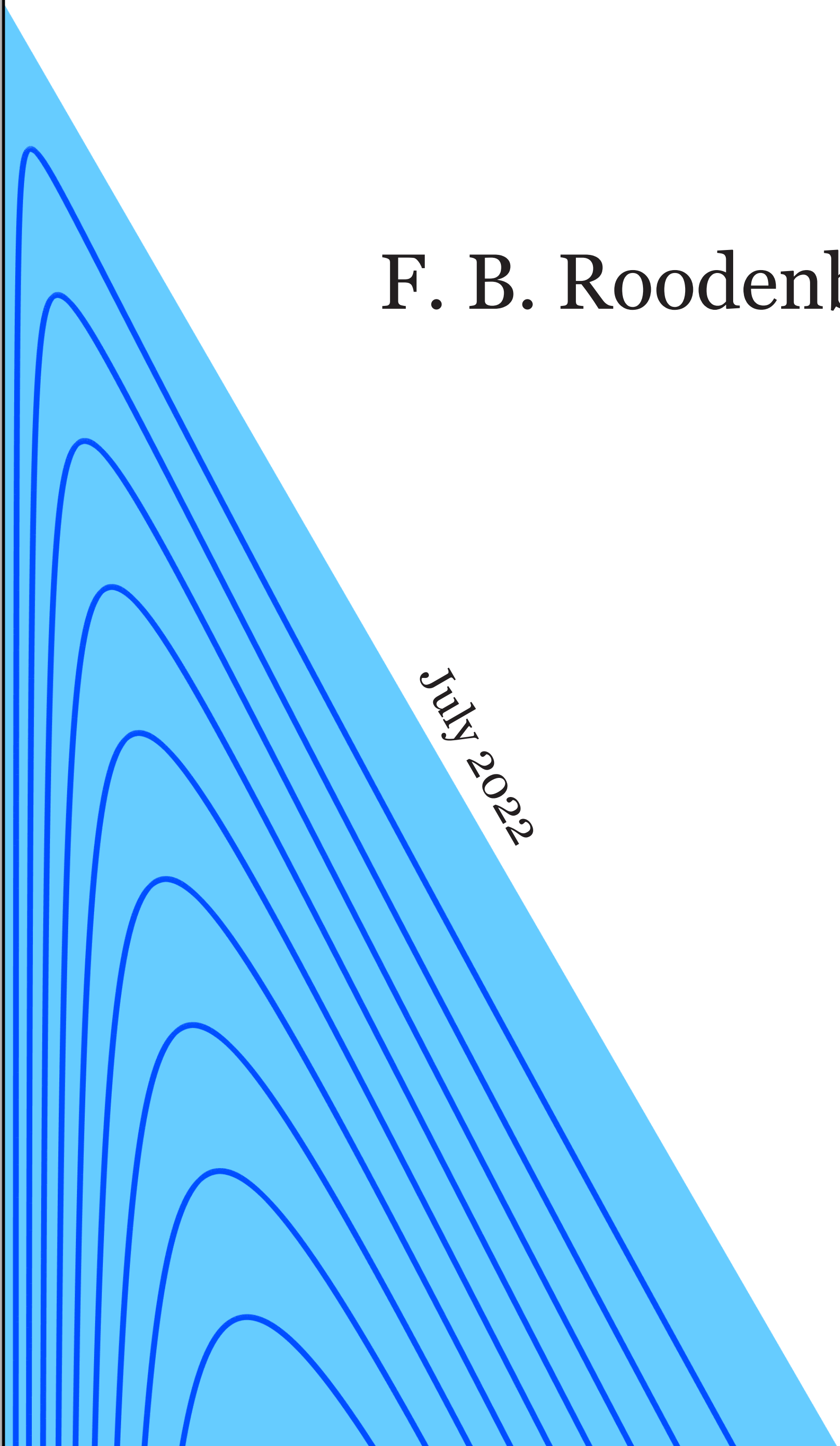
Strong Solutions to the Stokes Problem with Navier Slip on a Wedge

F. B. Roodenburg



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Cover image: streamlines corresponding to the Stokes flow in a moving wedge with no-slip boundary conditions as calculated by Huh & Scriven, 1971 [26].

MSc thesis APPLIED MATHEMATICS

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by

Floris Bart Roodenburg

Delft University of Technology

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Supervisor

Dr. M. V. Gnann

Committee members

Prof. dr. ir. M. C. Veraar

Dr. K. Marynets

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Abstract

In this thesis we consider the incompressible and stationary Stokes problem with Navier-slip boundary conditions on an infinite two-dimensional wedge with opening angle θ. As is common for differential equations on domains with corners, the problem is decomposed into a singular expansion near the corner (polynomial problem) and a regular remainder (smooth problem). We prove existence and uniqueness of solutions to the smooth problem related to the Stokes equation which is given by $-\mathbb{P}\Delta\mathbf{u} = \mathbf{f}$, where $\mathbb{P}$ is the Helmholtz projection. By means of the Lax-Milgram theorem it is found that this problem has a unique strong solution in a certain class of weighted Sobolev spaces if the opening angle θ is small enough. Direct application of the Lax-Milgram theorem would normally only yield a weak solution. However, by introducing additional bilinear forms we gain control on all second order derivatives and therewith obtain a strong solution. Finally, we touch upon the time-dependent Stokes problem and the polynomial problem.

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Contents

Chapter 1

Introduction

The incompressible Stokes and Navier-Stokes equations both form a system of partial differential equations (PDEs) that models the motion of a fluid. Within applied sciences, these equations are therefore widely used in practical problems such as weather prediction or aeroplane design. Theoretical research into the governing equations in fluid dynamics is important for the understanding of phenomena in fluid flows and especially phenomena concerning turbulence. However, a complete mathematical understanding of solutions to the (Navier-)Stokes equations is lacking until today. Many questions about existence, uniqueness and regularity of solutions are still to be answered and these questions are formulated in the third Millennium Prize Problem of the Clay Mathematics Institute [8]. In fluid dynamics there is also interest in moving boundary problems with fluid-fluid interfaces such as the spreading of droplets on a solid. These wetting and spreading phenomena play a role in applications as drainage of water from highways, inkjet printing or the wetting of leaf surfaces for deposition of pesticides.

In this thesis we will study the mathematics of a simplified problem related to those wetting and spreading phenomena. To explain the mathematical setting, consider a droplet of liquid with a free surface $h(t,x)$ on a perfectly flat solid substrate, see Figure 1.1. For simplicity we concern ourselves with a two-dimensional droplet and assume translation invariance in the third physical direction perpendicular to the (x,y)-plane. The region $\Omega_t \subset \mathbb{R}^2$ filled with the liquid may change over time $t \in [0,\infty)$. Furthermore, we assume that the liquid is a Newtonian fluid which is in addition incompressible and homogeneous, i.e. the density ρ of the liquid is constant in $[0,\infty) \times \Omega_t$. Then the governing equations for the velocity $\mathbf{u} = (u_x, u_y)^\top(t,x,y)$ and the pressure (divided by the density) $p(t,x,y)$ follow from conservation of mass and momentum, see for a detailed derivation e.g. [4, 40] or [7, 23, 44] for a more mathematical treatment. The

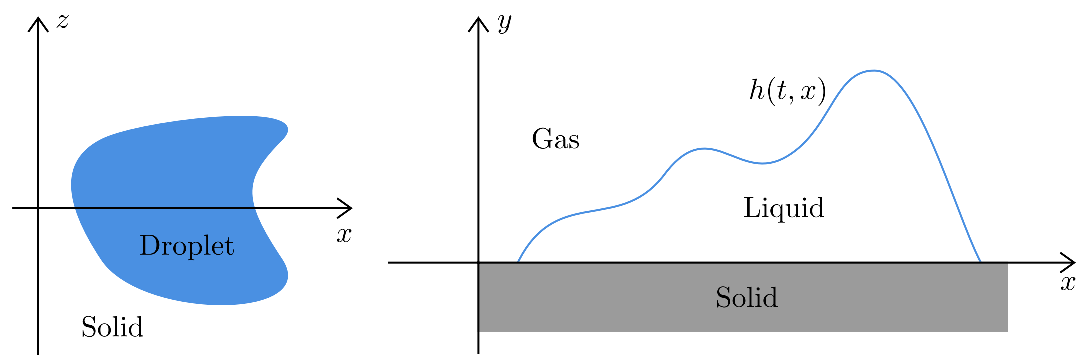


Figure 1.1: A liquid droplet on a solid substrate seen from above (left) and a two-dimensional cross-section (right).

resulting equations are the Navier-Stokes equations given by

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu_{\mathrm{k}}\Delta\mathbf{u} + \nabla p &= \mathbf{f}_{\mathrm{b}} && \text{in } [0,\infty)\times\Omega_t, \\ \operatorname{div}\mathbf{u} &= 0 && \text{in } [0,\infty)\times\Omega_t, \end{aligned} \tag{NSE}$$

where $\nu_{\mathrm{k}} = \eta/\rho$ is the kinematic viscosity (with η the shear viscosity) and $\mathbf{f}_{\mathrm{b}}$ is a body force density. To complete the initial boundary value problem (IBVP) we impose an initial condition $\mathbf{u}(0,x,y) = \mathbf{u}_0(x,y)$ and boundary conditions. The conditions required on the liquid-gas interface are shortly discussed in the following section. On the solid-liquid interface it is first of all assumed that the fluid cannot penetrate the boundary, i.e.

$$\mathbf{u}\cdot\mathbf{n} = 0, \tag{1.1}$$

where $\mathbf{n}$ is the outward pointing normal vector. In addition, a condition is required that determines the velocity in the tangential direction of the solid-liquid interface. Three possible types of boundary conditions are shown in Figure 1.2.

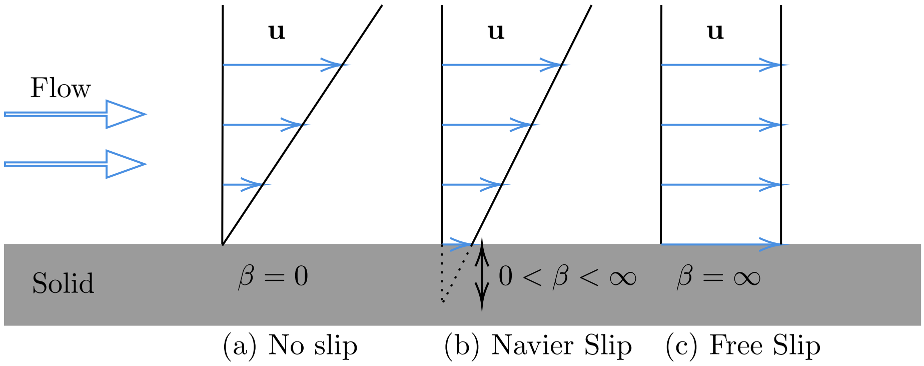


Figure 1.2: Three different types of boundary conditions on the solid-liquid interface, where $\beta \geq 0$ is the slip length.

The most standard boundary condition is the no-slip condition (Figure 1.2a) in which it is assumed that the fluid sticks to the solid and that the velocity is simply zero at the boundary, i.e. $\mathbf{u} = 0$.

If there is some non-zero velocity at the boundary which is proportional to the normal derivative of the velocity, we have a so-called Navier-slip condition (Figure 1.2b)

$$\mathbf{u}\cdot\tau + \beta\partial_{\mathbf{n}}(\mathbf{u}\cdot\tau) = 0, \tag{1.2}$$

where $\mathbf{n}$ is the outward pointing normal vector and τ is the tangential normal vector. The slip length β is a measure for the slippage and depends on the liquid and the surface structure of the solid. For instance, the slip length for water on a graphite surface is in the order of 10nm [27, 42]. Since the slip length is in general very small, the no-slip condition is more often used for modelling of fluids because it is easier to deal with. However, in the case of moving boundaries the no-slip conditions is unnatural as we will discuss below.

In case of a free-slip condition (Figure 1.2c), the solid wall has no influence on the velocity of the fluid and the slip length is infinite.

1.1 Spreading of Droplets and the No-Slip Paradox

If we consider a static droplet on a solid substrate, then at the contact line, which is the point where gas, liquid and solid meet, there is a balance of surface tensions

$$\gamma_{\mathrm{gs}} = \gamma_{\mathrm{ls}} + \cos(\theta)\gamma_{\mathrm{gl}},$$

where γ_{gs}, γ_{ls} and γ_{gl} denote the gas-solid, liquid-solid and gas-liquid surface tensions, respectively, and θ is the microscopic contact angle, see also Figure 1.3. This force balance is known as Young's law [61]. If the tensions are not balanced, then two regimes can occur: if $\gamma_{gs} < \gamma_{ls} + \gamma_{gl}$, then the contact angle θ is non-zero and the liquid partially wets the solid. If on the other hand $\gamma_{gs} \geq \gamma_{ls} + \gamma_{gl}$, then $\theta = 0$ and the liquid eventually wets the complete solid.

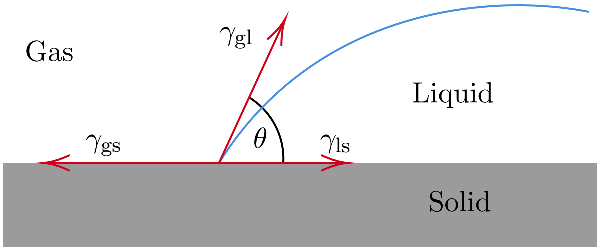


Figure 1.3: Surface tensions at the contact line.

In addition to the boundary conditions on the solid-liquid interface, also conditions are required for the gas-liquid interface, i.e. the free surface. The surface tension γ_{gl} induces a pressure jump across the free surface, which is also known as the Laplace pressure

$$\mathrm{d}p = \gamma_{gl}\kappa,$$

where $\mathrm{d}p$ is the pressure difference and κ is the mean curvature of the free surface

$$\kappa = \nabla \cdot \mathbf{n} = -\frac{\partial_x^2 h}{(1 + (\partial_x h)^2)^{3/2}}.$$

Imposing that the total stress across the free surface is continuous gives the dynamic boundary conditions

$$\begin{aligned} \mathbf{n} \cdot \mathrm{d}T \cdot \mathbf{n} &= \gamma_{gl}\kappa, \\ \tau \cdot \mathrm{d}T \cdot \mathbf{n} &= 0, \end{aligned} \tag{1.3}$$

where $T = -pI + \nu_k \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top\right)$ is the Cauchy stress tensor and $\mathrm{d}T$ is the difference between the gas and liquid stress tensors. Furthermore, to ensure that the fluid remains on the free surface we have the kinematic boundary condition

$$\frac{\partial h}{\partial t} + u_x \frac{\partial h}{\partial x} = u_y. \tag{1.4}$$

For more details on these boundary conditions we refer to [17, 50].

The No-Slip Paradox

The situation drastically complicates if the contact line can move. Huh and Scriven studied the moving contact line problem with the no-slip condition in [26]. They consider the two-dimensional Stokes problem

$$\begin{aligned} -\nu_k \Delta \mathbf{u} + \nabla p &= 0, \\ \operatorname{div} \mathbf{u} &= 0, \end{aligned} \tag{1.5}$$

which is a linearisation of the Navier-Stokes equations (NSE). We assume to have two fluids with kinematic viscosity ν_1 and ν_2 on a solid which moves with a constant velocity U, see Figure 1.4. On the solid-fluid interface we have the no-slip condition, i.e. the fluid on the boundary moves with speed U. Furthermore, we assume that the fluid-fluid interface is a straight line under the angle θ from the solid. On the fluid-fluid interface we assume continuity of the velocity and

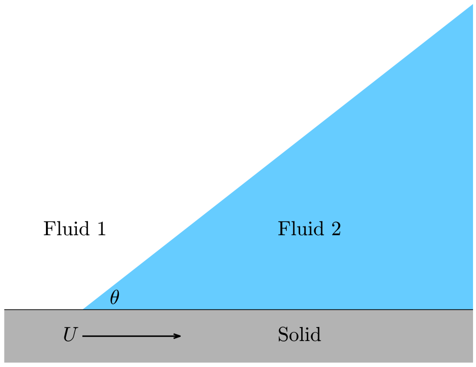


Figure 1.4: The situation of two fluids on a solid as considered by Huh and Scriven [26].

stress, i.e. (1.3) and (1.4), which simplify due to the geometry.

Because of this geometry it is convenient to use polar coordinates (r, φ) with the contact line as origin. In terms of the streamfunction $\psi(r, \varphi)$, which is everywhere parallel to the flow, the Stokes problem reduces to the biharmonic equation

$$\nabla^4 \psi = 0. \tag{1.6}$$

The relation between the velocity field in polar coordinates $\mathbf{u} = (u_r, u_\varphi)^\top$ and the streamfunction ψ is given by

$$u_r = -r^{-1}\frac{\partial \psi}{\partial \varphi} \quad \text{and} \quad u_\varphi = \frac{\partial \psi}{\partial r}.$$

The solution to the biharmonic equation (1.6) is

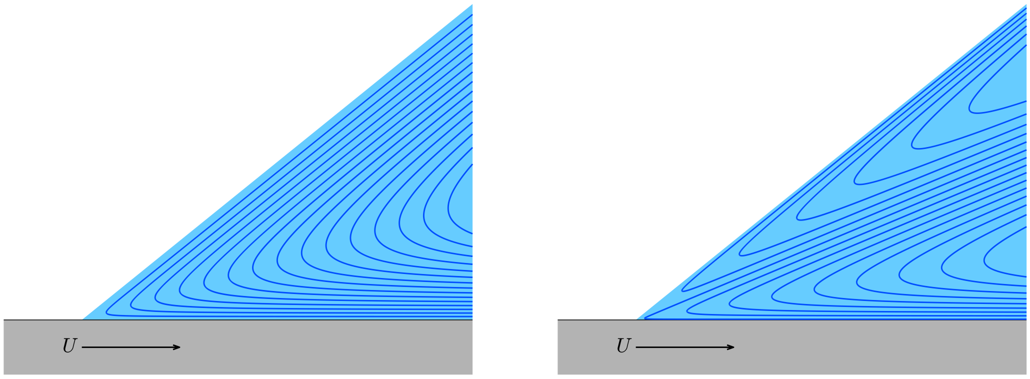
$$\psi(r, \varphi) = r\big(a \sin\varphi + b\cos\varphi + c\varphi\sin\varphi + d\varphi\cos\varphi\big), \tag{1.7}$$

where the constants a, b, c and d are determined by the boundary conditions. The corresponding streamlines for $U = 1$, $\theta = \pi/6$ and different ratios of the viscosities are shown in Figure 1.5. However, from the solution (1.7) we derive that the shear stress of the fluid is given by

$$\frac{2\nu_2}{r}\big(c\cos\theta - d\sin\theta\big),$$

which clearly diverges as $r \downarrow 0$. More generally, the stress tensor always diverges as $1/r$ near the contact line if there is no slip on the fluid-solid interface [12]. This problem remains if the Navier-Stokes equations (NSE) are used instead of only the Stokes equations (1.5). This divergence of the stress leads to a logarithmic divergence of the energy dissipation rate, $\|\nabla \mathbf{u}\|_{L^2} = \infty$, which is non-physical. Hence, by modelling the moving contact line problem with the no-slip condition "not even Herakles could sink a solid if the physical model were entirely true" as Huh and Scriven phrase it [26].

As the moving contact line problem with no slip is non-physical (which is also referred to as the no-slip paradox), the question arises how to solve this problem. Numerous mathematical models have been proposed to solve the singularity of the stress tensor, see e.g. [5, 16, 41, 51, 56]. One possible way of doing that is by modifying the governing equations, for instance by assuming that the fluid behaves in a non-Newtonian way near the contact line or by taking into account that the liquid-gas interface has a finite width with non-constant density. A second possibility,



(a) $\nu_1/\nu_2 = 1$ (b) $\nu_1/\nu_2 = 100$

Figure 1.5: The streamlines ψ in the fluid with viscosity ν_2 in the case that $U = 1$, $\theta = \pi/6$ and for different viscosity ratios ν_1/ν_2.

which we will discuss in more detail below, would be to replace the no-slip boundary condition.

As was first predicted by Maxwell [45], gases do stick to the wall on a very small scale and the slip length is proportional to the mean free path. Hence, from a physical perspective it would make sense to use a slip boundary condition. The option we will consider is the Navier-slip condition (Figure 1.2b). However, also other slip conditions are possible such as nonlinear slip [59]. Mathematically, it also makes sense to impose a slip condition because the introduced slip length removes the divergence of the stress near the contact line. The question remains how well this mathematical model with slip describes the physics very close to the contact line, see [51] for a discussion. Nevertheless, the Navier-slip condition can be derived rigorously under certain conditions [28].

The Microscopic and Macroscopic Contact Angle

As we have seen, the contact angle of a spreading droplet is microscopically determined by Young's law. However, the liquid-gas interface is highly curved near the contact line [11] and the microscopic contact angle θ_{mic} is much smaller than the macroscopic contact angle θ_{mac}, which can be obtained by a measurement at a macroscopic distance from the contact line. The macroscopic contact angle is dependent on the flow and the velocity of the contact line [55].

Since the microscopic contact angle is in general very small, it makes sense to apply a lubrication approximation in which the Navier-Stokes equations (NSE) are simplified to a scalar equation for the film height $h(t, x)$. This leads to the so called thin-film equation which is valid for small angles

$$\partial_t h + \partial_x\big((h^3 + \beta^{3-n}h^n)\partial_x^3 h\big) = 0, \tag{1.8}$$

where β is the slip length and $n \in [1, 3)$ represents the physically relevant boundary condition of the solid-liquid interface (see also Figure 1.2). The value $n = 1$ corresponds to the free-slip condition, $n = 2$ corresponds to Navier slip and $n = 3$ corresponds to no slip which is not physical as the no-slip paradox showed. For a formal derivation of this equation from the Navier-Stokes equations see for instance [19, 50].

The relation between the microscopic and macroscopic contact angle in the lubrication approximation is determined by the Cox-Voinov law which states that θ_{mac}^3 is proportional to the velocity of the free boundary up to a logarithmic correction [9, 60]. For more details on the thin-film equation and the relation between the contact angles we refer to the literature [5] and references therein. For mathematical results in both the complete and partial wetting regime, see e.g. [6, 18, 19, 21, 24, 30].

1.2 Mathematical Problem

The free boundary value problem (BVP) of a spreading droplet has been studied for Navier slip on the solid-liquid interface in [38, 57] and for special values for the contact angle leading to additional symmetries (i.e. $0, \pi/2$ and π) results on well-posedness are known [15, 53, 58]. A general procedure would be to transform the free boundary problem to a fixed wedge-shaped domain. However, this leads to complicated nonlinear problems which are hard to deal with. Instead of dealing with the free BVP, we will in this thesis restrict the domain to a fixed infinite wedge denoted by Ω with opening angle $\theta > 0$ and the tip at the origin, see Figure 1.6. In polar coordinates the domain is given by

$$\Omega := \{(r\cos\varphi, r\sin\varphi) : r > 0, \varphi \in (0,\theta)\}$$

and the boundary of the wedge is $\partial\Omega = \partial_0\Omega \cup \partial_1\Omega \cup \{(0,0)\}$, where

$$\begin{aligned} \partial_0\Omega &:= \{(r\cos\varphi, r\sin\varphi) : r > 0, \varphi = 0\} \quad \text{and} \\ \partial_1\Omega &:= \{(r\cos\varphi, r\sin\varphi) : r > 0, \varphi = \theta\}. \end{aligned}$$

Furthermore, we will denote the part of the boundary where a normal vector can be defined as $\partial\Omega' := \partial_0\Omega \cup \partial_1\Omega$.

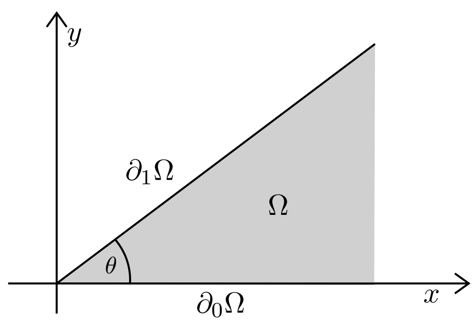


Figure 1.6: The wedge-shaped domain with opening angle θ.

The problem that we will mainly study in this thesis is the incompressible and stationary Stokes problem with no-penetration (1.1) and Navier-slip (1.2) boundary conditions on both the lower and upper boundary of the wedge, i.e.

$$-\nu_{\mathrm{k}}\Delta\mathbf{u} + \nabla p = \mathbf{f}_{\mathrm{b}} \qquad \text{in } \Omega, \tag{1.9a}$$

$$\operatorname{div}\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{1.9b}$$

$$\mathbf{u}\cdot\mathbf{n} = 0 \qquad \text{on } \partial\Omega', \tag{1.9c}$$

$$\mathbf{u}\cdot\tau + \beta\partial_{\mathbf{n}}(\mathbf{u}\cdot\tau) = 0 \qquad \text{on } \partial\Omega', \tag{1.9d}$$

where $\mathbf{n}$ and τ are the outward and tangential normal vector, respectively. The final goal is to prove existence, uniqueness and regularity of solutions to the Stokes problem (1.9). In this thesis we will mainly study the first two problems and leave the higher regularity for future work.

The Stokes problem (1.9a)-(1.9b) has already been studied in the literature, which we will discuss in Chapter 3, for a wide range of domains and boundary conditions. In the problem described above, two main issues occur and the combination of those two has not yet been studied in the existing literature. The difficulties arise from the boundary of the wedge, which is not smooth, and the Navier-slip condition. The non-smoothness of the domain can cause irregular behaviour of the solution in the vicinity of the corner. Therefore, we will decompose the Stokes problem into a singular expansion near the tip (polynomial problem) and a regular remainder (smooth problem). The difficulty with the Navier-slip boundary condition, which can be compared to a Robin condition, is that it is not scaling invariant. This makes the analysis more complicated since no explicit solution formulas for the polynomial problem can be found. Those complications and how to overcome them will be discussed in more detail in Chapter 3.

The core of this thesis is concerned with proving existence and uniqueness of solutions to the smooth problem related to the Stokes equations (1.9), which is after rescaling and projecting given by

$$\begin{aligned} -\mathbb{P}\Delta\mathbf{u} &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u}\cdot\mathbf{n} &= 0 && \text{on } \partial\Omega', \\ \mathbf{u}\cdot\tau + \partial_{\mathbf{n}}(\mathbf{u}\cdot\tau) &= 0 && \text{on } \partial\Omega', \end{aligned} \tag{1.10}$$

where $\mathbb{P}$ is the Helmholtz projection that eliminates the pressure and ensures that the solution $\mathbf{u}$ is divergence free (1.9b). With the aid of the Lax-Milgram theorem we will show that there exists a unique solution $\mathbf{u}$ to (1.10) in a weighted Sobolev space which is the right setting for proving higher regularity. A direct application of the Lax-Milgram theorem would only yield a weak solution which is once weakly differentiable. However, we will introduce two additional bilinear forms which are derived from the Stokes problem for $r\partial_r\mathbf{u}$ and the vorticity $\operatorname{curl}\mathbf{u}$ instead of $\mathbf{u}$. By making use of this approach, we get control on all second order derivatives and we show that if the angle θ is small enough, then there exists a unique strong solution which is twice weakly differentiable.

We emphasise that the Navier-slip condition on the upper boundary $\partial\Omega_1$ is essential for proving our results. The underlying idea is that if θ is small, then the two boundaries $\partial\Omega_0$ and $\partial\Omega_1$ are close together and we know what the solution looks like on the boundaries by the boundary conditions (1.9c) and (1.9d). Intuitively, in this case there is no possibility for the solution in the interior of the wedge to deviate from the boundary behaviour. For large contact angles or other boundary conditions on $\partial\Omega_1$ such as the free-slip condition (Figure 1.2c), it can be much harder to gain control on the solution.

Therefore, we restrict ourselves to small angles θ. This is not too restrictive since the contact angle θ_{mic} of a spreading droplet near the contact line is in general very small. In this case the thin-film equation (1.8) is an approximation of the Navier-Stokes equations (NSE). The thin-film equation only depends on the film height h and is independent of the angle. This makes the analysis in general easier than for the (Navier-)Stokes equations in a wedge with non-zero angle. Compared to the thin-film equation our problem is more difficult since it deals with a two-dimensional problem. However, the thin-film equation is more general in the sense that it allows for a moving boundary while we currently only consider a fixed boundary.

1.3 Overview of this Thesis

The outline of this thesis is as follows: in Chapter 2 the required theory on two-point BVPs and functional analysis is provided. In addition, we prove Hardy's inequality on the wedge and introduce the Mellin transform, which form the basic tools for proving our results in later chapters. In Chapter 3 the mathematical challenges of the non-smoothness of the domain and the Navier-slip boundary conditions are discussed. Moreover, we state the main theorem on the existence and uniqueness of strong solutions to the Stokes problem.

In Chapter 4 the Helmholtz projection is introduced which enables us to exclude treating the pressure explicitly. Moreover, we prove certain estimates on the Helmholtz projection. Chapters 5 and 6 contain the proof of the main result. In Chapter 5 three bilinear forms are derived and with the Lax-Milgram theorem we find a solution to the sum of those bilinear forms. Furthermore, we prove that the solution to this bilinear form is also a solution to the Stokes problem and satisfies the Navier-slip boundary condition. To apply the Lax-Milgram theorem, we need a coercivity and boundedness estimate. However, obtaining these estimates is quite cumbersome and therefore Chapter 6 is entirely devoted to the proof of the conditions for the Lax-Milgram theorem.

Finally, in Chapter 7 we discuss some incomplete results which can serve as a starting point for future work. Firstly, it is shown that the time-dependent Stokes problem is easier to treat than the stationary problem. Secondly, we solve the polynomial problem related to the stationary Stokes problem which captures the behaviour of the solution near the tip of the wedge.

Chapter 2

Preliminary Theory

The aim of this preliminary chapter is to give a short overview of the concepts and theorems that are used in this thesis. In Section 2.1 we introduce the notation that will be used and in the rest of this chapter, which can be omitted on first reading, certain results are stated that are required in later chapters.

2.1 Notation

We write $f \lesssim_P g$ (resp. $f \gtrsim_P g$) if there exists a constant $C \in (0, \infty)$ depending on the set of parameters P such that $f \leq Cg$ (resp. $f \geq Cg$). Furthermore, $f \sim_P g$ means $f \lesssim_P g$ and $g \lesssim_P f$. If $P = \emptyset$, then we just write $f \lesssim g$, $f \gtrsim g$ or $f \sim g$.

For $U \subset \mathbb{R}^n, V \subset \mathbb{R}^m$ $(n, m \geq 1)$ and $k \in \mathbb{N}_0 \cup \{\infty\}$, the space $C^k(U; V)$ denotes the k-times continuously differentiable functions from U to V. The space $C_\mathrm{c}^k(U; V)$ denotes the space of test functions on U, i.e. the set of k-times differentiable functions with compact support contained in U. If there is no confusion about the set V, we simply write $C^k(U)$ or $C_\mathrm{c}^k(U)$.

Throughout this thesis, the two-dimensional wedge with origin $(0, 0)$ and opening angle $0 < \theta < \pi/2$ will always be denoted by Ω with boundary $\partial\Omega = \partial_0\Omega \cup \partial_1\Omega \cup \{(0, 0)\}$, where $\partial_0\Omega$ and $\partial_1\Omega$ are the lower and upper boundary, respectively, see Figure 2.1. Furthermore, we define $\partial\Omega' := \partial_0\Omega \cup \partial_1\Omega$ as the part of the boundary where we can define the normal vectors $\mathbf{n}$ (outward) and τ (tangential). For example, on $\partial\Omega'$ the outward normal vector in polar coordinates is given by $\mathbf{n} = (0, \pm 1)^\top$, where the notation $\pm$ (resp. $\mp$) will mean $-$ (resp. $+$) on the lower boundary $\partial_0\Omega$ and $+$ (resp. $-$) on the upper boundary $\partial_1\Omega$.

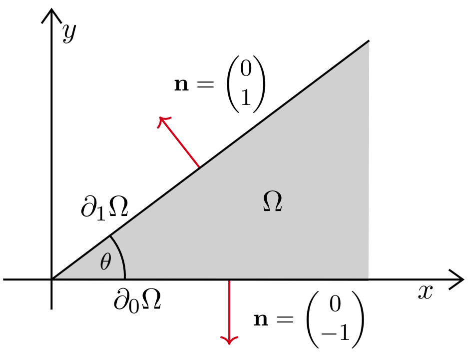


Figure 2.1: The domain Ω with lower and upper boundary $\partial_0\Omega$ and $\partial_1\Omega$, respectively, and the corresponding outward normal vectors in polar coordinates.

For any $\mathbf{u} : \Omega \to \mathbb{R}^2$ we write in polar coordinates $\mathbf{u}(r,\varphi) = (u_r(r,\varphi), u_\varphi(r,\varphi))^\top$ where u_r and u_φ denote the radial and angular component, respectively (see Appendix A.2 for more details). The derivatives in the radial and angular direction are written as ∂_r and ∂_φ.

Finally, we already remark that $\widehat{f}(\lambda)$ will always denote the Mellin transform of f and λ is the Mellin variable. The Mellin transform will be introduced in more detail in Section 2.5.

2.2 Two-Point Boundary Value Problems

Below we recall the theory for solving (systems of) non-homogeneous linear ordinary differential equations with boundary conditions, so-called two-point boundary value problems. For a complete introduction and proofs see e.g. [2, 14, 22]. Throughout this section we write ∂_φ for the ordinary derivative $\frac{\mathrm{d}}{\mathrm{d}\varphi}$.

2.2.1 Green's Matrix

Let $n \in \mathbb{N}$ and consider the n-dimensional two-point boundary value problem for $\mathbf{u}(\varphi)$

$$\partial_\varphi \mathbf{u} - A(\varphi)\mathbf{u} = \mathbf{b}(\varphi) \qquad \text{for } \varphi \in (0,\theta), \tag{2.1a}$$

$$R_0\mathbf{u}(0) + R_\theta \mathbf{u}(\theta) = \mathbf{c}, \tag{2.1b}$$

where $A \in C^0([0,\theta];\mathbb{C}^{n\times n})$, $\mathbf{b} \in C^0([0,\theta];\mathbb{C}^{n\times 1})$, $R_0, R_\theta \in \mathbb{C}^{n\times n}$ and $\mathbf{c} \in \mathbb{C}^{n\times 1}$. Recall that a fundamental matrix $V \in C^1([0,\theta];\mathbb{C}^{n\times n})$ has columns with the n linearly independent solutions to the homogeneous equation

$$\partial_\varphi \mathbf{u} = A(\varphi)\mathbf{u} \qquad \text{for } \varphi \in (0,\theta). \tag{2.2}$$

The most important properties of the fundamental matrix are collected below.

Lemma 2.2.1. *The fundamental matrix of* (2.2) *satisfies the following properties:*

1. *$V(\varphi)$ is a fundamental matrix if and only if $\partial_\varphi V = AV$ and $\det V(0) \neq 0$.*
2. *For two fundamental matrices V_1 and V_2, there exists a constant invertible matrix D such that $V_2 = V_1 D$.*
3. *If A is constant, the matrix $e^{\varphi A}$ is a fundamental matrix and $e^{\varphi A} = V(\varphi)V^{-1}(0)$.*

To determine whether (2.1) has a unique solution $\mathbf{u} \in C^1([0,\theta];\mathbb{C}^{n\times 1})$, we introduce the notion of the characteristic matrix.

Definition 2.2.2. *For any fundamental matrix V of* (2.2)*, define the corresponding characteristic matrix*

$$C := R_0 V(0) + R_\theta V(\theta).$$

From Lemma 2.2.1, property 2, it follows that for any two characteristic matrices C_1, C_2 of (2.2), there exists a constant invertible matrix $\tilde{D}$ such that $C_1 = C_2\tilde{D}$. Using the characteristic matrix and its properties we obtain the following result for solving (2.1).

Proposition 2.2.3. *For Problem* (2.1) *the following statements are equivalent:*

1. *For every $\mathbf{b} \in C^0([0,\theta];\mathbb{C}^{n\times 1})$ and $\mathbf{c} \in \mathbb{C}^{n\times 1}$ the solution $\mathbf{u} \in C^1([0,\theta];\mathbb{C}^{n\times 1})$ exists and is unique.*
2. *The homogeneous problem with $\mathbf{b} = 0$ and $\mathbf{c} = 0$ is only satisfied by the trivial solution.*

3. There exists a characteristic matrix with full rank.

4. All characteristic matrices have full rank.

In addition to the existence and uniqueness we would like to have an expression for the solution of (2.1). This is achieved by means of Green's matrix.

Proposition 2.2.4. *Let V be a fundamental matrix of* (2.2) *and let C be its corresponding characteristic matrix satisfying* $\det C \neq 0$. *Then the unique solution $\mathbf{u} \in C^1([0,\theta];\mathbb{C}^{n\times 1})$ of Problem* (2.1) *is given by*

$$\mathbf{u}(\varphi) = V(\varphi)C^{-1}\mathbf{c} + \int_0^\theta \Gamma(\varphi,\tilde{\varphi})\mathbf{b}(\tilde{\varphi})\,\mathrm{d}\tilde{\varphi} \qquad \textit{for } \varphi \in (0,\theta),$$

where $\Gamma : [0,\theta]\times[0,\theta] \to \mathbb{C}^{n\times n}$ is called the Green's matrix and is almost everywhere defined by

$$\Gamma(\varphi,\tilde{\varphi}) := \begin{cases} V(\varphi)\left[\mathrm{id} - C^{-1}R_\theta V(\theta)\right]V^{-1}(\tilde{\varphi}) & \textit{for } 0 \le \tilde{\varphi} < \varphi \le \theta, \\ -V(\varphi)C^{-1}R_\theta V^{-1}(\tilde{\varphi}) & \textit{for } 0 \le \varphi \le \tilde{\varphi} < \theta. \end{cases}$$

2.2.2 Green's Function

Let $n \in \mathbb{N}$ and consider the linear n-th order differential equation for $u(\varphi)$

$$\partial_\varphi^n u + a_{n-1}(\varphi)\partial_\varphi^{n-1}u + \cdots + a_1(\varphi)\partial_\varphi u + a_0(\varphi)u = g(\varphi) \qquad \text{for } \varphi \in (0,\theta), \tag{2.3}$$

where $a_j \in C^0([0,\theta];\mathbb{C})$ for $j = 0,\dots,n-1$ and $g \in C^0([0,\theta];\mathbb{C})$. Furthermore, we impose linear boundary conditions at $\varphi \in \{0,\theta\}$. Define

$$\mathbf{u} := \begin{pmatrix} u \\ \partial_\varphi u \\ \vdots \\ \partial_\varphi^{n-1}u \end{pmatrix}, \qquad A := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix} \quad \text{and} \quad \mathbf{b} := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g \end{pmatrix},$$

and rewrite the boundary conditions in the matrix-vector form $R_0\mathbf{u}(0) + R_\theta\mathbf{u}(\theta) = \mathbf{c}$, so that we obtain a special case of Problem (2.1). If $u_1(\varphi),\dots,u_n(\varphi)$ are linearly independent solutions to the homogeneous equation (2.3) with $g = 0$, then the corresponding Wronski matrix $W \in C^1([0,\theta];\mathbb{C}^{n\times n})$ is defined by

$$W(\varphi) := \begin{pmatrix} u_1 & \cdots & u_n \\ \partial_\varphi u_1 & \cdots & \partial_\varphi u_n \\ \vdots & & \vdots \\ \partial_\varphi^{n-1}u_1 & \cdots & \partial_\varphi^{n-1}u_n \end{pmatrix},$$

which is just the fundamental matrix of $\partial_\varphi\mathbf{u} = A\mathbf{u}$ in this special case. Similarly as before, the characteristic matrix is defined as $C := R_0W(0) + R_\theta W(\theta)$. Existence and uniqueness of the solution $u \in C^m([0,\theta],\mathbb{C})$ to (2.3) with boundary conditions can simply be determined with Proposition 2.2.3. To find an expression for the solution, we now make use of the Green's function.

Proposition 2.2.5. *Assume that the homogeneous problem of Equation* (2.3) *with boundary conditions has only the trivial solution. Then the solution $u \in C^m([0,\theta],\mathbb{C})$ is given by*

$$u(\varphi) = c_1u_1(\varphi) + \cdots + c_nu_n(\varphi) + \int_0^\theta G(\varphi,\tilde{\varphi})g(\tilde{\varphi})\,\mathrm{d}\tilde{\varphi} \qquad \textit{for } \varphi \in (0,\theta),$$

where $G : [0,\theta] \times [0,\theta] \to \mathbb{C}$ *is called the Green's function and is given by*

$$G(\varphi,\tilde{\varphi}) := \begin{cases} \sum_{j=1}^{n} \big(\beta_j(\tilde{\varphi}) + \gamma_j(\tilde{\varphi})\big) u_j(\varphi) & \text{for } 0 \le \tilde{\varphi} < \varphi \le \theta, \\ \sum_{j=1}^{n} \big(\beta_j(\tilde{\varphi}) - \gamma_j(\tilde{\varphi})\big) u_j(\varphi) & \text{for } 0 \le \varphi < \tilde{\varphi} \le \theta, \end{cases}$$

where $\boldsymbol{\beta}(\tilde{\varphi}) = (\beta_1,\dots,\beta_n)^\top$ *and* $\boldsymbol{\gamma}(\tilde{\varphi}) = (\gamma_1,\dots,\gamma_n)^\top$ *are determined by*

$$W(\tilde{\varphi})\boldsymbol{\gamma}(\tilde{\varphi}) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \big(R_0 W(0) + R_\theta W(\theta)\big)\boldsymbol{\beta}(\tilde{\varphi}) = \big(R_0 W(0) - R_\theta W(\theta)\big)\boldsymbol{\gamma}(\tilde{\varphi}).$$

Finally, the constants $c_1,\dots,c_n$ *can be determined from the boundary conditions.*

2.3 Functional Analysis

Tools from functional analysis play an important role within the study of partial differential equation. We shortly recall some standard results and introduce the L^p and Sobolev spaces. For a complete introduction to the topic see e.g. [1, 3, 13, 49].

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two non-trivial Banach spaces and let $T : X \to Y$ be a bounded linear operator, i.e.

$$\|Tx\|_Y \le \|T\| \|x\|_X,$$

where

$$\|T\| := \sup_{\|x\|_X \le 1} \|Tx\|_Y < \infty.$$

We denote the set of all bounded linear operators from X to Y as $\mathscr{L}(X,Y)$.

Definition 2.3.1. *Let* $\mathbb{K}$ *be* $\mathbb{R}$ *or* $\mathbb{C}$*. Then the set of all bounded linear operators* $\mathscr{L}(X,\mathbb{K})$ *is the (topological) dual space of* X *which is denoted by* X'*. Furthermore, for* $x \in X$ *and* $x' \in X'$ *we denote* $x'(x) \in \mathbb{K}$ *as the dual pairing* $\langle x', x\rangle$*.*

Proposition 2.3.2. *For* $x' \in X'$ *the norm*

$$\|x'\|_{X'} := \sup_{x \in X\setminus\{0\}} \frac{|\langle x', x\rangle|}{\|x\|_X} \tag{2.4}$$

turns $X' = \mathscr{L}(X,\mathbb{K})$ *into a Banach space.*

We turn to the special case of a Hilbert space H with inner product $(\cdot,\cdot)$ and the induced norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. Recall that an inner product on $H \times H$ is a sesquilinear mapping (i.e. linear in its first argument and conjugate linear in its second argument) satisfying

1. $(u,u) \ge 0$ for all $u \in H$ and if $(u,u) = 0$, then $u = 0$,
2. $(u,v) = \overline{(v,u)}$ for all $u, v \in H$.

For Hilbert spaces we have the following characterisation of its dual space.

Theorem 2.3.3 (Riesz Representation Theorem)**.** *Let* H *be a Hilbert space with inner product* $(\cdot,\cdot)$ *and let* $f \in H'$*. Then there exists a unique* $u \in H$ *such that*

$$(u,v) = \langle f, v\rangle \quad \text{for every } v \in H.$$

Within the study of differential equations, this theorem is used to show existence and uniqueness of a weak solution in a suitable Hilbert space. However, the Riesz representation theorem requires that the bilinear form arising from the original differential equation is an inner product. In particular, this means that the bilinear form should be (conjugate) symmetric which is in general a too restrictive condition. Fortunately, this symmetry condition can be relaxed and there is a generalisation of the Riesz representation theorem.

Theorem 2.3.4 (Lax-Milgram Theorem). *Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and let $B : H \times H \to \mathbb{K}$ be a sesquilinear mapping for which there exist constants $C, D \in (0,\infty)$ such that*

$$|B(u,v)| \leq C\|u\|\|v\| \quad \text{for all } u, v \in H, \qquad \text{(Boundedness)}$$
$$\operatorname{Re} B(u,u) \geq D\|u\|^2 \quad \text{for all } u \in H. \qquad \text{(Coercivity)}$$

In addition, assume that $f \in H'$. Then there exists a unique element $u \in H$ such that

$$B(u,v) = \langle f, v\rangle \quad \text{for all } v \in H.$$

Function Spaces

Consider a set $U \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ and let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$. We define the following function spaces.

Definition 2.3.5 (L^p-spaces). *For $1 \leq p < \infty$ we define $L^p(U)$ as the set of all measurable functions $f : U \to \mathbb{K}$ such that*

$$\|f\|_{L^p(U)}^p := \int_U |f(x)|^p \,\mathrm{d}x < \infty.$$

For $p = \infty$ we define $L^\infty(U)$ as the set of all measurable functions $f : U \to \mathbb{K}$ such that

$$\|f\|_{L^\infty(U)} := \operatorname*{ess\,sup}_{x\in U} |f(x)| < \infty.$$

For $p \in [1,\infty]$ the space $L^p(U)$ is a Banach space and for $p = 2$ the space $L^2(U)$ is even a Hilbert space with inner product

$$(f,g)_{L^2(U)} = \int_U f(x)\overline{g(x)} \,\mathrm{d}x \quad \text{for } f, g \in L^2(U).$$

Let $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}_0^n$ be a multi-index of order $|\beta| = \beta_1 + \dots + \beta_n$.

Definition 2.3.6 (Sobolev spaces). *For $1 \leq p < \infty$ and $k \in \mathbb{N}_0$ we define the Sobolev space $W^{k,p}(U)$ of all locally integrable functions on U for which all weak derivatives of order $|\beta| \leq k$ exist and are in $L^p(U)$. As proved in [47], we can equivalently define this space as the closure of all $f \in C^\infty(U)$ with $\partial^\beta f \in L^p(U)$ ($|\beta| \leq k$) with respect to the norm*

$$\|f\|_{W^{k,p}(U)}^p = \sum_{0\leq|\beta|\leq k} \|\partial^\beta f\|_{L^p(U)}^p.$$

For $1 \leq p < \infty$ the space $W^{k,p}(U)$ is a Banach space and for $p = 2$ the space $H^k(U) := W^{k,2}(U)$ is even a Hilbert space with inner product

$$(f,g)_{H^k(U)} = \sum_{0\leq|\beta|\leq k} (\partial^\beta f, \partial^\beta g)_{L^2(U)} \quad \text{for } f, g \in H^k(U).$$

2.4 Hardy's Inequality

Below we recall the famous Hardy inequalities and we prove a useful form of this inequality on the wedge.

Lemma 2.4.1 (Hardy's inequalities). *The unweighted Hardy inequality for a measurable non-negative function f is given by*

$$\int_0^\infty \left(\frac{1}{y}\int_0^y f(z)\,\mathrm{d}z\right)^p \mathrm{d}y \le \left(\frac{p}{p-1}\right)^p \int_0^\infty |f(y)|^p\,\mathrm{d}y.$$

The weighted inequality for $\alpha < 1$ and $p \in [1,\infty)$ is

$$\int_0^\infty y^{p(\alpha-1)-1}\left(\int_0^y |f(z)|\,\mathrm{d}z\right)^p \mathrm{d}y \le \left(\frac{1}{1-\alpha}\right)^p \int_0^\infty |f(z)|^p z^{\alpha p-1}\,\mathrm{d}z \tag{2.5}$$

and the weighted "dual" inequality for $\alpha > 1$ and $p \in [1,\infty)$ reads

$$\int_0^\infty y^{p(\alpha-1)-1}\left(\int_y^\infty |f(z)|\,\mathrm{d}z\right)^p \mathrm{d}y \le \left(\frac{1}{\alpha-1}\right)^p \int_0^\infty |f(z)|^p z^{\alpha p-1}\,\mathrm{d}z. \tag{2.6}$$

Proof. See [25]. □

For the two-dimensional wedge Ω with opening angle θ we can use the Hardy inequality to prove the following.

Lemma 2.4.2 (Hardy's inequality on the wedge). *For all $\delta \neq 0$ and $\psi \in C_\mathrm{c}^1(\overline{\Omega}\setminus\{0\})$ it holds*

$$\int_0^\theta\int_0^\infty r^{2\delta}|\psi(r,\varphi)|^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \le \frac{1}{\delta^2}\int_0^\theta\int_0^\infty r^{2\delta+2}|\partial_r\psi(r,\varphi)|^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi.$$

Proof. It suffices to show that for all $\varphi \in (0,\theta)$

$$\int_0^\infty r^{2\delta-1}|\psi(r,\varphi)|^2\,\mathrm{d}r \le \frac{1}{\delta^2}\int_0^\infty r^{2\delta+1}|\partial_r\psi(r,\varphi)|^2\,\mathrm{d}r.$$

Note that

$$|\psi(r,\varphi)| = \left|\int_r^\infty \partial_z\psi(z,\varphi)\,\mathrm{d}z\right| \le \int_r^\infty |\partial_z\psi(z,\varphi)|\,\mathrm{d}z,$$

which implies that for $\delta > 0$

$$\begin{aligned}\int_0^\infty r^{2\delta-1}|\psi(r,\varphi)|^2\,\mathrm{d}r &\le \int_0^\infty r^{2\delta-1}\left(\int_r^\infty |\partial_z\psi(z,\varphi)|\,\mathrm{d}z\right)^2 \mathrm{d}r\\ &\le \frac{1}{\delta^2}\int_0^\infty r^{2\delta+1}|\partial_r\psi(r,\varphi)|^2\,\mathrm{d}r,\end{aligned}$$

where in the last step we applied Hardy's inequality (2.6) on $\partial_z\psi$ with $\alpha = \delta+1 > 1$ and $p = 2$.

