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Control of Reaction-Diffusion Processes under Communication Delays

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Abstract— **In this paper we investigate the design of optimal spatially distributed controllers for a linear and spatially invariant reaction-diffusion process over the real line. The controller receives state measurements from different spatial locations with non-negligible delays. In this set-up and for the class of proportional spatially invariant state feedback controllers, the optimal control synthesis problem is equivalent to a feedback gain optimization for a spatially distributed delay system. We show that the spatial locality of optimal feedback gains is affected not only by diffusion and reaction coefficients, but also by the parameter representing communication time-delay that causes a sharp flattening of the control gains. In the expensive control regime, the optimal controller is solved analytically, yielding some practical design guidelines.**

I. Introduction

In large-scale networked and spatially distributed systems, the all-to-all communications required by centralized controllers is typically unfeasible. As a consequence, distributed control techniques have long been used in such systems to trade performance for the reduced complexity of the controller architecture. However, the synthesis of optimal distributed controllers poses inherent challenges, as it usually involves solving non-convex optimization problems. Nonetheless, some specific classes of systems are amenable to tractable formulations. Spatially invariant plants are one of these.

Spatial invariance is a useful abstraction to model multi-agent systems with homogeneous agents or distributed sensing applications. As shown in [1], optimal centralized controllers for spatially invariant systems are both spatially invariant and localized in space. Such a spatial localization implies that the importance for control of measurements from far away decays exponentially fast with the distance to the actuator. The inherent spatial locality of optimal centralized controllers raises the natural question of whether using only “close-by” measurements for feedback control could provide a viable means of designing distributed controllers with near-optimal performance. This could be achieved, for example, by setting the optimal centralized feedback gains to zero beyond a prescribed distance from the actuator, naming, performing *spatial truncation* of the optimal controller. However, such spatial truncation cannot always be arbitrary: as proved in [2], truncation can degrade control performance severely and even make the closed-loop system unstable. Consequently, characterizing the spatial localization of centralized control feedback operators is essential to guide the selection of an appropriate “level of truncation” yielding admissible closed-loop performance. Such a characterization has been quite well studied in the standard delay-free setting [1], [3].

Structural constraints are commonly imposed on controllers for large-scale systems: either measurements from a subsystem are available to the controller or not; if they are, typically the same communication latency is assumed between all accessible subsystems and the controller. A different line of work studies the control problem accounting for the communication latency: measurements from subsystems that are spatially afar from the actuator can only be obtained with a higher communication latency. Alternatively, the scenario in which communications of the actuator with additional subsystems incur a cost can be analyzed. These alternative formulations can lead to a distributed structure on the feedback controller (rather than a controller with an all-to-all communication structure). Within the context of spatially invariant systems, early work [4], [5] studied optimal controllers for cone- and funnel-causal plants where information propagates with a delay that depends on inter-site distance, imposing the same communication structure to the controller and proving structural properties of optimal $\mathcal{H}_2$ proportional controllers. The above works were extended in [6] to state-space representations, which provided algorithmic solutions to design suboptimal controllers. Recent work [7], [8] studied the performance trade-off between all-to-all and decentralized controller architecture under the assumption that delays depend on the number of communication links, showing optimality of distributed controllers with sparse interconnections. Departing from spatially invariant systems, but under similar assumptions for TDMA communication, the line of work [9], [10] proved optimality of decentralized controllers for distributed plants communicating over regular lattices of dimension greater than one. There is also work available [11] on studying the optimal structure for such a controller under specific assumptions. Reference [12] and follow-up work studied robust control for delay systems. Works on control of parabolic PDEs considered input and output delays via cascades of PDEs whose stability is analytically assessed [13], [14]. However, these works do not address spatial locality properties of the feedback controller.

