The modulus of a vector measure

de Pagter, Ben; Ricker, Werner J.

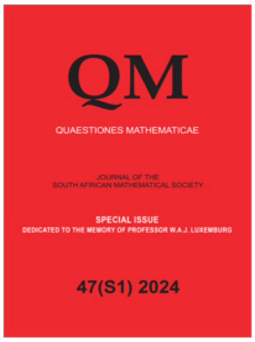
DOI
10.2989/16073606.2023.2287823

Publication date
2024

Document Version
Final published version

Published in
Quaestiones Mathematicae

Citation (APA)
de Pagter, B., & Ricker, W. J. (2024). The modulus of a vector measure. *Quaestiones Mathematicae*, *47*(sup1), 121-136. https://doi.org/10.2989/16073606.2023.2287823



Quaestiones Mathematicae

ISSN: (Print) (Online) Journal homepage: www.tandfonline.com/journals/tqma20

The modulus of a vector measure

Ben de Pagter & Werner J. Ricker

Quaestiones Mathematicae 2024, 47(S1): S121–S136

https://doi.org/10.2989/16073606.2023.2287823

THE MODULUS OF A VECTOR MEASURE

Ben de Pagter*
Delft Institute of Applied Mathematics, Faculty EEMCS, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands.
E-Mail b.depagter@tudelft.nl

Werner J. Ricker
Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt, D-85072 Eichstätt, Germany.
E-Mail werner.ricker@ku.de

Dedicated to the memory of our good friend Wim Luxemburg

Abstract. It is known that if L is a Dedekind complete Riesz space and (Ω, Σ) is a measurable space, then the partially ordered linear space of all L-valued, finitely additive and order bounded vector measures m on Σ is also a Dedekind complete Riesz space (for the natural operations). In particular, the modulus $|m|_o$ of m exists in this space of measures *and* $|m|_o$ is given by a well known formula. Some 20 years ago L. Drewnowski and W. Wnuk asked the question (for L not Dedekind complete) if there is an m for which $|m|_o$ exists but, $|m|_o$ is *not* given by the usual formula? We show that such a measure m does indeed exist.

Mathematics Subject Classification (2020): Primary: 46A40, 46G10; Secondary: 06F20.
Key words: Vector measure, finitely additive, order bounded, modulus.

1. Introduction. Let L be an Archimedean (real) Riesz space and (Ω, Σ) be a measurable space, that is, Σ is a σ-algebra of subsets of some non-empty set Ω. The partially ordered vector space of all L-valued, finitely additive, order bounded vector measures on Σ is denoted by $M_{ob}(\Sigma, L)$; see Section 2. Whenever it exists in $M_{ob}(\Sigma, L)$, denote by $|m|_o = m \vee (-m)$ the *modulus* (also called the absolute value) of $m \in M_{ob}(\Sigma, L)$. If, in addition, the formula

$$|m|_o(A) = \sup_{\pi \in \Pi(A)} \sum_{B \in \pi} |m(B)|, \quad A \in \Sigma, \tag{1}$$

is valid, meaning that for each $A \in \Sigma$ the supremum in the right-side of (1) exists in L and equals $|m|_o(A)$, then m is said to have a *proper modulus.* Here, for each $A \in \Sigma$, the family of all finite partitions of A in Σ is denoted by $\Pi(A)$. Whenever L is Dedekind complete, it is known that *every* element of $M_{ob}(\Sigma, L)$ has a proper

*Corresponding author.

modulus (cf. Section 2). If L is *not* Dedekind complete, then there may exist vector measures in $M_{ob}(\Sigma, L)$ which have no modulus at all and others for which the modulus does exist; see Section 2. Some 20 years ago L. Drewnowski and W. Wnuk [3] asked the question of whether there exist vector measures $m \in M_{ob}(\Sigma, L)$ which have a modulus $|m|_o$ in $M_{ob}(\Sigma, L)$ but, the formula (1) *fails* to hold. The aim of this note is to show that this can indeed happen; see Theorem 4.3. The m that we exhibit is even countably additive and has finite variation. Some explanation of how such an m could arise is relevant.

A regular linear operator $T : L \to M$ between Archimedean Riesz spaces L and M may or may not have a modulus $|T|$, that is, $|T| = T \vee (-T)$ exists for the natural order in the space of all regular operators from L into M. Here, regular means that the operator is the difference of two positive operators. An early and fundamental result, due to L.V. Kantorovich [6] under the assumption that M is Dedekind complete, states that any regular operator $T : L \to M$ has a modulus *and* this modulus is given by the so called Riesz-Kantorovich formula

$$|T|(x) = \sup \{Ty : y \in L, \quad |y| \leq x\}, \quad x \in L^+; \tag{2}$$

here $L^+ = \{x \in L : x \geq 0\}$ is the positive cone of L. Until recently, for every known example of an operator T for which the modulus $|T|$ exists, this modulus is given by (2). But, is this *always* the case? This issue is elegantly settled by M. Elliott in [4], where a regular operator $T : L^1([0,1]) \to E$ with E isometrically isomorphic to a $C(K)$-space, is constructed for which $|T|$ exists but, the Riesz-Kantorovich formula *fails* to hold. The features of this Banach lattice and the operator T suggest that the order bounded, E-valued vector measure m defined by $A \longmapsto T(\chi_A)$, for each Borel set A in $[0,1]$, is a good candidate to have the desired properties. It turns out that this is indeed the case (see Section 4). In Section 5, using the theory of integration with respect to a countably additive vector measure, we analyze further the close connection between m and T. In particular, it is shown that the space $L^1(m)$ of all m-integrable functions coincides with $L^1([0,1])$ and consequently, T has an integral representation with respect to m.

For the basic theory of Riesz spaces (i.e., vector lattices) we refer the reader to any of the books [7], [12], [8] or [1].

2. Order bounded vector measures and their moduli. In this section we discuss various properties of order bounded vector measures. Let L be an Archimedean Riesz space and (Ω, Σ) be a measurable space. A set function $m : \Sigma \to L$ is called a *finitely additive vector measure* if $m(A_1 \cup A_2) = m(A_1) + m(A_2)$ whenever $A_1, A_2 \in \Sigma$ are disjoint. A set $A \in \Sigma$ is said to be *m-null* if $m(B) = 0$ for every $B \in \Sigma$ with $B \subseteq A$. Furthermore, m is called *positive* if $m(A) \geq 0$ for all $A \in \Sigma$. It should be observed that if $m : \Sigma \to L$ is a finitely additive, positive vector measure and $A, B \in \Sigma$ satisfy $A \subseteq B$, then $m(A) \leq m(B)$.

