[10]

$$\begin{aligned}
x^p d_f(x) &= \int_0^{2\pi} x^p \mathbb{1}_{\{|f(t)|>x\}}(x,t) \, d\lambda(t) \\
&\leq \int_0^{2\pi} |f(t)|^p \mathbb{1}_{\{|f(t)|>x\}}(x,t) \, d\lambda(t) + \int_0^{2\pi} |f(t)|^p \mathbb{1}_{\{|f(t)|\leq x\}}(x,t) \, d\lambda(t) \\
&= 2\pi\|f\|^p_{L^p(\mathbb{T})},
\end{aligned}$$

which yields the inequality

$$d_f(x) \leq \frac{2\pi\|f\|^p_{L^p(\mathbb{T})}}{x^p}.$$

The concept of $L^p(\mathbb{T})$ spaces can be generalized by replacing the numerator of the fraction on the right side of this inequality by some constant.

Definition 2.3. *For $0 < p < \infty$, the space* weak $L^p(\mathbb{T})$ *is defined as the space $L^{p,\infty}(\mathbb{T})$ of Lebesgue measurable functions f such that there exists $C > 0$ such that*

$$d_f(x) \leq \left(\frac{C}{x}\right)^p$$

for all $x > 0$.
For $p = \infty$, we simply define $L^{\infty,\infty}(\mathbb{T}) := L^\infty(\mathbb{T})$.

[10] This is Chebyshev's inequality right here.

Two functions in $L^{p,\infty}(\mathbb{T})$ will be considered equal if they are equal almost everywhere, as taken from its subset $L^p(\mathbb{T})$. For $0 < p < \infty$, we can find the tightest upper bound and define this as the $L^{p,\infty}(\mathbb{T})$-norm[11] of f:

$$\|f\|_{L^{p,\infty}(\mathbb{T})} := \inf\left\{C > 0 : d_f(x) \leq \frac{C^p}{x^p} \text{ for all } x > 0\right\}$$

Rewriting $d_f(x) \leq \frac{C^p}{x^p}$ as $x d_f(x)^{\frac{1}{p}} \leq C$ and noting that the left part is the tightest lower bound of C exactly when C is the tightest upper bound of the left part, we may write the $L^{p,\infty}(\mathbb{T})$-norm in a more explicit way:

$$\|f\|_{L^{p,\infty}(\mathbb{T})} = \sup_{x>0} x d_f(x)^{\frac{1}{p}}$$

It is obvious from the first definition of the $L^{p,\infty}(\mathbb{T})$-norm that $\|f\|^p_{L^{p,\infty}(\mathbb{T})} \leq 2\pi\|f\|^p_{L^p(\mathbb{T})}$, but this can also quickly be seen from the explicit definition, recalling the estimate in the beginning of this subsection.
If $p = \infty$, then clearly we take the norm $\|f\|_{L^{\infty,\infty}(\mathbb{T})} := \|f\|_{L^\infty(\mathbb{T})} = \operatorname{ess\,sup}_{t\in\mathbb{T}}|f(t)|$ of $L^\infty(\mathbb{T})$.

Weak $L^p(\mathbb{T})$ spaces are in fact complete spaces, and for $p > 1$ there exists another functional $N : L^{p,\infty}(\mathbb{T}) \to \mathbb{R}$ that is in fact a norm, and hence $(L^{p,\infty}(\mathbb{T}), N)$ is a Banach space, as shown in [4]. However, we will only focus on the quasi-norm $\|\cdot\|_{L^{p,\infty}(\mathbb{T})}$.

For $0 < p < \infty$ we have that $L^p(\mathbb{T})$ is a strict subset of $L^{p,\infty}(\mathbb{T})$: if we look at the function $f : \mathbb{T} \to \mathbb{C}$ defined by $f(t) = t^{-\frac{1}{p}}$ (the 2π-periodic extension of the same function on $(0, 2\pi]$), we clearly have that $f \notin L^p(\mathbb{T})$, but we do have $d_f(x) = \lambda(\{t \in (0, 2\pi] : t < \frac{1}{x^p}\}) = \frac{1}{x^p}$ for all $x > 0$, so $f \in L^{p,\infty}(\mathbb{T})$. Another good example that proves this strict inclusion is the function $g(t) = |\sin(t)|^{-\frac{1}{p}}$: we have $d_g(x) = \lambda(\{t \in \mathbb{T} : |\sin(t)| < x^{-p}\})$, which is clearly equal to 2π for $0 < x < 1$, while for $x \geq 1$ we observe that $|\sin(t)| < x^{-p}$ is equivalent to $t < \arcsin(x^{-p})$ for $0 \leq t \leq \frac{\pi}{2}$, and thus by symmetry we get[12] $\lambda(\{t \in \mathbb{T} : |\sin(t)| < x^{-p}\}) = 4\arcsin(x^{-p}) \leq 2\pi x^{-p}$. So both cases taken together, we have $d_g(x) \leq \frac{2\pi}{x^p}$ for all $x > 0$. However, $g \notin L^p(\mathbb{T})$ due to the estimate $g(t) \geq |f(t)|$.

2.3 Marcinkiewicz interpolation theorem

We will now prove the *Marcinkiewicz interpolation theorem* for sublinear operators $L^p(\mathbb{T}) \to L^{p,\infty}(\mathbb{T})$ with $0 < p \leq \infty$. The theorem says that if the operator is bounded for $p = p_1$ and $p = p_2$, then it is also bounded for all p between p_1 and p_2. But we first recall some needed concepts and basic definitions[13] from functional analysis.

Definition 2.4. *Let $0 < p, q < \infty$ and suppose that T maps $L^p(\mathbb{T})$ to $L^q(\mathbb{T})$:*

(i) T is called linear *if $T(f + g) = T(f) + T(g)$ and $T(\lambda f) = \lambda T(f)$ for all $f, g \in L^p(\mathbb{T})$ and $\lambda \in \mathbb{C}$;*

[11] This is actually not a norm, but a quasi-norm, i.e. a norm that doesn't necessarily satisfy the triangle inequality, but instead satisfies the inequality $\|f + g\|_{L^{p,\infty}(\mathbb{T})} \leq C_p(\|f\|_{L^{p,\infty}(\mathbb{T})} + \|f\|_{L^{p,\infty}(\mathbb{T})})$ for some $C_p > 0$: see [4] for details.

[12] Recall that $\frac{2}{\pi}x \leq \sin(x)$ for $0 \leq x \leq \frac{\pi}{2}$, so $x \leq \sin\left(\frac{\pi}{2}x\right)$ for $0 \leq x \leq 1$ and thus we see that $\arcsin(x) \leq \frac{\pi}{2}x$ for $0 \leq x \leq 1$.

[13] These definitions work for any normed vector space: we are just stating them for $L^p(\mathbb{T})$ and $L^q(\mathbb{T})$, since we will only use them for these spaces (and their weak version).

(ii) T *is called* sublinear *if* $|T(f+g)| \leq |T(f)| + |T(g)|$ *and* $|T(\lambda f)| = |\lambda||T(f)|$ *for all* $f, g \in L^p(\mathbb{T})$ *and* $\lambda \in \mathbb{C}$ *(all in the pointwise sense on* $\mathbb{T}$*);*

(iii) In the case that T *is linear,* T *is called* bounded *if there exists* $C > 0$ *such that* $\|T(f)\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^q(\mathbb{T})}$ *for all* $f \in L^p(\mathbb{T})$*. In that case we can define the norm* $\|T\|_{L^p \to L^q} := \inf\left\{C > 0 : \|T(f)\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^q(\mathbb{T})} \text{ for all } f \in L^p(\mathbb{T})\right\}$ *(usually we just write* $\|T\|$ *if the domain and codomain of* T *have already been specified).*

If T is an operator that maps $L^p(\mathbb{T})$ to $L^q(\mathbb{T})$, then T is called an operator of *strong type (p,q)*. If operator T maps $L^p(\mathbb{T})$ to $L^{q,\infty}(\mathbb{T})$, then T is called an operator of *weak type (p,q)*.

Note that if T is a linear operator mapping measurable functions to measurable functions that is bounded (on the space of measurable functions, but using the norms as above), then these two assumptions are enough on the simple functions on $\mathbb{T}$ to obtain that T is bounded in $L^p(\mathbb{T})$, since the simple functions are dense in $L^p(\mathbb{T})$ and thus T can be uniquely extended to a bounded linear operator on $L^p(\mathbb{T})$.

We will now state and prove the main theorem of this section.

Theorem 2.5 (Marcinkiewicz interpolation theorem)**.** *Let* $0 < p_1 < p_2 \leq \infty$*. Suppose that* T *is a sublinear operator mapping functions in* $L^{p_1}(\mathbb{T})$ *to measurable funtions and assume that there exist* $C_1, C_2 > 0$ *such that the following weak type* (p_1, p_1) *and weak type* (p_2, p_2) *estimates hold:*

$$\begin{aligned} \|T(f)\|_{L^{p_1,\infty}(\mathbb{T})} &\leq C_1\|f\|_{L^{p_1}(\mathbb{T})} && \text{for all } f \in L^{p_1}(\mathbb{T}), \\ \|T(f)\|_{L^{p_2,\infty}(\mathbb{T})} &\leq C_2\|f\|_{L^{p_2}(\mathbb{T})} && \text{for all } f \in L^{p_2}(\mathbb{T}). \end{aligned}$$

Then for all $p_1 < p < p_2$*, there exists* $C > 0$ *such that the following strong type* (p,p) *estimate holds:*

$$\|T(f)\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})}$$

for all $f \in L^p(\mathbb{T})$*.*

Proof. The idea of the proof is to split f into two cut off functions, the larger one being in $L^{p_1}(\mathbb{T})$ and the smaller one being in $L^{p_2}(\mathbb{T})$, on which we can apply the sublinearity property and then estimate them.
We first prove the case $p_2 < \infty$. Fix $f \in L^p(\mathbb{T})$ and $x > 0$. For some $\delta > 0$ that we will determine later on[14], define

$$f_{1,x}(t) = \begin{cases} 0, & |f(t)| \leq \delta x, \\ f(t), & |f(t)| > \delta x \end{cases}$$

and

$$f_{2,x}(t) = \begin{cases} f(t), & |f(t)| \leq \delta x, \\ 0, & |f(t)| > \delta x. \end{cases}$$

[14]A lot of $\delta > 0$ will work, including simply $\delta = 1$, but we go on with this version of the proof, since the case $p_2 = \infty$ will have the structure of this proof anyway, and this way the constants in both cases will be related, as we will see at the end of the proof. In other words, this proof is more elegant than just directly taking $\delta = 1$.

We have $p_1 - p < 0$ and $p_2 - p > 0$, and so observe that

$$\|f_{1,x}\|^{p_1}_{L^{p_1}(\mathbb{T})} = \frac{1}{2\pi}\int_{|f|>\delta x} |f|^p |f|^{p_1-p}\,d\lambda \leq \frac{1}{2\pi}\int_{|f|>\delta x} |f|^p (\delta x)^{p_1-p}\,d\lambda \leq (\delta x)^{p_1-p}\|f\|^p_{L^p(\mathbb{T})}$$

and

$$\|f_{2,x}\|^{p_2}_{L^{p_2}(\mathbb{T})} = \frac{1}{2\pi}\int_{|f|\leq\delta x} |f|^p |f|^{p_2-p}\,d\lambda \leq \frac{1}{2\pi}\int_{|f|\leq\delta x} |f|^p (\delta x)^{p_2-p}\,d\lambda \leq (\delta x)^{p_2-p}\|f\|^p_{L^p(\mathbb{T})},$$

so $f_{1,x} \in L^{p_1}(\mathbb{T})$ and $f_{2,x} \in L^{p_2}(\mathbb{T}) \subseteq L^{p_1}(\mathbb{T})$. By sublinearity of T we have $|T(f)| \leq |T(f_{1,x})| + |T(f_{2,x})|$ and so, if $|T(f)(t)| > x$, then we must have that $|T(f_{1,x})(t)| > \frac{x}{2}$ or $|T(f_{2,x})(t)| > \frac{x}{2}$, which yields the inclusion

$$\{t \in \mathbb{T} : |T(f)(t)| > x\} \subseteq \left\{t \in \mathbb{T} : |T(f_{1,x})(t)| > \frac{x}{2}\right\} \cup \left\{t \in \mathbb{T} : |T(f_{2,x})(t)| > \frac{x}{2}\right\},$$

which again gives us the inequality

$$d_{T(f)}(x) \leq d_{T(f_{1,x})}\left(\frac{x}{2}\right) + d_{T(f_{2,x})}\left(\frac{x}{2}\right)$$

by subadditivity of λ. Recalling the definition of $\|\cdot\|_{L^{p,\infty}(\mathbb{T})}$ and using the weak type assumptions we now get

$$\begin{aligned} d_{T(f)}(x) &\leq d_{T(f_{1,x})}\left(\frac{x}{2}\right) + d_{T(f_{2,x})}\left(\frac{x}{2}\right) \\ &\leq \left(\frac{2}{x}\right)^{p_1} \|T(f_{1,x})\|^{p_1}_{L^{p_1,\infty}(\mathbb{T})} + \left(\frac{2}{x}\right)^{p_2} \|T(f_{2,x})\|^{p_2}_{L^{p_2,\infty}(\mathbb{T})} \\ &\leq \left(\frac{2C_1}{x}\right)^{p_1} \|f_{1,x}\|^{p_1}_{L^{p_1}(\mathbb{T})} + \left(\frac{2C_2}{x}\right)^{p_2} \|f_{2,x}\|^{p_2}_{L^{p_2}(\mathbb{T})}. \end{aligned}$$

We can now directly apply this estimate after using Lemma 2.2:

$$\begin{aligned} \|T(f)\|^p_{L^p(\mathbb{T})} &= \frac{p}{2\pi}\int_0^\infty x^{p-1} d_{T(f)}(x)\,dx \\ &\leq \frac{p}{2\pi}\int_0^\infty x^{p-1}\left(\frac{2C_1}{x}\right)^{p_1} \frac{1}{2\pi}\int_{|f(t)|>\delta x} |f(t)|^{p_1}\,d\lambda(t)\,dx \\ &\quad + \frac{p}{2\pi}\int_0^\infty x^{p-1}\left(\frac{2C_2}{x}\right)^{p_2} \frac{1}{2\pi}\int_{|f(t)|\leq\delta x} |f(t)|^{p_2}\,d\lambda(t)\,dx \\ &= \frac{p2^{p_1-1}C_1^{p_1}}{\pi}\cdot\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} x^{p-p_1-1}|f(t)|^{p_1}\mathbb{1}_{\{|f(t)|/\delta>x\}}(x,t)\,d\lambda(t)\,dx \\ &\quad + \frac{p2^{p_2-1}C_2^{p_2}}{\pi}\cdot\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} x^{p-p_2-1}|f(t)|^{p_2}\mathbb{1}_{\{|f(t)|/\delta\leq x\}}(x,t)\,d\lambda(t)\,dx \\ &= \frac{p2^{p_1-1}C_1^{p_1}}{\pi}\cdot\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^{p_1}\int_0^{|f(t)|/\delta} x^{p-p_1-1}\,dx\,d\lambda(t) \\ &\quad + \frac{p2^{p_2-1}C_2^{p_2}}{\pi}\cdot\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^{p_2}\int_{|f(t)|/\delta}^\infty x^{p-p_2-1}\,dx\,d\lambda(t) \\ &= \frac{p2^{p_1-1}C_1^{p_1}}{\pi(p-p_1)}\cdot\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^{p_1}\left(\frac{|f(t)|}{\delta}\right)^{p-p_1} d\lambda(t) \\ &\quad - \frac{p2^{p_2-1}C_2^{p_2}}{\pi(p-p_2)}\cdot\frac{1}{2\pi}\int_0^{2\pi}|f(t)|^{p_2}\left(\frac{|f(t)|}{\delta}\right)^{p-p_2} d\lambda(t) \\ &= \frac{p}{2\pi}\left(\frac{2^{p_1}C_1^{p_1}}{(p-p_1)\delta^{p-p_1}} + \frac{2^{p_2}C_2^{p_2}\delta^{p_2-p}}{(p_2-p)}\right)\|f\|^p_{L^p(\mathbb{T})}, \end{aligned}$$

where we used Fubini's theorem to swap integrals. We can now choose an appropriate $\delta > 0$. Since $\frac{1}{p-p_1}, \frac{1}{p_2-p} > 0$, we may want to choose a δ that solves the equation

$$\frac{2^{p_1} C_1^{p_1}}{\delta^{p-p_1}} = 2^{p_2} C_2^{p_2} \delta^{p_2-p},$$

so we find $\delta = \left(\frac{2^{p_1-p_2} C_1^{p_1}}{C_2^{p_2}}\right)^{\frac{1}{p_2-p_1}} > 0$ and after some simple calculations we can finally see that

$$\|T(f)\|_{L^p(\mathbb{T})} \leq 2\left(\frac{p}{2\pi(p-p_1)} + \frac{p}{2\pi(p_2-p)}\right)^{\frac{1}{p}} C_1^{\frac{\frac{1}{p}-\frac{1}{p_2}}{\frac{1}{p_1}-\frac{1}{p_2}}} C_2^{\frac{\frac{1}{p_1}-\frac{1}{p}}{\frac{1}{p_1}-\frac{1}{p_2}}} \|f\|_{L^p(\mathbb{T})}.$$