In this work, we study the impact of communication delays on the spatial localization of an optimal feedback

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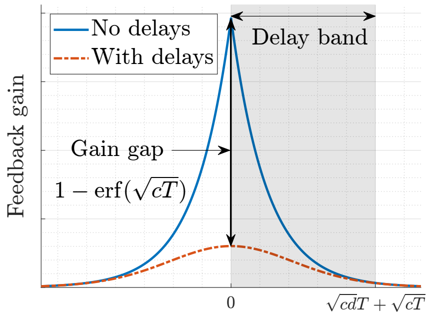


Fig. 1: Optimal feedback controller for a reaction-diffusion process over $\mathbb{R}$. The feedback gains that weigh state measurements from spatially distant locations are smaller and flatter across space in the presence of delays, making measurements from far away more important than in the delay-free setting. The figure shows the region where delays induce major differences between the optimal control gains in the delay-free setting and in the presence of delays ("Delay band"), whose width grows with both delay T and coefficients of reaction c and diffusion d, and the gap between the feedback gains that multiply the state measurement at the same location of the control ("Gain gap"), that grows with reaction coefficient and delay.

controller for a spatially invariant reaction-diffusion process. We consider the case where a feedback controller has access to delayed measurements of the state with the delay representing communication latency. We assume that the communication latency is constant across subsystems. We show that communication delays have a significant impact on the spatial structure of the optimal feedback gains, and characterize how the structure depends on the parameters of the system dynamics and delay. As illustrated in Fig. 1, the optimal feedback gains experience a sharp flattening as compared to the optimal controller without delays. This implies that the importance between close-by and far-away measurements for control becomes more uniform as a response to delays. This characterization may yield useful design guidelines in terms of selecting the appropriate truncation of the optimal feedback gains for distributed control implementations.

This paper is organized as follows. Section II introduces the dynamical system and the optimal control design problem considered. Section III presents background and optimal control of scalar retarded equations. Section IV provides the main results: an analytical and numerical characterization of the spatial localization properties of the optimal centralized controller for reaction-diffusion equations, in the setting where the feedback controller has access to delayed measurements of the state. Conclusions are drawn in Section V.

II. Setup

For technical preliminaries on translation/spatial invariance and spatial Fourier Transform, see preprint [15, Section II-A].

A. System Model

Dynamics. We consider the following reaction-diffusion dynamics that evolves on the real line $\mathbb{R}$:

$$\frac{\partial \psi}{\partial t}(x,t) = d\frac{\partial^2 \psi}{\partial x^2}(x,t) - c\psi(x,t) + u(x,t) + n(x,t), \quad \text{(II.1)}$$

where $\psi(\cdot,t) \in L^2(\mathbb{R})$ is the state of the system at time $t \geq 0$, $u(\cdot,t) \in L^2(\mathbb{R})$ is the spatially distributed control input at time t, and $n(\cdot,t) \in L^2(\mathbb{R})$ is an exogenous bounded disturbance. The symbol $x \in \mathbb{R}$ denotes the spatial coordinate. The constants $d > 0$ and $c > 0$ are diffusion and reaction coefficients, respectively. The state $\psi(\cdot,t)$ is fully observed, but only time-delayed measurements are available for control.

Controller. We consider the following proportional feedback controller that is fed with the delayed state of the system:

$$\begin{aligned} u(x,t) &= -[\mathcal{K}\psi(t-T)](x) \\ &= -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} K(x-y)\psi(y,t-T)\,\mathrm{d}y. \end{aligned} \quad \text{(II.2)}$$

The operator $\mathcal{K} : D \subseteq L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a spatial convolution (*i.e.,* spatially invariant) between the kernel $K(\cdot)$ and the delayed state $\psi(\cdot, t-T)$, and describes how the controller utilizes state measurements. The time-delay $T > 0$ represents communication delay incurred when collecting state measurements from different spatial locations.

For the sake of analysis, we assume what follows.

Assumption 1 (Spatial properties of the controller)**.**
1) *The convolutional kernel $K(\cdot)$ is an even function.*
2) *The controller has an infinite communication range,* i.e., *no restriction on the spatial spread of the convolutional kernel K is enforced. Hence, the controller at location x has access to the full (time-delayed) state $\psi(\cdot, t-T)$ to compute $u(x,t)$.*

Control (II.2) modifies dynamics (II.1) into the following closed-loop system, which is spatially invariant for every T:

$$\frac{\partial \psi}{\partial t}(x,t) = d\frac{\partial^2 \psi}{\partial x^2}(x,t) - c\psi(x,t) - [\mathcal{K}\psi(t-T)](x) + n(x,t). \quad \text{(II.3)}$$

