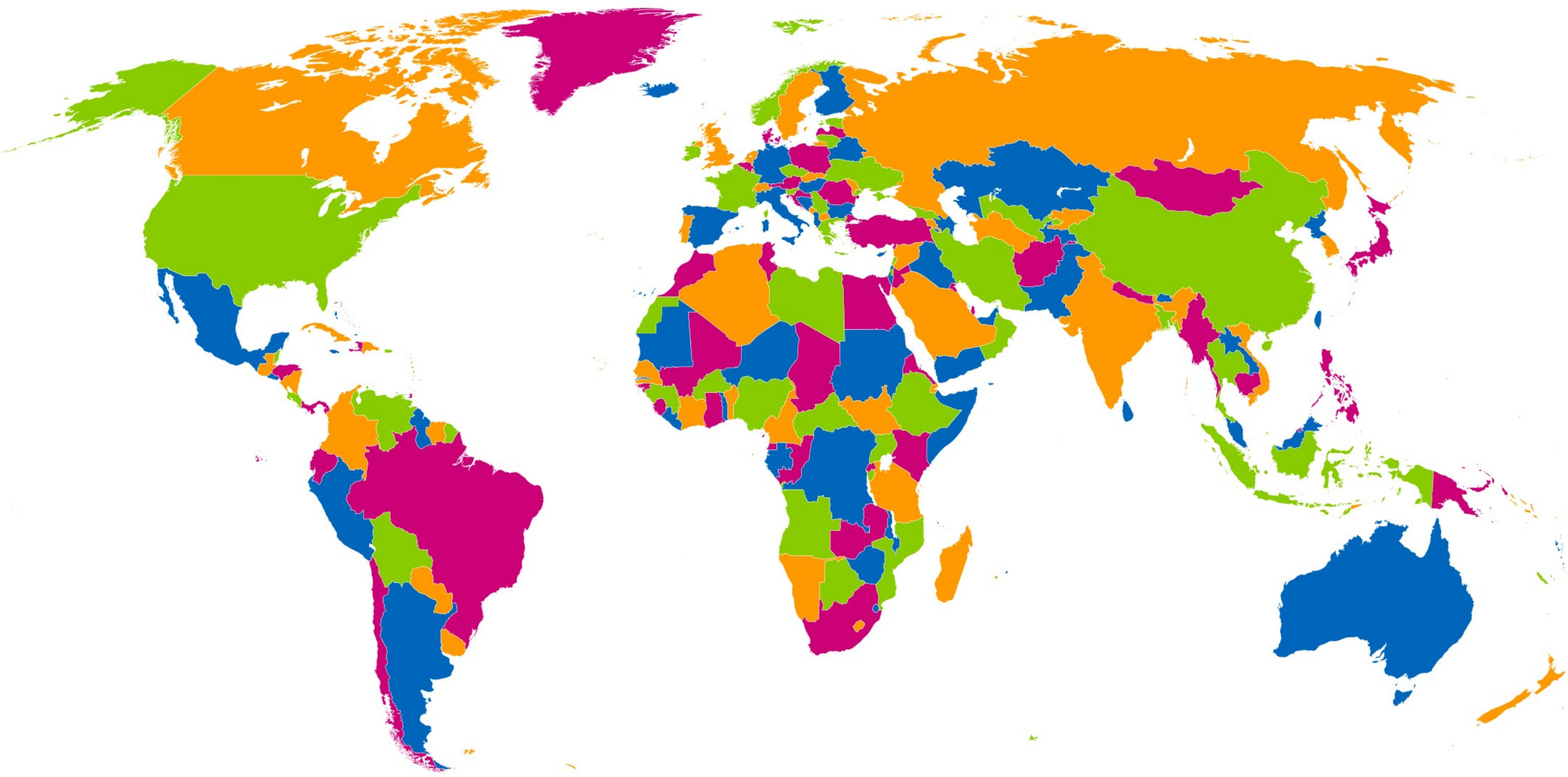
The Four-Colour Theorem

History and Proof

Jurre Kalff



TU Delft

The Four-Colour Theorem

History and Proof

by

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Abstract

The four-colour conjecture (4CC) is a question that asks whether any map can be coloured using only four colours, with the constraint that neighboring countries must have distinct colours. This conjecture has remained unanswered for over 170 years, with a rich history of various attempts to prove it. In 1879, Kempe put forward a proof, but it was invalidated by Heawood after 11 years. Heawood did succeed, however, in proving the weaker five-colour theorem. It wasn't until 1976 that the first genuine proof was discovered by Appel and Haken. This proof sparked controversy due to its reliance on approximately 1200 hours of computer computation, making it unverifiable by hand.

The purpose of this report is to provide a comprehensive overview of the historical background and recent advancements in the four-colour theorem. It will delve into the numerous failed attempts to prove the conjecture and discuss the groundbreaking proof by Appel and Haken. Additionally, it will explore a recent endeavor by Dr. Xiang, who approached the problem from a different perspective but also encountered a fallacy in his work. The main contribution of this thesis involves a new proof that builds upon the research of Dr. Yeh, who attempted to prove the four-colour theorem by transforming it into a system of linear equations. The missing crucial steps will be addressed for whose correction new ideas will be introduced, using an integral version of Farkas' lemma and superadditivity. The ideas being researched do not finalize the proof but contribute towards a possible elegant proof in the future. Finally, related topics such as maps on different surfaces and maps with disconnected regions will be considered, broadening the scope of the discussion.

Contents

1
Introduction

Can a graph be coloured using only four colours such that neighbouring countries have a different colour? At first sight, this statement seems a very straightforward and easy to solve problem, but nothing could be more wrong. This question that is known as the four-colour problem has kept the mathematical world occupied since a long time and even until today there has not been an unambiguous answer to whether or not the statement has been proved correctly. Various questions still remain amongst which the most important says: "Is a proof by computer a valid proof?" Until now, no one has ever managed to redo the calculations that the current accepted proof is based on for the fact that these are too tedious and complicated. Obviously this raises concerns among mathematicians and therefore many others have tried their own approach in solving the four-colour theorem. Many articles and reformulations of the original proof followed but the main question remains: is there a worldwide accepted proof of the four-colour theorem?

This research aims to provide a comprehensive overview of the 170-year history and advancements in the four-colour theorem, with the objective of offering a clear understanding to the reader. It will cover the initial emergence of the problem and trace its development through various attempts at a proof, ultimately exploring the latest advancements pertaining to the theorem's proof. The research will shed light on the reasons why a definitive, computer-unassisted proof for the four-colour theorem has yet to be established. It will thoroughly investigate all notable attempts made thus far and discuss the contributions that have been made towards proving the theorem. Additionally, the report will explore the possibility of introducing new contributions to existing attempts, aiming to bring mathematicians closer to their collective goal of establishing an elegant proof for the four-colour theorem.

The report will be presented in the following structure. First of all the historical background of the four-colour theorem will be provided in Chapter 2. Chapter 3 will then present a general introduction to graph theory. The main part of the report consists of Kempe's attempted proof in Chapter 4, Heawoods counterexample in Chapter 5 and his proof of the weaker five-colour theorem in Chapter 6. Chapter 7 examines the first real proof of the four-colour theorem by Appel and Haken and after that the two most recent attempts in finding a new proof will be discussed in Chapters 8 and 9. Finally the results of my own contribution to the proof will be presented in Chapter 10 and Chapter 11 covers the related topics to the four-colour theorem. The report will conclude with a summary of the findings and discussions presented throughout the chapters in Chapter 12. The next page contains the statement that we know today as the four-colour theorem [19].

Any map in a plane can be coloured using four colours in such a way that regions sharing a common boundary (other than a single point) do not share the same colour

THE FOUR-COLOUR THEOREM

2

History

Throughout history, there have been a handful of long-standing problems that have eluded solutions for over a century. One such problem is the four-colour theorem, which has captivated the world of mathematics since its inception in 1852. Francis Guthrie, a lawyer, botanist, and primarily a mathematician, stumbled upon this intriguing concept while colouring a map of England (Figure 2.1). He proposed that a mere four colours would be sufficient to colour any arbitrary map. Guthrie's brother, Frederick, relayed this observation to their professor, Augustus de Morgan, on October 23rd of that year. De Morgan then penned a letter outlining the conjecture and shared it with his friend and mathematician, Sir William Rowan Hamilton. This letter (Figure 2.2) marks the earliest recorded mention of the four-colour conjecture. Hamilton however was not as fascinated by the problem as De Morgan was, therefore the latter is regarded as its primary originator. The problem faded into the background for several years until Arthur Cayley resurrected it in a mathematical journal in 1878, inquiring if anyone had managed to prove it. The problem gained wider recognition, and in 1879, Sir Alfred Bray Kempe, a lawyer and former student of Cayley, presented what was initially accepted as a proof. This proof enjoyed acceptance worldwide for the next 11 years until Percy John Heawood discovered a counterexample, effectively refuting Kempe's proof. However, Heawood did succeed in proving the weaker variant of the theorem, known as the five-colour theorem. For nearly a century, the pursuit of a proof for the four-colour theorem remained fruitless. Then, in 1976, Kenneth Appel and Wolfgang Haken finally delivered the first "real" proof of the theorem, 124 years after Guthrie first posed the problem. Prior to their breakthrough, only incremental updates had been made to the proof. Over time, different mathematicians introduced clever structures and solutions, each contributing to the proof we now recognize. Appel and Haken's proof marked the first instance of a computer-aided proof in mathematical history. Despite the release of their proof, not everyone was entirely convinced, leading to further refinements. In 1996, Robertson, Sanders, Seymour, and Thomas simplified the proof over the course of a year. Even today, new attempts persist in search of a more elegant proof for the four-colour theorem.

Figure 2.1: The map of the counties of England that Francis Guthrie coloured in 1852 using only four colours

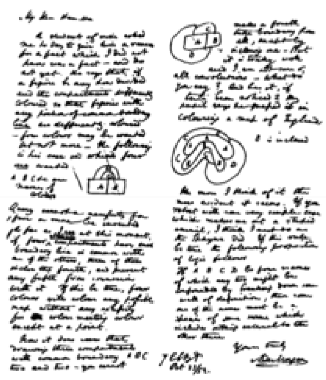


Figure 2.2: The letter of De Morgan sent to Hamilton marking the earliest recorded mention of the four-colour conjecture

3
Introduction to Graph theory

Before we are able to dive into the real problem of colouring a map, we must make sure to be speaking the same language. The study of graphs is called graph theory and in this section there will be a short introduction to this field of study, which is relevant towards the four colour problem.

First of all, a graph G consists of a set of **vertices** V and a set of **edges** E. A graph is called simple if it does not have parallel or multiple edges and loops. In the case of four-colouring a map we only consider simple graphs. Relating this to the four-colour problem: a country will be represented by a vertex and two neighbouring countries will have an edge connecting them in the graph. This is also called the **dual** of a graph. Figure 3.1 shows an example of the above. A part of Europe is drawn and every country is replaced by a vertex which still need to be coloured. Whenever two countries share a border, an edge is drawn.

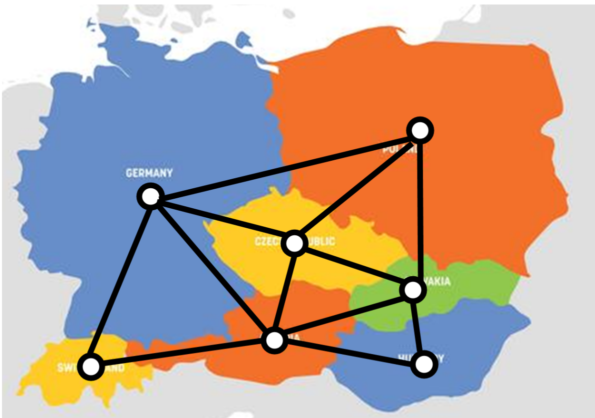


Figure 3.1: An example of a map represented as a graph with vertices and edges

A couple definitions will be introduced:

degree of a vertex	The number of edges incident to a vertex, noted by $deg(v)$
walk	An alternating sequence of vertices and edges
connected graph	A graph in which there exists a walk in between any two vertices
face	A region of a graph bounded by edges
plane graph	A graph that is drawn with no two edges crossing each other. A graph is planar if it is isomorphic to a plane graph, i.e. it can be drawn in a way such that no two edges cross each other.

In other words, if an edge crosses another edge there has to be a vertex at that crossing for a graph to be plane. The notion of planarity is crucial towards the four-colour theorem and we will later see how. In Figure 3.1 can be seen that no edges cross each other which makes this graph planar.