Now suppose that $p_2 = \infty$. Taking the same $f_{1,x}$ and $f_{2,x}$ (with a new undetermined $\delta > 0$), we again have that $f_{1,x} \in L^{p_1}(\mathbb{T})$ and clearly $f_{2,x} \in L^\infty(\mathbb{T})$, since $|f_{2,x}|$ is bounded everywhere by $x\delta$. We see that $\|T(f_{2,x})\|_{L^\infty(\mathbb{T})} \leq C_2 \|f_{2,x}\|_{L^\infty(\mathbb{T})} \leq C_2 x\delta$. Now choosing[15] $\delta = \frac{1}{2C_2}$ we have that $\|T(f_{2,x})\|_{L^\infty(\mathbb{T})} \leq \frac{x}{2}$, so $\lambda(\{t \in \mathbb{T} : |T(f_{2,x})(t)| > \frac{x}{2}\}) = 0$, giving us the simple inequality

$$d_{T(f)}(x) \leq d_{T(f_{1,x})}\left(\frac{x}{2}\right).$$

Just as in the previous case, this implies

$$d_{T(f)}(x) \leq \left(\frac{2C_1}{x}\right)^{p_1} \|f_{1,x}\|_{L^{p_1}(\mathbb{T})}^{p_1}$$

and repeating the last chain of inequalities in the previous case with only the integral of p_1 and $\delta = \frac{1}{2C_2}$, we have

$$\begin{aligned} \|T(f)\|_{L^p(\mathbb{T})}^p &\leq \frac{p2^{p_1-1}C_1^{p_1}}{\pi(p-p_1)} \cdot \frac{1}{2\pi}\int_0^{2\pi} |f(t)|^{p_1}\left(\frac{|f(t)|}{\delta}\right)^{p-p_1} d\lambda(t) \\ &= \frac{p2^{p_1}C_1^{p_1}(2C_2)^{p-p_1}}{2\pi(p-p_1)}\|f\|_{L^p(\mathbb{T})}^p, \end{aligned}$$

yielding

$$\|T(f)\|_{L^p(\mathbb{T})} \leq 2\left(\frac{p}{2\pi(p-p_1)}\right)^{\frac{1}{p}} C_1^{\frac{p_1}{p}} C_2^{1-\frac{p_1}{p}} \|f\|_{L^p(\mathbb{T})}.$$

We see that this indeed agrees with the previous case after letting $p_2 \to \infty$. $\square$

If p approaches $p_1 < \infty$ or $p_2 < \infty$, then we see that C approaches infinity. Of course this is because we only assumed a weak type estimate for p_1 and p_2. Note that this theorem holds for the stronger assumption by replacing weak type estimate by strong type (since we have that $\|f\|_{L^{p,\infty}(\mathbb{T})}^p \leq 2\pi\|f\|_{L^p(\mathbb{T})}^p$ for $0 < p < \infty$).

[15] Note how this agrees with the δ of the previous case after letting $p_2 \to \infty$.

3 Boundedness of the conjugate function

In this section we will introduce the *conjugate function* and the *maximal function of Hardy–Littlewood.* We will look at some properties of these functions, in particular by applying the Marcinkiewicz interpolation theorem on the last one. Furthermore, we will discuss how one could prove the boundedness of the mapping $f \mapsto \tilde{f}$ in $L^p(\mathbb{T})$ for $p > 1$ with the help of these functions. In Section 4 we will give a rigorous proof for this using a different method.

3.1 Conjugate series

We recall the trigonometric series

$$S(t) := \sum_{n=-\infty}^{\infty} c_n e^{int}$$

from Fourier Analysis, defined as the limit of the partial sums

$$S_N(t) := \sum_{n=-N}^{N} c_n e^{int}.$$

with $c_n \in \mathbb{C}$. For now we ignore any form of convergence. We define the signum function $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$ by

$$\operatorname{sgn}(x) := \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Definition 3.1. *The* conjugate series *of the trigonometric series* $S(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ *is the series*

$$\tilde{S}(t) := \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) c_n e^{int}.$$

In particular, the Fourier series $S[f](t) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int}$ of a function $f \in L^1(\mathbb{T})$ has a conjugate series $\tilde{S}[f](t) := \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) \hat{f}(n) e^{int}$. The question is whether there exists a function $g \in L^1(\mathbb{T})$ such that g has Fourier series $\tilde{S}[f]$. Obviously, we must then have $\hat{g}(n) = -i \operatorname{sgn}(n) \hat{f}(n)$.

We see that $\tilde{S}[f]$ is the conjugate series not only to the Fourier series of f, but also to that of $f + C$ for any constant $C \in \mathbb{C}$, since we have $(\widehat{f+C})(n) = \hat{f}(n) + \widehat{C}(n) = \hat{f}(n)$ for $n \neq 0$ and $-i \operatorname{sgn}(0) \hat{f}(0) = 0 = -i \operatorname{sgn}(0)(\widehat{f+C})(0)$. If f is an odd function or more generally, if we take $f := h - \int_0^{2\pi} h(t)\, dt$ for some $h \in L^1(\mathbb{T})$, then we have $\hat{f}(0) = 0$, so we see that

$$\tilde{\tilde{S}}[f](t) := \sum_{n=-\infty}^{\infty} (-i \operatorname{sgn}(n))^2 \hat{f}(n) e^{int} = \sum_{n=-\infty}^{\infty} -\hat{f}(n) e^{int} = -S[f](t),$$

which means that there is a bijection between Fourier series and conjugate series (of functions up to a constant) and since $f \in L^1(\mathbb{T})$ has a unique Fourier series due to the identity theorem (Theorem 4.20) in [3], f will have at most one function $g \in L^1(\mathbb{T})$ with Fourier series $\tilde{S}[f]$ and vice versa (all up to a constant).

3.2 Conjugate functions

Let $f_{bd} \in L^1(\mathbb{T})$ and $0 \leq r < 1$. We recall that we defined its extension f to $\mathbb{D}$ by $f(re^{it}) = \sum_{n=-\infty}^{\infty} r^{|n|} \widehat{f_{bd}}(n) e^{int}$ as a Poisson integral. As inspired by the definition of the conjugate series in the previous subsection, we may define

$$\tilde{f}(re^{it}) := \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) \widehat{f_{bd}}(n) r^{|n|} e^{int}.$$

for $0 \leq r < 1$. We rewrite this series as a convolution:

$$\begin{aligned}
\tilde{f}(re^{it}) &= \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) \frac{1}{2\pi} \int_0^{2\pi} f_{bd}(\tau) e^{-in\tau} \, d\tau \; r^{|n|} e^{int} \\
&= \frac{1}{2\pi} \int_0^{2\pi} f_{bd}(\tau) \left(\sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) r^{|n|} e^{in(t-\tau)} \right) d\tau \\
&= (f_{bd} * Q_r)(t),
\end{aligned}$$

where we swapped series and integral due to uniform convergence in a closed (and thus compact) disc with radius greater than r contained within $\mathbb{D}$, and defined $Q_r(t) := \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) r^{|n|} e^{int}$. Now we rewrite $Q_r(t)$, using uniform convergence to split the series and using the geometric series:

$$\begin{aligned}
Q_r(t) &= -i \sum_{n=1}^{\infty} (re^{it})^n + i \sum_{n=1}^{\infty} (re^{-it})^n \\
&= -i \sum_{n=0}^{\infty} (re^{it})^n + i \sum_{n=0}^{\infty} (re^{-it})^n \\
&= \frac{-i}{1 - re^{it}} + \frac{i}{1 - re^{-it}} \\
&= \frac{i(1 - re^{it}) - i(1 - re^{-it})}{(1 - re^{it})(1 - re^{-it})} \\
&= \frac{-ir(e^{it} - e^{-it})}{1 - r(e^{it} + e^{-it}) + r^2} \\
&= \frac{2r \sin(t)}{1 - 2r \cos(t) + r^2}.
\end{aligned}$$

Looking at Definition 6 in the appendix, we rewrite the holomorphic function $\frac{1+z}{1-z}$ in $\mathbb{D}$:

$$\begin{aligned}
\frac{1 + re^{it}}{1 - re^{it}} &= \frac{1 + re^{it}}{1 - re^{it}} \cdot \frac{1 - re^{-it}}{1 - re^{-it}} \\
&= \frac{1 + r(e^{it} - e^{-it}) - r^2}{1 - r(e^{it} + e^{-it}) + r^2} \\
&= \frac{1 - r^2}{1 - 2r \cos(t) + r^2} + i \frac{2r \sin(t)}{1 - 2r \cos(t) + r^2} \\
&= P_r(t) + iQ_r(t)
\end{aligned}$$

and thus we see that the integral kernel Q_r is the harmonic conjugate[16] of the Poisson kernel P_r. That's where the term conjugate series comes from. Now we will define the

[16] Recall that the Poisson kernel is harmonic, as we proved at the end of Section 1, but now taking all Fourier coefficients equal to 1.

conjugate function, but first we need to prove the existence.

Proposition 3.2. *Suppose $f_{bd} \in L^1(\mathbb{T})$ and let $\tilde{f}(re^{it}) = (f_{bd} * Q_r)(t)$ as above for $0 \leq r < 1$. Then we have that $\lim_{r\to 1} \tilde{f}(re^{it})$ exists for almost every $t \in \mathbb{T}$.*

Proof. Without loss of generality[17], we assume f_{bd} is real-valued and non-negative. In Lemma 6.3 in [3] we saw that $P_r(t)$ is non-negative, so f will be non-negative as a convolution of two non-negative functions. Similarly, Q_r is real-valued, so $\tilde{f}$ will be real-valued as a convolution of two real-valued functions. We note that by uniform convergence we have

$$\begin{aligned}
(f + i\tilde{f})(re^{it}) &= \sum_{n=-\infty}^{\infty} \widehat{f_{bd}}(n) r^{|n|} e^{int} - \sum_{n=-\infty}^{\infty} \operatorname{sgn}(n) \widehat{f_{bd}}(n) r^{|n|} e^{int} \\
&= \sum_{n=-\infty}^{\infty} (1 - \operatorname{sgn}(n)) \widehat{f_{bd}}(n) r^{|n|} e^{int} \\
&= -\widehat{f_{bd}}(0) + 2 \sum_{n=0}^{\infty} \widehat{f_{bd}}(n) (re^{it})^n,
\end{aligned}$$

so $f + i\tilde{f}$ is holomorphic in $\mathbb{D}$. We define $F(z) = \exp(-f(z) - i\tilde{f}(z))$, which is a holomorphic function as a composition of two holomorphic functions and thus harmonic. Since f is non-negative and $\tilde{f}$ is real-valued, we have $|F(z)| = |\exp(-f(z))| \leq 1$ in $\mathbb{D}$, so F is harmonic and bounded in $\mathbb{D}$. By Lemma 9 in the appendix we conclude that $F(re^{it})$ is the Poisson integral of some function $F_{bd} \in L^\infty(\mathbb{T}) \subseteq L^1(\mathbb{T})$, and by Theorem 3.11 we have convergence almost everywhere (meaning $F_{bd}(t) = \lim_{r\to 1} F(re^{it})$ a.e.). Since f is integrable, we have that $|f_{bd}(t)| < \infty$ almost everywhere, so $\lim_{r\to 1} |F(re^{it})| = \lim_{r\to 1} |\exp(-f(re^{it}))| \neq 0$ almost everywhere. At every $t \in \mathbb{T}$ where $F_{bd}(t)$ exists and is non-zero, we see that $F_{bd}(t) = \lim_{r\to 1} \exp(-f(re^{it}) - i\tilde{f}(re^{it})) = \exp(-\lim_{r\to 1} f(re^{it})) \exp(-i \lim_{r\to 1} \tilde{f}(re^{it}))$, so $\lim_{r\to 1} \tilde{f}(re^{it})$ must exist, since $F_{bd}(t)$ has a fixed argument. We conclude that $\lim_{r\to 1} \tilde{f}(re^{it})$ exists for almost every $t \in \mathbb{T}$. □

Let $p \geq 1$. Looking back at the theory of Section 1, and specifically the proof of Theorem 1.15, for $f_{bd} \in L^p(\mathbb{T})$ we have by Minkowski's inequality the estimate

$$\begin{aligned}
\|(f + i\tilde{f})(re^{it})\|_{L^p(\mathbb{T})} &= \|(f_{bd} * (P_r + iQ_r))(t)\|_{L^p(\mathbb{T})} \\
&\leq \|(P_r + iQ_r)(t)\|_{L^1(\mathbb{T})} \|f_{bd}(t)\|_{L^p(\mathbb{T})} < \infty,
\end{aligned}$$

since $P_r + iQ_r \in C(\mathbb{T}) \subseteq L^1(\mathbb{T})$ as $\frac{1+z}{1-z}$ is holomorphic in $\mathbb{D}$, and by the monotone convergence theorem for sequences of real numbers we now obtain that $h_p(f, r) = \|f(re^{it})\|^p_{L^p(\mathbb{T})}$ converges as $r \to 1$, which gives us that

$$\lim_{r\to 1} h_p(f + i\tilde{f}, r) = \lim_{r\to 1} \|(f + i\tilde{f})(re^{it})\|^p_{L^p(\mathbb{T})} < \infty,$$

and so $f + i\tilde{f} \in H^p$ due to being holomorphic as we proved in Proposition 3.2. For $f \in L^p(\mathbb{T})$ we have already seen that we can solve the harmonic case of the Dirichlet

[17] We have $f_{bd} = f_1 - f_2 + if_3 + if_4$ with $f_j \in L^1(\mathbb{T})$ real-valued and non-negative, and we know that convolution with the integral kernel Q_r is linear, so if the functions $\lim_{r\to 1} (f_j * Q_r)(t)$ exist a.e., then so does $\lim_{r\to 1} \tilde{f}(re^{it})$.

problem, but by adding the harmonic conjugate of P_r (as an imaginary part) to the Poisson kernel, we can change f such that it solves the stronger version of the Dirichlet problem, the holomorphic case.

Definition 3.3. *Let $f_{bd} \in L^1(\mathbb{T})$. We define the* conjugate function *of f_{bd} by $\widetilde{f_{bd}}(t) := \lim_{r\to 1} \tilde{f}(re^{it})$, where f is the Poisson integral of f_{bd}.*

When we work only with the function and its conjugate, so if we don't consider its Poisson integral, we may write $\tilde{f}$ as the conjugate function of $f \in L^1(\mathbb{T})$. If the series conjugate to the Fourier series of a function $f \in L^1(\mathbb{T})$ is the Fourier series of some function $g \in L^1(\mathbb{T})$, then we clearly have that the Poisson integral of g is $\tilde{f}(re^{it})$.

For $f \in C^\infty(\mathbb{T})$ (the space of infinitely differentiable functions/smooth functions), we recall from Corollary 3.12 in [3] that $\hat{f}(n) = \mathcal{O}\left(\frac{1}{n^k}\right)$ for any $k \in \mathbb{N}$ (so in particular for $k = 2$), so by Corollary 4.18 in [3] we have that the Fourier series $S_N(f)$ converges uniformly to f in $\mathbb{T}$. By uniform convergence of the conjugate function we observe that

$$\begin{aligned}
\tilde{f}(t) &= \lim_{r\to 1} \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\hat{f}(n)r^{|n|}e^{int} \\
&= \sum_{n=-\infty}^{\infty} \lim_{r\to 1} -i\,\mathrm{sgn}(n)\hat{f}(n)r^{|n|}e^{int} \\
&= \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\hat{f}(n)e^{int},
\end{aligned}$$

so $\hat{\tilde{f}}(n) = -i\,\mathrm{sgn}(n)\hat{f}(n)$ and thus we have $S[\tilde{f}] = \tilde{S}[f]$. This answers "the question" in Subsection 3.1, at least for the subspace $C^\infty(\mathbb{T})$ of $L^1(\mathbb{T})$. Due to uniform convergence, we may also split the series into two separate series. Using uniform convergence to swap series and derivative and using that $\hat{f}(n) = \mathcal{O}\left(\frac{1}{n^{k+2}}\right)$, we get

$$\begin{aligned}
\frac{d^k}{dt^k}\tilde{f}(t) &= \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\hat{f}(n)\frac{d^k}{dt^k}e^{int} \\
&= \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\hat{f}(n)(in)^k e^{int} \\
&= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \mathcal{O}\left(\frac{1}{n^2}\right) e^{int}
\end{aligned}$$

and by the triangle inequality this will be finite. This can be done analogously on the same series with (possibly infinitely many) fewer terms. We put these findings into the following proposition.