If $\delta < 0$, then

$$|\psi(r,\varphi)| = \left|\int_0^r \partial_z\psi(z,\varphi)\,\mathrm{d}z\right| \le \int_0^r |\partial_z\psi(z,\varphi)|\,\mathrm{d}z$$

and Hardy's inequality (2.5) with $\alpha = \delta+1 < 1$ and $p = 2$ gives the desired estimate for $\delta < 0$. □

2.5 The Mellin Transform

Before introducing the Mellin transform, we recall the closely related Fourier transform. Let $\mathscr{S}(\mathbb{R})$ be the Schwartz space defined as the set of all $f \in C^\infty(\mathbb{R};\mathbb{C})$ such that for all $k, \ell \in \mathbb{N}_0$

$$\sup_{x \in \mathbb{R}} |x^k \partial^\ell f(x)| < \infty.$$

On this space the Fourier transform, defined by

$$(\mathcal{F}f)(\xi) := (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} f(x) e^{-ix\xi} \,\mathrm{d}x \qquad \text{for } \xi \in \mathbb{R},$$

is a bijection and its inverse is given by

$$(\mathcal{F}^{-1}f)(x) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} f(\xi) e^{ix\xi} \,\mathrm{d}\xi \qquad \text{for } x \in \mathbb{R}.$$

By density of the Schwartz space in $L^p(\mathbb{R};\mathbb{C})$ for $1 \leq p < \infty$, we can define the Fourier transform on the Hilbert space $L^2(\mathbb{R};\mathbb{C})$. The Fourier transform is a bijection on L^2 and we have Plancherel's identity

$$(\mathcal{F}f, \mathcal{F}g)_{L^2(\mathbb{R};\mathbb{C})} = (f, g)_{L^2(\mathbb{R};\mathbb{C})} \qquad \text{for } f, g \in L^2(\mathbb{R};\mathbb{C}). \tag{2.7}$$

Subsequently, we define the Mellin transform.

Definition 2.5.1. *For $f \in C_c^\infty((0,\infty))$ the Mellin transform is defined as*

$$(\mathcal{M}f)(\lambda) = \widehat{f}(\lambda) := (2\pi)^{-\frac{1}{2}} \int_0^\infty r^{-\lambda} f(r) \,\frac{\mathrm{d}r}{r},$$

on some strip of absolute convergence $\gamma_1 < \mathrm{Re}\lambda < \gamma_2$. For any $\gamma \in (\gamma_1, \gamma_2)$ the inverse Mellin transform is

$$f(r) = \frac{1}{i\sqrt{2\pi}} \int_{\mathrm{Re}\lambda = \gamma} r^\lambda \widehat{f}(\lambda) \,\mathrm{dIm}\lambda,$$

where the integral is taken with increasing $\mathrm{Im}\lambda$*.*

Note that the Mellin transform is analytic on the strip $\gamma_1 < \mathrm{Re}\lambda < \gamma_2$ and therefore the inverse transform does not depend on the choice of γ by Cauchy's integral theorem.

Remark 2.5.2. *With the substitution $x = \log(r)$ we get*

$$\int_{-\infty}^\infty e^{-ix\xi} f(x) \,\mathrm{d}x = \int_0^\infty r^{-i\xi} f(\log(r)) \,\frac{\mathrm{d}r}{r}$$

and thus we have the following relation between the Fourier and Mellin transform

$$(\mathcal{F}f)(\xi) = \big(\mathcal{M}f(\log(\cdot))\big)(i\xi) \qquad \textit{for } \xi \in \mathbb{R},$$

which shows that the Fourier transform is equivalent to the Mellin transform for $\lambda \in i\mathbb{R}$. Similarly, we also have

$$(\mathcal{M}f)(\lambda) = (\mathcal{F}f(e^{(\cdot)}))(-i\lambda) \qquad \textit{for } \lambda \in \mathbb{C},$$

meaning that the Mellin transform can be formulated in terms of the complex Fourier transform.

Finally, we summarise the most important properties of the Mellin transform.

Lemma 2.5.3. *Let $f \in C_c^\infty((0,\infty))$. Then the Mellin transform satisfies the following properties:*

1. *For any $a \in \mathbb{R}$*
$$\widehat{r^{-a} f}(\lambda) = \widehat{f}(\lambda + a).$$

2. *For any $n \in \mathbb{N}$*
$$\widehat{\partial_r^n f}(\lambda) = (\lambda+1)\cdots(\lambda+n)\widehat{f}(\lambda+n)$$
and
$$\widehat{(r\partial_r)^n f}(\lambda) = \lambda^n \widehat{f}(\lambda).$$

Remark 2.5.4. *From the above properties we see that $r\partial_r$ corresponds to λ in Mellin representation and therefore the Mellin transform is scaling invariant under taking $r\partial_r$ derivatives.*

Furthermore, we need Plancherel's identity for the Mellin transform.

Lemma 2.5.5. *Let $\alpha \in \mathbb{R}$ and $f, g \in L^2\big((0,\infty), r^{-2\alpha-1}\,\mathrm{d}r\big)$. Then*
$$\int_0^\infty r^{-2\alpha}\overline{f(r)}g(r)\,\frac{\mathrm{d}r}{r} = \int_{\mathrm{Re}\lambda=\alpha} \overline{\widehat{f}(\lambda)}\widehat{g}(\lambda)\,\mathrm{dIm}\lambda$$
and in particular
$$\int_0^\infty r^{-2\alpha}|f(r)|^2\,\frac{\mathrm{d}r}{r} = \int_{\mathrm{Re}\lambda=\alpha} |\widehat{f}(\lambda)|^2\,\mathrm{dIm}\lambda.$$

Proof. The identities follow from Plancherel's identity for the Fourier transform (2.7):
$$\begin{aligned}
\|r^{-\alpha}f(r)\|_{L^2((0,\infty),\,\frac{\mathrm{d}r}{r})} &= \|e^{-\alpha x}f(e^x)\|_{L^2(\mathbb{R},\,\mathrm{d}x)} \\
&= \|\big(\mathcal{F}e^{-\alpha\cdot}f(e^{(\cdot)})\big)(\xi)\|_{L^2(\mathbb{R},\,\mathrm{d}\xi)} \\
&= \|(\mathcal{M}f)(\alpha+i\xi)\|_{L^2(\mathbb{R},\,\mathrm{d}\xi)} \\
&= \|(\mathcal{M}f)(\lambda)\|_{L^2(\mathrm{Re}\lambda=\alpha,\,\mathrm{dIm}\lambda)}
\end{aligned}$$
and similarly for the inner products. □

Chapter 3

Setting and Main Results

In this chapter we will first derive the Stokes equations from the Navier-Stokes equations in Section 3.1 and in Section 3.2 we outline the mathematical challenges of the Stokes problem with Navier slip on the wedge. In Section 3.3 we introduce weighted Sobolev spaces and this chapter is concluded with the main results obtained in this thesis.

3.1 The Stokes Equations

The homogeneous and incompressible Navier-Stokes equations (NSE), i.e.

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu_{\mathrm{k}} \Delta \mathbf{u} + \nabla p &= \mathbf{f}_{\mathrm{b}} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \end{aligned} \tag{NSE}$$

can be reduced by the creeping flow approximation to the Stokes equations if the viscosity ν_{k} is large. We set

$$\mathbf{u} = U\mathbf{u}^*, \qquad p = \frac{\nu_{\mathrm{k}} U}{L} p^*, \qquad \mathbf{x} = L\mathbf{x}^*, \qquad t = \frac{L}{U} t^*, \tag{3.1}$$

where L and U are typical length and velocity scales, respectively. Substituting this in (NSE) gives the dimensionless equations

$$\begin{aligned} \mathrm{Re} \cdot (\partial_{t^*} + \mathbf{u}^* \cdot \nabla^*)\, \mathbf{u}^* - \Delta^* \mathbf{u}^* + \nabla^* p^* &= \mathbf{f}_{\mathrm{b}}^* && \text{in } \Omega, \\ \operatorname{div}^* \mathbf{u}^* &= 0 && \text{in } \Omega, \end{aligned}$$

where $\mathrm{Re} = \frac{UL}{\nu_{\mathrm{k}}}$ is the Reynolds number which is small since the viscosity ν_{k} is large. Thus we obtain, after dropping the superscripts, the time-independent Stokes equations

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}_{\mathrm{b}} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega. \end{aligned} \tag{3.2}$$

Using the same scaling as in (3.1) gives the rescaled boundary conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{and} \quad \mathbf{u} \cdot \tau + \partial_{\mathbf{n}}(\mathbf{u} \cdot \tau) = 0 \qquad \text{on } \partial\Omega', \tag{3.3}$$

by choosing the length scale equal to the slip length, i.e. $L = \beta$. Therefore, the stationary Stokes problem on the wedge with Navier-slip boundary condition is given by

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}_{\mathrm{b}} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega', \\ \mathbf{u} \cdot \tau + \partial_{\mathbf{n}}(\mathbf{u} \cdot \tau) &= 0 && \text{on } \partial\Omega'. \end{aligned} \tag{S-St}$$

In addition, one can also study the time-dependent Stokes flow for $t > 0$

$$\begin{aligned} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla p &= \mathbf{f}_{\mathrm{b}} && \text{in } \Omega \times [0,\infty), \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega \times [0,\infty), \\ \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega' \times [0,\infty), \\ \mathbf{u} \cdot \tau + \partial_{\mathbf{n}}(\mathbf{u} \cdot \tau) &= 0 && \text{on } \partial\Omega' \times [0,\infty), \\ \mathbf{u} &= \mathbf{u}_{\mathrm{ic}} && \text{in } \Omega \times \{0\}. \end{aligned} \tag{N-St}$$

In this thesis we will mainly consider the stationary Stokes problem (S-St). This is an elliptic problem and it is more difficult to analyse than the parabolic non-stationary Stokes problem (N-St). We will discuss the non-stationary problem shortly in Chapter 7.

Because of the wedge-shaped domain it is convenient to consider the Stokes problem in polar coordinates (see also Appendix A.2). Writing $\mathbf{u} = (u_r, u_\varphi)^\top$ and $\mathbf{f}_{\mathrm{b}} = (f_r, f_\varphi)^\top$, the Stokes problem (S-St) on the wedge becomes

$$-r^{-2}\left[\left((r\partial_r)^2 + \partial_\varphi^2\right) u_r - 2\partial_\varphi u_\varphi - u_r\right] + \partial_r p = f_r \qquad \text{for } r > 0,\ \varphi \in (0,\theta), \tag{3.4a}$$

$$-r^{-2}\left[\left((r\partial_r)^2 + \partial_\varphi^2\right) u_\varphi + 2\partial_\varphi u_r - u_\varphi\right] + r^{-1}\partial_\varphi p = f_\varphi \qquad \text{for } r > 0,\ \varphi \in (0,\theta), \tag{3.4b}$$

$$(r\partial_r + 1)u_r + \partial_\varphi u_\varphi = 0 \qquad \text{for } r > 0,\ \varphi \in (0,\theta), \tag{3.4c}$$

$$u_\varphi = 0 \qquad \text{for } r > 0,\ \varphi \in \{0,\theta\}, \tag{3.4d}$$

$$u_r + \partial_{\mathbf{n}} u_r = 0 \qquad \text{for } r > 0,\ \varphi \in \{0,\theta\}. \tag{3.4e}$$

Equations (3.4a)-(3.4c) correspond to the mass and momentum equations (3.2). By noting that u_φ is orthogonal to the boundary and $\mathbf{u} \cdot \tau = u_r$ on the boundary, we find that the boundary conditions (3.3) become (3.4d) and (3.4e). Finally, using that in polar coordinates $\mathbf{n} = (0,-1)^\top$ for $\varphi = 0$ and $\mathbf{n} = (0,1)^\top$ for $\varphi = \theta$, the last equation can be written as two equations

$$u_r \pm r^{-1}\partial_\varphi u_r = 0,$$

where the notation $\pm$ will in this thesis always mean $+$ for $\varphi = \theta$ and $-$ for $\varphi = 0$ and a similar definition holds for the notation $\mp$.

3.2 Decomposition of the Problem

There are two main issues that will cause difficulties in the analysis of the stationary Stokes problem (S-St). These two issues are that the domain has no smooth boundary and that the Navier-slip condition is not scaling invariant. We will discuss this in more detail below.

Domains with Conical Points

First of all, the boundary of the wedge-shaped domain Ω has a corner (also called conical point) at the tip $(0,0)$. This can cause the solution to behave in an irregular way in the vicinity of a corner and the elliptic regularity results known for smooth domains do not hold in general for non-smooth domains. Nonetheless, the behaviour of solutions near conical points is well-studied for both elliptic and parabolic problems, see e.g. the monographs [33, 34, 46, 48]. Because of the irregular behaviour of the solution near the corner we decompose the problem into an expansion near the tip and a regular remainder. Hence, we write

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \quad \text{and} \quad p = p_0 + p_1,$$

where $\mathbf{u}_0, p_0$ are regular at the tip of the wedge and $\mathbf{u}_1, p_1$ are the expansions at the tip. Introducing the cut-off function $\zeta = \zeta(r) \in C^\infty([0,\infty))$ with $\zeta(r) = 1$ for $r \leq \frac{1}{2}$ and $\zeta(r) = 0$ for $r \geq 1$, we have that

$$\mathbf{u}_1 = \zeta \mathcal{P}_u, \qquad \mathbf{u}_0 = \mathbf{u} - \zeta \mathcal{P}_u, \qquad p_1 = \zeta \mathcal{P}_p, \qquad p_0 = p - \zeta \mathcal{P}_p,$$

where $\mathcal{P}_u$ and $\mathcal{P}_p$ are polynomials. These polynomials can be seen as a (generalised) Taylor polynomial around the tip. The right hand side of the Stokes problem (3.2) can be decomposed as $\mathbf{f}_\mathrm{b} = \mathbf{f}_0 + \mathbf{f}_1$, where

$$\mathbf{f}_1 := \zeta \mathcal{P}_f := -\Delta \mathbf{u}_1 + \nabla p_1 \quad \text{and} \quad \mathbf{f}_0 := -\Delta \mathbf{u}_0 + \nabla p_0.$$

By linearity of the Stokes operator the problem is decomposed in two problems: one for the regular part and one for the polynomial part. Away from the tip ($r \geq 1$) the cut-off function is zero, so that

$$\begin{aligned} -\Delta \mathbf{u}_0 + \nabla p_0 &= \mathbf{f}_0, \\ \operatorname{div} \mathbf{u}_0 &= 0. \end{aligned} \tag{3.5}$$

Near the tip of the wedge, where the cut-off function is one, the following polynomial problem should be solved

$$\begin{aligned} -\Delta \mathcal{P}_u + \nabla \mathcal{P}_p &= \mathcal{P}_f, \\ \operatorname{div} \mathcal{P}_u &= 0. \end{aligned} \tag{3.6}$$

To prove (higher) regularity for the Stokes problem in a wedge, both problems (3.5) and (3.6) have to be analysed. Results are already known in many different situations. The stationary and the non-stationary (Navier-)Stokes problem in two and three dimensions with no-slip boundary condition have been studied in [10, 35, 36, 37, 46, 52]. Moreover, the Navier-Stokes equations with the free-slip boundary condition have been studied [43]. A general treatment of the (Navier-)Stokes equations in domains with corners can be found in the trilogy [33, 34, 46] where also scaling invariant slip boundary conditions are treated.
For other moving boundary problems in fluid dynamics with a non-smooth boundary there are also results known. In [30] the thin-film equation (1.8) with Navier slip is studied and in [31, 32] the Darcy flow with governing equation $\mathbf{u} = -\nabla p$ is considered.

The Navier-Slip Boundary Condition

The second issue is that the Navier-slip condition complicates the analysis of the problem. By applying the Mellin transform on the Navier-slip condition (see Section 2.5 and Lemma 2.5.3) we get

$$\widehat{u_r}(\lambda, \varphi) \pm \partial_\varphi \widehat{u_r}(\lambda + 1, \varphi) = 0,$$

which is not scaling invariant due to the shift in the argument λ to $\lambda + 1$. In the existing literature for non-smooth domains only scaling-invariant boundary conditions are considered, which makes the analysis in general easier. Even the slip condition in [46] is scaling invariant and to our knowledge scaling-variant boundary conditions have not been studied yet in the literature. For a treatment of the Navier-Stokes equations in a bounded two-dimensional domain with a C^2-boundary and Navier slip see [29] and references therein.

One of the difficulties with the Navier-slip condition in a non-smooth domain arises in solving the polynomial problem (3.6). The solution $\mathcal{P}_u$ is assumed to be a generalised Taylor polynomial around the tip of the form

$$\mathcal{P}_u(r, \varphi) = \sum_{(j,\ell) \in \mathcal{I}} \mathbf{u}^{(j,\ell)}(\varphi)\, r^j \log^\ell r,$$

where $\mathbf{u}^{(j,\ell)}(\varphi)$ are unknown coefficients and $\mathcal{I} \subset \mathbb{Z}^2$ is some appropriate index set. By inserting these kind of expansions in (3.4), we obtain an ordinary BVP of infinitely many equations in the angle φ to solve the coefficients of the Taylor expansions.
In the case of a scaling-invariant boundary condition, this BVP is uncoupled and it is possible to write down explicit solution formulas. However, with the Navier-slip condition the system of equations is coupled and it will not be possible to write down an explicit solution representation.

3.3 Choice of Spaces

Because of the non-smooth domain the setting of Sobolev spaces as defined in Section 2.3 is insufficient. This is because of Hardy's inequality (Lemma 2.4.2), i.e.

$$\int_0^\theta \int_0^\infty r^{2\delta} |\mathbf{u}|^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \leq \frac{1}{\delta^2} \int_\Omega r^{2\delta} |\partial_r \mathbf{u}|^2 \,\mathrm{d}x,$$

which is not valid for $\delta = 0$. Hence, in unweighted Sobolev spaces it is not possible to apply this inequality and there is little control of the solution $\mathbf{u}$. Therefore, we will work with weighted Sobolev spaces where $\delta \neq 0$, so that Hardy's inequality can be applied. For a general introduction to weighted Sobolev spaces see e.g. [39].

Definition 3.3.1. *Let $U \subset \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Moreover, let $\beta \in \mathbb{N}_0^n$ be a multi-index of order $k \in \mathbb{N}_0$. The weighted Sobolev space $H_{k,\alpha}(U)$ is defined as the space of all functions f such that*

$$\|f\|^2_{H_{k,\alpha}(U)} := \sum_{0 \leq |\beta| \leq k} \int_U r^{-2\alpha} |\partial^\beta f(x)|^2 \,\mathrm{d}x < \infty,$$

where $r = |x|$. If $k = 0$ we write $H_\alpha(U)$ and if $\alpha = 0$ we recover the unweighted Sobolev spaces. In particular we have that $H_{0,0}(U) = L^2(U)$.

The space $H_{k,\alpha}(U)$ is a Hilbert space with inner product

$$(f,g)_{H_{k,\alpha}(U)} = \sum_{0 \leq |\beta| \leq k} \int_U r^{-2\alpha} \partial^\beta f(x) \overline{\partial^\beta g(x)} \,\mathrm{d}x \qquad \text{for } f, g \in H_{k,\alpha}(U).$$

In the case of the wedge $\Omega := \{(r\cos\varphi, r\sin\varphi) : r > 0, \varphi \in (0,\theta)\}$ we write for $\mathbf{u} \in \big(H_{k,\alpha}(\Omega)\big)^2$ with $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$ the norms as follows

$$\begin{aligned}
\|\mathbf{u}\|^2_\alpha &:= \int_\Omega r^{-2\alpha} |\mathbf{u}|^2 \,\mathrm{d}x = \int_0^\theta \int_0^\infty r^{-2(\alpha-1)} |\mathbf{u}|^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi, \\
\|\mathbf{u}\|^2_{k,\alpha} &:= \sum_{0 \leq |\beta| \leq k} \|\partial^\beta \mathbf{u}\|^2_\alpha \sim_k \sum_{0 \leq j+\ell \leq k} \int_0^\theta \int_0^\infty r^{-2(\alpha+j+\ell-1)} \big|(r\partial_r)^j \partial_\varphi^\ell \mathbf{u}\big|^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi.
\end{aligned}$$

In addition, on the boundary of the wedge Ω we write for $u \in H_{k,\alpha}((0,\infty))$ with $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$ the norm as

$$\begin{aligned}
|u|^2_{k,\alpha} &:= \sum_{j=0}^k \int_0^\infty r^{-2\alpha} \Big(\big|\partial^j u|_{\varphi=0}\big|^2 + \big|\partial^j u|_{\varphi=\theta}\big|^2 \Big) \,\mathrm{d}r \\
&\sim_k \sum_{j=0}^k \int_0^\infty r^{-2(\alpha+j-\frac{1}{2})} \Big(\big|(r\partial_r)^j u|_{\varphi=0}\big|^2 + \big|(r\partial_r)^j u|_{\varphi=\theta}\big|^2 \Big) \,\frac{\mathrm{d}r}{r},
\end{aligned}$$

where again $|u|_\alpha := |u|_{0,\alpha}$. Moreover, these norms have the following Mellin representation (see Section 2.5)

$$\begin{aligned}\|\mathbf{u}\|_{k,\alpha}^2 &= \sum_{0\le j+\ell\le k}\int_0^\theta\int_{\mathrm{Re}\lambda=\alpha+j+\ell-1}|\lambda|^{2j}|\partial_\varphi^\ell\widehat{\mathbf{u}}(\lambda,\varphi)|^2\,\mathrm{dIm}\lambda\,\mathrm{d}\varphi,\\ |u|_{k,\alpha}^2 &= \sum_{0\le j\le k}\int_{\mathrm{Re}\lambda=\alpha+j-\frac{1}{2}}|\lambda|^{2j}\left(|\widehat{u}(\lambda,0)|^2+|\widehat{u}(\lambda,\theta)|^2\right)\mathrm{dIm}\lambda.\end{aligned}$$

Although we will not need it, this Mellin representation of the norms allows to define weighted Sobolev spaces for $k\in\mathbb{R}$.

3.4 The Main Results

Below we state the main theorem on solutions to the stationary Stokes problem with Navier slip on the wedge with opening angle θ.

Smooth Problem

First, consider the smooth problem (cf. (3.5)) and recall the definition of a strong solution.

Definition 3.4.1 ([20])**.** *Let L be a second order linear differential operator. We call u a strong solution to the differential equation $Lu = f$ on some domain U if*

1. *u is twice weakly differentiable in U,*
2. *u satisfies the equation $Lu = f$ in U almost everywhere.*

After projecting the Stokes problem (S-St) on a divergence-free space, the problem is given by

$$\begin{aligned}-\mathbb{P}\Delta\mathbf{u} &= \mathbf{f} && \text{in } \Omega,\\ u_\varphi &= 0 && \text{on } \partial\Omega',\\ u_r+\partial_{\mathbf{n}}u_r &= 0 && \text{on } \partial\Omega',\end{aligned}\tag{P-S-St}$$

where $\mathbb{P}$ is the Helmholtz projection as will be defined in Chapter 4. Define the space $\mathscr{H}_\alpha := \overline{\mathcal{T}}^{\|\cdot\|_{\mathscr{H}_\alpha}}$, where $\mathcal{T}$ is the space of test functions given by

$$\begin{aligned}\mathcal{T} := \Big\{&\mathbf{v}\in C_c^2(\overline{\Omega}\setminus\{0\}) : \operatorname{div}\mathbf{v}=0 \text{ in } \Omega,\ v_\varphi = 0 \text{ on } \partial\Omega' \text{ and}\\ &(r\partial_r)^j\partial_\varphi^\ell\mathbf{v} \text{ is locally integrable with } \|(r\partial_r)^j\partial_\varphi^\ell\mathbf{v}\|_\alpha<\infty \text{ for } j+\ell=3\Big\}\end{aligned}$$

and the norm is

$$\begin{aligned}\|\mathbf{v}\|_{\mathscr{H}_\alpha}^2 &:= |v_r|_\alpha^2+|r\partial_r v_r|_\alpha^2+\|\nabla\mathbf{v}\|_\alpha^2+\|\nabla r\partial_r\mathbf{v}\|_\alpha^2+\|r\nabla\omega_{\mathbf{v}}\|_\alpha^2\\ &\sim\sum_{j=0}^1\int_0^\infty r^{-2\alpha}\Big(\big|(r\partial_r)^j v_r|_{\varphi=0}\big|^2+\big|(r\partial_r)^j v_r|_{\varphi=\theta}\big|^2\Big)\,\mathrm{d}r\\ &\quad+\sum_{0\le j+\ell\le 2}\int_0^\theta\int_0^\infty r^{-2\alpha}\big|(r\partial_r)^j\partial_\varphi^\ell\mathbf{v}\big|^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi.\end{aligned}\tag{3.7}$$

We prove in this thesis the following main theorem on existence and uniqueness of a strong solution to (P-S-St).

Theorem. *Let $\alpha < 0$ and $\theta > 0$ with $|\alpha|$ and θ small enough and assume $\mathbf{f} \in \mathscr{H}'_\alpha$. Then there exists a unique $\mathbf{u} \in \mathscr{H}_\alpha$ that satisfies*

1. $-\mathbb{P}\Delta\mathbf{u} = \mathbf{f}$ *almost everywhere in the wedge Ω with opening angle θ,*
2. $u_r + \partial_{\mathbf{n}} u_r = 0$ *almost everywhere on the boundary $\partial\Omega'$.*

In particular, the Stokes problem with Navier-slip boundary conditions on the wedge has a strong solution in weighted Sobolev spaces. Moreover, this theorem is valid for $\theta \le \frac{2\pi^2}{2.0672\cdot 10^6}$.

Although we do not have the ambition to find the largest possible angle θ, we can nonetheless show that the theorem is valid for $\theta \le \theta_0$ with

$$\theta_0 \le \frac{2\pi^2}{2.0672 \cdot 10^6}.$$

The angle θ_0 corresponds to 0.0005 degrees, however it is expected that it is possible to optimise so that it also applies for larger angles.

Polynomial Problem

Consider the polynomial problem (3.6), i.e.

$$\begin{aligned} -\Delta\mathcal{P}_u + \nabla\mathcal{P}_p &= \mathcal{P}_f, \\ \operatorname{div}\mathcal{P}_u &= 0, \end{aligned} \tag{3.8}$$

with no-penetration (3.4d) and Navier-slip (3.4e) boundary conditions. Because of the static domain Ω we use Taylor expansions of the form

$$\begin{aligned} \mathcal{P}_u(r,\varphi) &= \sum_{j\ge 0} \mathbf{u}^{(j)}(\varphi)\, r^j, \\ \mathcal{P}_p(r,\varphi) &= \sum_{j\ge -1} p^{(j)}(\varphi)\, r^j, \\ \mathcal{P}_f(r,\varphi) &= \sum_{j\ge -2} \mathbf{f}^{(j)}(\varphi)\, r^j, \end{aligned}$$

where $\mathbf{f}^{(j)}$ are known coefficients and $\mathbf{u}^{(j)}$, $p^{(j)}$ are unknown coefficients. To solve this problem we derive for every $j \ge 0$ a boundary value problem for $\mathbf{u}^{(j)}$, $p^{(j-1)}$. Due to the Navier-slip condition the boundary value problems need to be solved in increasing order of j and this makes it difficult to find explicit solution representations for the coefficients. However, we derive the following result on unique solvability of the polynomial problem

Theorem. *Consider the polynomial problem* (3.8) *related to the Stokes problem with Navier slip. Assume that $\mathbf{f} = (f_r, f_\varphi)^\top$ has a Taylor expansion around the tip of the wedge Ω of the form*

$$\mathcal{P}_f(r,\varphi) = \sum_{\ell\ge -1} \mathbf{f}^{(\ell)}(\varphi) r^\ell$$

and satisfies the compatibility condition

$$\int_0^\theta f_r^{(-1)}(\varphi)\,\mathrm{d}\varphi = 0.$$

Furthermore, let $0 < \theta < \frac{\pi}{2}$ and assume for $j \ge 2$ that

$$\theta \ne \frac{n\pi}{j+1} \quad \text{and} \quad \theta \ne \frac{n\pi}{j-1} \quad \text{for all } n \ge 1.$$

Then there exists a unique solution up to an additive constant for the pressure to the polynomial problem (3.8)*. Furthermore, the solution to the polynomial problem has a Taylor expansion around the tip of the form*

$$\mathcal{P}_u(r,\varphi) = \sum_{j\geq 0} \mathbf{u}^{(j)}(\varphi)\, r^j \quad \text{and} \quad \mathcal{P}_p(r,\varphi) = \sum_{j\geq 0} p^{(j-1)}(\varphi)\, r^{j-1},$$

where without loss of generality $p^{(0)} = 0$ *and all other coefficients* $\mathbf{u}^{(j)}, p^{(j-1)}$ *for* $j \geq 0$ *are uniquely determined. In particular, for any fixed* θ *it is possible to solve the polynomial problem up to order* $j < \frac{\pi-\theta}{\theta}$.

Higher Regularity

With the main result of this thesis described above, it would be possible to derive higher regularity for solutions to the Stokes problem with Navier-slip on a wedge. The idea is to reduce the Navier-slip boundary condition to a Neumann boundary condition in the Stokes problem (P-S-St) and to study the following problem for $\mathbf{w}$

$$\begin{aligned} -\mathbb{P}\Delta \mathbf{w} &= \mathbf{f} && \text{in } \Omega, \\ w_\varphi &= 0 && \text{on } \partial\Omega', \\ \partial_{\mathbf{n}} w_r &= -u_r && \text{on } \partial\Omega', \end{aligned} \tag{3.10}$$

where u_r is the solution to the Stokes problem with Navier slip (P-S-St) which can now be assumed to be data. Studying the above problem with Neumann boundary conditions will be easier than proving higher regularity directly for (P-S-St), namely (3.10) is scaling invariant while the original problem (P-S-St) with Navier slip is not. The Mellin representation of the weighted norms with higher derivatives have a shift in the line of integration (see Section 3.3). Therefore, singularities can be picked up and here the polynomial problem comes into play. We leave the treatment of higher regularity to solutions of the Stokes problem on a wedge for future work.

Chapter 4

Helmholtz Projection

The Stokes problem (S-St) contains the pressure p which can only be determined up to an additive constant. To simplify the problem we will apply the so-called Helmholtz projection to eliminate the pressure. Let $\mathbf{w} \in C_c^2(\overline{\Omega} \setminus \{0\})$, then the Helmholtz projection $\mathbb{P}$ is defined, at least formally, as

$$\mathbb{P}\mathbf{w} := \mathbf{w} - \nabla\Phi,$$

where Φ satisfies the elliptic problem

$$\begin{aligned} \Delta\Phi &= \operatorname{div}\mathbf{w} && \text{in } \Omega, \\ \partial_{\mathbf{n}}\Phi &= \mathbf{n}\cdot\mathbf{w} && \text{on } \partial\Omega'. \end{aligned} \tag{4.1}$$

It should be noted that in the case of a bounded domain this definition would make perfectly sense because then the Neumann problem (4.1) has a unique solution up to an additive constant. This would imply that $\nabla\Phi$ is uniquely determined and therefore $\mathbb{P}$ is uniquely defined as well. However, the wedge Ω is unbounded and therefore (4.1) has no unique solution unless the decay of the solution is specified as $r \downarrow 0$ and $r \to \infty$.

To uniquely define the Helmholtz projection in the case of the wedge-shaped domain, we will make use of the Mellin transform. In Mellin variables it is possible to find an explicit solution representation for $\widehat{\Phi}$ satisfying (4.1). Then, we apply the inverse Mellin transform and as long as this inversion is uniquely defined, we can also uniquely define the Helmholtz projection on Ω in polar coordinates.

Moreover, we remark that for proving existence and uniqueness of solutions to the Stokes problem (S-St) in weighted Sobolev spaces, it is natural to use the Mellin transform. The norms corresponding to those weighted spaces have convenient representations in Mellin variables (see Section 3.3) and those representations we will use.

In Section 4.1 problem (4.1) is solved in Mellin variables and it is shown that we can invert this solution under certain conditions. This allows to uniquely define the Helmholtz projection and derive some properties which is done in Section 4.2. Finally, in Section 4.3 another representation of the Helmholtz projection is established which makes it possible to easily derive certain estimate that are needed in Chapters 5, 6 and 7.

4.1 Green's Function Representation

Consider the Neumann problem (4.1) and recall that in polar coordinates $\mathbf{n} = (0, \pm 1)^\top$ and $\partial_\mathbf{n} = \pm r^{-1}\partial_\varphi$. Therefore, (4.1) reads in polar coordinates

$$\begin{aligned} \big((r\partial_r)^2 + \partial_\varphi^2\big)\Phi &= r\big((r\partial_r + 1)w_r + \partial_\varphi w_\varphi\big) && \text{in } \Omega, \\ \partial_\varphi \Phi &= r w_\varphi && \text{on } \partial\Omega'. \end{aligned} \tag{4.2}$$

Then applying the Mellin transform and using its properties (see Section 2.5), we obtain problem (4.1) in Mellin variables

$$\big(\lambda^2 + \partial_\varphi^2\big)\widehat{\Phi}(\lambda,\varphi) = \lambda\widehat{w_r}(\lambda-1,\varphi) + \partial_\varphi\widehat{w_\varphi}(\lambda-1,\varphi) =: \widehat{g}(\lambda,\varphi) \qquad \text{in } \Omega, \tag{4.3a}$$

$$\partial_\varphi\widehat{\Phi}(\lambda,\varphi) = \widehat{w_\varphi}(\lambda-1,\varphi) \qquad \text{on } \partial\Omega'. \tag{4.3b}$$

Proposition 4.1.1. *The solution to problem* (4.3) *is for* $\mathrm{Re}\lambda \cdot \theta \notin \pi\mathbb{Z}$ *given by*

$$\widehat{\Phi}(\lambda,\varphi) = \frac{\widehat{w_\varphi}(\lambda-1,0)}{\lambda\sin(\lambda\theta)}\cos\big(\lambda(\theta-\varphi)\big) - \frac{\widehat{w_\varphi}(\lambda-1,\theta)}{\lambda\sin(\lambda\theta)}\cos(\lambda\varphi) + \int_0^\theta G(\varphi,\tilde{\varphi},\lambda)\widehat{g}(\lambda,\tilde{\varphi})\,\mathrm{d}\tilde{\varphi},$$

where the Green's function is

$$G(\varphi,\tilde{\varphi},\lambda) = \begin{cases} \frac{\cos(\lambda\tilde{\varphi})\cos\big(\lambda(\theta-\varphi)\big)}{\lambda\sin(\lambda\theta)} & \text{for } 0 \le \tilde{\varphi} < \varphi \le \theta, \\ \frac{\cos\big(\lambda(\theta-\tilde{\varphi})\big)\cos(\lambda\varphi)}{\lambda\sin(\lambda\theta)} & \text{for } 0 \le \varphi < \tilde{\varphi} \le \theta, \end{cases}$$

and

$$\widehat{g}(\lambda,\varphi) = \lambda\widehat{w_r}(\lambda-1,\varphi) + \partial_\varphi\widehat{w_\varphi}(\lambda-1,\varphi).$$

Proof. Consider the homogeneous equation for (4.3a), i.e.

$$(\lambda^2 + \partial_\varphi^2)\widehat{\Phi}_\mathrm{h}(\lambda,\varphi) = 0 \quad \text{in } \Omega,$$

which has $\cos(\lambda\varphi)$ and $\cos(\lambda(\theta-\varphi))$ as linearly independent solutions. Using the boundary conditions (4.3b) we find the solution

$$\widehat{\Phi}_\mathrm{h}(\lambda,\varphi) = \frac{\widehat{w_\varphi}(\lambda-1,0)}{\lambda\sin(\lambda\theta)}\cos\big(\lambda(\theta-\varphi)\big) - \frac{\widehat{w_\varphi}(\lambda-1,\theta)}{\lambda\sin(\lambda\theta)}\cos(\lambda\varphi).$$

To find a particular solution we use a Green's function (see Section 2.2.2). Define

$$\mathbf{x} := \begin{pmatrix} \widehat{\Phi} \\ \partial_\varphi\widehat{\Phi} \end{pmatrix}, \quad A := \begin{pmatrix} 0 & 1 \\ -\lambda^2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{b} := \begin{pmatrix} 0 \\ \widehat{g}(\lambda,\varphi) \end{pmatrix}.$$

The boundary conditions (4.3b) can be written as

$$R_0\mathbf{x}(0) + R_\theta\mathbf{x}(\theta) = \mathbf{c} \quad \text{with} \quad R_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; R_\theta = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{c} = \begin{pmatrix} \widehat{w_\varphi}(\lambda-1,0) \\ \widehat{w_\varphi}(\lambda-1,\theta) \end{pmatrix}.$$

Furthermore, the Wronski matrix is given by

$$W(\varphi) = \begin{pmatrix} \cos\big(\lambda(\theta-\varphi)\big) & \cos(\lambda\varphi) \\ \lambda\sin\big(\lambda(\theta-\varphi)\big) & -\lambda\sin(\lambda\varphi) \end{pmatrix}.$$

With the terminology of Proposition 2.2.5 we find

$$\boldsymbol{\gamma}(\tilde{\varphi}) = W^{-1}(\varphi)\begin{pmatrix}0\\ \frac{1}{2}\end{pmatrix} = \frac{1}{2\lambda\sin(\lambda\theta)}\begin{pmatrix}\cos(\lambda\tilde{\varphi})\\ -\cos\big(\lambda(\theta-\tilde{\varphi})\big)\end{pmatrix}$$

and

$$\begin{aligned}\boldsymbol{\beta}(\tilde{\varphi}) &= \big[R_0W(0)+R_\theta W(\theta)\big]^{-1}\big(R_0W(0)-R_\theta W(\theta)\big)\boldsymbol{\gamma}(\tilde{\varphi})\\ &= \frac{1}{2\lambda\sin(\lambda\theta)}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos(\lambda\tilde{\varphi})\\ -\cos\big(\lambda(\theta-\tilde{\varphi})\big)\end{pmatrix}\\ &= \frac{1}{2\lambda\sin(\lambda\theta)}\begin{pmatrix}\cos(\lambda\tilde{\varphi})\\ \cos\big(\lambda(\theta-\tilde{\varphi})\big)\end{pmatrix},\end{aligned}$$

from which we can determine the Green's function

$$G(\varphi,\tilde{\varphi},\lambda) = \begin{cases}\frac{\cos(\lambda\tilde{\varphi})\cos\big(\lambda(\theta-\varphi)\big)}{\lambda\sin(\lambda\theta)} & \text{for } 0\le\tilde{\varphi}<\varphi\le\theta,\\ \frac{\cos\big(\lambda(\theta-\tilde{\varphi})\big)\cos(\lambda\varphi)}{\lambda\sin(\lambda\theta)} & \text{for } 0\le\varphi<\tilde{\varphi}\le\theta.\end{cases}$$

Hence, a particular solution is given by

$$\widehat{\Phi}_{\mathrm{p}}(\lambda,\varphi) = \int_0^\theta G(\varphi,\tilde{\varphi},\lambda)\widehat{g}(\lambda,\tilde{\varphi})\,\mathrm{d}\tilde{\varphi}.$$ □

To define the Helmholtz projection in polar coordinates we want to invert $\widehat{\mathbb{P}\mathbf{w}} = \widehat{\mathbf{w}} - \widehat{\nabla\Phi}$ and therefore we need to make sure that the inverse Mellin transform of $\widehat{\nabla\Phi}$ is uniquely defined. For the inverse Mellin transform we can integrate over any vertical line in the strip of convergence to find the same solution. However, if we integrate over a line which lies outside the strip of convergence, then singularities are picked up and we obtain a different solution after applying the inverse Mellin transform that can be calculated by residue calculus. This strip in the complex plane, in which we can shift the line of integration without changing the solution in polar coordinates, will be denoted by Σ.

Definition 4.1.2. *Let $a, b \in \mathbb{R}$ and $\lambda \in \mathbb{C}$. The strip of convergence $a < \mathrm{Re}\lambda < b$ where the line of integration can be shifted without picking up any singularities will be written as*

$$\Sigma(a,b) := \big\{s\in\mathbb{C} : \mathrm{Re}(s)\in(a,b),\ \mathrm{Im}(s)\in(-\infty,\infty)\big\}.$$

Note that

$$\widehat{\nabla\Phi}(\lambda,\varphi) = \begin{pmatrix}(\lambda+1)\widehat{\Phi}(\lambda+1,\varphi)\\ \partial_\varphi\widehat{\Phi}(\lambda+1,\varphi)\end{pmatrix}$$

and from the representation for $\widehat{\Phi}$ in Proposition 4.1.1 we find with integration by parts

$$\begin{aligned}\widehat{\Phi}(\lambda,\varphi) = &\underbrace{\left(G(\varphi,\theta,\lambda)-\frac{\cos(\lambda\varphi)}{\lambda\sin(\lambda\theta)}\right)}_{=0}\widehat{w_\varphi}(\lambda,\theta) + \underbrace{\left(\frac{\cos\big(\lambda(\theta-\varphi)\big)}{\lambda\sin(\lambda\theta)} - G(\varphi,0,\lambda)\right)}_{=0}\widehat{w_\varphi}(\lambda,0)\\ &+\int_0^\theta\big(G(\varphi,\tilde{\varphi},\lambda)\lambda\widehat{w_r}(\lambda-1,\tilde{\varphi}) - \partial_{\tilde{\varphi}}G(\varphi,\tilde{\varphi},\lambda)\widehat{w_\varphi}(\lambda-1,\tilde{\varphi})\big)\,\mathrm{d}\tilde{\varphi}.\end{aligned}$$

From the Green's function $G(\varphi,\tilde{\varphi},\lambda)$ in Proposition 4.1.1 it now follows that

$$\lambda\widehat{\Phi}(\lambda,\varphi) \quad \text{and} \quad \partial_\varphi\widehat{\Phi}(\lambda,\varphi)$$

only have singularities at $\lambda\theta = k\pi$ for $k\in\mathbb{Z}\setminus\{0\}$. Hence, as required we can uniquely define $\nabla\Phi$ as the inverse Mellin transform of $\widehat{\nabla\Phi}(\lambda,\varphi)$ if one integrates over any vertical line such that $\mathrm{Re}\lambda+1$ lies within the interval $\big(-\frac{\pi}{\theta},\frac{\pi}{\theta}\big)$, i.e. we can integrate over vertical lines in $\Sigma(-\frac{\pi+\theta}{\theta},\frac{\pi-\theta}{\theta})$. Note that as $\theta\downarrow 0$ these singularities move to plus and minus infinity. Since we assume θ to be small anyway, we have that the strip $\Sigma(-\frac{\pi+\theta}{\theta},\frac{\pi-\theta}{\theta})$ is large.

4.2 Properties of the Helmholtz Projection

Definition 4.2.1 (Helmholtz projection)**.** *Let* $\mathbf{w} \in C_c^2(\overline{\Omega} \setminus \{0\})$. *We define the Helmholtz projection* $\mathbb{P}$ *by*

$$\mathbb{P}\mathbf{w} := \mathbf{w} - \nabla\Phi.$$

Here, $\nabla\Phi$ *is the inverse Mellin transform of* $\widehat{\nabla\Phi}(\lambda, \varphi)$ *where is integrated over any vertical line in the strip* $\Sigma(-\frac{\pi+\theta}{\theta}, \frac{\pi-\theta}{\theta})$, *i.e.*

$$\nabla\Phi(r, \varphi) = \frac{1}{i\sqrt{2\pi}} \int_{\mathrm{Re}\lambda=\gamma} r^{\lambda} \widehat{\nabla\Phi}(\lambda, \varphi)\, \mathrm{dIm}\lambda \qquad \text{with } \gamma \in \big(-\tfrac{\pi+\theta}{\theta}, \tfrac{\pi-\theta}{\theta}\big).$$

Furthermore, $\widehat{\Phi}(\lambda, \varphi)$ *with* $\mathrm{Re}\lambda \in \big(-\frac{\pi}{\theta}, \frac{\pi}{\theta}\big)$ *is the solution to the Mellin transform of the following elliptic problem*

$$\Delta\Phi = \operatorname{div}\mathbf{w} \qquad \text{in } \Omega, \tag{4.4a}$$

$$\partial_{\mathbf{n}}\Phi = \mathbf{n}\cdot\mathbf{w} \qquad \text{on } \partial\Omega'. \tag{4.4b}$$

Remark 4.2.2. *Throughout this thesis, if an elliptic problem as* (4.4) *is written, then the solution* Φ *should always be understood as the inverse Mellin transform of* $\widehat{\Phi}$ *where is integrated over a suitable vertical line in the strip of convergence. This will ensure that the solution is unique, at least, up to an additive constant.*

Note that by applying the Helmholtz projection we project onto a divergence free space, namely by definition

$$\operatorname{div}\mathbb{P}\mathbf{w} = \operatorname{div}\mathbf{w} - \Delta\Phi \overset{(4.4a)}{=} 0.$$

In addition, the normal component of the projection on the boundary is zero

$$\mathbf{n}\cdot\mathbb{P}\mathbf{w} = \mathbf{n}\cdot\mathbf{w} - \mathbf{n}\cdot\nabla\Phi = \mathbf{n}\cdot\mathbf{w} - \partial_{\mathbf{n}}\Phi \overset{(4.4b)}{=} 0.$$

Projection of the Stokes Problem

Projection of the stationary Stokes problem (S-St) leads to the problem that we will study in the upcoming chapters

$$-\mathbb{P}\Delta\mathbf{u} = \mathbf{f} \qquad \text{in } \Omega, \tag{P-S-St.a}$$

$$u_\varphi = 0 \qquad \text{on } \partial\Omega', \tag{P-S-St.b}$$

$$u_r + \partial_{\mathbf{n}} u_r = 0 \qquad \text{on } \partial\Omega', \tag{P-S-St.c}$$

where without loss of generality $\mathbb{P}\mathbf{f}$ is replaced by $\mathbf{f}$. The pressure vanishes from the equation because $\mathbb{P}\nabla p = \nabla p - \nabla\Phi_p$, where Φ_p satisfies

$$\begin{aligned} \Delta\Phi_p &= \operatorname{div}\nabla p = \Delta p &&\text{in } \Omega,\\ \partial_{\mathbf{n}}\Phi_p &= \mathbf{n}\cdot\nabla p = \partial_{\mathbf{n}} p &&\text{on } \partial\Omega'. \end{aligned}$$

So Φ_p and p satisfy the same Poisson problem with Neumann boundary conditions and have the same decay as $r \downarrow 0$ and $r \to \infty$. Hence, Φ_p and p are equal up to an additive constant and it follows that $\mathbb{P}\nabla p = \nabla p - \nabla\Phi = 0$.

Assuming that we have solved the problem for the velocity, we can find the pressure by solving the problem

$$\begin{aligned} \Delta p &= \operatorname{div}\mathbf{f} + \operatorname{div}\Delta\mathbf{u} = \operatorname{div}\mathbf{f} &&\text{in } \Omega,\\ \partial_{\mathbf{n}} p &= \mathbf{n}\cdot\mathbf{f} + \mathbf{n}\cdot\Delta\mathbf{u} &&\text{on } \partial\Omega'. \end{aligned}$$

Remark 4.2.3. *Note that in the projected problem* (P-S-St) *the expression* $\mathbb{P}\Delta\mathbf{u} = \Delta\mathbf{u} - \nabla\Phi$ *is well-defined due to the divergence free condition* $\operatorname{div}\mathbf{u} = 0$. *In Mellin variables* $\widehat{\Phi}$ *satisfies*

$$\Delta\widehat{\Phi} = \operatorname{div}\Delta\mathbf{u} = 0 \qquad \text{in } \Omega, \tag{4.6a}$$

$$\partial_{\mathbf{n}}\widehat{\Phi} = \mathbf{n}\cdot\Delta\mathbf{u} \qquad \text{on } \partial\Omega'. \tag{4.6b}$$

On the boundary $\partial\Omega'$ *we have that* $\mathbf{n} = (0, \pm 1)^\top$ *and*

$$\widehat{(\Delta\mathbf{u})_\varphi}(\lambda,\varphi) \overset{\text{(A.9),(P-S-St.b)}}{=} \partial_\varphi^2\widehat{u_\varphi}(\lambda+2,\varphi) + 2\partial_\varphi\widehat{u_r}(\lambda+2,\varphi).$$

However, $\partial_\varphi^2\widehat{u_\varphi}$ *is not defined on the boundary. This problem is resolved by using the divergence free condition* (3.4c) *which reads in Mellin variables* $\partial_\varphi\widehat{u_\varphi}(\lambda,\varphi) = -(\lambda+1)\widehat{u_r}(\lambda,\varphi)$. *Therefore,* (4.6) *becomes*

$$(\lambda^2 + \partial_\varphi^2)\widehat{\Phi}(\lambda,\varphi) = 0 \qquad \text{in } \Omega,$$
$$\partial_\varphi\widehat{\Phi}(\lambda,\varphi) = -\lambda\partial_\varphi\widehat{u_r}(\lambda+1,\varphi) \qquad \text{on } \partial\Omega',$$

and there is only one ∂_φ *on the boundary. By integrating into the interior of the wedge with the fundamental theorem of calculus we obtain at most two derivatives* $\partial_\varphi^2\widehat{u_r}$ *in* Ω. *Two derivatives in* Ω *are controlled by the* $\mathscr{H}_\alpha$*-norm* (3.7) *and therefore the boundary condition* (4.6b) *is defined in a negative weighted Sobolev space. This implies that the problem for* $\widehat{\Phi}$ *and in addition also the expression* $\mathbb{P}\Delta\mathbf{u}$ *are well-defined.*

Properties of the Helmholtz Projection

Below we prove some properties of $\mathbb{P}$, namely that it is indeed a projection and that it is self-adjoint with respect to $(\cdot,\cdot)_{L^2(\Omega)}$.