B. Problem Formulation

Given a parameter $r > 0$ that weighs control effort, we address the performance output

$$z(x,t) \doteq \begin{bmatrix} \psi(x,t) \\ \sqrt{r}\,u(x,t) \end{bmatrix} = \begin{bmatrix} \psi(x,t) \\ -\sqrt{r}\,[\mathcal{K}\psi(t-T)](x) \end{bmatrix} \quad \text{(II.4)}$$

and focus on minimizing the $\mathcal{H}_2$-norm of system (II.3) from n to z, which we denote by J:

$$\mathcal{K}_T^{\text{opt}} = \arg\min_{\mathcal{K}} \quad J(\mathcal{K}) \quad \text{(II.5a)}$$

$$\text{subject to} \quad \text{(II.3)}. \quad \text{(II.5b)}$$

In (II.5), the operator $\mathcal{K}$ is a spatial convolution with even time-invariant kernel K according to (II.2) and Assumption 1. Moreover, we use the convention that the $\mathcal{H}_2$-norm J is infinite for unstable systems.

Problem decoupling. Because the closed-loop system (II.3) is spatially invariant by construction, applying the spatial Fourier transform decouples the spatiotemporal dynamics into a family of decoupled subsystems parameterized by $\lambda \in \mathbb{R}$:

$$\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t}(\lambda,t) = -(c+d\lambda^2)\hat{\psi}(\lambda,t) - \hat{K}_\lambda\hat{\psi}(\lambda,t-T) + \hat{n}(\lambda,t)\,. \quad \text{(II.6)}$$

The functions $\hat{\psi}(\cdot,t)$ and $\hat{n}(\cdot,t)$ are the spatial Fourier transforms of $\psi(\cdot,t)$ and $n(\cdot,t)$, respectively, and $\hat{K}_\lambda$ is the Fourier symbol of $\mathcal{K}$ – *i.e.,* the Fourier transform of the kernel K, see (II.2). By virtue of Assumption 1, it holds $\hat{K}_\lambda \in \mathbb{R}$. The Transfer Function from $\hat{n}(\lambda,\cdot)$ to $\hat{\psi}(\lambda,\cdot)$ is

$$\hat{H}_\lambda(s) \doteq \frac{1}{s+c+d\lambda^2+\hat{K}_\lambda \mathrm{e}^{-Ts}} \tag{II.7}$$

with $s \in \mathbb{C}$, so that (II.6) is equivalent in Laplace domain to

$$\hat{\psi}_L(\lambda,s) = \hat{H}_\lambda(s)\hat{n}_L(\lambda,s) \tag{II.8}$$

where $\hat{\psi}_L(\lambda,\cdot)$ and $\hat{n}_L(\lambda,\cdot)$ denote (one-sided) Laplacian transforms w.r.t. time of $\hat{\psi}(\lambda,\cdot)$ and $\hat{n}(\lambda,\cdot)$, respectively. Note that the time-delay T is reflected into the exponential term e^{-Ts} in the denominator of $\hat{H}_\lambda(s)$. The spatial Fourier transform of the performance output $z(\cdot,t)$ is

$$\hat{z}(\lambda,t) = \begin{bmatrix} \hat{\psi}(\lambda,t) \\ \sqrt{r}\,\hat{u}(\lambda,t) \end{bmatrix} = \begin{bmatrix} \hat{\psi}(\lambda,t) \\ -\sqrt{r}\,\hat{K}_\lambda\hat{\psi}(\lambda,t-T) \end{bmatrix} \tag{II.9}$$

and the Laplace transform of $\hat{z}(\lambda,\cdot)$ w.r.t. time is

$$\hat{z}_L(\lambda,s) = \begin{bmatrix} 1 \\ -\sqrt{r}\,\hat{K}_\lambda \mathrm{e}^{-Ts} \end{bmatrix} \hat{H}_\lambda(s)\hat{n}_L(\lambda,s)\,. \tag{II.10}$$

Invoking Parseval theorem, we evaluate the $\mathcal{H}_2$-norm as

$$J(\hat{K}) = \int_{-\infty}^{\infty} \left(1+r\hat{K}_\lambda^2\right) \int_{-\infty}^{\infty} |\hat{H}_\lambda(j\omega)|^2 \mathrm{d}\omega\mathrm{d}\lambda \tag{II.11}$$

where $\omega \in \mathbb{R}$ denotes temporal frequency.