Proposition 3.4. *For $f \in C^\infty(\mathbb{T})$, we can write f as*

$$f(t) = \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int}$$

and the conjugate function of f as

$$\tilde{f}(t) = \sum_{n=-\infty}^{\infty} -i\,\mathrm{sgn}(n)\hat{f}(n)e^{int}.$$

We define the Riesz projections P_+ *and* P_- *by*

$$\begin{aligned} P_+[f](t) &= \sum_{n=1}^{\infty} \hat{f}(n)e^{int}, \\ P_-[f](t) &= \sum_{n=-\infty}^{-1} \hat{f}(n)e^{int}, \end{aligned}$$

and we have $f = \hat{f}(0) + P_+[f] + P_-[f]$ *and* $\tilde{f} = -iP_+[f] + iP_-[f]$. *Furthermore, we have* $\tilde{f}, P_+[f], P_-[f] \in C^\infty(\mathbb{T})$.

It is clear that the map $f \mapsto \tilde{f}$ is linear in $C^\infty(\mathbb{T})$ due to linearity of Fourier coefficients and uniform convergence. Boundedness[18] of this map in $L^p(\mathbb{T})$ (considered as a map $L^p(\mathbb{T}) \to L^p(\mathbb{T})$) is a key part in showing that the Fourier series of f converges back to f in $L^p(\mathbb{T})$ for $1 < p < \infty$. In the following subsections, we will work towards the boundedness of this map.

3.3 Maximal functions

We will now define the Hardy–Littlewood maximal function, or just maximal function, which is the the greatest average value of a function taken on all intervals with a specific point as center.

Definition 3.5. *Let* $f_{bd} \in L^1(\mathbb{T})$. *We define the* maximal function *of* f *by*

$$M_f(t) := \sup_{0<h<\pi} \left| \frac{1}{2h} \int_{t-h}^{t+h} f(\tau)\, d\tau \right|.$$

Note that $M_f : \mathbb{T} \to [0, \infty]$ is a well-defined function. Next, we will prove that $M_f \in L^{1,\infty}$. For this we will need the Vitali covering lemma, stated for the space $\mathbb{T}$. If $I = (x-\varepsilon, x+\varepsilon)$ is an interval[19] in $\mathbb{T}$ and $n \in \mathbb{N}$ a natural number, then we define $nI := (x-n\varepsilon, x+n\varepsilon)$ as the interval n times larger than I with the same center. Note that $|nI| = \min\{2\pi, n|I|\} \le n|I|$.

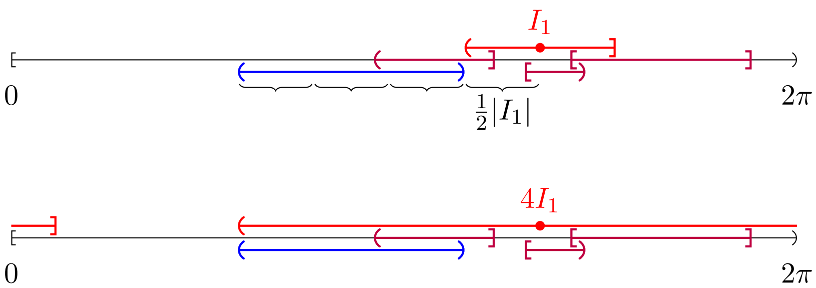

Figure 2: Intervals in $\mathbb{T}$. The red interval is the interval I_1 that will be enlarged by factor 4 in order to cover all other intervals. The blue interval is the open interval $\frac{3}{2}$ times as large as I_1 lying right next to I_1 on the left. The purple intervals intersect I_1 and are at most $\frac{4}{3}$ times as large as I_1.

[18]Note that we only need to show this boundedness for smooth functions, since bounded linear operators extend to $L^p(\mathbb{T})$ due to continuity of this operator and density of $C^\infty(\mathbb{T})$ in $L^p(\mathbb{T})$: density is explained in Footnote 32.

[19]With interval we mean a non-degenerate interval, i.e. $\varepsilon > 0$. These intervals may be (half-)open or closed.

Lemma 3.6 (Vitali covering lemma)**.** *Let $\mathcal{F}$ be a family of intervals in $\mathbb{T}$. Then $\mathcal{F}$ has a countable subfamily $(I_n)_{n=1}^{\infty}$ of disjoint intervals such that*

$$\bigcup_{I\in\mathcal{F}} I \subseteq \bigcup_{n=1}^{\infty} 4I_n.$$

Subsequently, we have

$$\left|\bigcup_{n=1}^{\infty} I_n\right| \geq \frac{1}{4}\left|\bigcup_{I\in\mathcal{F}} I\right|.$$

Proof. Define $a_1 := \sup_{I\in\mathcal{F}}|I|$. By the definition of supremum we can choose some interval $I_1 \in \mathcal{F}$ such that $|I_1| > \frac{3}{4}a_1$. Now define $\mathcal{F}_2$ as the subfamily[20] of intervals in $\mathcal{F}$ that don't intersect I_1. Define $a_2 := \sup_{I\in\mathcal{F}_2}|I|$ and choose some interval $I_2 \in \mathcal{F}_2$ such that $|I_2| > \frac{3}{4}a_2$. By induction we will have the intervals $I_1, \ldots, I_k$, and we can define the subfamily $\mathcal{F}_{k+1}$ of intervals in $\mathcal{F}_k$ that don't intersect $I_1, \ldots, I_k$. We define $a_{k+1} := \sup_{I\in\mathcal{F}_{k+1}}|I|$ and again we can choose some interval $I_{k+1} \in \mathcal{F}_{k+1}$ such that $|I_{k+1}| > \frac{3}{4}a_{k+1}$. Assuming that this never stops (else we were already done), we note that $a_n \to 0$ as $n \to \infty$ (else the whole of $\mathbb{T}$ would already be covered after finitely many steps), and since the intervals have positive length (in the Lebesgue measure), we see that $\bigcap_{n=1}^{\infty}\mathcal{F}_n = \varnothing$. Let $I \in \mathcal{F}$ and let k be the first natural number such that $I \notin \mathcal{F}_{k+1}$, then $I \cap I_k \neq \varnothing$ and $|I_k| > \frac{3}{4}a_k \geq \frac{3}{4}|I|$ by the definition of supremum, so by the same argument as in Footnote 20 we have that $I \subseteq 4I_k$. This means that for any interval $I \in \mathcal{F}$ we have $I \subseteq \bigcup_{n=1}^{\infty} 4I_n$, and so we obtain the inclusion $\bigcup_{I\in\mathcal{F}} I \subseteq \bigcup_{n=1}^{\infty} 4I_n$.
For the last part, we note that

$$\left|\bigcup_{I\in\mathcal{F}} I\right| \leq \left|\bigcup_{n=1}^{\infty} 4I_n\right| \leq \sum_{n=1}^{\infty}|4I_n| \leq 4\sum_{n=1}^{\infty}|I_n| = 4\left|\bigcup_{n=1}^{\infty} I_n\right|$$

by σ-subadditivity and then σ-additivity due to the disjointness of $(I_n)_{n=1}^{\infty}$ (intuitively, if we enlarge $\bigcup_{n=1}^{\infty} I_n$ by factor 4, it might happen that an interval intersects with new parts of enlarged intervals, but this will not be measured, in contrary to the right side of the inequality), so we obtain the inequality

$$\left|\bigcup_{n=1}^{\infty} I_n\right| \geq \frac{1}{4}\left|\bigcup_{I\in\mathcal{F}} I\right|$$

and the lemma is proved. ☐

Now we can use the lemma to prove that M_f is a function in weak $L^1(\mathbb{T})$ whenever f is integrable.

Theorem 3.7. *Suppose that $f \in L^1(\mathbb{T})$, then we have $M_f \in L^{1,\infty}(\mathbb{T})$.*

Proof. Let $f \in L^1(\mathbb{T})$ and choose any $x > 0$. For each $t \in \mathbb{T}$ with $M_f(t) > x$, we can find some $0 < h < \pi$ such that $\left|\frac{1}{2h}\int_{t-h}^{t+h} f(\tau)\,d\tau\right| > x$ by the definition of supremum, where we

[20] If this subfamily is empty, then we already have $\bigcup_{\alpha\in A} I_\alpha \subseteq 4I_1$: other intervals can at most be $\frac{4}{3}$ times larger than I_1, since $|I_1| > \frac{3}{4}\sup_{I\in\mathcal{F}}|I|$, and they intersect I_1, so they must lie within $4I_1$, as $4I_1$ would still contain an open interval $\frac{3}{2}$ times as large as I_1 lying right next to I_1. See figure 2. This argument will hold for the union of the intervals $I_1, \ldots, I_k$ for any $k \in \mathbb{N}$ in the proof.

write $I_t := [t-h, t+h]$ in order to rewrite the inequality as $\left|\frac{1}{|I_t|}\int_{I_t} f(\tau)\,d\tau\right| > x$. In other words, we cover the set $S := \{t \in \mathbb{T} : M_f(t) > x\}$ by a family of intervals $(I_t)_{t\in S}$ satisfying $\int_{I_t}|f(\tau)|\,d\tau \geq \left|\int_{I_t} f(\tau)\,d\tau\right| > x|I_t|$ for each $t \in S$. By the Vitali covering lemma we can find a countable subfamily $(I_{t_n})_{n=1}^{\infty}$ of disjoint intervals such that $|\bigcup_{n=1}^{\infty} I_{t_n}| \geq \frac{1}{4}\left|\bigcup_{t\in S} I_t\right|$, so we obtain the estimate

$$\begin{aligned}
\lambda(\{t \in \mathbb{T} : M_f(t) > x\}) &\leq \left|\bigcup_{t\in S} I_t\right| \leq 4\left|\bigcup_{n=1}^{\infty} I_{t_n}\right| = 4\sum_{n=1}^{\infty}|I_{t_n}| \\
&\leq \frac{4}{x}\sum_{n=1}^{\infty}\int_{I_{t_n}}|f(\tau)|\,d\tau = \frac{4}{x}\int_{\bigcup_{n=1}^{\infty} I_{t_n}}|f(\tau)|\,d\tau \\
&\leq \frac{4}{x}\int_{\mathbb{T}}|f(\tau)|\,d\tau = \frac{4\|f\|_{L^1(\mathbb{T})}}{x},
\end{aligned}$$

and thus we have that $M_f \in L^{1,\infty}(\mathbb{T})$. □

The proof of this theorem actually says that the map $f \mapsto M_f$ is of weak type $(1,1)$ and satisfies the estimate $\|M_f\|_{L^{1,\infty}(\mathbb{T})} \leq 4\|f\|_{L^1(\mathbb{T})}$ for all $f \in L^1(\mathbb{T})$. In fact, if $f \in L^p(\mathbb{T})$ for $1 < p \leq \infty$, then we even have $M_f \in L^p(\mathbb{T})$, as we will show in the following theorem.

Theorem 3.8. *Let $1 < p \leq \infty$. Then there exists $C > 0$ such that the strong (p,p) estimate $\|M_f\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})}$ holds for all $f \in L^p(\mathbb{T})$. In particular, we have $M_f \in L^p(\mathbb{T})$.*

Proof. We will first prove this for $p = \infty$. Then we can combine this with the previous theorem and apply the Marcinkiewicz interpolation theorem.
Let $f \in L^p(\mathbb{T})$. We easily obtain the estimate

$$\begin{aligned}
\|M_f\|_{L^\infty(\mathbb{T})} &= \operatorname*{ess\,sup}_{t\in\mathbb{T}} \sup_{0<h<\pi}\left|\frac{1}{2h}\int_{t-h}^{t+h} f(\tau)\,d\tau\right| \\
&\leq \operatorname*{ess\,sup}_{t\in\mathbb{T}} \sup_{0<h<\pi}\frac{1}{2h}\int_{t-h}^{t+h}|f(\tau)|\,d\tau \\
&\leq \operatorname*{ess\,sup}_{t\in\mathbb{T}} \sup_{0<h<\pi}\frac{1}{2h}\int_{t-h}^{t+h}\operatorname*{ess\,sup}_{s\in\mathbb{T}}|f(s)|\,d\tau \\
&= \operatorname*{ess\,sup}_{t\in\mathbb{T}} \sup_{0<h<\pi}\|f\|_{L^\infty(\mathbb{T})} \\
&= \|f\|_{L^\infty(\mathbb{T})},
\end{aligned}$$

so for $p = \infty$ we have $C = 1$. Now let $f, g \in L^1(\mathbb{T})$: we note that for each $t \in \mathbb{T}$, we have

$$\begin{aligned}
|M_{f+g}(t)| &= \sup_{0<h<\pi}\left|\frac{1}{2h}\int_{t-h}^{t+h} f(\tau)+g(\tau)\,d\tau\right| \\
&= \sup_{0<h<\pi}\left|\frac{1}{2h}\int_{t-h}^{t+h} f(\tau)\,d\tau + \frac{1}{2h}\int_{t-h}^{t+h} g(\tau)\,d\tau\right| \\
&\leq \sup_{0<h<\pi}\left(\left|\frac{1}{2h}\int_{t-h}^{t+h} f(\tau)\,d\tau\right| + \left|\frac{1}{2h}\int_{t-h}^{t+h} g(\tau)\,d\tau\right|\right) \\
&\leq \sup_{0<h<\pi}\left|\frac{1}{2h}\int_{t-h}^{t+h} f(\tau)\,d\tau\right| + \sup_{0<h<\pi}\left|\frac{1}{2h}\int_{t-h}^{t+h} g(\tau)\,d\tau\right| \\
&= |M_f(t)| + |M_g(t)|,
\end{aligned}$$

so the map $f \mapsto M_f$ is sublinear. Furthermore, this map satisfies the conditions of the Marcinkiewicz interpolation theorem, where $p_1 = 1$, $p_2 = \infty$, $C_1 = 4$ and $C_2 = 1$, and thus the statement follows for $1 < p < \infty$. $\square$

With Theorem 3.7 we can also prove that both the Fejér kernel and the Poisson kernel convolved with an integrable function f converge almost everywhere to f: this second fact has been used frequently throughout Section 1 and the first fact will be used in Section 4.

Definition 3.9. *Let $(k_n)_{n=0}^\infty$ be a summability kernel. We say that $(k_n)_{n=0}^\infty$ is* radially bounded *if there exists a sequence of 2π-periodic functions $(\Psi_n)_{n=0}^\infty$ such that for all $n \geq 0$ we have*

(i) $|k_n| \leq \Psi_n$ *(so we call $(\Psi_n)_{n=0}^\infty$ a* dominating sequence*),*

(ii) Ψ_n *is even on $[-\pi, \pi]$ and non-increasing on $[0, \pi]$,*

(iii) $\|\Psi_n\|_{L^1(\mathbb{T})} \leq C$ *for some $C > 0$ independent of n.*

Lemma 3.10. *Let $f \in L^1(\mathbb{T})$. If $\Psi \in C(\mathbb{T})$ is even and non-negative on $[-\pi, \pi]$ and non-increasing on $[0, \pi]$, then we have that $(f * \Psi)(t) \leq \|\Psi\|_{L^1(\mathbb{T})} M_{|f|}(t)$ for all $t \in \mathbb{T}$.*

Proof. We first note that $M_{|f|}(t) = \sup_{0<h<\pi} \int_{\mathbb{T}} \frac{1}{2h} \mathbb{1}_{(-h,h)}(t-\tau)|f(\tau)|\, d\tau$. From the course Real Analysis we recall that non-increasing functions that are bounded can be uniformly approximated by step functions, which are simple functions on $\mathbb{T}$. This means that we may uniformly approximate Ψ from below by linear combinations of the form $L_n = \sum_{k=1}^n \frac{a_k}{2h_k} \mathbb{1}_{(-h_k,h_k)}$ with each $0 \leq h_k \leq \pi$ and $a_k \geq 0$. We now note that $\sum_{k=1}^n a_k = \|L_n\|_{L^1(\mathbb{T})} \leq \|\Psi\|_{L^1(\mathbb{T})}$, since L_n approximates (non-negative) Ψ from below, so we get the estimate

$$\begin{aligned}
\int_{\mathbb{T}} L_n(t-\tau)|f(\tau)|\, d\tau &= \sum_{k=1}^n a_k \int_{\mathbb{T}} \frac{1}{2h_k} \mathbb{1}_{(-h_k,h_k)}(t-\tau)|f(\tau)|\, d\tau \\
&\leq \sum_{k=1}^n a_k M_{|f|}(t) \\
&\leq \|\Psi\|_{L^1(\mathbb{T})} M_{|f|}(t),
\end{aligned}$$

and by uniform convergence we may swap limit and integral to obtain

$$(f * \Psi)(t) \leq (\Psi * |f|)(t) = \lim_{n\to\infty} \int_{\mathbb{T}} L_n(t-\tau)|f(\tau)|\, d\tau \leq \|\Psi\|_{L^1(\mathbb{T})} M_{|f|}(t),$$

where the first inequality holds since Ψ is non-negative. $\square$

Theorem 3.11. *Let $f \in L^1(\mathbb{T})$ and suppose that $(k_n)_{n=0}^\infty$ is a radially bounded summability kernel with a dominating sequence $(\Psi_n)_{n=0}^\infty$ consisting of functions in $C(\mathbb{T})$. Then we have that $f * k_n \to f$ almost everywhere as $n \to \infty$.*