Lemma 4.2.4. *The Helmholtz projection* $\mathbb{P}$, *as defined in Definition 4.2.1, is a projection and satisfies for* $\mathbf{w}_1, \mathbf{w}_2 \in C_c^2(\overline{\Omega}\setminus\{0\})$

$$(\mathbb{P}\mathbf{w}_1, \mathbf{w}_2)_{L^2(\Omega)} = (\mathbf{w}_1, \mathbb{P}\mathbf{w}_2)_{L^2(\Omega)}.$$

Proof. Note that

$$\mathbb{P}^2\mathbf{w}_1 = \mathbf{w}_1 - \nabla\Phi_1 - \nabla\Psi,$$

where Φ_1 and Ψ satisfy

$$\begin{cases}\Delta\Phi_1 = \operatorname{div}\mathbf{w}_1 & \text{in } \Omega,\\ \partial_{\mathbf{n}}\Phi_1 = \mathbf{n}\cdot\mathbf{w}_1 & \text{on } \partial\Omega',\end{cases} \qquad \begin{cases}\Delta\Psi = \operatorname{div}\mathbb{P}\mathbf{w}_1 = 0 & \text{in } \Omega,\\ \partial_{\mathbf{n}}\Psi = \mathbf{n}\cdot\mathbb{P}\mathbf{w}_1 = 0 & \text{on } \partial\Omega'.\end{cases}$$

Hence, Ψ satisfies the Laplace equation with homogeneous Neumann boundary conditions which has only the trivial solution up to an additive constant and therefore

$$\mathbb{P}^2\mathbf{w}_1 = \mathbf{w}_1 - \nabla\Phi_1 - \nabla\Psi = \mathbf{w}_1 - \nabla\Phi_1 = \mathbb{P}\mathbf{w}_1,$$

implying that $\mathbb{P}$ is indeed a projection. For the self-adjointness note that

$$(\mathbb{P}\mathbf{w}_1, \mathbf{w}_2)_{L^2(\Omega)} = (\mathbf{w}_1 - \nabla\Phi_1, \mathbf{w}_2)_{L^2(\Omega)},$$

where Φ_1 satisfies the same BVP as above and let $\mathbb{P}\mathbf{w}_2 = \mathbf{w}_2 - \nabla\Phi_2$, where Φ_2 satisfies

$$\Delta\Phi_2 = \operatorname{div}\mathbf{w}_2 \qquad \text{in } \Omega,$$
$$\partial_{\mathbf{n}}\Phi_2 = \mathbf{n}\cdot\mathbf{w}_2 \qquad \text{on } \partial\Omega'.$$

Then by using the problem for Φ_2 we have that

$$\begin{aligned}(\nabla\Phi_1, \mathbf{w}_2)_{L^2(\Omega)} &= \int_\Omega \nabla\Phi_1 \cdot \mathbf{w}_2 \,\mathrm{d}x \\ &= \int_\Omega \operatorname{div}(\Phi_1 \mathbf{w}_2)\,\mathrm{d}x - \int_\Omega \Phi_1 \operatorname{div}\mathbf{w}_2\,\mathrm{d}x \\ &= \int_{\partial\Omega'} \Phi_1 \mathbf{w}_2 \cdot \mathbf{n}\,\mathrm{d}s - \int_\Omega \Phi_1 \Delta\Phi_2\,\mathrm{d}x \\ &= \int_\Omega \operatorname{div}(\Phi_1\nabla\Phi_2)\,\mathrm{d}x - \int_\Omega \operatorname{div}(\Phi_1\nabla\Phi_2)\,\mathrm{d}x + \int_\Omega \nabla\Phi_1\cdot\nabla\Phi_2\,\mathrm{d}x \\ &= \int_\Omega \nabla\Phi_1\cdot\nabla\Phi_2\,\mathrm{d}x\end{aligned}$$

and hence by symmetry we obtain $(\nabla\Phi_1, \mathbf{w}_2)_{L^2(\Omega)} = (\mathbf{w}_1, \nabla\Phi_2)_{L^2(\Omega)}$. From this it follows that

$$(\mathbb{P}\mathbf{w}_1, \mathbf{w}_2)_{L^2(\Omega)} = (\mathbf{w}_1 - \nabla\Phi_1, \mathbf{w}_2)_{L^2(\Omega)} = (\mathbf{w}_1, \mathbf{w}_2 - \nabla\Phi_2)_{L^2(\Omega)} = (\mathbf{w}_1, \mathbb{P}\mathbf{w}_2)_{L^2(\Omega)}. \qquad \square$$

For later reference we also prove that the Helmholtz projection commutes with the $r\partial_r$ derivative in the case of a divergence free function which does not necessarily have a homogeneous boundary condition (cf. (4.4b)). This is for example the case for $\Delta\mathbf{u}$, since $\operatorname{div}\Delta\mathbf{u} = 0$ ($\mathbf{u}$ is divergence free), but $\mathbf{n}\cdot\Delta\mathbf{u} \neq 0$ in general.

Lemma 4.2.5. *Assume that the vector field* $\mathbf{w} \in C^2_\mathrm{c}(\overline{\Omega}\setminus\{0\})$ *satisfies* $\operatorname{div}\mathbf{w} = 0$ *in* Ω. *Then*

$$\mathbb{P}r\partial_r\mathbf{w} = r\partial_r\mathbb{P}\mathbf{w}.$$

Proof. Note that

$$r\partial_r\mathbb{P}\mathbf{w} = r\partial_r\mathbf{w} - \nabla(r\partial_r - 1)\Psi \quad \text{and} \quad \mathbb{P}(r\partial_r\mathbf{w}) = r\partial_r\mathbf{w} - \nabla\Phi,$$

where

$$\begin{cases}\Delta\Psi = \operatorname{div}\mathbf{w} = 0 & \text{in } \Omega,\\ \partial_\mathbf{n}\Psi = \mathbf{n}\cdot\mathbf{w} & \text{on } \partial\Omega',\end{cases} \quad \text{and} \quad \begin{cases}\Delta\Phi = \operatorname{div}(r\partial_r\mathbf{w}) = (r\partial_r + 1)\operatorname{div}\mathbf{w} = 0 & \text{in } \Omega,\\ \partial_\mathbf{n}\Phi = \mathbf{n}\cdot r\partial_r\mathbf{w} & \text{on } \partial\Omega'.\end{cases}$$

Both problems have a unique solution up to an additive constant by Definition 4.2.1. It is straightforward to check that $(r\partial_r - 1)\Psi$ satisfies the problem for Φ and therefore $(r\partial_r - 1)\Psi$ and Φ are equal up to an additive constant which proves the lemma. $\square$

4.3 Estimates on the Helmholtz Projection

In the next chapters we will look for weak solutions and therefore we need (coercivity) estimates. However, it is easier to find estimates with a Fourier expansion than with the Green's function that we derived above in Proposition 4.1.1.

4.3.1 Fourier Series Representation

We derive a Fourier expansion for $\widehat{\Phi}$ in the angle φ in a special case which is needed for finding weak solutions in weighted spaces. If the projected Stokes problem (P-S-St) is tested against a test function $\mathbf{v} \in C^2_\mathrm{c}(\overline{\Omega}\setminus\{0\})$ satisfying $\operatorname{div}\mathbf{v} = 0$ in Ω and $v_\varphi = 0$ on $\partial\Omega'$ in the weighted inner product $(\cdot,\cdot)_\alpha$ with $\alpha \neq 0$, then

$$(-\mathbb{P}\Delta\mathbf{u}, \mathbf{v})_\alpha = (-\mathbb{P}\Delta\mathbf{u}, r^{-2\alpha}\mathbf{v})_{L^2(\Omega)} = (-\Delta\mathbf{u}, \mathbb{P}r^{-2\alpha}\mathbf{v})_{L^2(\Omega)},$$

since the Helmholtz projection is only self-adjoint in the unweighted case (Lemma 4.2.4). Therefore, we want to find the Fourier expansion of $\widehat{\Phi}$ where Φ satisfies

$$\begin{aligned} \Delta\Phi &= \operatorname{div} r^{-2\alpha}\mathbf{v} = -2\alpha r^{-2\alpha-1} v_r && \text{in } \Omega, \\ \partial_{\mathbf{n}}\Phi &= \mathbf{n}\cdot r^{-2\alpha}\mathbf{v} = 0 && \text{on } \partial\Omega', \end{aligned}$$

which can be written more conveniently in polar coordinates as

$$\left((r\partial_r)^2 + \partial_\varphi^2\right)\Phi = -2\alpha r^{-2\alpha+1} v_r =: g \qquad \text{in } \Omega, \tag{4.7a}$$

$$\partial_\varphi \Phi = 0 \qquad \text{on } \partial\Omega'. \tag{4.7b}$$

Before deriving the Fourier expansion, we introduce two orthonormal systems.

Definition 4.3.1. *For $k \in \mathbb{N}_0$ define the orthonormal system of cosines and sines*

$$\mathbf{e}_k(\varphi) := \begin{cases} \frac{1}{\sqrt{\theta}} & k = 0, \\ \sqrt{\frac{2}{\theta}}\cos\left(\frac{k\pi\varphi}{\theta}\right) & k \geq 1, \end{cases} \quad \text{and} \quad \tilde{\mathbf{e}}_k(\varphi) := \sqrt{\tfrac{2}{\theta}}\sin\left(\tfrac{k\pi\varphi}{\theta}\right) \quad k \geq 1,$$

which satisfy

$$\int_0^\theta \mathbf{e}_k(\varphi)\mathbf{e}_\ell(\varphi)\,\mathrm{d}\varphi = \delta_{k\ell} \quad \text{and} \quad \int_0^\theta \tilde{\mathbf{e}}_k(\varphi)\tilde{\mathbf{e}}_\ell(\varphi)\,\mathrm{d}\varphi = \delta_{k\ell},$$

where δ_{kl} is the Kronecker delta.

Note that by using this notation the Mellin transform $\widehat{g}(\lambda,\varphi)$ of the right hand side of (4.7a) can be extended as an even function on $(-\theta,\theta)$ and therefore admits a Fourier expansion in the angle φ of the form

$$\widehat{g}(\lambda,\varphi) = \sum_{k=0}^\infty \widehat{g}_k(\lambda)\mathbf{e}_k(\varphi),$$

where the Fourier coefficients are given by

$$\widehat{g}_k(\lambda) = -2\alpha\int_0^\theta \widehat{v}_r(\lambda+2\alpha-1,\tilde{\varphi})\mathbf{e}_k(\tilde{\varphi})\,\mathrm{d}\tilde{\varphi} = -2\alpha\widehat{v}_{rk}(\lambda+2\alpha-1), \tag{4.8}$$

where $\widehat{v}_{rk}$ is the Fourier coefficient

$$\widehat{v}_{rk}(\lambda) = \int_0^\theta \widehat{v}_r(\lambda,\tilde{\varphi})\mathbf{e}_k(\tilde{\varphi})\,\mathrm{d}\tilde{\varphi}$$

and

$$\widehat{v}_r(\lambda,\varphi) = \sum_{k=0}^\infty \widehat{v}_{rk}(\lambda)\mathbf{e}_k(\varphi) \quad \text{in } L^2(0,\theta).$$

In addition, we remark that Bessel's identity (also called Parseval's identity) holds

$$\sum_{k=1}^\infty |\widehat{v}_{rk}(\lambda)|^2 = \int_0^\theta |\widehat{v}_r(\lambda,\varphi)|^2\,\mathrm{d}\varphi. \tag{4.9}$$

Lemma 4.3.2. *The Mellin transform of the solution Φ of problem (4.7) can be written as*

$$\widehat{\Phi}(\lambda,\varphi) = -2\alpha\sum_{k=1}^\infty \frac{\widehat{v}_{rk}(\lambda+2\alpha-1)}{\lambda^2-\left(\frac{k\pi}{\theta}\right)^2}\mathbf{e}_k(\varphi) \quad \text{for } \varphi\in[0,\theta],$$

and its derivative is given by

$$\partial_\varphi\widehat{\Phi}(\lambda,\varphi) = 2\alpha\sum_{k=1}^\infty \frac{k\pi}{\theta}\cdot\frac{\widehat{v}_{rk}(\lambda+2\alpha-1)}{\lambda^2-\left(\frac{k\pi}{\theta}\right)^2}\tilde{\mathbf{e}}_k(\varphi) \quad \text{for } \varphi\in[0,\theta],$$

with $\mathrm{Re}\lambda \in \left(-\frac{\pi}{\theta},\frac{\pi}{\theta}\right)$ *and* $\lambda \neq -2\alpha$.

Proof. Taking the Mellin transform of (4.7) gives

$$(\lambda^2 + \partial_\varphi^2)\widehat{\Phi}(\lambda,\varphi) = \widehat{g}(\lambda,\varphi) \qquad \text{in } \Omega, \tag{4.10a}$$

$$\partial_\varphi \widehat{\Phi}(\lambda,\varphi) = 0 \qquad \text{on } \partial\Omega', \tag{4.10b}$$

which is a non-homogeneous second order ordinary differential equation (ODE) with homogeneous Neumann boundary conditions and has a series solution of the form

$$\widehat{\Phi}(\lambda,\varphi) = \sum_{k=0}^{\infty} \widehat{\Phi}_k(\lambda)\mathbf{e}_k(\varphi).$$

Inserting this into Equation (4.10a) and using the orthogonality of the cosines gives that the coefficients satisfy

$$\left(\lambda^2 - \left(\tfrac{k\pi}{\theta}\right)^2\right)\widehat{\Phi}_k(\lambda) = \widehat{g}_k(\lambda,\varphi).$$

By Equation (4.8) this leads to the following series representation

$$\widehat{\Phi}(\lambda,\varphi) = -2\alpha \sum_{k=0}^{\infty} \frac{\widehat{v_r}_k(\lambda + 2\alpha - 1)}{\lambda^2 - \left(\frac{k\pi}{\theta}\right)^2}\mathbf{e}_k(\varphi).$$

The condition $\operatorname{div}\mathbf{v} = 0$ reads in Mellin variables $(\lambda+1)\widehat{v_r}(\lambda,\varphi) + \partial_\varphi\widehat{v_\varphi}(\lambda,\varphi) = 0$ and integrating this over the angle gives

$$(\lambda+1)\int_0^\theta \widehat{v_r}(\lambda,\varphi)\,\mathrm{d}\varphi = -\int_0^\theta \partial_\varphi\widehat{v_\varphi}(\lambda,\varphi)\,\mathrm{d}\varphi = \widehat{v_\varphi}(\lambda,0) - \widehat{v_\varphi}(\lambda,\theta) = 0,$$

by the boundary condition $v_\varphi = 0$ on $\partial\Omega'$. Therefore,

$$\widehat{v_r}_0(\lambda) = \frac{1}{\sqrt{\theta}}\int_0^\theta \widehat{v_r}(\lambda,\varphi)\,\mathrm{d}\varphi = 0 \qquad \text{for } \lambda \neq -1$$

and the Fourier expansion reduces for $\mathrm{Re}\lambda \cdot \theta \neq k\pi$ with $k \in \mathbb{N}$ to

$$\widehat{\Phi}(\lambda,\varphi) = -2\alpha \sum_{k=1}^{\infty} \frac{\widehat{v_r}_k(\lambda + 2\alpha - 1)}{\lambda^2 - \left(\frac{k\pi}{\theta}\right)^2}\mathbf{e}_k(\varphi) \qquad \text{for } \lambda \neq -2\alpha.$$

In view of Definition 4.2.1 we consider this solution only for $\mathrm{Re}\lambda \in \left(-\frac{\pi}{\theta}, \frac{\pi}{\theta}\right)$. For $\lambda = -2\alpha$ the zeroth Fourier coefficient does not necessarily vanish. However, we will always consider $\widehat{\Phi}$ in some norm, so that a contribution on a measure zero set does not matter. Note that

$$\sum_{k=1}^{\infty} |k^2\widehat{\Phi}_k(\lambda)|^2 < \infty$$

and therefore the series converges in $H^2(0,\theta)$ which embeds into $C^1([0,\theta])$. Therefore, the series converges pointwise and the series can be differentiated to obtain

$$\partial_\varphi\widehat{\Phi}(\lambda,\varphi) = 2\alpha \sum_{k=1}^{\infty} \frac{k\pi}{\theta} \cdot \frac{\widehat{v_r}_k(\lambda + 2\alpha - 1)}{\lambda^2 - \left(\frac{k\pi}{\theta}\right)^2}\tilde{\mathbf{e}}_k(\varphi),$$

which converges in $H^1(0,\theta)$ and therefore $\partial_\varphi\widehat{\Phi}$ also converges pointwise. $\square$

4.3.2 Estimates on the Helmholtz Projection

Below in Lemma 4.3.3 and 4.3.4 we derive certain estimates on the Helmholtz projection which are needed to prove (coercivity) estimates for the elliptic and parabolic Stokes problem.
To obtain coercivity in the elliptic case in Chapter 6, we will need that $\theta > 0$ and $\alpha < 0$ with θ and $|\alpha|$ small. Therefore, in Lemma 4.3.3 it is already assumed that $-\frac{1}{4} < \alpha < 0$. For the parabolic case a larger range of α should be considered to obtain coercivity.

Lemma 4.3.3. *Let Φ satisfy problem (4.7) and assume that $-\frac{1}{4} < \alpha < 0$ and $\theta < \frac{\pi}{2}$. Then for $\ell \in \{0,1\}$ we have the estimates*

$$\int_0^\theta \int_0^\infty r^{2\alpha}(\partial_r \partial_\varphi^\ell \Phi)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \lesssim \theta^{4-2\ell}\|\nabla \mathbf{v}\|_\alpha^2, \tag{4.11}$$

$$\int_0^\theta \int_0^\infty r^{2\alpha}(r\partial_r^2 \partial_\varphi^\ell \Phi)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \lesssim \alpha^2\theta^{2-2\ell}\|\nabla \mathbf{v}\|_\alpha^2, \tag{4.12}$$

$$\int_0^\theta \int_0^\infty r^{2\alpha}(r^{-1}\partial_\varphi^{\ell+1} \Phi)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \lesssim \theta^{2-2\ell}\|\nabla \mathbf{v}\|_\alpha^2. \tag{4.13}$$

Proof. We first prove some preliminary estimates. Note that for $-\frac{1}{4} < \alpha < 0$ and $\theta < \frac{\pi}{2}$ we have

$$(1-\alpha)^2 \le \frac{25}{16} < 2 \le \frac{1}{2}\frac{\pi^2}{\theta^2}.$$

Then, for $k \ge 1$ and $t \in \mathbb{R}$ we have on the one hand

$$t^2 - (1-\alpha)^2 + \left(\frac{k\pi}{\theta}\right)^2 \ge t^2 - (1-\alpha)^2 + \frac{\pi^2}{\theta^2} \ge t^2 + \frac{1}{2}\frac{\pi^2}{\theta^2} \tag{4.14}$$

and on the other hand

$$t^2 - (1-\alpha)^2 + \left(\frac{k\pi}{\theta}\right)^2 \ge -(1-\alpha)^2 + \left(\frac{k\pi}{\theta}\right)^2 \ge \frac{1}{2}\left(\frac{k\pi}{\theta}\right)^2. \tag{4.15}$$

For $\lambda = it$ with $t \in \mathbb{R}$ we get the estimates

$$\frac{1}{\left|(\lambda-\alpha+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right|^2} \le \frac{1}{\left(t^2 - (1-\alpha)^2 + \left(\frac{k\pi}{\theta}\right)^2\right)^2} \overset{(4.15)}{\le} 4\left(\frac{\theta}{k\pi}\right)^4 \tag{4.16}$$

and

$$\begin{aligned}\frac{1}{\left|(\lambda-\alpha+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right|^2} &\le \frac{1}{\left(t^2 - (1-\alpha)^2 + \left(\frac{k\pi}{\theta}\right)^2\right)^2} \overset{(4.14),(4.15)}{\le} \frac{2}{\left(t^2 + \frac{1}{2}\frac{\pi^2}{\theta^2}\right)\left(\frac{k\pi}{\theta}\right)^2} \\ &\le \frac{2}{\left(t^2 + (1-\alpha)^2\right)\left(\frac{k\pi}{\theta}\right)^2} = \frac{2}{|\lambda-\alpha+1|^2\left(\frac{k\pi}{\theta}\right)^2}.\end{aligned} \tag{4.17}$$

Estimates (4.11)-(4.13) follow by transforming to Mellin variables, inserting the Fourier representation of $\widehat{\Phi}$ from Lemma 4.3.2, and then using one of the estimates (4.16) or (4.17). Note that by Lemma 4.3.2 we are only allowed to integrate over vertical lines such that $\mathrm{Re}\lambda \in \left(-\frac{\pi}{\theta}, \frac{\pi}{\theta}\right)$ and for $\theta < \frac{\pi}{2}$ we at least have $\mathrm{Re}\lambda \in (-2,2)$. As is clear from (4.18) below, the line of integration is $\mathrm{Re}\lambda = 1-\alpha$ and for the considered values of α it holds that $1-\alpha \in (-2,2)$.

Consider (4.11) with $\ell = 0$. Note that $|\lambda - \alpha + 1|^2 \lesssim \alpha^{-2}|\lambda + \alpha|^2$, so we get

$$\begin{aligned}\int_0^\theta \int_0^\infty r^{2\alpha}(\partial_r \Phi)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &= \int_0^\theta \int_{\mathrm{Re}\lambda=1-\alpha} |\lambda|^2 |\widehat{\Phi}(\lambda,\varphi)|^2 \,\mathrm{dIm}\lambda\,\mathrm{d}\varphi \\ &= \int_0^\theta \int_{\mathrm{Re}\lambda=0} |\lambda-\alpha+1|^2 |\widehat{\Phi}(\lambda-\alpha+1,\varphi)|^2 \,\mathrm{dIm}\lambda\,\mathrm{d}\varphi \\ &\le 4\alpha^2 \int_{\mathrm{Re}\lambda=0} |\lambda-\alpha+1|^2 \sum_{k=1}^\infty \frac{|\widehat{v}_{rk}(\lambda+\alpha)|^2}{\left|(\lambda-\alpha+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right|^2} \,\mathrm{dIm}\lambda \\ &\overset{(4.16)}{\lesssim} \theta^4 \int_{\mathrm{Re}\lambda=\alpha} |\lambda|^2 \sum_{k=1}^\infty |\widehat{v}_{rk}(\lambda)|^2 \,\mathrm{dIm}\lambda \\ &\overset{(4.9)}{=} \theta^4 \int_0^\theta \int_{\mathrm{Re}\lambda=\alpha} |\lambda|^2 |\widehat{v}_r(\lambda,\varphi)|^2 \,\mathrm{dIm}\lambda\,\mathrm{d}\varphi \le \theta^4 \|\nabla \mathbf{v}\|_\alpha^2.\end{aligned} \tag{4.18}$$

If in (4.11) we have $\ell = 1$, then there is a φ derivative which produces an extra $\frac{k\pi}{\theta}$ in the numerator of the Fourier expansion, see Lemma 4.3.2. Therefore, we get with a similar estimate as above that

$$\begin{aligned}\int_0^\theta \int_0^\infty r^{2\alpha}|\partial_r\partial_\varphi \Phi(r,\varphi)|^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &\le 4\alpha^2 \int_{\mathrm{Re}\lambda=0} |\lambda-\alpha+1|^2 \sum_{k=1}^\infty \frac{\left(\frac{k\pi}{\theta}\right)^2 |\widehat{v}_{rk}(\lambda+\alpha)|^2}{\left|(\lambda-\alpha+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right|^2} \,\mathrm{dIm}\lambda \\ &\overset{(4.16)}{\lesssim} \theta^2 \int_{\mathrm{Re}\lambda=\alpha} |\lambda|^2 \sum_{k=1}^\infty |\widehat{v}_{rk}(\lambda)|^2 \,\mathrm{dIm}\lambda \le \theta^2 \|\nabla \mathbf{v}\|_\alpha^2,\end{aligned}$$

which finishes the proof of (4.11). For (4.12) with $\ell \in \{0,1\}$ we get

$$\begin{aligned}\int_0^\theta \int_0^\infty r^{2\alpha}(r\partial_r^2\partial_\varphi^\ell \Phi)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &= \int_0^\theta \int_{\mathrm{Re}\lambda=0} |\lambda-\alpha|^2 |\lambda-\alpha+1|^2 |\partial_\varphi^\ell \widehat{\Phi}(\lambda-\alpha+1,\varphi)|^2 \,\mathrm{dIm}\lambda\,\mathrm{d}\varphi \\ &\le 4\alpha^2 \int_{\mathrm{Re}\lambda=0} |\lambda-\alpha|^2|\lambda-\alpha+1|^2 \sum_{k=1}^\infty \frac{\left(\frac{k\pi}{\theta}\right)^{2\ell} |\widehat{v}_{rk}(\lambda+\alpha)|^2}{\left|(\lambda-\alpha+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right|^2} \,\mathrm{dIm}\lambda \\ &\overset{(4.17)}{\lesssim} \alpha^2\theta^{2-2\ell} \int_{\mathrm{Re}\lambda=0} |\lambda+\alpha|^2 \sum_{k=1}^\infty |\widehat{v}_{rk}(\lambda+\alpha)|^2 \,\mathrm{dIm}\lambda \\ &\overset{(4.9)}{=} \alpha^2\theta^{2-2\ell} \int_0^\theta \int_{\mathrm{Re}\lambda=\alpha} |\lambda|^2 |\widehat{v}_r(\lambda,\varphi)|^2 \,\mathrm{dIm}\lambda\,\mathrm{d}\varphi \le \alpha^2\theta^{2-2\ell}\|\nabla \mathbf{v}\|_\alpha^2.\end{aligned}$$

Finally, we have similarly as above for $\ell \in \{0,1\}$

$$\begin{aligned}\int_0^\theta \int_0^\infty r^{2\alpha}(r^{-1}\partial_\varphi^{\ell+1} \Phi)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &\le 4\alpha^2 \int_{\mathrm{Re}\lambda=0} \sum_{k=1}^\infty \frac{\left(\frac{k\pi}{\theta}\right)^{2\ell} |\widehat{v}_{rk}(\lambda+\alpha)|^2}{\left|(\lambda-\alpha+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right|^2} \,\mathrm{dIm}\lambda \\ &\overset{(4.16)}{\lesssim} \alpha^2\theta^{2-2\ell} \int_0^\theta \int_0^\infty r^{-2\alpha} v_r^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\end{aligned} \tag{4.19}$$

and the desired estimate (4.13) follows by applying Hardy's inequality (Lemma 2.4.2). □

Lemma 4.3.4. *Let Φ satisfy problem (4.7) and let $\alpha \neq 0$. Then there exists a $\theta > 0$ small enough such that*

$$\|r^{2\alpha}(\nabla \otimes \nabla \Phi)\|_\alpha^2 \lesssim_{\alpha,\theta} \|r^{-1}v_r\|_\alpha^2.$$

Proof. First, note that by (A.10)

$$\nabla \otimes \nabla \Phi = \begin{pmatrix} \partial_r^2 \Phi & r^{-1}\partial_\varphi \partial_r \Phi - r^{-2}\partial_\varphi \Phi \\ r^{-1}\partial_\varphi \partial_r \Phi - r^{-2}\partial_\varphi \Phi & r^{-2}\partial_\varphi^2 \Phi + r^{-1}\partial_r \Phi \end{pmatrix}. \tag{4.20}$$

Hence, it suffices to prove for $j, \ell \in \{0, 1\}$ (excluding $j = \ell = 1$) the estimates

$$\int_0^\theta \int_0^\infty r^{2\alpha}(r^j \partial_r^{j+1} \partial_\varphi^\ell \Phi)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \lesssim_{\alpha,\theta} \|r^{-1}v_r\|_\alpha^2, \tag{4.21}$$

$$\int_0^\theta \int_0^\infty r^{2\alpha}(r^{-1} \partial_\varphi^{\ell+1} \Phi)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \lesssim_{\alpha,\theta} \|r^{-1}v_r\|_\alpha^2. \tag{4.22}$$

For any $\alpha \neq 0$, there exists an $n \geq 2$ such that $(1-\alpha)^2 \leq \frac{1}{2}n^2$. Then, if $\theta < \frac{\pi}{n}$ we obtain

$$(1-\alpha)^2 - \frac{\pi^2}{\theta^2} \leq \frac{\pi^2\big((1-\alpha)^2 - n^2\big)}{n^2\theta^2} \leq -\frac{1}{2}\frac{\pi^2}{\theta^2} \implies (1-\alpha)^2 \leq \frac{1}{2}\frac{\pi^2}{\theta^2}.$$

Therefore, we can reuse the preliminary estimates (4.16) and (4.17) from Lemma 4.3.3 and in addition we have for $k \geq 1$ and $\lambda = it$ with $t \in \mathbb{R}$

$$\begin{aligned} \frac{1}{\left|(\lambda - \alpha + 1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right|^2} &\leq \frac{1}{\left(t^2 - (1-\alpha)^2 + \left(\frac{k\pi}{\theta}\right)^2\right)^2} \overset{(4.15)}{\leq} \frac{1}{\left(t^2 + \frac{1}{2}\frac{\pi^2}{\theta^2}\right)^2} \\ &\leq \frac{1}{\left(t^2 + (1-\alpha)^2\right)^2} = \frac{1}{|\lambda - \alpha + 1|^4}. \end{aligned} \tag{4.23}$$

Note that (4.22) follows immediately from (4.19). The estimates (4.21) can be proved with the same strategy as in the proof of Lemma 4.3.3. Furthermore, note that with the given conditions on α and θ it is still possible to integrate over the line $\mathrm{Re}\lambda = 1 - \alpha$ as required for using Lemma 4.3.2. We obtain the following estimates which may depend on α and θ

$$\begin{aligned} \int_0^\theta \int_0^\infty r^{2\alpha}(r\partial_r^2 \Phi)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &\lesssim \int_{\mathrm{Re}\lambda=0} |\lambda - \alpha + 1|^4 \sum_{k=1}^\infty \frac{|\widehat{v}_{rk}(\lambda+\alpha)|^2}{\left|(\lambda - \alpha + 1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right|^2} \,\mathrm{dIm}\lambda \\ &\overset{(4.23)}{\lesssim} \int_{\mathrm{Re}\lambda=\alpha} \sum_{k=1}^\infty |\widehat{v}_{rk}(\lambda+\alpha)|^2 \,\mathrm{dIm}\lambda \\ &= \int_0^\theta \int_0^\infty r^{-2\alpha} v_r^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi = \|r^{-1}v_r\|_\alpha^2 \end{aligned}$$

and for $\ell \in \{0, 1\}$

$$\begin{aligned} \int_0^\theta \int_0^\infty r^{2\alpha}|\partial_r \partial_\varphi^\ell \Phi(r,\varphi)|^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &\lesssim \int_{\mathrm{Re}\lambda=0} |\lambda - \alpha + 1|^2 \sum_{k=1}^\infty \frac{\left(\frac{k\pi}{\theta}\right)^{2\ell} |\widehat{v}_{rk}(\lambda+\alpha)|^2}{\left|(\lambda - \alpha + 1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right|^2} \,\mathrm{dIm}\lambda \\ &\overset{(4.17)}{\lesssim} \int_{\mathrm{Re}\lambda=\alpha} \sum_{k=1}^\infty |\widehat{v}_{rk}(\lambda)|^2 \,\mathrm{dIm}\lambda = \|r^{-1}v_r\|_\alpha^2, \end{aligned}$$

which proves (4.21) and this finishes the proof. □

Chapter 5

Weak and Strong Solutions

In this chapter we search for solutions of the projected stationary Stokes problem (P-S-St) given by

$$-\mathbb{P}\Delta \mathbf{u} = \mathbf{f} \qquad \text{in } \Omega, \tag{P-S-St.a}$$
$$u_\varphi = 0 \qquad \text{on } \partial\Omega', \tag{P-S-St.b}$$
$$u_r + \partial_{\mathbf{n}} u_r = 0 \qquad \text{on } \partial\Omega', \tag{P-S-St.c}$$

where Ω is the wedge-shaped domain with opening angle θ and $\partial\Omega' = \partial\Omega \setminus \{(0,0)\}$ on which we can define the outward pointing normal vector $\mathbf{n}$.

In Section 5.1 weak solutions in unweighted Sobolev spaces are discussed which are straightforward to establish. However, weak solutions in unweighted spaces are not sufficient for proving higher regularity. Therefore, the rest of this chapter is devoted to finding solutions in weighted Sobolev spaces. The obtained result is based on a coercivity estimate which is proved in the next chapter.

In any case we need suitable test functions to test the equation (P-S-St.a) with. Therefore, we define the space of test functions given by

$$\begin{aligned}\mathcal{T} := \Big\{&\mathbf{v} \in C_c^2(\overline{\Omega} \setminus \{0\}) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ v_\varphi = 0 \text{ on } \partial\Omega' \text{ and} \\ &(r\partial_r)^j \partial_\varphi^\ell \mathbf{v} \text{ is locally integrable with } \|(r\partial_r)^j \partial_\varphi^\ell \mathbf{v}\|_\alpha < \infty \text{ for } j + \ell = 3\Big\}.\end{aligned} \tag{5.2}$$

The condition that the third order derivatives should exist as distributional derivatives and are finite in the α-norm is required for applying integration by parts, the necessity of this condition is discussed in the proof of Theorem 5.4.10. Note that the Navier-slip boundary condition is a natural boundary condition and is thus not included in the definition of the space $\mathcal{T}$.

5.1 Weak Solutions in Unweighted Spaces

The Stokes problem (P-S-St) tested with a test function $\mathbf{v} \in \mathcal{T}$ in the unweighted $(\cdot,\cdot)_{L^2(\Omega)}$ inner product becomes

$$(-\mathbb{P}\Delta \mathbf{u}, \mathbf{v})_{L^2(\Omega)} = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)}.$$

By using Lemma 4.2.4, the notation from Appendix A.1, and the Navier-slip condition (P-S-St.c), we obtain

$$\begin{aligned}(-\mathbb{P}\Delta\mathbf{u},\mathbf{v})_{L^2(\Omega)} &= (-\Delta\mathbf{u},\mathbb{P}\mathbf{v})_{L^2(\Omega)} = (-\Delta\mathbf{u},\mathbf{v})_{L^2(\Omega)} \\ &= -\int_\Omega \sum_{j=1}^{2} \operatorname{div}(v_j \nabla u_j)\,\mathrm{d}x + \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}x \\ &= -\int_{\partial\Omega'} \mathbf{v}\cdot\partial_{\mathbf{n}}\mathbf{u}\,\mathrm{d}s + \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}x \\ &\overset{(5.2)}{=} -\int_{\partial\Omega'} v_r \partial_{\mathbf{n}} u_r\,\mathrm{d}s + \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}x \\ &\overset{\text{(P-S-St.c)}}{=} \int_{\partial\Omega'} v_r u_r\,\mathrm{d}s + \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}x,\end{aligned}$$

so that the weak formulation reads

$$B_0(\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}x \quad \text{for all } \mathbf{v}\in\mathcal{T},$$

where the bilinear form is

$$B_0(\mathbf{u},\mathbf{v}) := \int_{\partial\Omega'} v_r u_r\,\mathrm{d}s + \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\,\mathrm{d}x.$$

Consider the Hilbert space $\mathscr{H}_0$ given by

$$\mathscr{H}_0 := \overline{\mathcal{T}}^{\|\cdot\|_{\mathscr{H}_0}}$$

with $\mathcal{T}$ as in (5.2) and

$$B_0(\mathbf{v},\mathbf{v}) = \|\mathbf{v}\|^2_{\mathscr{H}_0} = \int_{\partial\Omega'} v_r^2\,\mathrm{d}s + \int_\Omega |\nabla\mathbf{v}|^2\,\mathrm{d}x.$$

Thus, B_0 defines an inner product on the space $\mathscr{H}_0$ and therefore the Riesz representation theorem (see Theorem 2.3.3) gives existence and uniqueness of the solution in $\mathscr{H}_0$ for any $\mathbf{f}\in\mathscr{H}_0'$.

Proposition 5.1.1. *For any $\mathbf{f}\in\mathscr{H}_0'$ there exists a unique weak solution $\mathbf{u}\in\mathscr{H}_0$ of the Stokes problem* (P-S-St).

5.2 Bilinear Form in the Weighted Case

Equation (P-S-St.a) tested against $\mathbf{v}\in\mathcal{T}$ in the weighted inner product $(\cdot,\cdot)_\alpha$ with $\alpha\neq 0$ reads

$$(-\mathbb{P}\Delta\mathbf{u},\mathbf{v})_\alpha = (\mathbf{f},\mathbf{v})_\alpha.$$

Then again by Lemma 4.2.4

$$\begin{aligned}(-\mathbb{P}\Delta\mathbf{u},\mathbf{v})_\alpha &= (-\mathbb{P}\Delta\mathbf{u}, r^{-2\alpha}\mathbf{v})_{L^2(\Omega)} = (-\Delta\mathbf{u},\mathbb{P}r^{-2\alpha}\mathbf{v})_{L^2(\Omega)} \\ &= \int_\Omega (-\Delta\mathbf{u})\cdot(r^{-2\alpha}\mathbf{v} - \nabla\Phi_1)\,\mathrm{d}x \\ &= \underbrace{\int_\Omega (-\Delta\mathbf{u})\cdot(r^{-2\alpha}\mathbf{v})\,\mathrm{d}x}_{=:I_1} + \underbrace{\int_\Omega (-\Delta\mathbf{u})\cdot(-\nabla\Phi_1)\,\mathrm{d}x}_{=:I_2},\end{aligned}$$

where Φ_1 (in the sense of Definition 4.2.1) satisfies

$$\Delta\Phi_1 = \operatorname{div}(r^{-2\alpha}\mathbf{v}) = -2\alpha r^{-2\alpha-1}v_r \qquad \text{in } \Omega, \tag{5.3a}$$

$$\partial_{\mathbf{n}}\Phi_1 = \mathbf{n}\cdot r^{-2\alpha}\mathbf{v} = 0 \qquad \text{on } \partial\Omega'. \tag{5.3b}$$

With the theorem of Gauß the first integral I_1 becomes

$$\begin{aligned} I_1 &= -\int_\Omega \sum_{j=1}^{2} \operatorname{div}(r^{-2\alpha}v_j\nabla u_j)\,\mathrm{d}x + \int_\Omega (\nabla r^{-2\alpha}\mathbf{v}) : \nabla\mathbf{u}\,\mathrm{d}x \\ &= -\int_{\partial\Omega'} (r^{-2\alpha}\mathbf{v})\cdot\partial_{\mathbf{n}}\mathbf{u}\,\mathrm{d}s + \int_\Omega (\nabla r^{-2\alpha}\mathbf{v}) : \nabla\mathbf{u}\,\mathrm{d}x. \end{aligned}$$

By the product rule for vector fields (see Appendix A.1)

$$\begin{aligned} \nabla(r^{-2\alpha}\mathbf{v}) : \nabla\mathbf{u} &= r^{-2\alpha}\nabla\mathbf{v} : \nabla\mathbf{u} + \left(\mathbf{v}\otimes\nabla r^{-2\alpha}\right) : \nabla\mathbf{u} \\ &= r^{-2\alpha}\nabla\mathbf{v} : \nabla\mathbf{u} + \left[\begin{pmatrix} v_r \\ v_\varphi \end{pmatrix} \otimes \begin{pmatrix} -2\alpha r^{-2\alpha-1} \\ 0 \end{pmatrix}\right] : \begin{pmatrix} \partial_r u_r & r^{-1}(\partial_\varphi u_r - u_\varphi) \\ \partial_r u_\varphi & r^{-1}(\partial_\varphi u_\varphi + u_r) \end{pmatrix} \\ &= r^{-2\alpha}\nabla\mathbf{v} : \nabla\mathbf{u} - 2\alpha r^{-2\alpha-1}\mathbf{v}\cdot\partial_r\mathbf{u}, \end{aligned} \tag{5.4}$$

and the boundary conditions (5.2) and (P-S-St.c), we obtain

$$I_1 = \int_{\partial\Omega'} r^{-2\alpha}v_r u_r\,\mathrm{d}s + \int_\Omega r^{-2\alpha}\nabla\mathbf{v} : \nabla\mathbf{u}\,\mathrm{d}x - 2\alpha\int_\Omega r^{-2\alpha-1}\mathbf{v}\cdot\partial_r\mathbf{u}\,\mathrm{d}x. \tag{5.5}$$

Again by the theorem of Gauß, the second integral I_2 becomes

$$\begin{aligned} I_2 &= \int_\Omega (\Delta\mathbf{u})\cdot(\nabla\Phi_1)\,\mathrm{d}x \\ &= \int_\Omega \sum_{j=1}^{2} \operatorname{div}\left((\partial_j\Phi_1)\nabla u_j\right)\mathrm{d}x - \int_\Omega \left(\nabla\otimes\nabla\Phi_1\right) : \nabla\mathbf{u}\,\mathrm{d}x \\ &= \int_{\partial\Omega'} \partial_{\mathbf{n}}\mathbf{u}\cdot\nabla\Phi_1\,\mathrm{d}s - \int_\Omega \left(\nabla\otimes\nabla\Phi_1\right) : \nabla\mathbf{u}\,\mathrm{d}x \\ &\overset{(5.3b)}{=} \int_{\partial\Omega'} (\partial_{\mathbf{n}}u_r)(\partial_r\Phi_1)\,\mathrm{d}s - \int_\Omega \left(\nabla\otimes\nabla\Phi_1\right) : \nabla\mathbf{u}\,\mathrm{d}x. \end{aligned} \tag{5.6}$$

By combining the expressions for I_1 and I_2 in (5.5) and (5.6), we obtain the bilinear form

$$\begin{aligned} B_1(\mathbf{u},\mathbf{v}) = &\int_{\partial\Omega'} r^{-2\alpha}v_r u_r\,\mathrm{d}s + \int_\Omega r^{-2\alpha}\nabla\mathbf{v} : \nabla\mathbf{u}\,\mathrm{d}x - 2\alpha\int_\Omega r^{-2\alpha-1}\mathbf{v}\cdot\partial_r\mathbf{u}\,\mathrm{d}x \\ &+ \int_{\partial\Omega'} (\partial_{\mathbf{n}}u_r)(\partial_r\Phi_1)\,\mathrm{d}s - \int_\Omega \left(\nabla\otimes\nabla\Phi_1\right) : \nabla\mathbf{u}\,\mathrm{d}x = \sum_{i=1}^{5} T_i^{(1)}, \end{aligned} \tag{B1}$$

where Φ_1 is defined as in (5.3).

Remark 5.2.1. *In order to get a coercivity estimate consider $B_1(\mathbf{u},\mathbf{u})$ that contains the positive terms*

$$\int_{\partial\Omega'} r^{-2\alpha}u_r^2\,\mathrm{d}s \qquad \text{and} \qquad \int_\Omega r^{-2\alpha}|\nabla\mathbf{u}|^2\,\mathrm{d}x, \tag{5.7}$$

which can be used to absorb the remaining terms. It will turn out that the fourth term $T_4^{(1)}$ with the boundary integral arising from I_2 is the most problematic. Note that $\partial_{\mathbf{n}} = \pm r^{-1}\partial_\varphi$, but there is no control of derivatives on the boundary. A natural thing to do would be to apply the Navier-slip condition (P-S-St.c)*. Gaining control is then possible by using the estimate from the unweighted case. However, this approach leads to the additional condition $\alpha \geq \frac{1}{2}$, while it will turn out that the coercivity estimate for the other terms requires $\alpha < 0$ (see Chapter 6).*

For obtaining a coercivity estimate, we can integrate the fourth term into the interior of the wedge with the fundamental theorem of calculus:

$$\begin{aligned}\int_{\partial\Omega'} (\partial_{\mathbf{n}} u_r)(\partial_r \Phi_1)\, \mathrm{d}s &= \int_0^\infty \Big((\partial_\varphi u_r)(\partial_r \Phi_1)\big|_{\varphi=\theta} - (\partial_\varphi u_r)(\partial_r \Phi_1)\big|_{\varphi=0} \Big) \frac{\mathrm{d}r}{r} \\ &= \int_0^\theta \int_0^\infty (\partial_\varphi^2 u_r)(\partial_r \Phi_1) \frac{\mathrm{d}r}{r}\, \mathrm{d}\varphi + \int_0^\theta \int_0^\infty (\partial_\varphi u_r)(\partial_\varphi \partial_r \Phi_1) \frac{\mathrm{d}r}{r}\, \mathrm{d}\varphi.\end{aligned}$$

This now requires control on the second order derivative $\partial_\varphi^2 u_r$ in Ω, but we only have control on the first derivatives $\|\nabla \mathbf{u}\|_\alpha^2$. We can obtain control on all second order derivatives by introducing two additional bilinear forms arising from testing the equation (P-S-St.a) with particularly chosen test functions. Those two bilinear forms are derived in the following sections.

5.2.1 The Second Bilinear Form

For the second bilinear form we test (P-S-St.a) with

$$\mathbf{v}_2 := (-r\partial_r + 2\alpha)(r\partial_r)\mathbf{v}, \tag{5.8}$$

where $\mathbf{v} \in \mathcal{T}$ is as before. Through integration by parts and commutation of $r\partial_r$ with $\mathbb{P}$ (Lemma 4.2.5) we obtain

$$\big(- \mathbb{P}\Delta \mathbf{u}, \mathbf{v}_2 \big)_\alpha = \big(- (r\partial_r + 2)\mathbb{P}\Delta \mathbf{u}, r\partial_r \mathbf{v} \big)_\alpha = \big(- \mathbb{P}\Delta r\partial_r \mathbf{u}, r\partial_r \mathbf{v} \big)_\alpha. \tag{5.9}$$

We can carry out the same derivation as for B_1 but with $\mathbf{u}$ and $\mathbf{v}$ replaced by $r\partial_r \mathbf{u}$ and $r\partial_r \mathbf{v}$, respectively. The bilinear form that will arise looks similar to B_1 and it contains the term (cf. $T_2^{(1)}$ in (B1))

$$\int_\Omega r^{-2\alpha} (\nabla r\partial_r \mathbf{v}) : (\nabla r\partial_r \mathbf{u})\, \mathrm{d}x,$$

which will give control on certain second order derivatives as desired.

Note that in (5.9) we put one derivative from the test function $\mathbf{v}_2$ on $\mathbb{P}\Delta \mathbf{u}$, so that there are three derivatives on $\mathbf{u}$. For the derivation of the bilinear form it does not matter how many derivatives there are on $\mathbf{u}$, but for later purposes (see the proof of Theorem 5.4.10) we do not want more than two derivatives on $\mathbf{u}$. Therefore, the second bilinear form B_2 is derived in an alternative way without having more than two derivatives on $\mathbf{u}$.