DEFINITION 2.1. A finitely additive vector measure $m : \Sigma \to L$ is called *order bounded* if its range

$$m(\Sigma) = \{m(A) : A \in \Sigma\}$$

is an order bounded subset of L, that is, there exists $u \in L^+$ such that $|m(A)| \leq u$ for all $A \in \Sigma$.

The set of all L-valued, finitely additive, order bounded vector measures on Σ will be denoted by $M_{ob}(\Sigma, L)$, which is a real vector space with respect to the "natural operations". Any positive vector measure belongs to $M_{ob}(\Sigma, L)$. The set of all positive, L-valued measures is denoted by $M_{ob}(\Sigma, L)^+$, which is a proper cone in $M_{ob}(\Sigma, L)$. The linear space $M_{ob}(\Sigma, L)$ is a partially ordered vector space with respect to this cone (i.e., if $m_1, m_2 \in M_{ob}(\Sigma, L)$, then $m_1 \leq m_2$ if and only if $m_1(A) \leq m_2(A)$ for all $A \in \Sigma$).

It is well known that $M_{ob}(\Sigma, L)$ is a Dedekind complete Riesz space whenever the Riesz space L is Dedekind complete. In fact, the following theorem holds, which may be deduced from its more abstract analogue [11], Theorem 2.1.3 (see also [5]).

THEOREM 2.2. *Let L be a Dedekind complete Riesz space and (Ω, Σ) be a measurable space. With respect to the above partial ordering, $M_{ob}(\Sigma, L)$ is a Dedekind complete Riesz space where, for any $m_1, m_2 \in M_{ob}(\Sigma, L)$, the supremum $m_1 \vee m_2$ is given by*

$$(m_1 \vee m_2)(A) = \sup\{m_1(B) + m_2(A \setminus B) : B \in \Sigma,\ B \subseteq A\}, \quad A \in \Sigma. \tag{3}$$

For any upwards directed, order bounded system $0 \leq m_\alpha \uparrow_\alpha \leq m_0$ in $M_{ob}(\Sigma, L)$, its supremum $m \in M_{ob}(\Sigma, L)$ is given by the formula

$$m(A) = \sup_\alpha m_\alpha(A), \quad A \in \Sigma. \tag{4}$$

It follows, in particular, from the above theorem that for each $m \in M_{ob}(\Sigma, L)$ the *absolute value* $|m|_o = m \vee (-m)$ of m exists, whenever L is Dedekind complete (we denote the absolute value of m by $|m|_o$, whereas we reserve the notation $|m|$ for the variation of m; see Section 5). The formulae in the next result also appear in [2], [3] and [5]. For each $A \in \Sigma$, the collection of all finite partitions of A in Σ is denoted by $\Pi(A)$.

COROLLARY 2.3. *Let L be a Dedekind complete Riesz space and (Ω, Σ) be a measurable space.*

(i) *Let $m \in M_{ob}(\Sigma, L)$. For each $A \in \Sigma$, we have that*

$$\begin{aligned} |m|_o(A) &= \sup\{m(B) - m(A \setminus B) : B \in \Sigma,\ B \subseteq A\} \\ &= \sup\{|m(B) - m(A \setminus B)| : B \in \Sigma,\ B \subseteq A\}. \end{aligned} \tag{5}$$

(ii) *Let $m \in M_{ob}(\Sigma, L)$. Then $|m|_o$ is also given by the formula*

$$|m|_o(A) = \sup_{\pi \in \Pi(A)} \sum_{B \in \pi} |m(B)|, \quad A \in \Sigma. \tag{6}$$

If the Riesz space L is *not* Dedekind complete and $m \in M_{ob}(\Sigma, L)$, then the absolute value $|m|_o = m \vee (-m)$ of m may or may not exist in the partially ordered vector space $M_{ob}(\Sigma, L)$; see Example 2.7 (a) below. Note that if $|m|_o$ exists, then $|m(A)| \leq |m|_o(A)$ for each $A \in \Sigma$. The following notion appears in [2], p. 223.

Definition 2.4. Let L be an Archimedean Riesz space and $m \in M_{ob}(\Sigma, L)$. We say that the modulus $|m|_o$ *exists properly* if $|m|_o = m \vee (-m)$ exists in $M_{ob}(\Sigma, L)$ and if $|m|_o(A)$ is given by the formula (5) for each $A \in \Sigma$.

Of course, if L is Dedekind complete, then $|m|_o$ exists properly for every $m \in M_{ob}(\Sigma, L)$; cf. Corollary 2.3. However, there are several important cases in which $|m|_o$ exists properly without the assumption that L is Dedekind complete; see Example 2.7 (c), (d) below.

The following result (without proof) is stated on pp. 222–223 of [2] and on p. 363 of [3]. We include a proof for the sake of completeness.

Lemma 2.5. *Let L be an Archimedean Riesz space and $m \in M_{ob}(\Sigma, L)$. The following three statements are equivalent.*

(i) *The modulus $|m|_o$ exists properly in $M_{ob}(\Sigma, L)$.*

(ii) *For each $A \in \Sigma$, the supremum*

$$\sup\{m(B) - m(A \setminus B) : B \in \Sigma,\ B \subseteq A\}$$

exists in L.

(iii) *For each $A \in \Sigma$, the supremum*

$$\sup\left\{\sum\nolimits_{B\in\pi} |m(B)| : \pi \in \Pi(A)\right\} \tag{7}$$

exists in L.

If any one of (i)-(iii) is satisfied, then $|m|_o(A)$ is also given by (7) for each $A \in \Sigma$.

Proof. The implication (i)⇒(ii) is evident from Definition 2.4.

(ii)⇒(i). Defining $m_0 : \Sigma \to L$ by

$$m_0(A) = \sup\{m(B) - m(A \setminus B) : B \in \Sigma,\ B \subseteq A\}, \quad A \in \Sigma,$$

it is routine to verify that m_0 is finitely additive and that $m_0 = |m|_o$.

(i)⇒(iii). Fix $A \in \Sigma$. If $\pi \in \Pi(A)$, then

$$\sum\nolimits_{B\in\pi} |m(B)| \le \sum\nolimits_{B\in\pi} |m|_o(B) = |m|_o(A).$$

Hence, $|m|_o(A)$ is an upper bound of the set $\left\{\sum_{B\in\pi} |m(B)| : \pi \in \Pi(A)\right\}$.