Proof. Let $f \in L^1(\mathbb{T})$ and choose any $\delta > 0$. Now let $\varepsilon > 0$. Since $C(\mathbb{T})$ is dense[21] in $L^1(\mathbb{T})$, we can find $g \in C(\mathbb{T})$ such that $\|f - g\|_{L^1(\mathbb{T})} < \varepsilon$. Note that $g * k_n \to g$ uniformly

[21] Think of trigonometric polynomials: see Footnote 32.

as $n \to \infty$ by Corollary 4.23 in [3]. Implicitly adding and substracting $\lim\limits_{n\to\infty} g * k_n$, we obtain[22]

$$\begin{aligned}
&|\{t \in \mathbb{T} : \limsup_{n\to\infty}(f * k_n)(t) - \liminf_{n\to\infty}(f * k_n)(t) > \delta\}| \\
=\ &|\{t \in \mathbb{T} : \limsup_{n\to\infty}((f-g) * k_n)(t) - \liminf_{n\to\infty}((f-g) * k_n)(t) > \delta\}|.
\end{aligned}$$

Lemma 3.10 (or rather the last line and its reasoning in the proof) now gives us the inequality $|((f-g)*k_n)(t)| \leq (|f-g|*|k_n|)(t) \leq (|f-g|*\Psi_n)(t) \leq \|\Psi_n\|_{L^1(\mathbb{T})} M_{|f-g|}(t)$ and thus $\limsup\limits_{n\to\infty}((f-g)*k_n)(t) \leq \limsup\limits_{n\to\infty}\|\Psi_n\|_{L^1(\mathbb{T})} M_{|f-g|}(t) \leq CM_{|f-g|}(t)$ with C the uniform bound of $(\Psi_n)_{n=0}^{\infty}$ (similarly we have $-\liminf\limits_{n\to\infty}((f-g)*k_n)(t) = \limsup\limits_{n\to\infty} -((f-g)*k_n)(t) \leq CM_{|f-g|}(t)$), so we get

$$\begin{aligned}
&|\{t \in \mathbb{T} : \limsup_{n\to\infty}((f-g) * k_n)(t) - \liminf_{n\to\infty}((f-g) * k_n)(t) > \delta\}| \\
\leq\ &|\{t \in \mathbb{T} : CM_{|f-g|}(t) > \frac{\delta}{2}\}| \\
=\ &|\{t \in \mathbb{T} : M_{|f-g|}(t) > \frac{\delta}{2C}\}|,
\end{aligned}$$

where we reasoned similarly as in the proof of the Marcinkiewicz interpolation theorem. Theorem 3.7 now yields

$$|\{t \in \mathbb{T} : M_{|f-g|}(t) > \frac{\delta}{2C}\}| = \frac{8C\||f-g|\|_{L^1(\mathbb{T})}}{\delta} = \frac{8C\|f-g\|_{L^1(\mathbb{T})}}{\delta} < \frac{8C}{\delta}\varepsilon.$$

By approximation with g this will hold for all $\varepsilon > 0$, so $|\{t \in \mathbb{T} : \limsup\limits_{n\to\infty}(f * k_n)(t) - \liminf\limits_{n\to\infty}(f * k_n)(t) > \delta\}| = 0$, which again implies that $\limsup\limits_{n\to\infty}(f * k_n)(t) = \liminf\limits_{n\to\infty}(f * k_n)(t)$ for almost every $t \in \mathbb{T}$. In other words, $\lim\limits_{n\to\infty} f * k_n$ exists almost everywhere, and since $f * k_n \to f$ in $L^1(\mathbb{T})$ as $n \to \infty$ (so convergence almost everywhere to f for a subsequence), we must have that $\lim\limits_{n\to\infty} f * k_n = f$ almost everywhere. $\square$

In particular, this holds for the Fejér kernel and the Poisson kernel.
For the Fejér kernel $(F_n)_{n=1}^{\infty}$, we recall from [3] that we can write $F_n(t) = \frac{\sin^2(\frac{1}{2}nt)}{2\pi n \sin^2(\frac{1}{2}t)}$ (its continuous extension to 0). Note that $F_n(t) \leq \frac{1}{2\pi n \sin^2(\frac{1}{2}t)} \leq \frac{1}{2\pi n(\frac{1}{\pi}t)^2} = \frac{\pi}{2nt^2}$ on $(-\pi, \pi]$ and that $|F_n(t)| \leq \left|\frac{1}{n}\sum_{k=0}^{n-1} D_k\right| = \left|\frac{1}{n}\sum_{k=0}^{n-1}\sum_{m=-k}^{k} e^{imt}\right| \leq \frac{1}{n}\sum_{k=0}^{n-1}\sum_{m=-k}^{k} 1 = \frac{1}{n}\sum_{k=0}^{n-1}(2k+1) = \frac{1}{n}\left(2\frac{n(n-1)}{2} + n\right) = n$, so we can choose $\Psi_n = \min\{n, \frac{\pi}{2nt^2}\}$ (actually the 2π-periodic extension of this function on $(-\pi, \pi]$) as candidate for the dominating function. These bounds intersect in $\sqrt{\frac{\pi}{2n^2}}$, so by symmetry we obtain

$$\int_{\mathbb{T}} \Psi_n(t)\,dt = 2\int_0^{\sqrt{\frac{\pi}{2n^2}}} n\,dt + 2\int_{\sqrt{\frac{\pi}{2n^2}}}^{\pi} \frac{\pi}{2nt^2}\,dt = \sqrt{2\pi} - \frac{1}{n} + \sqrt{2\pi} = 2\sqrt{2\pi} - \frac{1}{n}.$$

We see that $C = 2\sqrt{2\pi}$, so $(\Psi_n)_{n=1}^{\infty}$ is indeed a (continuous) dominating sequence and therefore $(F_n)_{n=1}^{\infty}$ is radially bounded and thus the theorem applies.

[22] We use the inequalities $\limsup\limits_{n\to\infty} a_n + \liminf\limits_{n\to\infty} b_n \leq \limsup\limits_{n\to\infty}(a_n + b_n) \leq \limsup\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$ and $\liminf\limits_{n\to\infty} a_n + \liminf\limits_{n\to\infty} b_n \leq \liminf\limits_{n\to\infty}(a_n + b_n) \leq \liminf\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$ to obtain the equalities $\limsup\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = \limsup\limits_{n\to\infty}(a_n + b_n)$ and $\liminf\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = \liminf\limits_{n\to\infty}(a_n + b_n)$ in the case that $\lim\limits_{n\to\infty} b_n$ exists.

For the Poisson kernel $(P_r)_{0\leq r<1}$, we simply note that P_r is itself a continuous dominating function.

3.4 Hilbert transforms and duality

In this subsection we will not prove everything, but rather discuss the results of the theorems. As mentioned at the end of Subsection 3.2, the goal here is to give several different arguments that $f \mapsto \tilde{f}$ is bounded as a map from $L^p(\mathbb{T})$ to $L^p(\mathbb{T})$ for $p > 1$, which will yield the convergence of the Fourier series in $L^p(\mathbb{T})$ as we will prove in Section 4.

We recall the definition of the principal value of an integral from [3] and restate it for the space $\mathbb{T}$.

Definition 3.12. *Let $f : \mathbb{T} \to \mathbb{C}$ be integrable on $\mathbb{T} \setminus (-\varepsilon, \varepsilon)$ for each $\varepsilon > 0$. We define the* principal value *of $\int_{\mathbb{T}} f(t)\,dt$ by*

$$\text{p. v.} \int_{\mathbb{T}} f(t)\,dt := \lim_{\varepsilon \downarrow 0} \int_{\varepsilon}^{2\pi-\varepsilon} f(t)\,dt.$$

We note that $\cot(\frac{t}{2}) := \frac{1}{\tan(\frac{t}{2})}$ is continuous and thus integrable on the compact set $\mathbb{T}_\varepsilon := \mathbb{T} \setminus (-\varepsilon, \varepsilon)$ for every $\varepsilon > 0$. In particular it's bounded and thus in $L^\infty(\mathbb{T}_\varepsilon)$, so by Minkowski's inequality the convolution with some $f \in L^1(\mathbb{T})$ is also in $L^\infty(\mathbb{T}_\varepsilon)$ and therefore in $L^1(\mathbb{T}_\varepsilon)$. Also in the space $\mathbb{T}$ we have a Hilbert transform, as we will see in the following proposition.

Proposition 3.13. *Let $f \in L^1(\mathbb{T})$. Then the principal value of $\frac{1}{2\pi}\int_{\mathbb{T}} f(t-\tau)\cot\left(\frac{\tau}{2}\right)\,d\tau$ exists almost everywhere. Furthermore, we have that*

$$\tilde{f}(t) = \text{p. v.} \frac{1}{2\pi} \int_{\mathbb{T}} f(t-\tau)\cot\left(\frac{\tau}{2}\right)\,d\tau$$

for almost every $t \in \mathbb{T}$.
We call $H[f](t) := \text{p. v.} \frac{1}{2\pi}\int_{\mathbb{T}} f(t-\tau)\cot\left(\frac{\tau}{2}\right)\,d\tau$ the Hilbert transform *on $\mathbb{T}$.*

Proof. A proof can be found in Section III.2.7 of [1]. □

Before showing boundedness of the map $f \mapsto \tilde{f}$ (or the Hilbert transform H), we will prove a relationship between this bound and the bound of its adjoint. For this we need some more functional analysis.

Lemma 3.14. *Let $1 \leq p < \infty$. For every functional $F : L^p(\mathbb{T}) \to \mathbb{C}$ there exists a unique $g \in L^{p'}(\mathbb{T})$ such that*

$$F(f) = \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\overline{g(t)}\,dt$$

for all $f \in L^p(\mathbb{T})$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, we have that $\|F\| = \|g\|_{L^{p'}(\mathbb{T})}$. In other words, $L^p(\mathbb{T})'$ (the dual space of $L^p(\mathbb{T})$) and $L^{p'}(\mathbb{T})$ are isometrically isomorphic: $L^p(\mathbb{T})' \cong L^{p'}(\mathbb{T})$.

Proof. This is proved as a theorem[23] in Appendix B of [5]. □

[23] We stated the result in a different way, but it's essentially the same: for example, the complex conjugate of a function is unique to the function, so the statement with a complex conjugate works too. This way suits the style of this project.

For $L^\infty(\mathbb{T})$ this is not true: if it were, then we would have that $L^\infty(\mathbb{T})' \cong L^1(\mathbb{T})$, which implies that $L^\infty(\mathbb{T})'$ separable due to isometry, but this implies[24] that $L^\infty(\mathbb{T})$ is separable, which is contradictory with the fact that $L^\infty(\mathbb{T})$ is not separable.

Let $f \in L^p(\mathbb{T})$ and $g \in L^{p'}(\mathbb{T})$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Then by Hölder's inequality, we have

$$\begin{aligned} \left| \frac{1}{2\pi} \int_{\mathbb{T}} f(t)\overline{g(t)}\, dt \right| &\leq \frac{1}{2\pi} \int_{\mathbb{T}} |f(t)g(t)|\, dt \\ &\leq \frac{1}{2\pi} \left(\int_{\mathbb{T}} |f(t)|^p\, dt \right)^{\frac{1}{p}} \left(\int_{\mathbb{T}} |g(t)|^{p'}\, dt \right)^{\frac{1}{p'}} \\ &= \|f\|_{L^p(\mathbb{T})} \|g\|_{L^{p'}(\mathbb{T})}, \end{aligned}$$

which means that we may define the bilinear map $\langle \cdot, \cdot \rangle : L^p(\mathbb{T}) \times L^{p'}(\mathbb{T}) \to \mathbb{C}$ by

$$\langle f, g \rangle := \frac{1}{2\pi} \int_{\mathbb{T}} f(t)\overline{g(t)}\, dt.$$

Note how this agrees with the inner product for $p = 2$. From now on, for $1 \leq p \leq \infty$, p' denotes the Hölder conjugate of p (the element in $[1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{p'} = 1$).

Definition 3.15. *Let $1 \leq p, q < \infty$. For $T : L^p(\mathbb{T}) \to L^q(\mathbb{T})$ a bounded linear operator, we define the* adjoint operator *of T by the map $T^* : L^{q'}(\mathbb{T}) \to L^{p'}(\mathbb{T})$ that satisfies*

$$\langle T(f), g \rangle = \langle f, T^*(g) \rangle$$

for all $f \in L^p(\mathbb{T})$ and $g \in L^{q'}(\mathbb{T})$.

An equivalent[25] equality in the definition is

$$\langle T^*(g), f \rangle = \overline{\langle f, T^*(g) \rangle} = \overline{\langle T(f), g \rangle} = \langle g, T(f) \rangle.$$

Lemma 3.16. *Let $1 \leq p, q < \infty$ and $T : L^p(\mathbb{T}) \to L^q(\mathbb{T})$ a bounded linear operator. Then its adjoint operator T^* is unique, bounded linear and $\|T^*\| = \|T\|$.*

Proof. Suppose T_1^* and T_2^* are adjoint operators of T, then for all $f \in L^p(\mathbb{T})$ and $g \in L^{q'}(\mathbb{T})$ we have

$$\langle f, T_1^*(g) - T_2^*(g) \rangle = \langle f, T_1^*(g) \rangle - \langle f, T_2^*(g) \rangle = \langle T(f), g \rangle - \langle T(f), g \rangle = 0,$$

so we must have $T_1^* - T_2^* = 0$, or $T_1^* = T_2^*$. This proves uniqueness. Analogously, we have that

$$\langle f, T^*(g + h) - T^*(g) - T^*(h) \rangle = 0,$$

for all $f \in L^p(\mathbb{T})$ and $g, h \in L^{q'}(\mathbb{T})$ and this proves linearity. By the Hahn-Banach theorem[26] we have that $\|T(f)\|_{L^q(\mathbb{T})} = \sup\{|F(T(f))| : F \in L^q(\mathbb{T})', \|F\| = 1\}$. By lemma 3.14 this translates to $\|T(f)\|_{L^q(\mathbb{T})} = \sup\{|\langle T(f), g \rangle| : g \in L^{q'}(\mathbb{T}), \|g\|_{L^{q'}(\mathbb{T})} = 1\}$. By the equivalent equality for Definition 3.15 we obtain similarly that $\|T^*(g)\|_{L^{p'}(\mathbb{T})} = \sup\{|\langle T^*(g), f \rangle| :$

[24] If X', the dual space of a real or complex normed space X, is separable, then X is separable. See Theorem 5.24 in [6] for the proof.

[25] Both here and in the definition, note that each $\langle \cdot, \cdot \rangle$ has a different input: notation-wise this is easier.

[26] The fact that $\|x\| = \sup\{|F(x)| : F \in X', \|F\| = 1\}$ for any $x \in X$ with X a real or complex normed space is a consequence of the Hahn-Banach theorem. See Corollary 5.22 in [6] for the proof.

$f \in L^p(\mathbb{T}), \|f\|_{L^p(\mathbb{T})} = 1\}$. Using the definition of the adjoint operator, Hölder's inequality and the boundedness of T, we see that

$$\begin{aligned}
\|T^*(g)\|_{L^{p'}(\mathbb{T})} &= \sup_{\|f\|_{L^p(\mathbb{T})}=1} |\langle T^*(g), f\rangle| \\
&= \sup_{\|f\|_{L^p(\mathbb{T})}=1} |\langle g, T(f)\rangle| \\
&\leq \sup_{\|f\|_{L^p(\mathbb{T})}=1} \|g\|_{L^{q'}(\mathbb{T})}\|T(f)\|_{L^q(\mathbb{T})} \\
&\leq \sup_{\|f\|_{L^p(\mathbb{T})}=1} \|T\|\|f\|_{L^p(\mathbb{T})}\|g\|_{L^{q'}(\mathbb{T})} \\
&= \|T\|\|g\|_{L^{q'}(\mathbb{T})},
\end{aligned}$$

and so $\|T^*\| \leq \|T\|$, which implies that T^* is indeed bounded. Analogously, we obtain the inequality $\|T\| \leq \|T^*\|$, and we conclude $\|T^*\| = \|T\|$. □

Note once again how the case $p = 2$ agrees with the adjoint in a Hilbert space.