Testing (P-S-St.a) with $\mathbf{v}_2$ gives with Lemma 4.2.4

$$\begin{aligned}(-\mathbb{P}\Delta \mathbf{u}, \mathbf{v}_2)_\alpha &= (-\mathbb{P}\Delta \mathbf{u}, r^{-2\alpha} \mathbf{v}_2)_{L^2(\Omega)} = (-\Delta \mathbf{u}, \mathbb{P} r^{-2\alpha} \mathbf{v}_2)_{L^2(\Omega)} \\ &= \underbrace{\int_\Omega (-\Delta \mathbf{u}) \cdot (r^{-2\alpha} \mathbf{v}_2)\, \mathrm{d}x}_{=: I_1^{(2)}} + \underbrace{\int_\Omega (-\Delta \mathbf{u}) \cdot (-\nabla \Phi_{\mathbf{v}_2})\, \mathrm{d}x}_{=: I_2^{(2)}},\end{aligned}$$

where $\Phi_{\mathbf{v}_2}$ (in the sense of Definition 4.2.1) satisfies

$$\Delta \Phi_{\mathbf{v}_2} = \operatorname{div}(r^{-2\alpha} \mathbf{v}_2) = -2\alpha r^{-2\alpha-1} (-r\partial_r + 2\alpha)(r\partial_r) v_r \qquad \text{in } \Omega, \tag{5.10a}$$

$$\partial_{\mathbf{n}} \Phi_{\mathbf{v}_2} = \mathbf{n} \cdot r^{-2\alpha} \mathbf{v}_2 = 0 \qquad \text{on } \partial\Omega'. \tag{5.10b}$$

With the theorem of Gauß and (5.4) the first integral $I_1^{(2)}$ becomes

$$\begin{aligned} I_1^{(2)} &= -\int_\Omega \sum_{j=1}^2 \operatorname{div}\big(r^{-2\alpha}\big((-r\partial_r + 2\alpha)(r\partial_r)v_j\big)\nabla u_j\big)\,\mathrm{d}x + \int_\Omega (\nabla r^{-2\alpha}\mathbf{v}_2) : \nabla \mathbf{u}\,\mathrm{d}x \\ &= -\int_{\partial\Omega'} (r^{-2\alpha}\mathbf{v}_2)\cdot\partial_{\mathbf{n}}\mathbf{u}\,\mathrm{d}s + \int_\Omega r^{-2\alpha}\nabla\mathbf{v}_2 : \nabla\mathbf{u}\,\mathrm{d}x - 2\alpha\int_\Omega r^{-2\alpha-1}\mathbf{v}_2\partial_r\mathbf{u}\,\mathrm{d}x \\ &= -\int_{\partial\Omega'} r^{-2\alpha}(r\partial_r v_r)\big((r\partial_r+1)\partial_{\mathbf{n}}u_r\big)\,\mathrm{d}s + \int_\Omega r^{-2\alpha}(\nabla r\partial_r\mathbf{v}) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x \\ &\quad - 2\alpha\int_\Omega r^{-2\alpha-1}(r\partial_r\mathbf{v})\cdot(\partial_r r\partial_r\mathbf{u})\,\mathrm{d}x, \end{aligned}$$

where in the last step we used the commutation relations (A.11) and applied integration by parts. Using the Navier-slip boundary condition (P-S-St.c) gives that

$$\begin{aligned} I_1^{(2)} &= \int_{\partial\Omega'} r^{-2\alpha}(r\partial_r v_r)\big((r\partial_r+1)u_r\big)\,\mathrm{d}s + \int_\Omega r^{-2\alpha}(\nabla r\partial_r\mathbf{v}) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x \\ &\quad - 2\alpha\int_\Omega r^{-2\alpha-1}(r\partial_r\mathbf{v})\cdot(\partial_r r\partial_r\mathbf{u})\,\mathrm{d}x. \end{aligned} \tag{5.11}$$

Again by the theorem of Gauß, the second integral $I_2^{(2)}$ becomes

$$\begin{aligned} I_2^{(2)} &= \int_\Omega (\Delta\mathbf{u})\cdot(\nabla\Phi_{\mathbf{v}_2})\,\mathrm{d}x \\ &= \int_\Omega \sum_{j=1}^2 \operatorname{div}\big((\partial_j\Phi_{\mathbf{v}_2})\nabla u_j\big)\,\mathrm{d}x - \int_\Omega \big(\nabla\otimes\nabla\Phi_{\mathbf{v}_2}\big) : \nabla\mathbf{u}\,\mathrm{d}x \\ &\overset{(5.10b)}{=} \int_{\partial\Omega'} (\partial_{\mathbf{n}}u_r)(\partial_r\Phi_{\mathbf{v}_2})\,\mathrm{d}s - \int_\Omega \big(\nabla\otimes\nabla\Phi_{\mathbf{v}_2}\big) : \nabla\mathbf{u}\,\mathrm{d}x. \end{aligned}$$

To simplify this expression we use the potential Φ_2 which (in the sense of Definition 4.2.1) satisfies

$$\begin{aligned} \Delta\Phi_2 &= \operatorname{div}(r^{-2\alpha}r\partial_r\mathbf{v}) = -2\alpha r^{-2\alpha-1}r\partial_r v_r && \text{in } \Omega, \\ \partial_{\mathbf{n}}\Phi_2 &= \mathbf{n}\cdot r^{-2\alpha}r\partial_r\mathbf{v} = 0 && \text{on } \partial\Omega', \end{aligned} \tag{5.12}$$

and is related to $\Phi_{\mathbf{v}_2}$ in (5.10) via $\Phi_{\mathbf{v}_2} = -(r\partial_r - 1)\Phi_2$. This problem for Φ_2 would appear if we derive the bilinear form using (5.9). To rewrite $I_2^{(2)}$ in terms of Φ_2, we apply again the commutation relations (A.11) and integration by parts to obtain

$$\begin{aligned} I_2^{(2)} &= -\int_{\partial\Omega'} (\partial_{\mathbf{n}}u_r)(\partial_r(r\partial_r-1)\Phi_2)\,\mathrm{d}s + \int_\Omega \big(\nabla\otimes\nabla(r\partial_r-1)\Phi_2\big) : \nabla\mathbf{u}\,\mathrm{d}x \\ &= \int_{\partial\Omega'} \big((r\partial_r+1)\partial_{\mathbf{n}}u_r\big)\partial_r\Phi_2\,\mathrm{d}s - \int_\Omega (\nabla\otimes\nabla\Phi_2) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x. \end{aligned} \tag{5.13}$$

By combining the expressions for $I_1^{(2)}$ and $I_2^{(2)}$ in (5.11) and (5.13), we obtain the bilinear form

$$\begin{aligned} B_2(\mathbf{u},\mathbf{v}) &= \int_{\partial\Omega'} r^{-2\alpha}(r\partial_r v_r)((r\partial_r+1)u_r)\,\mathrm{d}s + \int_\Omega r^{-2\alpha}(\nabla r\partial_r\mathbf{v}) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x \\ &\quad - 2\alpha\int_\Omega r^{-2\alpha-1}(r\partial_r\mathbf{v})\cdot(\partial_r r\partial_r\mathbf{u})\,\mathrm{d}x + \int_{\partial\Omega'} ((r\partial_r+1)\partial_{\mathbf{n}}u_r)(\partial_r\Phi_2)\,\mathrm{d}s \\ &\quad - \int_\Omega (\nabla\otimes\nabla\Phi_2) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x = \sum_{i=1}^5 T_i^{(2)}, \end{aligned} \tag{B2}$$

where Φ_2 satisfies (5.12). With this second bilinear form (B2) we see from $B_2(\mathbf{u},\mathbf{u})$ that there is control in Ω on terms with $\|\nabla r\partial_r \mathbf{u}\|_\alpha^2$, i.e. the second order derivatives in r and the mixed derivatives. By using the divergence free condition we also gain control on $\partial_\varphi^2 u_\varphi$. However, still control on $\partial_\varphi^2 u_r$ is missing and, even worse, B_2 gives an extra $\partial_\varphi^2 u_r$ term. Therefore, a third bilinear form is required to get control on those problematic terms.

Remark 5.2.2. *Note that in the above derivation of B_2 there are not more than two derivatives on $\mathbf{u}$. Alternatively, one can apply integration by parts in a different order by first putting one derivative from the test function $\mathbf{v}_2$ on $\mathbb{P}\Delta\mathbf{u}$ as in* (5.9)*. Then carrying out the same derivation as for B_1, but replacing $\mathbf{u}$ and $\mathbf{v}$ by $r\partial_r\mathbf{u}$ and $r\partial_r\mathbf{v}$, respectively, gives*

$$\begin{aligned}\big(-\mathbb{P}\Delta r\partial_r\mathbf{u}, r\partial_r\mathbf{v}\big)_\alpha = &-\int_{\partial\Omega'} r^{-2\alpha}(r\partial_r v_r)(\partial_\mathbf{n} r\partial_r u_r)\,\mathrm{d}s + \int_\Omega r^{-2\alpha}(\nabla r\partial_r\mathbf{v}) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x\\ &- 2\alpha\int_\Omega r^{-2\alpha-1}(r\partial_r\mathbf{v})\cdot(\partial_r r\partial_r\mathbf{u})\,\mathrm{d}x + \int_{\partial\Omega'}(\partial_\mathbf{n} r\partial_r u_r)(\partial_r\Phi_2)\,\mathrm{d}s\\ &-\int_\Omega(\nabla\otimes\nabla\Phi_2) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x,\end{aligned}$$

where Φ_2 satisfies (5.12)*. Then rewriting $\partial_\mathbf{n} r\partial_r u_r = (r\partial_r + 1)\partial_\mathbf{n} u_r$ and using the Navier-slip condition* (P-S-St.c) *on the first boundary integral (but not on the other boundary integral to avoid problems with the scaling), leads to the same bilinear form* (B2)*.*

5.2.2 The Vorticity Bilinear Form

Recall that the curl in polar coordinates (see also (A.7)) is given by

$$\omega_\mathbf{u} := \operatorname{curl}\mathbf{u} = r^{-1}\big((r\partial_r + 1)u_\varphi - \partial_\varphi u_r\big). \tag{5.14}$$

To derive the third bilinear form we should test (P-S-St.a) against a suitable test function in $(\cdot,\cdot)_\alpha$ such that after integration by parts we gain control on $\|r\nabla\omega_\mathbf{u}\|_\alpha^2$. This term also contains the derivative $\partial_\varphi^2 u_r$ and we finally get control on all second order derivatives which will be required for a coercivity estimate. To obtain the third bilinear form, which we will also call the vorticity bilinear form, we use the test function

$$\mathbf{v}_3 := r\begin{pmatrix}\partial_\varphi\\ -r\partial_r - 2 + 2\alpha\end{pmatrix}\omega_\mathbf{v} \quad \text{with } \mathbf{v}\in\mathcal{T}. \tag{5.15}$$

Furthermore, recall from Appendix A that for $\mathbf{a} = (a_1, a_2)^\top$ we define $\mathbf{a}^\perp := (-a_2, a_1)^\top$ which satisfies

$$\mathbf{a}^\perp\cdot\mathbf{b} = -\mathbf{a}\cdot\mathbf{b}^\perp \quad \text{for } \mathbf{a},\mathbf{b}\in\mathbb{C}^{2\times 1}, \tag{5.16}$$

and for $\mathbf{w} : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ satisfying $\operatorname{div}\mathbf{w} = 0$ we have

$$\nabla^\perp\cdot\mathbf{w} = \omega_\mathbf{w} \quad\text{and}\quad \Delta\mathbf{w} = \nabla^\perp\omega_\mathbf{w}, \tag{5.17}$$

where $\nabla^\perp = (-r^{-1}\partial_\varphi, \partial_r)^\top$ is the rotated gradient.

Testing (P-S-St.a) with the test function (5.15) gives with Lemma 4.2.4 and the theorem of Gauß

$$\begin{aligned}(-\mathbb{P}\Delta\mathbf{u}, \mathbf{v}_3)_\alpha &= -\int_\Omega \Delta\mathbf{u}\cdot\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)\,\mathrm{d}x \overset{(5.16),(5.17)}{=} \int_\Omega(\nabla\omega_\mathbf{u})\cdot\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)^\perp\,\mathrm{d}x\\ &\overset{(5.16)}{=} \int_\Omega \operatorname{div}\Big(\omega_\mathbf{u}\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)^\perp\Big)\,\mathrm{d}x + \int_\Omega \omega_\mathbf{u}\nabla^\perp\cdot\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)\,\mathrm{d}x\\ &= -\int_\Omega r^{-2\alpha}\omega_\mathbf{u}\big((r\partial_r + 2 - 2\alpha)^2 + \partial_\varphi^2\big)\omega_\mathbf{v}\,\mathrm{d}x\\ &\quad + \int_0^\infty \Big(\omega_\mathbf{u}\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)_r\big|_{\varphi=\theta} - \omega_\mathbf{u}\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)_r\big|_{\varphi=0}\Big)\,\mathrm{d}r,\end{aligned} \tag{5.18}$$

where in the last step we used the identity (which holds by the representation of $\mathbb{P}$ in Definition 4.2.1)

$$\begin{aligned}\nabla^\perp \cdot \big(\mathbb{P}r^{-2\alpha}\mathbf{v_3}\big) &= \nabla^\perp \cdot (r^{-2\alpha}\mathbf{v_3}) \overset{(5.17)}{=} \operatorname{curl}(r^{-2\alpha}\mathbf{v_3}) \\ &= r^{-1}\left[(r\partial_r+1)r^{-2\alpha+1}(-r\partial_r-2+2\alpha)r^{-1}\omega_{\mathbf{v}} - r^{-2\alpha+1}\partial_\varphi^2\omega_{\mathbf{v}}\right] \\ &= -r^{-2\alpha}\big((r\partial_r+2-2\alpha)^2+\partial_\varphi^2\big)\omega_{\mathbf{v}}.\end{aligned}$$

Through integration by parts in r we obtain

$$\begin{aligned}&-\int_0^\theta\int_0^\infty r^{-2\alpha+2}\omega_{\mathbf{u}}(r\partial_r+2-2\alpha)^2\omega_{\mathbf{v}}\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ &= \int_0^\theta\int_0^\infty \big((r\partial_r-2+2\alpha)r^{-2\alpha+2}\omega_{\mathbf{u}}\big)\big((r\partial_r+2-2\alpha)\omega_{\mathbf{v}}\big)\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ &= \int_0^\theta\int_0^\infty r^{-2\alpha+2}(r\partial_r\omega_{\mathbf{u}})\big((r\partial_r+2-2\alpha)\omega_{\mathbf{v}}\big)\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\end{aligned} \tag{5.19}$$

and through integration by parts in φ we obtain

$$\begin{aligned}-\int_0^\theta\int_0^\infty r^{-2\alpha+2}\omega_{\mathbf{u}}\partial_\varphi^2\omega_{\mathbf{v}}\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &= \int_0^\theta\int_0^\infty r^{-2\alpha+2}(\partial_\varphi\omega_{\mathbf{u}})(\partial_\varphi\omega_{\mathbf{v}})\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ &\quad -\int_0^\infty r^{-2\alpha+2}\left(\omega_{\mathbf{u}}\partial_\varphi\omega_{\mathbf{v}}\big|_{\varphi=\theta} - \omega_{\mathbf{u}}\partial_\varphi\omega_{\mathbf{v}}\big|_{\varphi=0}\right)\frac{\mathrm{d}r}{r}.\end{aligned} \tag{5.20}$$

Substituting (5.19) and (5.20) in (5.18) gives

$$\begin{aligned}(-\mathbb{P}\Delta\mathbf{u},\mathbf{v_3})_\alpha &= \int_\Omega r^{-2\alpha+2}\nabla\omega_{\mathbf{u}}\cdot\nabla\omega_{\mathbf{v}}\,\mathrm{d}x + (2-2\alpha)\int_\Omega r^{-2\alpha+1}\omega_{\mathbf{v}}\partial_r\omega_{\mathbf{u}}\,\mathrm{d}x \\ &\quad + \int_0^\infty\left(\omega_{\mathbf{u}}\big(\mathbb{P}r^{-2\alpha}\mathbf{v_3}\big)_r\big|_{\varphi=\theta} - \omega_{\mathbf{u}}\big(\mathbb{P}r^{-2\alpha}\mathbf{v_3}\big)_r\big|_{\varphi=0}\right)\mathrm{d}r \\ &\quad - \int_0^\infty r^{-2\alpha+2}\left(\omega_{\mathbf{u}}\partial_\varphi\omega_{\mathbf{v}}\big|_{\varphi=\theta} - \omega_{\mathbf{u}}\partial_\varphi\omega_{\mathbf{v}}\big|_{\varphi=0}\right)\frac{\mathrm{d}r}{r}.\end{aligned} \tag{5.21}$$

To rewrite the boundary integrals we need a representation of $(\mathbb{P}r^{-2\alpha}\mathbf{v_3})$ on the boundary.

Lemma 5.2.3. *For* $\mathrm{Re}\lambda \in \Sigma(-\frac{\pi+\theta}{\theta}, \frac{\pi-\theta}{\theta})$ *we have that*

$$\begin{aligned}\mathcal{M}\left(\big(\mathbb{P}r^{-2\alpha}\mathbf{v_3}\big)_r\right)(\lambda,0) &= (\lambda+2\alpha+1)\partial_\varphi\widehat{v_\varphi}(\lambda+2\alpha,0) - \partial_\varphi^2\widehat{v_r}(\lambda+2\alpha,0) \\ &\quad -(\lambda+1)\frac{\partial_\varphi\widehat{v_r}(\lambda+2\alpha,0)}{\sin\big((\lambda+1)\theta\big)}\cos\big((\lambda+1)\theta\big) + (\lambda+1)\frac{\partial_\varphi\widehat{v_r}(\lambda+2\alpha,\theta)}{\sin\big((\lambda+1)\theta\big)}, \\ \mathcal{M}\left(\big(\mathbb{P}r^{-2\alpha}\mathbf{v_3}\big)_r\right)(\lambda,\theta) &= (\lambda+2\alpha+1)\partial_\varphi\widehat{v_\varphi}(\lambda+2\alpha,\theta) - \partial_\varphi^2\widehat{v_r}(\lambda+2\alpha,0), \\ &\quad -(\lambda+1)\frac{\partial_\varphi\widehat{v_r}(\lambda+2\alpha,0)}{\sin\big((\lambda+1)\theta\big)} + (\lambda+1)\frac{\partial_\varphi\widehat{v_r}(\lambda+2\alpha,\theta)}{\sin\big((\lambda+1)\theta\big)}\cos\big((\lambda+1)\theta\big).\end{aligned}$$

Proof. By Definition 4.2.1 it follows that

$$\mathcal{M}\left(\mathbb{P}r^{-2\alpha}\mathbf{v_3}\right)(\lambda,\varphi) = \widehat{\mathbf{v_3}}(\lambda+2\alpha,\varphi) - \begin{pmatrix}(\lambda+1)\widehat{\Phi}_{\mathbf{v_3}}(\lambda+1,\varphi) \\ \partial_\varphi\widehat{\Phi}_{\mathbf{v_3}}(\lambda+1,\varphi)\end{pmatrix}, \tag{5.22}$$

where $\widehat{\Phi}_{\mathbf{v_3}}$ (in the sense of Definition 4.2.1) satisfies

$$\begin{aligned}\Delta\Phi_{\mathbf{v_3}} &= \operatorname{div}(r^{-2\alpha}\mathbf{v_3}) = 0 &&\text{in } \Omega, \\ \partial_{\mathbf{n}}\Phi_{\mathbf{v_3}} &= \mathbf{n}\cdot r^{-2\alpha}\mathbf{v_3} &&\text{on } \partial\Omega'.\end{aligned}$$

We already solved this elliptic problem in the general case in Proposition 4.1.1, but now the equation is homogeneous. Hence, we will not get the term involving the Green's function, so that

$$\widehat{\Phi}_{\mathbf{v}_3}(\lambda,\varphi) = \frac{\widehat{(\mathbf{v}_3)_\varphi}(\lambda+2\alpha-1,0)}{\lambda\sin(\lambda\theta)}\cos\big(\lambda(\theta-\varphi)\big) - \frac{\widehat{(\mathbf{v}_3)_\varphi}(\lambda+2\alpha-1,\theta)}{\lambda\sin(\lambda\theta)}\cos(\lambda\varphi). \tag{5.23}$$

Substituting (5.23) in the r component of (5.22), evaluating at $\varphi\in\{0,\theta\}$, and using that

$$\widehat{\mathbf{v}_3}(\lambda,\varphi) = \begin{pmatrix} (\lambda+1)\partial_\varphi\widehat{v_\varphi}(\lambda,\varphi) - \partial_\varphi^2\widehat{v}_r(\lambda,\varphi) \\ -(\lambda-2\alpha+1)\partial_\varphi\widehat{v}_r(\lambda,\varphi)\end{pmatrix} \quad \text{for } \varphi\in\{0,\theta\}$$

gives the result. □

Note that for the vorticity we have the boundary condition

$$\omega_{\mathbf{u}} \overset{(5.14)}{=} r^{-1}((r\partial_r+1)u_\varphi - \partial_\varphi u_r) \overset{\text{(P-S-St.b)}}{=} -r^{-1}\partial_\varphi u_r \quad \text{on } \partial\Omega', \tag{5.24}$$

where we do not apply the Navier-slip condition to avoid problems with the scaling in r. Consider the boundary terms at $\varphi=0$ in (5.21), which become with Plancherel's identity (Lemma 2.5.5)

$$\begin{aligned}
&-\int_0^\infty r^{-2\alpha+1}\omega_{\mathbf{u}} r^{2\alpha}\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)_r\big|_{\varphi=0}\,\frac{\mathrm{d}r}{r} + \int_0^\infty r^{-2\alpha+2}\omega_{\mathbf{u}}\partial_\varphi\omega_{\mathbf{v}}\big|_{\varphi=0}\,\frac{\mathrm{d}r}{r}\\
&= -\int_{\mathrm{Re}\lambda=\alpha}\widehat{r\omega_{\mathbf{u}}}(\lambda,0)\overline{\mathcal{M}\left(\left(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\right)_r\right)(\lambda-2\alpha,0)}\,\mathrm{dIm}\lambda + \int_{\mathrm{Re}\lambda=\alpha}\widehat{r\omega_{\mathbf{u}}}(\lambda,0)\overline{\partial_\varphi\widehat{r\omega_{\mathbf{v}}}(\lambda,0)}\,\mathrm{dIm}\lambda\\
&= \int_{\mathrm{Re}\lambda=\alpha}\partial_\varphi\widehat{u}_r(\lambda,0)\frac{\overline{\lambda}-2\alpha+1}{\sin\big((\overline{\lambda}-2\alpha+1)\theta\big)}\Big[-\overline{\partial_\varphi\widehat{v}_r(\lambda,0)}\cos\big((\overline{\lambda}-2\alpha+1)\theta\big) + \overline{\partial_\varphi\widehat{v}_r(\lambda,\theta)}\Big]\,\mathrm{dIm}\lambda,
\end{aligned}$$

where we have used (5.24) and inserted the result from Lemma 5.2.3. Note that the $\partial_\varphi\widehat{v_\varphi}$ and $\partial_\varphi^2\widehat{v}_r$ terms in Lemma 5.2.3 cancel with the second integral. Similarly, for the boundary terms in (5.21) at $\varphi=\theta$

$$\begin{aligned}
&\int_0^\infty r^{-2\alpha+1}\omega_{\mathbf{u}} r^{2\alpha}\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)_r\big|_{\varphi=\theta}\,\frac{\mathrm{d}r}{r} - \int_0^\infty r^{-2\alpha+2}\omega_{\mathbf{u}}\partial_\varphi\omega_{\mathbf{v}}\big|_{\varphi=\theta}\,\frac{\mathrm{d}r}{r}\\
&= -\int_{\mathrm{Re}\lambda=\alpha}\partial_\varphi\widehat{u}_r(\lambda,\theta)\frac{\overline{\lambda}-2\alpha+1}{\sin\big((\overline{\lambda}-2\alpha+1)\theta\big)}\Big[-\overline{\partial_\varphi\widehat{v}_r(\lambda,0)} + \overline{\partial_\varphi\widehat{v}_r(\lambda,\theta)}\cos\big((\overline{\lambda}-2\alpha+1)\theta\big)\Big]\,\mathrm{dIm}\lambda.
\end{aligned}$$

By rewriting we obtain

$$\begin{aligned}
&\int_0^\infty\Big(\omega_{\mathbf{u}}\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)_r\big|_{\varphi=\theta} - \omega_{\mathbf{u}}\big(\mathbb{P}r^{-2\alpha}\mathbf{v}_3\big)_r\big|_{\varphi=0}\Big)\,\mathrm{d}r\\
&-\int_0^\infty r^{-2\alpha+2}\Big(\omega_{\mathbf{u}}\partial_\varphi\omega_{\mathbf{v}}\big|_{\varphi=\theta} - \omega_{\mathbf{u}}\partial_\varphi\omega_{\mathbf{v}}\big|_{\varphi=0}\Big)\,\frac{\mathrm{d}r}{r}\\
&= \int_{\mathrm{Re}\lambda=\alpha}\overline{\partial_\varphi\widehat{v}_r(\lambda,0)}\frac{\overline{\lambda}-2\alpha+1}{\sin\big((\overline{\lambda}-2\alpha+1)\theta\big)}\Big[-\partial_\varphi\widehat{u}_r(\lambda,0)\cos\big((\overline{\lambda}-2\alpha+1)\theta\big) + \partial_\varphi\widehat{u}_r(\lambda,\theta)\Big]\,\mathrm{dIm}\lambda\\
&-\int_{\mathrm{Re}\lambda=\alpha}\overline{\partial_\varphi\widehat{v}_r(\lambda,\theta)}\frac{\overline{\lambda}-2\alpha+1}{\sin\big((\overline{\lambda}-2\alpha+1)\theta\big)}\Big[-\partial_\varphi\widehat{u}_r(\lambda,0) + \partial_\varphi\widehat{u}_r(\lambda,\theta)\cos\big((\overline{\lambda}-2\alpha+1)\theta\big)\Big]\,\mathrm{dIm}\lambda\\
&= \int_{\mathrm{Re}\lambda=\alpha}(\lambda-1)\overline{\partial_\varphi\widehat{v}_r(\lambda,0)}\left[-\frac{\partial_\varphi\widehat{u}_r(\lambda,0)}{\sin\big((\lambda-1)\theta\big)}\cos\big((\lambda-1)\theta\big) + \frac{\partial_\varphi\widehat{u}_r(\lambda,\theta)}{\sin\big((\lambda-1)\theta\big)}\right]\,\mathrm{dIm}\lambda\\
&-\int_{\mathrm{Re}\lambda=\alpha}(\lambda-1)\overline{\partial_\varphi\widehat{v}_r(\lambda,\theta)}\left[-\frac{\partial_\varphi\widehat{u}_r(\lambda,0)}{\sin\big((\lambda-1)\theta\big)} + \frac{\partial_\varphi\widehat{u}_r(\lambda,\theta)}{\sin\big((\lambda-1)\theta\big)}\cos\big((\lambda-1)\theta\big)\right]\,\mathrm{dIm}\lambda,
\end{aligned}$$

where in the last step one can write $\lambda = \alpha + it$ with $t \in \mathbb{R}$ to see that $\overline{\lambda} - 2\alpha + 1 = -(\lambda - 1)$. Substituting this representation for the boundary integrals in (5.21) gives for $\alpha \in \big(-\frac{\pi-\theta}{\theta}, \frac{\pi+\theta}{\theta}\big)$ the vorticity bilinear form

$$\begin{aligned} B_3(\mathbf{u},\mathbf{v}) &= \int_\Omega r^{-2\alpha+2}\nabla\omega_{\mathbf{v}}\cdot\nabla\omega_{\mathbf{u}}\,\mathrm{d}x + (2-2\alpha)\int_\Omega r^{-2\alpha+1}\omega_{\mathbf{v}}\partial_r\omega_{\mathbf{u}}\,\mathrm{d}x \\ &\quad + \int_{\mathrm{Re}\lambda=\alpha}(\lambda-1)\Big[\overline{\partial_\varphi\widehat{v_r}(\lambda,0)}\widehat{\Phi}_3(\lambda-1,0) - \overline{\partial_\varphi\widehat{v_r}(\lambda,\theta)}\widehat{\Phi}_3(\lambda-1,\theta)\Big]\,\mathrm{dIm}\lambda \\ &= \sum_{i=1}^{3} T_i^{(3)}, \end{aligned} \tag{B3}$$

where Φ_3 is the potential of $-\mathbb{P}\Delta\mathbf{u} = -\Delta\mathbf{u} + \nabla\Phi_3$ and satisfies (in the sense of Definition 4.2.1)

$$\begin{aligned} \Delta\Phi_3 &= \operatorname{div}\Delta\mathbf{u} = 0 && \text{in } \Omega, \\ \partial_{\mathbf{n}}\Phi_3 &= \mathbf{n}\cdot\Delta\mathbf{u} && \text{on } \partial\Omega'. \end{aligned} \tag{5.25}$$

With the aid of Proposition 4.1.1 one finds that

$$\begin{aligned} \widehat{\Phi}_3(\lambda,\varphi) &= \frac{\widehat{(\Delta\mathbf{u})_\varphi}(\lambda-1,\varphi)}{\lambda\sin(\lambda\theta)}\cos\big(\lambda(\theta-\varphi)\big) - \frac{\widehat{(\Delta\mathbf{u})_\varphi}(\lambda-1,\varphi)}{\lambda\sin(\lambda\theta)}\cos(\lambda\varphi) \\ &= -\frac{\partial_\varphi\widehat{u_r}(\lambda+1,0)}{\sin(\lambda\theta)}\cos\big(\lambda(\theta-\varphi)\big) + \frac{\partial_\varphi\widehat{u_r}(\lambda+1,\theta)}{\sin(\lambda\theta)}\cos(\lambda\varphi), \end{aligned}$$

since

$$\begin{aligned} (\Delta\mathbf{u})_\varphi &\overset{(\mathrm{A.9})}{=} r^{-2}\big[((r\partial_r)^2+\partial_\varphi^2)u_\varphi + 2\partial_\varphi u_r - u_\varphi\big] \\ &\overset{(\text{P-S-St.b})}{=} r^{-2}(\partial_\varphi^2 u_\varphi + 2\partial_\varphi u_r) \overset{(3.4\mathrm{c})}{=} -r^{-2}(r\partial_r - 1)\partial_\varphi u_r. \end{aligned}$$

Evaluating in $\varphi \in \{0,\theta\}$ gives that $\widehat{\Phi}_3$ is for $\mathrm{Re}\lambda \in \big(-\frac{\pi}{\theta}, \frac{\pi}{\theta}\big)$ given by

$$\widehat{\Phi}_3(\lambda,0) = -\frac{\partial_\varphi\widehat{u_r}(\lambda+1,0)}{\sin(\lambda\theta)}\cos(\lambda\theta) + \frac{\partial_\varphi\widehat{u_r}(\lambda+1,\theta)}{\sin(\lambda\theta)}, \tag{5.26a}$$

$$\widehat{\Phi}_3(\lambda,\theta) = -\frac{\partial_\varphi\widehat{u_r}(\lambda+1,0)}{\sin(\lambda\theta)} + \frac{\partial_\varphi\widehat{u_r}(\lambda+1,\theta)}{\sin(\lambda\theta)}\cos(\lambda\theta). \tag{5.26b}$$

Remark 5.2.4. *Note that the test function* $\mathbf{v}_3$ (5.15) *contains two derivatives and to derive the bilinear form* B_3 *from* $(-\mathbb{P}\Delta\mathbf{u}, \mathbf{v}_3)_\alpha$*, we first applied integration by parts to get one derivative from* $\Delta\mathbf{u}$ *on the test function (see* (5.18)*). Subsequently, by another integration by parts the derivatives are again evenly distributed on* $\mathbf{u}$ *and* $\mathbf{v}$ *(see* (5.19) *and* (5.19)*). Alternatively, we could first apply integration by parts to get one derivative from the test function on* $\mathbb{P}\Delta\mathbf{u}$*, so that there are three derivatives on* $\mathbf{u}$*. In this case there are also boundary terms appearing and to evaluate them it requires to solve* (5.25)*. By another integration by parts one obtains the same expression* (B3)*. Therefore, it is not a surprise that* $\widehat{\Phi}_3$ *appears in* (B3) *and we will use* (5.26) *to shorten the notation.*

5.3 Solution to the Bilinear Form

By introducing the additional bilinear forms, we obtain control on all second order derivatives, meaning that we can absorb the problematic $\partial_\varphi^2 u_r$ terms. Therefore, define for some appropriate constants $c_2, c_3 > 0$ the bilinear form

$$B(\mathbf{u},\mathbf{v}) := B_1(\mathbf{u},\mathbf{v}) + c_2\theta^2 B_2(\mathbf{u},\mathbf{v}) + c_3\theta^4 B_3(\mathbf{u},\mathbf{v}),$$

which thus arises from testing the equation $-\mathbb{P}\Delta\mathbf{u} = \mathbf{f}$ in $(\cdot,\cdot)_\alpha$ with the test function

$$\mathbf{v}_{\text{test}} := \mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3, \tag{5.27}$$

where $\mathbf{v} \in \mathcal{T}$ and $\mathbf{v}_2, \mathbf{v}_3$ are defined as in (5.8) and (5.15), i.e.

$$\mathbf{v}_2 = (-r\partial_r + 2\alpha)(r\partial_r)\mathbf{v} \quad \text{and} \quad \mathbf{v}_3 = r\begin{pmatrix} \partial_\varphi \\ -r\partial_r - 2 + 2\alpha \end{pmatrix}\omega_{\mathbf{v}}, \tag{5.28}$$

with $\omega_{\mathbf{v}} := \operatorname{curl}\mathbf{v}$, see (5.14). The extra factors $c_2\theta^2$ and $c_3\theta^4$ in $\mathbf{v}_{\text{test}}$ are included to make the estimates work as we will see in Chapter 6.

With the bilinear form B all second order derivatives are controlled but note that the bilinear forms B_2 and B_3 come together with additional terms which also need to be absorbed to obtain coercivity. These remainder terms should be absorbed by the other forms and this will be done in detail in Chapter 6. For now we will state the coercivity and boundedness estimate which we need to obtain a solution.

Consider for $\alpha \neq 0$ the space

$$\mathscr{H}_\alpha = \overline{\mathcal{T}}^{\|\cdot\|_{\mathscr{H}_\alpha}} \tag{5.29}$$

with $\mathcal{T}$ the space of test functions as in (5.2) and

$$\begin{aligned} \|\mathbf{v}\|_{\mathscr{H}_\alpha}^2 &:= |v_r|_\alpha^2 + |r\partial_r v_r|_\alpha^2 + \|\nabla\mathbf{v}\|_\alpha^2 + \|\nabla r\partial_r\mathbf{v}\|_\alpha^2 + \|r\nabla\omega_{\mathbf{v}}\|_\alpha^2 \\ &= \int_{\partial\Omega'} r^{-2\alpha}v_r^2\,\mathrm{d}s + \int_{\partial\Omega'} r^{-2\alpha}(r\partial_r v_r)^2\,\mathrm{d}s \\ &\quad + \int_\Omega r^{-2\alpha}|\nabla\mathbf{v}|^2\,\mathrm{d}x + \int_\Omega r^{-2\alpha}|\nabla r\partial_r\mathbf{v}|^2\,\mathrm{d}x + \int_\Omega r^{-2\alpha+2}|\nabla\omega_{\mathbf{v}}|^2\,\mathrm{d}x. \end{aligned}$$

Proposition 5.3.1 (Coercivity estimate)**.** *Let $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Moreover, let $\mathbf{u} \in \mathcal{T}$. Then there are constants $c_2, c_3 > 0$ independent of α and θ for which there exists an $\alpha_0 \in (-\frac{1}{4}, 0)$ large enough such that for all $\alpha \in (\alpha_0, 0)$ there exists a $\theta_0 \in (0, \frac{\pi}{2})$ small enough such that for all $\theta \in (0, \theta_0)$ we have the coercivity estimate*

$$B(\mathbf{u},\mathbf{u}) = B_1(\mathbf{u},\mathbf{u}) + c_2\theta^2B_2(\mathbf{u},\mathbf{u}) + c_3\theta^4B_3(\mathbf{u},\mathbf{u}) \gtrsim \|\mathbf{u}\|_{\mathscr{H}_\alpha}^2.$$

Specifically, this estimate is valid for $\theta \leq \frac{2\pi^2}{2.0672\cdot 10^6}$.

Proposition 5.3.2 (Boundedness)**.** *Let $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Then for any $\mathbf{u}, \mathbf{v} \in \mathcal{T}$ the bilinear form is bounded, i.e.*

$$B(\mathbf{u},\mathbf{v}) \lesssim_{\alpha,\theta} \|\mathbf{u}\|_{\mathscr{H}_\alpha}\|\mathbf{v}\|_{\mathscr{H}_\alpha}.$$

The proofs of these estimates, especially for coercivity, are quite cumbersome and therefore the proofs of those two propositions are postponed to the next chapter.

With these two estimates we can apply the Lax-Milgram theorem to find a solution to the bilinear form.

Theorem 5.3.3 (Solution to the bilinear form)**.** *Let $c_2, c_3 > 0$, $\alpha_0 \in (-\frac{1}{4}, 0)$ and $\theta_0 \in (0, \frac{\pi}{2})$ be as determined from Proposition 5.3.1 and let $\alpha \in (\alpha_0, 0)$ and $\theta \in (0, \theta_0)$. Moreover, assume that $\mathbf{f} \in \mathscr{H}_\alpha'$. Then there exists a unique $\mathbf{u} \in \mathscr{H}_\alpha$ satisfying*

$$B_1(\mathbf{u},\mathbf{v}) + c_2\theta^2B_2(\mathbf{u},\mathbf{v}) + c_3\theta^4B_3(\mathbf{u},\mathbf{v}) = \langle\mathbf{f}, \mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3\rangle \quad \text{for all } \mathbf{v} \in \mathscr{H}_\alpha, \tag{5.30}$$

where the bilinear forms B_1, B_2 and B_3 are defined as in (B1),(B2) *and* (B3)*, and $\mathbf{v}_2$ and $\mathbf{v}_3$ are the test functions* (5.28)*.*

Proof. This result follows immediately from Propositions 5.3.1, 5.3.2 and the Lax-Milgram theorem (Theorem 2.3.4), since those estimates also hold in $\mathscr{H}_\alpha$ by density of the test functions $\mathcal{T}$ in $\mathscr{H}_\alpha$, see (5.29). □

5.4 Strong Solutions to the Stokes Problem

With Theorem 5.3.3 we have derived a solution in $\mathscr{H}_\alpha$ to equation (5.30) which arises from testing (P-S-St.a) with the specific test function $r^{-2\alpha}(\mathbf{v}+c_2\theta^2\mathbf{v}_2+c_3\theta^4\mathbf{v}_3)$ in $(\cdot,\cdot)_{L^2(\Omega)}$. However, it is not clear yet whether this solution is also a solution to the original Stokes problem with Navier-slip (P-S-St) since we have introduced additional bilinear forms. Therefore, we will in the remainder of this chapter verify that indeed the solution from Theorem 5.3.3 is also a solution to the Stokes problem. In Section 5.4.1 it is shown that still enough test functions are generated so that the equation $-\mathbb{P}\Delta\mathbf{u} = \mathbf{f}$ is satisfied. Furthermore, we check that the Navier-slip condition still holds and in Section 5.4.2 we conclude that the solution of Theorem 5.3.3 is a strong solution to the Stokes problem.

5.4.1 Generating Test Functions

For the solution from Theorem 5.3.3 to be also a solution to the Stokes equation we need that with the special chosen test function $r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)$ still enough test functions are generated to apply the fundamental lemma of calculus of variations. That means that the set

$$\left\{\mathbb{P}\left[r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)\right] : \mathbf{v} \in \mathcal{T}\right\} \tag{5.31}$$

should lie dense in $\{\mathbb{P}\mathbf{v} : \mathbf{v} \in L^2(\Omega)\}$ where $\mathcal{T}$ is as defined in (5.2). For this purpose we study the surjectivity of the mapping

$$\mathbf{v} \mapsto \mathbb{P}\left[r^{-2\alpha}\left(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3\right)\right], \tag{5.32}$$

by solving the problem

$$\mathbb{P}\left[r^{-2\alpha}\left(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3\right)\right] = \mathbf{w}, \quad \mathbf{w} \in \mathcal{T}. \tag{5.33}$$

Remark 5.4.1. *In the above we additionally have to apply the Helmholtz projection to the test function because $r^{-2\alpha}\left(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3\right)$ with $\mathbf{v} \in \mathcal{T}$ is not divergence free. From the definition of $\mathbf{v}_2$ and $\mathbf{v}_3$ in (5.28) and the fact that $\mathbf{v} \in \mathcal{T}$, it follows that $\mathbf{v}, \mathbf{v}_2$ and $r^{-2\alpha}\mathbf{v}_3$ are divergence free. Therefore,*

$$\begin{aligned}\operatorname{div}\left[r^{-2\alpha}\left(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3\right)\right] &= \operatorname{div}\left[r^{-2\alpha}\left(\mathbf{v} + c_2\theta^2\mathbf{v}_2\right)\right] \\ &= -2\alpha r^{-2\alpha-1}\big(1 + c_2\theta^2(-r\partial_r + 2\alpha)r\partial_r\big)v_r.\end{aligned} \tag{5.34}$$

To ensure that $\mathbf{v}$ and $\mathbf{w}$ can both be divergence free we apply the Helmholtz projection. Note that it does not matter if we test the equation with a test function ϕ or $\mathbb{P}\phi$ since in the derivation of the bilinear form we used that $\mathbb{P}$ is self-adjoint to put the projection on the test function to obtain $\mathbb{P}\phi$ or $\mathbb{P}^2\phi$, respectively. Thus, it does not matter which test function we use since $\mathbb{P} = \mathbb{P}^2$. Moreover, problem (5.33) has no unique solution since it involves a projection. However, we want $\mathbf{v}$ to satisfy $\operatorname{div}\mathbf{v} = 0$ in Ω and $v_\varphi = 0$ on $\partial\Omega'$. Therefore, for solving problem (5.33) we can assume that $\mathbf{v}$ has those two properties.

We proceed as follows: first problem (5.33) will be solved by means of the Fourier-Mellin representation (Lemma 5.4.2) and then in Proposition 5.4.4 it is shown that the mapping (5.32) is surjective from $\mathcal{C} \to \mathcal{T}$, where

$$\begin{aligned}\mathcal{C} := \Big\{&\mathbf{v} \in C^2(\overline{\Omega}) : \operatorname{div}\mathbf{v} = 0 \text{ in } \Omega,\ v_\varphi = 0 \text{ on } \partial\Omega' \text{ and} \\ &(r\partial_r)^j\partial_\varphi^\ell\mathbf{v} \text{ is locally integrable with } \|(r\partial_r)^j\partial_\varphi^\ell\mathbf{v}\|_\alpha < \infty \text{ for } j + \ell = 3\Big\}.\end{aligned}$$

By a cut-off of the streamfunction it is proved in Lemma 5.4.8 that $\mathcal{C}$ is dense in $\mathcal{T}$. Finally, it follows that (5.31) is also dense in $\{\mathbb{P}\mathbf{v} : \mathbf{v} \in L^2(\Omega)\}$ as will be shown in Corollary 5.4.9.

Lemma 5.4.2. *Problem* (5.33) *has a solution* $\mathbf{v}$ *which satisfies* $\operatorname{div} \mathbf{v} = 0$ *in* Ω *and* $v_\varphi = 0$ *on* $\partial\Omega'$*. This solution* $\mathbf{v}$ *has a Fourier-Mellin representation which is given by*

$$\widehat{v_r}(\lambda, \varphi) = \sum_{k=1}^{\infty} m_{rk}(\lambda)\widehat{w_r}_k(\lambda - 2\alpha)\mathbf{e}_k(\varphi) \quad \text{and} \quad \widehat{v_\varphi}(\lambda, \varphi) = \sum_{k=1}^{\infty} m_{\varphi k}(\lambda)\widehat{w_\varphi}_k(\lambda - 2\alpha)\tilde{\mathbf{e}}_k(\varphi),$$

where the multipliers m_{rk} *and* $m_{\varphi k}$ *are given by*

$$m_{rk}(\lambda) = \frac{(\lambda - 2\alpha + 1)^2 - \left(\frac{k\pi}{\theta}\right)^2}{\left(X - c_3\theta^4\left((\lambda+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right)\right) \cdot \left((\lambda - 2\alpha + 1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right) + 2\alpha X(\lambda - 2\alpha + 1)}, \tag{5.35}$$

$$m_{\varphi k}(\lambda) \tag{5.36}$$

$$= \frac{(\lambda + 1)\left[(\lambda - 2\alpha + 1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right]}{\left[(\lambda+1)X - c_3\theta^4(\lambda - 2\alpha + 1)\left((\lambda+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right)\right] \cdot \left((\lambda - 2\alpha + 1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right) + 2\alpha X\left(\frac{k\pi}{\theta}\right)^2},$$

with $X := 1 + c_2\theta^2(-\lambda + 2\alpha)\lambda$.

In particular, m_{rk} *and* $m_{\varphi k}$ *decay as* k^{-2} *as* $k \to \infty$ *and both Fourier series for* $\widehat{v_r}$ *and* $\widehat{v_\varphi}$ *converge pointwise.*

Proof. Let $\mathbf{v}$ be divergence free in Ω and let $v_\varphi = 0$ on $\partial\Omega'$. Then we have the Fourier expansions (see Definition 4.3.1)

$$\widehat{v_r}(\lambda, \varphi) = \sum_{k=1}^{\infty} \widehat{v_r}_k(\lambda)\mathbf{e}_k(\varphi) \quad \text{and} \quad \widehat{v_\varphi}(\lambda, \varphi) = \sum_{k=1}^{\infty} \widehat{v_\varphi}_k(\lambda)\tilde{\mathbf{e}}_k(\varphi), \tag{5.37}$$

where the coefficients are related by

$$\widehat{v_r}_k(\lambda) \overset{\operatorname{div}\mathbf{v}=0}{=} -\frac{k\pi}{\theta(\lambda+1)}\widehat{v_\varphi}_k(\lambda), \quad \lambda \neq -1,\ k > 0. \tag{5.38}$$

In Mellin variables problem (5.33) reads using (5.28)

$$\left(1 + c_2\theta^2(-\lambda + 2\alpha)\lambda\right)\hat{\mathbf{v}}(\lambda, \varphi) + c_3\theta^4\widehat{\mathbf{v}_3}(\lambda, \varphi) - \widehat{\nabla\Phi}(\lambda - 2\alpha, \varphi) = \hat{\mathbf{w}}(\lambda - 2\alpha, \varphi), \tag{5.39}$$

where Φ (in the sense of Definition 4.2.1) satisfies

$$\begin{aligned} &\Delta\Phi \overset{(5.34)}{=} -2\alpha r^{-2\alpha-1}\left(1 + c_2\theta^2(-r\partial_r + 2\alpha)r\partial_r\right)v_r && \text{in } \Omega, \\ &\partial_{\mathbf{n}}\Phi = \mathbf{n} \cdot r^{-2\alpha}\left(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\alpha^2\theta^4\mathbf{v}_3\right) = c_3\alpha^2\theta^4 r^{-2\alpha}\mathbf{n} \cdot \mathbf{v}_3 && \text{on } \partial\Omega', \end{aligned} \tag{5.40}$$

where we used that $v_\varphi = 0$ on $\partial\Omega'$ and the expression for $\mathbf{v}_2$ in (5.28) to simplify the boundary condition for Φ. In case of homogeneous boundary data the Fourier-Mellin representation from Section 4.3.1 can be used to find a series solution to the problem. Hence, instead of solving the above problem we solve the problem with the homogeneous boundary condition

$$\begin{aligned} &\Delta\Phi = -2\alpha r^{-2\alpha-1}\left(1 + c_2\theta^2(-r\partial_r + 2\alpha)r\partial_r\right)v_r && \text{in } \Omega, \\ &\partial_{\mathbf{n}}\Phi = 0 && \text{on } \partial\Omega', \end{aligned} \tag{5.41}$$

and if we have a solution to (5.33) using (5.41) we verify that indeed $\partial_{\mathbf{n}}\Phi = c_3\alpha^2\theta^4 r^{-2\alpha}\mathbf{n}\cdot\mathbf{v}_3 = 0$ on $\partial\Omega'$. From Lemma 4.3.2 we obtain the solution for problem (5.41) in Fourier-Mellin representation

$$\widehat{\Phi}(\lambda,\varphi) = -2\alpha\big(1 + c_2\theta^2(-\lambda+1)(\lambda+2\alpha-1)\big)\sum_{k=1}^{\infty}\frac{\widehat{v_r}_k(\lambda+2\alpha-1)}{\lambda^2-\left(\frac{k\pi}{\theta}\right)^2}\mathbf{e}_k(\varphi), \tag{5.42}$$

for $\mathrm{Re}\lambda \in \left(-\frac{\pi}{\theta},\frac{\pi}{\theta}\right)$. Furthermore, by (5.28) and (5.14) we have in Mellin variables

$$\widehat{\mathbf{v}_3}(\lambda,\varphi) = \begin{pmatrix} (\lambda+1)\partial_\varphi\widehat{v_\varphi}(\lambda,\varphi) - \partial_\varphi^2\widehat{v_r}(\lambda,\varphi) \\ (-\lambda-1+2\alpha)\big[(\lambda+1)\widehat{v_\varphi}(\lambda,\varphi) - \partial_\varphi\widehat{v_r}(\lambda,\varphi)\big]\end{pmatrix}$$

and

$$\widehat{\nabla\Phi} = \begin{pmatrix}(\lambda+1)\widehat{\Phi}(\lambda+1,\varphi)\\ \partial_\varphi\widehat{\Phi}(\lambda+1,\varphi)\end{pmatrix},$$

so that problem (5.39) simplifies to

$$\begin{aligned}\widehat{w_r}(\lambda-2\alpha,\varphi) &= \big(1+c_2\theta^2(-\lambda+2\alpha)\lambda\big)\widehat{v_r}(\lambda,\varphi) + c_3\theta^4\big[(\lambda+1)\underbrace{\partial_\varphi\widehat{v_\varphi}(\lambda,\varphi)}_{\overset{(3.4c)}{=}-(\lambda+1)\widehat{v_r}(\lambda,\varphi)}-\partial_\varphi^2\widehat{v_r}(\lambda,\varphi)\big]\\ &\quad-(\lambda-2\alpha+1)\widehat{\Phi}(\lambda-2\alpha+1,\varphi), \qquad (5.43)\\ \widehat{w_\varphi}(\lambda-2\alpha,\varphi) &= \big(1+c_2\theta^2(-\lambda+2\alpha)\lambda\big)\widehat{v_\varphi}(\lambda,\varphi)\\ &\quad + c_3\theta^4(-\lambda+2\alpha-1)\big[(\lambda+1)\widehat{v_\varphi}(\lambda,\varphi)-\partial_\varphi\widehat{v_r}(\lambda,\varphi)\big] - \partial_\varphi\widehat{\Phi}(\lambda-2\alpha+1,\varphi). \qquad (5.44)\end{aligned}$$

Substituting the Fourier expansions (5.37) in (5.43) and using (5.42) gives

$$\begin{aligned}\widehat{w_r}(\lambda-2\alpha,\varphi) = \sum_{k=1}^{\infty}\Bigg[&1+c_2\theta^2(-\lambda+2\alpha)\lambda - c_3\theta^4\left((\lambda+1)^2-\left(\tfrac{k\pi}{\theta}\right)^2\right) \\ &+\big(1+c_2\theta^2(-\lambda+2\alpha)\lambda\big)\frac{2\alpha(\lambda-2\alpha+1)}{(\lambda-2\alpha+1)^2-\left(\frac{k\pi}{\theta}\right)^2}\Bigg]\widehat{v_r}_k(\lambda)\mathbf{e}_k(\varphi).\end{aligned} \tag{5.45}$$

Since $\mathbf{w}\in C_c^2(\overline{\Omega}\setminus\{0\})$, we can expand $\widehat{w_r}$ and $\widehat{w_\varphi}$ in a similar manner as for $\mathbf{v}$ (Equation (5.37)). The Fourier coefficients $\widehat{w_r}_k$ are then square summable, meaning that they decay faster than $k^{-\frac{1}{2}}$ as $k\to\infty$. Using that $\operatorname{div}\mathbf{w}=0$ (i.e. Equation (5.38) for $\mathbf{w}$) gives that in addition $\widehat{w_\varphi}_k$ decays as $o(k^{-\frac{3}{2}})$ as $k\to\infty$.