Problem (II.5) is *decoupled* in spatial frequency, as shown in [1]: the cost (II.11) is an integral over λ and the dynamics constraint (II.5b) is decoupled in λ (II.6). Hence, problem (II.5) is equivalent to a family of decoupled problems parameterized by $\lambda \in \mathbb{R}$:

$$\underset{\hat{K}_\lambda}{\text{minimize}} \quad J_\lambda(\hat{K}_\lambda) \doteq \left(1+r\hat{K}_\lambda^2\right) \int_{-\infty}^{\infty} |\hat{H}_\lambda(j\omega)|^2 \mathrm{d}\omega. \tag{II.12}$$

The constraint (II.5b) is implicitly enforced in (II.12) through the transfer function $\hat{H}_\lambda$. The optimal controller $\mathcal{K}_T^{\text{opt}}$ is retrieved by taking the inverse spatial Fourier transform of the solution to (II.12).

In view of the problem decoupling discussed above, we next focus on the optimal control of the scalar system (II.6).

III. Optimal Control of Scalar Delay Systems

In this section, we revise background on the class of systems (II.6) with $T > 0$ and analyze their optimal control.

For a fixed $\lambda \in \mathbb{R}$, system (II.6) is a standard delay differential equation with real coefficients, so that the real and imaginary parts of $\hat{\psi}(\lambda,\cdot)$ evolve independently overtime.

Hence, in this section we address the following equation with $x(t) \in \mathbb{R}$, exogenous input $n(t) \in \mathbb{R}$, and constant coefficients $a \in \mathbb{R}$ and $k \in \mathbb{R}$:

$$\dot{x}(t) = ax(t) - kx(t-T) + n(t). \tag{III.1}$$

System (III.1) coincides with (II.6) by setting $x(t) = \hat{\psi}(\lambda,t)$, $n(t) = \hat{n}(\lambda,t)$, $a = -c - d\lambda^2$, and $k = \hat{K}_\lambda$. Using the result

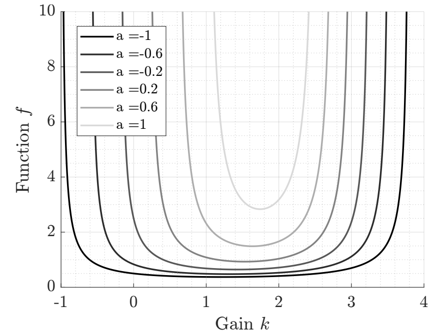


Fig. 2: Graphic of function $f(k)$ defined in (III.2) with delay $T = 0.5$ and dynamics coefficient $a \in \{\pm 1, \pm 0.6, \pm 0.2\}$.

in [16] for the steady-state solution of stochastic equations such as (III.1) where n is a Wiener process, the integral of $|\hat{H}_\lambda(j\omega)|^2$ in (II.12) is given by $f(k)$ defined as

$$f(k) = \begin{cases} \dfrac{-k\sinh(\ell T) - \ell}{2\ell\,(a - k\cosh(\ell T))}, & |k| < -a \\ \dfrac{T}{4} + \dfrac{1}{4|a|}, & k = |a|, a \neq 0 \\ \dfrac{-k\sin(\ell T) - \ell}{2\ell\,(a - k\cos(\ell T))}, & |a| < k < k^{\text{u}}, \end{cases} \tag{III.2}$$

where $\ell \doteq \sqrt{|k^2 - a^2|}$ and k^{u} is the unique solution of the following equation with variable k subject to $k > |a|$:

$$T\sqrt{k^2 - a^2} = \arccos\left(\frac{a}{k}\right). \tag{III.3}$$

The typical profile of $f(k)$ is shown in Fig. 2. More details on this derivation are given in [15, Section IV].

The optimal control design problem (II.12) with the current notation reduces to the following instantiation:

$$k_T^{\text{opt}} \doteq \arg\min_{k} \quad J(k) = \left(1 + rk^2\right) f(k) \tag{III.4a}$$

$$\text{subject to} \quad a < k < k^{\text{u}} \tag{III.4b}$$

where the constraint (III.4b) highlights the interval where the function $f(k)$ is defined, which we name *stability region* because it corresponds to systems with bounded $\mathcal{H}_2$-norm.

The next result characterizes the behavior of the minimizer of J for stable systems when the dynamics coefficient a becomes large in magnitude, corresponding to stable autonomous dynamics with small time constants.