Suppose now that $u \in L^+$ satisfies $\sum_{B\in\pi} |m(B)| \le u$ for all $\pi \in \Pi(A)$. This implies, in particular, that

$$|m(B) - m(A \setminus B)| \le |m(B)| + |m(A \setminus B)| \le u$$

for all $B \in \Sigma$ with $B \subseteq A$. It follows from (5) that $|m|_o(A) \le u$. Consequently, $|m|_o(A)$ is the supremum in L of the set

$$\left\{\sum\nolimits_{B\in\pi} |m(B)| : \pi \in \Pi(A)\right\}.$$

(iii)⇒(i). Since the supremum in (7) exists for every $A \in \Sigma$, we can define $m_1 : \Sigma \to L^+$ by setting

$$m_1(A) = \sup\left\{\sum\nolimits_{B\in\pi} |m(B)| : \pi \in \Pi(A)\right\}, \quad A \in \Sigma.$$

It is readily verified that m_1 is finitely additive, i.e., $m_1 \in M_{ob}(\Sigma, L)^+$. Since $|m(A)| \le m_1(A)$ for $A \in \Sigma$, it is clear that m_1 is an upper bound of $\{m, -m\}$ in $M_{ob}(\Sigma, L)$.

Suppose that $m_2 \in M_{ob}(\Sigma, L)^+$ is also an upper bound of $\{m, -m\}$, i.e., $|m(B)| \le m_2(B)$ for all $B \in \Sigma$. Given $A \in \Sigma$ and $\pi \in \Pi(A)$, it follows that

$$\sum\nolimits_{B\in\pi} |m(B)| \le \sum\nolimits_{B\in\pi} m_2(B) = m_2(A).$$

By the definition of $m_1(A)$, this implies that $m_1(A) \le m_2(A)$. Hence, $m_1 \le m_2$. We conclude that m_1 is the supremum of $\{m, -m\}$, that is, $m_1 = |m|_o$.

It remains to show that $|m|_o$ is also given by (5). Let $A \in \Sigma$ be fixed. If $B \in \Sigma$ with $B \subseteq A$, then

$$m(B) - m(A \backslash B) \le |m(B)| + |m(A\backslash B)| \le m_1(A) = |m|_o(A).$$

Therefore, $|m|_o(A)$ is an upper bound of the set

$$\{m(B) - m(A\backslash B) : B \in \Sigma, \quad B \subseteq A\}.$$

Suppose now that $w \in L^+$ is any upper bound of this set and let $\pi = \{B_1, \dots, B_n\} \in \Pi(A)$. Recall (cf. Proposition 1 in [2]) that

$$\sum\nolimits_{j=1}^{n} |m(B_j)| = \sup\left\{\sum\nolimits_{j=1}^{n} \varepsilon_j m(B_j) : \varepsilon_j \in \{-1, 1\} \text{ for } 1 \le j \le n\right\}.$$

Given any $\varepsilon_j \in \{-1, 1\}$, for $1 \le j \le n$, define $B^+ = \bigcup\{B_j : \varepsilon_j = 1\}$. Then

$$\sum\nolimits_{j=1}^{n} \varepsilon_j m(B_j) = m\left(B^+\right) - m\left(A\backslash B^+\right) \le w.$$

Consequently, $\sum_{j=1}^{n} |m(B_j)| \le w$. This shows that w is an upper bound of the set $\left\{\sum_{B\in\pi} |m(B)| : \pi \in \Pi(A)\right\}$ and so $|m|_o(A) = m_1(A) \le w$. We can conclude that (5) holds, that is, $|m|_o$ exists properly. This suffices for the proof of the lemma. □

COROLLARY 2.6. *Let L be an Archimedean Riesz space and $m \in M_{ob}(\Sigma, L)$. The modulus $|m|_o$ of m exists properly if and only if $|m|_o$ exists in $M_{ob}(\Sigma, L)$ and is given by*

$$|m|_o(A) = \sup_{\pi\in\Pi(A)} \sum_{B\in\pi} |m(B)|, \quad A \in \Sigma. \tag{8}$$

Let E be a Banach lattice. A finitely additive vector measure $m : \Sigma \to E$ is called *countably additive* if

$$m\left(\bigcup\nolimits_{n=1}^{\infty} A_n\right) = \sum\nolimits_{n=1}^{\infty} m(A_n),$$

whenever $(A_n)_{n=1}^{\infty}$ is a pairwise disjoint sequence in Σ, where the series $\sum_{n=1}^{\infty} m(A_n)$ is (unconditionally) convergent in E. If this is the case, then m is simply called an E-valued vector measure. We denote by $M_{obc}(\Sigma, E)$ the subset of $M_{ob}(\Sigma, E)$ consisting of all the *order bounded vector measures*. It is readily verified that $M_{obc}(\Sigma, E)$ is a linear subspace of $M_{ob}(\Sigma, E)$. In general, even for E a Dedekind complete Banach lattice, $m \in M_{obc}(\Sigma, E)$ need *not* imply that $|m|_o \in M_{obc}(\Sigma, E)$; see Example 2.7 (b) below. In other words, if E is a Dedekind complete Banach lattice and $m : \Sigma \to E$ is an order bounded vector measure, then its modulus $|m|_o$ exists as a finitely additive positive vector measure but, $|m|_o$ need not be a vector measure.

EXAMPLE 2.7. (a) Let $\Omega = [0,1]$ and $\Sigma = \mathcal{B}([0,1])$, the Borel σ-algebra of $[0,1]$. Denote by c the Banach lattice of all convergent sequences (equipped with the norm $\|\cdot\|_{\infty}$). Note that c is not Dedekind complete. There exists an order bounded vector measure $m : \Sigma \to c$ for which $|m|_o$ does not exist in $M_{ob}(\Sigma, c)$; see [10]. For related examples see also Examples 1.9, 2.4 and 7.1 in [3].

(b) There exist Dedekind complete Banach lattices E and order bounded vector measures $m : \Sigma \to E$ for which the modulus $|m|_o$ is not countably additive. See [5], Ch. III, Examples 4.5 and 5.14, for instance, [2], Example 3 and [3], Example 7.10.

(c) Let E be any Banach lattice (not necessarily Dedekind complete) and (Ω, Σ, μ) be a σ-finite measure space. Let $f : \Omega \to E$ be a Bochner μ-integrable function and define $\mu_f : \Sigma \to E$ by

$$\mu_f(A) = \int_A^{(B)} f \, d\mu, \quad A \in \Sigma$$

(here $\int^{(B)}$ denotes the Bochner integral). Then μ_f is an order bounded vector measure. The modulus $|\mu_f|_o$ of μ_f exists properly and is given by $|\mu_f|_o(A) = \int_A^{(B)} |f| \, d\mu$, for $A \in \Sigma$, where the Bochner μ-integrable function $|f| : \Omega \to E$ is defined by $|f|(t) = |f(t)|$, for $t \in \Omega$. Note that $|\mu_f|_o$ is countably additive. For the details we refer to Theorem 1 of [2].