Using the Fourier expansion for $\widehat{w_r}$ in (5.45) we find by orthogonality that

$$\widehat{v_r}_k(\lambda) = m_{rk}(\lambda)\widehat{w_r}_k(\lambda-2\alpha),$$

where

$$m_{rk}(\lambda) = \frac{(\lambda-2\alpha+1)^2-\left(\frac{k\pi}{\theta}\right)^2}{\left(X-c_3\theta^4\left((\lambda+1)^2-\left(\frac{k\pi}{\theta}\right)^2\right)\right)\cdot\left((\lambda-2\alpha+1)^2-\left(\frac{k\pi}{\theta}\right)^2\right)+2\alpha X(\lambda-2\alpha+1)},$$

with $X := 1+c_2\theta^2(-\lambda+2\alpha)\lambda$.

Similarly, substituting the Fourier expansions (5.37) in (5.44), using (5.38) and (5.42) gives that $\widehat{v_{\varphi}}_k(\lambda) = m_{\varphi k}(\lambda)\widehat{w_{\varphi}}_k(\lambda - 2\alpha)$, where

$$m_{\varphi k}(\lambda) = \frac{(\lambda+1)\left[(\lambda-2\alpha+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right]}{\left[(\lambda+1)X - c_3\theta^4(\lambda-2\alpha+1)\left((\lambda+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right)\right]\cdot\left((\lambda-2\alpha+1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right) + 2\alpha X\left(\frac{k\pi}{\theta}\right)^2}.$$

Both sequences m_{rk} and $m_{\varphi k}$ decay as k^{-2} as $k \to \infty$ and are therefore summable. Furthermore, the Fourier coefficients satisfy the following decay

$$\widehat{v_r}_k(\lambda) = o\left(k^{-\frac{5}{2}}\right) \quad \text{and} \quad \widehat{v_{\varphi}}_k(\lambda) = o\left(k^{-\frac{7}{2}}\right) \quad \text{as } k \to \infty, \tag{5.46}$$

and therefore the Fourier series for $\widehat{v_r}$ and $\widehat{v_\varphi}$ converge pointwise. It remains to check whether this solution satisfies the boundary condition $\partial_{\mathbf{n}}\Phi = c_3\alpha^2\theta^4 r^{-2\alpha}\mathbf{n}\cdot\mathbf{v}_3 = 0$, which reads in Mellin

$$\pm\, c_3\alpha^2\theta^4\lambda\partial_\varphi\widehat{v_r}(\lambda+2\alpha-1,\varphi) = 0 \quad \text{for } \varphi \in \{0,\theta\}. \tag{5.47}$$

Note that

$$\partial_\varphi\widehat{v_r}(\lambda,\varphi) = -\sum_{k=1}^{\infty}\frac{k\pi}{\theta}\widehat{v_r}_k(\lambda)\tilde{\mathbf{e}}_k(\varphi),$$

where by (5.46) the coefficients satisfy

$$\frac{k\pi}{\theta}\widehat{v_r}_k(\lambda) = o\left(k^{-\frac{3}{2}}\right).$$

and thus also the series for $\partial_\varphi\widehat{v_r}$ converges pointwise. Therefore, the series can be evaluated at the boundary $\varphi \in \{0,\theta\}$ and there we obtain $\partial_\varphi\widehat{v_r} = 0$ meaning that (5.47) holds. □

Remark 5.4.3. *In view of Definition 4.2.1 we need to ensure that the inverse Mellin transform of the Fourier-Mellin representations for $\widehat{v_r}$ and $\widehat{v_\varphi}$ are uniquely defined. Therefore, we do not want to pick up the singularities from the multipliers m_{rk} and $m_{\varphi k}$. It is straightforward to check that the multipliers do not have a singularity at* $\mathrm{Re}\lambda = 0$ *and as $\theta \downarrow 0$ the strip of convergence $\Sigma(a,b)$ satisfies $a \to -\infty$ and $b \to \infty$. Thus, for θ small, there is a large strip in which we can integrate over any line* $\mathrm{Re}\lambda$ *for the inverse Mellin transform to get a uniquely defined solution in polar coordinates. This strip will be denoted by Σ_m.*

Proposition 5.4.4. *Let* $\mathbf{v}$ *be the solution to problem* (5.33) *as given in Proposition 5.4.2. Then we have for $j,\ell \in \{0,1,2,3\}$ the estimate*

$$\sum_{0\le j+\ell\le 4}\|(r\partial_r)^j\partial_\varphi^\ell\mathbf{v}\|_\beta^2 \lesssim \sum_{0\le j+\ell\le 2}\|(r\partial_r)^j\partial_\varphi^\ell\mathbf{w}\|_{\beta-2\alpha}^2, \quad \mathbf{w} \in \mathcal{T},$$

for some weight β such that $\beta - 1 \in \Sigma_m$. Moreover, we have that $\mathbf{v} \in C^2(\overline{\Omega})$.

Proof. The estimates are proved by transforming to Mellin variables and using the solution representation from Proposition 5.4.2 and the fact that the multipliers m_{rk} and $m_{r\varphi}$ decay as k^{-2} as $k \to \infty$. The estimates for v_r are presented and for v_φ the proof is similar since m_{rk} and $m_{\varphi k}$ share the same properties. By the expression of m_{rk} (5.35) and its decay it follows that for $p \in \{0,1\}$

$$\sum_{k=1}^{\infty}\left(\tfrac{k\pi}{\theta}\right)^{2p}|m_{rk}(\lambda)|^2 \lesssim 1 \quad \text{and} \quad \sum_{k=1}^{\infty}\left(\tfrac{k\pi}{\theta}\right)^{2p}|\lambda|^2|m_{rk}(\lambda)|^2 \lesssim 1. \tag{5.48}$$

For $\ell = 0$ and $j \in \{0,1,2\}$ we obtain

$$\begin{aligned}\|(r\partial_r)^j v_r\|_\beta^2 &= \int_0^\theta \int_{\mathrm{Re}\lambda=\beta-1} |\lambda|^{2j} \left|\sum_{k=1}^\infty m_{rk}(\lambda)\widehat{w_r}_k(\lambda-2\alpha)\mathbf{e}_k(\varphi)\right|^2 \,\mathrm{dIm}\lambda\,\mathrm{d}\varphi \\ &\leq \int_{\mathrm{Re}\lambda=\beta-1} |\lambda|^{2j} \left(\sum_{k=1}^\infty |m_{rk}(\lambda)|^2\right)\left(\sum_{k=1}^\infty |\widehat{w_r}_k(\lambda-2\alpha)|^2\right) \mathrm{dIm}\lambda \\ &\lesssim \|(r\partial_r)^j w_r\|_{\beta-2\alpha}^2.\end{aligned}$$

In the case that $\ell = 0$ and $j = 3$ we get in a similar way that

$$\begin{aligned}\|(r\partial_r)^3 v_r\|_\beta^2 &\leq \int_{\mathrm{Re}\lambda=\beta-1} |\lambda|^4 \left(\sum_{k=1}^\infty |\lambda m_{rk}(\lambda)|^2\right)\left(\sum_{k=1}^\infty |\widehat{w_r}_k(\lambda-2\alpha)|^2\right) \mathrm{dIm}\lambda \\ &\overset{(5.48)}{\lesssim} \|(r\partial_r)^2 w_r\|_{\beta-2\alpha}^2.\end{aligned}$$

If $\ell = 1$ and $j \in \{0,1\}$, then an extra $\frac{k\pi}{\theta}$ arises from differentiating the Fourier series and therefore

$$\begin{aligned}\|(r\partial_r)^j \partial_\varphi v_r\|_\beta^2 &\leq \int_{\mathrm{Re}\lambda=\beta-1} |\lambda|^{2j} \left(\sum_{k=1}^\infty \left|\frac{k\pi}{\theta} m_{rk}(\lambda)\right|^2\right)\left(\sum_{k=1}^\infty |\widehat{w_r}_k(\lambda-2\alpha)|^2\right) \mathrm{dIm}\lambda \\ &\overset{(5.48)}{\lesssim} \|(r\partial_r)^j w_r\|_{\beta-2\alpha}^2.\end{aligned}$$

If $\ell = 1$ and $j \in \{2,3\}$, then there is an extra $\left(\frac{k\pi}{\theta}\right)^2$ and λ^2

$$\begin{aligned}\|(r\partial_r)^j \partial_\varphi v_r\|_\beta^2 &\leq \int_{\mathrm{Re}\lambda=\beta-1} |\lambda|^{2(j-1)} \left(\sum_{k=1}^\infty \left(\tfrac{k\pi}{\theta}\right)^2 |\lambda m_{rk}(\lambda)|^2\right)\left(\sum_{k=1}^\infty |\widehat{w_r}_k(\lambda-2\alpha)|^2\right) \mathrm{dIm}\lambda \\ &\overset{(5.48)}{\lesssim} \|(r\partial_r)^{j-1} w_r\|_{\beta-2\alpha}^2.\end{aligned}$$

For $\ell = 2$ and $j \in \{0,1,2\}$ we get by orthonormality of $\mathbf{e}_k(\varphi)$

$$\begin{aligned}\|(r\partial_r)^j \partial_\varphi^2 v_r\|_\beta^2 &= \int_{\mathrm{Re}\lambda=\beta-1} |\lambda|^{2j} \sum_{k=1}^\infty \underbrace{\left(\tfrac{k\pi}{\theta}\right)^4 |m_{rk}(\lambda)|^2}_{\lesssim 1} |\widehat{w_r}_k(\lambda-2\alpha)|^2 \,\mathrm{dIm}\lambda \\ &\lesssim \|(r\partial_r)^j w_r\|_{\beta-2\alpha}^2.\end{aligned}$$

Finally, for $\ell = 3$ and $j \in \{0,1\}$ consider

$$\partial_\varphi^3 \widehat{v_r}(\lambda,\varphi) = \sum_{k=1}^\infty \left(\tfrac{k\pi}{\theta}\right)^3 m_{rk}(\lambda)\widehat{w_r}_k(\lambda-2\alpha)\tilde{\mathbf{e}}_k(\varphi)$$

and with integration by parts we can identify $-\frac{k\pi}{\theta}\widehat{w_r}_k$ as the k-th Fourier coefficient of $\partial_\varphi \widehat{w_r}$:

$$-\frac{k\pi}{\theta}\widehat{w_r}_k(\lambda) = -\frac{k\pi}{\theta}\int_0^\theta \widehat{w_r}(\lambda,\tilde\varphi)\mathbf{e}_k(\tilde\varphi)\,\mathrm{d}\tilde\varphi = \int_0^\theta (\partial_\varphi \widehat{w_r})(\lambda,\tilde\varphi)\tilde{\mathbf{e}}_k(\tilde\varphi)\,\mathrm{d}\tilde\varphi. \tag{5.49}$$

Therefore,

$$\begin{aligned}\|(r\partial_r)^j \partial_\varphi^3 v_r\|_\beta^2 &= \int_{\mathrm{Re}\lambda=\beta-1} |\lambda|^{2j} \sum_{k=1}^\infty \underbrace{\left(\tfrac{k\pi}{\theta}\right)^4 |m_{rk}(\lambda)|^2}_{\lesssim 1} |(\partial_\varphi \widehat{w_r})_k(\lambda-2\alpha)|^2 \,\mathrm{dIm}\lambda \\ &\lesssim \|(r\partial_r)^j \partial_\varphi w_r\|_{\beta-2\alpha}^2.\end{aligned}$$

Since $\mathbf{w} \in C^2_{\mathrm{c}}(\overline{\Omega} \setminus \{0\})$ it follows by combining all the above estimates that

$$\sum_{\substack{0 \leq j+\ell \leq 4 \\ j,\ell \in \{0,\ldots,3\}}} \|(r\partial_r)^j \partial_\varphi^\ell \mathbf{v}\|_\beta^2 < \infty, \tag{5.50}$$

for some weight β with $\beta - 1 \in \Sigma_m$.

Remark 5.4.5. *We remark that if one wants to prove the above estimate for more than three derivatives in φ, then boundary terms from the integration by parts are occurring which do not necessarily vanish (cf. (5.49)). This is the reason that we only prove that $\mathbf{v} \in C^2(\overline{\Omega})$ and not $\mathbf{v} \in C^\infty(\overline{\Omega})$ even if $\mathbf{w}$ would be in $C^\infty_{\mathrm{c}}(\overline{\Omega} \setminus \{0\})$. Since we do not need the estimates for higher derivatives, it suffices to show that $\mathbf{v} \in C^2(\overline{\Omega})$.*

To complete the proof we still show that indeed $\mathbf{v} \in C^2(\overline{\Omega})$, for that purpose we apply the Sobolev embedding twice: first in the angle and then in the radius. This is shown schematically in Figure 5.1. Note that for $\theta \ll 1$ the strip Σ_m is large (Remark 5.4.3) and therefore there exists a β such that $\beta - 1, -(\beta - 1) \in \Sigma_m$. Hence, by Morrey's inequality (see e.g. [13, Section 5.6.3, Theorem 6]) and the inequality $1 \leq r^{2\beta} + r^{-2\beta}$ we obtain

$$\|\mathbf{v}\|_{C^{n-1,\frac{1}{2}}(\overline{I})} \lesssim_n \|\mathbf{v}\|_{H^n(I)} \leq \|\mathbf{v}\|_{H_{n,\beta}(I)} + \|\mathbf{v}\|_{H_{n,-\beta}(I)} \quad \text{for } n \in \mathbb{N}, \tag{5.51}$$

where $C^{n,\gamma}(I)$ is the space of n-times continuously differentiable functions on the interval $I \subset \mathbb{R}$ of which the n-th partial derivatives are γ-Hölder continuous.

Fix $j \in \{1,2,3\}$, then by (5.50) and (5.51) we obtain

$$\|\mathbf{v}\|_{C^{3-j,\frac{1}{2}}([0,\theta])} \lesssim_j \|\mathbf{v}\|_{H^{4-j}((0,\theta))} \leq \|\mathbf{v}\|_{H_{4-j,\beta}((0,\theta))} + \|\mathbf{v}\|_{H_{4-j,-\beta}((0,\theta))} < \infty.$$

Similarly, for fixed $\ell \in \{1,2,3\}$

$$\|\mathbf{v}\|_{C^{3-\ell,\frac{1}{2}}([0,\infty))} \lesssim_\ell \|\mathbf{v}\|_{H^{4-\ell}((0,\infty))} \leq \|\mathbf{v}\|_{H_{4-\ell,\beta}((0,\infty))} + \|\mathbf{v}\|_{H_{4-\ell,-\beta}((0,\infty))} < \infty.$$

This shows that $\mathbf{v}$ is twice continuously differentiable in both the angle and radius, i.e. $\mathbf{v} \in C^2(\overline{\Omega})$. □

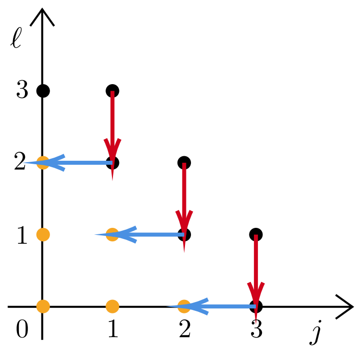


Figure 5.1: Schematic representation of applying the Sobolev embedding: first in the angle with ℓ derivatives (red arrows) and then in the radius with j derivatives (blue arrows) so that we end up in $C^2(\overline{\Omega})$ (orange dots).

From Proposition 5.4.4 it thus follows that problem (5.33), i.e.

$$\mathbb{P}\left[r^{-2\alpha}\left(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3\right)\right] = \mathbf{w}, \quad \mathbf{w} \in \mathcal{T},$$

is a surjective mapping from $\mathcal{C} \to \mathcal{T}$, where

$$\begin{aligned}\mathcal{C} := \Big\{ &\mathbf{v} \in C^2(\overline{\Omega}) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ v_\varphi = 0 \text{ on } \partial\Omega' \text{ and} \\ &(r\partial_r)^j \partial_\varphi^\ell \mathbf{v} \text{ is locally integrable with } \|(r\partial_r)^j \partial_\varphi^\ell \mathbf{v}\|_\alpha < \infty \text{ for } j + \ell = 3 \Big\}.\end{aligned} \tag{5.52}$$

To prove the desired density of $\mathbb{P}\big[r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)\big]$ with $\mathbf{v} \in \mathcal{T}$ in $\{\mathbb{P}\mathbf{v} : \mathbf{v} \in L^2(\Omega)\}$, it is thus required to approximate functions in $\mathcal{C}$ with a sequence of functions in $\mathcal{T}$. A natural thing to do would be to cut off $\mathbf{v} \in C^2(\overline{\Omega})$ to get a compactly supported function. However, by naively cutting off a C^2-function the divergence free condition is destroyed. Instead, consider the streamfunction $\psi(r,\varphi)$ corresponding to $\mathbf{v} \in \mathcal{C}$:

$$\mathbf{v} = \nabla^\perp \psi = \begin{pmatrix} -r^{-1}\partial_\varphi \psi \\ \partial_r \psi \end{pmatrix},$$

which satisfies the boundary condition $v_\varphi = \partial_r \psi = 0$ on $\partial\Omega'$. By construction the rotated gradient $\nabla^\perp \psi$ always satisfies the divergence free condition, namely

$$\operatorname{div} \nabla^\perp \psi = (r\partial_r + 1)(-r^{-1}\partial_\varphi \psi) + \partial_\varphi \partial_r \psi = 0.$$

Hence, the idea is to cut off ψ and to consider the rotated gradient of this cut-off in order to preserve the divergence free condition. Furthermore, ψ satisfies the elliptic problem

$$\begin{aligned} \Delta\psi &= \operatorname{curl} \mathbf{v} && \text{in } \Omega, \\ \partial_r \psi &= 0 && \text{on } \partial\Omega'. \end{aligned} \tag{5.53}$$

In addition to the divergence free condition, we also want that the condition $v_\varphi = 0$ on $\partial\Omega'$ is preserved. Therefore, we need that $\psi = 0$ on the boundary, so instead of (5.53) we consider the problem with homogeneous Dirichlet boundary conditions

$$\Delta\psi = \operatorname{curl} \mathbf{v} \qquad \text{in } \Omega, \tag{5.54a}$$

$$\psi = 0 \qquad \text{on } \partial\Omega'. \tag{5.54b}$$

Lemma 5.4.6. *The solution ψ of problem* (5.54) *is also a streamfunction for $\mathbf{v} \in \mathcal{C}$, i.e. $\mathbf{v} = \nabla^\perp \psi$.*

Proof. In Mellin variables the solution ψ of problem (5.54) satisfies

$$\big((\lambda+2)^2 + \partial_\varphi^2\big)\widehat{\psi}(\lambda+2,\varphi) = (\lambda+2)\widehat{v_\varphi}(\lambda+1,\varphi) - \partial_\varphi \widehat{v_r}(\lambda+1,\varphi).$$

Using the divergence free condition for $\mathbf{v}$

$$(\lambda+2)\partial_\varphi \widehat{v_r}(\lambda+1,\varphi) + \partial_\varphi^2 \widehat{v_\varphi}(\lambda+1,\varphi) = 0$$

we obtain for $\lambda \neq -2$ the ODE for $\widehat{v_\varphi}$

$$\big((\lambda+2)^2 + \partial_\varphi^2\big)\widehat{v_\varphi}(\lambda+1,\varphi) = (\lambda+2)\big((\lambda+2)^2 + \partial_\varphi^2\big)\widehat{\psi}(\lambda+2,\varphi),$$

with boundary conditions $\widehat{v_\varphi} = 0$ for $\varphi \in \{0,\theta\}$. Furthermore, $\psi = 0$ on the boundary and there is sufficient decay on $\widehat{v_\varphi}$, namely $\|\partial_\varphi^\ell v_\varphi\|_\beta^2 < \infty$ for $\ell \in \{0,1,2\}$ and $\beta - 1 \in \Sigma_m$, see Proposition 5.4.4. So the ODE has a unique solution and thus

$$\widehat{v_\varphi}(\lambda+1,\varphi) = (\lambda+2)\widehat{\psi}(\lambda+2,\varphi). \tag{5.55}$$

Applying the inverse Mellin transform gives $v_\varphi = \partial_r \psi$. Furthermore, for $\lambda \neq -2$

$$\widehat{v_r}(\lambda+1,\varphi) = -(\lambda+2)^{-1}\partial_\varphi \widehat{v_\varphi}(\lambda+1,\varphi) \overset{(5.55)}{=} -\partial_\varphi \widehat{\psi}(\lambda+2,\varphi)$$

and thus $v_r = -r^{-1}\partial_\varphi \psi$. $\square$

Lemma 5.4.7. *The solution ψ of problem (5.54) can be written in Fourier-Mellin representation as*

$$\widehat{\psi}(\lambda,\varphi) = \sum_{k=1}^{\infty} \frac{1}{\lambda} \widehat{v_{\varphi}}_k(\lambda-1)\tilde{\mathbf{e}}_k(\varphi),$$

for $\mathrm{Re}\lambda \in \big(-\frac{\pi}{\theta}, \frac{\pi}{\theta}\big)$.

Proof. The proof is similar as in Lemma 4.3.2, but because of the Dirichlet condition we now use the orthonormal basis $\tilde{\mathbf{e}}_k(\varphi)$ instead of $\mathbf{e}_k(\varphi)$, see Definition 4.3.1. By (A.5) and (A.7) we have in Mellin variables that $\widehat{\psi}$ satisfies

$$\big(\lambda^2 + \partial_{\varphi}^2\big)\widehat{\psi}(\lambda,\varphi) = \lambda\widehat{v_{\varphi}}(\lambda-1,\varphi) - \partial_{\varphi}\widehat{v_r}(\lambda-1,\varphi).$$

Expanding the streamfunction as

$$\widehat{\psi}(\lambda,\varphi) = \sum_{k=1}^{\infty} \widehat{\psi}_k(\lambda)\tilde{\mathbf{e}}_k(\varphi)$$

and inserting the Fourier expansion for $\widehat{v_r}$ and $\widehat{v_{\varphi}}$ (5.37) gives using (5.38) and orthogonality

$$\widehat{\psi}_k(\lambda) = \frac{1}{\lambda}\widehat{v_{\varphi}}_k(\lambda-1).$$ □

We are now able to prove that any divergence free C^2-function can be approximated in $L^2(\Omega)$ by a sequence of compactly supported and divergence free functions.

Lemma 5.4.8. *For any $\mathbf{v} \in C^2(\overline{\Omega})$ satisfying $\operatorname{div}\mathbf{v} = 0$ in Ω and $v_{\varphi} = 0$ on $\partial\Omega'$ there exists a sequence of functions $\mathbf{v}^{(k)}$ satisfying $\operatorname{div}\mathbf{v}^{(k)} = 0$ in Ω and $v_{\varphi}^{(k)} = 0$ on $\partial\Omega'$ such that*

$$\lim_{k\to\infty} \|\mathbf{v}^{(k)} - \mathbf{v}\|_{L^2(\Omega)} = 0.$$

Proof. Introduce the cut-off function

$$\zeta \in C_{\mathrm{c}}^{\infty}(\mathbb{R}) \quad \text{satisfying} \quad \zeta\big|_{[-1,1]} = 1 \text{ and } \zeta\big|_{(-\infty,-2]\cup[2,\infty)} = 0.$$

Furthermore, define for $k \in \mathbb{N}$

$$\eta_k(r) := \zeta\left(\frac{\log(r)}{k}\right), \qquad r > 0.$$

This cut-off function η_k has its support in $[e^{-2k}, e^{2k}]$ and $\eta_k(r) \to 1$ pointwise as $k \to \infty$ for all $r > 0$. Moreover, its derivative is given by

$$r\partial_r\eta_k(r) = \frac{\partial}{\partial(\log r)}\zeta\left(\tfrac{\log r}{k}\right) = \frac{1}{k}\zeta'\left(\tfrac{\log r}{k}\right). \tag{5.56}$$

Let ψ be the corresponding streamfunction to $\mathbf{v} \in \mathcal{C}$, which by Lemma 5.4.6 satisfies problem (5.54). We now choose the approximating sequence

$$\mathbf{v}^{(k)}(r,\varphi) := \nabla^{\perp}\big(\eta_k(r)\psi(r,\varphi)\big).$$

Then $\mathbf{v}^{(k)} \in C_{\mathrm{c}}^2(\overline{\Omega}\setminus\{0\})$, $\operatorname{div}\mathbf{v}^{(k)} = 0$ in Ω and on the boundary $\partial\Omega'$ we have

$$v_{\varphi}^{(k)}(r,\varphi) = \partial_r\big(\eta_k(r)\psi(r,\varphi)\big) \overset{(5.54b)}{=} 0 \quad \text{for } \varphi \in \{0,\theta\},$$

so $\mathbf{v}^{(k)}$ has the required properties. Furthermore, since $\eta_k \to 1$ we in addition have that $\mathbf{v}^{(k)} - \mathbf{v} = \nabla^\perp(\eta_k - 1)\psi \to 0$ pointwise as $k \to \infty$. Because of $|\eta_k - 1| \le 1$ and (5.56) we find for $k \ge 1$

$$\begin{aligned} |\nabla^\perp(\eta_k - 1)\psi|^2 &= \big((\eta_k - 1)r^{-1}\partial_\varphi\psi\big)^2 + \big((\eta_k - 1)\partial_r\psi + \psi\partial_r\eta_k\big)^2 \\ &\lesssim |\mathbf{v}|^2 + r^{-2}\psi^2 k^{-2}\big(\zeta'(\tfrac{\log r}{k})\big)^2. \end{aligned} \tag{5.57}$$

Note that for $\beta \neq 0$ we obtain by Lemma 5.4.7

$$\begin{aligned} \|r^{-1}\psi\|_\beta^2 &= \int_0^\theta \int_{\mathrm{Re}\lambda=\beta-1} |\widehat{\psi}(\lambda+1,\varphi)|^2 \,\mathrm{dIm}\lambda\,\mathrm{d}\varphi \\ &= \int_{\mathrm{Re}\lambda=\beta-1} \sum_{k=1}^\infty \underbrace{|\lambda+1|^{-2}}_{\lesssim 1 \text{ if } \mathrm{Re}\lambda\neq -1} |\widehat{v_\varphi}_k(\lambda)|^2 \,\mathrm{dIm}\lambda \\ &\lesssim_\beta \|v_\varphi\|_\beta^2. \end{aligned} \tag{5.58}$$

Let β be such that $\beta - 1, -(\beta - 1) \in \Sigma_m$, then by the inequality $1 \le r^{2\beta} + r^{-2\beta}$ we obtain

$$\|r^{-1}\psi\|_{L^2(\Omega)}^2 \le \|r^{-1}\psi\|_\beta^2 + \|r^{-1}\psi\|_{-\beta}^2 \overset{(5.58)}{\lesssim_\beta} \|v_\varphi\|_\beta^2 + \|v_\varphi\|_{-\beta}^2 \overset{(5.50)}{<} \infty,$$

so that

$$\|\nabla^\perp(\eta_k - 1)\psi\|_{L^2(\Omega)}^2 \overset{(5.57)}{\lesssim} \|\mathbf{v}\|_{L^2(\Omega)}^2 + \|\zeta'\|_{L^\infty(\mathbb{R})}^2 \|r^{-1}\psi\|_{L^2(\Omega)}^2 < \infty.$$

By the dominated convergence theorem the approximation follows

$$\lim_{k\to\infty} \|\mathbf{v}^{(k)} - \mathbf{v}\|_{L^2(\Omega)} = \lim_{k\to\infty} \|\nabla^\perp(\eta_k - 1)\psi\|_{L^2(\Omega)}^2 = 0. \qquad \square$$

With the above approximation the required density result for the test functions (5.31) is obtained.

Corollary 5.4.9. *The following density result holds*

$$\overline{\big\{\mathbb{P}\big[r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)\big] : \mathbf{v} \in \mathcal{T}\big\}}^{\|\cdot\|_{L^2(\Omega)}} = \big\{\mathbb{P}\mathbf{v} : \mathbf{v} \in L^2(\Omega)\big\}.$$

Proof. Throughout this proof we consider density with respect to the L^2-norm. As already mentioned, from Proposition 5.4.4 it follows that the mapping

$$\mathbb{P}\left[r^{-2\alpha}\left(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3\right)\right] = \mathbf{w}, \quad \mathbf{w} \in \mathcal{T}.$$

is surjective from $\mathcal{C} \to \mathcal{T}$, where $\mathcal{T}$ and $\mathcal{C}$ are defined in (5.2) and (5.52). Moreover, Lemma 5.4.8 implies that $\mathcal{T}$ is dense in $\mathcal{C}$ and therefore

$$\mathcal{T} \subset \overline{\big\{\mathbb{P}\big[r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)\big] : \mathbf{v} \in \mathcal{T}\big\}}^{\|\cdot\|_{L^2(\Omega)}}.$$

By a similar cut-off argument as in Lemma 5.4.8 it can be shown that $\mathcal{T}$ is dense in $\{\mathbb{P}\mathbf{v} : \mathbf{v} \in C^2_c(\overline{\Omega} \setminus \{0\})\}$. Continuity of the Helmholtz projection $\mathbb{P}$ in $L^2(\Omega)$ then implies that

$$\overline{\big\{\mathbb{P}\mathbf{v} : \mathbf{v} \in C^2_c(\overline{\Omega} \setminus \{0\})\big\}}^{\|\cdot\|_{L^2(\Omega)}} = \big\{\mathbb{P}\mathbf{v} : \mathbf{v} \in L^2(\Omega)\big\}.$$

By combining the above density results we obtain the result

$$\overline{\big\{\mathbb{P}\big[r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)\big] : \mathbf{v} \in \mathcal{T}\big\}}^{\|\cdot\|_{L^2(\Omega)}} = \big\{\mathbb{P}\mathbf{v} : \mathbf{v} \in L^2(\Omega)\big\}. \qquad \square$$

5.4.2 The Final Result

Recall from (5.29) that the space $\mathscr{H}_\alpha$ is defined as $\mathscr{H}_\alpha := \overline{\mathcal{T}}^{\|\cdot\|_{\mathscr{H}_\alpha}}$, where

$$\begin{aligned}\mathcal{T} := \Big\{&\mathbf{v} \in C_c^2(\overline{\Omega} \setminus \{0\}) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ v_\varphi = 0 \text{ on } \partial\Omega' \text{ and} \\ &(r\partial_r)^j \partial_\varphi^\ell \mathbf{v} \text{ is locally integrable with } \|(r\partial_r)^j \partial_\varphi^\ell \mathbf{v}\|_\alpha < \infty \text{ for } j + \ell = 3\Big\}\end{aligned} \tag{5.59}$$

and

$$\begin{aligned}\|\mathbf{v}\|_{\mathscr{H}_\alpha}^2 &:= |v_r|_\alpha^2 + |r\partial_r v_r|_\alpha^2 + \|\nabla \mathbf{v}\|_\alpha^2 + \|\nabla r\partial_r \mathbf{v}\|_\alpha^2 + \|r\nabla\omega_{\mathbf{v}}\|_\alpha^2 \\ &\sim \sum_{j=0}^{1} \int_0^\infty r^{-2\alpha}\Big(\big|(r\partial_r)^j v_r|_{\varphi=0}\big|^2 + \big|(r\partial_r)^j v_r|_{\varphi=\theta}\big|^2\Big)\,\mathrm{d}r \\ &\quad + \sum_{0\le j+\ell\le 2} \int_0^\theta \int_0^\infty r^{-2\alpha}\big|(r\partial_r)^j \partial_\varphi^\ell \mathbf{v}\big|^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi.\end{aligned}$$

In Theorem 5.3.3 we found a solution $\mathbf{u} \in \mathscr{H}_\alpha$ to the equation (5.30) with the bilinear form $B(\mathbf{u}, \mathbf{v})$ which arises from testing (P-S-St.a) with the test function $r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)$ in $(\cdot,\cdot)_{L^2(\Omega)}$. In the previous subsection it was shown that with $\mathbb{P}\big[r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)\big]$ still enough test functions are generated. Therefore, as we will show below, the solution from Theorem 5.3.3 is a solution to the stationary Stokes problem (P-S-St), i.e.

$$-\mathbb{P}\Delta\mathbf{u} = \mathbf{f} \qquad \text{in } \Omega, \tag{P-S-St.a}$$

$$u_\varphi = 0 \qquad \text{on } \partial\Omega', \tag{P-S-St.b}$$

$$u_r + \partial_{\mathbf{n}} u_r = 0 \qquad \text{on } \partial\Omega'. \tag{P-S-St.c}$$

Recall that the bilinear form is

$$B(\mathbf{u}, \mathbf{v}) = B_1(\mathbf{u}, \mathbf{v}) + c_2\theta^2 B_2(\mathbf{u}, \mathbf{v}) + c_3\theta^4 B_3(\mathbf{u}, \mathbf{v}), \tag{B}$$

where B_1, B_2 and B_3 are given as in (B1), (B2) and (B3), i.e.

$$\begin{aligned}B_1(\mathbf{u}, \mathbf{v}) = &\int_{\partial\Omega'} r^{-2\alpha} v_r u_r \,\mathrm{d}s + \int_\Omega r^{-2\alpha}\nabla\mathbf{v} : \nabla\mathbf{u}\,\mathrm{d}x - 2\alpha\int_\Omega r^{-2\alpha-1}\mathbf{v}\cdot\partial_r\mathbf{u}\,\mathrm{d}x \\ &+ \int_{\partial\Omega'} (\partial_{\mathbf{n}} u_r)(\partial_r\Phi_1)\,\mathrm{d}s - \int_\Omega \big(\nabla\otimes\nabla\Phi_1\big) : \nabla\mathbf{u}\,\mathrm{d}x =: \sum_{i=1}^{5} T_i^{(1)},\end{aligned} \tag{B1}$$

$$\begin{aligned}B_2(\mathbf{u}, \mathbf{v}) = &\int_{\partial\Omega'} r^{-2\alpha}(r\partial_r v_r)((r\partial_r + 1)u_r)\,\mathrm{d}s + \int_\Omega r^{-2\alpha}(\nabla r\partial_r\mathbf{v}) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x \\ &- 2\alpha\int_\Omega r^{-2\alpha-1}(r\partial_r\mathbf{v})\cdot(\partial_r r\partial_r\mathbf{u})\,\mathrm{d}x + \int_{\partial\Omega'} ((r\partial_r + 1)\partial_{\mathbf{n}} u_r)(\partial_r\Phi_2)\,\mathrm{d}s \\ &- \int_\Omega (\nabla\otimes\nabla\Phi_2) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x =: \sum_{i=1}^{5} T_i^{(2)},\end{aligned} \tag{B2}$$

$$\begin{aligned}B_3(\mathbf{u}, \mathbf{v}) = &\int_\Omega r^{-2\alpha+2}\nabla\omega_{\mathbf{v}}\cdot\nabla\omega_{\mathbf{u}}\,\mathrm{d}x + (2 - 2\alpha)\int_\Omega r^{-2\alpha+1}\omega_{\mathbf{v}}\partial_r\omega_{\mathbf{u}}\,\mathrm{d}x \\ &+ \int_{\mathrm{Re}\lambda=\alpha} (\lambda - 1)\left[\overline{\partial_\varphi\widehat{v_r}(\lambda, 0)}\widehat{\Phi}_3(\lambda - 1, 0) - \overline{\partial_\varphi\widehat{v_r}(\lambda, \theta)}\widehat{\Phi}_3(\lambda - 1, \theta)\right]\mathrm{dIm}\lambda =: \sum_{i=1}^{3} T_i^{(3)}.\end{aligned} \tag{B3}$$

Theorem 5.4.10. *Let $\alpha < 0$ and $\theta > 0$ with $|\alpha|$ and θ small enough and assume $\mathbf{f} \in \mathscr{H}_\alpha'$. Then there exists a unique $\mathbf{u} \in \mathscr{H}_\alpha$ that satisfies*

1. $-\mathbb{P}\Delta\mathbf{u} = \mathbf{f}$ *almost everywhere in the wedge Ω with opening angle θ,*
2. $u_r + \partial_{\mathbf{n}} u_r = 0$ *almost everywhere on the boundary $\partial\Omega'$.*

In particular, the Stokes problem with Navier-slip boundary conditions on the wedge has a strong solution in weighted Sobolev spaces. Moreover, this theorem is valid for $\theta \leq \frac{2\pi^2}{2.0672\cdot 10^6}$.

Proof. It remains to show that the solution $\mathbf{u} \in \mathscr{H}_\alpha$ from Theorem 5.3.3 satisfies the equation (P-S-St.a) in Ω and the Navier-slip boundary condition (P-S-St.c). The conditions $\operatorname{div}\mathbf{u} = 0$ in Ω and $u_\varphi = 0$ on $\partial\Omega'$ are ensured by the space $\mathscr{H}_\alpha$.

To show that $-\mathbb{P}\Delta\mathbf{u} = \mathbf{f}$ in Ω, take a test function $\mathbf{v} \in C^2_{\mathrm{c}}(\Omega)$ satisfying $\operatorname{div}\mathbf{v} = 0$ in Ω and $v_\varphi = 0$ on $\partial\Omega'$ instead of the usual test function from $\mathcal{T}$. Inserting this test function $\mathbf{v}$ into the bilinear form (B) gives that the boundary integrals $T_1^{(1)}, T_1^{(2)}$ and $T_3^{(3)}$ vanish, i.e.

$$\sum_{i=2}^{5} T_i^{(1)} + c_2\theta^2 \sum_{i=2}^{5} T_i^{(2)} + c_3\theta^4 \sum_{i=1}^{2} T_i^{(3)} = \langle \mathbf{f}, \mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\alpha^2\theta^4\mathbf{v}_3\rangle. \tag{5.61}$$

Corollary 5.4.9 ensures that still enough test functions are generated with our special chosen test function. Therefore, by the fundamental lemma of calculus of variations the solution $\mathbf{u}$ from Theorem 5.3.3 satisfies $-\mathbb{P}\Delta\mathbf{u} = \mathbf{f}$ almost everywhere in Ω.

Lastly, we verify that the Navier-slip condition (P-S-St.c) holds. Take $\mathbf{v} \in \mathcal{T}$, and consider

$$T_1^{(1)} + c_2\theta^2 T_1^{(2)} + \sum_{i=2}^{5} T_i^{(1)} + c_2\theta^2 \sum_{i=2}^{5} T_i^{(2)} + c_3\theta^4 \sum_{i=1}^{3} T_i^{(3)} = \langle \mathbf{f}, \mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\alpha^2\theta^4\mathbf{v}_3\rangle. \tag{5.62}$$

Recall from the derivation of the bilinear forms in Section 5.2 that we only applied the Navier-slip boundary condition (P-S-St.c) to get $T_1^{(1)}$ and $T_1^{(2)}$. The idea is to undo the integration by parts on the remaining terms on the left hand side of (5.62). However, we need that the regularity of $\mathbf{u} \in \mathscr{H}_\alpha$ and $\mathbf{v} \in \mathcal{T}$ allow for applying integration by parts. Recall that for deriving the bilinear forms B_1, B_2 and B_3 we do not need more than two derivatives on $\mathbf{u}$ if the integration by parts is applied in the right order, see Remark 5.2.2 and 5.2.4. This, however, does require three derivatives on the test function $\mathbf{v}$ and this is ensured in the test space (5.59) by the condition

$$\|(r\partial_r)^j \partial_\varphi^\ell \mathbf{v}\|_\alpha < \infty \quad \text{for } j + \ell = 3. \tag{5.63}$$

Therefore, we can undo the integration by parts for B_1, B_2 and B_3 as in Section 5.2 but then in the opposite direction to obtain

$$\begin{aligned}\sum_{i=2}^{5} T_i^{(1)} + c_2\theta^2 \sum_{i=2}^{5} T_i^{(2)} + c_3\theta^4 \sum_{i=1}^{3} T_i^{(3)} &= \big(-\mathbb{P}\Delta\mathbf{u}, r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)\big)_{L^2(\Omega)} \\ &\quad + \int_{\partial\Omega'} r^{-2\alpha} v_r \partial_{\mathbf{n}} u_r \,\mathrm{d}s \\ &\quad + c_2\theta^2 \int_{\partial\Omega'} r^{-2\alpha}(r\partial_r v_r)(\partial_{\mathbf{n}} r\partial_r u_r)\,\mathrm{d}s.\end{aligned}$$

Obviously, we cannot apply the Navier-slip condition on the last two boundary integrals as we did in the derivation of the bilinear forms since we are trying to show that $\mathbf{u}$ satisfies it. Substituting this into (5.62) and using that the equation is satisfied in Ω, i.e.

$$\big(-\mathbb{P}\Delta\mathbf{u}, r^{-2\alpha}(\mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\theta^4\mathbf{v}_3)\big)_{L^2(\Omega)} = \langle \mathbf{f}, \mathbf{v} + c_2\theta^2\mathbf{v}_2 + c_3\alpha^2\theta^4\mathbf{v}_3\rangle,$$

gives that

$$\int_{\partial\Omega'} r^{-2\alpha} v_r (u_r + \partial_{\mathbf{n}} u_r)\,\mathrm{d}s + c_2\theta^2 \int_{\partial\Omega'} r^{-2\alpha}(r\partial_r v_r)\big[(r\partial_r + 1)(u_r + \partial_{\mathbf{n}} u_r)\big]\,\mathrm{d}s = 0.$$

We obtain that $u_r + \partial_{\mathbf{n}} u_r = 0$ on $\partial\Omega'$ almost everywhere if enough test functions are generated. Therefore, we must have that for $v_r \in C_{\mathrm{c}}^2((0,\infty))$ the test functions

$$r^{-2\alpha+1}\big(1 + c_2\theta^2(-r\partial_r + 2\alpha)(r\partial_r)\big)v_r = \big(1 + c_2\theta^2(-r\partial_r + 1)(r\partial_r + 2\alpha - 1)\big)r^{-2\alpha+1}v_r$$

are dense in $L^2\big((0,\infty), \frac{\mathrm{d}r}{r}\big)$. Note that $r^{-2\alpha+1}v_r \in C_{\mathrm{c}}^2((0,\infty))$ and we can do the transformation $r = e^s$, apply the Fourier transform and use the density of the Schwartz space $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$ to see that it suffices to show that

$$\big\{\big(1 + c_2\theta^2(-i\xi + 1)(i\xi + 2\alpha - 1)\big)\phi : \phi \in \mathcal{S}(\mathbb{R})\big\}$$

is dense in $L^2(\mathbb{R})$. We will thus show that the mapping

$$\phi \mapsto \big(1 + c_2\theta^2(-i\xi + 1)(i\xi + 2\alpha - 1)\big)\phi$$

is surjective from $\mathcal{S}(\mathbb{R})$ to $\mathcal{S}(\mathbb{R})$. Therefore, assume that

$$\tilde{\phi}(\xi) = \big(1 + c_2\theta^2(-i\xi + 1)(i\xi + 2\alpha - 1)\big)\phi \in \mathcal{S}(\mathbb{R}),$$

then

$$\phi(\xi) = \frac{\tilde{\phi}(\xi)}{1 + c_2\theta^2(-i\xi + 1)(i\xi + 2\alpha - 1)}$$

is in the Schwartz space if the denominator has no real roots. We find that

$$\mathrm{Re}\big(1 + c_2\theta^2(-i\xi + 1)(i\xi + 2\alpha - 1)\big) = 1 + c_2\theta^2(\xi^2 + 2\alpha - 1),$$

which attains its minimum at $\xi = 0$, so to stay away from the roots we must have

$$1 + c_2\theta^2(2\alpha - 1) > 0.$$

This condition is satisfied in the case that θ is small enough. Hence, enough test functions are generated and therefore with the fundamental lemma of variational calculus we can conclude that $u_r + \partial_{\mathbf{n}} u_r = 0$ almost everywhere on $\partial\Omega'$. □

Remark 5.4.11. *Note that the condition on the third order derivatives* (5.63) *in the test space* (5.59) *can be avoided if one uses test functions which are at least in* $C_{\mathrm{c}}^3(\overline{\Omega} \setminus \{0\})$. *However, in Proposition 5.4.4 we have only shown that* $\mathbf{v} \in C^2(\overline{\Omega})$ *instead of* $C^3(\overline{\Omega})$. *Therefore, using test functions* $\mathbf{v} \in C_{\mathrm{c}}^3(\overline{\Omega} \setminus \{0\})$ *is possible, but one must then perform an additional mollification step to approximate* $C^2(\overline{\Omega})$ *with* $C_{\mathrm{c}}^3(\overline{\Omega} \setminus \{0\})$.

Chapter 6

Proof of the Coercivity and Boundedness Estimate

This chapter is devoted to the proof of the coercivity and boundedness estimate as stated in Propositions 5.3.1 and 5.3.2. Recall that the space of test functions is given by

$$\mathcal{T} := \Big\{ \mathbf{v} \in C^2_c(\overline{\Omega} \setminus \{0\}) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega,\ v_\varphi = 0 \text{ on } \partial\Omega' \text{ and } (r\partial_r)^j \partial_\varphi^\ell \mathbf{v} \text{ is locally integrable with } \|(r\partial_r)^j \partial_\varphi^\ell \mathbf{v}\|_\alpha < \infty \text{ for } j + \ell = 3 \Big\},$$

and that the original bilinear form B_1 arose from testing $-\mathbb{P}\Delta\mathbf{u} = \mathbf{f}$ with $\mathbf{v} \in \mathcal{T}$ in $(\cdot,\cdot)_\alpha$. As we have seen in Section 5.2 control on the second order derivatives was missing. Therefore, we tested (P-S-St.a) with

$$\mathbf{v}_2 := (-r\partial_r + 2\alpha)(r\partial_r)\mathbf{v} \quad \text{and} \quad \mathbf{v}_3 := r \begin{pmatrix} \partial_\varphi \\ -r\partial_r - 2 + 2\alpha \end{pmatrix} \operatorname{curl} \mathbf{v}$$

to obtain two additional bilinear forms B_2 and B_3 (Sections 5.2.1 and 5.2.2) with which control on all the second order derivatives was obtained. For the resulting bilinear form

$$B(\mathbf{u},\mathbf{v}) := B_1(\mathbf{u},\mathbf{v}) + c_2\theta^2 B_2(\mathbf{u},\mathbf{v}) + c_3\theta^4 B_3(\mathbf{u},\mathbf{v}) \quad \text{for some } c_2, c_3 > 0, \tag{B}$$

we prove the coercivity and boundedness estimates required for the Lax-Milgram theorem:

$$\begin{aligned} B(\mathbf{u},\mathbf{u}) &\gtrsim \|\mathbf{u}\|^2_{\mathscr{H}_\alpha} && \text{for } \mathbf{u} \in \mathcal{T}, \\ B(\mathbf{u},\mathbf{v}) &\lesssim_{\alpha,\theta} \|\mathbf{u}\|_{\mathscr{H}_\alpha} \|\mathbf{v}\|_{\mathscr{H}_\alpha} && \text{for } \mathbf{u},\mathbf{v} \in \mathcal{T}, \end{aligned}$$

where $\mathscr{H}_\alpha = \overline{\mathcal{T}}^{\|\cdot\|_{\mathscr{H}_\alpha}}$ is defined in (5.29). As mentioned earlier we want to prove coercivity for a small opening angle θ and in a certain range of weights with $\alpha < 0$ and $|\alpha|$ small. Therefore, we assume throughout this whole chapter that

$$0 < \theta < \tfrac{\pi}{2} \quad \text{and} \quad -\tfrac{1}{4} < \alpha < 0, \tag{6.1}$$

although most of the estimates apply for larger ranges of $\alpha \neq 0$ and this condition on α is only crucial in Lemma 6.1.6. Moreover, (6.1) will also ensure that if the Mellin transform is employed, the integration will be over some line $\operatorname{Re}\lambda$ which lies in the strip of convergence.

In Sections 6.1, 6.2 and 6.3 we focus on the coercivity estimates of the bilinear forms B_1, B_2 and B_3 separately. Finally, in Section 6.4 we combine all the estimates to prove Propositions 5.3.1 and 5.3.2.