Lemma 3.17. *Let $1 \leq p, q < \infty$. Suppose that bounded linear operator $T : L^p(\mathbb{T}) \to L^q(\mathbb{T})$ is an integral operator with kernel $K(t, s)$. Then its adjoint operator is an integral operator with kernel $K^*(t, s) := \overline{K(s, t)}$. In particular, the adjoint operator of H is $-H$.*

Proof. Let $f \in L^p(\mathbb{T})$ and $g \in L^{q'}(\mathbb{T})$. We write $T[f](s) = \int_{\mathbb{T}} K(t, s)f(s)\, ds$. Applying Fubini once, we obtain

$$\begin{aligned}
\langle T[f](t), g(t)\rangle &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_{\mathbb{T}} K(t,s)f(s)\,ds\,\overline{g(t)}\,dt \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} f(s)\int_{\mathbb{T}} \overline{\overline{K(t,s)}g(t)}\,dt\,ds \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} f(s)\overline{\int_{\mathbb{T}} \overline{K(t,s)}g(t)\,dt}\,ds \\
&= \langle f(s), \int_{\mathbb{T}} \overline{K(t,s)}g(t)\,dt\rangle \\
&= \langle f(s), \int_{\mathbb{T}} K^*(s,t)g(t)\,dt\rangle,
\end{aligned}$$

so indeed $T^*[g](t) = \int_{\mathbb{T}} K^*(t, s)g(s)\, ds$ with $K^*(t, s) = \overline{K(s, t)}$.
Now observe that if $K(t, \tau) = \cot\left(\frac{t-\tau}{2}\right)$, then $K^*(t, \tau) = \overline{K(\tau, t)} = \cot\left(\frac{\tau-t}{2}\right) = -\cot\left(\frac{t-\tau}{2}\right)$. Recalling the commutativity of the convolution operator, we have that

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{T}_\varepsilon} g(\tau)\cot\left(\frac{\tau-t}{2}\right)\,d\tau &= -\frac{1}{2\pi}\int_{\mathbb{T}_\varepsilon} g(\tau)\cot\left(\frac{t-\tau}{2}\right)\,d\tau \\
&= -\frac{1}{2\pi}\int_{\mathbb{T}_\varepsilon} g(t-\tau)\cot\left(\frac{\tau}{2}\right)\,d\tau
\end{aligned}$$

holds[27] for every $\varepsilon > 0$ and $g \in L^{p'}(\mathbb{T})$, and we see that the same must hold for the principal value as a limit, so $H^* = -H$. □

[27]This is actually just an integral over $\mathbb{T}$ of two 2π-periodic functions that are equal to zero on $(-\varepsilon, \varepsilon)$, so the commutative property does indeed hold.

Combining the last two lemmas, we obtain a very interesting result: if we know that $H : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ is bounded with norm $\|H\|_{L^p \to L^p}$ for some $1 \leq p < \infty$, then its adjoint $H^* : L^{p'}(\mathbb{T}) \to L^{p'}(\mathbb{T})$ is bounded (with the same norm), but then so is $H = -H^*$ as a map $L^{p'}(\mathbb{T}) \to L^{p'}(\mathbb{T})$. This means that boundedness of H for all $p \in (1,2)$ implies boundedness of H for all $p \in (2,\infty)$ and vice versa. We will refer to this as "the duality argument".

We can also prove more directly that the adjoint operator of $f \mapsto \tilde{f}$ is $f \mapsto -\tilde{f}$.

Lemma 3.18. *Let $1 \leq p, q < \infty$. Suppose that bounded linear operator $T : L^p(\mathbb{T}) \to L^q(\mathbb{T})$ is a Fourier multiplier operator with Fourier multiplier*[28] *$m(n)$. Then its adjoint operator is a Fourier multiplier operator with Fourier multiplier $m^*(n) := \overline{m(n)}$. In particular, the adjoint operator of $f \mapsto \tilde{f}$ is $f \mapsto -\tilde{f}$.*

Proof. We write $T[f](s) = \sum_{n=-\infty}^{\infty} m(n)\hat{f}(n)e^{int}$. By density[29] of $C^\infty(\mathbb{T})$ in $L^p(\mathbb{T})$ and $L^{q'}(\mathbb{T})$ and Proposition 3.4 we may suppose that $f \in C^\infty(\mathbb{T})$ and $g \in C^\infty(\mathbb{T})$. By uniform convergence of the series we may swap series and integral, so we see that

$$\begin{aligned}
\langle T[f](t), g(t)\rangle &= \frac{1}{2\pi}\int_{\mathbb{T}} \sum_{n=-\infty}^{\infty} m(n)\hat{f}(n)e^{int} \sum_{k=-\infty}^{\infty} \overline{\hat{g}(k)}e^{-ikt}\,dt \\
&= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} m(n)\hat{f}(n)\overline{\hat{g}}(k)\frac{1}{2\pi}\int_{\mathbb{T}} e^{i(n-k)t}\,dt \\
&= \sum_{n=-\infty}^{\infty} m(n)\hat{f}(n)\overline{\hat{g}}(n) \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\overline{m(n)}\hat{g}(n)} \\
&= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} \hat{f}(n)\overline{\overline{m(k)}\hat{g}(k)}\frac{1}{2\pi}\int_{\mathbb{T}} e^{i(n-k)t}\,dt \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \sum_{n=-\infty}^{\infty} \hat{f}(n)e^{int} \sum_{k=-\infty}^{\infty} \overline{\overline{m(k)}\hat{g}(k)}e^{-ikt}\,dt \\
&= \langle f(t), \sum_{k=-\infty}^{\infty} \overline{m(k)}\hat{g}(k)e^{ikt}\rangle
\end{aligned}$$

so indeed $T^*[g](t) = \sum_{k=-\infty}^{\infty} m^*(n)\hat{g}(n)e^{int}$ with $m^*(n) = \overline{m(n)}$.
Now we simply observe that the Fourier multiplier of the map $f \mapsto \tilde{f}$ is $m(n) = -i\,\mathrm{sgn}(n)$, so $m^*(n) = i\,\mathrm{sgn}(n) = -m(n)$. This implies that the adjoint operator of $f \mapsto \tilde{f}$ is $f \mapsto -\tilde{f}$ and by density this holds for $f \in L^p(\mathbb{T})$. □

Replacing Lemma 3.17 by Lemma 3.18 in the discussion above, we have that the duality argument still holds, but this time without needing the Hilbert transform.

Now we look at the boundedness of the map $f \mapsto \tilde{f}$ for $p = 2$.

[28] On $\mathbb{T}$ this a function $m : \mathbb{Z} \to \mathbb{C}$.

[29] We extend the bounded bilinear operator $\langle T(\cdot), \cdot\rangle$ by density in the first argument and then in the second argument.

Lemma 3.19. *$f \mapsto \tilde{f}$ is bounded as a mapping $L^2(\mathbb{T}) \to L^2(\mathbb{T})$.*

Proof. We use the Lemma 4.28 from [3] (as we did in the proof of Lemma 1.6 from Section 1: the equality is better known as the Plancherel theorem). Let $f \in L^2(\mathbb{T})$. Recall from the last lemma that the Fourier multiplier of the map $f \mapsto \tilde{f}$ is $m(n) = -i\,\mathrm{sgn}(n)$. We obtain

$$\begin{aligned}
\|\tilde{f}\|_{L^2(\mathbb{T})} &= 2\pi\| \sum_{n=-\infty}^{\infty} m(n)\hat{f}(n)\frac{1}{\sqrt{2\pi}}e^{int}\|_{L^2(\mathbb{T})} = \sqrt{\sum_{n=-\infty}^{\infty} |m(n)\hat{f}(n)|^2} \\
&\leq \sqrt{\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2} = 2\pi\| \sum_{n=-\infty}^{\infty} \hat{f}(n)\frac{1}{\sqrt{2\pi}}e^{int}\|_{L^2(\mathbb{T})} = \|f\|_{L^2(\mathbb{T})},
\end{aligned}$$

or written shortly (noting that $\|m\|_{\ell^\infty(\mathbb{Z})} = 1$),

$$\|\tilde{f}\|_{L^2(\mathbb{T})} = \|m\hat{f}\|_{\ell^2(\mathbb{Z})} \leq \|\|m\|_{\ell^\infty(\mathbb{Z})}\hat{f}\|_{\ell^2(\mathbb{Z})} = \|m\|_{\ell^\infty}\|\hat{f}\|_{\ell^2(\mathbb{Z})} = \|f\|_{L^2(\mathbb{T})}.$$

□

Theorem 3.20. *$f \mapsto \tilde{f}$ is bounded as a mapping $L^p(\mathbb{T}) \to L^p(\mathbb{T})$ for any $1 < p < \infty$.*

Proof. Since we will prove this rigorously (in another way) in the next section, we will present the next fact without proving it (however, the proof can be found in [7] as proof of Kolmogorov's theorem in Section V.C.1): there exists $C > 0$ such that $\|\tilde{f}\|_{L^{1,\infty}(\mathbb{T})} \leq C\|f\|_{L^1(\mathbb{T})}$. Applying the Marcinkiewicz interpolation theorem on this fact and Lemma 3.19, we obtain boundedness for $1 < p < 2$. By the duality argument this now holds for $1 < p < \infty$. □

Boundedness of $f \mapsto \tilde{f}$ can also be proved using the maximal function. The theorem in Section VIII.C.4 of [7] (the $\mathbb{T}$ version, which still holds as noted in the remark below the theorem) shows that the maximal Hilbert transform

$$\check{f}(t) := \sup_{0<\varepsilon<\pi} \left| \frac{1}{2\pi} \int_{\varepsilon}^{2\pi-\varepsilon} f(t-\tau)\cot\left(\frac{\tau}{2}\right)\, d\tau \right|$$

is bounded as a mapping $L^p(\mathbb{T}) \to L^p(\mathbb{T})$ for $1 < p < \infty$, one of the main ingredients in the proof being the boundedness of M_f for $1 < p < \infty$. Then we note that $|H[f](t)| \leq \check{f}(t)$, so $|H[f](t)|^p \leq |\check{f}(t)|^p$ for $p > 1$, and due to monotonicity of the integral over $\mathbb{T}$ we obtain $\|Hf\|^p_{L^p(\mathbb{T})} \leq \|\check{f}\|^p_{L^p(\mathbb{T})}$, which implies that H is bounded.

4 Convergence of Fourier Series

In this section the main question is for which $1 \le p \le \infty$ we have that the Fourier series of a function in $L^p(\mathbb{T})$ converges back to that function: we will prove that this is only true for $1 < p < \infty$. For this we will need the boundedness of $f \mapsto \tilde{f}$, which we will prove in a different way, this time not shifting the burden of proof to other literature.

4.1 Equivalent formulations of convergence of Fourier series

We will prove that boundedness of the map $f \mapsto \tilde{f}$ is equivalent to the convergence of Fourier series in $L^p(\mathbb{T})$ for $1 \le p < \infty$, where the case $p = 1$ allows us to easily disprove that every Fourier series converges in $L^1(\mathbb{T})$. Note how the case $p = \infty$ is excluded: this is no problem, because we can directly disprove convergence of Fourier series in $L^\infty(\mathbb{T})$. First we need the following lemma before we prove the equivalence.

Lemma 4.1. *Let $1 \le p < \infty$ and let $(a_n)_{n=-\infty}^{\infty}$ be a sequence in $\ell^\infty(\mathbb{Z})$. For each $R \in \mathbb{Z}_{\ge 0}$, let $(a_n(R))_{n=1}^{\infty}$ be some sequence[30] in c_{00} such that $\lim_{R\to\infty} a_n(R) = a_n$. For $f \in L^p(\mathbb{T})$, we define $T_R[f](t) := \sum_{n=-\infty}^{\infty} a_n(R)\widehat{f}(n)e^{int}$ by the Fourier multiplier operator corresponding to f with Fourier multiplier $(a_n(R))_{n=1}^{\infty}$. Then for all $f \in L^p(\mathbb{T})$, we have that $T_R[f]$ converges in $L^p(\mathbb{T})$ as $R \to \infty$ if and only if there exists $K > 0$ such that*

$$\sup_{R\ge 0} \|T_R\|_{L^p\to L^p} \le K. \tag{3}$$

In particular, the limit of $T_R[f]$ is a bounded operator $L^p(\mathbb{T}) \to L^p(\mathbb{T})$.

Proof. Suppose that $T_R[f]$ converges in $L^p(\mathbb{T})$ as $R \to \infty$, let's say to $T_f \in L^p(\mathbb{T})$, then taking $\varepsilon = \|T_f\|_{L^p(\mathbb{T})}$ we can find some $N > 0$ such that $|\|T_R[f]\|_{L^p(\mathbb{T})} - \|T_f\|_{L^p(\mathbb{T})}| \le \|T_R[f] - T_f\|_{L^p(\mathbb{T})} < \|T_f\|_{L^p(\mathbb{T})}$ for all $R > N$, so $\|T_R[f]\|_{L^p(\mathbb{T})} \le 2\|T_f\|_{L^p(\mathbb{T})}$, and noting that each $T_R[f]$ is a finite sum for $R \le N$, we choose C_f as the maximum of $2\|T_f\|_{L^p(\mathbb{T})}$ and $\|T_R[f]\|_{L^p(\mathbb{T})}$ for $R \le N$, so $\|T_R[f]\|_{L^p(\mathbb{T})} \le C_f$ for some constant $C_f > 0$. Now the uniform boundedness principle[31] tells us that $\sup_{R\ge 0}\|T_R\|_{L^p\to L^p} \le K$ for some $K > 0$.
Conversely, assume that Inequality (3) holds. We define $T[h](t) := \sum_{n=-\infty}^{\infty} a_n\widehat{h}(n)e^{int}$ by the Fourier multiplier operator corresponding to $h \in C^\infty(\mathbb{T})$ with Fourier multiplier $(a_n)_{n=1}^{\infty}$. Then for $h \in C^\infty(\mathbb{T})$ we obtain the estimate

$$\begin{aligned} \|T[h]\|_{L^p(\mathbb{T})} &= \|\lim_{R\to\infty} T_R[h]\|_{L^p(\mathbb{T})} \le \liminf_{R\to\infty}\|T_R[h]\|_{L^p(\mathbb{T})} \\ &\le \sup_{R\ge 0}\|T_R\|_{L^p\to L^p}\|h\|_{L^p(\mathbb{T})} \le K\|h\|_{L^p(\mathbb{T})}. \end{aligned}$$

by Proposition 3.4 and Fatou's lemma. By density of $C^\infty(\mathbb{T})$ in $L^p(\mathbb{T})$ we can extend T to a bounded linear operator $\widetilde{T}$ on $L^p(\mathbb{T})$. Now we claim that for each $f \in L^p(\mathbb{T})$ we have that $T_R[f]$ converges to $\widetilde{T}[f]$ in $L^p(\mathbb{T})$ as $R \to \infty$. So let $f \in L^p(\mathbb{T})$ and $\varepsilon > 0$. Since the trigonometric polynomials are dense[32] in $L^p(\mathbb{T})$, we can find a trigonometric polynomial

[30] Recall from the course Real Analysis (extending the definition from $\ell^\infty(\mathbb{N})$ to $\ell^\infty(\mathbb{Z})$) that $c_{00} \subseteq \ell^\infty(\mathbb{Z})$ is the space of compactly supported sequences, in other words we can find some $N > 0$ such that $a_n = 0$ for all $|n| > N$.

[31] As seen in the course Linear Analysis: see theorem 4.52 in [6].

[32] In the course Fourier Analysis we have seen that the trigonometric polynomials are dense in $L^p(\mathbb{T})$, since we know that $\sigma_n[f] \to f$ as $n \to \infty$ in $L^p(\mathbb{T})$ by Corollary 4.23 in [3]. This is the part that would fail for $p = \infty$.

P (let's say with degree d) such that $\|f - P\|_{L^p(\mathbb{T})} < \varepsilon$. Since $P \in C^\infty(\mathbb{T})$, we also have that $\widetilde{T}[P] = T[P]$, so

$$\begin{aligned}
\|T_R[P] - \widetilde{T}[P]\|_{L^p(\mathbb{T})} &\leq \|T_R[P] - \widetilde{T}[P]\|_{L^\infty(\mathbb{T})} \\
&= \|\sum_{n=-d}^{d} (a_n(R) - a_n)\hat{P}(n)e^{int}\|_{L^\infty(\mathbb{T})} \\
&\leq \sum_{n=-d}^{d} \|(a_n(R) - a_n)\hat{P}(n)e^{int}\|_{L^\infty(\mathbb{T})} \\
&= \sum_{n=-d}^{d} |(a_n(R) - a_n)||\hat{P}(n)| < \varepsilon
\end{aligned}$$

for all $R > N, d$ for some $N > 0$, because $\lim\limits_{R\to\infty} a_n(R) = a_n$. This yields

$$\begin{aligned}
\|T_R[f] - \widetilde{T}[f]\|_{L^p(\mathbb{T})} \leq \quad & \|T_R[f] - T_R[P]\|_{L^p(\mathbb{T})} + \|T_R[P] - \widetilde{T}[P]\|_{L^p(\mathbb{T})} \\
+ \quad & \|\widetilde{T}[P] - \widetilde{T}[f]\|_{L^p(\mathbb{T})} \\
< \quad & \|T_R[f - P]\|_{L^p(\mathbb{T})} + \varepsilon + \|\widetilde{T}[P - f]\|_{L^p(\mathbb{T})} \\
\leq \quad & K\|f - P\|_{L^p(\mathbb{T})} + \varepsilon + K\|P - f\|_{L^p(\mathbb{T})} \\
< \quad & (1 + 2K)\varepsilon
\end{aligned}$$

for all $R > N$ and thus the claim is proven. □

Corollary 4.2. *Let $1 \leq p < \infty$. Then we have that $S_N[f] \to f$ as $N \to \infty$ in $L^p(\mathbb{T})$ for all $f \in L^p(\mathbb{T})$ if and only if $\sup\limits_{N\geq 0} \|S_N\|_{L^p\to L^p} < \infty$.*

Proof. We simply take $a_n(N) := \begin{cases} 1, & |n| \leq N, \\ 0, & \text{else} \end{cases}$ to obtain $T_N := S_N$. We also note that if $S_N[f]$ converges in $L^p(\mathbb{T})$, then it must converge to f: recalling Theorem 3.11, $\sigma_N[f](t)$ converges to $\sigma[f](t) = f(t)$ for almost all $t \in \mathbb{T}$ as $N \to \infty$, and for these t we have that $\lim\limits_{N\to\infty} S_N[f](t) = \sigma[f](t)$ by Lemma 4.7 in [3], so $S_N[f](t)$ converges to $f(t)$ for almost all $t \in \mathbb{T}$ as $N \to \infty$, and thus $S_N[f]$ must converge to f in $L^p(\mathbb{T})$ as $N \to \infty$. □

Using Corollary 4.2 we can disprove convergence of the Fourier series in $L^1(\mathbb{T})$ very easily without having to give an explicit counterexample. We note that $D_N \in L^1(\mathbb{T})$ for each $N \in \mathbb{Z}_{\geq 0}$ as a continuous function, so $\lim\limits_{M\to\infty} \|D_N * F_M\|_{L^1(\mathbb{T})} = \|D_N\|_{L^1(\mathbb{T})}$ by Corollary 4.23 in [3] and the reverse triangle inequality. We also have $\|F_M * D_N\|_{L^1(\mathbb{T})} = \|S_N[F_M]\|_{L^1(\mathbb{T})} \leq \|S_N\|_{L^p\to L^p}\|F_M\|_{L^1(\mathbb{T})} = \|S_N\|_{L^p\to L^p}$ since $\|F_M\|_{L^1(\mathbb{T})} = 1$ as a non-negative summability kernel. We obtain the estimate

$$\|S_N\|_{L^p\to L^p} \geq \lim_{M\to\infty} \|D_N * F_M\|_{L^1(\mathbb{T})} = \|D_N\|_{L^1(\mathbb{T})},$$

and from Lemma 5.1 in [3] we know that $\sup\limits_{N\geq 0} \|D_N\|_{L^1(\mathbb{T})} = \infty$, so we can now see that $\sup\limits_{N\geq 0} \|S_N\|_{L^p\to L^p} = \infty$, and thus there are functions in $L^1(\mathbb{T})$ whose Fourier series don't converge (at all, because it either converges to f or diverges as noted in the proof of Corollary 4.2).