Two important properties of planar graphs will now be discussed. Firstly, Euler's formula [20, pp. 20-22] which says:

Theorem 1 (Euler's formula)**.** *For any plane, connected graph v - e + f = 2 for v the number of vertices, e the number of edges and f the number of faces of a graph.*

Proof. We prove this by induction on v. For $v = 1$ we have a collection of loops at that one vertex. With 0 edges we have exactly 1 face, and the formula is satisfied. Each loop introduces a new face, and thus the formula is true for any number of edges. Now let G be an arbitrary connected plane graph on $v > 1$ vertices, and assume that each connected plane graph on $< v$ vertices satisfies Euler's formula. As it is connected there is an edge ab in the graph that is not a loop. Let G' be the graph obtained by contracting this edge, i.e., replacing the two endpoints a, b by a new vertex, removing the edge ab, and for each edge that has exactly one endpoint in a, b changing the endpoint to the new vertex. This process does not change the face number, as it merely decreases the length of the two faces that e was adjacent to by 1. Moreover, it reduces both v and e by 1. Therefore, the graph G' has $v-1$ vertices, $e-1$ edges and f faces. G' is also a connected graph and thus by the induction hypothesis it satisfies Euler's formula, that is, $v-1-(e-1)+f=2$. This implies the Euler's formula for G as the left hand side is equal to $v-e+f$. This proof is copied from [5, p. 33]. □

Before the second property is introduced, we need to define an upper bound on the number of edges

Theorem 2. *If G is a simple planar graph on $v \geq 3$ vertices and e edges, then $e \leq 3v-6$.*

Proof. This proof is mostly copied from [5, p. 33]. Assume that the graph is connected. Consider a planar embedding of G, with faces $F_1, ..., F_k$, and $\ell(F_i)$ being the length of face F_i. Note that $2e = \sum_{i=1}^{k} \ell(F_i)$ [5, p. 32], because two times the number of edges is equal to the sum of the degrees of all vertices and this is in its turn equal to the sum of the lengths of all faces. The length of a face is equal to the degree of the corresponding vertex in the dual, based on the way the dual is constructed.

Since the graph is simple each bounded face has length at least three, and the outer face also has length at least three because $v \geq 3$. Therefore, we can conclude that $3k \leq \sum_{i=1}^{k} \ell(F_i) = 2e$. If the graph is connected, then by Euler's formula 1 we have $k = e-v+2$, and hence $2e \geq 3(e-v+2)$, which implies the bound $e \leq 3v-6$. If the graph is not connected then we can add extra edges to make it connected while keeping it planar, thus implying that the bound still holds. □

Now the upper bound on the number of edges is defined, we are able to introduce the following lemma.

Lemma 1. *For any plane connected graph, there is always a vertex with degree at most five.*

Proof. Let v be the number of vertices in the graph. The number of edges is at most $3v-6$ by Theorem 2. If every vertex has degree at least six, then the number of edges will be at least $3v$ by the Handshaking lemma [20, p. 23] which says that two times the number of edges is equal to the sum of the degrees of all vertices. This reaches a contradiction. □

In other words, there is no planar graph where the degree of every vertex is greater or equal to 6. Later on we will see that this is a very important property in proving the four-colour theorem.

4

Kempe's Approach

In 1879, Alfred J. Kempe published his first "proof" of the four-colour theorem, which remained valid for 11 years until Heawood found a counterexample to a specific subcase of the proof. In this section, the general idea of the proof will be portrayed. The first publication of the proof by Kempe himself can be found in [14].

First of all Kempe introduced a new definition:

Definition 1 (Kempe chain [20])**.** *The largest set of countries you can get to from a given place by keeping to countries of a particular two colours, and crossing at edges, not vertices.*

Figure 4.1 shows an example of a Kempe chain. Kempe's proof made heavily use of those chains, by using the fact that you can swap all the colours in that chain and still keeping a valid colouring of the graph. In the same Figure, all countries that are part of the red-blue Kempe chain can swap colours. Hence all countries coloured blue will be coloured red and similarly all countries coloured red will be coloured blue. In this way the colours have been "swapped" while the colouring of the graph is still valid.

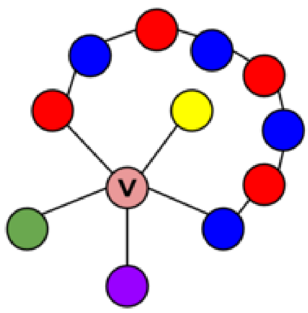


Figure 4.1: An Example of a blue and red Kempe chain [13]

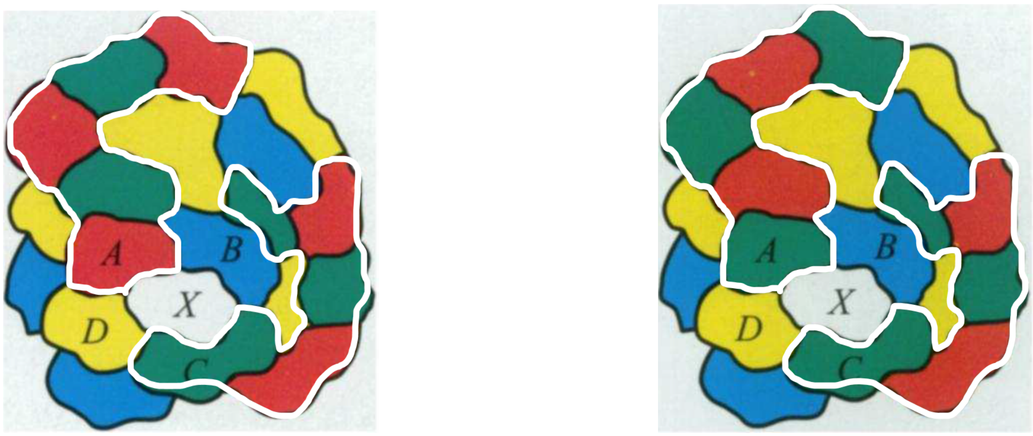
In his proof he used the principle of strong induction, i.e. he assumes that a graph on less than n vertices is four-colourable and then shows that a graph on n vertices is also four-colourable. He did this by looking at a **maximal planar graph** which is a graph that has the maximum number of edges such that if one edge would be added, the graph would not be planar anymore. The reason for this is that if a maximal planar graph on n vertices would be four-colourable, then any graph that is formed by removing any number of edges would also be four-colourable. Also he used Lemma 1 to deduce that in any maximal planar graph there has to be a vertex with maximum degree five. This is the vertex that he focused on, call it X, which is surrounded by a number of other countries. He divided the proof in four different cases based on the degree of the vertex X.

Case 1: the degree of X is smaller than or equal to 3. Then obviously you will only need at most four colours to get a valid colouring of X and its neighbouring countries.

Case 2: X is surrounded by four or more countries coloured with 3 or less different colours. Then there will always be a colour remaining to colour X.

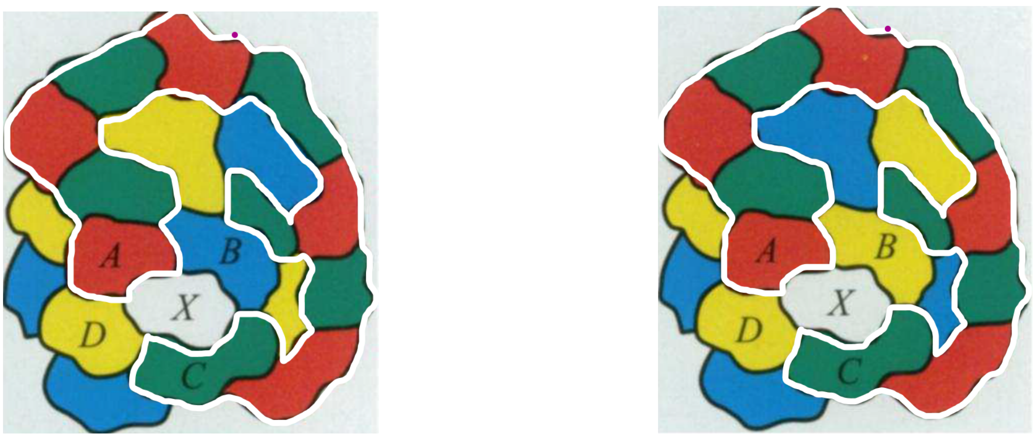
Those were the two easy cases, however if X is surrounded by four or more countries with four different colours, there might be a conflict which Kempe addressed in the following way. In the remainder of this section the maps will not be represented as graphs but as real countries in order for the reader to experience a nice graphical representation of Kempe's proof.

Case 3: X is surrounded by four countries coloured with four different colours. Suppose X is surrounded by four countries with colours red, blue, green, yellow counted clockwise as can be seen in Figure 4.2. Now the notion of a Kempe chain comes in handy. There are two subcases to be distinguished, one where red and green belong to a different chain and the other where red and green belong to the same chain. In the first case (Figure 4.2), the colours of the first red-green chain can be interchanged such that A becomes green and the colour red will be available to colour X. In the second case (Figure 4.3), it is not possible to interchange the colours to get either green or red available to colour X. Note however that in this case the blue country B and the yellow country D are separated by this red-green chain and therefore they cannot belong to the same blue-yellow chain. Swap one of the two chains, for example the blue-yellow chain starting in B, such that blue becomes available to colour X.



(a) Before (b) After

Figure 4.2: The third case of Kempe's proof where X is surrounded by four countries coloured with four different colours, where countries A and C belong to a different red-green Kempe chain, before and after swapping the colours of the Kempe chain starting in A [18]

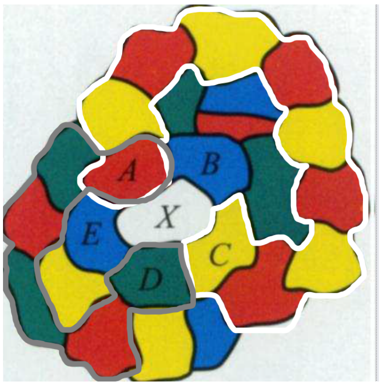


(a) Before (b) After

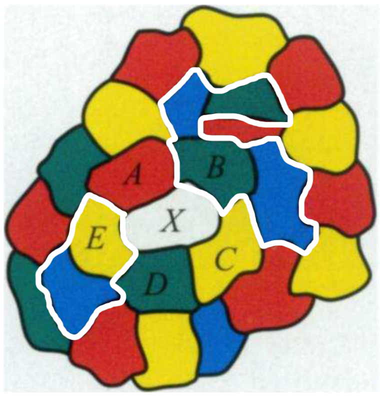
Figure 4.3: The third case of Kempe's proof where X is surrounded by four countries with four different colours, where countries A and C belong to the same red-green Kempe chain before and after swapping the colours of the Kempe chain starting in B [18]

The final and fallacious part of Kempe's proof is the following.

Case 4: X is surrounded by five countries coloured with four different colours. Suppose the countries are coloured red, blue, yellow, green, blue in clockwise order as in Figure 4.4. There must be one colour repeated, which is blue in this case. If country A (red) and country C (yellow) do not belong to the same red-yellow chain or country A (red) and country D (green) do not belong to the same red-green chain, the colours in one of those chains can be swapped and there will be a colour remaining for X. This process is similar to case 3. What if both red and yellow belong to the same red-yellow chain and red and green to the same red-green chain, isolating both the blue countries? Then by the argument similar to the previous case, the two blue countries do not belong to the same chain, because it is separated by the two other chains. Therefore we can swap the colours in the blue-green and the blue-yellow chain, making country B green and country E yellow leaving blue as the colour for X (Figure 4.4b).



(a) Before



(b) After

Figure 4.4: The final case of Kempe's proof where X is surrounded by five countries coloured with four different colours, before and after the blue-green and blue-yellow chain are swapped [18]

Following this procedure every map can be coloured using only four colours, according to Kempe. The four colour theorem was believed to be solved but we all know by now that unfortunately this proof contains a mistake.

5

Heawood's counterexample

For over a decade, Kempe's proof was widely accepted as a valid proof of the four-colour theorem. However, in 1890, Heawood shattered this illusion by presenting a counterexample that undermined Kempe's proof. It took Heawood a remarkable 11 years to find this counterexample, which exposed an error within Kempe's reasoning. It is important to note that the existence of a counterexample does not invalidate the truth of the four-colour theorem itself; it simply indicates a flaw in the presented proof. Specifically, Heawood's counterexample targeted the final case of Kempe's proof, which dealt with a country surrounded by five other countries coloured with four different colours. Figure 5.1 contains the map illustrating this counterexample. Now, what precisely is the issue with this map? Why does Kempe's construction fail? Note in this map the two Kempe chains, red-green ($A-D$) and red-yellow ($A-C$) both isolate a chain starting from the blue countries B and E. One blue-green chain starting from B and one blue-yellow chain starting from E (illustrated by the white lines). However, the problem is that those chains touch each other at the point where Y and Z meet. Now if we follow the construction of Kempe's proof, we are dealing with the last case: X is surrounded by five countries coloured with four different colours. As we have seen there is a red-green chain between A and D and a red-yellow chain between A and C. By Kempe, we can now freely swap the colours of the blue-green chain from B and the blue-yellow chain from E. Then B becomes green and E becomes yellow, leaving blue as the designated colour for X. However, Y and Z will both be coloured blue in this case and they touch each other, which results in an invalid colouring of the map.