(d) Let (Ω, Σ, μ) be a σ-finite measure space, E be any Banach lattice and $f : \Omega \to E$ be a strongly μ-measurable, Pettis μ-integrable function. Define $\mu_f^P : \Sigma \to E$ by

$$\mu_f^P(A) = \int_A^{(P)} f \, d\mu, \quad A \in \Sigma$$

(where $\int^{(P)}$ denotes the Pettis integral). Then μ_f^P is a vector measure. In general, μ_f^P need not be order bounded. However, if the function $|f| : \Omega \to E$ is also Pettis μ-integrable, then μ_f^P is order bounded, its modulus $\left|\mu_f^P\right|_o$ exists properly and is given by the formula $\left|\mu_f^P\right|_o(A) = \int_A^{(P)} |f| \, d\mu$, for $A \in \Sigma$. In particular, $\left|\mu_f^P\right|_o$ is countably additive. For the details we refer to Theorem 2 in [2].

(e) Let E be an AM-space, that is, E is a Banach lattice in which the norm satisfies $\|u \vee v\|_E = \max\{\|u\|_E, \|v\|_E\}$ for all $u, v \in E^+$. It is well known (see e.g. Theorem 2.1.12 in [8]) that for any relatively compact subset $D \subseteq E$, its supremum $\sup D$ exists in E (and belongs to the norm closure of D). In particular, every relatively compact subset of E is order bounded.

Let (Ω, Σ) be a measurable space and $m : \Sigma \to E$ be a finitely additive vector measure with relatively compact range $m(\Sigma) = \{m(A) : A \in \Sigma\}$. Then m is order bounded and, for each $A \in \Sigma$, the set

$$\{m(B) - m(A \backslash B) : B \subseteq A,\ B \in \Sigma\}$$

is also relatively compact. Consequently, for each $A \in \Sigma$, the supremum

$$\sup\{m(B) - m(A \backslash B) : B \subseteq A,\ B \in \Sigma\}$$

exists in E. Hence, by Lemma 2.5, the modulus $|m|_o$ exists properly. This example is also exhibited in Example 1 (c) of [2].

3. Elliott's construction. In this section we introduce some notation and preliminaries that will be needed in the sequel and describe the Banach lattice constructed by M. Elliott in [4]. For convenience of the reader, we follow the notation used in [4]. For proofs of the stated facts we also refer to Section 3 in [4].

We write $\mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\}$. Given a non-empty set A and $n \in \mathbb{N}_0$, consider the set A^n of all n-tuples $(\tau_0, \ldots, \tau_{n-1})$ of elements from A, except for A^0, which is interpreted to be the singleton set $A^0 = \{\emptyset\}$. Let

$$\mathbb{T}_n(A) = A^n, \quad n \in \mathbb{N}_0,$$

and define

$$\mathbb{T}(A) = \bigcup_{n \in \mathbb{N}} \mathbb{T}_n(A). \tag{9}$$

For $\tau \in \mathbb{T}_n(A)$, we call n the *length* of τ and write $|\tau| = n$. Elements of $\mathbb{T}_n(A)$ for $n \geq 1$ can be thought of as sequences of length n (with elements from A), whereas $\mathbb{T}_0(A)$ is the set consisting of the "empty sequence". Given $\tau = (\tau_0, \ldots, \tau_{n-1}) \in \mathbb{T}_n(A)$ and $\sigma = (\sigma_0, \ldots, \sigma_{m-1}) \in \mathbb{T}_m(A)$ with $n \geq 1$ and $m \geq 1$, we define the $(n+m)$-tuple $\tau \oplus \sigma \in \mathbb{T}_{n+m}(A) \subseteq \mathbb{T}(A)$ via concatenation, that is,

$$\tau \oplus \sigma = (\tau_0, \ldots, \tau_{n-1,}\sigma_0, \ldots, \sigma_{m-1}).$$

For the remaining cases we define $\emptyset \oplus \emptyset = \emptyset$ and $\emptyset \oplus \tau = \tau \oplus \emptyset = \tau$ for $\tau \in \bigcup_{n \geq 1} \mathbb{T}_n(A)$.

The two sets A that we will be using are $A = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $A =$ "3" $= \{0, 1, 2\}$.

For each $n \in \mathbb{N}_0$, let

$$\varphi_n : \{0, 1, \ldots, 3^n - 1\} \to \{0, 1, 2\}^n = \mathbb{T}_n(3)$$

be a bijection, where we interpret $\mathbb{T}_0(3) = \{\emptyset\}$ with $\varphi_0 : \{0\} \to \{\emptyset\}$ uniquely defined. Now define the map $\Phi_n : \mathbb{N}_0 \to \{0,1,2\}^n \subseteq \mathbb{T}(3)$ by setting

$$\Phi_n(k) = \varphi_n(m), \quad (m \in \{0,1,\ldots,3^n-1\}, \quad m = k \bmod 3^n). \tag{10}$$

It is important to note that the values of $\Phi_n(k)$ cycle, with period 3^n, through the elements of $\{0,1,2\}^n$. The maps Φ_n will be used later.

Next, for $A = \mathbb{N}_0$, consider the Banach lattice $\ell^\infty(\mathbb{T}(\mathbb{N}_0))$ consisting of all bounded, $\mathbb{R}$-valued functions defined on $\mathbb{T}(\mathbb{N}_0)$ (equipped with the sup-norm $\|\cdot\|_\infty$). Since $\mathbb{T}(\mathbb{N}_0)$ is countable, the Banach lattice $\ell^\infty(\mathbb{T}(\mathbb{N}_0))$ is isometrically isomorphic to the Banach lattice $\ell^\infty(\mathbb{N})$ (only a different labeling of the elements is involved).

It is readily verified that the set E, defined by

$$E = \left\{ x = (x_\tau)_{\tau \in \mathbb{T}(\mathbb{N}_0)} \in \ell^\infty(\mathbb{T}(\mathbb{N}_0)) : \lim_{k\to\infty} x_{\tau\oplus(k)} = x_\tau \ \forall \tau \in \mathbb{T}(\mathbb{N}_0) \right\}, \tag{11}$$

is a norm closed Riesz subspace of $\ell^\infty(\mathbb{T}(\mathbb{N}_0))$. Hence, E is itself a Banach lattice with respect to $\|\cdot\|_\infty$. Evidently, E contains all the constant sequences.