Proposition 1 (Optimal gain for fast stable dynamics)**.** *Let $T > 0$ be fixed and $a < 0$. Then, for every constant $r > 0$, it holds*

$$\lim_{|a|\to+\infty} \frac{1}{k_T^{opt}} \frac{\mathrm{e}^{Ta}}{2r|a|} = 1. \tag{III.5}$$

Proof. See preprint [15, Appendix B]. □

The main insight of Proposition 1 is that, if the control is penalized by a constant $r > 0$, limit (III.5) shows that the optimal gain k_T^{opt} decreases exponentially with $|a|$. This is in stark contrast with the standard optimal controller in the absence of delays, which we recall below for convenience,

$$k_0^{\text{opt}} = a + \sqrt{a^2 + r^{-1}}, \tag{III.6}$$

and at the limit for $|a| \to \infty$, $a < 0$ has Taylor expansion

$$k_0^{\text{opt}} = \frac{1}{2r|a|} + o\left(\frac{1}{|a|}\right). \tag{III.7}$$

In particular, the optimal delay-free gain is *inversely proportional* to $|a|$ according to (III.7).

Asymptotically, the optimal time-delayed gain (III.5) and optimal delay-free gain (III.7) differ in the exponential factor e^{Ta}. We define the characteristic timescale of the open-loop dynamics as $t_* := {}^1/_{|a|}$. *Dimensional analysis* shows that the ratio $\alpha := {}^T/_{t_*}$ is a dimensionless parameter which provides physical interpretation [17], [18] of the value of delayed measurements for optimal feedback control. Since the exponential $e^{-\alpha}$ dictates the magnitude of the gain k_T^{opt}, the effect of time delays is enlightened by the following two regimes.

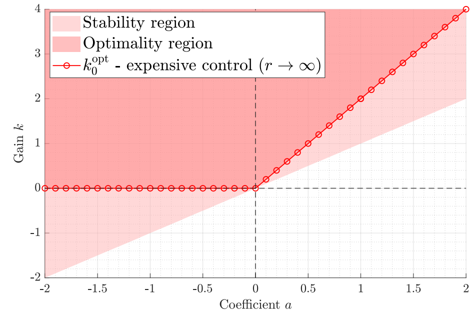
Regime $\alpha \ll 1$ (*i.e.,* $t_* \gg T$): the time-delay T in the feedback control is small compared to the characteristic timescale t_* of the stable open-loop dynamics: despite delayed measurements, the feedback control signal is useful to optimally stabilize the system. The optimal delay-aware gain k_T^{opt} resembles k_0^{opt}.

Regime $\alpha \gg 1$ (*i.e.,* $t_* \ll T$): in this regime the open-loop stable dynamics are characterized by a timescale that is much shorter than communication delays, yielding the control input fed with delayed measurements useless for optimal stabilization. Consequently, $k_T^{\text{opt}} \to 0$ to avoid an unnecessary control cost in the performance objective J.

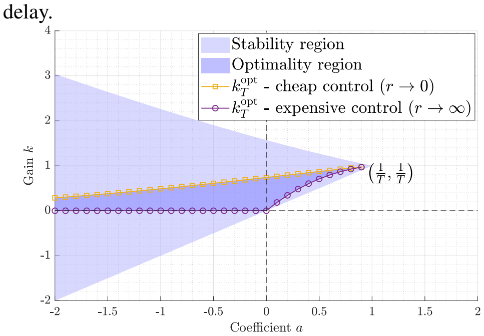
A. Numerical Solutions

To complement the insights obtained through asymptotic approximations, we solve problem (III.4) numerically for fixed $T > 0$ and varying parameter a. We plot the stability region (III.4b) in Fig. 3b, while the stability region for the delay-free problem is $(a, +\infty)$ and is depicted in Fig. 3a. As previously noted, the presence of a nonzero delay makes it possible to stabilize a system only if a is sufficiently small (smaller than ${}^1/_T$), while in the delay-free case every system can be stabilized by suitably increasing the feedback gain. Moreover, under delays, the stabilizing gains are upper bounded by $k^{\text{u}} \in \mathbb{R}$, contrarily to the delay-free case that allows arbitrarily large gains. It can be seen that k^{u} steadily increases as a decreases and tends to the value ${}^1/_T$ as $a \to {}^1/_T$.