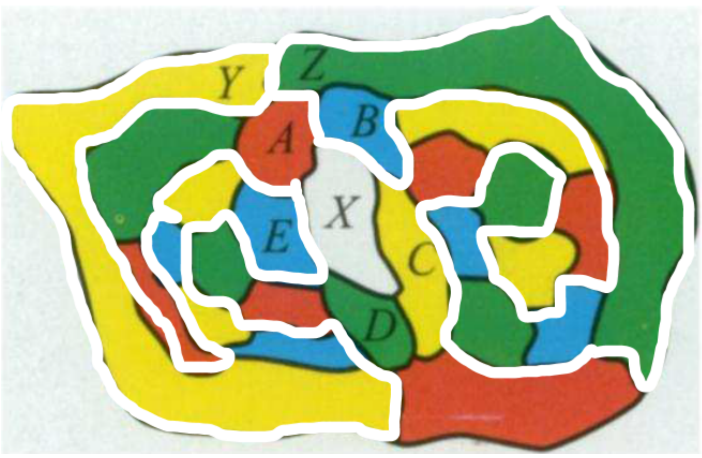


Figure 5.1: A map constructed by Heawood that forms a counterexample to a specific case of the proof of Kempe [18]

6

The five-colour theorem

Although an error was found in the proof of the four-colour theorem, Heawood did manage to prove the weaker five-colour theorem based on the work of Kempe. In this section, this proof will be given [5].

Theorem 3 (Five-colour theorem)**.** *Every planar graph is five-colourable*

The five colour theorem is proved by induction.

Proof. Base case: Any planar graph with five or less vertices is five-colourable.

Induction step: Let G be a graph with $n > 5$ vertices. By the Induction Hypothesis (IH) we can assume that every planar graph on $< n$ vertices is five-colourable. We have to show that G is also five-colourable. By Lemma 1, G has a vertex of degree ≤ 5. Select such a vertex and call it v. By the IH, $G - v$ can be coloured with five colours. Now if $\deg(v) < 5$ there will be at least one colour available to colour v. Therefore assume that $\deg(v) = 5$ and let v_1, v_2, v_3, v_4, v_5 be the neighbours of v, see Figure 6.1.

Figure 6.1: A vertex v of degree five with its five neighbours

For the sake of contradiction, assume that no proper five-colouring of G exist.
Claim: For all $i \neq j$ there exists a path P_{ij} in $G - v$ between v_i and v_j consisting entirely of vertices that are coloured i or j, in other words a Kempe Chain.
Proof. We have already seen that we can unconditionally swap the colours in a Kempe chain to still have a proper five-colouring of a graph. Consider now the induced subgraph of $G - v$ on vertices that are coloured i or j. If v_i and v_j are in different components, then we can change the colouring of $G - v$ such that the colour of v_i is the same as the colour of v_j. And then there is a colour left to colour v such that G has a proper five-colouring.

Consider the cycle formed by P_{13} and the edges vv_1 and vv_3. Note that v_2 lies inside the circle and v_4 outside. Therefore the path P_{24} must intersect path P_{13}. However, the graph is planar so such a intersection can only occur at a vertex of G. This is a contradiction since the set of vertices coloured 1 or 3 is disjoint from the set of vertices coloured 2 or 4. □

7
Proof of Appel and Haken

In a report about the four-colour theorem, the names Kenneth Appel and Wolfgang Haken cannot miss (Figure 7.1). These two mathematicians have come up with the first "accepted" proof of the four-colour theorem. It was also the first computer-assisted proof in history, using more than 1200 hours of computer calculations. This chapter contains a short introduction to how they came up with the proof. Next the general idea of the proof itself will be given, for the reason that the whole proof would be too long and complicated to examine in this report. For the interested reader, the complete proof can be found in [2] and [3]. One should be warned that these two articles account for a total of 139 pages, of which the last 63 pages contain the set of 1834 configurations. In Section 7.3 a simplified version of the proof of Appel and Haken will be discussed. It was in 1997 that four mathematicians concluded that the existing proof was too complex for anyone to understand. In turn, they came up with a simplified version that also requires way less computing time.

7.1. Introduction to the proof

Three concepts that are crucial to the proof are the discharging procedure, the unavoidable set and reducible configurations. These will be explained later on in this Chapter, but they will be stated already to emphasize who was working on what part. It was in 1971 that Appel became interested in the problem through Haken. Appel contributed mainly by programming the discharging procedures. Together they focused on developing more powerful discharging procedures to study sets up to one million elements [10]. In 1975 they worked towards an unavoidable set of 2000 reducible configurations. A third member got included, a graduate student named John Koch that focused on the reducibility calculations. A year later, the proof was finalized. Unfortunately many errors were found by other mathematicians, but Appel and Haken came up with a clever way to deal with those errors and published a corrected version of their proof in 1989. Problem solved.

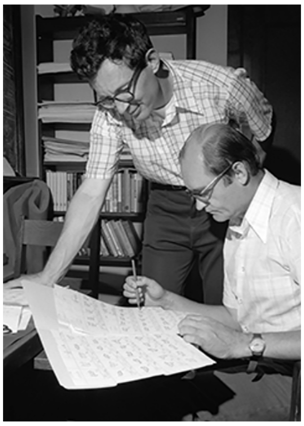


Figure 7.1: Kenneth Appel and Wolfgang Haken working on the proof of the four-colour theorem [8]

The next section will be spent on the proof itself and then it will become clear why the proof is so controversial. Not only the computer assistance that it requires but also the part that can, in their eyes, be checked by hand is very complex. It needs to be mentioned that it was not just Appel and Haken that came up with the proof. A lot of mathematicians started the search for a correct proof, after the proof of Kempe appeared fallacious. The proof that is valid nowadays was a joint effort of multiple mathematicians over multiple decades. In the end it were Appel and Haken that finalized the proof and therefore it is in their name, however they do mention the use of ideas of other mathematicians in their proof. Also in the proof that follows in the next section, some parts will talk about contributions of other mathematicians.

7.2. General structure of the proof

This section is based on an article [4] that contains a very comprehensible digest of the four colour theorem which explains the proof of Appel and Haken in a much more logical way than the original article containing the proof of Appel and Haken.

Consider an arbitrary map M in the plane. First two terms need to be defined. To construct a graph G from a map M, the **dualization** of M is performed by representing each region of M with one vertex in G and vice versa. When two countries are neighbours in M, the corresponding vertices are connected by an edge in G, what results in a planar graph G. Colouring the vertices of this graph G is equivalent to colouring the regions of the map M.

After the dualization, a **triangulation** T is obtained from G. A triangulation of a graph G is the addition of edges to the original graph G in order to make every face bounded by three edges. If the original graph is not a triangulation, two operations can be applied: (1) introducing an edge between unconnected vertices without crossing other edges (in order to keep the graph planar), and (2) merging two vertices into one if the constraint of crossing edges in (1) is violated. If either of the actions result in creating multiple edges between the same vertices, these are reduced to a single edge. The process continues until a complete triangulation T of the graph G is found, i.e. all faces are bounded by three edges. The operations can be found in Figure 7.2.

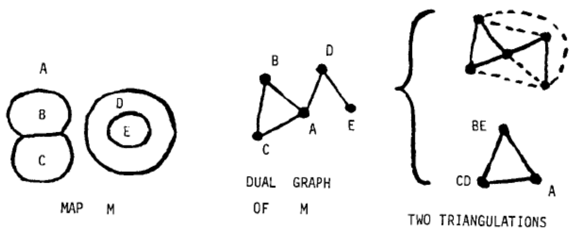


Figure 7.2: The dualization of a map M [4]

A claim is introduced that states the following:

Claim There exists a finite triangulation T in the plane, and a correspondence between regions of M and vertices of T, where every vertex colouring of T yields a proper colouring of M.

Once the triangulation T is obtained, the approach is based on induction. Assume T has n vertices. First a smaller triangulation T' with less than n vertices is created by removal of one or more vertices from T. Assuming a colouring exists for the smaller triangulation T', the goal is to transfer this colouring to T by making necessary additions and changes. The process involves reducing the size of T step by step until a colouring is immediate, and then passing the result back up the line to fit the larger triangulations. As of right now this "process" seems very vague and indeed it is a very complex process which will be explained in the best way possible. First of all, there are two restrictions to creating the graph T'.

- T' needs to be smaller than T, smaller in the sense that T' has less vertices and consequently less edges as well.
- There should be an absolutely reliable method such that any colouring of T' can be modified to produce

a valid colouring of T.

The method that incorporates these two constraints is a technique called **classical reduction**, introduced by Kempe in 1879 and generalized by Birkhoff in 1913. First a suitable vertex or cluster of vertices is selected in T. Those vertices along with their edges are deleted to create a hole (the empty space where the deleted vertices and edges used to be) in T. The boundary of the hole is called the ring. The ring together with its interior before deletion make up a **primary configuration**. The hole is then filled with a **secondary configuration**, which is smaller (less vertices and edges) than the primary configuration. T is then changed to a reduced graph T'. Now if after the substitution of the secondary configuration, T' is four-colourable and the colouring of T' can be modified to produce a valid colouring of T, then this configuration is said to be **reducible**. The details will be left out, but this primary configuration is said to be reducible if it can be substituted by a secondary configuration which makes the smaller graph T' four-colourable by the induction hypothesis. The primary configuration should be chosen in such a way that any colouring of T' can be modified to produce a colouring of T, making T four-colourable as well by re-substituting the primary configuration.

This procedure will be illustrated with an example. In Figure 7.3 there is an initial graph T that contains a primary configuration A inside the ring that is drawn dotted. The primary configuration consists of a cluster of six 5-vertices. This configuration is then substituted by a secondary configuration B (also called the effective reducer) consisting of only one 5-vertex. Next, a four-colouring of the new graph T' is found and the colouring of the ring is transferred to the initial graph T. Any valid colour pattern of the ring can then be extended to create a valid four-colouring of the primary configuration as well. The term extended requires some additional attention. If no colour changes to the ring need to be performed, then this configuration is said to be reducible and B is its reducer. If this is not the case, then the colouring of the ring and part of the graph T outside of the ring can be changed using Kempe chains. Once again it can be tried to find a valid colouring of the primary configuration and if this works then the configuration is said to be reducible as well. In conclusion, a colouring of the ring can be extended to the primary configuration A if there exists a valid colouring of the graph T as a whole only using Kempe chain-swapping on and outside the ring.

Figure 7.3: An example of classical reduction. A configuration A is found, consisting of one 5-vertex surrounded by five others. The ring is drawn dotted around the cluster of 5-vertices. This configuration is then substituted by an effective reducer B (one 5-vertex). This new triangulation T' can be four-coloured and after transferring the ring pattern to the initial graph T it can be concluded that also the colouring of T can be completed using four colours.[4]

What is left to prove is the following:

- Find configurations with specific properties that are reducible
- Demonstrate the presence of those configurations in an arbitrary triangulation T

The flaw in Kempe's proof was that one specific case, a vertex with degree five, was not reducible. The vertex of degree five was removed from T to end up with a smaller graph T'. However the colouring of T' could not logically be extended to the original graph T.

What is meant exactly by an arbitrary triangulation T? Appel and Haken were looking for a minimal counterexample to the four-colour theorem, minimal in the sense that if one vertex or edge would be removed, the new graph would be four-colourable. If they could prove this counterexample did not exist, then the four-colour theorem would be true. To find this counterexample, they came up with a lot of constraints that this counterexample had to satisfy. This counterexample makes up the arbitrary triangulation T. The details will not be given here, but in the end they proved that every minimal counterexample contains a configuration that can be replaced by a smaller configuration (less vertices and edges) to construct a smaller counterexample. This process is described above, sing the primary and secondary configurations. The smaller counterexample that was founds contradicts minimality of the counterexample and therefore the four-colour theorem holds.