Next we discuss an indexation for certain subintervals of $[0,1]$. We begin with $I_\emptyset = [0,1]$ ($\emptyset \in \mathbb{T}_0(3)$). Next, define $I_{(0)} = [0,1/3]$, $I_{(1)} = (1/3,2/3]$ and $I_{(2)} = (2/3,1]$ (this defines I_τ for $\tau \in \mathbb{T}_1(3)$). Then define the intervals $I_{(0,0)} = [0,1/9]$, $I_{(0,1)} = (1/9,2/9]$, ..., $I_{(2,2)} = (8/9,1]$ (this defines the sets I_τ for $\tau \in \mathbb{T}_2(3)$). We continue in this way and define (via induction) the sets I_τ for all $\tau \in \mathbb{T}(3)$. It is clear that $\lambda(I_\tau) = 3^{-|\tau|}$ for all $\tau \in \mathbb{T}(3)$, where λ denotes Lebesgue measure on $[0,1]$.

Observe, given $m,n \in \mathbb{N}_0$ with $m < n$ and $\tau \in \mathbb{T}(3)$ with $|\tau| = n$, that there exists a (unique) $\sigma \in \mathbb{T}(3)$ such that $|\sigma| = m$ and $I_\tau \subseteq I_\sigma$. Indeed, if τ is given by $\tau = (\tau_0,\cdots,\tau_{n-1})$, then σ is given by $\sigma = (\tau_0,\ldots,\tau_{m-1})$. It is also clear that $I_\tau \cap I_\sigma = \emptyset$ for all σ satisfying $|\sigma| = m$ and $\sigma \neq (\tau_0,\ldots,\tau_{m-1})$.

For $n \in \mathbb{N}_0$, denote by $\mathcal{E}_n$ the algebra of subsets of $[0,1]$ generated by $\{I_\tau : \tau \in \mathbb{T}(3), |\tau| = n\}$. That is, $\mathcal{E}_n$ is the collection of all subsets of $[0,1]$ which are finite unions of intervals I_τ with $\tau \in \mathbb{T}(3)$, $|\tau| = n$. Then the cardinality $|\mathcal{E}_n| = 2^{3^n}$. Note that $\mathcal{E}_m \subseteq \mathcal{E}_n$ whenever $m < n$ in $\mathbb{N}_0$. It should also be observed that if $m < n$, $F \in \mathcal{E}_m$ and $G = I_\tau$ for some $\tau \in \mathbb{T}(3)$ with $|\tau| = n$, then either $G \subseteq F$ or $G \cap F = \emptyset$.

Next, define the collection $\{F_\tau : \tau \in \mathbb{T}(\mathbb{N}_0)\}$ of subsets of $[0,1]$ by induction on the length of $\tau \in \mathbb{T}(\mathbb{N}_0)$ as follows. For $|\tau| = 0$ we set $F_\emptyset = \emptyset$. Suppose now that $n \in \mathbb{N}_0$ with $n \geq 1$ and that F_τ has already been defined for all $\tau \in \mathbb{T}(\mathbb{N}_0)$ satisfying $|\tau| = n-1$. Given $\tau \in \mathbb{T}(\mathbb{N}_0)$ with $|\tau| = n-1$ and $k \in \mathbb{N}_0$, define the interval $G_{\tau\oplus(k)}$ by setting

$$G_{\tau\oplus(k)} = \begin{cases} I_{\Phi_n(k)} & \text{if } I_{\Phi_n(k)} \cap F_\tau = \emptyset \\ \emptyset & \text{if } I_{\Phi_n(k)} \cap F_\tau \neq \emptyset \end{cases} \tag{12}$$

(recall the definition of $\Phi_n(k)$ as given in (10)). Now define

$$F_{\tau\oplus(k)} = F_\tau \cup G_{\tau\oplus(k)}. \tag{13}$$

Since the function Φ_n has period 3^n, it follows that the sequence $\left\{F_{\tau\oplus(k)}\right\}_{k=0}^{\infty}$ also has period 3^n.

Some properties of the sets F_τ, for $\tau \in \mathbb{T}(\mathbb{N}_0)$, are formulated in the following result (for a proof, see Section 3 in [4]).

LEMMA 3.1. (i) *The set* $F_\tau \in \mathcal{E}_{|\tau|}$ *for all* $\tau \in \mathbb{T}(\mathbb{N}_0)$.

(ii) *Let* $\tau \in \mathbb{T}(\mathbb{N}_0)$ *with* $|\tau| = n-1$ $(1 \leq n \in \mathbb{N}_0)$ *and* $k \in \mathbb{N}_0$. *Then, either* $I_{\Phi_n(k)} \cap F_\tau = \emptyset$ *or* $I_{\Phi_n(k)} \subseteq F_\tau$. *Consequently,* $F_{\tau\oplus(k)} = F_\tau \cup I_{\Phi_n(k)}$ *holds for all* $k \in \mathbb{N}_0$.

(iii) *For every* $\tau \in \mathbb{T}(\mathbb{N}_0)$ *it is the case that*

$$F_\tau = \bigcup_{i=1}^{|\tau|} I_{\Phi_i}(\tau_{i-1}). \tag{14}$$

(iv) *For every* $\tau \in \mathbb{T}(\mathbb{N}_0)$ *we have that* $\lambda(F_\tau) \leq 1/2$.

(v) *Given* $\sigma \in \mathbb{T}(3)$ *and* $\tau \in \mathbb{T}(\mathbb{N}_0)$ *such that* $|\sigma| > |\tau|$, *we have* $I_\sigma \subseteq F_{\tau\oplus(k)}$ *for infinitely many values of* $k \in \mathbb{N}_0$.

Next we define a system $\{s_\tau : \tau \in \mathbb{T}(\mathbb{N}_0)\}$ of functions in $L^\infty([0,1])$, where $[0,1]$ is equipped with Lebesgue measure λ defined on the Borel σ-algebra Σ in $[0,1]$. The definition is by induction on the length $|\tau|$ of $\tau \in \mathbb{T}(\mathbb{N}_0)$.

Denote by $(r_n)_{n=0}^{\infty}$ the sequence of Rademacher functions on $[0,1]$, that is, $r_n(x) = sgn(\sin(2^n \pi x))$, $x \in [0,1]$. Observe that:

- $|r_n(x)| = 1$ for all $x \in [0,1]$ and $n \in \mathbb{N}_0$;
- for each $f \in L^1([0,1])$ we have that $\int_0^1 f(x) r_n(x)\,dx \to 0$ as $n \to \infty$.

Identifying $L^\infty([0,1])$ with the dual space of $L^1([0,1])$, the latter property may also be formulated as: $r_n \to 0$ weak* in $L^\infty([0,1])$ as $n \to \infty$. Note that also $r_n g \to_n 0$ weak* for all $g \in L^\infty([0,1])$.