For the case $p = \infty$, we pick the 2π-periodic extension (which we will also denote as f) of the function $f(t) := \begin{cases} -2, & t \in [0, \pi), \\ 2, & t \in [\pi, 2\pi). \end{cases}$ Since D_N is real-valued for each $N \in \mathbb{Z}_{\geq 0}$, we have that the continuous function $S_N[f](t) = (f * D_N)(t)$ is real-valued as a convolution of two real-valued functions. If $S_N[f](t)$ were to converge to f in $L^\infty(\mathbb{T})$, then we would have convergence almost everywhere, so we can pick $t_1, t_2 \in \mathbb{T}$ such that $S[f](t_1) = -2$ and $S[f](t_2) = 2$, so we can find a fixed $N \in \mathbb{Z}_{\geq 0}$ such that $S_n[f](t_1) \leq -1$ and $S_n[f](t_2) \geq 1$ for all $n \geq N$. For each $n \geq N$ we can find two points a and b between t_1 and t_2 such that $S_n[f](a) = -1$ and $S_n[f](b) = 1$ by the intermediate value theorem (we can choose the a and b that are closest to each other by continuity). Again by continuity, the closed interval with endpoints a and b has a positive Lebesgue measure, and each point in this interval gives $-1 \leq S_n[f] \leq 1$, so $\|S_n[f] - f\|_{L^\infty(\mathbb{T})} \geq 1$ for each $n \geq N$, which contradicts convergence in $L^\infty(\mathbb{T})$. This argument also works for the last statement in Footnote 32.

Now we can prove the equivalence between convergence of Fourier series in $L^p(\mathbb{T})$ and boundedness of the map $f \mapsto \tilde{f}$.

Theorem 4.3. *Let $1 \leq p < \infty$. Then we have that $S_N[f] \to f$ as $N \to \infty$ in $L^p(\mathbb{T})$ for all $f \in L^p(\mathbb{T})$ if and only if there exists $C_p > 0$ such that $\|\tilde{f}\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p(\mathbb{T})}$ for all $f \in C^\infty(\mathbb{T})$.*

Proof. First we note that for $f \in C^\infty$ we have $P_+[f] = \frac{1}{2}(f + i\tilde{f}) - \frac{1}{2}\hat{f}(0)$ and $P_-[f] = f - P_+[f] - \hat{f}(0)$, so $\|P_+[f]\|_{L^p(\mathbb{T})} \leq \|f\|_{L^p(\mathbb{T})} + \frac{1}{2}\|\tilde{f}\|_{L^p(\mathbb{T})}$ and $\|\tilde{f}\|_{L^p(\mathbb{T})} \leq \|P_+[f]\|_{L^p(\mathbb{T})} + \|P_-[f]\|_{L^p(\mathbb{T})} \leq 2(\|P_+[f]\|_{L^p(\mathbb{T})} + \|f\|_{L^p(\mathbb{T})})$ by the triangle inequality[33]. So $L^p(\mathbb{T})$ boundedness on C^∞ of $f \mapsto \tilde{f}$ is equivalent to that of $f \mapsto P_+[f]$. By density this will automatically also hold on $L^p(\mathbb{T})$.
We define $S'_N[f] := \sum_{n=0}^{2N} \hat{f}(n)e^{int}$ and then also note that

$$\sum_{n=-N}^{N} \hat{f}(n)e^{int} = e^{-iNt}\sum_{n=0}^{2N} \hat{f}(n-N)e^{int} = e^{-iNt}\sum_{n=0}^{2N} \big(\widehat{e^{iN(\cdot)}f(\cdot)}\big)(n)e^{int}, \tag{4}$$

so $\|\sum_{n=-N}^{N} \hat{f}(n)e^{int}\|_{L^p(\mathbb{T})} = \|\sum_{n=0}^{2N} \big(\widehat{e^{iN(\cdot)}f(\cdot)}\big)(n)e^{int}\|_{L^p(\mathbb{T})}$. This means that $\|S_N[f]\|_{L^p(\mathbb{T})} \leq C\|f\|_{L^p(\mathbb{T})}$ for all $f \in L^p(\mathbb{T})$ if and only if $\|S'_N[f]\|_{L^p(\mathbb{T})} \leq C\|e^{-iN(\cdot)}f(\cdot)\|_{L^p(\mathbb{T})} = C\|f\|_{L^p(\mathbb{T})}$ for all[34] $f \in L^p(\mathbb{T})$. Thus S_N and S'_N share the same $L^p(\mathbb{T})$ bound, which implies that $\sup_{N\geq 0}\|S_N\|_{L^p\to L^p} < \infty$ and $\sup_{N\geq 0}\|S'_N\|_{L^p\to L^p} < \infty$ are equivalent.
Now we start with the main part of the proof. Suppose that $S_N[f] \to f$ as $N \to \infty$ in $L^p(\mathbb{T})$ for all $f \in L^p(\mathbb{T})$. By Corollary 4.2 and the equivalence above we obtain $\sup_{N\geq 0}\|S'_N\|_{L^p\to L^p} < \infty$. Applying Lemma 4.1 to the sequence $a_n(N) := \begin{cases} 1, & 0 \leq n \leq 2N, \\ 0, & \text{else}, \end{cases}$ we see that $\tilde{T}[f] = P_+[f] + \hat{f}(0)$ is a bounded operator and by the triangle inequality P_+ is bounded (on $L^p(\mathbb{T})$, so in particular on $C^\infty(\mathbb{T})$). Now this holds too for $f \mapsto \tilde{f}$.
Conversely, suppose that there exists $C_p > 0$ such that $\|\tilde{f}\|_{L^p(\mathbb{T})} \leq C_p\|f\|_{L^p(\mathbb{T})}$ for all $f \in C^\infty(\mathbb{T})$. As noted above we have that P_+ is bounded and by density we extend this

[33] Using the estimate $\|\hat{f}(0)\|_{L^p(\mathbb{T})} = |\hat{f}(0)| \leq \|f\|_{L^1(\mathbb{T})} \leq \|f\|_{L^p(\mathbb{T})}$.

[34] Actually for all $e^{-iN(\cdot)}f(\cdot) \in L^p(\mathbb{T})$, but the statements are equivalent by a simple bijection in the space $L^p(\mathbb{T})$ between the functions $e^{-iN(\cdot)}f(\cdot)$ and f.

to $L^p(\mathbb{T})$. For $f \in C^\infty$ we have

$$\begin{aligned} S'_N[f](t) &= \sum_{n=0}^{\infty} \hat{f}(n)e^{int} - \sum_{n=2N+1}^{\infty} \hat{f}(n)e^{int} \\ &= \sum_{n=0}^{\infty} \hat{f}(n)e^{int} - e^{i2Nt}\sum_{n=1}^{\infty} \hat{f}(n+2N)e^{int} \\ &= \sum_{n=0}^{\infty} \hat{f}(n)e^{int} - e^{i2Nt}\sum_{n=1}^{\infty} \left(\widehat{e^{-2iN(\cdot)}f(\cdot)}\right)(n)e^{int} \\ &= P_+[f](t) - e^{i2Nt}P_+\left[e^{-2iN(\cdot)}f(\cdot)\right](t) + \hat{f}(0), \end{aligned}$$

so just as we reasoned for Equation (4), we obtain that the operators $e^{i2Nt}P_+\left[e^{-2iN(\cdot)}f(\cdot)\right](t)$ and $P_+[f](t)$ have the same bound $\|P_+\|_{L^p\to L^p}$. We then have

$$\begin{aligned} \|S'_N[f]\|_{L^p(\mathbb{T})} &\leq \|P_+[f]\|_{L^p(\mathbb{T})} + \|e^{i2Nt}P_+\left[e^{-2iN(\cdot)}f(\cdot)\right](t)\|_{L^p(\mathbb{T})} + \|f\|_{L^p(\mathbb{T})} \\ &\leq (2\|P_+\|_{L^p\to L^p} + 1)\|f\|_{L^p(\mathbb{T})} \end{aligned}$$

independently of $N \geq 0$ for all $f \in C^\infty(\mathbb{T})$, so we can extend it to $L^p(\mathbb{T})$ by density and we get $\sup\limits_{N\geq 0}\|S'_N[f]\|_{L^p(\mathbb{T})} \leq (2\|P_+\|_{L^p\to L^p} + 1)\|f\|_{L^p(\mathbb{T})}$ for each $f \in L^p(\mathbb{T})$, and thus we have $\sup\limits_{N\geq 0}\|S_N\|_{L^p\to L^p} < \infty$. By Corollary 4.2 the convergence in $L^p(\mathbb{T})$ of $S_N[f]$ to f follows. $\square$

4.2 Boundedness of $f \mapsto \tilde{f}$

We will end this section with a "proper" proof of the boundedness of $f \mapsto \tilde{f}$ for $1 < p < \infty$.

Theorem 4.4. *Let $1 < p < \infty$. Then there exists $C_p > 0$ such that*

$$\|\tilde{f}\|_{L^p(\mathbb{T})} \leq C_p\|f\|_{L^p(\mathbb{T})}$$

for all $f \in L^p(\mathbb{T})$.
Subsequently, we have that the Fourier series of $f \in L^p(\mathbb{T})$ converges back to f in $L^p(\mathbb{T})$.

Proof. This slick proof by S. Bochner uses the density of the trigonometric polynomials[35] in $L^p(\mathbb{T})$.
Let $f(t) = \sum_{n=-N}^{N} c_n e^{int}$ be a trigonometric polynomial (of degree N), then we can clearly rewrite this as

$$f(t) = \left(\sum_{n=-N}^{N} \frac{c_n + \overline{c_{-n}}}{2}e^{int}\right) + i\left(\sum_{n=-N}^{N} \frac{c_n - \overline{c_{-n}}}{2i}e^{int}\right) =: P(t) + iQ(t),$$

and recalling the first exercise of the course Fourier Analysis, we have that

$$a_n := \frac{c_n + \overline{c_{-n}}}{2} + \frac{c_{-n} + \overline{c_n}}{2} = \frac{(c_{-n} + \overline{c_{-n}}) + (c_n + \overline{c_n})}{2} \in \mathbb{R}$$

[35] As argued in Footnote 32.

and

$$b_n := i\left(\frac{c_n + \overline{c_{-n}}}{2} - \frac{c_{-n} + \overline{c_n}}{2}\right) = \frac{i(\overline{c_{-n}} - c_{-n}) + i(c_n - \overline{c_n})}{2} \in \mathbb{R}$$

for all n, so P is real-valued. Analogously we can check that the same holds for the right sum within the square brackets, so Q is real-valued too. So we can temporarily suppose that f is real-valued. Also we will assume for now that $\hat{f}(0) = 0$. Since f is real, we have $\hat{f}(-n) = \overline{\hat{f}(n)}$, and then using Proposition 3.4, noting that $\hat{f}(n) = 0$ for $|n| > N$, we obtain the equality

$$\begin{aligned}
\tilde{f}(t) &= -i\sum_{n=1}^{N} \hat{f}(n)e^{int} + i\sum_{n=1}^{N} \hat{f}(-n)e^{-int} \\
&= -i\sum_{n=1}^{N} \hat{f}(n)e^{int} + i\sum_{n=1}^{N} \overline{\hat{f}(n)e^{int}} \\
&= -i\sum_{n=1}^{N} \hat{f}(n)e^{int} + \overline{\left(-i\sum_{n=1}^{N} \hat{f}(n)e^{int}\right)} \\
&= 2\,\mathrm{Re}\left(-i\sum_{n=1}^{N} \hat{f}(n)e^{int}\right),
\end{aligned}$$

so $\tilde{f}$ is real-valued as well. As we can see, $(f + i\tilde{f})(t) = 2\sum_{n=1}^{N} \hat{f}(n)e^{int}$ contains only positive frequencies, since $\hat{f}(n) = 0$. This implies that we have

$$\int_{\mathbb{T}} (f(t) + i\tilde{f}(t))^{2k}\, dt = 0$$

for all $k \in \mathbb{N}$, since the integrand also has only positive frequencies. Now fix $k \in \mathbb{N}$. The binomial theorem with the fact that f and $\tilde{f}$ are real-valued then gives us

$$\begin{aligned}
0 = &\sum_{m=0}^{2k} \binom{2k}{m} i^{2k-m} \int_{\mathbb{T}} f(t)^m \tilde{f}(t)^{2k-m}\, dt \\
= &\sum_{m=0}^{k-1} \binom{2k}{2m+1} i^{2(k-m)-1} \int_{\mathbb{T}} f(t)^{2m+1} \tilde{f}(t)^{2(k-m)-1}\, dt \\
+ &\sum_{m=0}^{k} \binom{2k}{2m} i^{2(k-m)} \int_{\mathbb{T}} f(t)^{2m} \tilde{f}(t)^{2(k-m)}\, dt \\
= - &\, i\sum_{m=0}^{k-1} \binom{2k}{2m+1} (-1)^{k-m} \int_{\mathbb{T}} f(t)^{2m+1} \tilde{f}(t)^{2(k-m)-1}\, dt \\
+ &\sum_{m=0}^{k} \binom{2k}{2m} (-1)^{k-m} \int_{\mathbb{T}} f(t)^{2m} \tilde{f}(t)^{2(k-m)}\, dt,
\end{aligned}$$

which implies that the real part is equal to zero, and multiplying this by $\frac{1}{2\pi}$, we get

$$\begin{aligned}
0 &= \sum_{m=0}^{k} \binom{2k}{2m} \frac{(-1)^{k-m}}{2\pi} \int_{\mathbb{T}} f(t)^{2m} \tilde{f}(t)^{2(k-m)}\, dt \\
&= \frac{1}{2\pi}\int_{\mathbb{T}} \tilde{f}(t)^{2k}\, dt + \sum_{m=1}^{k} \binom{2k}{2m} \frac{(-1)^{k-m}}{2\pi} \int_{\mathbb{T}} f(t)^{2m} \tilde{f}(t)^{2(k-m)}\, dt.
\end{aligned}$$