Moving on, different mathematicians started working on the two processes stated above: (1) finding configurations in the minimal counterexample that are reducible and (2) showing that a reducible configuration always exists in an arbitrary graph T. With regards to (1) during the years leading up to the proof of Appel and Haken the list of known reductions was augmented by different mathematicians. Appel and Haken started working on (2): constructing an **unavoidable set**, meaning at least one member of the set could be found in an arbitrary graph T. Appel and Haken came up with two theories: one of unavoidability and one of **reduction obstacles**. First, it was relatively easy to construct a finite set of unavoidable configurations. This set was small and it was therefore easy to prove that any graph contained at least one of these members. On the other hand, some of the configurations contained reduction obstacles, meaning that if such a configuration was found, it was hard to prove it was actually reducible: a problem that Kempe encounterd as well. Appel and Haken therefore enlarged the set of unavoidable configurations, minimizing the number of obstacles making it easier to prove reducibility of the configurations. In the end they had a very large set of 1834 configurations. These configurations form the primary configurations, i.e. the part of the graph that will be removed. A few members of the set can be found in Figure 7.4, where black dots represent five-degree vertices. The real proof contains a section of 63 pages only covering different configurations. The years following the proof, some configurations were found to be redundant leaving 1482 unavoidable configurations.

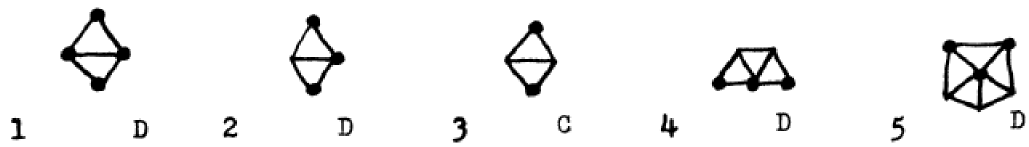


Figure 7.4: The first five members of the set of unavoidable configurations that was constructed by Appel and Haken [3]

The most complicated part in the proof is to show that every configuration in the unavoidable set is reducible. In other words the unavoidable set has to be a subset of the reducible set. If the unavoidable set is small, then the reducible set can be small as well. However, because of the reduction obstacles it was not always possible to prove reduction of some members of the unavoidable set. As a result, more members were added to the unavoidable set making the set of reducible configurations larger as well but at the same time the reduction argument was simplified. How exactly does this argument get less complex when more members are added? For example, take the same vertex that formed the flaw in Kempe's proof: a vertex of degree five. This vertex is part of the unavoidable set but not of the set of the reducible configurations. It was not possible to prove a vertex of degree five to be reducible. However, if we take this vertex out of the unavoidable set and replace it with a couple different situations, namely more vertices of degree five or one vertex of degree five and a neighbour of degree six, it appears that these configurations can be proved to be reducible (See Figure 7.3). In conclusion, the unavoidable set got enlarged but the reducibility argument simplified. If we keep on trying and finding new configurations and simplifying the reducibility argument, logically the unavoidable set gets incredibly big at one point.

A question that remains is: how did Appel and Haken come up with a set of almost 2000 configurations? A mechanism that is used for producing this unavoidable set is called **discharging**, initially introduced by H. Heesch [11]. Appel and Haken came up with an incredibly difficult charge function, which assigns values to faces, vertices, and edges of a triangulation. Then this charge is redistributed based on this charge function. Intuitively, this charging procedure works in the following way. A charge of $6 - d(v_i)$ is assigned to every vertex v_i where $d(v_i)$ defines the degree of a vertex. For vertices with high degree, there is a negative charge assigned to the vertex. Vertices with a degree less than six receive a positive charge. Now positive charge is redistributed to the neighbours of a positively charged vertex if possible in order to try to make the charge of that vertex negative. Eventually there will be vertices or clusters of vertices where there is still some positive charge. These areas (called overcharged vertices) form the complications for four-colouring a graph. Indeed there are a lot of vertices that are five-connected grouped together, which may cause difficulties in four-colouring the graph. These areas will form the members of the unavoidable set. The charge function that Appel and Haken used consisted of 487 different charging rules.

All in all, applying the discharging procedure and checking the set of almost 2000 configurations involved a significant amount of computational work. For that reason the proof was met with some controversy due to its reliance on computer assistance and the inability for human mathematicians to manually verify all the cases involved. However, subsequent studies and independent verifications have provided support for the validity of their proof.

The possibility of finding a much smaller unavoidable set of reductions for four-colour theorem is a hope that has been raised. However, it is not a straightforward task. While there may be some potential for improving the efficiency of the discharging algorithm, it would come at the cost of introducing numerous additional provisions and significantly increasing the time required for checking the associated unavoidable set. The current discharge method is already highly refined and represents a systematic approach built upon earlier counting arguments. The version used by Haken and Appel, based on Heesch's work, incorporated crucial technical modifications. Without these changes, the idea may still have potential, but Haken predicts that the unavoidable set would become very large, reaching up to 10,000 or even 50,000 (or more) members. In addition, Haken has provided generalized estimation and probability arguments that support the conclusion that the unavoidable set should consist of at least 1000 or 1500 members.

In summary, Appel and Haken constructed a set of 1834 configuration and showed all of these configuration to be reducible, i.e. if one of those configurations appeared in an arbitrary graph G, it could be replaced by a smaller secondary configuration such that the smaller graph (less vertices and edges) is four-colourable by induction. The four-colouring of the smaller graph was then proved to extend to the four-colouring of the original graph by re-substituting the primary configuration. Furthermore, using a procedure called discharging they proved that every minimal counterexample should contain at least one of the 1834 configurations making up their unavoidable set. Combining these results proves that no minimal counterexample exists and therefore the four-colour theorem holds.

7.3. Simplified proof by Robertson, Sanders, Seymour and Thomas

Luckily after the publication of the proof of Appel and Haken, there were more mathematicians that found the proof too unclear to be checked by hand and tried to come up with their own version. Others thoroughly went through the proof of Appel and Haken (A&H) and tried to explain it in a clearer way. The latter is what Neil Robertson, Daniel P. Sanders, Paul Seymour and Robin Thomas did [17]. They came up with a new proof in 1996 that is faster, easier to check by hand and it is easier to validate the computer calculations. This section will discuss their proof and highlight the differences between the original proof and their simplified version.

The four mathematicians started by reading the proof of A&H, however very soon gave this up. It would require an immense amount of time and programming to input by hand all the 1482 configurations. Instead, they decided to come up with their own version of the proof following the same general approach as A&H. This essentially describes why the proof of A&H was not fully accepted, for two reasons: [17]

1. The part of A&H that uses a computer and cannot be checked by hand.
2. The part of the proof that is supposed to be checked by hand is extraordinarily complicated and tedious

The approach of the new proof is the same to the proof of A&H, however every step has been optimized. The set of configurations consists of only 633 members. It is proved that none of these can appear in a minimal counterexample to the four-colour theorem. If one would appear, then it could be replaced by a smaller secondary configuration proving reducibility of the primary configuration. This would make a smaller counterexample, contradicting minimality of the counterexample. They proved every minimal counterexample is an internally 6-connected triangulation and that at least one of the 633 primary configuration appears in every internally 6-connected triangulation. As a result there is no minimal counterexample so the four-colour theorem has to be true. In this Section the differences between A&H and the new proof will be highlighted.

- In the new proof, unavoidability of some reducible set is proved without looking beyond the second neighbours of overcharged vertices, whereas A&H does. With overcharged vertices are meant the vertices that still contain positive charge after the discharging procedure. This adjustment avoided a lot of complications that A&H did have.
- The set of configurations only counts 633 members instead of the 1482 that A&H used.

- The discharging procedure involves 32 charging rules instead of the 487 rules that A&H use.
- In the new proof, they came up with a quadratic time algorithm to find a four-colouring of a planar graph instead of the quartic time algorithm of A&H.
- The hand-checking of unavoidability is replaced by another hand-checkable proof that can be read and checked by a computer in a few minutes, which is not possible in the proof of A&H.

This all seems very positive, however the proof is still not checkable by hand. About four hours of computer-checking time is required, which is less than the 1200 hours that A&H needed. The difference is that all the computer programs are provided and can be checked by anyone to test and guarantee validity.

8
Xiang's formal proof of the four-colour theorem

In 2009 Limin Xiang attempted to come up with a formal proof of the four-colour theorem that is comprehensible for the greater public with some background in mathematics [21]. His approach was again finding a way to get around that vertex with degree five. Kempe tried it using Kempe chains, which failed in this specific case. Therefore Xiang split up the last case into different subcases hoping to find a construction that deals with all different cases that one may encounter when colouring a graph.

First of all there is some notation to introduce.

$\mathbf{G(i,j)}$	Not necessarily connected subgraph of G consisting of the vertices that are coloured with colours i and j only, and edges connecting two of them.
$\mathbf{G^c(i,j,v)}$	Connected component of $G(i,j)$ containing vertex v.
$\mathbf{Ch(i,j,u,v)}$	Kempe chain (its definition is described in Section 4) from vertex u to vertex v coloured with colours i and j
$\mathbf{d(v_i)}$	Degree of vertex v_i
$\mathbf{c(G_i)}$	The number of colours needed to colour graph G_i
$\mathbf{C(v_i)}$	$\{c(u)\|c(u)$ is the colour of vertex u in G_{i-1} and u is adjacent to v_i in $G_i\}$
Kempe circle	Combination of a Kempe chain and a vertex w (not in the kempe chain) with edges connecting to both ends of the kempe chain such that it becomes a closed path.

Now all the new notation is introduced, it is time for the first Lemma which says:

Lemma 2. *For any planar graph G_n with $n(n \geq 6)$ vertices, there are vertices $v_n, v_{n-1}, ..., v_6$ such that $d(v_i) \leq 5$ and $G_{i-1} = G_i - v_i$ are also planar graphs for i from n down to 6.*

Xiang does not specifically proof this Lemma, but uses an earlier mentioned result (Lemma 1) and deduces that this Lemma therefore holds as well.

Next up, Xiang actually starts proving the four-colour theorem. The construction can be best explained with the help of a flowchart, drawn by Xiang himself. The flowchart can be found in Figure 8.1. At first sight, the flowchart seems scary and overly complicated with unknown notation, but clearly there is some structure which will be explained now. Consider an arbitrary graph G_n on n vertices. There are three different cases to consider. First, start at the top left in the circle that says "begin". There is a decision node that depends on the number of vertices, if this number is between one and four go to case 1, if this number is equal to five go to case 2 and if this number is greater or equal to six set i equal to six and go to case 3. In this box $c(G_i)$ defines the number of colours needed to colour graph G_i, as defined before. Case 1 and 2 will end at the bottom where it says "end". Case 3 is treated in a different chart on the right side. Again start in the circle that says "begin". The situations in which the vertices are located can be subdivided into eight different subcases. In short, the different cases depend on the following:

- How many colours are needed to colour the neighbours
- How many neighbours the vertex has
- How the colours of the neighbours are oriented
- If the colours of the neighbours belong to the same Kempe chain or not