For $|\tau| = 0$, define $s_\tau = 0$. Suppose now that $s_\tau \in L^\infty([0,1])$ has already been defined for every $\tau \in \mathbb{T}(\mathbb{N}_0)$ with $|\tau| = n-1$ for some $1 \leq n \in \mathbb{N}_0$. For each $k \in \mathbb{N}_0$, define the function $s_{\tau\oplus(k)}$ by setting

$$s_{\tau\oplus(k)} = s_\tau + r_k \chi_{G_{\tau\oplus(k)}}, \tag{15}$$

where the set $G_{\tau\oplus(k)}$ is defined by (12). For the proof of the following result we also refer to Section 3 in [4].

LEMMA 3.2. (i) *The modulus of* s_τ *satisfies* $|s_\tau| = \chi_{F_\tau}$ *for all* $\tau \in \mathbb{T}(\mathbb{N}_0)$.

(ii) *Let* $\tau \in \mathbb{T}(\mathbb{N}_0)$. *Then* $s_{\tau\oplus(k)} \to s_\tau$ *weak** *in* $L^\infty([0,1])$ *as* $k \to \infty$.

REMARK 3.3. Some further remarks are of interest. Define a set $O \subseteq \mathbb{T}(\mathbb{N}_0)$ to be open if for every $\tau \in O$ there exists $K \in \mathbb{N}_0$ such that $\tau \oplus (k) \in O$ for all $k \geq K$. It is readily verified that these open sets constitute a topology in $\mathbb{T}(\mathbb{N}_0)$. It is not difficult to show that the space $C_b(\mathbb{T}(\mathbb{N}_0))$ consisting of all the bounded $\mathbb{R}$-valued continuous functions defined on $\mathbb{T}(\mathbb{N}_0)$ is precisely the space E. It can be shown that $\mathbb{T}(\mathbb{N}_0)$ is completely regular and normal but, $\mathbb{T}(\mathbb{N}_0)$ is not metrizable. Furthermore, $\mathbb{T}(\mathbb{N}_0)$ is not an F-space. We leave the details to the interested reader.

4. A vector measure with non-proper modulus. Let the Banach lattice E (see (11)) and the functions

$$\{s_\tau : \tau \in \mathbb{T}(\mathbb{N}_0)\} \subseteq L^\infty([0,1])$$

(see (15)) be as specified in Section 3. Let $\Sigma = \mathcal{B}([0,1])$ be the Borel σ-algebra of $[0,1]$. For $A \in \Sigma$ define $m(A) \in \ell^\infty(\mathbb{T}(\mathbb{N}_0))$ by

$$(m(A))_\tau = \int_A s_\tau \, d\lambda, \quad \tau \in \mathbb{T}(\mathbb{N}_0). \tag{16}$$

Note, via Lemma 3.2 (i), that for each $\tau \in \mathbb{T}(\mathbb{N}_0)$ we have

$$|(m(A))_\tau| \leq \int_A |s_\tau| \, d\lambda = \int_A \chi_{F_\tau} d\lambda = \lambda(A \cap F_\tau) \leq \lambda(A), \quad A \in \Sigma.$$

This implies, in particular, that

$$\|m(A)\|_\infty \leq \lambda(A), \quad A \in \Sigma. \tag{17}$$

Since $s_{\tau \oplus (k)} \to s_\tau$ weak* as $k \to \infty$, for each $\tau \in \mathbb{T}(\mathbb{N}_0)$ (see Lemma 3.2 (ii)), it follows that $m(A) \in E$ for each $A \in \Sigma$. Hence, $m : \Sigma \to E$ is a finitely additive vector measure. It follows from (17) that m is actually countably additive. Since $-\chi_{[0,1]} \leq s_\tau \leq \chi_{[0,1]}$ for $\tau \in \mathbb{T}(\mathbb{N}_0)$, it is clear from (16) that

$$|m(A)| \leq \lambda(A) \chi_{\mathbb{T}(\mathbb{N}_0)}, \quad A \in \Sigma. \tag{18}$$

In particular, m is order bounded, that is, $m \in M_{ob}(\Sigma, E)$. Defining the positive, countably additive vector measure $m_0 : \Sigma \to E$ by

$$m_0(A) = \lambda(A) \chi_{\mathbb{T}(\mathbb{N}_0)}, \quad A \in \Sigma, \tag{19}$$

inequality (18) may also be written as $-m_0 \leq m \leq m_0$ in $M_{ob}(\Sigma, E)$. Hence, m_0 is an upper bound for $\{m, -m\}$. In particular, m can be written as the difference of two positive vector measures (indeed, $m = m_0 - (m_0 - m)$).

For each $\tau \in \mathbb{T}(\mathbb{N}_0)$ define $\delta_\tau \in E^*$ by $\delta_\tau(x) = \langle x, \delta_\tau \rangle = x_\tau$, for $x \in E$. Note that $\|\delta_\tau\|_{E^*} = 1$. The scalar measure $\langle m, \delta_\tau \rangle$ is given by

$$\langle m, \delta_\tau \rangle(A) = \langle m(A), \delta_\tau \rangle = (m(A))_\tau = \int_A s_\tau \, d\lambda, \quad A \in \Sigma.$$

Accordingly, its variation measure $|\langle m, \delta_\tau \rangle|$ is given (see Lemma 3.2 (i)) by

$$|\langle m, \delta_\tau \rangle| (A) = \int_A |s_\tau| \, d\lambda = \int_A \chi_{F_\tau} d\lambda = \lambda (A \cap F_\tau), \quad A \in \Sigma. \tag{20}$$

LEMMA 4.1. *The modulus $|m|_o$ of m exists in $M_{ob}(\Sigma, E)$ and is precisely m_0.*

Proof. It has already been observed that m_0 is an upper bound for $\{m, -m\}$ in $M_{ob}(\Sigma, E)$. Suppose that $m_1 \in M_{ob}(\Sigma, E)^+$ is any upper bound of $\{m, -m\}$, i.e., $-m_1 \le m \le m_1$, which is equivalent to saying that $|m(A)| \le m_1(A)$, for $A \in \Sigma$. We need to show that $m_0 \le m_1$, that is, $\lambda(A) \le (m_1(A))_\tau$ for all $A \in \Sigma$ and every $\tau \in \mathbb{T}(\mathbb{N}_0)$; see (19).