Putting the sum on the other side of the equality and noting that both sides are non-negative, we can start estimating[36]:

$$
\begin{aligned}
\|\tilde{f}\|_{L^{2k}(\mathbb{T})}^{2k} &= \left| -\sum_{m=1}^{k} \binom{2k}{2m} \frac{(-1)^{k-m}}{2\pi} \int_{\mathbb{T}} f(t)^{2m} \tilde{f}(t)^{2(k-m)}\, dt \right| \\
&\leq \sum_{m=1}^{k} \binom{2k}{2m} \frac{1}{2\pi} \int_{\mathbb{T}} \left| f(t)^{2m} \tilde{f}(t)^{2(k-m)} \right| \, dt \\
&\leq \sum_{m=1}^{k-1} \binom{2k}{2m} \left(\frac{1}{2\pi} \int_{\mathbb{T}} \left(f(t)^{2m}\right)^{\frac{2k}{2m}} dt \right)^{\frac{2m}{2k}} \left(\frac{1}{2\pi} \int_{\mathbb{T}} \left(\tilde{f}(t)^{2(k-m)}\right)^{\frac{2k}{2(k-m)}} dt \right)^{\frac{2(k-m)}{2k}} \\
&+ \frac{1}{2\pi} \int_{\mathbb{T}} f(t)^{2k}\, dt \\
&= \sum_{m=1}^{k} \binom{2k}{2m} \|f\|_{L^{2k}(\mathbb{T})}^{2m} \|\tilde{f}\|_{L^{2k}(\mathbb{T})}^{2(k-m)},
\end{aligned}
$$

where we used Hölder's inequality multiple times with conjugate exponents $\frac{2k}{2m}$ and $\frac{2k}{2(k-m)}$ for $m = 1, \ldots, k-1$. We divide the inequality by $\|f\|_{L^{2k}(\mathbb{T})}^{2k}$ and write $R := \frac{\|\tilde{f}\|_{L^{2k}(\mathbb{T})}}{\|f\|_{L^{2k}(\mathbb{T})}}$, so we can write the inequality more nicely as

$$
R^{2k} \leq \sum_{m=1}^{k} \binom{2k}{2m} R^{2(k-m)}. \tag{5}
$$

Since the left side has the greatest exponent, it would dominate as $R \to \infty$, which means that we can find some $C_{2k} > 0$ such that Inequality (5) does not hold for all $R > C_{2k}$. In other words, if $R > 0$ satisfies Inequality (5), then we must have that $R \leq C_{2k}$. Since this happens for each real-valued trigonometric polynomial f with $\hat{f}(0) = 0$ as we just proved, we get

$$
\|\tilde{f}\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p(\mathbb{T})}
$$

for these described f as $p = 2k$.
Now we remove the assumption $\hat{f}(0) = 0$, and see that $g := f - \hat{f}(0)$ is a trigonometric polynomial as described in the previous case. We also observe that $\tilde{g} = \tilde{f}$ as argued in Subsection 3.1, so

$$
\|\tilde{f}\|_{L^p(\mathbb{T})} = \|\tilde{g}\|_{L^p(\mathbb{T})} \leq C_p \|g\|_{L^p(\mathbb{T})} \leq 2C_p \|f\|_{L^p(\mathbb{T})}
$$

for $p = 2k$ by the triangle inequality and Footnote 33.
Now removing the assumption that f is real-valued, we may write $f = P + iQ$ with P and Q two real-valued trigonometric polynomials as we just showed. By linearity of $f \mapsto \tilde{f}$ and again the triangle inequality[37] we now have

$$
\begin{aligned}
\|\tilde{f}\|_{L^p(\mathbb{T})} &= \|\tilde{P} + i\tilde{Q}\|_{L^p(\mathbb{T})} \leq \|\tilde{P}\|_{L^p(\mathbb{T})} + \|\tilde{Q}\|_{L^p(\mathbb{T})} \\
&\leq 2C_p \|P\|_{L^p(\mathbb{T})} + 2C_p \|Q\|_{L^p(\mathbb{T})} \leq 4C_p \|f\|_{L^p(\mathbb{T})}
\end{aligned}
$$

[36] Clearly the absolute value within the integral is unnecessary, but this is to show how we estimated and makes us recognize Hölder's inequality.

[37] Also using that $|P|, |Q| \leq |P + iQ| = |f|$ since P and Q are real-valued, so $|P|^p, |Q|^p \leq |f|^p$ as $p > 1$ and due to monotonicity of the integral over $\mathbb{T}$ we obtain $\|P\|_{L^p(\mathbb{T})}^p, \|Q\|_{L^p(\mathbb{T})}^p \leq \|f\|_{L^p(\mathbb{T})}^p$.

for $p = 2k$. By density of the trigonometric polynomials in $L^p(\mathbb{T})$ we now have that $\|\tilde{f}\|_{L^p(\mathbb{T})} \leq C_p \|f\|_{L^p(\mathbb{T})}$ for all $f \in L^p(\mathbb{T})$, where $p = 2k$. We have this for all $k \in \mathbb{N}$, so by the Marcinkiewicz interpolation theorem the boundedness also holds for p in any interval $(2k, 2k+2)$ with $k \in \mathbb{N}$. Therefore the boundedness holds for each $p \geq 2$, and by the duality argument this now also holds for $1 < p < 2$. By Theorem 4.3 we have that the Fourier series of $f \in L^p(\mathbb{T})$ converges back to f in $L^p(\mathbb{T})$ for $1 < p < \infty$. $\square$

Conclusion

We have seen that in all discussed cases the Dirichlet problem can be solved by taking the Poisson integral of the given boundary function:

$$u(re^{it}) = \begin{cases} \frac{1}{2\pi} \int\limits_0^{2\pi} f_{bd}(t-\tau)P_r(\tau)\,d\tau, & r < 1, \\ f_{bd}(t), & r = 1. \end{cases}$$

Therefore we only need to consider the boundary function f_{bd} that is given in the Dirichlet problem to see if we can solve the problem. We will now, in a very distilled and succinct style, discuss the different cases of the Dirichlet problem, the different spaces of the boundary function, and the type of convergence of the Poisson integral to the boundary function.

Harmonic Dirichlet problem

If $f_{bd} \in C(\mathbb{T})$, then we have uniform convergence on $\mathbb{T}$ of $u(re^{it})$ to $f_{bd}(t)$ as $r \to 1$. Let $1 \leq p < \infty$, then in the more general setting that $f_{bd} \in L^p(\mathbb{T})$, we have convergence in $L^p(\mathbb{T})$ of $u(re^{it})$ to $f_{bd}(t)$ as $r \to 1$.

Holomorphic Dirichlet problem

If $f_{bd} \in C(\mathbb{T})$, then we have uniform convergence on $\mathbb{T}$ of $u(re^{it})$ to $f_{bd}(t)$ as $r \to 1$ just like in the harmonic case. Let $1 \leq p < \infty$, then for $f_{bd} \in L^p(\mathbb{T})$ we have the following sufficient condition on the Fourier coefficients of f_{bd}: $\widehat{f_{bd}}(n) = 0$ for all $n < 0$.
Consequently, if we only have $f_{bd} \in L^p(\mathbb{T})$, then we actually do have that $u(re^{it}) := (f_{bd} * (P_r + iQ_r))\,(t) = (f + i\tilde{f})(re^{it})$ solves the holomorphic case.

Convergence of Fourier series

We have seen that for $1 < p < \infty$, the Fourier series of each $f_{bd} \in L^p(\mathbb{T})$ converges back to f_{bd} in $L^p(\mathbb{T})$. This is not true for $p = 1$ and $p = \infty$.
Subsequently, for $1 < p < \infty$ we may reconstruct f_{bd} in the $L^p(\mathbb{T})$ sense if we only know its Fourier coefficients, which may be more efficient than using convergence twice by means of its Poisson integral. For $p = 1$ we will have to solve the Dirichlet problem first before reconstructing the boundary function f_{bd}.

Appendix

As mentioned in the preface, we will use the definitions in the style of [1]. For example the one-dimensional torus $\mathbb{T}$ will be coming forth from $[0, 2\pi)$ instead of $(0, 1]$ (or $(-\pi, \pi]$, although it wouldn't make any difference for functions on $\mathbb{T}$). A function $f : \partial\mathbb{D} \to \mathbb{C}$ will be identified with the equivalence class of boundary functions $f_{bd} : \mathbb{T} \to \mathbb{C}$ with representative $f(e^{it})$ on $[0, 2\pi)$. We denote the Lebesgue measure by λ (we will use $|\cdot|$ when we explicitly measure intervals), while the notation of the Lebesgue integral will usually be $\int f(t)dt := \int f(x)d\lambda(x)$ whenever it's convenient (for example when the Lebesgue measure isn't mentioned). When zeros of a function are explicitly given, then the zeros are given again if their multiplicities are greater than 1. We denote the set of natural numbers by $\mathbb{N} := \{1, 2, 3, \ldots\}$. The first few propositions and definitions are some facts from basic Fourier analysis while clearing up notation and 2π-scaling ambiguities. In general, definitions with an integral will be scaled by $\frac{1}{2\pi}$.

Fourier analysis

Proposition 1. *For $1 \leq p < \infty$, the normed space $(L^p(\mathbb{T}), \|\cdot\|_{L^p(\mathbb{T})})$ of p-integrable functions on $\mathbb{T}$ with norm $\|f\|_{L^p(\mathbb{T})} := \left(\frac{1}{2\pi}\int\limits_{\mathbb{T}}|f(t)|^p dt\right)^{\frac{1}{p}}$ is a Banach space.*

Definition 2. *If $f, g \in L^1(\mathbb{T})$, then we define their* convolution *by*

$$(f * g)(t) := \frac{1}{2\pi}\int_0^{2\pi} f(t-\tau)g(\tau)\,d\tau,$$

which is also a function in $L^1(\mathbb{T})$ and we have $\|f * g\|_{L^1(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})}\|g\|_{L^1(\mathbb{T})}$. Note that $*$ is a commutative operator.

The following lemma generalizes this inequality to have one of the two functions in $L^p(\mathbb{T})$.

Lemma 3 (Minkowski's inequality)**.** *For $1 \leq p \leq \infty$, let $f \in L^1(\mathbb{T})$ and $g \in L^p(\mathbb{T})$, then $f * g \in L^p(\mathbb{T})$, and we have the inequality*

$$\|f * g\|_{L^p(\mathbb{T})} \leq \|f\|_{L^1(\mathbb{T})}\|g\|_{L^p(\mathbb{T})}.$$

Proof. Let $1 \leq p < \infty$. By Hölder's inequality, we get

$$\begin{aligned}\int_{\mathbb{T}}|f(t-\tau)g(\tau)|d\tau &\leq \left(\int_{\mathbb{T}}|f(t-\tau)|^{\frac{p}{p}}|g(\tau)|^p d\tau\right)^{\frac{1}{p}}\left(\int_{\mathbb{T}}|f(t-\tau)|^{\frac{p'}{p'}}d\tau\right)^{\frac{1}{p'}}\\ &= \left(2\pi(|f| * |g|^p)(t)\right)^{\frac{1}{p}}\left(2\pi\|f\|_{L^1(\mathbb{T})}\right)^{\frac{1}{p'}}\\ &= 2\pi\left((|f| * |g|^p)(t)\right)^{\frac{1}{p}}\|f\|_{L^1(\mathbb{T})}^{\frac{1}{p'}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, and so

$$\begin{aligned}
\frac{1}{2\pi}\int_{\mathbb{T}}\left|\frac{1}{2\pi}\int_{\mathbb{T}} f(t-\tau)g(\tau)d\tau\right|^p dt &\leq \frac{1}{2\pi}\int_{\mathbb{T}}\left(\frac{1}{2\pi}\int_{\mathbb{T}}|f(t-\tau)g(\tau)|d\tau\right)^p dt \\
&\leq \frac{1}{2\pi}\int_{\mathbb{T}}\left(((|f|*|g|^p)(t))^{\frac{1}{p}}\,\|f\|_{L^1(\mathbb{T})}^{\frac{1}{p'}}\right)^p dt \\
&= \|f\|_{L^1(\mathbb{T})}^{\frac{p}{p'}}\frac{1}{2\pi}\int_{\mathbb{T}}(|f|*|g|^p)(t)dt \\
&= \|f\|_{L^1(\mathbb{T})}^{\frac{p}{p'}}\||f|*|g|^p\|_{L^1(\mathbb{T})} \\
&\leq \|f\|_{L^1(\mathbb{T})}^{\frac{p}{p'}}\|f\|_{L^1(\mathbb{T})}\||g|^p\|_{L^1(\mathbb{T})} \\
&= \|f\|_{L^1(\mathbb{T})}^{\frac{p}{p'}+\frac{p}{p}}\|g\|_{L^p(\mathbb{T})}^p \\
&= \|f\|_{L^1(\mathbb{T})}^p\|g\|_{L^p(\mathbb{T})}^p.
\end{aligned}$$

Now if $p=\infty$, then, by translation invariance of the essential supremum on $\mathbb{T}$, we simply have

$$\begin{aligned}
\operatorname*{ess\,sup}_{t\in\mathbb{T}}\left|\frac{1}{2\pi}\int_{\mathbb{T}} g(t-\tau)f(\tau)d\tau\right| &\leq \operatorname*{ess\,sup}_{t\in\mathbb{T}}\frac{1}{2\pi}\int_{\mathbb{T}}|g(t-\tau)||f(\tau)|d\tau \\
&\leq \operatorname*{ess\,sup}_{t\in\mathbb{T}}\frac{1}{2\pi}\int_{\mathbb{T}}|f(\tau)|\operatorname*{ess\,sup}_{s\in\mathbb{T}}|g(s-\tau)|d\tau \\
&= \operatorname*{ess\,sup}_{s\in\mathbb{T}}|g(s)|\frac{1}{2\pi}\int_{\mathbb{T}}|f(\tau)|d\tau \\
&= \|f\|_{L^1(\mathbb{T})}\|g\|_{L^\infty(\mathbb{T})}.
\end{aligned}$$

□

Definition 4. *The sequence of functions* $(D_n)_{n=0}^{\infty}$*, where* $D_n := \sum_{k=-n}^{n} e^{ikt}$*, is called the* Dirichlet kernel*. For* $1\leq p\leq\infty$*, if* $f\in L^p(\mathbb{T})$*, then we define the* Fourier series $S[f](t) := \sum_{n=-\infty}^{\infty}\widehat{f}(n)e^{int}$ *by the limit of the partial sums* $S_N[f](t) := (f*D_N)(t) = \sum_{n=-N}^{N}\widehat{f}(n)e^{int}$*, where* $\widehat{f}(n) := \frac{1}{2\pi}\int\limits_{\mathbb{T}} f(t)e^{-int}dt$ *is the* n-th Fourier coefficient *of* f*.*

Definition 5. *The sequence of functions* $(F_n)_{n=1}^{\infty}$*, where* $F_n := \frac{1}{n}\sum_{k=0}^{n-1} D_k$*, is called the* Fejér kernel*. For* $1\leq p\leq\infty$*, if* $f\in L^p(\mathbb{T})$*, then we define the* Cesàro sum $\sigma[f](t)$ *of the Fourier series by the limit of the Cesàro means* $\sigma_N[f](t) := (f*F_N)(t) = \frac{1}{N}\sum_{n=0}^{N-1} S_n[f](t)$ *of the Fourier series.*

Definition 6. *The family of functions* $(P_r)_{0\leq r<1}$*, where* $P_r(t) := \sum_{n=-\infty}^{\infty} r^{|n|}e^{int} = \frac{1-r^2}{1-2r\cos(t)+r^2} = \operatorname{Re}\left(\frac{1+re^{it}}{1-re^{it}}\right)$*, is called the* Poisson kernel*. For* $1\leq p\leq\infty$*, if* $f\in L^p(\mathbb{T})$*, then we call* $(f*P_r)(t) = \sum_{n=-\infty}^{\infty}\widehat{f}(n)r^{|n|}e^{int}$ *the* Poisson integral *of* f*.*

Definition 7. *A sequence of* 2π*-periodic continuous functions* $(k_n)_{n=0}^{\infty}$ *is a* summability kernel *(or* Dirac kernel*) if*

(i) $\frac{1}{2\pi}\int\limits_{\mathbb{T}} k_n(t)\,dt = 1,$

(ii) $\|k_n\|_{L^1(\mathbb{T})}\leq C$ *for some* $C>0$ *independent of* n*,*

(iii) $\lim\limits_{n\to\infty} \int\limits_{\delta}^{2\pi-\delta} |k_n(t)|dt = 0$ *for each* $0 < \delta < \pi$.

It is a well-known fact that the Fejér kernel and the Poisson kernel are non-negative, even, continuous summability kernels. Furthermore, the Poisson kernel is strictly decreasing on $[0,\pi]$. Details for these facts can be found in [3].

Lemma 8. *Suppose f is holomorphic in $\mathbb{D}$ and let $0 < r, \rho < 1$. Then we have that*

$$f(r\rho e^{it}) = f(re^{it}) * P_\rho(t).$$

Proof. Write $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, then we have that both f and P are uniformly convergent on the closed disc with radius r, and so we may swap series and integrals:

$$\begin{aligned} f(re^{it}) * P_\rho(t) &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n=0}^{\infty} a_n r^n e^{in\tau} \sum_{m=-\infty}^{\infty} \rho^{|m|} e^{im(t-\tau)}\, d\tau \\ &= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} a_n r^n \rho^{|m|} e^{imt} \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)\tau}\, d\tau \\ &= \sum_{n=0}^{\infty} a_n r^n \rho^{|n|} e^{int} \\ &= f(r\rho e^{it}). \end{aligned}$$

□

If f is a harmonic function, then $f = \mathrm{Re}(g)$ for some holomorphic function g, since $\mathbb{D}$ is simply connected. Now we see that

$$\begin{aligned} &\mathrm{Re}\left(g(re^{it})\right) * P_\rho(t) + i\,\mathrm{Im}\left(g(re^{it})\right) * P_\rho(t) \\ =\ & \frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\left(g(re^{i(t-\tau)})\right) P_\rho(\tau)\, d\tau + \frac{1}{2\pi}\int_0^{2\pi} i\,\mathrm{Im}\left(g(re^{i(t-\tau)})\right) P_\rho(\tau)\, d\tau \\ =\ & \frac{1}{2\pi}\int_0^{2\pi} g(re^{i(t-\tau)}) P_\rho(\tau)\, d\tau \\ =\ & g(re^{it}) * P_\rho(t) \\ =\ & g(r\rho e^{it}) \\ =\ & \mathrm{Re}\left(g(r\rho e^{it})\right) + i\,\mathrm{Im}\left(g(r\rho e^{it})\right). \end{aligned}$$

We recall that P_ρ is a real-valued function, and so we must have that $f(re^{it}) * P_\rho(t) = f(r\rho e^{it})$. We conclude that Lemma 8 also holds for harmonic functions in $\mathbb{D}$.