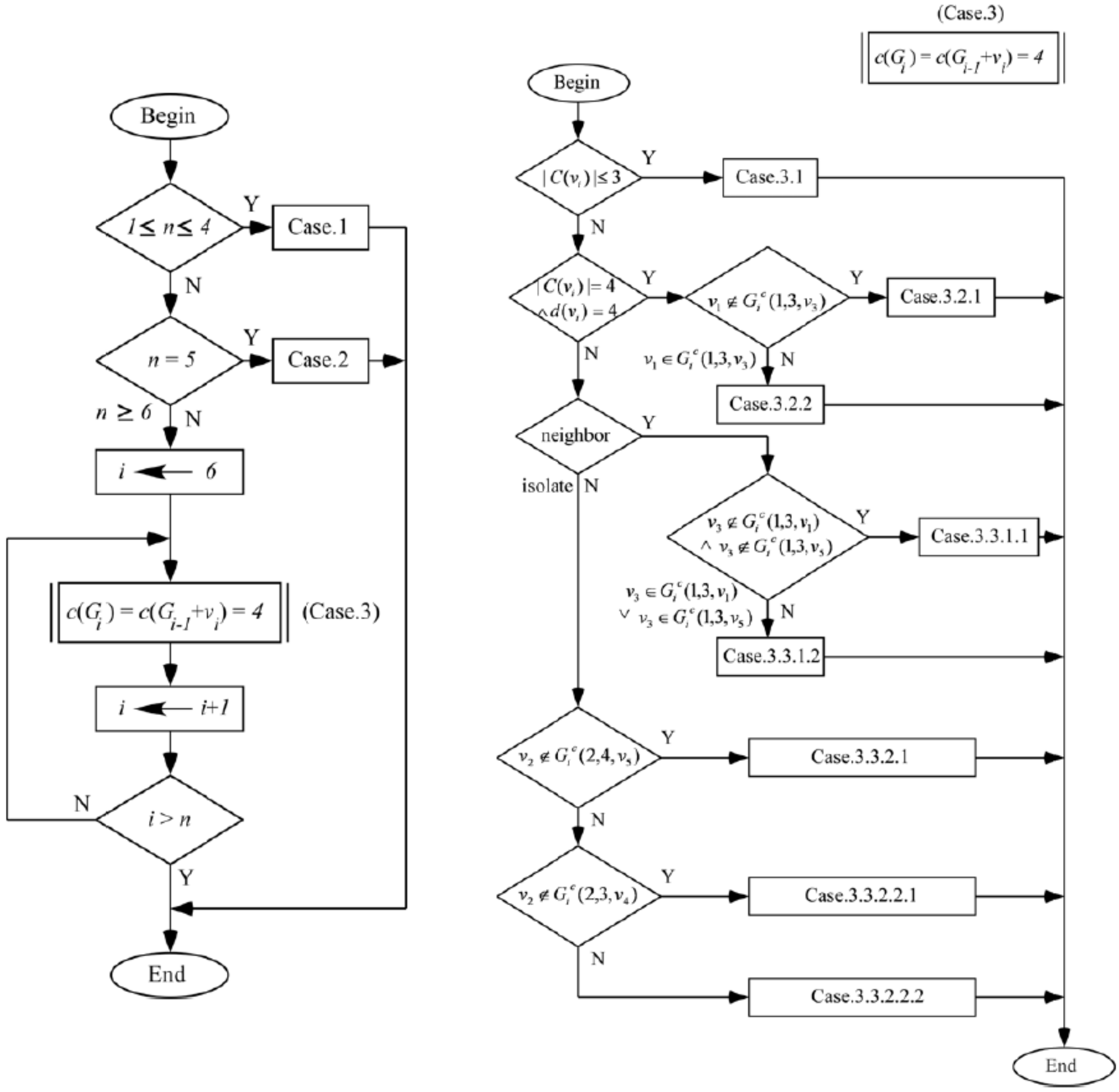


Figure 8.1: The flowchart that shows the construction of the proof of the four-colour theorem by Limin Xiang [21]

Now we have all the necessary mathematical instruments to construct the proof, which is a proof by induction: for the number of countries n we show the result holds for small n, then we assume it holds for arbitrary n and eventually we show it holds for $n+1$ countries. The proof is divided into three cases by the number of countries n.

1. $1 \leq n \leq 4$

2. $n = 5$

3. $n \geq 6$

In the first case, the result holds obviously. For the second case, note that a maximal planar graph with five vertices has nine edges. This specific case can be coloured with four colours, but then also every subgraph of this graph only uses four colours and the result follows. The third subcase deals with graphs that have six or more vertices. Lemma 1 then tells us that there is at least one vertex with degree smaller than or equal to five. Assume that the graph without this vertex v_i can be coloured by four colours by the induction hypothesis. Then we have to show that adding this vertex does not increase the amount of colours needed. Now there are a lot of different cases to consider, and here is where the flowchart in Figure 8.1 comes in handy.

3.1 v_i is surrounded by five or less countries that are coloured with three different colours

3.2 v_i is surrounded by four countries that are coloured with four different colours

3.3 v_i is surrounded by five countries that are coloured with four different colours

Again, the first subcase is obvious since there will be a colour left to colour vertex v_i. The second subcase is treated as in Kempe's proof (Section 4) with the help of Kempe chains. The third subcase is the most difficult and can be divided up into more subcases, depending on if the two vertices with the same colour are neighbours or isolated. Here Xiang mentions the word "neighbours", however this would be a violation of the four-colouring of the graph G_{i-1} since then two neighbours would have the same colour. Assumed is that Xiang means that these vertices are oriented in a way that they appear next to each other around v_i, however there is no edge connecting them.

Treating those subcases is again done by looking at the Kempe chains, where the general recipe is to swap the colours of the countries in a Kempe chain if the two countries of the same colour are not part of the same Kempe chain. If the countries are of the same Kempe chain however, then this Kempe chain together with the v_i forms a Kempe circle which seperates a Kempe chain between two other neighbours of v_i. Then the colours of one of those Kempe chains can be swapped in order to create a 3-colouring of the five neighbours leaving a fourth colour available for v_i. The details can be found in the article itself [21], however the difference in this proof compared to Kempe's proof will be emphasized here.

Kempe's fallacy was indeed in the case that there were Kempe chains from v_2 to v_4 and from v_2 to v_5, see Figure 8.2 for an overview of the vertex v_i with five vertices and their colours 1 to 4. Xiang deals with this case as follows. The circle formed by $Ch(2,3,v_2,v_4)+v_i$ separates $G_i^c(1,4,v_3)$ from $G_i^c(1,3,v_5)$. Reverse then this colouring on $G_i^c(1,4,v_3)$. Then there is a Kempe chain $Ch(2,4,v_2,v_5)$. The Kempe circle formed by this Chain and v_i separates $G_i^c(1,3,v_1)$ from $G_i^c(1,3,v_4)$. Reverse now the colouring of $G_i^c(1,3,v_1)$. Now colour 1 is available for v_i. This construction is slightly different from Kempe's approach: it considers more cases and involves another step in some subcase. Would the four-colour theorem actually be proved then?

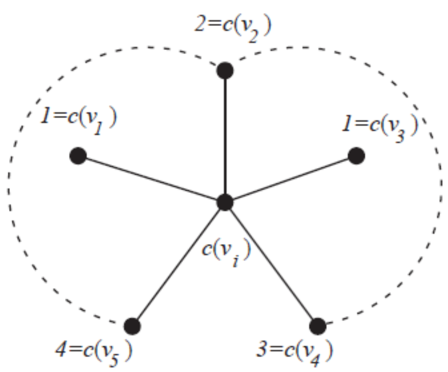


Figure 8.2: Subcase where vertex v_i is surrounded by five neighbours that are coloured with four different colours 1 to 4, such that there are two Kempe chains both connecting 2 neighbours of vertex v_i [21]

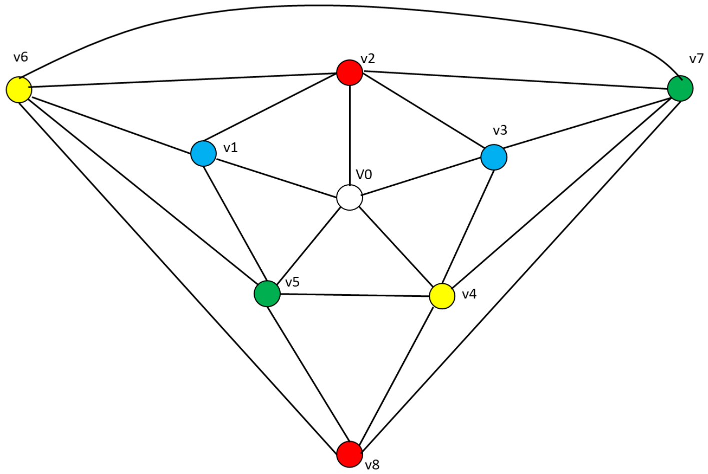


Figure 8.3: The counterexample to the proof of Kempe

Unfortunately, the answer is no. At first it might seem that this construction works perfectly because every case is carefully considered except for the one case that also formed the flaw in Kempe's proof more than a century ago. This is the case where the two Kempe chains cross each other, as in the example $Ch(2,3,v_2,v_4)$ and $Ch(2,4,v_2,v_5)$. Consider the same counterexample that Heawood found against Kempe's construction, see Figure 8.3. Following Xiang's flowchart we find ourselves in case 3.3.2.2.2. There are two Kempe chains starting from the top vertex v_2, one red-green chain and one red-yellow chain which cross at v_8. The red-yellow chain separates the blue-green chain starting in v_3 from the blue-green chain starting in v_5. Therefore we can swap the colouring of the chain starting in v_3. Now Xiang tells us there is a Kempe chain $Ch(2,4,v_2,v_5)$, however this is not possible since we just swapped the colouring of v_3 and v_7. There is no red-green Kempe chain anymore from v_2 to v_5 since v_7 is now coloured blue. The construction fails and we do not have a colour available for v_0.

In the end, there is once again no short proof to the four-colour theorem.

9

Yeh's simple proof of the four-colour theorem

10 years later in 2019 another mathematician tried yet another approach to the more than one and a half century old problem. It was Wei-Chang Yeh of the National Tsing Hua University who, according to my research, followed a completely different route than any other mathematician had ever done before [22]. The goal was clear: "provide the first correct proof of this 170-year-old mathematical problem composed with the human brain and without computer assistance in only five pages" [22]. Since this proof is so recent (in the mathematical world), no reaction has to this attempt has been documented, nor that it is valid nor that it is false. For that reason, I will give a summary of his simple proof with personal comments. In the next Chapter (10), further research on this article will be discussed and missing details to Yeh's article will be provided.

First of all, Yeh introduces the following lemma:

Lemma 3. *A planar graph is four-colourable if its related triangulated graph is four-colourable*

To understand this, the definition of a **triangulated graph** needs to be given.

Definition 2 (Triangulated graph)**.** *A triangulated graph is a graph in which each bounded face is surrounded by three edges.*

A triangulated graph G' is formed from a graph G by adding extra edges to make every bounded face be surrounded by 3 edges except for the outer, unbounded face. Any planar graph can be extended to a triangular graph G' by adding extra edges.

The proof of Lemma 3 is not that difficult. If a triangulated graph $G' = (V, E')$ of $G = (V, E)$ where $E \subseteq E'$ is four-colourable, then upon removal of the added edges the four-colouring remains unchanged. Still all adjacent nodes will have a different colour. Hence there also exists a valid four-colouring of G.

Combining the fact that any planar graph can be extended to a triangulated graph and Lemma 3, we can continue to prove the four-colour theorem for triangulated graphs. Yeh introduces another definition.

Definition 3 (Complementary coloured face)**.** *A complementary coloured face is defined as the face having the only colour that is not used to colour all three nodes in the related triangle.*

Then he deduces that all four colours are needed exactly once to colour one triangle: its three vertices and its complementary face. Now the crux of his proof is introduced: the four-colour linear system with variables as follows

$$x_{i,j} = \begin{cases} 1 & \text{if the } i\text{th component (a node or face) is coloured in the } j\text{th colour in the triangular graph} \\ 0 & \text{otherwise} \end{cases}$$

There are a total of $4 \cdot (|V| + |F| - 1)$ variables, where $|V|$ and $|F|$ are the number of nodes and faces, respectively. The -1 does not appear out of nothing but subtracts the outer unbounded face which is always present in a graph. These variables are bound to some constraints, namely that every component, i.e. a node or a triangular face, can have one and only one colour (**component equation**) and that all colours appear exactly once in every triangle including the complementary coloured face (**colour equation**). See the below equations:

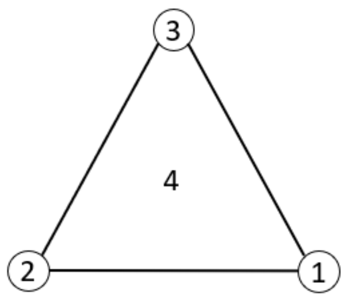
$$\text{Component eq.: } \sum_{j=1}^{4} x_{i,j} = 1 \text{ for all components i in the same triangle} \tag{9.1}$$

$$\text{Colour eq.: } \sum_{i} x_{i,j} = 1 \text{ for all components i in the same triangle and } j = 1,2,3,4 \tag{9.2}$$

For every triangle, there is a total of 8 linear equations that are divided in two groups: four component and four colour equations. See Figure 9.1a for an example of the linear equations of a single triangle. Next, remove every last colour equation and remove duplicates. Duplicates appear for example when nodes appear in multiple triangles. The component equation corresponding to that vertex will appear multiple times. In the example in Figure 9.1 there is just one single triangle so no duplicates to be removed that appear in other triangles. Then following this procedure for every triangle results in a linearly independent system of linear equations and all linear equations are equal to a constant 1 on the right side. Then Dr. Yeh concludes that the linear system has integer feasible solutions. In other words he states that the triangulated graph is four-colourable, and by Lemma 3 also the original graph is four-colourable by removal of the added edges.