For this purpose, observe that the inequality $|m(A)| \le m_1(A)$, for $A \in \Sigma$, implies that

$$|\langle m, \delta_\tau \rangle (A)| = |(m(A))_\tau| \le (m_1(A))_\tau, \quad A \in \Sigma, \quad \tau \in \mathbb{T}(\mathbb{N}_0).$$

Since $A \longmapsto (m_1(A))_\tau$, for $A \in \Sigma$, is a finitely additive, positive (scalar) measure on Σ, for each $\tau \in \mathbb{T}(\mathbb{N}_0)$, this yields (via (20)) that

$$\lambda (A \cap F_\tau) = |\langle m, \delta_\tau \rangle| (A) \le (m_1(A))_\tau, \quad A \in \Sigma. \tag{21}$$

Let $A \in \Sigma$ and $\tau \in \mathbb{T}(\mathbb{N}_0)$ be fixed and set $n = |\tau|$. Select any $\sigma \in \mathbb{T}_{n+1}(3)$. Since $|\sigma| > |\tau|$, it follows from Lemma 3.1 (v) that $I_\sigma \subseteq F_{\tau \oplus (k)}$ holds for infinitely many values of $k \in \mathbb{N}_0$. Moreover, (20) and (21) imply, for infinitely many values of k, that

$$\begin{aligned} (m_1(A \cap I_\sigma))_{\tau \oplus (k)} &\ge \left| \langle m, \delta_{\tau \oplus (k)} \rangle \right| (A \cap I_\sigma) \\ &= \lambda \left(A \cap I_\sigma \cap F_{\tau \oplus (k)} \right) = \lambda (A \cap I_\sigma). \end{aligned}$$

Since $m_1(A \cap I_\sigma) \in E$, it follows that

$$(m_1(A \cap I_\sigma))_\tau = \lim_{k \to \infty} (m_1(A \cap I_\sigma))_{\tau \oplus (k)} \ge \lambda (A \cap I_\sigma).$$

The sets $\{I_\sigma : \sigma \in \mathbb{T}_{n+1}(3)\}$ form a partition of $[0,1]$ and so,

$$\begin{aligned} (m_1(A))_\tau &= \sum_{\sigma \in \mathbb{T}_{n+1}(3)} (m_1(A \cap I_\sigma))_\tau \\ &\ge \sum_{\sigma \in \mathbb{T}_{n+1}(3)} \lambda (A \cap I_\sigma) = \lambda (A). \end{aligned}$$

Since $A \in \Sigma$ and $\tau \in \mathbb{T}(\mathbb{N}_0)$ are arbitrary, this suffices to complete the proof of the lemma. □

The following result is analogous to Proposition 3 in [4].

LEMMA 4.2. *Let m be the vector measure given by (16). Its modulus $|m|_o$ in $M_{ob}(\Sigma, E)$ is* not *given by the formula (8). Actually, for any partition $\pi \in \Pi([0,1])$, we have that*

$$\sum_{B\in\pi} |m(B)| \leq \frac{1}{2}\chi_{\mathbb{T}(\mathbb{N}_0)}, \tag{22}$$

whereas $|m|_o([0,1]) = \chi_{\mathbb{T}(\mathbb{N}_0)}$.

Proof. Let $\pi \in \Pi([0,1])$ and $B \in \pi$. It follows from (20) that

$$|(m(B))_\tau| \leq |\langle m, \delta_\tau\rangle|(B) = \lambda(B \cap F_\tau), \quad \tau \in \mathbb{T}(\mathbb{N}_0),$$

and so, by Lemma 3.1 (iv), we have that

$$\sum_{B\in\pi} |m(B)_\tau| \leq \sum_{B\in\pi} \lambda(B\cap F_\tau) = \lambda(F_\tau) \leq 1/2.$$

This shows that (22) holds. On the other hand, by Lemma 4.1 we know that $|m|_o = m_0$ and hence, from (19), it is clear that $m_0([0,1]) = \chi_{\mathbb{T}(\mathbb{N}_0)}$. The proof is thereby complete. □

Let us summarize what has been established, namely, the main result of the paper.

THEOREM 4.3. *The modulus of the order bounded vector measure $m : \Sigma \to E$, as defined in (16), exists in $M_{ob}(\Sigma, E)$ and is given by $|m|_o(A) = \lambda(A)\chi_{\mathbb{T}(\mathbb{N}_0)}$ for each $A \in \Sigma$. In particular, $|m|_o$ is countably additive. However, $|m|_o$ is* not *given by the formula (8), that is, the modulus of m* exists *but, it* does not exist properly.

According to Theorem 2.2 (and Corollary 2.3), it follows from Theorem 4.3 that the Banach lattice E is *not* Dedekind complete. However, E is order separable. Indeed, the set $\mathbb{T}(\mathbb{N}_0)$ is countable and hence, every disjoint system in E is at most countable. Consequently, E is not even Dedekind σ-complete (actually, E does not have the σ-interpolation property; cf. Remark 3.3).

5. Relation between m and Elliott's operator. As alluded to in the Introduction the vector measure m, as defined in Section 4, is generated by Elliott's operator $T : L^1([0,1]) \to E$ (in [4] T is denoted by R) via the formula

$$m(A) = T(\chi_A), \quad A \in \Sigma = \mathcal{B}([0,1]). \tag{23}$$

Here T is defined by

$$Tf = \left(\int_0^1 s_\tau f \, d\lambda\right)_{\tau\in\mathbb{T}(\mathbb{N}_0)}, \quad f \in L^1([0,1]). \tag{24}$$

According to (23) and (24) one would expect a close interaction between the properties of m and those of T. These connections are exposed in this final section.

The space E given by (11) is an AM-space. Therefore, it follows from Example 2.7 (e) and Theorem 4.3 that the range $m(\Sigma)$ of m is *not* relatively compact. It should be noted that this implies, in particular, that the operator T is *not* a Dunford-Pettis operator (as order intervals in $L^1([0,1])$ are weakly compact; see Theorem 2.4.2 in [8]). Furthermore, T is *not* weakly compact, as $L^1([0,1])$ has the Dunford-Pettis property (see Proposition 3.7.9 in [8], for example). It is evident from the formula for $|m|_o$ (see Theorem 4.3) that $m(\Sigma)$ is compact.

Since $m : \Sigma \to E$ is countably additive, there is available a well developed theory of integration with respect to m; see Ch. 3 of [9], for example, and the references therein. We summarize the relevant aspects from there which are needed here.