The figure also shows the *optimality regions* which collect all solutions to problem (II.12) as the parameter r in the performance index is varied. Such analysis closely relates to the *inverse optimal control problem* [19] that searches the parameter values of the performance index that make a specific choice of the controller optimal. The boundaries of the (open) optimality region are defined by the optimal control gains in the cheap and expensive control regimes, which feature respectively $r \to 0$ and $r \to \infty$. The boundaries in Fig. 3b have been numerically computed by setting



(a) Zero delay.



(b) Nonzero delay ($T = 1$).

Fig. 3: Stability and optimality regions for problem (III.4).

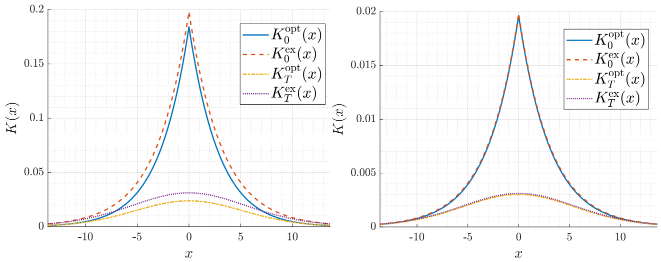
respectively $J(k) = f(k)$ and $J(k) = k^2 f(k)$ in (III.4). The presence of communication delays induces a fundamentally different behavior compared to the delay-free scenario: the optimal feedback gains lie within a region that is strictly inside the stability region, and decrease to zero as the dynamics coefficient a decreases according to Proposition 1, for every value of r. Moreover, given any fixed value of a, the optimal gain appears to be bounded for all values of r. This contrasts with the delay-free controller: indeed, according to (III.6), the optimal gain k_0^{opt} grows unbounded for fixed a as the control weight r tends to zero (cheap control regime).

IV. Optimal Control of Reaction-Diffusion Process under Communication Delays

We turn back to the original problem (II.5) and study the control of the spatiotemporal dynamics (II.3). An approximation to the optimal control gain can be found by numerically solving (II.12) (*i.e.,* problem (III.4) after replacing a and k with $\hat{A}_\lambda$ and $\hat{K}_\lambda$, respectively) and taking the inverse Fourier transform of the solution. However, an analytic expression of the optimal gain is in general challenging to obtain. To gain some understanding on the optimal control gain, in Section IV-A we address the expensive control regime, which makes the analysis tractable. Then, in Section IV-B, we numerically compute the optimal controller and derive analytical approximations of the optimal kernel in the expensive regime, which are tailored to design.

A. Expensive Control Regime

Assume that the control effort is much more penalized than the state. We denote the optimal controller in the expensive regime by $\mathcal{K}_T^{\text{ex}}$, with convolutional kernel K_T^{ex} and Fourier symbol $\hat{K}_T^{\text{ex}}$. We have the following result.



(a) Control weight $r = 1$.

(b) Control weight $r = 10$.

Fig. 4: Optimal controllers without delay and with delay $T = 1$ and the optimal controllers in the expensive-control regime with $d = 10$ and $c = 1$.

Theorem 1 (Optimal controller in expensive regime)**.** *For $r \gg 1$, let the convolution kernel of the optimal control gain in the expensive regime be defined as*

$$K_T^{ex}(x) \doteq \frac{1}{2r}\sqrt{\frac{\pi}{2dc}}\left(\phi(x) + \phi(-x)\right) \tag{IV.1}$$

where

$$\phi(x) \doteq \frac{\mathrm{e}^{\sqrt{\frac{c}{d}}x}}{2}\left(1 + \operatorname{erf}\left(-\frac{x}{2\sqrt{dT}} - \sqrt{cT}\right)\right). \tag{IV.2}$$

Then, it holds

$$\lim_{r\to\infty}\frac{K_T^{opt}(x)}{K_T^{ex}(x)} = 1 \qquad \forall x \in \mathbb{R}. \tag{IV.3}$$