Triangle	Component equations(1) and node equations (2)	Last item removed
$\Delta_{1,2,3,4}$	$x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4}=1$	$x_{1,1}+x_{1,2}+x_{1,3}+x_{1,4}=1$
	$x_{2,1}+x_{2,2}+x_{2,3}+x_{2,4}=1$	$x_{2,1}+x_{2,2}+x_{2,3}+x_{2,4}=1$
	$x_{3,1}+x_{3,2}+x_{3,3}+x_{3,4}=1$	$x_{3,1}+x_{3,2}+x_{3,3}+x_{3,4}=1$
	$x_{4,1}+x_{4,2}+x_{4,3}+x_{4,4}=1$	$x_{4,1}+x_{4,2}+x_{4,3}+x_{4,4}=1$
	$x_{1,1}+x_{2,1}+x_{3,1}+x_{4,1}=1$	$x_{1,1}+x_{2,1}+x_{3,1}+x_{4,1}=1$
	$x_{1,2}+x_{2,2}+x_{3,2}+x_{4,2}=1$	$x_{1,2}+x_{2,2}+x_{3,2}+x_{4,2}=1$
	$x_{1,3}+x_{2,3}+x_{3,3}+x_{4,3}=1$	$x_{1,3}+x_{2,3}+x_{3,3}+x_{4,3}=1$
	$x_{1,4}+x_{2,4}+x_{3,4}+x_{4,4}=1$	

(a) The component (blue) and colour equations (red) corresponding to a triangle with vertices 1, 2 and 3 and complementary coloured face 4 [22].



(b) Triangle composed of vertices 1,2 and 3 with triangular face 4 that the linear equations in Figure 9.1a correspond to.

Figure 9.1: Example of a triangle with its corresponding linear equations.

At first sight, this whole proof seems easy to understand and true. It is indeed partially correct what Yeh is saying. However there is definitely a problem with the proof, namely that it is too short. A lot of details have been left out making the proof incomplete. For example the most essential part of the proof where Yeh claims that the linear system has integer feasible solutions lacks every bit of detail. Indeed, we know that it should have solutions because the four-colour theorem is true however no proof is included as to why the system does have integer feasible solutions (that are only 0's and 1's). Upon emailing Dr. Yeh he shared with me that he has not yet published his proof, hence so far there is still not a "correct proof of this 170-year-old mathematical problem composed with the human brain and without computer assistance in only five pages", restating Dr. Yeh's words. In the next section we will try to fill in the gaps.

10

Ideas towards a new proof

After multiple tries, I have not come up with a new proof of the four-colour theorem. However, some progress has been made and once again the four-colour theorem knows yet another attempt to prove its statement. Hopefully these developments bring mathematicians closer to that one common goal: finding an elegant proof of the four-colour theorem.

The developments that have been made are built off of the proof of Dr. Yeh, which is described in Section 9. There were some issues with his proof, of which the biggest one was that the most important part of his proof has been left out, to know (1) if the system of linear equations has a solution and (2) whether this solution is an integral solution only containing values in the set $\{0,1\}$. First of all, the construction of a matrix from the linear equations is explained in Section 10.1, then (1) is proved in Section 10.2 and (2) will be attempted to proved in Section 10.3. Another approach in proving (2) will be presented in Section 10.4.

10.1. Creation of the matrix A

First of all, solving the system of linear equations is translated into solving the system $A\mathbf{x} = \mathbf{b}$, where:

- A is an $m \times n$ matrix which corresponds to the linear equations corresponding to a triangulated graph G. Its structure will be explained thoroughly in this Section.
- $\mathbf{x}$ is an n-dimensional column vector containing the variables $x_{i,j}$
- $\mathbf{b}$ is an m-dimensional column vector containing only 1's, i.e. the right side of all the linear equations corresponding to a triangulated graph G.

The structure of the $m \times n$ matrix A is as follows. Every row corresponds to either a component or a colour equation and every column corresponds to a variable $x_{i,j}$. A only takes integer values in the set $\{0,1\}$. For every vertex and triangle there is a row of four subsequent 1's corresponding to the variables that decide the colour of the three vertices and the face of that triangle, and 0's elsewhere. The matrix below corresponds to the linear equations of the triangle in the example of Figure 9.1, composed of three vertices that enclose one triangular face. The first four rows then correspond to the component equations ($comp$) and the last three rows to the colour equations (col). For a definition of the component and colour equations, see Section 9. The structure of the colour equations is a row with four non-subsequent entries equal to 1: one in every column corresponding to that colour, but only the columns that have variables corresponding to that triangle. So in the fifth row of the matrix below there is a one in the columns corresponding to variables $x_{1,1}, x_{2,1}, x_{3,1}$ and $x_{4,1}$. This means that the triangle (4) formed by vertices 1, 2 and 3, either vertex 1, 2 or 3 or triangle 4 has colour 1.

	x_{11}	x_{12}	x_{13}	x_{14}	x_{21}	x_{22}	x_{23}	x_{24}	x_{31}	x_{32}	x_{33}	x_{34}	x_{41}	x_{42}	x_{43}	x_{44}
$comp_1$	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
$comp_2$	0	0	0	0	1	1	1	1	0	0	0	0	0	0	0	0
$comp_3$	0	0	0	0	0	0	0	0	1	1	1	1	0	0	0	0
$comp_4$	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
col_1	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0
col_2	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0
col_3	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0

The following properties hold for general A (where # defines the number):

1. m = dimension of the rows of A = 4 * #triangles + #vertices
2. n = dimension of the columns of A = 4 * (#triangles + #vertices)
3. d = #component equations = #triangles + #vertices
4. m - d = #colour equations = 3 * #triangles

Finally there are three other properties that are noticeable in the structure of this matrix. First of all

Property 1. *In every row corresponding to a component equation, there is one entry equal to 1 which is the only entry in that entire column that is nonzero.*

Note that there are $n/4$ columns that have exactly one nonzero entry. Furthermore,

Property 2. *The number of 1's in a column is at least one and is directly proportional to the #triangles that the vertex or triangle corresponding to that column is incident to.*

For example if a vertex is incident to three different triangles, then the columns that correspond to this vertex have four nonzero entries: one for the component equation and three for the different colour equations of the different triangles that this vertex is incident to. The last property is the following:

Property 3. *The variables corresponding to the colour of a triangle appear in exactly one colour equation and in one component equation.*

Of the four columns corresponding to the variables that decide the colour of this triangle, three columns have two nonzero entries and one column has one nonzero entry. The reason for this is the way the colour equations are structured. Vertices should have a colour different from its enclosed triangle and its neighbouring vertices. Triangle faces can easily have the same colour as any neighbouring triangle face and therefore only appear in one colour equation corresponding to the vertices that form that triangle.

10.2. The system of linear equations has infinitely many solutions

Let A be a $m \times n$ matrix containing the system of linear equations of an arbitrary map M. It is claimed by dr. Yeh that the rows of A are linearly independent, since every last variable of all linear equations connected to a triangle does not appear in any other equation of another triangle thanks to the way the linear equations are structured. Then he concludes that the system of linear equations always has a solution. This will be more properly proved in this section.

By Property 1 all rows corresponding to a component equations have one entry equal to 1 which is the only nonzero entry in that column. This means that all those rows contain a pivot: one entry in that column equal to 1 and only zeros everywhere else. Furthermore by Property 3 all variables corresponding to the colour of a triangle appear in exactly one colour and one component equation. Subtracting the column containing the pivot for this component equation from the 3 different columns corresponding to the variables that decide the colour of the triangle face results in a column containing a pivot for that colour equation. As a result, all rows contain a pivot and there are more columns than rows implying there are columns without a pivot. Therefore using basic linear algebra the system of linear equations has infinitely many solutions.

To illustrate this process with an example, consider the matrix in Section 10.1. Note that columns 4, 8, 12 and 16 only contain one entry equal to 1 and 0's elsewhere. Therefore the first four rows corresponding to the component equations already contain a pivot. In order for the colour equations to contain a pivot as well, subtract column 16 from columns 13, 14 and 15. These four columns correspond to the variables that decide the colour of the triangle face. As a result, columns 13, 14 and 15 also contain only one entry equal to 1 which provide a pivot for rows 5, 6 and 7. In the end every row contains a pivot and we can conclude the system has infinitely many solutions. The reason this works for every single map and system of linear equations is that every fourth column contains a pivot resulting in every row corresponding to a component equation containing a pivot. Additionally, a block of 3 columns in which only two entries equal 1 and all other entries equal 0 as in columns 13, 14 and 15 in the example exists for every triangle, resulting in every row corresponding to a colour equation also having a pivot. Concluding, we can construct a pivot in every row using this procedure resulting in infinitely many solutions.

10.3. The solution to the system of linear equations only contains integer values in the set $\{0,1\}$

By the previous result we know that there are infinitely many solutions. We want to prove there is a solution that only consists of integer values 0 and 1. In other words, we want to prove that the system $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b} = \mathbf{1}$ and A an $m \times n$ matrix as described in Subsection 10.1 only containing 0's and 1's has a solution $\mathbf{x}$ taking integer values in $\{0,1\}$. This Section will initiate an approach but not completely prove this.

To start off, we use the following lemma.

Lemma 4 (Farkas' lemma [9])**.** *For every $m \times n$ matrix A and every vector $\mathbf{b} \in \mathbb{R}^m$, exactly one of the following two statements is true:*

i) $\exists \mathbf{x} \in \mathbb{R}^n$ *such that* $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$

ii) $\exists \mathbf{y} \in \mathbb{R}^m$ *such that* $\mathbf{y}^T A \geq \mathbf{0}^T$ *and* $\mathbf{y}^T \mathbf{b} < 0$

There are various ways to prove Farkas' lemma, of which the original one dates back to 1902 [9], but many other proofs are available nowadays. One of these proofs that makes use of duality will be given here [1]. A quick introduction to optimization and duality is required. An optimization problem is structured in the following way. The objective function is written down at the top and states the function and if that function needs to be minimized or maximized. The following lines contain the constraints that the solution should adhere to. A feasible solution to the optimization problem exists if there is a solution that complies with all the constraints. This solution is an optimal solution if this is the minimal/maximal (depending on the kind of problem) feasible solution to the optimization problem. Every optimization problem has a corresponding dual form with specific rules to construct the dual problem from the primal (original) problem. The details for constructing a dual are globally available and will not be discussed here but can be found in [1, pp. 64–65]. After constructing this primal-dual pair, there are certain theorems stating that a property holds simultaneously for both the primal and dual problem. One of these theorems that is necessary for the proof of Farkas' lemma is the following.

Theorem 4 ([1])**.** *If $\mathbf{x}^*$ is an optimal solution to the primal problem, then π^* is an optimal solution to the dual problem and the objective function of the primal with the optimal solution $z(\mathbf{x}^*) = w(\pi^*)$, the objective function of the dual with the optimal solution*

Proof of Farkas' lemma. We will prove the lemma by considering the following primal-dual pair.

The primal problem (P):	The dual problem (D):
$\max z = \mathbf{0}^T \mathbf{x}$	$\min w = \mathbf{b}^T \mathbf{y}$
s.t. $A\mathbf{x} = \mathbf{b}$	s.t. $A^T \mathbf{y} \geq \mathbf{0}$
$\mathbf{x} \geq \mathbf{0}$	$\mathbf{y} \in \mathbb{R}^m$

Note that $A^T \mathbf{y} \geq \mathbf{0}$ is equivalent to $\mathbf{y}^T A \geq \mathbf{0}^T$ in statement (ii) of Farkas' lemma. The proof consists of two parts. First we show that if statement (i) holds, statement (ii) cannot hold. Hence, assume that statement (i) holds, i.e. there exists an $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. This means that $\mathbf{x}$ is a feasible solution to problem (P). Since the objective function of (P) is always 0, the optimal value is $z^* = 0$. It then follows by Theorem 4 that problem (D) also has an optimal solution with optimal value $w^* = z^* = 0$. If the optimal value of the minimization problem (D) is 0, then every feasible solution will have an objective value that is at least 0. In other words, for every $\mathbf{y} \in \mathbb{R}^m$ with $A^T\mathbf{y} \geq \mathbf{0}$, we have that $\mathbf{b}^T\mathbf{y} \geq 0$. This means that there exists no $\mathbf{y} \in \mathbb{R}^m$ with $A^T\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} < 0$, i.e., statement (ii) does not hold. At this point, we have shown that statements (i) and (ii) cannot both hold simultaneously. However, we haven't shown yet that at least one of the two statements needs to hold. We will now prove this, by showing that if statement (i) does not hold, statement (ii) needs to hold.

Assume that statement (i) does not hold, i.e., there exists no $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. This means that problem (P) has no feasible solution. Then by a result about the primal-dual pair that is not discussed in this paper [1, p. 66] there are two possibilities for problem (D): either it is also infeasible, or it is unbounded. However, note that it is not possible that (D) is infeasible because $\mathbf{y} = \mathbf{0}$ is always a feasible solution. Hence, we conclude that (D) is unbounded. This means that there are feasible solutions with arbitrarily low objective function values. In particular, there must be feasible solutions with negative objective function values. Hence, there exists an $\mathbf{y} \in \mathbb{R}^m$ with $A^T\mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^T\mathbf{y} < 0$, i.e. statement (ii) does hold. □

If we can prove that the second statement of Farkas' lemma does not hold, then we can conclude that (i) is true and the system $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ such that $\mathbf{x} \geq \mathbf{0}$. We will show that for every vector $\mathbf{y}$ such that $\mathbf{y}^T A \geq \mathbf{0}^T$, $\mathbf{y}^T\mathbf{b} < 0$ cannot hold.