6.1 An Incomplete Coercivity Estimate for B_1

Consider the first bilinear form (B1)

$$B_1(\mathbf{u},\mathbf{u}) = \int_{\partial\Omega'} r^{-2\alpha} u_r^2 \,\mathrm{d}s + \int_\Omega r^{-2\alpha} |\nabla \mathbf{u}|^2 \,\mathrm{d}x - 2\alpha \int_\Omega r^{-2\alpha-1} \mathbf{u}\cdot\partial_r \mathbf{u}\,\mathrm{d}x \\ + \int_{\partial\Omega'} (\partial_{\mathbf{n}} u_r)(\partial_r \Phi_1)\,\mathrm{d}s - \int_\Omega \big(\nabla\otimes\nabla\Phi_1\big) : \nabla\mathbf{u}\,\mathrm{d}x =: \sum_{i=1}^5 T_i^{(1)},$$

where Φ_1 (in the sense of Definition 4.2.1) is determined by

$$\begin{aligned} \Delta\Phi_1 &= -2\alpha r^{-2\alpha-1} u_r && \text{in } \Omega, \\ \partial_{\mathbf{n}}\Phi_1 &= 0 && \text{on } \partial\Omega'. \end{aligned} \tag{6.2}$$

The terms $T_1^{(1)}$ and $T_2^{(2)}$ are already positive, hence these terms can be used to absorb other terms which are possibly negative. Throughout this chapter, we use that the gradient in polar coordinates (see also Appendix A.2) can be expressed as

$$\nabla\mathbf{u} = \begin{pmatrix} \partial_r u_r & r^{-1}(\partial_\varphi u_r - u_\varphi) \\ \partial_r u_\varphi & r^{-1}(\partial_\varphi u_\varphi + u_r) \end{pmatrix}. \tag{6.3}$$

Before deriving estimates for $T_3^{(1)}, T_4^{(1)}$ and $T_5^{(1)}$, we first prove an auxiliary estimate.

Lemma 6.1.1. *For $\alpha \neq 0$ and $\mathbf{u}\in\mathcal{T}$ the following estimate holds*

$$\int_0^\theta \int_0^\infty r^{-2\alpha} u_\varphi^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \le 2\theta^2(1+\alpha^{-2})\|\nabla\mathbf{u}\|_\alpha^2.$$

Proof. Note that $u_\varphi(r,\varphi)=0$ for $\varphi\in\{0,\theta\}$ and hence by the fundamental theorem of calculus

$$u_\varphi(r,\varphi) = \int_0^\varphi (\partial_\varphi u_\varphi)(r,\tilde\varphi)\,\mathrm{d}\tilde\varphi,$$

which gives that

$$\begin{aligned} \int_0^\theta u_\varphi^2\,\mathrm{d}\varphi &= \int_0^\theta \left(\int_0^\varphi (\partial_\varphi u_\varphi)(r,\tilde\varphi)\,\mathrm{d}\tilde\varphi\right)^2 \mathrm{d}\varphi \\ &\le \int_0^\theta \theta \int_0^\theta \big((\partial_\varphi u_\varphi)(r,\tilde\varphi)\big)^2 \,\mathrm{d}\tilde\varphi\,\mathrm{d}\varphi \\ &= \theta^2 \int_0^\theta \big((\partial_\varphi u_\varphi)(r,\tilde\varphi)\big)^2\,\mathrm{d}\tilde\varphi \\ &\le 2\theta^2 \int_0^\theta \big(\partial_\varphi u_\varphi + u_r\big)^2\,\mathrm{d}\varphi + 2\theta^2\int_0^\theta u_r^2\,\mathrm{d}\varphi. \end{aligned}$$

Multiplying with $r^{-2\alpha-1}$ and integrating over r leads to the following estimate

$$\begin{aligned} \int_0^\theta\int_0^\infty r^{-2\alpha}u_\varphi^2\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &\le 2\theta^2\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi u_\varphi+u_r)^2\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + 2\theta^2\int_0^\theta\int_0^\infty r^{-2\alpha}u_r^2\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ &\le 2\theta^2\int_\Omega r^{-2\alpha}\big(r^{-1}(\partial_\varphi u_\varphi+u_r)\big)^2\,\mathrm{d}x + \frac{2\theta^2}{\alpha^2}\int_\Omega r^{-2\alpha}(\partial_r u_r)^2\,\mathrm{d}x, \end{aligned}$$

where we have applied Hardy's inequality (Lemma 2.4.2). □

We start with the estimate for the boundary integral $T_4^{(1)}$ in (6.2).

Lemma 6.1.2 (Estimate of $T_4^{(1)}$). *Let $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Then for $\mathbf{u} \in \mathcal{T}$ the following estimate holds*

$$\left|T_4^{(1)}\right| = \left|\int_{\partial\Omega'} (\partial_{\mathbf{n}} u_r)(\partial_r \Phi_1)\,\mathrm{d}s\right| \le \frac{\gamma_1}{2}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \left(\frac{C_1\theta^4}{2\gamma_1} + \frac{C_2\theta^2}{2\gamma_2} + \gamma_2 + 2\theta^2\gamma_2(1+\alpha^{-2})\right)\|\nabla\mathbf{u}\|_\alpha^2,$$

where $C_1, C_2 > 0$ are universal constants arising from Lemma 4.3.3 and $\gamma_1, \gamma_2 > 0$ are arbitrary constants.

Proof. Note that $\partial_{\mathbf{n}} u_r = \pm r^{-1}\partial_\varphi u_r$. Hence, by the fundamental theorem of calculus

$$\begin{aligned} T_4^{(1)} &= \int_0^\infty \left((\partial_\varphi u_r)(\partial_r\Phi_1)\big|_{\varphi=\theta} - (\partial_\varphi u_r)(\partial_r\Phi_1)\big|_{\varphi=0}\right)\frac{\mathrm{d}r}{r} \\ &= \int_0^\theta\int_0^\infty (\partial_\varphi^2 u_r)(\partial_r\Phi_1)\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \int_0^\theta\int_0^\infty (\partial_\varphi u_r)(\partial_r\partial_\varphi\Phi_1)\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi. \end{aligned}$$

For the first term, applying the Cauchy-Schwarz inequality, Young's inequality, and Lemma 4.3.3 gives

$$\begin{aligned} \left|\int_0^\theta\int_0^\infty (\partial_\varphi^2 u_r)(\partial_r\Phi_1)\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\right| &\le \frac{\gamma_1}{2}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{1}{2\gamma_1}\int_0^\theta\int_0^\infty r^{2\alpha}(\partial_r\Phi_1)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ &\lesssim \frac{\gamma_1}{2}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{\theta^4}{2\gamma_1}\|\nabla\mathbf{u}\|_\alpha^2. \end{aligned}$$

Similarly, for the second term

$$\begin{aligned} \int_0^\theta\int_0^\infty (\partial_\varphi u_r)(\partial_r\partial_\varphi\Phi)\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &\le \frac{\gamma_2}{2}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{1}{2\gamma_2}\int_0^\theta\int_0^\infty r^{2\alpha}(\partial_r\partial_\varphi\Phi)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ &\lesssim \frac{\gamma_2}{2}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{\theta^2}{2\gamma_2}\|\nabla\mathbf{u}\|_\alpha^2 \end{aligned}$$

and with Lemma 6.1.1 the first integral can be bounded by

$$\begin{aligned} \frac{1}{2}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi &\le \int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi u_r - u_\varphi)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \int_0^\theta\int_0^\infty r^{-2\alpha}u_\varphi^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ &\le \left(1 + 2\theta^2(1+\alpha^{-2})\right)\|\nabla\mathbf{u}\|_\alpha^2. \end{aligned} \tag{6.4}$$

Combining all the estimates above and introducing the universal constants $C_1, C_2 > 0$ from applying Lemma 4.3.3 gives the result. □

We are left with the terms $T_3^{(1)}$ and $T_5^{(1)}$, where $T_3^{(1)}$ can be split into a φ and an r part:

$$T_3^{(1)} = -2\alpha\int_0^\theta\int_0^\infty r^{-2\alpha}\left(u_\varphi\partial_r u_\varphi + u_r\partial_r u_r\right)\,\mathrm{d}r\,\mathrm{d}\varphi =: T_{3,\varphi}^{(1)} + T_{3,r}^{(1)}.$$

The term $T_{3,\varphi}^{(1)}$ will be treated in the next lemma and the remaining part $T_{3,r}^{(1)} + T_5^{(1)}$ will be considered together in Lemma 6.1.5 and 6.1.6.

Lemma 6.1.3 (Estimate of $T_{3,\varphi}^{(1)}$). *For $\alpha \neq 0$ and $\mathbf{u} \in \mathcal{T}$ the following estimate holds*

$$\left|T_{3,\varphi}^{(1)}\right| = \left|2\alpha\int_0^\theta\int_0^\infty r^{-2\alpha}u_\varphi\partial_r u_\varphi\,\mathrm{d}r\,\mathrm{d}\varphi\right| \le 4\theta^2(\alpha^2+1)\|\nabla\mathbf{u}\|_\alpha^2.$$

Proof. Using that $u_\varphi \partial_r u_\varphi = \frac{1}{2}\partial_r u_\varphi^2$ gives with integration by parts

$$T_{3,\varphi}^{(1)} = -\alpha \int_0^\theta \int_0^\infty r^{-2\alpha} \partial_r u_\varphi^2 \, \mathrm{d}r \, \mathrm{d}\varphi = -2\alpha^2 \int_0^\theta \int_0^\infty r^{-2\alpha} u_\varphi^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi$$

and the desired estimate follows from Lemma 6.1.1. □

Remark 6.1.4. *Note that if we apply integration by parts on $T_{3,r}^{(1)}$ similarly as in the proof of Lemma 6.1.3 for $T_{3,\varphi}^{(1)}$, then we would end up with $\left|T_{3,r}^{(1)}\right| \leq 2\|\nabla \mathbf{u}\|_\alpha^2$ where the α^2 has been cancelled since we have applied Hardy's inequality. Thus there is no α or θ in front of this term to make it small so that it can be absorbed in $\|\nabla \mathbf{u}\|_\alpha^2$. Therefore, we have to find another way to absorb $T_{3,r}^{(1)}$.*

We are left with $T_{3,r}^{(1)} + T_5^{(1)}$ which we will treat together. For $T_5^{(1)}$ note that

$$\nabla \otimes \nabla \Phi_1 \overset{\text{(A.10)}}{=} \underbrace{\begin{pmatrix} \partial_r^2 \Phi_1 & r^{-1}\partial_r \partial_\varphi \Phi_1 \\ r^{-1}\partial_\varphi \partial_r \Phi_1 & r^{-2}\partial_\varphi^2 \Phi_1 \end{pmatrix}}_{=:A_1} + \underbrace{\begin{pmatrix} 0 & -r^{-2}\partial_\varphi \Phi_1 \\ -r^{-2}\partial_\varphi \Phi_1 & r^{-1}\partial_r \Phi_1 \end{pmatrix}}_{=:A_2}, \tag{6.5}$$

and we write $T_{3,r}^{(1)} + T_5^{(1)} = T_{3,r}^{(1)} + T_{5,A_1}^{(1)} + T_{5,A_2}^{(1)}$. We first treat $T_{3,r}^{(1)} + T_{5,A_1}^{(1)}$ below in Lemma 6.1.5 and 6.1.6. After that, we prove coercivity for $T_{5,A_2}^{(1)}$ in Lemma 6.1.7.

To obtain an estimate for $T_{3,r}^{(1)} + T_{5,A_1}^{(1)}$ we first rewrite using (6.3)

$$\begin{aligned}
&A_1 : \nabla \mathbf{u} \\
= \;& (\partial_r^2 \Phi_1)(\partial_r u_r) + r^{-1}(\partial_\varphi \partial_r \Phi_1)(\partial_r u_\varphi) + r^{-2}(\partial_r \partial_\varphi \Phi_1)(\partial_\varphi u_r - u_\varphi) + r^{-3}(\partial_\varphi^2 \Phi_1)(\partial_\varphi u_\varphi + u_r) \\
\overset{\text{(3.4c)}}{=} & (\partial_r^2 \Phi_1)(\partial_r u_r) + r^{-1}(\partial_\varphi \partial_r \Phi_1)(\partial_r u_\varphi) + r^{-2}(\partial_r \partial_\varphi \Phi_1)(\partial_\varphi u_r - u_\varphi) - r^{-2}(\partial_\varphi^2 \Phi_1)(\partial_r u_r) \\
= \;& (\partial_r \partial_\varphi \Phi_1)\left[r^{-1}(\partial_r u_\varphi) + r^{-2}(\partial_\varphi u_r - u_\varphi)\right] + \left[(\partial_r^2 \Phi_1) + r^{-2}(r\partial_r)^2 \Phi_1 + 2\alpha r^{-2\alpha-1} u_r\right](\partial_r u_r),
\end{aligned}$$

where in the last step we have used problem (6.2) for Φ_1 which is in polar $((r\partial_r)^2 + \partial_\varphi^2)\Phi_1 = -2\alpha r^{-2\alpha+1} u_r$. Thus $T_{3,r}^{(1)} + T_{5,A_1}^{(1)}$ can be rewritten as

$$\begin{aligned}
&-2\alpha \int_\Omega r^{-2\alpha-1} u_r \partial_r u_r \, \mathrm{d}x - \int_\Omega A_1 : \nabla \mathbf{u} \, \mathrm{d}x \\
=& -4\alpha \int_\Omega r^{-2\alpha-1} u_r \partial_r u_r \, \mathrm{d}x \\
&- \int_\Omega (\partial_r \partial_\varphi \Phi_1)\left[r^{-1}(\partial_r u_\varphi) + r^{-2}(\partial_\varphi u_r - u_\varphi)\right] + \left[(\partial_r^2 \Phi) + r^{-2}(r\partial_r)^2 \Phi_1\right](\partial_r u_r) \, \mathrm{d}x \\
=& -4\alpha \int_0^\theta \int_0^\infty r^{-2\alpha+1} u_r \partial_r u_r \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi - \int_0^\theta \int_0^\infty (\partial_r \partial_\varphi \Phi_1)(r\partial_r u_\varphi - u_\varphi) \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \\
&- \int_0^\theta \int_0^\infty (\partial_r \partial_\varphi \Phi_1)(\partial_\varphi u_r) \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi + \int_0^\theta \int_0^\infty \left[r^2 \partial_r^2 \Phi + (r\partial_r)^2 \Phi_1\right](\partial_r u_r) \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \\
=:& \; S_1 + \tilde{S} + S_2 + S_3.
\end{aligned}$$

In the above rewriting there occurs one term $\tilde{S}$ which only contains u_φ and therefore can be dealt with by aid of Lemmas 4.3.3 and 6.1.1. This is done below in Lemma 6.1.5. The remaining terms with u_r are treated in Lemma 6.1.6.

Lemma 6.1.5 (Estimate of $\tilde{S}$)**.** *Let $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Then for $\mathbf{u} \in \mathcal{T}$ the following estimate holds*

$$\begin{aligned}
&\left| \int_0^\theta \int_0^\infty (\partial_r \partial_\varphi \Phi_1)(r\partial_r u_\varphi - u_\varphi) \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \right| \\
&\leq \left(\frac{C_3 \gamma_3 \theta^2}{2} + \frac{\theta^2}{\gamma_3}(1+\alpha^{-2}) + \frac{C_4 \gamma_4 \alpha^2}{2} + \frac{\theta^2}{\gamma_4}(1+\alpha^{-2}) \right) \|\nabla \mathbf{u}\|_\alpha^2,
\end{aligned}$$

where $C_3, C_4 > 0$ are universal constants arising from Lemma 4.3.3 and $\gamma_3, \gamma_4 > 0$ are arbitrary constants.

Proof. Applying the Cauchy-Schwarz inequality, Young's inequality and Lemma 4.3.3 and 6.1.1 on the term with u_φ gives

$$\begin{aligned}
\left| \int_0^\theta \int_0^\infty (\partial_r \partial_\varphi \Phi) u_\varphi \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \right| &\leq \frac{\gamma_3}{2} \int_0^\theta \int_0^\infty r^{2\alpha} (\partial_r \partial_\varphi \Phi)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi + \frac{1}{2\gamma_3} \int_0^\theta \int_0^\infty r^{-2\alpha} u_\varphi^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \\
&\leq \left(\frac{C_3 \gamma_3 \theta^2}{2} + \frac{\theta^2}{\gamma_3}(1+\alpha^{-2}) \right) \|\nabla \mathbf{u}\|_\alpha^2
\end{aligned}$$

and similarly for the term with $r\partial_r u_\varphi$ after additionally applying integration by parts

$$\begin{aligned}
\left| \int_0^\theta \int_0^\infty (\partial_r \partial_\varphi \Phi) r \partial_r u_\varphi \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \right| &= \left| \int_0^\theta \int_0^\infty (r \partial_r^2 \partial_\varphi \Phi) u_\varphi \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \right| \\
&\leq \left(\frac{C_4 \gamma_4 \alpha^2}{2} + \frac{\theta^2}{\gamma_4}(1+\alpha^{-2}) \right) \|\nabla \mathbf{u}\|_\alpha^2,
\end{aligned}$$

where the universal constants $C_3, C_4 > 0$ appear from applying Lemma 4.3.3 and $\gamma_3, \gamma_4 > 0$ come from Young's inequality. ☐

The remaining terms $S_1 + S_2 + S_3$ will also be absorbed in the gradient term $T_2^{(1)} = \|\nabla \mathbf{u}\|_\alpha^2$. Note that for $\mathbf{u} \in \mathcal{T}$ we have $\operatorname{div} \mathbf{u} = 0$, so that

$$\|\nabla \mathbf{u}\|_\alpha^2 = \int_0^\theta \int_0^\infty r^{-2\alpha} \left[2(r\partial_r u_r)^2 + (\partial_\varphi u_r - u_\varphi)^2 + (r\partial_r u_\varphi)^2 \right] \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \tag{6.6}$$

and in the next lemma we show the terms S_1, S_2 and S_3 can directly be absorbed into the integral

$$2 \int_0^\theta \int_0^\infty r^{-2\alpha} (r\partial_r u_r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi.$$

Lemma 6.1.6 (Estimate of $S_1 + S_2 + S_3$)**.** *Let $\mathbf{u} \in \mathcal{T}$ and assume that $0 < \theta < \frac{\pi}{2}$ and $-\frac{1}{4} < \alpha < 0$. Then there exists a constant $C > 0$ such that*

$$2 \int_0^\theta \int_0^\infty r^{-2\alpha} (r\partial_r u_r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi + S_1 + S_2 + S_3 \geq C \int_0^\theta \int_0^\infty r^{-2\alpha} (r\partial_r u_r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi.$$

Specifically, for the specified values of α and θ, $C = \frac{1}{2}$ is an admissible choice.

Proof. The strategy of the proof is as follows: first the terms S_1, S_2 and S_3 will be transformed to Mellin variables, using the Fourier representation for $\widehat{\Phi}$ and $\partial_\varphi \widehat{\Phi}$ (Lemma 4.3.2). Then setting $\lambda = it$ ($t \in \mathbb{R}$) leads to polynomial fractions and since the original integrals are real, we can take the real part and rewrite the fractions using

$$\mathrm{Re} \frac{z}{w} = \mathrm{Re} \frac{z\overline{w}}{|w|^2} = \frac{\mathrm{Re} z \mathrm{Re} w + \mathrm{Im} z \mathrm{Im} w}{|w|^2}, \qquad z, w \in \mathbb{C}. \tag{6.7}$$

Finally, it is shown that for the resulting fraction there exists a positive lower bound.

By using Bessel's identity (4.9) we obtain for S_1

$$\begin{aligned} S_1 =& -4\alpha \int_{\mathrm{Re}\lambda=0} (\overline{\lambda}+\alpha) \sum_{k=1}^{\infty} |\widehat{u_r}_k(\lambda+\alpha)|^2 \,\mathrm{dIm}\lambda \\ \overset{\lambda=it}{=}& -4\alpha \mathrm{Re} \int_{-\infty}^{\infty} (-it+\alpha) \sum_{k=1}^{\infty} |\widehat{u_r}_k(it+\alpha)|^2 \,\mathrm{d}t \\ =& -4\alpha^2 \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} |\widehat{u_r}_k(it+\alpha)|^2 \,\mathrm{d}t \end{aligned}$$

and define $F_1 := -4\alpha^2$. For S_2 we obtain

$$\begin{aligned} S_2 =& -\int_0^{\theta} \int_{\mathrm{Re}\lambda=0} (\lambda-\alpha+1)\partial_{\varphi}\widehat{\Phi}(\lambda-\alpha+1,\varphi)[\partial_{\varphi}\overline{\widehat{u_r}(\lambda+\alpha,\varphi)}] \,\mathrm{dIm}\lambda \,\mathrm{d}\varphi \\ =& 2\alpha \int_0^{\theta} \int_{\mathrm{Re}\lambda=0} (\lambda-\alpha+1) \left(\sum_{k=1}^{\infty} \frac{\widehat{u_r}_k(\lambda+\alpha)\left(\frac{k\pi}{\theta}\right)\tilde{\mathbf{e}}_k(\varphi)}{(\lambda-\alpha+1)^2-\left(\frac{k\pi}{\theta}\right)^2} \right) \\ & \cdot \left(\sum_{\ell=1}^{\infty} \overline{\widehat{u_r}_\ell(\lambda+\alpha)} \left(\frac{\ell\pi}{\theta}\right) \tilde{\mathbf{e}}_\ell(\varphi) \right) \,\mathrm{dIm}\lambda \,\mathrm{d}\varphi \\ =& 2\alpha \mathrm{Re} \int_{-\infty}^{\infty} (it-\alpha+1) \sum_{k=1}^{\infty} \frac{|\widehat{u_r}_k(it+\alpha)|^2}{(it-\alpha+1)^2-\left(\frac{k\pi}{\theta}\right)^2} \left(\frac{k\pi}{\theta}\right)^2 \,\mathrm{d}t \\ =& \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} F_2 \,|\widehat{u_r}_k(it+\alpha)|^2 \,\mathrm{d}t, \end{aligned}$$

where by (6.7) we obtain

$$2\alpha\mathrm{Re}\frac{\left(\frac{k\pi}{\theta}\right)^2(it-\alpha+1)}{(it-\alpha+1)^2-\left(\frac{k\pi}{\theta}\right)^2} = 2\alpha\frac{-(\alpha-1)^3\left(\frac{k\pi}{\theta}\right)^2-(\alpha-1)\left(\frac{k\pi}{\theta}\right)^2t^2+(\alpha-1)\left(\frac{k\pi}{\theta}\right)^4}{t^4+\left[2(\alpha-1)^2+2\left(\frac{k\pi}{\theta}\right)^2\right]t^2+\left((\alpha-1)^2-\left(\frac{k\pi}{\theta}\right)^2\right)^2} =: F_2.$$

In a similar manner converting S_3 to Mellin variables gives

$$\begin{aligned} & -\int_0^{\theta} \int_{\mathrm{Re}\lambda=0} [(\lambda-\alpha)(\lambda-\alpha+1)+(\lambda-\alpha+1)^2]\widehat{\Phi}(\lambda-\alpha+1,\varphi)(\overline{\lambda}+\alpha)\overline{\widehat{u_r}(\lambda+\alpha,\varphi)} \,\mathrm{dIm}\lambda \,\mathrm{d}\varphi \\ &= 2\alpha\mathrm{Re}\int_{-\infty}^{\infty}(it-\alpha+1)(-it+\alpha)\left[2it-2\alpha+1\right]\sum_{k=1}^{\infty}\frac{|\widehat{u_r}_k(it+\alpha)|^2}{(it-\alpha+1)^2-\left(\frac{k\pi}{\theta}\right)^2}\,\mathrm{d}t \\ &=: \int_{-\infty}^{\infty}\sum_{k=1}^{\infty} F_3\,|\widehat{u_r}_k(it+\alpha)|^2 \,\mathrm{d}t, \end{aligned}$$

where again by (6.7)

$$\begin{aligned} F_3 :=& \ 2\alpha\frac{F_3^{(4)}t^4+F_3^{(2)}t^2+F_3^{(0)}}{t^4+\left[2(\alpha-1)^2+2\left(\frac{k\pi}{\theta}\right)^2\right]t^2+\left((\alpha-1)^2-\left(\frac{k\pi}{\theta}\right)^2\right)^2} \quad \text{with} \\ F_3^{(4)} :=& \ 2\alpha+1, \\ F_3^{(2)} :=& \ (6\alpha-3)\left(\frac{k\pi}{\theta}\right)^2+\alpha(4\alpha^2-6\alpha+1)+1, \\ F_3^{(0)} :=& \ \alpha(2\alpha^4-7\alpha^3+9\alpha^2-5\alpha+1)+(-2\alpha^3+3\alpha^2-\alpha)\left(\frac{k\pi}{\theta}\right)^2. \end{aligned}$$

Hence,

$$S_1 + S_2 + S_3 = \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} (F_1 + F_2 + F_3) \, |\widehat{u_r}_k(it + \alpha)|^2 \, \mathrm{d}t$$

and by bringing everything under the same denominator the expression simplifies to

$$F_1 + F_2 + F_3 := 2\alpha \frac{t^4 + F^{(2)} t^2 + F^{(0)}}{t^4 + \left[2(\alpha-1)^2 + 2\left(\frac{k\pi}{\theta}\right)^2\right] t^2 + \left((\alpha-1)^2 - \left(\frac{k\pi}{\theta}\right)^2\right)^2} \quad \text{with} \tag{6.8}$$

$$F^{(2)} := (2\alpha - 1)(\alpha - 1) + (\alpha - 2)\left(\frac{k\pi}{\theta}\right)^2,$$

$$F^{(0)} := \alpha(\alpha-1)^3 + (\alpha^3 - 2\alpha^2)\left(\frac{k\pi}{\theta}\right)^2 + \left(1 - (\alpha+1)\left(\frac{k\pi}{\theta}\right)^2\right)\left(\frac{k\pi}{\theta}\right)^2. \tag{6.9}$$

Using that we want to absorb this into (see Equation (6.6))

$$2\int_0^\theta \int_0^\infty r^{-2\alpha} (r\partial_r u_r)^2 \, \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi = 2\int_{-\infty}^{\infty} \underbrace{|it+\alpha|^2}_{=t^2+\alpha^2} \sum_{k=0}^{\infty} |\widehat{u_r}_k(it+\alpha)|^2 \, \mathrm{d}t$$

it is convenient to factor out $t^2 + \alpha^2$ from $F_1 + F_2 + F_3$ so that we can write

$$\begin{aligned} &2\int_0^\theta \int_0^\infty r^{-2\alpha} (r\partial_r u_r)^2 \, \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi + S_1 + S_2 + S_3 \\ &= \int_{-\infty}^{\infty} (t^2 + \alpha^2) \sum_{k=1}^{\infty} \left(2 + \frac{F_1 + F_2 + F_3}{t^2 + \alpha^2}\right) |\widehat{u_r}_k(it+\alpha)|^2 \, \mathrm{d}t. \end{aligned}$$

It suffices to find a lower bound $C > 0$ uniformly in t and k for the expression

$$F := 2 + \frac{F_1 + F_2 + F_3}{t^2 + \alpha^2}, \tag{6.10}$$

because

$$\begin{aligned} &2\int_0^\theta \int_0^\infty r^{-2\alpha} (r\partial_r u_r)^2 \, \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi + S_1 + S_2 + S_3 \\ &\geq C \int_{-\infty}^{\infty} (t^2 + \alpha^2) \sum_{k=0}^{\infty} |\widehat{u_r}_k(it+\alpha)|^2 \, \mathrm{d}t = C \int_0^\theta \int_0^\infty r^{-2\alpha} (r\partial_r u_r)^2 \, \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi. \end{aligned}$$

The remainder of the proof is to find that lower bound C. Fix any t and the limit of F as $Y := \frac{k\pi}{\theta} \to \infty$ is

$$2 - 2\alpha \frac{\alpha + 1}{t^2 + \alpha^2}.$$

This limit is positive for each $t \in \mathbb{R}$ if and only if $\alpha < 0$ holds.

Furthermore, for fixed k, we have that the limit as $t \to \pm\infty$ equals 2. The expression F has a singularity at $(t, Y) = (0, 1 - \alpha)$. The idea is that for $k \geq 1$ we jump over the singularity by choosing θ small enough. The condition $\alpha < 0$ ensures that the limit for any fixed t as $Y \to \infty$ is positive. Hence, if θ is small enough, the fraction becomes positive and we can find a positive lower bound C for it. It suffices to jump over the singularity already for $k = 1$, i.e. the value of F should be positive already for $k = 1$ for all t. Then for $k > 1$ the value of the fraction will only be larger for all t. Figure 6.1 depicts contour plots of F for different values of α.

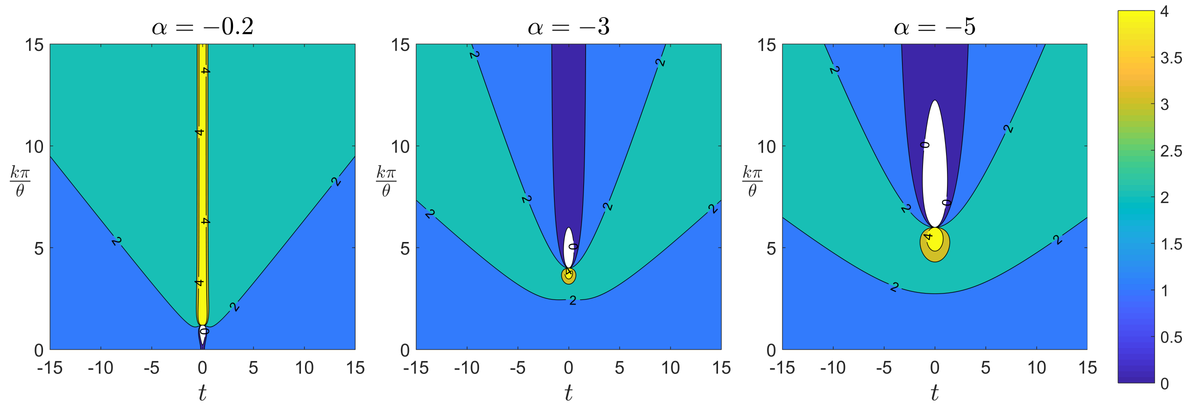


Figure 6.1: Contour plots of the expression F in (6.10) for different values of α. Moreover, F is negative in the white area.

From the contour plots it appears that for any $\alpha < 0$ the angle θ can be chosen small enough such that a positive lower bound C can be found. However, we are only interested in $|\alpha|$ small and we assumed that $-\frac{1}{4} < \alpha < 0$. Below we prove that indeed for these values of α a lower bound exists and therefore it suffices to choose $\theta < \frac{\pi}{2}$, i.e. $Y := \frac{k\pi}{\theta} \geq 2$. Note that under these assumptions we have for the last term in $F^{(0)}$ (see (6.9))

$$2\alpha Y^2\big(1-(\alpha+1)Y^2\big) > 0$$

and therefore by leaving out all positive terms in the numerator of $F_1 + F_2 + F_3$ in (6.8) we obtain

$$F_1 + F_2 + F_3 \geq 2\alpha \frac{t^4 + (2\alpha-1)(\alpha-1)t^2 + \alpha(\alpha-1)^3}{t^4 + [2(\alpha-1)^2 + 2Y^2]\, t^2 + \big((\alpha-1)^2 - Y^2\big)^2}.$$

For the three terms separately we obtain by using $1 \leq (\alpha-1)^2 \leq \frac{25}{16}$ the bounds

$$\frac{2\alpha t^4}{t^4 + [2(\alpha-1)^2 + 2Y^2]\, t^2 + \big((\alpha-1)^2 - Y^2\big)^2} \geq \frac{2\alpha t^4}{2(\alpha-1)^2 t^2} \geq \alpha t^2,$$

$$\frac{2\alpha(2\alpha-1)(\alpha-1)t^2}{t^4 + [2(\alpha-1)^2 + 2Y^2]\, t^2 + \big((\alpha-1)^2 - Y^2\big)^2} \geq \frac{15}{4}\frac{\alpha t^2}{\big(Y^2-(\alpha-1)^2\big)^2} \geq \alpha t^2,$$

$$\frac{-2\alpha^2(1-\alpha)^3}{t^4 + [2(\alpha-1)^2 + 2Y^2]\, t^2 + \big((\alpha-1)^2 - Y^2\big)^2} \geq \frac{-4\alpha^2}{\big(Y^2-(\alpha-1)^2\big)^2} \geq -\alpha^2,$$

where we used for the second and third estimate that

$$\frac{15}{4}\frac{1}{\big(Y^2-(\alpha-1)^2\big)^2} \leq \frac{15}{4}\frac{1}{\big(4-\frac{25}{16}\big)^2} \leq 1 \quad \text{and} \quad \frac{4}{\big(Y^2-(\alpha-1)^2\big)^2} \leq \frac{4}{\big(4-\frac{25}{16}\big)^2} \leq 1.$$

Therefore we obtain

$$F = 2 + \frac{F_1+F_2+F_3}{t^2+\alpha^2} \geq 2 + \frac{2\alpha t^2 - \alpha^2}{t^2+\alpha^2} \geq 1 + 2\alpha > \frac{1}{2},$$

meaning that $C = \frac{1}{2}$ is an admissible choice. □

Finally, we have to absorb the term $T^{(1)}_{5,A_2}$ (see Equation (6.5)). Recall that

$$\begin{aligned} T^{(1)}_{5,A_2} &= -\int_\Omega A_2 : \nabla \mathbf{u}\, \mathrm{d}x \\ &= \int_\Omega r^{-2}(\partial_\varphi \Phi_1)\left(r^{-1}(\partial_\varphi u_r - u_\varphi) + \partial_r u_\varphi\right)\, \mathrm{d}x - \int_\Omega (r^{-1}\partial_r \Phi_1)\big(r^{-1}(\partial_\varphi u_\varphi + u_r)\big)\, \mathrm{d}x. \end{aligned}$$

Lemma 6.1.7 (Estimate of $T^{(1)}_{5,A_2}$). *Let $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Then for $\mathbf{u} \in \mathcal{T}$ the following estimates hold*

$$\left|\int_\Omega r^{-2}(\partial_\varphi \Phi_1)\left(r^{-1}(\partial_\varphi u_r - u_\varphi) + \partial_r u_\varphi\right)\,\mathrm{d}x\right| \le \gamma_5 \|\nabla \mathbf{u}\|_\alpha^2 + \frac{C_5\theta^2}{2\gamma_5}\|\nabla \mathbf{u}\|^2_\alpha,$$
$$\left|\int_\Omega (r^{-1}\partial_r\Phi_1)\big(r^{-1}(\partial_\varphi u_\varphi + u_r)\big)\,\mathrm{d}x\right| \le \frac{\gamma_6}{2}\|\nabla \mathbf{u}\|^2_\alpha + \frac{C_6\theta^4}{2\gamma_6}\|\nabla\mathbf{u}\|^2_\alpha,$$

where $C_5, C_6 > 0$ are universal constants arising from Lemma 4.3.3 and $\gamma_5, \gamma_6 > 0$ are arbitrary constants.

Proof. By the Cauchy-Schwarz inequality, Young's inequality and Lemma 4.3.3 we have

$$\begin{aligned}\left|\int_\Omega r^{-2}(\partial_\varphi \Phi_1)\left(r^{-1}(\partial_\varphi u_r - u_\varphi) + \partial_r u_\varphi\right)\,\mathrm{d}x\right| &\le \gamma_5\|\nabla\mathbf{u}\|^2_\alpha + \frac{1}{2\gamma_5}\int_0^\theta\int_0^\infty r^{2\alpha}(r^{-1}\partial_\varphi\Phi_1)\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\\ &\lesssim \gamma_5\|\nabla\mathbf{u}\|^2_\alpha + \frac{\theta^2}{2\gamma_5}\|\nabla\mathbf{u}\|^2_\alpha\end{aligned}$$

and

$$\left|\int_\Omega (r^{-1}\partial_r\Phi_1)r^{-1}(\partial_\varphi u_\varphi + u_r)\,\mathrm{d}x\right| \lesssim \frac{\gamma_6}{2}\|\nabla\mathbf{u}\|^2_\alpha + \frac{\theta^4}{2\gamma_6}\|\nabla\mathbf{u}\|^2_\alpha. \qquad \square$$

Combining the results of Lemma 6.1.2, 6.1.3, 6.1.5, 6.1.6 and 6.1.7 gives an incomplete coercivity estimate for B_1. Everything can be absorbed apart from the term with $\partial^2_\varphi u_r$ in Lemma 6.1.2.

Corollary 6.1.8. *Let $\mathbf{u} \in \mathcal{T}$ and assume $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Then there exists a constant $C > 0$ such that*

$$\begin{aligned}B_1(\mathbf{u},\mathbf{u}) \ge |u_r|^2_\alpha + \Bigg(&C - 4\theta^2(\alpha^2+1) - \frac{C_1\theta^4}{2\gamma_1} - \frac{C_2\theta^2}{2\gamma_2} - \gamma_2 - 2\theta^2\gamma_2(1+\alpha^{-2})\\ &- \frac{C_3\gamma_3\theta^2}{2} - \frac{\theta^2}{\gamma_3}(1+\alpha^{-2}) - \frac{C_4\gamma_4\alpha^2}{2} - \frac{\theta^2}{\gamma_4}(1+\alpha^{-2}) - \gamma_5 - \frac{C_5\theta^2}{2\gamma_5} - \frac{\gamma_6}{2} - \frac{C_6\theta^4}{2\gamma_6}\Bigg)\|\nabla\mathbf{u}\|^2_\alpha\\ &- \frac{\gamma_1}{2}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial^2_\varphi u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi,\end{aligned}$$

where $C_i > 0$ $(i = 1,\ldots,6)$ are universal constants arising from Lemma 4.3.3 and $\gamma_i > 0$ $(i = 1,\ldots,6)$ are constants arising from Young's inequality and can be chosen later.

6.2 An Incomplete Coercivity Estimate for B_2

The second bilinear form is quite similar to the first, so that we can reuse most of the work from the previous section. However, the commutation of $\partial_\mathbf{n}$ and $r\partial_r$ in T_1 and T_4 causes some extra boundary terms which should be absorbed by B_1. Consider the second bilinear form (B2)

$$\begin{aligned}B_2(\mathbf{u},\mathbf{u}) = &\int_{\partial\Omega'} r^{-2\alpha}(r\partial_r u_r)((r\partial_r+1)u_r)\,\mathrm{d}s + \int_\Omega r^{-2\alpha}|\nabla r\partial_r\mathbf{u}|^2\,\mathrm{d}x\\ &- 2\alpha\int_\Omega r^{-2\alpha-1}(r\partial_r\mathbf{u})\cdot(\partial_r r\partial_r\mathbf{u})\,\mathrm{d}x + \int_{\partial\Omega'}((r\partial_r+1)\partial_\mathbf{n} u_r)(\partial_r\Phi_2)\,\mathrm{d}s\\ &- \int_\Omega(\nabla\otimes\nabla\Phi_2) : (\nabla r\partial_r\mathbf{u})\,\mathrm{d}x =: \sum_{i=1}^5 T_i^{(2)},\end{aligned}$$

where Φ_2 (in the sense of Definition 4.2.1) satisfies

$$\begin{aligned} \Delta\Phi_2 &= -2\alpha r^{-2\alpha-1} r\partial_r u_r, && \text{in } \Omega,\\ \partial_{\mathbf{n}}\Phi_2 &= 0, && \text{on } \partial\Omega'. \end{aligned}$$

For $T_3^{(2)} + T_5^{(2)}$ we can reuse Lemma 6.1.3, 6.1.5, 6.1.6 and 6.1.7 to get the following estimate.

Lemma 6.2.1 (Estimate of $T_3^{(2)} + T_5^{(2)}$)**.** *Let $\mathbf{u} \in \mathcal{T}$ and assume that $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Then there exists a constant $D > 0$ such that*

$$\begin{aligned} T_3^{(2)} + T_5^{(2)} \geq \Bigg(D - 4\theta^2(\alpha^2+1) - \frac{D_3\delta_3\theta^2}{2} - \frac{\theta^2}{\delta_3}(1+\alpha^{-2}) \\ - \frac{D_4\delta_4\alpha^2}{2} - \frac{\theta^2}{\delta_4}(1+\alpha^{-2}) - \delta_5 - \frac{D_5\theta^2}{2\delta_5} - \frac{\delta_6}{2} - \frac{D_6\theta^4}{2\delta_6} \Bigg) \|\nabla r\partial_r \mathbf{u}\|_\alpha^2, \end{aligned}$$

where $D_3, \ldots, D_6$ are the constants arising from Lemma 4.3.3 and $\delta_3, \ldots, \delta_6 > 0$ are arbitrary constants. Specifically, $D = \frac{1}{2}$ is an admissible choice.

Remark 6.2.2. *Note that the constants D and D_i, $i = 1, \ldots 6$ can be chosen equal to the constants C and C_i introduced for B_1. However, to make clear which constants come from which bilinear form we will use different notation.*

We now focus on the boundary terms which are slightly different. In $T_1^{(2)}$ the term with $(r\partial_r u_r)^2$ is positive and the other term can be absorbed in $T_1^{(1)}$ of B_1.

Lemma 6.2.3 (Reformulation of remainder $T_1^{(2)}$)**.** *For $\alpha \neq 0$ and $\mathbf{u} \in \mathcal{T}$ the following estimate holds*

$$\int_{\partial\Omega'} r^{-2\alpha}(r\partial_r u_r)u_r \,\mathrm{d}s = (\alpha - \tfrac{1}{2}) \int_{\partial\Omega'} r^{-2\alpha} u_r^2 \,\mathrm{d}s.$$

Proof. This is immediate with integration by parts

$$\int_{\partial\Omega'} r^{-2\alpha}(r\partial_r u_r)u_r \,\mathrm{d}s = \frac{1}{2}\int_{\partial\Omega'} r^{-2\alpha+1} r\partial_r(u_r^2)\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi = (\alpha - \tfrac{1}{2}) \int_{\partial\Omega'} r^{-2\alpha} u_r^2 \,\mathrm{d}s. \qquad \square$$

Lemma 6.2.4 (Estimate of $T_4^{(2)}$)**.** *Let $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Then for $\mathbf{u} \in \mathcal{T}$ the following estimate holds*

$$\begin{aligned} \left|T_4^{(2)}\right| \leq \frac{\delta_1}{2}\int_0^\theta \int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + (\delta_2 + 2\delta_2\theta^2(1+\alpha^{-2}))\|\nabla \mathbf{u}\|_\alpha^2 \\ + \left(\frac{D_1\alpha^2\theta^2}{2\delta_1} + \frac{D_2\alpha^2}{2\delta_2}\right)\|\nabla r\partial_r \mathbf{u}\|_\alpha^2, \end{aligned}$$

where $D_1, D_2 > 0$ are universal constants arising from Lemma 4.3.3 and $\delta_1, \delta_2 > 0$ are arbitrary constants.

Proof. By the fundamental theorem of calculus we obtain

$$\begin{aligned} T_4^{(2)} &= \int_{\partial\Omega'} (\partial_{\mathbf{n}} r\partial_r u_r)(\partial_r\Phi_2)\,\mathrm{d}s \\ &= \int_0^\infty \left((\partial_\varphi r\partial_r u_r)(\partial_r\Phi_2)\big|_{\varphi=\theta} - (\partial_\varphi r\partial_r u_r)(\partial_r\Phi_2)\big|_{\varphi=0} \right)\frac{\mathrm{d}r}{r} \\ &= \underbrace{\int_0^\theta \int_0^\infty (r\partial_r\partial_\varphi^2 u_r)(\partial_r\Phi_2)\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi}_{=:P_1} + \underbrace{\int_0^\theta \int_0^\infty (r\partial_r\partial_\varphi u_r)(\partial_r\partial_\varphi\Phi_2)\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi}_{=:P_2}. \end{aligned}$$

Then applying integration by parts gives

$$\begin{aligned}
|P_1| &= \left| - \int_0^\theta \int_0^\infty (\partial_\varphi^2 u_r)(r\partial_r^2 \Phi_2) \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \right| \\
&\leq \frac{\delta_1}{2} \int_0^\theta \int_0^\infty r^{-2\alpha} (\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi + \frac{1}{2\delta_1} \int_0^\theta \int_0^\infty r^{2\alpha} (r\partial_r^2 \Phi_2)^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \\
&\lesssim \frac{\delta_1}{2} \int_0^\theta \int_0^\infty r^{-2\alpha} (\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi + \frac{\alpha^2\theta^2}{2\delta_1} \|\nabla r\partial_r \mathbf{u}\|_\alpha^2,
\end{aligned}$$

where we have applied Lemma 4.3.3 with $\mathbf{v} = r\partial_r \mathbf{u}$. Similarly, for the second term we get

$$\begin{aligned}
|P_2| &= \left| - \int_0^\theta \int_0^\infty (\partial_\varphi u_r)(r\partial_r^2 \partial_\varphi \Phi_2) \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \right| \\
&\leq \frac{\delta_2}{2} \int_0^\theta \int_0^\infty r^{-2\alpha} (\partial_\varphi u_r)^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi + \frac{1}{2\delta_2} \int_0^\theta \int_0^\infty r^{2\alpha} (r\partial_r^2 \partial_\varphi \Phi_2)^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \\
&\overset{(6.4)}{\lesssim} (\delta_2 + 2\delta_2\theta^2(1+\alpha^{-2})) \|\nabla \mathbf{u}\|_\alpha^2 + \frac{\alpha^2}{2\delta_2} \|\nabla r\partial_r \mathbf{u}\|_\alpha^2. \qquad \square
\end{aligned}$$

Combining Lemma 6.2.1, 6.2.3 and 6.2.4 gives the following (still incomplete) coercivity estimate for B_2.

Corollary 6.2.5. *Let $\mathbf{u} \in \mathcal{T}$ and assume that $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Then there exists a constant $D > 0$ such that*

$$\begin{aligned}
B_2(\mathbf{u}, \mathbf{u}) \geq{}& |r\partial_r u_r|_\alpha^2 + (\alpha - \tfrac{1}{2})|u_r|_\alpha^2 + \bigg(D - 4\theta^2(\alpha^2+1) - \frac{D_1\alpha^2\theta^2}{2\delta_1} - \frac{D_2\alpha^2}{2\delta_2} \\
&- \frac{D_3\delta_3\theta^2}{2} - \frac{\theta^2}{\delta_3}(1+\alpha^{-2}) - \frac{D_4\delta_4\alpha^2}{2} - \frac{\theta^2}{\delta_4}(1+\alpha^{-2}) - \delta_5 - \frac{D_5\theta^2}{2\delta_5} \\
&- \frac{\delta_6}{2} - \frac{D_6\theta^4}{2\delta_6} \bigg) \|\nabla r\partial_r \mathbf{u}\|_\alpha^2 - (\delta_2 + 2\delta_2\theta^2(1+\alpha^{-2})) \|\nabla \mathbf{u}\|_\alpha^2 \\
&- \frac{\delta_1}{2} \int_0^\theta \int_0^\infty r^{-2\alpha} (\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi,
\end{aligned}$$

where $D_i > 0$ $(i = 1, \dots, 6)$ are universal constants arising from Lemma 4.3.3 and $\delta_i > 0$ $(i = 1, \dots, 6)$ are constants arising from Young's inequality and can be chosen later.

6.3 An Incomplete Coercivity Estimate for B_3

Recall the notation

$$\omega_\mathbf{u} := \operatorname{curl} \mathbf{u} = r^{-1}\big((r\partial_r + 1)u_\varphi - \partial_\varphi u_r\big).$$

For coercivity of the vorticity bilinear form (B3), consider

$$\begin{aligned}
B_3(\mathbf{u}, \mathbf{u}) ={}& \int_\Omega r^{-2\alpha+2} |\nabla \omega_\mathbf{u}|^2 \, \mathrm{d}x + (2 - 2\alpha) \int_\Omega r^{-2\alpha+1} \omega_\mathbf{u} \partial_r \omega_\mathbf{u} \, \mathrm{d}x \\
&+ \int_{\mathrm{Re}\lambda=\alpha} (\lambda - 1) \left[\overline{\partial_\varphi \widehat{u_r}(\lambda, 0)} \widehat{\Phi}_3(\lambda - 1, 0) - \overline{\partial_\varphi \widehat{u_r}(\lambda, \theta)} \Phi_3(\lambda - 1, \theta) \right] \mathrm{dIm}\lambda = \sum_{i=1}^3 T_i^{(3)},
\end{aligned}$$

where Φ_3 satisfies (in the sense of Definition 4.2.1)

$$\begin{aligned}
\Delta \Phi_3 &= 0 && \text{in } \Omega, \\
\partial_\mathbf{n} \Phi_3 &= \mathbf{n} \cdot \Delta \mathbf{u} && \text{on } \partial\Omega'.
\end{aligned}$$

Below we derive estimates for $T_2^{(3)}$ and $T_3^{(3)}$. Again we obtain some terms that can directly be absorbed into $T_1^{(3)}$, but also remainder terms appear which need to be absorbed into B_1 and B_2.