Proof. See preprint [15, Section VI-A]. □

B. Numerical Solutions and Approximations for Design

The convolutional kernel K_T^{ex} defined in (IV.1) is plotted in Fig. 4 (purple dotted) against the optimal kernel K_T^{opt} (yellow dashed-dotted), which is numerically computed as the solution of problem (II.12), the delay-free optimal kernel K_0^{opt} obtained as the inverse spatial Fourier transform of (III.6) (blue solid), and the delay-free optimal kernel in the expensive regime K_0^{ex} given by (red dashed)

$$K_0^{\text{ex}}(x) \doteq \mathcal{F}^{-1}\left[\hat{K}_0^{\text{ex}}(\cdot)\right](x) = \frac{\epsilon}{2}\sqrt{\frac{\pi}{2dc}}\mathrm{e}^{-\sqrt{\frac{c}{d}}|x|}. \tag{IV.4}$$

Visual inspection of Fig. 4 reveals that both in the delay-free and delay-aware cases the optimal kernels for the expensive control regime are good approximations of the optimal control kernels already with control weight $r = 10$. However, the presence of communication delays induces a fundamentally different shape of the control kernel about the origin, while the tails enjoy the same asymptotic exponential decay observed for delay-free spatially invariant systems in [1].

We next analyze the convolutional kernel (IV.1) to provide further insights into the spatial structure of the controller.

Lemma 1. *Let*

$$\begin{aligned} D_0 &\doteq 1 - \operatorname{erf}\left(\sqrt{cT}\right) \\ D_2 &\doteq \frac{cD_0}{2d} - \sqrt{\frac{c}{\pi T}}\frac{\mathrm{e}^{-cT}}{2d}. \end{aligned} \tag{IV.5}$$

Then, it holds

$$K_T^{ex}(x) = K_0^{ex}(0)\left(D_0 + D_2x^2\right) + o(x^3). \tag{IV.6}$$

Proof. See preprint [15, Section VI-B.1]. □

Lemma 1 shows that the convolutional kernel of the optimal controller in the expensive control regime can be approximated by a quadratic function within a suitable interval about the origin. In particular, the gap between the delay-free and delay-aware control kernels at $x = 0$ is given by

$$\Delta K_T^{\text{ex}}(0) \doteq \frac{K_T^{\text{ex}}(0)}{K_0^{\text{ex}}(0)} = 1 - \operatorname{erf}\left(\sqrt{cT}\right) < 1. \tag{IV.7}$$

The gap $\Delta K_T^{\text{ex}}(0)$ jointly decreases with parameters c and T. This suggests that, in the presence of communication delays, the optimal feedback gain that multiplies the state at the control location become smaller with both the time-delay and the reaction coefficient.

Similarly, the curvature $D_2 < 0$ increases with d and T, meaning that the kernel of the optimal controller is more spread out about the origin both with longer communication delays and higher diffusion coefficient. This suggests that either as diffusion or the time delays in the state feedback increase, measurements from further away from the actuator become increasingly relevant for control.

Overall, the behavior of D_0 and D_2 that increase with T reflects a "flattening" of the control kernel, which becomes more spread out as time delays in the feedback increase.

The next result quantifies how well the delay-free control kernel approximates the delay-aware one for any $x \in \mathbb{R}$.

Lemma 2. *It holds*

$$K_T^{ex}(x) = K_0^{ex}(x)(1 + R(x)) \tag{IV.8}$$

where function $R(\cdot)$ is bounded for all $x \in \mathbb{R}$ as

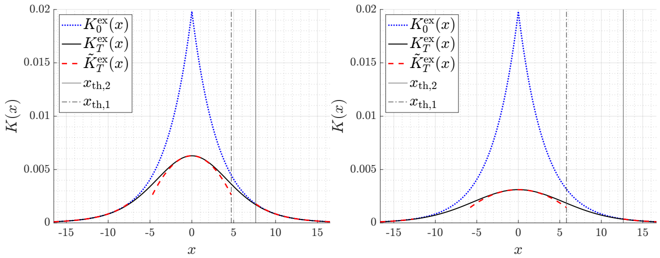
$$\begin{aligned} 0 > R(x) > &-\frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{x}{\sqrt{2dT}} - \sqrt{2cT}\right)^2}}{\sqrt{2\pi}}\frac{\sqrt{2dT}}{x - 2\sqrt{dcT}} \\ &+ \frac{\mathrm{e}^{-\frac{1}{2}\left(\frac{x^2}{2dT} + 2cT\right)}}{\sqrt{2\pi}}\left(\frac{\sqrt{2dT}}{x + 2\sqrt{dcT}} - \frac{\sqrt{2dT}^3}{(x + 2\sqrt{dcT})^3}\right) \end{aligned} \tag{IV.9}$$

and has limit

$$\lim_{|x|\to+\infty} R(x) = 0. \tag{IV.10}$$

Proof. See preprint [15, Section VI-B.2]. □

Practical design guidelines. The asymptotic approximations provided in Lemmas 1 and 2 lend themselves to practical guidelines for delay-aware controller design. A rule of thumb is given by imposing that the quadratic function of the Taylor-based approximation (IV.6) dominates the higher-order terms, whereas the difference term R in (IV.8) can be approximately set to zero by imposing that the exponents of the two exponential functions are sufficiently large. A possible practical design is formalized by the following corollary.