Let $\mathbf{1}$ be the m-dimensional column vector consisting of just 1's. Then by the structure of A: $A\mathbf{1} = 4 \cdot \mathbf{1}$, since every row of A has exactly four entries equal to 1. Then

$$(\mathbf{y}^T)A\mathbf{1} = 4 \cdot (\mathbf{y}^T)\mathbf{1} = 4 \cdot (\mathbf{y}^T)\mathbf{b}.$$

Therefore we have shown that there does not exists a vector $\mathbf{y}$ such that $\mathbf{y}^T A \geq \mathbf{0}$ and $\mathbf{y}^T\mathbf{b} < 0$. If for example $\mathbf{y}^T A \geq \mathbf{0}$, then all its entries are greater than or equal to 0, implying that the sum of all its entries is also greater than or equal to 0. This is equivalent to: $\mathbf{y}^T A\mathbf{1} \geq \mathbf{0}$ which implies $4 \cdot (\mathbf{y}^T)\mathbf{b} \geq 0$ as well and the result follows.

In summary, we have shown that (ii) cannot hold and therefore (i) holds by Farkas' lemma. The system $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \geq \mathbf{0}$.

The final part of the proof is to show that the solution $\mathbf{x}$ only takes integer values. Since we know that $\mathbf{x}$ only takes nonnegative values, still these values can be real numbers which does not give us the required results as we are looking for integer values in the set $\{0, 1\}$. For this we need the following theorem

Theorem 5 (Integer Farkas' lemma[6]). *Let $A\mathbf{x} = \mathbf{b}$ be a rational linear system. There exists an integral solution $\mathbf{x}$ satisfying $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{y}^T\mathbf{b}$ is an integer for each rational vector $\mathbf{y}$ such that $\mathbf{y}^T A$ is integral.*

Note that a vector $\mathbf{y}$ is integral if every entry of the vector is integer. The proof of this lemma can be found in Appendix A.

We have to show that for every rational $\mathbf{y}$ such that $\mathbf{y}^T A$ integral, $\mathbf{y}^T\mathbf{b}$ is integer as well. First we examine when $\mathbf{y}^T A$ is integral. Recall Property 1 that states that every component equation has exactly one entry equal to 1, which is the single nonzero entry in its entire column. This is always the column corresponding to the fourth entry that is equal to 1. For $\mathbf{y}^T A$ to be integer, the entry in $\mathbf{y}$ corresponding to this row has to be integer. In other words, all the entries of $\mathbf{y}$ corresponding to the rows that represent the component equations have to be integer (1).

Furthermore, by Property 3, the variables corresponding to colouring a triangle always appear in exactly one colour equation. By (1) we have already seen that $\mathbf{y}$ has an integer entry for every row corresponding to a component equation. So now the entries of $\mathbf{y}$ corresponding to those colour equations also have be integer. Namely $f + g$ with f integer can only be integer itself if g is integer as well. Since every colour equation appears in exactly one triangle, this holds for all colour equations and therefore all entries of $\mathbf{y}$ must be integer for $\mathbf{y}^T A$ to be integer. Finally, since $\mathbf{b}$ is just the $\mathbf{1}$ vector and therefore integer, $\mathbf{y}^T b$ is integer as well. Therefore Theorem 5 holds and we can conclude that there exists an integer solution.

Summarizing, we have shown the following:

1. The system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions

2. The system $A\mathbf{x} = \mathbf{b}$ has a solution that is greater than or equal to 0

3. The system $A\mathbf{x} = \mathbf{b}$ has a solution that is integral

We want to conclude that the system $A\mathbf{x} = \mathbf{b}$ has an integer solution $\mathbf{x}$ in the set $\{0, 1\}$. Unfortunately there is a problem with this proof. Indeed it is shown that there exists a solution $\mathbf{x}$ such that $\mathbf{x} \geq 0$ and there exists a solution $\mathbf{x}$ that is integral. However, this does not necessarily have to be the same solution. We therefore cannot conclude that there exists a solution taking solely integer values in the set $\{0, 1\}$, which is what we set out to prove. Hopefully these results have further progressed the search for an elegant proof of the four-colour theorem and maybe someone can supply the missing link to finalize the proof. In the end, there still does not exist an elegant proof to the four colour theorem and we should settle for the proof of Appel and Haken.

10.4. Using superadditivity to show an integral solution in the set $\{0, 1\}$ exists

After the realization that the previous additions to the proof contain a fallacy, yet another idea to complete the proof is introduced. This idea makes use of a discrete version of Farkas' lemma that is based on superadditivity. This will all be explained in this section.

Before we introduce a new theorem, some notation needs to be clarified:

$$\begin{aligned}
\mathcal{M} &:= \{1, \ldots, m\} \\
\mathcal{N} &:= \{1, \ldots, n\} \\
\mathcal{F} &:= \{F : \mathbb{R}^m \to \mathbb{R} | F(\mathbf{0}_m) = 0, F \text{ superadditive}\} \\
\mathbb{Z}^n_+ &:= \{(x_1, ..., x_n)^T : x_1, ..., x_n \in \mathbb{Z}_{\geq 0}\}
\end{aligned}$$

The theorem makes use of superadditive functions. A function F is superadditive if the following holds for all x and y in the domain of F

$$F(x + y) \geq F(x) + F(y)$$

Finally, the theorem itself will be stated which is a variant on Farkas' lemma.

Theorem 6 (A discrete version of Farkas lemma [7])**.** *Let the matrix $A \in \mathbb{Z}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$ arbitrary. Exactly one of the following two statements is true*

i $\{\mathbf{x} \in \mathbb{Z}^n_+ : A\mathbf{x} = \mathbf{b}\} \neq \emptyset$

ii $\{F \in \mathcal{F} : F(\mathbf{b}) < 0, F(\mathbf{a}_j) \geq 0$ *for all* $j \in \mathcal{N}\} \neq \emptyset$

Proof. Consider the integer program:

$$\max \{\mathbf{0}_n^T x : A\mathbf{x} = \mathbf{b}, \mathbf{x} \in \mathbb{Z}^n_+, \mathbf{b} \in \mathbb{R}^m\}$$

and its superadditive dual

$$\min \{F(\mathbf{b}) : F \in \mathcal{F}, F(\mathbf{a}_j) \geq 0, \forall j \in \mathcal{N}\}$$

These integer programs are a primal-dual pair due to rules of constructing a dual that are explained in [1, pp. 64–65]. For the forward direction, suppose the primal is feasible and has optimal objective 0. Then by Theorem 4 the optimal solution to the superadditive dual is 0. Therefore, for every $F \in \mathcal{F}$ such that $F(\mathbf{a_j}) \geq 0$ for all $j \in \mathcal{N}$, $F(\mathbf{b}) \geq 0$ as well.

For the other direction let $\bar{F} \in \mathcal{F}, \bar{F}(\mathbf{b}) < 0, \bar{F}(\mathbf{a}_j) \geq 0, \forall j \in \mathcal{N}$. Suppose $\bar{\mathbf{x}} \in \mathbb{Z}^n_+$, and $A\bar{\mathbf{x}} = \mathbf{b}$. Then $0 > \bar{F}(\mathbf{b}) = \bar{F}(A\bar{\mathbf{x}}) \geq \sum_{j=1}^n \bar{F}(\mathbf{a}_j)\bar{x}_j \geq 0$, where the second inequality is verified by superadditivity of the function F. This clearly leads to a contradiction $0 > 0$. We have to conclude that (i) cannot hold. □

To use this theorem, the constraints need to be satisfied and furthermore we want to prove that the system $A\mathbf{x} = \mathbf{b}$ has nonnegative integer solutions. Therefore, we want to show that the second statement cannot be true and therefore the first statement should hold. To prove that the second statement does not hold we must show that there does not exist a superadditive function for which $F(\mathbf{b}) < 0$ and $F(\mathbf{a}_j) \geq 0 \forall j \in \mathcal{N}$.

In our case, the matrix A indeed only takes values in $\mathbb{Z}^{m \times n}$ and $\mathbf{b}$ in $\mathbb{R}^m$. The constraints to use this theorem are met and we continue by showing that the set in the second statement is empty. First of all, note that every $\mathbf{a_j}$ corresponds to a column in the matrix A. Recall that every row in the matrix A has exactly four entries equal to 1 and all other entries equal to 0. On the other side, $\mathbf{b}$ is a column vector of straight ones. Therefore the element-wise sum of all the columns of A is equal to four times the vector $\mathbf{b}$.

$$\sum_{j=1}^{n} \mathbf{a_j} = 4 \cdot \mathbf{b}$$

Also

$$F(\sum_{j=1}^{n} \mathbf{a_j}) = F(4 \cdot \mathbf{b})$$

Using superadditivity and the constraints on an arbitrary function F in statement 2 leads to the following:

$$F(4\mathbf{b}) = F(\sum_{j=1}^{n} \mathbf{a_j}) \geq \sum_{j=1}^{n} F(\mathbf{a_j}) \geq 0$$

The inequaility follows from the superadditivity of F. There is however one issue that makes the proof incomplete: we cannot assume that $F(4\mathbf{b}) < 0$. If that would be true then the above statement would reach a contradiction $0 > 0$ and (ii) of Theorem 6 would not hold implying there is a solution in the set $\mathbb{Z}^n_+$, essentially proving the four-colour theorem. The only problem is that we don't know whether $0 > F(\mathbf{b}) \geq F(4\mathbf{b})$. If F would be a function that acts on every entry of the input independently, then using a similar construction to the second statement of Farkas' lemma (Lemma 4), the second statement could be disproved. However there are too many different functions to consider in order to prove that the function F needs to take on this specific form.

Also this approach reaches a dead end. So far, there has not been any success in proving that $F(\mathbf{b}) \geq F(4\mathbf{b})$. Again, if this step can be completed then the second statement of Theorem 6 does not hold, implying that the first statement should hold proving there is an integral solution that is greater than or equal to 0 and hence proving the four-colour theorem.

11

Related topics

While most researchers working on the four-colour theorem are still occupied with finding a proof that satisfies every mathematician, others have shifted their focus to different applications. In this section related topics to the four-colour theorem will be covered.

11.1. Torus

It would not come as a big surprise to tell that mathematicians always look at ways to change the problem in any dimension and see if it still holds. This is also the case with the four-colour theorem. It started off with colouring a 2-dimensional map, but the question of colouring other topological spaces is not far from home. One such topological space that has already been researched is the torus, see Figure 11.1. Intuitively one would say that more colours are needed to colour a map on a torus and this is true, 7 to be precise as can be seen in Figure 11.1: all 7 countries touch each other. This number did not magically come out of nothing, but was found by Heawood in 1890 when he came up with the Heawood conjecture [15] specifying a lower bound for the number of colours $\gamma(g)$ needed to colour a surface of genus g.

$$\gamma(g) = \lfloor \frac{7+\sqrt{1+48g}}{2} \rfloor \tag{11.1}$$

where g denotes the genus of a surface. The genus of a torus equals 1, that of a 2-dimensional map equals 0 implying that a map can be coloured using at least 4 colours. The 4-colour theorem in its turn proves that 4 is also the upper bound.

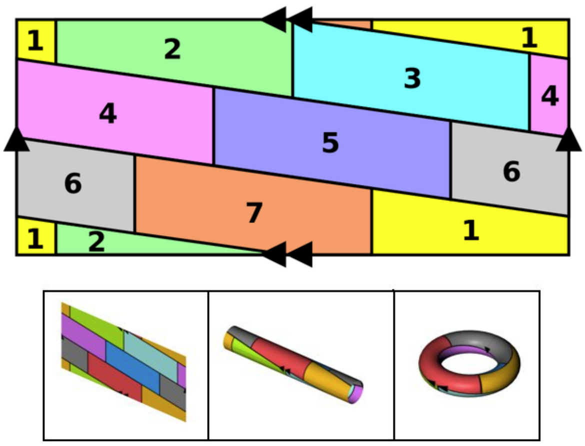


Figure 11.1: An example of colouring a map on a torus using 7 colours

It was not until 1986 that the formula was proved, by Gerhard Ringel and Ted Youngs. Their proof can be found in [16]. In the 15 years leading up to the proof, there were various mathematicians that independently proved different cases, i.e. the formula for different genera. Especially finding a lower bound for the cases with high genus formed a challenge. There is however one case that forms an exception to the formula: the Klein bottle (see Figure 11.2). The formula says that you will need 7 colours whereas in practice only 6 colours are required. This was discovered by Philip Franklin in 1930.