Lemma 9. *Every function that is harmonic and bounded in $\mathbb{D}$ is the Poisson integral of some bounded function on $\mathbb{T}$.*

Proof. Let F be harmonic and bounded in $\mathbb{D}$ (i.e. $\|F\|_\infty < \infty$) and define $f_n(e^{it}) := F(r_n e^{it})$ with $r_n \uparrow 1$ as $n \to \infty$. We note that $(f_n)_{n=1}^{\infty}$ is a bounded sequence of functions[38] in $L^\infty(\mathbb{T})$. As $\mathbb{T}$ is separable, $L^1(\mathbb{T})$ is separable, and so the closed unit ball in $L^\infty(\mathbb{T})$, the dual space of $L^1(\mathbb{T})$, is sequentially compact in the *weak-* topology* by the sequential

[38] Each function f_n is in $L^\infty(\mathbb{T})$, since $\|f_n\|_{L^\infty(\mathbb{T})} \le \|f_n\|_\infty \le \|F\|_\infty < \infty$, where the second norm is the supremum norm for continuous functions on $\mathbb{T}$ and the third the supremum norm for continuous functions on $\mathbb{D}$.

version of the *Banach-Alaoglu theorem.* After scaling the unit ball to the corresponding bound of $(f_n)_{n=1}^{\infty}$, this implies[39] that there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ that converges in the *weak-* topology* to some function $F_{bd}(t)$, i.e. for each $x \in L^1(\mathbb{T})$, we have that $\int_{\mathbb{T}} f_{n_k}(t)x(t)\,dt \to \int_{\mathbb{T}} F_{bd}(t)x(t)\,dt$ as $k \to \infty$ with $F_{bd} \in L^{\infty}(\mathbb{T})$ (a bounded function on $\mathbb{T}$). Now let $re^{it} \in \mathbb{D}$. We note that $P_r \in L^1(\mathbb{T})$ (the same obviously holds for the translated variants) and so we see that[40]

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} P_r(t-\tau)F_{bd}(\tau)\,d\tau &= \lim_{k\to\infty}\frac{1}{2\pi}\int_0^{2\pi} P_r(t-\tau)f_{n_k}(e^{i\tau})\,d\tau \\
&= \lim_{k\to\infty}\frac{1}{2\pi}\int_0^{2\pi} P_r(t-\tau)F(r_{n_k}e^{i\tau})\,d\tau \\
&= \lim_{k\to\infty} F(r_{n_k}re^{it}) \\
&= F(re^{it}),
\end{aligned}$$

where the third equality holds due to Lemma 8 and the last equality holds due to continuity of F in $\mathbb{D}$, as it is a harmonic function. ☐

Complex analysis

Proposition 10. *Let $D \subseteq \mathbb{C}$ be a domain. Then every holomorphic function $D \to \mathbb{C}$ is harmonic.*

Proof. Suppose $f : D \to \mathbb{C}$ is holomorphic. Write $f(x+iy) = u(x,y) + iv(x,y)$ with u and v real functions. Then by the Cauchy-Riemann equations, we see that

$$\begin{aligned}
\Delta f &= \frac{\partial}{\partial x}\frac{\partial u}{\partial x} + i\frac{\partial}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial}{\partial y}\frac{\partial u}{\partial y} + i\frac{\partial}{\partial y}\frac{\partial v}{\partial y} \\
&= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - i\frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial y}\frac{\partial v}{\partial x} + i\frac{\partial}{\partial y}\frac{\partial u}{\partial x} \\
&= \frac{\partial}{\partial x}\frac{\partial v}{\partial y} - i\frac{\partial}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + i\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = 0
\end{aligned}$$

in D, where the third equality holds due to smoothness of u and v. So f is indeed harmonic. ☐

Theorem 11. *Let $D \subseteq \mathbb{C}$ be a simply connected domain. Then for harmonic functions $D \to \mathbb{C}$, the mean value property holds, i.e. we have*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{it})dt \tag{6}$$

for $f : D \to \mathbb{C}$ harmonic, where z and $z+re^{it}$ (for all $t \in [0,2\pi)$) are in D.

Proof. Suppose $f : D \to \mathbb{C}$ is harmonic. Write $f(x+iy) = u(x,y) + iv(x,y)$ with u and v real functions. As we can see in the proof of Proposition 10, both u and v are harmonic in D, which means that both u and v are the real parts of holomorphic functions in D.

[39]This is all out of scope of this project: see Chapter 5 Theorem 3.1 in [5] for the details of the proof.
[40]Existence of this convolution is due to Lemma 3.

We consider u, which is the real part of some holomorphic function g, where v_g is the harmonic conjugate of u. Let $a \in D$ and $r > 0$ such that $\gamma(t) := a + re^{it}$ lies in D. By Cauchy's integral formula we have

$$\begin{aligned} g(a) &= \frac{1}{2\pi i} \int_\gamma \frac{g(z)}{z-a}\, dz \\ &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{g(a+re^{it})}{re^{it}} ire^{it}\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(a+re^{it})\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(a+re^{it})\, dt + i\frac{1}{2\pi} \int_0^{2\pi} v_g(a+re^{it})\, dt, \end{aligned}$$

but we also have $g(a) = u(a) + iv_g(a)$, and since both u and v_g are real-valued functions, we must have that $u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a+re^{it})\, dt$.
Analogously we get $v(a) = \frac{1}{2\pi} \int_0^{2\pi} v(a+re^{it})\, dt$, and so we have that

$$\begin{aligned} f(a) &= u(a) + iv(a) \\ &= \frac{1}{2\pi} \int_0^{2\pi} u(a+re^{it})\, dt + i\frac{1}{2\pi} \int_0^{2\pi} v(a+re^{it})\, dt \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{it})\, dt. \end{aligned}$$

$\square$

In this theorem, we actually proved the harmonic property first for holomorphic functions. If we were to prove the theorem in another way, for example using Green's theorem, we would still see that Theorem 11 holds for holomorphic functions in particular due to Proposition 10.

Infinite products

Definition 12. *Let $(p_n)_{n=1}^\infty$ be a sequence in $\mathbb{C}$. We say that the* infinite product

$$\prod_{n=1}^\infty p_n$$

converges if the limit of partial products $\prod_{n=1}^N p_n$ exists and is non-zero.

The assumption of being non-zero is justified by its use in Lemma 13 and Lemma 15. We note that if the elements of the sequence are non-negative real numbers, then the infinite product is equal to the absolute value of itself, while for series we only have an estimate. Recall that for a convergent series $\sum_{n=1}^\infty a_n$ we must have that $\lim_{n\to\infty} a_n = 0$: we have a similar statement for infinite products.

Lemma 13. *If the infinite product $\prod_{n=1}^\infty p_n$ converges, then we have*

$$\lim_{n\to\infty} p_n = 1.$$

Proof. Let's say the infinite product converges to p, then we simply observe that

$$p_n = \frac{\prod_{m=1}^{n} p_m}{\prod_{m=1}^{n-1} p_m} \to \frac{p}{p} = 1$$

as $n \to \infty$. □

Now we state a consequence of the comparison test, which we will use in the last lemma.

Lemma 14 (Limit comparison test). *Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two series with* $a_n \geq 0$ *and* $b > 0$ *for all* $n \in \mathbb{N}$. *If*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

for some $0 < c < \infty$, *then* $\sum_{n=1}^{\infty} a_n$ *converges if and only if* $\sum_{n=1}^{\infty} b_n$ *converges.*

Proof. For each $\varepsilon > 0$ we can find $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\left|\frac{a_n}{b_n} - c\right| < \varepsilon$, or in other words, $(c - \varepsilon)b_n < a_n < (c + \varepsilon)b_n$. Choosing $\varepsilon = \frac{c}{2}$, we obtain $\frac{c}{2}b_n < a_n < \frac{3c}{2}b_n$. Noting that $\sum_{n=1}^{\infty} \frac{c}{2}b_n = \frac{c}{2}\sum_{n=1}^{\infty} b_n$, we have that if $\sum_{n=1}^{\infty} a_n$ converges, then $\frac{c}{2}\sum_{n=1}^{\infty} b_n$ converges by the comparison test, and thus $\sum_{n=1}^{\infty} b_n$ converges. If $\sum_{n=1}^{\infty} b_n$ converges, then similarly we have that $\frac{2}{3c}\sum_{n=1}^{\infty} a_n$ converges by the comparison test, and thus $\sum_{n=1}^{\infty} a_n$ converges. □

Lemma 15. *Let* $(p_n)_{n=1}^{\infty}$ *be a sequence in* $\mathbb{C}$, *where* $0 < |p_n| < 1$. *Then we have that*

$$\prod_{n=1}^{\infty} p_n \text{ is convergent if } \sum_{n=1}^{\infty} |1 - p_n| \text{ is convergent.}$$

If the same sequence $(p_n)_{n=1}^{\infty}$ *is in* $\mathbb{R}$, *then a stronger statement holds:*

$$\prod_{n=1}^{\infty} p_n \text{ is convergent if and only if } \sum_{n=1}^{\infty} |1 - p_n| \text{ is convergent.}$$

Proof. We will first prove the stronger statement, so we suppose that $(p_n)_{n=1}^{\infty}$ a real sequence. Since $\lim_{n\to\infty} p_n = 1$ by Lemma 13 (this also holds if we assume that the series converges), we only have to consider the case that the sequence only has positive elements, as the first (finitely many) terms don't influence convergence.
We observe that $\prod_{n=1}^{\infty} p_n$ converges if and only if $\sum_{n=1}^{\infty} \log(p_n)$ converges (to some negative real number, since $0 < p_n < 1$) due to the continuity of log.
We see that

$$\lim_{n\to\infty} \frac{-\log(p_n)}{1 - p_n} = \lim_{n\to\infty} \frac{\log(p_n)}{p_n - 1} = 1,$$

since we have that

$$\lim_{x\to 1} \frac{\log(x)}{x - 1} = \lim_{y\to 0} \frac{\log(y + 1)}{y} = \lim_{y\to 0} \frac{y + \mathcal{O}\left(y^2\right)}{y} = 1 + \lim_{y\to 0} \mathcal{O}\left(y\right) = 1,$$

so by the limit comparison test we obtain that $-\sum_{n=1}^{\infty}\log(p_n)$ converges if and only if $\sum_{n=1}^{\infty}1-p_n$ converges: the first is equivalent to the convergence of $\sum_{n=1}^{\infty}\log(p_n)$. So indeed $\prod_{n=1}^{\infty}p_n$ converges if and only if $\sum_{n=1}^{\infty}1-p_n$ does.

Now we prove the (weaker) statement for $(p_n)_{n=1}^{\infty}$ a complex sequence. We write $p_n = x_n + iy_n$ with $x_n, y_n \in \mathbb{R}$. We have that the infinite product $\prod_{n=1}^{\infty}p_n$ converges if and only if $\sum_{n=1}^{\infty}\mathrm{Log}(p_n) = \sum_{n=1}^{\infty}\log|p_n| + i\sum_{n=1}^{\infty}\mathrm{Arg}(p_n)$ converges, by the continuity[41] of Log. By linear independence of real and imaginary part, the last series converges if and only if both $\sum_{n=1}^{\infty}\log|p_n|$ and $\sum_{n=1}^{\infty}\mathrm{Arg}(p_n)$ converge.

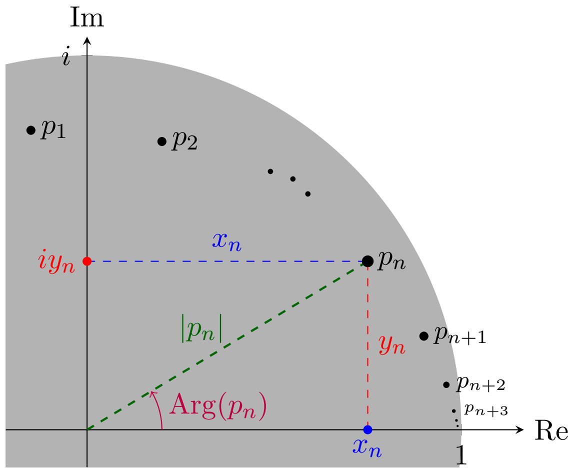

Figure 3: $p_n = x_n + iy_n$ in the complex unit disc as part of the sequence $(p_n)_{n=1}^{\infty}$.

We note that $y_n = |p_n|\sin(\mathrm{Arg}(p_n))$, so $|y_n| = |p_n||\sin(\mathrm{Arg}(p_n))| = |p_n|\sin(|\mathrm{Arg}(p_n)|)$. Again, in both parts of the statement of the lemma we have that $p_n \to 1$ as $n \to \infty$, which implies that $x_n \to 1$, so again we only have to consider the case that x_n is positive as argued in the real case. This also means that $\lim_{n\to\infty}\frac{|y_n|}{|\mathrm{Arg}(p_n)|} = \lim_{n\to\infty}|p_n|\lim_{n\to\infty}\frac{\sin(|\mathrm{Arg}(p_n)|)}{|\mathrm{Arg}(p_n)|} = 1$, since $\lim_{n\to\infty}\mathrm{Arg}(p_n) = 0$ as $\lim_{n\to\infty}p_n = 1$, so by the limit comparison test we have that $\sum_{n=1}^{\infty}|y_n|$ converges if and only if $\sum_{n=1}^{\infty}|\mathrm{Arg}(p_n)|$ converges. Now that we have proved the details, we can prove the main statement.
Suppose $\sum_{n=1}^{\infty}|1-p_n|$ is convergent. We have $|y_n| \le |1-x_n-iy_n| = |1-p_n|$ and from the reverse triangly inequality we obtain $1-|p_n| \le |1-p_n|$, so by the comparison test we have that $\sum_{n=1}^{\infty}|y_n|$ and $\sum_{n=1}^{\infty}1-|p_n|$ converge, the first implying that $\sum_{n=1}^{\infty}|\mathrm{Arg}(p_n)|$ converges, and then so does $\sum_{n=1}^{\infty}\mathrm{Arg}(p_n)$ due to absolute convergence. The second series agrees with the conditions of the real case, so we have that $\sum_{n=1}^{\infty}\log|p_n|$ converges. As noted at the beginning of the complex case, we now have that $\prod_{n=1}^{\infty}p_n$ converges. $\square$

If $p_n = 0$ for finitely many n, then we see that Lemma 15 still holds (if we consider an infinite product that is equal to zero due to finitely many zero-valued elements also convergent). Similarly, the second statement still holds if $(p_n)_{n=1}^{\infty}$ is real except for a finite number of elements.

[41] Continuity in complex numbers with their (principal) argument in $(-\pi,\pi)$. In the next lines we will see that we may assume $x_n > 0$.

We can show that the stronger statement does not necessarily hold in the complex case. We need that the series of both the argument and the logarithm of the modulus converge. This happens for $p_n = e^{\frac{1}{n^2}} \exp\left(i\frac{(-1)^n}{n}\right)$, since we have[42] $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = -\log(2)$, and so we have that $\prod_{n=1}^{\infty} p_n$ converges. Note that $p_n = e^{\frac{1}{n^2}} \cos\left(\frac{(-1)^n}{n}\right) + ie^{\frac{1}{n^2}} \sin\left(\frac{(-1)^n}{n}\right)$, so $y_n = e^{\frac{1}{n^2}} \sin\left(\frac{(-1)^n}{n}\right) = (-1)^n e^{\frac{1}{n^2}} \sin\left(\frac{1}{n}\right)$, and recalling the estimate $|y_n| \leq |1 - p_n|$, we see that[43]

$$|1 - p_n| \geq e^{\frac{1}{n^2}} \sin\left(\frac{1}{n}\right) \geq \sin\left(\frac{1}{n}\right) \geq \frac{2}{\pi} \cdot \frac{1}{n},$$

and recognizing the harmonic series on the right, we have that $\sum_{n=1}^{\infty} |1 - p_n|$ diverges by the comparison test.

[42] Or we note that the first converges as a p-series with $p = 2$ and the second converges by the alternating series test.

[43] Here we also use the estimate $\frac{2}{\pi}x \leq \sin(x)$ for $|x| \leq \frac{\pi}{2}$, since $\left|\frac{1}{n}\right| \leq 1 < \frac{\pi}{2}$ for $n \in \mathbb{N}$.

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