(a) Delay $T = 0.5$. (b) Delay $T = 1$.

Fig. 5: Exact controllers in the expensive-control regime vs. design approximation $\widetilde{K_T^{ex}}(x)$ with $d = 10$, $c = 1$, and $r = 10$.

Corollary 1. *Let D_0 and D_2 be defined as per (IV.5) and*

$$D_4 \doteq \frac{c^2 D_0}{d^2} - \sqrt{\frac{c}{\pi T}}\frac{e^{-cT}}{d^2}\left(c + \frac{1}{2T}\right) \tag{IV.11}$$

$$x_{th,1} \doteq \sqrt{\frac{12}{|D_4|}\left(D_2 + \sqrt{D_2^2 + \frac{D_0|D_4|}{6}}\right)} \tag{IV.12}$$

$$x_{th,2} \doteq 2(\sqrt{dT} + \sqrt{cdT}). \tag{IV.13}$$

Then, for $|x| \leq \alpha x_{th,1}$ and $|x| \geq \beta x_{th,2}$, where the parameters $\alpha, \beta \in (0,1)$ are such that $\alpha x_{th,1} \leq \beta x_{th,2}$, the kernel K_T^{ex} can be approximated by the function $\widetilde{K_T^{ex}}$ defined as

$$\widetilde{K_T^{ex}}(x) = \begin{cases} K_0^{ex}(0)\left(D_0 + D_2 x^2\right) & |x| \leq \alpha x_{th,1} \\ K_0^{ex}(x) & |x| \geq \beta x_{th,2}. \end{cases} \tag{IV.14}$$

Figure 5 illustrates the delay-free control kernel K_0^{ex} (dotted blue), the delay-aware kernel K_T^{ex} (solid black), and the approximation $\widetilde{K_T^{\text{ex}}}$ with $\alpha = \beta = 1$ (dashed red) together with the threshold values $x_{\text{th},1}$ and $x_{\text{th},2}$ defined in (IV.12)–(IV.13). The two branches of the design approximation $\widetilde{K_T^{\text{ex}}}$ closely approach the optimal control kernel K_T^{ex} as x approaches the origin (first case) or grows large in magnitude (second case). The intervals $(0, x_{\text{th},1})$ and $(x_{\text{th},2}, \infty)$ represent where the two asymptotic expressions derived from Lemmas 1 and 2 yield reasonable approximations, whereas the parameters α and β can be tuned so as to trade a simple design (large α and small β) for an accurate approximation (small α and large β). The feedback gains corresponding to the interval $(\alpha x_{\text{th},1}, \beta x_{\text{th},2})$ may be chosen in practice by interpolating the two cases in (IV.14), for instance via linear or polynomial interpolation.

Also, Fig. 5 confirms the intuition given by Lemma 1. As the delay increases, the feedback gains decrease and the kernel becomes flatter. This consideration urges attention in truncating the control kernel to implement a distributed architecture, because the gap between feedback gains about the origin ($x \approx 0$) and gains associated with far-away measurements ($|x| \gg 1$) is much smaller in the presence of delays. In particular, this gap becomes smaller with longer communication delay, suggesting that an effective truncation-based implementation should truncate the feedback gains of the centralized controller at an increasing distance with the delay. Similar observations hold as a function of the coefficients d and c in the dynamics, according to the discussion following Lemma 1.

V. Conclusion

We study the optimal proportional feedback control for a spatially distributed reaction-diffusion process subject to communication delays. We first establish that, for scalar delay systems, the optimal gains are always bounded regardless of the penalization on control effort. Then, we analytically and numerically study the optimal convolutional kernel, showing that it is fundamentally different about the origin than the delay-free case and that it gets flatter as delays increase.

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