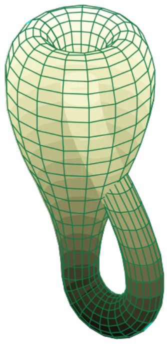


Figure 11.2: The klein bottle

11.2. Disconnected countries

Another interesting variant of the four-colour problem is the case where countries need not necessarily be connected. For example take the United States with one of its states, Alaska, which is not connected to the other states but has a country, Canada, disconnecting it. Is it still possible to four-colour this map? Well, in this specific case yes but in general no. If disconnected regions of the same country must be coloured the same then a four-colouring may not exist. See for example Figure 11.3 that contains a counterexample to four-colouring a map with disconnected regions. Is there anything useful to say about how many colours one may need then? The short answer is no, for the simple reason that there is no bound on how disconnected a country may be. If there are n countries that all consist of n disconnected regions that all need to have the same colour, then it is possible for them to all touch each other, resulting in a minimal n-colouring of the graph. Therefore, for the four-colour theorem to hold the constraint that all countries have to be connected cannot be ignored.

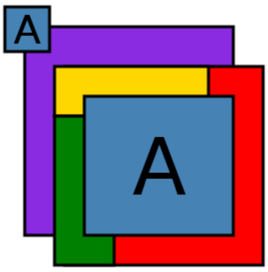
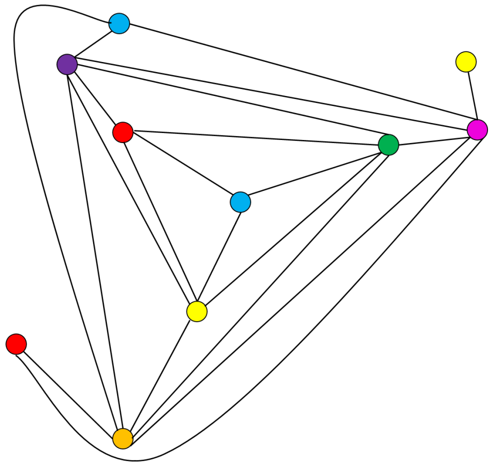
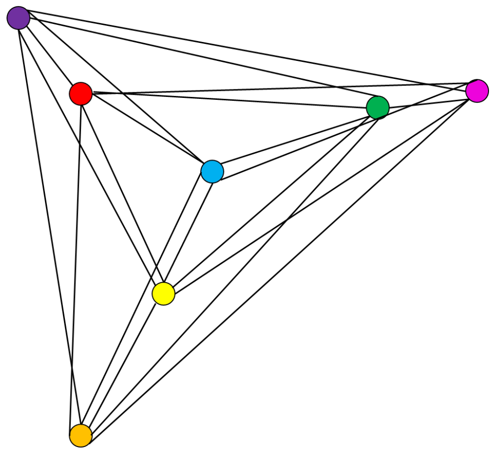


Figure 11.3: A map where one country (A) is disconnected and needs to be coloured in the same colour, resulting in a 5 instead of 4-colouring of the map [12]

This can be best explained by an example, see Figure 11.4a. As with a regular map M we convert M to a graph G. Only difference is that if a country is disconnected into different regions, represent this country as just one vertex. An edge is drawn in between vertices if their corresponding regions are neighbouring. Since the countries can be disconnected, the graph representing this map can contain more internally connected vertices than in a graph that represents a map without disconnected countries. This will result in the map not being planar anymore (Figure 11.4b. Planarity was a constraint of the four-colour theorem and therefore the four-colour theorem does not hold anymore. For this example there are 7 colours needed instead of 4, for the reason that the two blue countries, the two red countries and the two yellow countries should be coloured the same.



(a) A map containing disconnected countries represented as graph in which disconnected regions belonging to the same country that should have the same colour are coloured the same, i.e. red, blue and yellow



(b) A map containing disconnected countries represented as graph in which disconnected regions belonging to the same country are represented as one vertex. Note that the graph is not planar anymore

11.3. Applications

Maybe some wonder towards the end of this report, why we care so much about the proof of this theorem. There must be a ton of applications and its proof must be of great value to numerous sectors outside of mathematics. Well, I must disappoint you since there are not many applications of four-colouring graphs. Geographers say that they can easily use 5 colours to colour their maps and also outside of geography and mathematics there is no real use of the four-colour theorem. Although graph-colouring in general is of great importance to many daily optimization problems and is widely used, the specific four-colouring is not that valuable. Consider it the inner drive of mathematicians to find a satisfactory answer to this problem and be able to verify the problem by hand using a nice mathematical construction. No harm will be done if this problem would live on for another 100 years. On the other hand, while trying to come up with a proof for the four-colour theorem, many useful constructions and results have been established that can also be of great importance to other fields. For example the discharging procedure or a result like Heawood's equation (11.1) are significant results that know applications outside of the four-colour theorem as well. The four-colour theorem is therefore a good base to start researching and opens up to various possibilities for further research. Therefore it would be fair to conclude that a problem like the four-colour theorem is very important to mathematics in general and mainly the problems that arise in the field of graph theory.

12
Conclusion and discussions

The purpose of this report was to analyse the history and developments around the four-colour theorem. After a compact history to the problem, the research started with analyzing Kempe's proof in 1879 and the reason why his attempt failed thanks to Heawood 11 years later. Heawood then proved the five-colour theorem. It was tried to treat the comlex proof of Appel and Haken in a comprehensive way. More recent attempts of dr. Xiang and dr. Yeh were covered next. Dr. Xiang came up with a different approach to deal with the fallacious case in Kempe's proof, however this was unsuccessful. Dr. Yeh introduced a revolutionary idea of representing the four-colour theorem as a system of linear independent equations but lacked the details and most important reasoning to complete the proof. Some of those details were then provided: proving that the system of linear equations representing the colouring of a graph always has a solution. However, the proof that this solution only takes integer values in the set $\{0,1\}$ is still lacking. In the end a light was shed on variations on the four-colour theorem and its applications. To conclude, this research has contributed to constructing a new proof of the four-colour theorem by providing two new ideas. The first being the combination of Farkas' lemma and the integral version of Farkas' lemma to show an integral solution in the set $\{0,1\}$ exists. The second option is that the answer might be in the discrete version of Farkas' lemma using superadditivity to prove the same. Further research in this field may provide the missing link in reaching a simple proof of the four-colour theorem.

In short, after 170 years of the original statement there is still no consensus about the proof of the four-colour theorem. Until today new attempts are made to come up with a simple, elegant and logical proof of the four-colour theorem that does not require computer assistance. In the end, the goal is to reach a simple proof that is understandable for the greater public and does not use hours of computer calculations to check thousands of different cases. Even if this proof is to be found in the near future, is the Mathematics community ready to believe it can be proved in only 5 pages? Nevertheless, it is a beautiful problem that is a perfect example of a statement that is easy to understand for non-mathematicians but where an immensely complex proof lies in hiding.

Another question that might keep the reader busy is the purpose of this problem. Why not simply use five colours, one may ask. And this is a whole other side to the story that can be asked with many mathematical problems. Why is this specific problem so important that we have spent more than 170 years researching and arguing? Good question: for the beauty of math? One answer is that along the way a lot of useful results and arguments were discovered that have various applications outside of the four-colour theorem. Another possibility is that We know there has to be an answer and this answer can be logical and elegant with all the mathematical tools available, only it has to be found by someone. And the frustrating part is that no one has succeeded yet in finding a proof to this relatively "easy" problem. Although breakthroughs have been made and important results and properties have been discovered starting with Kempe-chains all the way to constructing an unavoidable set, still there is a key element missing to finish the proof. Lies the answer in discharging and reducibility [2]? Or is the system of independent linear equations the way to go [22]? Clearly the discussion is not over yet and the four-colour theorem will patiently await its formal satisfactory proof.

A

Proof of integral Farkas' lemma

Proof. The proof consists of two parts, the forward and the backward implication. For the forward implication we prove the inverse of the statement, namely if there exists a vector $\mathbf{y}$ such that $\mathbf{y}^T A$ is integral but $\mathbf{y}^T\mathbf{b}$ is not an integer then the system $A\mathbf{x} = \mathbf{b}$ has no integer solution. Say $A\mathbf{x} = \mathbf{b}$ had an integral solution $\bar{\mathbf{x}}$. We should have $\mathbf{y}^T A\bar{\mathbf{x}} = \mathbf{y}^T\mathbf{b}$ for every vector $\mathbf{y}$. Since $\bar{\mathbf{x}}$ is integral and the vector $\mathbf{y}$ is such that $\mathbf{y}^T A$ is integral, it follows that $\mathbf{y}^T\mathbf{b}$ (which is equal to $\mathbf{y}^T A\bar{\mathbf{x}}$) should be an integer. This contradicts the choice of $\mathbf{y}$ in the proposition and hence, $A\mathbf{x} = \mathbf{b}$ has no integral solution.

For the backward implication suppose $\mathbf{y}^T\mathbf{b}$ is an integer for all $\mathbf{y}$ such that $\mathbf{y}^T A$ is integral. This means that there is no y such that $\mathbf{y}^T A = \mathbf{0}$ and $\mathbf{y}^T\mathbf{b}$ is not an integer. This implies that there is no $\mathbf{y}$ such that $\mathbf{y}^T A = 0$ and $\mathbf{y}^T\mathbf{b} \neq 0$ (otherwise, scale $\mathbf{y}$ to violate the previous statement). Thus, for all $\mathbf{y}$ such that $\mathbf{y}^T A = \mathbf{0}$ implies $\mathbf{y}^T\mathbf{b} = 0$. In other words the null space of A^T is a subset of the orthogonal complement of $\mathbf{b}$: $\text{Nul}(A^T) \subseteq \{\mathbf{b}\}^\perp$. Also by taking the orthogonal complement on both sides and applying basic linear algebra we get: $\text{Nul}(A^T)^\perp = \text{Col}(A)$. Hence $\mathbf{b} \in \text{Col}(A)$ and therefore $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x} \in \mathbb{R}^n$.

Before we continue, there are some definitions and a theorem that need to be introduced.

Definition 4. *A matrix U is unimodular if it is integral and det(U)*$\in \{\pm 1\}$.

Definition 5 (Hermite normal form)**.** *The matrix A is in Hermite Normal Form (HNF) if* $A = [B\ 0]$ *where B is lower-triangular, has non-negative entries with the diagonal entry being the unique maximum entry in each row.*

Theorem 7. *If A has full rank, then*

(i) HNF(A) is unique and

(ii) there exists a unimodular matrix U such that HNF(A) = AU

We may assume that rows(A) are linearly independent (otherwise, work with a linearly independent subset of rows of A and the corresponding entries of $\mathbf{b}$), i.e., A has full row rank. Let $HNF(A) = AU$ for some unimodular matrix U. Then $AU = [B\ 0]$ where B is invertible. Then $B^{-1}[B\ 0] = [I\ 0]$. Therefore, by taking $\mathbf{y}^T = [B^{-1}]_j$ for any row j of the matrix B^{-1} makes $\mathbf{y}^T B$ integral. That is, $\mathbf{y}^T[B\ 0]$ is integral which means that $\mathbf{y}^T AU$ is integral, i.e., $\mathbf{y}^T AUU^{-1}$ is integral (since U is unimodular). Therefore, $\mathbf{y}^T A$ is integral and hence, $\mathbf{y}^T\mathbf{b}$ should be an integer. Thus, $\mathbf{y}^T\mathbf{b}$ is an integer for all rows $\mathbf{y}^T$ of B^{-1}.

Therefore, $B^{-1}\mathbf{b}$ is an integral vector. Now we observe that

$$HNF(A)\begin{bmatrix} B^{-1}\mathbf{b} \\ 0 \end{bmatrix} = [B\ 0]\begin{bmatrix} B^{-1}\mathbf{b} \\ 0 \end{bmatrix} = \mathbf{b} \tag{A.1}$$

This implies that

$$\begin{bmatrix} B^{-1}\mathbf{b} \\ 0 \end{bmatrix}$$

is an integral solution to the system $HNF(A)\mathbf{y} = \mathbf{b}$. Therefore, there exists an integral solution $\mathbf{y}$ to $AU\mathbf{y} = \mathbf{b}$. Consequently, there exists an integral solution $\mathbf{x}$ to $A\mathbf{x} = \mathbf{b}$ (by setting $\mathbf{x} = U\mathbf{y}$) [6]. □

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