Lemma 6.3.1 (Estimate of $T_2^{(3)}$)**.** *Let $\alpha \neq 0$ and $\mathbf{u} \in \mathcal{T}$. Then the following estimate holds*

$$\left|T_2^{(3)}\right| = \left|(2-2\alpha)\int_\Omega r^{-2\alpha+1}\omega_{\mathbf{u}}\partial_r\omega_{\mathbf{u}}\,\mathrm{d}x\right| \leq 4(\alpha-1)^2\|\nabla\mathbf{u}\|_\alpha^2.$$

Proof. Integration by parts gives

$$\begin{aligned}(2-2\alpha)\int_\Omega r^{-2\alpha+1}\omega_{\mathbf{u}}\partial_r\omega_{\mathbf{u}}\,\mathrm{d}x &= (1-\alpha)\int_0^\theta\int_0^\infty r^{-2\alpha+2}r\partial_r(\omega_{\mathbf{u}}^2)\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\\ &= -2(\alpha-1)^2\int_0^\theta\int_0^\infty r^{-2\alpha+2}\omega_{\mathbf{u}}^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\\ &= -2(\alpha-1)^2\int_0^\theta\int_0^\infty r^{-2\alpha}(r\partial_r u_\varphi-(\partial_\varphi u_r-u_\varphi))^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi.\end{aligned}$$

Hence,

$$\begin{aligned}\left|T_2^{(3)}\right| &\leq 4(\alpha-1)^2\int_0^\theta\int_0^\infty r^{-2\alpha}(r\partial_r u_\varphi)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + 4(\alpha-1)^2\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi u_r-u_\varphi)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\\ &\leq 4(\alpha-1)^2\int_\Omega r^{-2\alpha}|\nabla\mathbf{u}|^2\,\mathrm{d}x.\end{aligned}$$

□

Lemma 6.3.2 (Estimate of $T_3^{(3)}$)**.** *Let $-\frac{1}{4} < \alpha < 0$ and $0 < \theta < \frac{\pi}{2}$. Then for $\mathbf{u} \in \mathcal{T}$ the following estimate holds*

$$\begin{aligned}\left|T_3^{(3)}\right| \leq &\left(\frac{8\varepsilon_1}{\theta}+\frac{4\varepsilon_2}{\theta}\right)\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \left(\frac{12}{\theta^2}+\frac{8}{\varepsilon_1\theta}+\frac{16}{\varepsilon_2\theta}\right)\|\nabla r\partial_r\mathbf{u}\|_\alpha^2\\ &+\left(\frac{18}{\theta^2}+(1+\alpha^{-2})\left(24+\frac{16\varepsilon_1}{\theta}+\frac{32\theta}{\varepsilon_2}\right)+\frac{8}{\varepsilon_1\theta}+\frac{16}{\varepsilon_2\theta}\right)\|\nabla\mathbf{u}\|_\alpha^2,\end{aligned}$$

where $\varepsilon_1, \varepsilon_2 > 0$ are arbitrary constants.

Proof. By applying Plancherel's identity and using the expression (5.26) for $\widehat{\Phi}_3(\lambda,\varphi)$ with $\varphi \in \{0,\theta\}$ we obtain

$$\begin{aligned}T_3^{(3)} = &\int_{\mathrm{Re}\lambda=\alpha}(\lambda-1)\overline{\partial_\varphi\widehat{u_r}(\lambda,0)}\left[-\frac{\partial_\varphi\widehat{u_r}(\lambda,0)}{\sin\big((\lambda-1)\theta\big)}\cos\big((\lambda-1)\theta\big)+\frac{\partial_\varphi\widehat{u_r}(\lambda,\theta)}{\sin\big((\lambda-1)\theta\big)}\right]\mathrm{dIm}\lambda\\ &-\int_{\mathrm{Re}\lambda=\alpha}(\lambda-1)\overline{\partial_\varphi\widehat{u_r}(\lambda,\theta)}\left[-\frac{\partial_\varphi\widehat{u_r}(\lambda,0)}{\sin\big((\lambda-1)\theta\big)}+\frac{\partial_\varphi\widehat{u_r}(\lambda,\theta)}{\sin\big((\lambda-1)\theta\big)}\cos\big((\lambda-1)\theta\big)\right]\mathrm{dIm}\lambda\\ = &-\int_{\mathrm{Re}\lambda=\alpha}(\lambda-1)\frac{\cos\big((\lambda-1)\theta\big)}{\sin\big((\lambda-1)\theta\big)}\left[|\partial_\varphi\widehat{u_r}(\lambda,\theta)|^2+|\partial_\varphi\widehat{u_r}(\lambda,0)|^2\right]\mathrm{dIm}\lambda\\ &+\int_{\mathrm{Re}\lambda=\alpha}(\lambda-1)\frac{1}{\sin\big((\lambda-1)\theta\big)}\left[\overline{\partial_\varphi\widehat{u_r}(\lambda,0)}\partial_\varphi\widehat{u_r}(\lambda,\theta)+\overline{\partial_\varphi\widehat{u_r}(\lambda,\theta)}\partial_\varphi\widehat{u_r}(\lambda,0)\right]\mathrm{dIm}\lambda\\ =: &\,M_1+M_2.\end{aligned}\tag{6.11}$$

To estimate those integrals we integrate in the angle and use the estimates

$$\left|\frac{\cos\big((\lambda-1)\theta\big)}{\sin\big((\lambda-1)\theta\big)}\right| \leq \frac{2}{\theta} \quad \text{and} \quad \left|\frac{1}{\sin\big((\lambda-1)\theta\big)}\right| \leq \frac{2}{\theta}.\tag{6.12}$$

To prove these two preliminary estimates, write $\lambda = \alpha + it$ with $t \in \mathbb{R}$. Then

$$\begin{aligned}
|\cos\big((\lambda-1)\theta\big)|^2 &= \tfrac{1}{4}\big|e^{i(\alpha-1)\theta}e^{-t\theta} + e^{-i(\alpha-1)\theta}e^{t\theta}\big|^2 \\
&= \tfrac{1}{4}\big|\cos\big((\alpha-1)\theta\big)\big[e^{-t\theta}+e^{t\theta}\big] + i\sin\big((\alpha-1)\theta\big)\big[e^{t\theta}-e^{-t\theta}\big]\big|^2 \\
&= \cos^2\big((\alpha-1)\theta\big)\cosh^2(t\theta) + \sin^2\big((\alpha-1)\theta\big)\sinh^2(t\theta), \\
|\sin\big((\lambda-1)\theta\big)|^2 &= \tfrac{1}{4}\big|e^{i(\alpha-1)\theta}e^{-t\theta} - e^{-i(\alpha-1)\theta}e^{t\theta}\big|^2 \\
&= \tfrac{1}{4}\big|\cos\big((\alpha-1)\theta\big)\big[e^{-t\theta}-e^{t\theta}\big] + i\sin\big((\alpha-1)\theta\big)\big[e^{t\theta}+e^{-t\theta}\big]\big|^2 \\
&= \cos^2\big((\alpha-1)\theta\big)\sinh^2(t\theta) + \sin^2\big((\alpha-1)\theta\big)\cosh^2(t\theta),
\end{aligned}$$

and therefore

$$\begin{aligned}
\left|\frac{1}{\sin\big((\lambda-1)\theta\big)}\right|^2 &\le \frac{1}{\sin^2\big((\alpha-1)\theta\big)\cosh^2(t\theta)} \le \frac{1}{\sin^2\big((\alpha-1)\theta\big)} \le \frac{1}{\sin^2(\theta)} \\
\left|\frac{\cos\big((\lambda-1)\theta\big)}{\sin\big((\lambda-1)\theta\big)}\right|^2 &\le \frac{\cos^2\big((\alpha-1)\theta\big)}{\sin^2\big((\alpha-1)\theta\big)}\cdot\frac{\cosh^2(t\theta)}{\cosh^2(t\theta)} + \frac{\sin^2\big((\alpha-1)\theta\big)}{\sin^2\big((\alpha-1)\theta\big)}\cdot\frac{\sinh^2(t\theta)}{\cosh^2(t\theta)} \\
&\le \frac{2}{\sin^2\big((\alpha-1)\theta\big)} \le \frac{2}{\sin^2(\theta)}.
\end{aligned}$$

For $0 < \theta < \frac{\pi}{2}$ it holds that $\sin(\theta) \ge \frac{1}{2}\theta$ and in particular the estimates (6.12) hold true with the same constant.

Note that both M_1 and M_2 in (6.11) contain a term which is evaluated at 0 and one at θ, but both these terms can be treated similarly. We write $M_1 = M_1^{(0)} + M_1^{(\theta)}$ and consider $M_1^{(0)}$ which is the part of M_1 evaluated at $\varphi = 0$. Note that $\widehat{u_\varphi} = 0$ on the boundary, so we can add this to the integral. Then using the fundamental theorem of calculus gives

$$\begin{aligned}
M_1^{(0)} &= \int_{\mathrm{Re}\lambda=\alpha}\int_0^\theta -(\lambda-1)\frac{\cos\big((\lambda-1)\theta\big)}{\sin\big((\lambda-1)\theta\big)}\partial_\varphi\left(\frac{\varphi-\theta}{\theta}|\partial_\varphi\widehat{u_r}-\widehat{u_\varphi}|^2\right)\mathrm{d}\varphi\,\mathrm{dIm}\lambda \\
&= \int_{\mathrm{Re}\lambda=\alpha}\int_0^\theta -(\lambda-1)\frac{\cos\big((\lambda-1)\theta\big)}{\sin\big((\lambda-1)\theta\big)}\left(\frac{1}{\theta}|\partial_\varphi\widehat{u_r}-\widehat{u_\varphi}|^2 + \frac{\varphi-\theta}{\theta}\partial_\varphi|\partial_\varphi\widehat{u_r}-\widehat{u_\varphi}|^2\right)\mathrm{d}\varphi\,\mathrm{dIm}\lambda,
\end{aligned}$$

for which we have the estimate

$$\begin{aligned}
&\left|\frac{1}{\theta}\int_{\mathrm{Re}\lambda=\alpha}\int_0^\theta -(\lambda-1)\frac{\cos\big((\lambda-1)\theta\big)}{\sin\big((\lambda-1)\theta\big)}|\partial_\varphi\widehat{u_r}-\widehat{u_\varphi}|^2\,\mathrm{d}\varphi\,\mathrm{dIm}\lambda\right| \\
&\overset{(6.12)}{\le} \frac{2}{\theta^2}\int_{\mathrm{Re}\lambda=\alpha}\int_0^\theta |\lambda-1||\partial_\varphi\widehat{u_r}-\widehat{u_\varphi}|^2\,\mathrm{d}\varphi\,\mathrm{dIm}\lambda \\
&\le \frac{1}{\theta^2}\int_0^\theta\int_{\mathrm{Re}\lambda=\alpha}|\lambda-1|^2|\partial_\varphi\widehat{u_r}-\widehat{u_\varphi}|^2\,\mathrm{dIm}\lambda\,\mathrm{d}\varphi + \frac{1}{\theta^2}\int_0^\theta\int_{\mathrm{Re}\lambda=\alpha}|\partial_\varphi\widehat{u_r}-\widehat{u_\varphi}|^2\,\mathrm{dIm}\lambda\,\mathrm{d}\varphi \\
&= \frac{1}{\theta^2}\int_0^\theta\int_0^\infty r^{-2\alpha}|(r\partial_r-1)(\partial_\varphi u_r-u_\varphi)|^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{1}{\theta^2}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi u_r-u_\varphi)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\
&\le \frac{2}{\theta^2}\|\nabla r\partial_r\mathbf{u}\|_\alpha^2 + \frac{3}{\theta^2}\|\nabla\mathbf{u}\|_\alpha^2 \qquad (6.13)
\end{aligned}$$

and using that $\left|\frac{\varphi-\theta}{\theta}\right| \leq 1$ and $\partial_\varphi |f(\varphi)|^2 = f(\varphi)\partial_\varphi \overline{f(\varphi)} + \overline{f(\varphi)}\partial_\varphi f(\varphi)$ we get the estimate

$$
\begin{aligned}
&\left| \int_{\mathrm{Re}\lambda=\alpha} \int_0^\theta -(\lambda-1)\frac{\cos\big((\lambda-1)\theta\big)}{\sin\big((\lambda-1)\theta\big)} \frac{\varphi-\theta}{\theta} \partial_\varphi |\partial_\varphi \widehat{u_r} - \widehat{u_\varphi}|^2 \,\mathrm{d}\varphi \,\mathrm{dIm}\lambda \right| \\
&\overset{(6.12)}{\leq} \frac{2\varepsilon_1}{\theta} \int_{\mathrm{Re}\lambda=\alpha} \int_0^\theta |\partial_\varphi(\partial_\varphi \widehat{u_r} - \widehat{u_\varphi})|^2 \,\mathrm{d}\varphi \,\mathrm{dIm}\lambda + \frac{2}{\theta\varepsilon_1} \int_{\mathrm{Re}\lambda=\alpha} \int_0^\theta |\lambda-1|^2 |\partial_\varphi \widehat{u_r} - \widehat{u_\varphi}|^2 \,\mathrm{d}\varphi \,\mathrm{dIm}\lambda \\
&\leq \frac{4\varepsilon_1}{\theta} \int_0^\theta \int_0^\infty r^{-2\alpha} (\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi + \frac{4\varepsilon_1}{\theta} \int_0^\theta \int_0^\infty r^{-2\alpha} (\underbrace{\partial_\varphi u_\varphi}_{=-(r\partial_r+1)u_r})^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \\
&\quad + \frac{2}{\varepsilon_1\theta} \int_0^\theta \int_0^\infty r^{-2\alpha} |(r\partial_r - 1)(\partial_\varphi u_r - u_\varphi)|^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \\
&\leq \frac{4\varepsilon_1}{\theta} \int_0^\theta \int_0^\infty r^{-2\alpha} (\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi + \left(\frac{8\varepsilon_1}{\theta}(1+\alpha^{-2}) + \frac{4}{\varepsilon_1\theta} \right) \|\nabla \mathbf{u}\|_\alpha^2 + \frac{4}{\varepsilon_1\theta} \|\nabla r\partial_r \mathbf{u}\|_\alpha^2,
\end{aligned} \tag{6.14}
$$

where in the last step Hardy's inequality is used. The part $M_1^{(\theta)}$ of M_1 that is evaluated at $\varphi = \theta$ is similar (replace $\frac{\varphi-\theta}{\theta}$ by $\frac{\varphi}{\theta}$), so we get the above estimates (6.13) and (6.14) twice. Therefore,

$$
\begin{aligned}
|M_1| \leq{}& \frac{8\varepsilon_1}{\theta} \int_0^\theta \int_0^\infty r^{-2\alpha} (\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi + \left(\frac{6}{\theta^2} + \frac{16\varepsilon_1}{\theta}(1+\alpha^{-2}) + \frac{8}{\varepsilon_1\theta} \right) \|\nabla \mathbf{u}\|_\alpha^2 \\
&+ \left(\frac{4}{\theta^2} + \frac{8}{\varepsilon_1\theta} \right) \|\nabla r\partial_r \mathbf{u}\|_\alpha^2.
\end{aligned} \tag{6.15}
$$

For M_2 in (6.11) we write $M_2 = M_2^{(0)} + M_2^{(\theta)}$ and consider $M_2^{(0)}$ which is the part of M_2 where the function $\overline{\partial_\varphi \widehat{u_r}(\lambda,\varphi)} \partial_\varphi u_r(\lambda, \theta-\varphi)$ is evaluated at $\varphi = 0$. We get

$$
\begin{aligned}
M_2^{(0)} &= \int_{\mathrm{Re}\lambda=\alpha} (\lambda-1) \frac{1}{\sin\big((\lambda-1)\theta\big)} \overline{\partial_\varphi \widehat{u_r}(\lambda,0)} \partial_\varphi \widehat{u_r}(\lambda,\theta) \,\mathrm{dIm}\lambda \\
&= \int_{\mathrm{Re}\lambda=\alpha} \int_0^\theta (\lambda-1) \frac{1}{\sin\big((\lambda-1)\theta\big)} \partial_\varphi \left(\frac{\varphi-\theta}{\theta} \overline{\partial_\varphi \widehat{u_r}(\lambda,\varphi)} \partial_\varphi \widehat{u_r}(\lambda,\theta-\varphi) \right) \mathrm{d}\varphi \,\mathrm{dIm}\lambda \\
&= \int_{\mathrm{Re}\lambda=\alpha} \int_0^\theta (\lambda-1) \frac{1}{\sin\big((\lambda-1)\theta\big)} \left(\frac{1}{\theta} \overline{\partial_\varphi \widehat{u_r}(\lambda,\varphi)} \partial_\varphi \widehat{u_r}(\lambda,\theta-\varphi) + \right. \\
&\quad \left. \frac{\varphi-\theta}{\theta} (\partial_\varphi^2 \overline{\widehat{u_r}(\lambda,\varphi)}) \partial_\varphi \widehat{u_r}(\lambda,\theta-\varphi) - \frac{\varphi-\theta}{\theta} \overline{\partial_\varphi \widehat{u_r}(\lambda,\varphi)} (\partial_\varphi^2 \widehat{u_r})(\lambda,\theta-\varphi) \right) \mathrm{d}\varphi \,\mathrm{dIm}\lambda.
\end{aligned} \tag{6.16}
$$

In the following estimates it is used that

$$
\begin{aligned}
\int_0^\theta \int_0^\infty r^{-2\alpha} \big((r\partial_r - 1)\partial_\varphi u_r\big)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \leq{}& 4 \int_0^\theta \int_0^\infty r^{-2\alpha} \big(\partial_\varphi r\partial_r u_r - r\partial_r u_\varphi\big)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \\
&+ 4 \int_0^\theta \int_0^\infty r^{-2\alpha} \left((r\partial_r u_\varphi)^2 + (\partial_\varphi u_r - u_\varphi)^2 \right) \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \\
&+ 4 \int_0^\theta \int_0^\infty r^{-2\alpha} u_\varphi^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi \\
\overset{\text{Lemma 6.1.1}}{\leq}{}& 4\|\nabla r\partial_r \mathbf{u}\|_\alpha^2 + \big(4 + 8\theta^2(1+\alpha^{-2})\big) \|\nabla \mathbf{u}\|_\alpha^2.
\end{aligned} \tag{6.17}
$$

For the first term in (6.16) we have the estimate

$$
\begin{aligned}
&\left|\frac{1}{\theta}\int_{\mathrm{Re}\lambda=\alpha}\int_0^\theta \frac{\lambda-1}{\sin\left((\lambda-1)\theta\right)}\overline{\partial_\varphi\widehat{u_r}(\lambda,\varphi)}\partial_\varphi\widehat{u_r}(\lambda,\theta-\varphi)\,\mathrm{d}\varphi\,\mathrm{dIm}\lambda\right| \\
&\overset{(6.12)}{\leq} \frac{2}{\theta^2}\int_{\mathrm{Re}\lambda=\alpha}\int_0^\infty |\lambda-1||\overline{\partial_\varphi\widehat{u_r}(\lambda,\varphi)}\partial_\varphi\widehat{u_r}(\lambda,\theta-\varphi)|\,\mathrm{d}\varphi\,\mathrm{dIm}\lambda \\
&\leq \frac{1}{\theta^2}\int_0^\theta\int_0^\infty r^{-2\alpha}((r\partial_r-1)\partial_\varphi u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{1}{\theta^2}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\
&\leq \frac{4}{\theta^2}\|\nabla r\partial_r\mathbf{u}\|_\alpha^2 + \left(\frac{6}{\theta^2}+12(1+\alpha^{-2})\right)\|\nabla\mathbf{u}\|_\alpha^2,
\end{aligned}
\tag{6.18}
$$

using (6.4), (6.17) and Lemma 6.1.1. The second term in (6.16) can be estimated as follows

$$
\begin{aligned}
&\left|\int_{\mathrm{Re}\lambda=\alpha}\int_0^\theta \frac{\lambda-1}{\sin\left((\lambda-1)\theta\right)}\frac{\varphi-\theta}{\theta}(\partial_\varphi\overline{\partial_\varphi\widehat{u_r}(\lambda,\varphi)})\partial_\varphi\widehat{u_r}(\lambda,\theta-\varphi)\,\mathrm{d}\varphi\,\mathrm{dIm}\lambda\right| \\
&\overset{(6.12)}{\leq} \frac{2}{\theta}\int_{\mathrm{Re}\lambda=\alpha}\int_0^\theta |\lambda-1||(\partial_\varphi^2\overline{\widehat{u_r}(\lambda,\varphi)})(\partial_\varphi\widehat{u_r}(\lambda,\theta-\varphi))|\,\mathrm{d}\varphi\,\mathrm{dIm}\lambda \\
&\leq \frac{\varepsilon_2}{\theta}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{1}{\varepsilon_2\theta}\int_0^\theta\int_0^\infty r^{-2\alpha}((r\partial_r-1)\partial_\varphi u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\
&\overset{(6.17)}{\leq} \frac{\varepsilon_2}{\theta}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{4}{\varepsilon_2\theta}\|\nabla r\partial_r\mathbf{u}\|_\alpha^2 + \left(\frac{4}{\varepsilon_2\theta}+\frac{8\theta}{\varepsilon_2}(1+\alpha^{-2})\right)\|\nabla\mathbf{u}\|_\alpha^2
\end{aligned}
\tag{6.19}
$$

and similarly for the third term in (6.16)

$$
\begin{aligned}
&\left|\int_{\mathrm{Re}\lambda=\alpha}\int_0^\theta \frac{\lambda-1}{\sin\left((\lambda-1)\theta\right)}\frac{\varphi-\theta}{\theta}(\overline{\partial_\varphi\widehat{u_r}(\lambda,\varphi)})\partial_\varphi^2\widehat{u_r}(\lambda,\theta-\varphi)\,\mathrm{d}\varphi\,\mathrm{dIm}\lambda\right| \\
&\overset{(6.12)}{\leq} \frac{2}{\theta}\int_{\mathrm{Re}\lambda=\alpha}\int_0^\theta |\lambda-1||(\overline{\partial_\varphi\widehat{u_r}(\lambda,\varphi)})\partial_\varphi^2\widehat{u_r}(\lambda,\theta-\varphi)|\,\mathrm{d}\varphi\,\mathrm{dIm}\lambda \\
&\leq \frac{\varepsilon_2}{\theta}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{1}{\varepsilon_2\theta}\int_0^\theta\int_0^\infty r^{-2\alpha}((r\partial_r-1)\partial_\varphi u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\
&\overset{(6.17)}{\leq} \frac{\varepsilon_2}{\theta}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \frac{4}{\varepsilon_2\theta}\|\nabla r\partial_r\mathbf{u}\|_\alpha^2 + \left(\frac{4}{\varepsilon_2\theta}+\frac{8\theta}{\varepsilon_2}(1+\alpha^{-2})\right)\|\nabla\mathbf{u}\|_\alpha^2.
\end{aligned}
\tag{6.20}
$$

The part $M_2^{(\theta)}$ of M_2 where $\overline{\partial_\varphi\widehat{u_r}(\lambda,\varphi)}\partial_\varphi u_r(\lambda,\theta-\varphi)$ is evaluated at $\varphi=\theta$ is similar by symmetry. Therefore, we get the above estimates (6.18)-(6.20) twice which gives the following estimate for M_2

$$
\begin{aligned}
|M_2| \leq{}& \frac{4\varepsilon_2}{\theta}\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2\,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + \left(\frac{12}{\theta^2}+\left(24+\frac{32\theta}{\varepsilon_2}\right)(1+\alpha^{-2})+\frac{16}{\varepsilon_2\theta}\right)\|\nabla\mathbf{u}\|_\alpha^2 \\
&+\left(\frac{8}{\theta^2}+\frac{16}{\varepsilon_2\theta}\right)\|\nabla r\partial_r\mathbf{u}\|_\alpha^2.
\end{aligned}
\tag{6.21}
$$

Adding the estimates for M_1 and M_2 in (6.15) and (6.21) gives the result. □

From Lemma 6.3.1 and 6.3.2 above we find the last missing estimate required for proving coercivity in the next section.

Corollary 6.3.3. *Let* $\mathbf{u} \in \mathcal{T}$ *and assume* $-\frac{1}{4} < \alpha < 0$ *and* $0 < \theta < \frac{\pi}{2}$*. Then*

$$\begin{aligned} B_3(\mathbf{u},\mathbf{u}) \geq & \int_\Omega r^{-2\alpha+2}|\nabla\omega_{\mathbf{u}}|^2 \,\mathrm{d}x - \left(\frac{8\varepsilon_1}{\theta} + \frac{4\varepsilon_2}{\theta}\right)\int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ & - \left(\frac{12}{\theta^2} + \frac{8}{\varepsilon_1\theta} + \frac{16}{\varepsilon_2\theta}\right)\|\nabla r\partial_r\mathbf{u}\|_\alpha^2 \\ & - \left(\frac{18}{\theta^2} + (1+\alpha^{-2})\left(24 + \frac{16\varepsilon_1}{\theta} + \frac{32\theta}{\varepsilon_2}\right) + \frac{8}{\varepsilon_1\theta} + \frac{16}{\varepsilon_2\theta} - 4(\alpha-1)^2\right)\|\nabla\mathbf{u}\|_\alpha^2, \end{aligned}$$

where $\varepsilon_1, \varepsilon_2 > 0$ *are constants arising from Young's inequality and can be chosen later.*

6.4 Proof of Propositions 5.3.1 and 5.3.2

We are now in the position to prove the coercivity estimate in Proposition 5.3.1 by combining Corollaries 6.1.8, 6.2.5 and 6.3.3. The strategy is to absorb the problematic terms with $\partial_\varphi^2 u_r$ from B_1 and B_2 into B_3 using the estimate

$$\begin{aligned} & \int_0^\theta\int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \qquad (6.22) \\ & \leq 2\int_0^\theta\int_0^\infty r^{-2\alpha}\big(\partial_\varphi^2 u_r + (r\partial_r+1)^2 u_r\big)^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + 2\int_0^\theta\int_0^\infty r^{-2\alpha}\big((r\partial_r+1)^2 u_r\big)^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ & \overset{(3.4c)}{\leq} 2\int_0^\theta\int_0^\infty r^{-2\alpha}\big(\partial_\varphi^2 u_r - (r\partial_r+1)\partial_\varphi u_\varphi\big)^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi + 2\int_0^\theta\int_0^\infty r^{-2\alpha}\big(((r\partial_r)^2 + 2r\partial_r + 1)u_r\big)^2 \frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi \\ & \leq 2\|r\nabla\omega_{\mathbf{u}}\|_\alpha^2 + 6\|\nabla r\partial_r\mathbf{u}\|_\alpha^2 + 6(4+\alpha^{-2})\|\nabla\mathbf{u}\|_\alpha^2, \end{aligned}$$

where in the last step we used (A.7), the inequality $(a+b+c)^2 \leq 3a^2+3b^2+3c^2$ and Hardy's inequality. By estimating the $\partial_\varphi^2 u_r$ terms, also terms in B_1 and B_2 appear, but we already have control on $\|\nabla\mathbf{u}\|_\alpha^2$ and $\|\nabla r\partial_r\mathbf{u}\|_\alpha^2$ so this will not cause new problems. The other remainder terms that were created by introducing the additional bilinear forms are again absorbed into the other forms. This is schematically shown in Figure 6.2.

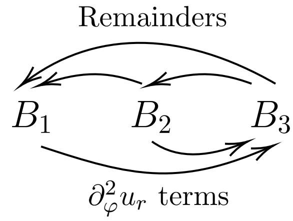


Figure 6.2: Schematic representation of the absorption of terms in other bilinear forms.

Proposition (Coercivity estimate)**.** *Let* $-\frac{1}{4} < \alpha < 0$ *and* $0 < \theta < \frac{\pi}{2}$*. Moreover, let* $\mathbf{u} \in \mathcal{T}$*. Then there are constants* $c_2, c_3 > 0$ *independent of* α *and* θ *such that there exists an* $\alpha_0 \in (-\frac{1}{4}, 0)$ *large enough such that for all* $\alpha \in (\alpha_0, 0)$ *there exists a* $\theta_0 \in (0, \frac{\pi}{2})$ *small enough such that for all* $\theta \in (0, \theta_0)$ *we have the coercivity estimate for the bilinear form* (B)*, i.e.*

$$B(\mathbf{u},\mathbf{u}) = B_1(\mathbf{u},\mathbf{u}) + c_2\theta^2 B_2(\mathbf{u},\mathbf{u}) + c_3\theta^4 B_3(\mathbf{u},\mathbf{u}) \gtrsim \|\mathbf{u}\|_{\mathscr{H}_\alpha}^2.$$

Specifically, this estimate is valid for $\theta \leq \frac{2\pi^2}{2.0672\cdot 10^6}$*.*

Proof. In the terminology of Corollaries 6.1.8, 6.2.5 and 6.3.3 we choose for instance the constants

$$\begin{aligned}
&\gamma_1 = \frac{c_3\theta^4}{4}, && \gamma_2 = \frac{C}{10}, && \gamma_3 = 1, && \gamma_4 = \theta\alpha^{-2}, && \gamma_5 = \theta, && \gamma_6 = \theta^2,\\
&\delta_1 = \frac{c_3\theta^2}{4c_2}, && \delta_2 = \frac{D_2\alpha^2}{D}, && \delta_3 = 1, && \delta_4 = \theta\alpha^{-2}, && \delta_5 = \theta, && \delta_6 = \theta^2,\\
&\varepsilon_1 = \frac{\theta}{80}, && \varepsilon_2 = \frac{\theta}{40}.
\end{aligned}$$

Then, combining the results from Corollaries 6.1.8, 6.2.5 and 6.3.3, using (6.22) and the above constants gives that

$$\begin{aligned}
&B_1(\mathbf{u},\mathbf{u}) + c_2\theta^2 B_2(\mathbf{u},\mathbf{u}) + c_3\theta^4 B_3(\mathbf{u},\mathbf{u})\\
\geq\ &\left[1 + c_2\theta^2(\alpha - \tfrac{1}{2})\right] |u_r|_\alpha^2 + c_2\theta^2 |r\partial_r u_r|_\alpha^2 + K_1\|\nabla \mathbf{u}\|_\alpha^2 + c_2\theta^2 K_2 \|r\partial_r \nabla \mathbf{u}\|_\alpha^2 + \frac{1}{10}c_3\theta^4\|r\nabla\omega_{\mathbf{u}}\|_\alpha^2,
\end{aligned}$$

where

$$\begin{aligned}
K_1 = {}& \frac{9C}{10} - 4c_3\theta^4(\alpha-1)^2 - \theta(\alpha^2+1)\left[1 + 4\theta + 2\theta^3\frac{c_2D_2}{D}\right] \qquad (6.23)\\
&- \theta^2\left[1298c_3 + \frac{5C_2}{C} + \frac{C_3+C_6+1}{2} + \frac{c_2D_2\alpha^2}{D}\right] - \theta\left[1 + \frac{C_4+C_5}{2}\right]\\
&- \frac{27}{10}c_3\theta^4(4+\alpha^{-2}) - \theta^2(1+\alpha^{-2})\left[\frac{C}{5} + 1 + c_3\theta^2(1304 + \tfrac{1}{5})\right] - \frac{2C_1}{c_3}\\
K_2 = {}& \frac{D}{2} - \theta^2\left[\frac{D_3+D_6+1}{2} + \frac{27}{10}\frac{c_3}{c_2}\right] - \theta(\alpha^2+1)(4\theta+1) - \theta\left[\frac{D_4+D_5}{2} + 1\right] \qquad (6.24)\\
&- \theta^2(1+\alpha^{-2}) - \frac{2D_1c_2\alpha^2}{c_3} - 1292\frac{c_3}{c_2}.
\end{aligned}$$

For coercivity we need $K_1, K_2 > 0$. Only the last term in both K_1 and K_2 are constant (i.e. do not depend on θ or α), however we can still choose the constants c_2 and c_3 in the definition of the bilinear form (B). We set

$$c_2 = \frac{32300C_1}{C\cdot D} \quad \text{and} \quad c_3 = \frac{5C_1}{C} \qquad (6.25)$$

so that

$$\frac{2C_1}{c_3} = \frac{C}{5} \quad \text{and} \quad 1292\frac{c_3}{c_2} = \frac{D}{5}.$$

Moreover, we choose $|\alpha|$ small enough such that the second but last term in K_2 becomes

$$\frac{D_1c_2\alpha^2}{c_3} \leq \frac{D}{5}. \qquad (6.26)$$

However, having $|\alpha|$ small will cause the terms with an α^{-2} in K_1 and K_2 to blow up. But we can in addition choose θ so small that $\theta \leq \alpha^2$ so that it cancels the α^{-2}, i.e.

$$\theta^2(1+\alpha^{-2}) \leq \theta(\alpha^2+1) \quad \text{and} \quad \theta^4(4+\alpha^{-2}) \leq \theta^3(4\alpha^2+1).$$

Finally, we choose θ small enough (if not small enough yet by the condition $\theta \leq \alpha^2$) such that $K_1, K_2 > 0$ and we obtain

$$B_1(\mathbf{u},\mathbf{u}) + c_2\theta^2 B_2(\mathbf{u},\mathbf{u}) + c_3\theta^4 B_3(\mathbf{u},\mathbf{u}) \gtrsim \|\mathbf{u}\|_{\mathscr{H}_\alpha}^2 \quad \text{for all } \mathbf{u} \in \mathcal{T}.$$

By keeping track of all the constants, a value of θ_0 (which will not be optimal) can be determined such that for all $\theta \in (0, \theta_0)$ the coercivity estimate holds. The most restrictive condition on θ is that $\theta \leq \alpha^2$ and α should be such that (6.26) holds. The constants C_1 and D_1 appear in Lemma 6.1.2 and 6.2.4 after applying the estimates (4.11) and (4.12) (both with $\ell = 0$) from Lemma 4.3.3. From the proof of Lemma 4.3.3 we find the values of the constants $C_1 = 16\pi^{-4}$ and $D_1 = 8\pi^{-2}$. From Lemma 6.1.6 and 6.2.1 it follows that $C = D = \frac{1}{2}$. Inserting this into (6.25) gives

$$c_2 = \frac{2.0672 \cdot 10^6}{\pi^4} \quad \text{and} \quad c_3 = \frac{160}{\pi^4},$$

so that by (6.26)

$$\theta \leq \alpha^2 \leq \frac{Dc_3}{5D_1c_2} = \frac{2\pi^2}{2.0672 \cdot 10^6} \approx 9.55 \cdot 10^{-6}.$$

Furthermore, it is elementary to check that for these values of α and θ the coefficients K_1 and K_2 in (6.23) and (6.24) are positive, so that the statement of this proposition is true for

$$\alpha_0^2 = \theta_0 = \frac{2\pi^2}{2.0672 \cdot 10^6}. \qquad \square$$

Finally, we give the proof of the boundedness of the bilinear form as stated in Proposition 5.3.2.

Proposition (Boundedness)**.** *Let* $-\frac{1}{4} < \alpha < 0$ *and* $0 < \theta < \frac{\pi}{2}$. *Then for any* $\mathbf{u}, \mathbf{v} \in \mathcal{T}$ *the bilinear form* (B) *is bounded, i.e.*

$$B(\mathbf{u}, \mathbf{v}) \lesssim_{\alpha,\theta} \|\mathbf{u}\|_{\mathscr{H}_\alpha} \|\mathbf{v}\|_{\mathscr{H}_\alpha}.$$

Proof. First consider the terms in bilinear form (B1). It is immediate that $T_1^{(1)}$, $T_2^{(1)}$ and $T_3^{(1)}$ can be bounded by applying the Cauchy-Schwarz inequality and in addition by Hardy's inequality for $T_3^{(1)}$. From the proof of Lemma 6.1.2 it follows that for $T_4^{(1)}$

$$\begin{aligned} \left|T_4^{(1)}\right| \leq &\left(\int_0^\theta \int_0^\infty r^{-2\alpha} (\partial_\varphi^2 u_r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi\right)^{\frac{1}{2}} \left(\int_0^\theta \int_0^\infty r^{2\alpha} (\partial_r \Phi_1)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi\right)^{\frac{1}{2}} \\ &+ \left(\int_0^\theta \int_0^\infty r^{-2\alpha} (\partial_\varphi u_r)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi\right)^{\frac{1}{2}} \left(\int_0^\theta \int_0^\infty r^{2\alpha} (\partial_r \partial_\varphi \Phi_1)^2 \frac{\mathrm{d}r}{r} \,\mathrm{d}\varphi\right)^{\frac{1}{2}}, \end{aligned}$$

where Φ_1 satisfies (5.3). Using Lemma 4.3.3 and Equations (6.4), (6.22) we obtain

$$\left|T_4^{(1)}\right| \lesssim_{\alpha,\theta} \left(\|\nabla \mathbf{u}\|_\alpha + \|\nabla r \partial_r \mathbf{u}\|_\alpha + \|r \nabla \omega_{\mathbf{u}}\|_\alpha\right) \|\nabla \mathbf{v}\|_\alpha \lesssim \|\mathbf{u}\|_{\mathscr{H}_\alpha} \|\mathbf{v}\|_{\mathscr{H}_\alpha}.$$

For the last term $T_5^{(1)}$ in B_1 recall that

$$\nabla \otimes \nabla \Phi_1 \overset{(A.10)}{=} \begin{pmatrix} \partial_r^2 \Phi_1 & r^{-1}\partial_r \partial_\varphi \Phi_1 - r^{-2}\partial_\varphi \Phi_1 \\ r^{-1}\partial_\varphi \partial_r \Phi_1 - r^{-2}\partial_\varphi \Phi_1 & r^{-2}\partial_\varphi^2 \Phi_1 + r^{-1}\partial_r \Phi_1 \end{pmatrix},$$

so that by the Cauchy-Schwarz inequality and Lemma 4.3.3, we obtain

$$\left|T_5^{(1)}\right| \leq \sum_{i,j=1}^{2} \left(\int_\Omega r^{2\alpha} |(\nabla \otimes \nabla \Phi_1)_{ij}|^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega r^{-2\alpha} (\nabla \mathbf{u})_{ij}^2 \,\mathrm{d}x\right)^{\frac{1}{2}} \lesssim_{\alpha,\theta} \|\mathbf{u}\|_{\mathscr{H}_\alpha} \|\mathbf{v}\|_{\mathscr{H}_\alpha}.$$

Boundedness for the terms in the second bilinear form (B2) follow in the same manner as for B_1, but with $\mathbf{u}$ and $\mathbf{v}$ replaced by $r\partial_r \mathbf{u}$ and $r\partial_r \mathbf{v}$, respectively. Again, boundedness of $T_1^{(2)}$,

$T_2^{(2)}$ and $T_3^{(2)}$ follows immediately from the Cauchy-Schwarz inequality and Hardy's inequality. From the proof of Lemma 6.2.4 it follows that for $T_4^{(2)}$

$$\left|T_4^{(2)}\right| \leq \left(\int_0^\theta \int_0^\infty r^{-2\alpha}(\partial_\varphi^2 u_r)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\right)^{\frac{1}{2}} \left(\int_0^\theta \int_0^\infty r^{2\alpha}(r\partial_r^2\Phi_2)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\right)^{\frac{1}{2}} + \left(\int_0^\theta \int_0^\infty r^{-2\alpha}(\partial_\varphi u_r)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\right)^{\frac{1}{2}} \left(\int_0^\theta \int_0^\infty r^{2\alpha}(r\partial_r^2\partial_\varphi\Phi_1)^2 \,\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi\right)^{\frac{1}{2}},$$

where Φ_2 satisfies (5.12). Using Lemma 4.3.3 and Equations (6.4), (6.22) we obtain

$$\left|T_4^{(2)}\right| \lesssim_{\alpha,\theta} \left(\|\nabla\mathbf{u}\|_\alpha + \|\nabla r\partial_r\mathbf{u}\|_\alpha + \|r\nabla\omega_{\mathbf{u}}\|_\alpha\right)\|\nabla\mathbf{v}\|_\alpha \lesssim \|\mathbf{u}\|_{\mathscr{H}_\alpha}\|\mathbf{v}\|_{\mathscr{H}_\alpha}.$$

Moreover, with the Cauchy-Schwarz inequality and Lemma 4.3.3 we obtain

$$\left|T_5^{(2)}\right| \leq \sum_{i,j=1}^{2} \left(\int_\Omega r^{2\alpha}|(\nabla\otimes\nabla\Phi_2)_{ij}|^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega r^{-2\alpha}(\nabla\mathbf{u})_{ij}^2\,\mathrm{d}x\right)^{\frac{1}{2}} \lesssim_{\alpha,\theta} \|\mathbf{u}\|_{\mathscr{H}_\alpha}\|\mathbf{v}\|_{\mathscr{H}_\alpha}.$$

Finally, for the vorticity bilinear form (B3) it is immediate that $T_1^{(3)}$ can be bounded by applying the Cauchy-Schwarz inequality. By in addition applying Hardy's inequality for $T_2^{(3)}$ gives

$$\left|T_2^{(3)}\right| \leq \left(\int_\Omega r^{-2\alpha}\omega_{\mathbf{v}}^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega r^{-2\alpha+2}(\partial_r\omega_{\mathbf{u}})^2\,\mathrm{d}x\right)^{\frac{1}{2}} \lesssim_\alpha \|r\nabla\omega_{\mathbf{v}}\|_\alpha\|r\nabla\omega_{\mathbf{u}}\|_\alpha.$$

The boundedness of the boundary term $T_3^{(3)}$ can be proved in a similar manner as for the coercivity in Lemma 6.3.2. Recall from (B3) and (5.26) that

$$\begin{aligned} T_3^{(3)} = &-\int_{\mathrm{Re}\lambda=\alpha}(\lambda-1)\frac{\cos\left((\lambda-1)\theta\right)}{\sin\left((\lambda-1)\theta\right)}\left[\partial_\varphi\widehat{u_r}(\lambda,\theta)\overline{\partial_\varphi\widehat{v_r}(\lambda,\theta)} + \partial_\varphi\widehat{u_r}(\lambda,0)\overline{\partial_\varphi\widehat{v_r}(\lambda,0)}\right]\,\mathrm{dIm}\lambda \\ &+\int_{\mathrm{Re}\lambda=\alpha}(\lambda-1)\frac{1}{\sin\left((\lambda-1)\theta\right)}\left[\partial_\varphi\widehat{u_r}(\lambda,\theta)\overline{\partial_\varphi\widehat{v_r}(\lambda,0)} + \partial_\varphi\widehat{u_r}(\lambda,0)\overline{\partial_\varphi\widehat{v_r}(\lambda,\theta)}\right]\,\mathrm{dIm}\lambda. \end{aligned}$$

Then integrating into the wedge with the fundamental theorem of calculus and applying the Cauchy-Schwarz inequality on each term gives with similar estimates as in Lemma 6.3.2 that $T_3^{(3)}$ is bounded. □

Chapter 7

The Parabolic and Polynomial Problem

In this final chapter we gather some incomplete results, which can serve as a starting point for future research into the (Navier-)Stokes equations with Navier slip on a wedge. In the previous chapters we made an effort to find a solution to the elliptic Stokes problem (P-S-St). Fortunately, our efforts were rewarded with even the existence and uniqueness of a strong solution. In Section 7.1 we will show that for the parabolic (i.e. time-dependent) Stokes problem it is much easier to find at least a weak solution.
As outlined in Chapter 3, the solution on a non-smooth domain is decomposed in a regular and singular part. For the elliptic Stokes problem we have dealt with the regular problem in the foregoing chapters and in Section 7.2 we will make a start with the polynomial problem.

7.1 The Parabolic Stokes Problem

Recall from Chapter 3 the time-dependent Stokes problem with Navier slip (N-St), i.e.

$$\begin{aligned} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \times [0,\infty), \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega \times [0,\infty), \\ \mathbf{u} \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega' \times [0,\infty), \\ \mathbf{u} \cdot \tau + \partial_{\mathbf{n}}(\mathbf{u} \cdot \tau) &= 0 && \text{on } \partial\Omega' \times [0,\infty), \\ \mathbf{u} &= \mathbf{u}_{\mathrm{ic}} && \text{in } \Omega \times \{0\}. \end{aligned} \tag{N-St}$$

For simplicity assume that $\mathbf{u}_{\mathrm{ic}} \equiv 0$, then after applying the Laplace transform in time and the Helmholtz projection, we obtain the resolvent equation

$$s\mathbb{P}\mathbf{u} - \mathbb{P}\Delta\mathbf{u} = \mathbf{f} \qquad \text{in } \Omega, \tag{P-N-St.a}$$

$$\operatorname{div} \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{P-N-St.b}$$

$$u_\varphi = 0, \qquad \text{on } \partial\Omega', \tag{P-N-St.c}$$

$$u_r + \partial_{\mathbf{n}} u_r = 0, \qquad \text{on } \partial\Omega', \tag{P-N-St.d}$$

for some arbitrary complex number $\operatorname{Re}(s) \geq 1$. In comparison with the elliptic case there is an extra term $s\mathbb{P}\mathbf{u}$. This term can be used to absorb other terms in the bilinear form by enlarging $\operatorname{Re}(s)$ and therefore we do not need additional bilinear forms with higher derivatives for coercivity.

Let $\mathcal{T}^{\mathrm{p}}$ be the space of test functions given by

$$\mathcal{T}^{\mathrm{p}} := \{\mathbf{v} \in C^2_{\mathrm{c}}(\overline{\Omega} \setminus \{0\}) : \operatorname{div} \mathbf{v} = 0 \text{ in } \Omega \text{ and } v_\varphi = 0 \text{ on } \partial\Omega'\}.$$

Testing the resolvent equation (P-N-St.a) against $\mathbf{v} \in \mathcal{T}^{\mathrm{p}}$ in the unweighted $L^2(\Omega)$ inner product gives the following weak formulation

$$(s\mathbb{P}\mathbf{u} - \mathbb{P}\Delta\mathbf{u}, \mathbf{v})_{L^2(\Omega)} = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)} \quad \text{for all } \mathbf{v} \in \mathcal{T}^{\mathrm{p}}, \quad \mathrm{Re}(s) \geq 1.$$

By using that $\mathbb{P}$ is self-adjoint (Lemma 4.2.4), a similar computation as in the elliptic case (Section 5.1) shows that

$$\begin{aligned}(s\mathbb{P}\mathbf{u} - \mathbb{P}\Delta\mathbf{u}, \mathbf{v})_{L^2(\Omega)} &= s(\mathbf{u}, \mathbf{v})_{L^2(\Omega)} - (\Delta\mathbf{u}, \mathbf{v})_{L^2(\Omega)} \\ &= s\int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}} \,\mathrm{d}x + \int_\Omega \nabla\mathbf{u} : \overline{\nabla\mathbf{v}} \,\mathrm{d}x + \int_{\partial\Omega'} u_r \overline{v_r} \,\mathrm{d}s \\ &=: B_{s,0}(\mathbf{u}, \mathbf{v}),\end{aligned}$$

where the Navier-slip condition (P-N-St.c) is applied in the boundary integral. Consider the Hilbert space

$$\mathscr{H}_0^{\mathrm{p}} := \overline{\mathcal{T}^{\mathrm{p}}}^{\|\cdot\|_{\mathscr{H}_0^{\mathrm{p}}}}, \tag{7.2}$$

where

$$\|\mathbf{v}\|^2_{\mathscr{H}_0^{\mathrm{p}}} = \int_\Omega |\mathbf{v}|^2 \,\mathrm{d}x + \int_\Omega |\nabla\mathbf{v}|^2 \,\mathrm{d}x + \int_{\partial\Omega'} v_r^2 \,\mathrm{d}s.$$

Proposition 7.1.1. *Let* $\mathrm{Re}(s) \geq 1$*. For any* $\mathbf{f} \in (\mathscr{H}_0^{\mathrm{p}})'$ *there exists a unique* $\mathbf{u} \in \mathscr{H}_0^{\mathrm{p}}$ *satisfying*

$$B_{s,0}(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad \textit{for all } \mathbf{v} \in \mathscr{H}_0^{\mathrm{p}}.$$

Proof. Let $\mathbf{u}, \mathbf{v} \in \mathcal{T}^{\mathrm{p}}$. With the Cauchy-Schwarz inequality it follows that

$$|B_{s,0}(\mathbf{u}, \mathbf{v})| \leq (|s| + 2)\|\mathbf{u}\|_{\mathscr{H}_0^{\mathrm{p}}} \|\mathbf{v}\|_{\mathscr{H}_0^{\mathrm{p}}}$$

and we trivially have

$$\begin{aligned}\mathrm{Re}B_{s,0}(\mathbf{u}, \mathbf{u}) &= \mathrm{Re}(s)\int_\Omega |\mathbf{u}|^2 \,\mathrm{d}x + \int_\Omega |\nabla\mathbf{u}|^2 \,\mathrm{d}x + \int_{\partial\Omega'} u_r^2 \,\mathrm{d}s \\ &\geq \|\mathbf{u}\|^2_{\mathscr{H}_0^{\mathrm{p}}} \quad \text{for } \mathrm{Re}(s) \geq 1.\end{aligned}$$

Hence, the result follows from the Lax-Milgram theorem (Theorem 2.3.4) and the density of $\mathcal{T}^{\mathrm{p}}$ in $\mathscr{H}_0^{\mathrm{p}}$, see (7.2). □

7.1.1 Solutions in the Weighted Case

Testing the resolvent equation (P-N-St.a) against $\mathbf{v} \in \mathcal{T}^{\mathrm{p}}$ in the weighted inner product gives

$$(\mathbf{f}, \mathbf{v})_\alpha = (s\mathbb{P}\mathbf{u} - \mathbb{P}\Delta\mathbf{u}, \mathbf{v})_\alpha = (s\mathbf{u}, \mathbf{v})_\alpha + (-\mathbb{P}\Delta\mathbf{u}, \mathbf{v})_\alpha = (s\mathbf{u}, \mathbf{v})_\alpha + (-\Delta\mathbf{u}, \mathbb{P}r^{-2\alpha}\mathbf{v})_{L^2(\Omega)},$$

where we used $\mathbb{P}\mathbf{u} = \mathbf{u}$ (by (P-N-St.b) and (P-N-St.c)) and Lemma 4.2.4. Recall that $\mathbb{P}r^{-2\alpha}\mathbf{v} = r^{-2\alpha}\mathbf{v} - \nabla\Phi$, where Φ satisfies (in the sense of Definition 4.2.1)

$$\begin{aligned}\Delta\Phi &= \mathrm{div}(r^{-2\alpha}\mathbf{v}) = -2\alpha r^{-2\alpha-1} v_r && \text{in } \Omega, \\ \partial_{\mathbf{n}}\Phi &= \mathbf{n} \cdot r^{-2\alpha}\mathbf{v} = 0 && \text{on } \partial\Omega'.\end{aligned}$$

The derivation of the rest of the bilinear form is similar as for the elliptic case in Section 5.2. We obtain

$$\begin{aligned}B_{s,\alpha}(\mathbf{u}, \mathbf{v}) = {}& \big(s\mathbf{u}, r^{-2\alpha}\mathbf{v}\big)_{L^2(\Omega)} + \big(u_r, r^{-2\alpha}v_r\big)_{L^2(\partial\Omega')} + \big(\nabla\mathbf{u}, r^{-2\alpha}\nabla\mathbf{v}\big)_{L^2(\Omega)} - 2\alpha\big(\partial_r\mathbf{u}, r^{-2\alpha-1}\mathbf{v}\big)_{L^2(\Omega)} \\ & - \big(u_r, \partial_r\Phi\big)_{L^2(\partial\Omega')} - \big(\nabla\mathbf{u}, \nabla \otimes \nabla\Phi\big)_{L^2(\Omega)} =: \sum_{i=0}^{5} T_i.\end{aligned}$$

In comparison with the bilinear form for the elliptic problem (B1), we have the extra T_0 term. We will show that for this bilinear form it is much easier to obtain a coercivity estimate only using the unweighted bilinear form $B_{s,0}$. We first prove an auxiliary result.

Lemma 7.1.2. *For $\alpha < -1$ the following estimate holds*

$$\|r^{-1}u_r\|_\alpha^2 \leq \varepsilon\|u_r\|_{L^2(\Omega)}^2 + C_\varepsilon\|u_r\|_\alpha^2,$$

where $C_\varepsilon > 0$ is some large constant depending on ε.

Proof. Write

$$r^{-2\alpha-2} = (r^{-2\alpha})^{\frac{1}{p}}(1)^{\frac{1}{q}}, \quad \frac{1}{p} = \frac{\alpha+1}{\alpha}, \quad q = -\frac{1}{\alpha},$$

so that $\frac{1}{p} + \frac{1}{q} = 1$ (and $\alpha < -1$ ensures that $p, q > 1$) and thus application of Hölder's and Young's inequality gives

$$\begin{aligned}
\int_\Omega r^{-2\alpha-2}|u_r|^2 \,\mathrm{d}x &= \int_\Omega (r^{-2\alpha})^{\frac{1}{p}}(|u_r|^2)^{\frac{1}{p}}(|u_r|^2)^{\frac{1}{q}} \,\mathrm{d}x \\
&\leq \left(\int_\Omega r^{-2\alpha}|u_r|^2 \,\mathrm{d}x\right)^{\frac{\alpha+1}{\alpha}} \left(\int_\Omega |u_r|^2 \,\mathrm{d}x\right)^{-\frac{1}{\alpha}} \\
&= \|u_r\|_{L^2(\Omega)}^{-\frac{2}{\alpha}}\|u_r\|_\alpha^{2\frac{\alpha+1}{\alpha}} \\
&\leq \varepsilon\|u_r\|_{L^2(\Omega)}^2 + C_\varepsilon\|u_r\|_\alpha^2.
\end{aligned}$$

□

Consider for some $C > 0$ the bilinear form

$$B_s(\mathbf{u}, \mathbf{v}) := CB_{s,0}(\mathbf{u}, \mathbf{v}) + B_{s,\alpha}(\mathbf{u}, \mathbf{v}),$$

for which we can now obtain the following coercivity estimate.

Proposition 7.1.3. *Let $\alpha < -1$ and $\theta > 0$ be small enough. Furthermore, let C and $\operatorname{Re}(s)$ be large enough. Then for all $\mathbf{u} \in \mathcal{T}^{\mathrm{p}}$*

$$\begin{aligned}
\operatorname{Re} B_s(\mathbf{u}, \mathbf{u}) &= \operatorname{Re} CB_{s,0}(\mathbf{u}, \mathbf{u}) + \operatorname{Re} B_{s,\alpha}(\mathbf{u}, \mathbf{u}) \\
&\gtrsim |u_r|_{L^2(\partial\Omega')}^2 + |u_r|_\alpha^2 + \|\mathbf{u}\|_{L^2(\Omega)}^2 + \|\nabla\mathbf{u}\|_{L^2(\Omega)}^2 + \|\mathbf{u}\|_\alpha^2 + \|\nabla\mathbf{u}\|_\alpha^2.
\end{aligned}$$

Proof. We only need to absorb the last three terms T_3, T_4 and T_5. Note that for T_3 the φ part can be dealt with similarly as in Lemma 6.1.3, i.e.

$$\left|\alpha\int_0^\theta\int_0^\infty r^{-2\alpha}\partial_r|u_\varphi|^2 \,\mathrm{d}r\,\mathrm{d}\varphi\right| \leq 4\theta^2(\alpha^2+1)\|\nabla\mathbf{u}\|_\alpha^2. \tag{7.3}$$

For the r part of T_3 we obtain with integration by parts and Lemma 7.1.2

$$\left|2\alpha\int_\Omega r^{-2\alpha-1}\partial_r|u_r|^2 \,\mathrm{d}x\right| = 2\alpha^2\|r^{-1}u_r\|_\alpha^2 \leq \varepsilon_1\|u_r\|_{L^2(\Omega)}^2 + C_{\varepsilon_1}\|u_r\|_\alpha^2. \tag{7.4}$$

For T_4 we can integrate into the interior of Ω to obtain

$$\begin{aligned}
-\big(u_r, \partial_r\Phi\big)_{L^2(\partial\Omega')} &= -\int_0^\infty \left(\overline{u_r}\partial_r\Phi\big|_{\varphi=0} + \overline{u_r}\partial_r\Phi\big|_{\varphi=\theta}\right) \mathrm{d}r \\
&= -\int_0^\theta\int_0^\infty \frac{2\varphi-\theta}{\theta}\big[(\partial_\varphi\overline{u_r})(r\partial_r\Phi) + \overline{u_r}(r\partial_r\partial_\varphi\Phi)\big]\frac{\mathrm{d}r}{r}\,\mathrm{d}\varphi.
\end{aligned} \tag{7.5}$$

To further estimate this integral we need estimates on Φ. However, applying the Navier-slip condition (P-N-St.d) on T_4 has changed the scaling of this term. Therefore, we cannot directly use Lemma 4.3.4 for estimating the integrals with Φ. Yet, with an analogous proof as the one of Lemma 4.3.4, we can nonetheless show the estimates

$$\int_0^\theta \int_0^\infty r^{2\alpha} |r\partial_r \partial_\varphi^\ell \Phi|^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \lesssim \alpha^2 \theta^{2-2\ell} \int_\Omega r^{-2\alpha} |\mathbf{u}|^2 \, \mathrm{d}x \quad \text{for } \ell \in \{0,1\}. \tag{7.6}$$

Then for the first term in (7.5) we obtain using Lemma 6.1.1 and (7.6)

$$\begin{aligned}
&\left| \int_0^\theta \int_0^\infty \frac{2\varphi - \theta}{\theta} (\partial_\varphi \overline{u_r})(r\partial_r \Phi) \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \right| \\
&\le \frac{\varepsilon_1}{2} \int_0^\theta \int_0^\infty r^{-2\alpha} |\partial_\varphi u_r|^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi + \frac{1}{2\varepsilon_1} \int_0^\theta \int_0^\infty r^{2\alpha} |r\partial_r \Phi|^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \\
&\le \varepsilon_1 \int_0^\theta \int_0^\infty r^{-2\alpha} |\partial_\varphi u_r - u_\varphi|^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi + \varepsilon_1 \int_0^\theta \int_0^\infty r^{-2\alpha} |u_\varphi|^2 \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi + \frac{C_1 \alpha^2 \theta^2}{2\varepsilon_1} \int_\Omega r^{-2\alpha} |\mathbf{u}|^2 \, \mathrm{d}x \\
&\le \big(\varepsilon_1 + 2\varepsilon_1 \theta^2 (1 + \alpha^{-2})\big) \|\nabla \mathbf{u}\|_\alpha^2 + \frac{C_1 \alpha^2 \theta^2}{2\varepsilon_1} \|\mathbf{u}\|_\alpha^2.
\end{aligned} \tag{7.7}$$

Similarly, we get for the second term in (7.5)

$$\left| \int_0^\theta \int_0^\infty \frac{2\varphi - \theta}{\theta} \overline{u_r} (r\partial_r \partial_\varphi \Phi) \frac{\mathrm{d}r}{r} \, \mathrm{d}\varphi \right| \le \frac{\varepsilon_2}{2\alpha^2} \|\nabla \mathbf{u}\|_\alpha^2 + \frac{C_2}{2\varepsilon_2} \alpha^2 \|\mathbf{u}\|_\alpha^2. \tag{7.8}$$

Therefore, from (7.5), (7.7) and (7.8) we obtain

$$\left| \big(u_r, \partial_r \Phi\big)_{L^2(\partial\Omega')} \right| \le \left(\varepsilon_1 + 2\varepsilon_1 \theta^2 (1 + \alpha^{-2}) + \frac{\varepsilon_2}{2\alpha^2} \right) \|\nabla \mathbf{u}\|_\alpha^2 + \left(\frac{C_1 \alpha^2 \theta^2}{2\varepsilon_1} + \frac{C_2}{2\varepsilon_2} \alpha^2 \right) \|\mathbf{u}\|_\alpha^2. \tag{7.9}$$

For T_5 we can use Lemma 4.3.4 and 7.1.2 to obtain

$$\begin{aligned}
\left| \big(\nabla \mathbf{u}, \nabla \otimes \nabla \Phi\big)_{L^2(\Omega)} \right| &\le \frac{\varepsilon_3}{2} \|\nabla \mathbf{u}\|_\alpha^2 + \frac{1}{2\varepsilon_3} \|r^{2\alpha} (\nabla \otimes \nabla \Phi)\|_\alpha^2 \\
&\le \frac{\varepsilon_3}{2} \|\nabla \mathbf{u}\|_\alpha^2 + \frac{1}{2\varepsilon_3} \|r^{-1} u_r\|_\alpha^2 \\
&\le \frac{\varepsilon_3}{2} \|\nabla \mathbf{u}\|_\alpha^2 + \frac{1}{2\varepsilon_3} \left(\varepsilon_4 \|u_r\|_{L^2(\Omega)}^2 + C_{\varepsilon_4} \|u_r\|_\alpha^2 \right).
\end{aligned} \tag{7.10}$$

Combining estimates (7.3),(7.4), (7.9) and (7.10), and choosing for instance $\varepsilon_1 = \varepsilon_2 = \alpha^2 \theta$ and $\varepsilon_3 = \varepsilon_4 = 1$ gives

$$\begin{aligned}
\operatorname{Re} B_s(\mathbf{u}, \mathbf{u}) \ge{}& C|u_r|_{L^2(\partial\Omega')}^2 + \left[C\operatorname{Re}(s) - \frac{3}{2} \right] \|\mathbf{u}\|_{L^2(\Omega)}^2 + C\|\nabla \mathbf{u}\|_{L^2(\Omega)}^2 \\
&+ |u_r|_\alpha^2 + \left[\operatorname{Re}(s) - C_{\varepsilon_1} - C_1 \theta - \frac{C_2}{2\theta} - \frac{C_{\varepsilon_4}}{2} \right] \|\mathbf{u}\|_\alpha^2 \\
&+ \left[\frac{1}{2} - 2\theta^2 (2 + \theta)(\alpha^2 + 1) - \alpha^2 \theta - \frac{\theta}{2} \right] \|\nabla \mathbf{u}\|_\alpha^2.
\end{aligned}$$

If $\operatorname{Re}(s) \sim \theta^{-1}$ and θ is small enough, then all coefficients are positive and we get the coercivity estimate. □

With this coercivity estimate for B_s in hand, one can (after transforming back in time) also construct a solution for the parabolic Stokes problem if enough test functions are generated. Recall that the bilinear form B_s was constructed by testing the equation with $C\mathbf{v} + r^{-2\alpha}\mathbf{v}$ ($\mathbf{v} \in \mathcal{T}^{\mathrm{p}}$) in $(\cdot,\cdot)_{L^2(\Omega)}$, so to determine whether enough test functions are generated with this special test function it is required to solve the dual problem (see also Remark 5.4.1)

$$\mathbb{P}\big(C\mathbf{v} + r^{-2\alpha}\mathbf{v}\big) = \mathbf{w}, \quad \mathbf{w} \in \mathcal{T}^{\mathrm{p}}.$$

7.2 The Polynomial Problem

Recall from Section 3.2 that the Stokes problem (S-St) on the wedge is decomposed into an expansion that captures the behaviour of the solution near the tip of the wedge and a regular remainder. We have dealt with the smooth problem in Chapter 5 and 6, and it remains to study the problem for the polynomial part. The polynomial problem that we have to solve (see also Section 3.2) is given by (3.6), i.e.

$$\begin{aligned} -\Delta \mathcal{P}_u + \nabla \mathcal{P}_p &= \mathcal{P}_f, \\ \operatorname{div} \mathcal{P}_u &= 0. \end{aligned} \tag{7.11}$$

In general one can consider generalised Taylor expansions which allow for logarithmic blow-up near the tip, e.g. for the velocity

$$\mathcal{P}_u(r,\varphi) = \sum_{(j,\ell)\,\in\,\mathbb{Z}^2} \mathbf{u}^{(j,\ell)}(\varphi)\, r^j \log^\ell r,$$

where $\mathbf{u}^{(j,\ell)}$ are unknown coefficients for which we will solve. However, since we have a fixed domain it is expected that the logarithms are not needed in the expansion [54]. Therefore, we will consider the Taylor expansions

$$\mathcal{P}_u(r,\varphi) = \sum_{j\geq 0} \mathbf{u}^{(j)}(\varphi)\, r^j, \tag{7.12a}$$

$$\mathcal{P}_p(r,\varphi) = \sum_{j\geq -1} p^{(j)}(\varphi)\, r^j, \tag{7.12b}$$

$$\mathcal{P}_f(r,\varphi) = \sum_{j\geq -2} \mathbf{f}^{(j)}(\varphi)\, r^j, \tag{7.12c}$$

where $\mathbf{f}^{(j)}$ are known coefficients and $\mathbf{u}^{(j)}$, $p^{(j)}$ are unknown coefficients. The shift in the summation index j for $\mathcal{P}_p$ and $\mathcal{P}_f$ with respect to $\mathcal{P}_u$ is basically because the velocity has a derivative more than the pressure and two derivatives more than the right hand side $\mathbf{f}$.

We remark that in the case of a moving domain it might be necessary to include logarithms in the expansion. This is for instance done for the thin-film equation (see Chapter 1) with Navier slip in [30].

Remark 7.2.1. *Note that by integrating the body force density* $\mathbf{f}^{(-2)}$ *we get*

$$\int \mathbf{f}^{(-2)} r^{-2}\,\mathrm{d}x\,\mathrm{d}y = \int \mathbf{f}^{(-2)} r^{-1}\,\mathrm{d}r\,\mathrm{d}\varphi,$$

which diverges as $r \downarrow 0$. *This is non-physical and we assume that only finite forces can act on finite volumes. Therefore, we in addition assume that* $\mathbf{f}^{(-2)} = 0$.

To solve the polynomial problem we need to determine the coefficients $\mathbf{u}^{(j)}$ and $p^{(j)}$. The strategy will be to substitute the expansions $\mathcal{P}_u, \mathcal{P}_p$ and $\mathcal{P}_f$ into the problem (7.11) and derive a system of ODEs. This is done below in Lemma 7.2.2 and after that in Section 7.2.1 we will concern ourselves with the solvability of the resulting BVP. Recall from Chapter 3 that the Stokes problem (3.4) in polar coordinates is given by

$$-r^{-2}\left[\left((r\partial_r)^2 + \partial_\varphi^2\right) u_r - 2\partial_\varphi u_\varphi - u_r\right] + \partial_r p = f_r, \qquad \text{for } r>0,\ \varphi\in(0,\theta), \tag{7.13a}$$

$$-r^{-2}\left[\left((r\partial_r)^2 + \partial_\varphi^2\right) u_\varphi + 2\partial_\varphi u_r - u_\varphi\right] + r^{-1}\partial_\varphi p = f_\varphi, \qquad \text{for } r>0,\ \varphi\in(0,\theta), \tag{7.13b}$$

$$(r\partial_r + 1)u_r + \partial_\varphi u_\varphi = 0, \qquad \text{for } r>0,\ \varphi\in(0,\theta), \tag{7.13c}$$

$$u_\varphi = 0, \qquad \text{for } r>0,\ \varphi\in\{0,\theta\}, \tag{7.13d}$$

$$u_r + \partial_{\mathbf{n}} u_r = 0, \qquad \text{for } r>0,\ \varphi\in\{0,\theta\}. \tag{7.13e}$$

Lemma 7.2.2. *The vector with the coefficients of the Taylor polynomials*

$$\mathbf{v}^{(j)}(\varphi) := \big(u_r^{(j)}, u_\varphi^{(j)}, \partial_\varphi u_r^{(j)}, p^{(j-1)}\big)^\top \quad \textit{for } j \geq 0,$$

satisfies the ordinary boundary value problem

$$\begin{aligned} &\partial_\varphi \mathbf{v}^{(j)}(\varphi) - A\mathbf{v}^{(j)}(\varphi) = \mathbf{g}^{(j)}(\varphi) \quad \textit{for } 0 < \varphi < \theta, \\ &R_0\mathbf{v}^{(j)}(0) + R_\theta \mathbf{v}^{(j)}(\theta) = \mathbf{c}^{(j)}, \end{aligned} \tag{BVP}$$

where

$$-A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ j+1 & 0 & 0 & 0 \\ (j+1)^2 & 0 & 0 & -(j-1) \\ 0 & -(j^2-1) & j-1 & 0 \end{pmatrix} \quad \textit{and} \quad \mathbf{g}^{(j)} = \begin{pmatrix} 0 \\ 0 \\ -f_r^{(j-2)} \\ f_\varphi^{(j-2)} \end{pmatrix}, \tag{7.14}$$

and the boundary conditions are given by

$$R_0 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad R_\theta := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \textit{and} \quad \mathbf{c}^{(j)} := \begin{pmatrix} 0 \\ 0 \\ u_r^{(j-1)}(0) \\ u_r^{(j-1)}(\theta) \end{pmatrix}. \tag{7.15}$$

Remark 7.2.3. *Throughout this section we write ∂_φ for the ordinary derivative $\frac{\mathrm{d}}{\mathrm{d}\varphi}$.*

Proof. Let $\mathcal{P}_{u_r}$, $\mathcal{P}_{u_\varphi}$ and $\mathcal{P}_{f_r}$, $\mathcal{P}_{f_\varphi}$ denote the radial and angular components of $\mathcal{P}_u$ and $\mathcal{P}_f$ in (7.12), respectively. First, substitute $\mathcal{P}_{u_r}$ and $\mathcal{P}_{u_\varphi}$ in the incompressibility constraint (7.13c) to obtain

$$(r\partial_r + 1)\sum_{j\geq 0} u_r^{(j)}\, r^j + \partial_\varphi \sum_{j\geq 0} u_\varphi^{(j)}\, r^j = \sum_{j\geq 0} \big[(j+1)u_r^{(j)} + \partial_\varphi u_\varphi^{(j)}\big] r^j = 0.$$

For fixed $j \geq 0$ we thus have the following relation

$$(j+1)u_r^{(j)} + \partial_\varphi u_\varphi^{(j)} = 0. \tag{7.16}$$

Substitution of the Taylor expansions $\mathcal{P}_{u_r}, \mathcal{P}_{u_\varphi}, \mathcal{P}_p$ and $\mathcal{P}_{f_r}$ in the momentum equation (7.13a) gives

$$-\sum_{j\geq 0}\Big((j^2-1)u_r^{(j)} + \partial_\varphi^2 u_r^{(j)} - 2\partial_\varphi u_\varphi^{(j)}\Big)\, r^{j-2} + \sum_{j\geq -1} j p^{(j)} r^{j-1} = \sum_{j\geq -2} f_r^{(j)}\, r^j,$$

and by shifting every term to r^{j-2} we obtain

$$-\sum_{j\geq 0}\Big((j^2-1)u_r^{(j)} + \partial_\varphi^2 u_r^{(j)} - 2\partial_\varphi u_\varphi^{(j)}\Big)\, r^{j-2} + \sum_{j\geq 0}(j-1)p^{(j-1)} r^{j-2} = \sum_{j\geq 0} f_r^{(j-2)}\, r^{j-2},$$

which leads for fixed $j \geq 0$ to the relation

$$-\Big[(j^2-1)u_r^{(j)} + \partial_\varphi^2 u_r^{(j)} - 2\partial_\varphi u_\varphi^{(j)}\Big] + (j-1)p^{(j-1)} = f_r^{(j-2)}. \tag{7.17}$$

Similarly, for the momentum equation (7.13b) we substitute $\mathcal{P}_{u_r}, \mathcal{P}_{u_\varphi}, \mathcal{P}_p$ and $\mathcal{P}_{f_\varphi}$ to obtain

$$-\sum_{j\geq 0}\Big((j^2-1)u_\varphi^{(j)} + \partial_\varphi^2 u_\varphi^{(j)} + 2\partial_\varphi u_r^{(j)}\Big)\, r^{j-2} + \sum_{j\geq -1} \partial_\varphi p^{(j)} r^{j-1} = \sum_{j\geq -2} f_\varphi^{(j)}\, r^j,$$

Shifting the indices such that each term contains a factor r^{j-2} gives for fixed index j the relation

$$-\Big[(j^2-1)u_\varphi^{(j)} + \partial_\varphi^2 u_\varphi^{(j)} + 2\partial_\varphi u_r^{(j)}\Big] + \partial_\varphi p^{(j-1)} = f_\varphi^{(j-2)}. \tag{7.18}$$

For $j \geq 0$ we rewrite Equations (7.16)-(7.18) into a system ODEs of the form

$$\partial_\varphi \mathbf{v}^{(j)} - A\mathbf{v}^{(j)} = \mathbf{g}^{(j)}, \tag{7.19}$$

where

$$\mathbf{v}^{(j)} = (v_1, v_2, v_3, v_4)^\top := \big(u_r^{(j)}, u_\varphi^{(j)}, \partial_\varphi u_r^{(j,)}, p^{(j-1)}\big)^\top. \tag{7.20}$$

It should be noted that Equation (7.17) contains a term $\partial_\varphi^2 u_r^{(j)}$ and therefore it is also required to include $\partial_\varphi u_r^{(j)}$ in the vector $\mathbf{v}^{(j)}$. For the term $\partial_\varphi^2 u_\varphi^{(j)}$ occurring in (7.18) it is not required to include $\partial_\varphi u_\varphi^{(j)}$ in $\mathbf{v}^{(j)}$ since we can rewrite $\partial_\varphi^2 u_\varphi^{(j)}$ with the aid of the incompressibility constraint (7.16).

We now derive A and $\mathbf{g}^{(j)}$ in the ODE (7.19) by rewriting (7.16)-(7.18). By definition of the vector $\mathbf{v}^j$ in (7.20) it is clear that

$$\partial_\varphi v_1 - v_3 = 0. \tag{7.21}$$

From Equation (7.16) it follows that

$$\partial_\varphi v_2 + (j+1)v_1 = 0. \tag{7.22}$$

Using Equation (7.17) we find

$$\partial_\varphi v_3 = -\Big[(j^2-1)v_1 - 2\partial_\varphi u_\varphi^{(j)}\Big] + (j-1)v_4 - f_r^{(j-2)}$$

and for the term with $\partial_\varphi u_\varphi^{(j)}$ we can apply the incompressibility (7.16) to obtain

$$\partial_\varphi v_3 + \big(\underbrace{(j^2-1) + 2(j+1)}_{=\,(j+1)^2}\big)v_1 - (j-1)v_4 = -f_r^{(j-2)}. \tag{7.23}$$

Furthermore, using Equation (7.18) we find

$$\partial_\varphi v_4 = (j^2-1)v_2 + \partial_\varphi^2 u_\varphi^{(j)} + 2v_3 + f_\varphi^{(j-2)}$$

and for the term with $\partial_\varphi^2 u_\varphi^{(j)}$ we can again apply the incompressibility (7.16) to obtain

$$\partial_\varphi^2 u_\varphi^{(j)} = -(j+1)\partial_\varphi v_1 \overset{(7.21)}{=} -(j+1)v_3$$

and finally we arrive at

$$\partial_\varphi v_4 - (j^2-1)v_2 + (j-1)v_3 = f_\varphi^{(j-2)}. \tag{7.24}$$

Combining the four Equations (7.21)-(7.24) gives that $-A$ and $\mathbf{g}^{(j)}$ are as in Equation (7.14) of the statement of the proposition.

The boundary conditions of the boundary value problem are determined from the boundary conditions in (7.13). Substituting the expansion $\mathcal{P}_{u_\varphi}$ in boundary condition (7.13d) gives

$$\sum_{j\geq 0} u_\varphi^{(j)}(\varphi) r^j = 0 \quad \text{for } \varphi \in \{0, \theta\},$$

and thus

$$u_\varphi^{(j)}(0) = u_\varphi^{(j)}(\theta) = 0. \tag{7.25}$$

Substituting the expansion $\mathcal{P}_{u_r}$ in the Navier-slip boundary condition (7.13e) gives

$$\sum_{j\geq 0} u_r^{(j)}(\varphi) r^j - \sum_{j\geq 0} \partial_\varphi u_r^{(j)}(\varphi) r^{j-1} = 0 \quad \text{for } \varphi \in \{0,\theta\},$$

and by shifting $j \mapsto j-1$ in the first sum we find

$$\partial_\varphi u_r^{(j)}(0) = u_r^{(j-1)}(0) \quad \text{and} \quad \partial_\varphi u_r^{(j)}(\theta) = u_r^{(j-1)}(\theta). \tag{7.26}$$

Writing the boundary conditions (7.25) and (7.26) in matrix form gives $R_0 \mathbf{v}^{(j)}(0) + R_\theta \mathbf{v}^{(j)}(\theta) = \mathbf{c}^{(j)}$ where R_0, R_θ and $\mathbf{c}^{(j)}$ are as given in the statement of the proposition, see Equation (7.15). □

Before studying the solvability of the boundary value problem (BVP), we collect some properties of the matrix A.

Lemma 7.2.4. *Let $j \geq 0$. The matrix A as in Lemma 7.2.2 has the eigenvalues*

$$i(j+1), \quad -i(j+1), \quad i(j-1) \quad \text{and} \quad -i(j-1).$$

If $j=0$, then the eigenvalues are i and $-i$, both with algebraic multiplicity 2 and geometric multiplicity 1. The corresponding eigenvectors are

$$\mathbf{v}^1_{\lambda=i} = \big(-i,1,1,0\big)^\top \quad \text{and} \quad \mathbf{v}^1_{\lambda=-i} = \big(i,1,1,0\big)^\top$$

and the generalised eigenvectors are

$$\mathbf{v}^2_{\lambda=i} = \big(1,2i,0,-2\big)^\top \quad \text{and} \quad \mathbf{v}^2_{\lambda=-i} = \big(1,-2i,0,-2\big)^\top.$$

If $j=1$, then the eigenvalues are $0,0,2i$ and $-2i$ and there are four eigenvectors given by

$$\mathbf{v}^1_{\lambda=0} = \big(0,1,0,0\big)^\top, \quad \mathbf{v}^2_{\lambda=0} = \big(0,0,0,1\big)^\top, \quad \mathbf{v}_{\lambda=2i} = \big(-i,1,2,0\big)^\top, \quad \mathbf{v}_{\lambda=-2i} = \big(i,1,2,0\big)^\top.$$

If $j \geq 2$, then the four eigenvalues are distinct and the corresponding eigenvectors are given by

$$\mathbf{v}_{\lambda=i(j+1)} = \Big(\tfrac{-i}{j+1}, \tfrac{1}{j+1}, 1, 0\Big)^\top, \qquad \mathbf{v}_{\lambda=-i(j+1)} = \Big(\tfrac{i}{j+1}, \tfrac{1}{j+1}, 1, 0\Big)^\top,$$
$$\mathbf{v}_{\lambda=i(j-1)} = \Big(\tfrac{j-1}{4j}, \tfrac{i(j+1)}{4j}, \tfrac{i(j-1)^2}{4j}, 1\Big)^\top, \qquad \mathbf{v}_{\lambda=-i(j-1)} = \Big(\tfrac{j-1}{4j}, \tfrac{-i(j+1)}{4j}, \tfrac{-i(j-1)^2}{4j}, 1\Big)^\top.$$

Proof. A straightforward calculation shows that

$$\det(A-\lambda I) = \lambda^4 + 2(j^2+1)\lambda^2 + (j^2-1)^2$$

has four roots given by $i(j+1), -i(j+1), i(j-1)$ and $-i(j-1)$. For $j=0$ we have thus two distinct eigenvalues, each with one eigenvector

$$\mathbf{v}^1_{\lambda=i} = \big(-i,1,1,0\big)^\top \quad \text{and} \quad \mathbf{v}^1_{\lambda=-i} = \big(i,1,1,0\big)^\top.$$

The generalised eigenvectors $\mathbf{v}^2_\lambda$ should satisfy

$$(A-\lambda I)\mathbf{v}^2_\lambda = \mathbf{v}^1_\lambda \quad \text{and} \quad (A-\lambda I)^2 \mathbf{v}^2_\lambda = 0,$$

and we find that

$$\mathbf{v}^2_{\lambda=i} = \big(1,2i,0,-2\big)^\top \quad \text{and} \quad \mathbf{v}^2_{\lambda=-i} = \big(1,-2i,0,-2\big)^\top.$$

For $j \geq 1$ an involved but straightforward calculation shows that for all eigenvalues the algebraic and geometric multiplicities are equal and that the eigenvectors are as in the statement of the proposition. □

7.2.1 Solvability of Polynomial Problem

Recall the boundary value problem (BVP), i.e.

$$\partial_\varphi \mathbf{v}^{(j)}(\varphi) - A\mathbf{v}^{(j)}(\varphi) = \mathbf{g}^{(j)}(\varphi) \quad \text{for } 0 < \varphi < \theta, \tag{BVP.a}$$

$$R_0\mathbf{v}^{(j)}(0) + R_\theta \mathbf{v}^{(j)}(\theta) = \mathbf{c}^{(j)}, \tag{BVP.b}$$

where $A, \mathbf{g}^{(j)}, R_0, R_\theta$ and $\mathbf{c}^{(j)}$ are given in (7.14) and (7.15). Note that for solving (BVP) for $j \in \mathbb{N}_0$ we need $u_r^{(j-1)}$ as data (see $\mathbf{c}^{(j)}$ in (7.15)). Therefore, we have to solve the problem in increasing order for j and this coupling of the equations due to the Navier-slip boundary condition makes it difficult to find explicit solution representations. We will make use of the Green's matrix (see Section 2.2.1) to show that a unique solution to (BVP) exists for every $j \geq 0$. In the Propositions 7.2.5-7.2.7 we study the solvability for $j = 0, j = 1$ and $j > 1$.

Proposition 7.2.5 (Solvability of (BVP) for $j = 0$)**.** *Let $0 < \theta < 2\pi$. Then the boundary value problem* (BVP) *for $j = 0$ is only satisfied by the trivial solution, i.e.*

$$\mathbf{v}^{(0)} = \left(u_r^{(0)}, u_\varphi^{(0)}, \partial_\varphi u_r^{(0)}, p^{(-1)}\right)^\top = 0.$$

Proof. For $j = 0$ the matrix A in (7.14) is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & -1 & 1 & 0 \end{pmatrix}$$

and in view of Remark 7.2.1 the right hand side is

$$\mathbf{g}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ -f_r^{(-2)} \\ f_\varphi^{(-2)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In addition, the boundary conditions (BVP.b) are also homogeneous, i.e. $\mathbf{c}^{(j)} = 0$ by the definition of $\mathbf{c}^{(j)}$ in (7.15). The Jordan form becomes $A = PJP^{-1}$, where by Lemma 7.2.4

$$P = \begin{pmatrix} -i & 1 & i & 1 \\ 1 & 2i & 1 & -2i \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & -2 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 1 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

The fundamental matrix $e^{\varphi A}$ is given by

$$e^{\varphi A} = Pe^{\varphi J}P^{-1} = P \begin{pmatrix} e^{\varphi} & \varphi e^{\varphi} & 0 & 0 \\ 0 & e^{\varphi} & 0 & 0 \\ 0 & 0 & e^{-\varphi} & \varphi e^{-\varphi} \\ 0 & 0 & 0 & e^{-\varphi} \end{pmatrix} P^{-1}$$

and the characteristic matrix corresponding to this fundamental matrix is

$$\begin{aligned} \mathbf{C} &= R_0 e^{0\cdot A} + R_\theta e^{\theta A} \\ &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\sin(\theta) & \cos(\theta) + \frac{1}{2}\theta\sin(\theta) & -\frac{1}{2}\theta\sin(\theta) & \frac{1}{2}\sin(\theta) - \frac{1}{2}\theta\cos(\theta) \\ 0 & 0 & 1 & 0 \\ -\sin(\theta) & \frac{1}{2}\theta\sin(\theta) & \cos(\theta) - \frac{1}{2}\sin(\theta) & -\frac{1}{2}\sin(\theta) - \frac{1}{2}\theta\cos(\theta) \end{pmatrix}. \end{aligned}$$

The determinant of the characteristic matrix is

$$\det \mathbf{C} = \frac{1}{4}\Big(e^{2i\theta} + e^{-2i\theta} - 2\Big) = \frac{1}{2}\cos(2\theta) - \frac{1}{2} < 0 \quad \text{for } 0 < \theta < 2\pi.$$

The determinant of the characteristic matrix is non-zero and therefore by Proposition 2.2.3 the (BVP) is only satisfied by the trivial solution. □

Next, consider $j = 1$, then we solve (BVP) for

$$\mathbf{v}^{(1)}(\varphi) = \big(u_r^{(1)}, u_\varphi^{(1)}, \partial_\varphi u_r^{(1)}, p^{(0)}\big)^\top$$

and the matrix A in (7.14) is given by

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -2 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is clear that A has determinant 0 and therefore (BVP) will not have a unique solution. However, note that we solve for $p^{(0)}$ which corresponds to the constant contribution of the pressure. Since the pressure is always determined up to an additive constant, it is to be expected that the system does not have a unique solution and without loss of generality we can set $p^{(0)} = 0$. We study the remaining three-dimensional system and show that this system satisfies the four boundary conditions if $\mathbf{f}$ satisfies some compatibility condition.

Proposition 7.2.6 (Solvability of (BVP) for $j = 1$)**.** *Let $0 < \theta < \frac{\pi}{2}$ and assume that $\mathbf{f}$ satisfies the compatibility condition*

$$\int_0^\theta f_r^{(-1)}(\varphi)\,\mathrm{d}\varphi = 0.$$

Then the boundary value problem (BVP) *for $j = 1$ has a unique solution up to an additive constant for the pressure.*

Proof. In this proof we ignore the pressure in the vector $\mathbf{v}^{(1)}$ and consider the remaining three-dimensional system

$$\partial_\varphi \tilde{\mathbf{v}} = \tilde{A}\tilde{\mathbf{v}} + \tilde{\mathbf{g}} \qquad \text{for } \varphi \in (0, \theta), \tag{7.28}$$

where

$$\tilde{\mathbf{v}}(\varphi) = \begin{pmatrix} u_r^{(1)} \\ u_\varphi^{(1)} \\ \partial_\varphi u_r^{(1)} \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ -4 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{g}} = \begin{pmatrix} 0 \\ 0 \\ -f_r^{(-1)} \end{pmatrix}.$$

The three boundary conditions to determine the solution are $\tilde{R}_0\tilde{\mathbf{v}}(0) + \tilde{R}_\theta\tilde{\mathbf{v}}(\theta) = \tilde{\mathbf{c}}$, where

$$\tilde{R}_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{R}_\theta = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\mathbf{c}} = \begin{pmatrix} 0 \\ 0 \\ u_r^{(0)}(0) \end{pmatrix} \overset{\text{Prop. 7.2.5}}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7.29}$$

By Lemma 7.2.4 the matrix $\tilde{A}$ has eigenvalues 0, $2i$ and $2i$ and can be written in Jordan form as $\tilde{A} = PJP^{-1}$, where

$$P = \begin{pmatrix} 0 & -i & i \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2i & 0 \\ 0 & 0 & -2i \end{pmatrix}. \tag{7.30}$$

The fundamental matrix is given by

$$e^{\varphi\tilde{A}} = \begin{pmatrix} \cos(2\varphi) & 0 & \frac{1}{2}\sin(2\varphi) \\ -\sin(2\varphi) & 1 & \frac{1}{2}\cos(2\varphi) - \frac{1}{2} \\ -2\sin(2\varphi) & 0 & \cos(2\varphi) \end{pmatrix} \tag{7.31}$$

and therefore the characteristic matrix is

$$\mathbf{C} = \tilde{R}_0 e^{0\cdot\tilde{A}} + \tilde{R}_\theta e^{\theta\tilde{A}} = \begin{pmatrix} 0 & 1 & 0 \\ -\sin(2\theta) & 1 & \frac{1}{2}\cos(2\theta) - \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}.$$

We obtain that $\det\mathbf{C} = \sin(2\theta)$ which is non-zero for $0 < \theta < \frac{\pi}{2}$ and by Proposition 2.2.3 the three-dimensional system (7.28) has a unique solution. The solution is given by

$$\tilde{\mathbf{v}}(\varphi) = e^{\varphi\tilde{A}}\mathbf{b} + \int_0^\varphi e^{(\varphi-\tilde{\varphi})\tilde{A}}\tilde{\mathbf{g}}(\tilde{\varphi})\,\mathrm{d}\tilde{\varphi}, \tag{7.32}$$

where $\mathbf{b} := \big(b_1, b_2, b_3\big)^\top$ is a constant vector determined by the boundary conditions (7.29). We will insert the solution $\tilde{\mathbf{v}}$ into the boundary conditions to find the constant vector $\mathbf{b}$. By (7.29) we obtain

$$\begin{aligned} 0 &= \tilde{R}_0\tilde{\mathbf{v}}(0) + \tilde{R}_\theta\tilde{\mathbf{v}}(\theta) \\ &= \begin{pmatrix} b_2 \\ 0 \\ b_3 \end{pmatrix} + \begin{pmatrix} 0 \\ -b_1\sin(2\theta) + b_2 + b_3\big(\frac{1}{2}\cos(2\theta) - \frac{1}{2}\big) - \int_0^\theta f_r^{(-1)}(\tilde{\varphi})\big[\frac{1}{2}\cos\big(2(\theta-\tilde{\varphi})\big) - \frac{1}{2}\big]\,\mathrm{d}\tilde{\varphi} \\ 0 \end{pmatrix} \end{aligned}$$

and it follows that $b_2 = b_3 = 0$ and that b_1 satisfies

$$b_1 = -\frac{1}{\sin(2\theta)}\int_0^\theta f_r^{(-1)}(\tilde{\varphi})\Big[\frac{1}{2}\cos\big(2(\theta-\tilde{\varphi})\big) - \frac{1}{2}\Big]\,\mathrm{d}\tilde{\varphi}. \tag{7.33}$$

Therefore, we have a unique solution $\tilde{\mathbf{v}}$ satisfying (7.28) with the three boundary conditions (7.29). However, this solution should also satisfy the fourth, still unused, boundary condition

$$\partial_\varphi u_r^{(1)}(\theta) = u_r^{(0)}(\theta) \overset{\text{Prop. 7.2.5}}{=} 0.$$

Writing out the third component of the solution $\tilde{\mathbf{v}}$ in (7.32) and substituting b_1, b_2 and b_3 gives

$$\begin{aligned} \partial_\varphi u_r^{(1)}(\theta) &= -2b_1\sin(2\theta) - \int_0^\theta f_r^{(-1)}(\tilde{\varphi})\cos\big(2(\theta-\tilde{\varphi})\big)\,\mathrm{d}\tilde{\varphi} \\ &= -\int_0^\theta f_r^{(-1)}(\tilde{\varphi})\,\mathrm{d}\tilde{\varphi}. \end{aligned}$$

Therefore, (BVP) has a unique solution up to an additive constant for the pressure if $\mathbf{f}$ satisfies the compatibility condition

$$\int_0^\theta f_r^{(-1)}(\tilde{\varphi})\,\mathrm{d}\tilde{\varphi} = 0.$$ □

Proposition 7.2.7 (Solvability of (BVP) for $j \geq 2$)**.** *Let $\theta > 0$ and*

$$\theta \neq \frac{n\pi}{j+1} \quad \text{and} \quad \theta \neq \frac{n\pi}{j-1} \quad \text{for all } n \geq 1.$$

Then the boundary value problem (BVP) *for $j \geq 2$ has a unique solution.*

Proof. Recall from Lemma 7.2.4 that for $j \geq 2$ the matrix A in (7.14) has four distinct eigenvalues and we can calculate the corresponding fundamental matrix $e^{\varphi A}$ and characteristic matrix $\mathbf{C}$ similarly to the previous propositions. A tedious but straightforward calculation shows that

$$\det \mathbf{C} = \frac{1}{2}\cos(2\theta) - \frac{1}{2}\cos(2j\theta).$$

This determinant is zero for

$$\theta = \frac{n\pi}{1-j} \quad \text{and} \quad \theta = \frac{(n+1)\pi}{j+1} \quad \text{for } n \in \mathbb{Z}.$$

Therefore, by Proposition 2.2.3 there is a unique solution to (BVP) for $\theta > 0$ and

$$\theta \neq \frac{n\pi}{j+1} \quad \text{and} \quad \theta \neq \frac{n\pi}{j-1} \quad \text{for all } n \geq 1.$$

Note that $\frac{\pi}{j+1}$ is the smallest root of $\det \mathbf{C}$ and therefore we can solve (BVP) for some fixed θ at least up to order $j < \frac{\pi-\theta}{\theta}$. □

By combining the above Propositions 7.2.5-7.2.7 we obtain the following result on the solvability of the polynomial problem (7.11), i.e.

$$\begin{aligned} -\Delta \mathcal{P}_u + \nabla \mathcal{P}_p &= \mathcal{P}_f, \\ \operatorname{div} \mathcal{P}_u &= 0, \end{aligned} \tag{7.34}$$

with Navier-slip boundary condition.

Theorem 7.2.8. *Consider the polynomial problem* (7.34) *related to the Stokes problem with Navier slip. Assume that* $\mathbf{f} = (f_r, f_\varphi)^\top$ *has a Taylor expansion around the tip of the wedge* Ω *of the form*

$$\mathcal{P}_f(r,\varphi) = \sum_{\ell \geq -1} \mathbf{f}^{(\ell)}(\varphi) r^\ell$$

and satisfies the compatibility condition

$$\int_0^\theta f_r^{(-1)}(\varphi)\,\mathrm{d}\varphi = 0.$$

Furthermore, let $0 < \theta < \frac{\pi}{2}$ *and assume for* $j \geq 2$ *that*

$$\theta \neq \frac{n\pi}{j+1} \quad \textit{and} \quad \theta \neq \frac{n\pi}{j-1} \quad \textit{for all } n \geq 1.$$

Then there exists a unique solution up to an additive constant for the pressure to the polynomial problem (7.34)*. Furthermore, the solution to the polynomial problem has a Taylor expansion around the tip of the form*

$$\mathcal{P}_u(r,\varphi) = \sum_{j\geq 0} \mathbf{u}^{(j)}(\varphi)\, r^j \quad \textit{and} \quad \mathcal{P}_p(r,\varphi) = \sum_{j \geq 0} p^{(j-1)}(\varphi)\, r^{j-1},$$

where without loss of generality $p^{(0)} = 0$ *and all other coefficients* $\mathbf{u}^{(j)}, p^{(j-1)}$ *for* $j \geq 0$ *are uniquely determined. In particular, for any fixed* θ *it is possible to solve the polynomial problem up to order* $j < \frac{\pi-\theta}{\theta}$.

Appendix A

Vector Identities and Polar Coordinates

A.1 Vector Identities

For a vector-valued function $\mathbf{u} = \big(u_1(x_1,x_2), u_2(x_1,x_2)\big)^\top : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ in Cartesian coordinates (x_1, x_2) we define

$$\operatorname{div}\mathbf{u} := \nabla \cdot \mathbf{u} = \partial_{x_1} u_1 + \partial_{x_2} u_2, \qquad \text{(divergence)} \tag{A.1}$$

$$\operatorname{curl}\mathbf{u} := \nabla^\perp \cdot \mathbf{u} = \partial_{x_1} u_2 - \partial_{x_2} u_1, \qquad \text{(curl or rotation)} \tag{A.2}$$

$$\nabla\mathbf{u} := \begin{pmatrix} \partial_{x_1} u_1 & \partial_{x_2} u_1 \\ \partial_{x_1} u_2 & \partial_{x_2} u_2 \end{pmatrix}, \qquad \text{(gradient)} \tag{A.3}$$

$$\Delta\mathbf{u} := \begin{pmatrix} \Delta u_1 \\ \Delta u_2 \end{pmatrix}, \qquad \text{(Laplacian).} \tag{A.4}$$

Moreover, for $\mathbf{u}, \mathbf{v} : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ we introduce the notation

$$\nabla\mathbf{u} : \nabla\mathbf{v} = \sum_{i,j=1}^{2} \partial_{x_i} u_j \partial_{x_i} v_j,$$

$$\mathbf{u} \otimes \mathbf{v} \in \mathbb{R}^{2\times 2} \text{ with } (\mathbf{u} \otimes \mathbf{v})_{ij} := u_i v_j \text{ for } 1 \le i, j \le 2.$$

With this notation we get the identities

$$\sum_{j=1}^{2} \operatorname{div}(v_j \nabla u_j) = \sum_{j=1}^{2} v_j \Delta u_j + \nabla v_j \cdot \nabla u_j = \mathbf{v} \cdot \Delta\mathbf{u} + \nabla\mathbf{v} : \nabla\mathbf{u},$$

$$\nabla(\phi\mathbf{u}) = \phi\nabla\mathbf{u} + \mathbf{u} \otimes \nabla\phi,$$

for any scalar field ϕ. Furthermore, we have the following properties.

Lemma A.1. *For* $\mathbf{u}$ *a vector field in* $\mathbb{R}^2$ *and* ϕ *a scalar field we have the following properties:*

1. *The curl of a gradient is zero:* $\operatorname{curl}\nabla\phi = 0$.

2. *The divergence of a curl is zero:* $\operatorname{div}\operatorname{curl}\mathbf{u} = 0$.

3. *The divergence and Laplacian commute:* $\operatorname{div}\Delta\mathbf{u} = \Delta\operatorname{div}\mathbf{u}$.

4. *If* $\operatorname{div}\mathbf{u} = 0$*, then the curl and Laplacian commute:* $\operatorname{curl}\Delta\mathbf{u} = \Delta\operatorname{curl}\mathbf{u}$.

A.2 Polar Coordinates

To switch from Cartesian coordinates (x, y) to polar coordinates (r, φ), set

$$x = r\cos\varphi \quad \text{and} \quad y = r\sin\varphi$$

and the unit vectors are

$$\text{Cartesian: } \mathbf{e}_\mathrm{x} = \begin{pmatrix}1\\0\end{pmatrix},\ \mathbf{e}_\mathrm{y} = \begin{pmatrix}0\\1\end{pmatrix}. \qquad \text{Polar: } \mathbf{e}_\mathrm{r} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\end{pmatrix},\ \mathbf{e}_\varphi = \begin{pmatrix}-\sin\varphi\\ \cos\varphi\end{pmatrix}.$$

We write $\mathbf{u} = u_r(r,\varphi)\mathbf{e}_\mathrm{r} + u_\varphi(r,\varphi)\mathbf{e}_\varphi$ as

$$\mathbf{u} = \begin{pmatrix}u_r\\ u_\varphi\end{pmatrix}.$$

The gradient and Laplace operators are in polar coordinates given by

$$\nabla = (\partial_r)\mathbf{e}_\mathrm{r} + \left(r^{-1}\partial_\varphi\right)\mathbf{e}_\varphi \quad \text{and} \quad \Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\varphi^2 = r^{-2}\big((r\partial_r)^2 + \partial_\varphi^2\big). \tag{A.5}$$

Therefore, for any vector-valued function $\mathbf{u} : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^2$ and scalar field ϕ we have

$$\operatorname{div}\mathbf{u} = r^{-1}\big((r\partial_r + 1)u_r + \partial_\varphi u_\varphi\big), \tag{A.6}$$

$$\operatorname{curl}\mathbf{u} = r^{-1}\big((r\partial_r + 1)u_\varphi - \partial_\varphi u_r\big), \tag{A.7}$$

$$\nabla\mathbf{u} = r^{-1}\begin{pmatrix} r\partial_r u_r & \partial_\varphi u_r - u_\varphi \\ r\partial_r u_\varphi & \partial_\varphi u_\varphi + u_r \end{pmatrix}, \tag{A.8}$$

$$\Delta\mathbf{u} = r^{-2}\begin{pmatrix} \big((r\partial_r)^2 + \partial_\varphi^2\big)u_r - 2\partial_\varphi u_\varphi - u_r \\ \big((r\partial_r)^2 + \partial_\varphi^2\big)u_\varphi + 2\partial_\varphi u_r - u_\varphi \end{pmatrix}, \tag{A.9}$$

$$\nabla\otimes\nabla\phi = \begin{pmatrix} \partial_r^2\phi & r^{-1}\partial_\varphi\partial_r\phi - r^{-2}\partial_\varphi\phi \\ r^{-1}\partial_\varphi\partial_r\phi - r^{-2}\partial_\varphi\phi & r^{-2}\partial_\varphi^2\phi + r^{-1}\partial_r\phi \end{pmatrix}. \tag{A.10}$$

Finally, we also have the commutation relations

$$r^\gamma r\partial_r = (r\partial_r - \gamma)r^\gamma \quad \text{and} \quad r\partial_r r^\gamma = r^\gamma(r\partial_r + \gamma) \quad \text{for } \gamma \in \mathbb{R}. \tag{A.11}$$

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