A Σ-measurable function $f : [0,1] \to \mathbb{R}$ is called *scalarly m-integrable* if $\int_{[0,1]} |f|\, d|\langle m, x^*\rangle| < \infty$ for all $x^* \in E^*$. The space $L^1_w(m)$ of all such (equivalence classes of) functions f is a Banach function space (with respect to any control measure for m) when it is equipped with the norm

$$\|f\|_{L^1_w(m)} = \sup_{\|x^*\|_{E^*} \le 1} \int_{[0,1]} |f|\, d|\langle m, x^*\rangle|, \quad f \in L^1_w(m),$$

and it has the Fatou property. A function $f \in L^1_w(m)$ is said to be *m-integrable* if, for every $A \in \Sigma$, there exists an element $\int_A f dm \in E$ (necessarily unique) which satisfies

$$\left\langle \int_A f dm, x^* \right\rangle = \int_A f d\langle m, x^*\rangle, \quad x^* \in E^*.$$

The space $L^1(m)$ of all m-integrable functions is a closed ideal in $L^1_w(m)$. Hence, $L^1(m)$ is also a Banach function space for the restriction of the norm $\|\cdot\|_{L^1_w(m)}$ to $L^1(m)$, which is denoted by $\|\cdot\|_{L^1(m)}$. The norm $\|\cdot\|_{L^1(m)}$ is order continuous. The integration operator $I_m : L^1(m) \to E$ is defined by

$$I_m f = \int_{[0,1]} f dm, \quad f \in L^1(m).$$

It is a continuous linear map satisfying $\|I_m\| = 1$.

Recall that the variation measure $|m| : \Sigma \to [0,\infty]$ of m is defined by

$$|m|(A) = \sup_{\pi \in \Pi(A)} \sum_{B \in \pi} \|m(B)\|_E, \quad A \in \Sigma.$$

It follows from (17) that $|m| \le \lambda$ on Σ and so, in particular, $|m|$ is finite. It is routine to verify that $|m|$ and m have the same null sets. Hence, $|m|$ is a control measure for m. The measure $|m|$ can be precisely identified.

Lemma 5.1. *The variation measure $|m|$ is equal to Lebesgue measure λ.*

Proof. As has already been observed, $|m| \le \lambda$ on Σ. To prove the reverse inequality we first establish that the inequalities

$$|m|(A) \ge \lambda(A \cap F_\tau), \quad \tau \in \mathbb{T}(\mathbb{N}_0), \tag{25}$$

are valid for each $A \in \Sigma$. So, fix $A \in \Sigma$ and $\tau \in \mathbb{T}(\mathbb{N}_0)$. Define the Borel sets $A_\tau^+ = \{x \in A : s_\tau(x) \geq 0\}$ and $A_\tau^- = \{x \in A : s_\tau(x) < 0\}$, which form a partition of A in Σ. Accordingly,

$$|m|(A) \geq \left\| m\left(A_\tau^+\right)\right\|_\infty + \left\| m\left(A_\tau^-\right)\right\|_\infty .$$

Since $s_\tau = |s_\tau|$ on A_τ^+ and $s_\tau = -|s_\tau|$ on A_τ^-, it follows from (16) that

$$\left\| m\left(A_\tau^+\right)\right\|_\infty \geq \left| \int_{A_\tau^+} s_\tau \, d\lambda \right| = \int_{A_\tau^+} |s_\tau| \, d\lambda$$

and that

$$\left\| m\left(A_\tau^-\right)\right\|_\infty \geq \left| \int_{A_\tau^-} s_\tau \, d\lambda \right| = \int_{A_\tau^-} |s_\tau| \, d\lambda.$$

Combining the previous three inequalities, in combination with the fact that $|s_\tau| = \chi_{F_\tau}$ (cf. Lemma 3.2 (i)), yields

$$|m|(A) \geq \int_{A_\tau^+} |s_\tau| \, d\lambda + \int_{A_\tau^-} |s_\tau| \, d\lambda = \lambda(A \cap F_\tau).$$

This completes the proof of (25).

Choose any $\sigma \in \mathbb{T}(3)$ with $|\sigma| \geq 1$. Then Lemma 3.1 (v) ensures that there exists $\tau \in \mathbb{T}(\mathbb{N}_0)$ satisfying $I_\sigma \subseteq F_\tau$. It follows from (25), applied to $A = I_\sigma$, that

$$|m|(I_\sigma) \geq \lambda(I_\sigma \cap F_\tau) = \lambda(I_\sigma).$$

Since the reverse inequality has already been established, we can conclude that

$$|m|(I_\sigma) = \lambda(I_\sigma), \quad \sigma \in \mathbb{T}(3), \ |\sigma| \geq 1. \tag{26}$$

It follows from (26) that $|m|$ and λ coincide on each algebra of sets $\mathcal{E}_n$ (cf. Section 3) for $n \in \mathbb{N}$ and hence, also on the algebra $\bigcup_{n=1}^\infty \mathcal{E}_n$. Since $\bigcup_{n=1}^\infty \mathcal{E}_n$ generates the σ-algebra $\Sigma = \mathcal{B}([0,1])$ and both $|m|$ and λ are finite measures, it follows that $|m| = \lambda$ on Σ. □

The previous result implies that λ is a control measure for m and hence, both $L^1(m)$ and $L_w^1(m)$ are Banach function spaces over $([0,1], \Sigma, \lambda)$. In the terminology of p. 187 in [9], the operator T is λ-*determined.* The following result is a direct consequence of Proposition 4.4 (iii) in [9].

Lemma 5.2. *The space* $L^1([0,1]) \subseteq L^1(m)$ *with a continuous inclusion and the integration operator* $I_m : L^1(m) \to E$ *satisfies*

$$Tf = I_m f = \int_{[0,1]} f dm, \quad f \in L^1([0,1]).$$

Perhaps somewhat surprising is the following fact.

PROPOSITION 5.3. *The following Banach function spaces satisfy*

$$L^1(|m|) = L^1(m) = L^1_w(m) = L^1([0,1]) \tag{27}$$

with equivalent norms.

Proof. Lemmas 5.1 and 5.2 imply that

$$L^1(|m|) = L^1([0,1]) \subseteq L^1(m) \subseteq L^1_w(m).$$

Let $f \in L^1_w(m)$. Then $f \in L^1(|\langle m, \delta_\tau\rangle|)$ for each $\tau \in \mathbb{T}(\mathbb{N}_0)$ (where the functionals $\delta_\tau \in E^*$ are defined in Section 4). Since $|\langle m, \delta_\tau\rangle|(A) = \int_A \chi_{F_\tau} d\lambda$ (see (20)), this implies that

$$\int_{[0,1]} |f| \chi_{F_\tau} d\lambda < \infty, \quad \tau \in \mathbb{T}(\mathbb{N}_0).$$

It was noted in the proof of Lemma 5.1 that for each $\sigma \in \mathbb{T}(3)$ with $|\sigma| \geq 1$, there exists $\tau \in \mathbb{T}(\mathbb{N}_0)$ satisfying $I_\sigma \subseteq F_\tau$. Consequently, $\int_{I_\sigma} |f| \, d\lambda < \infty$ for every $\sigma \in \mathbb{T}(3)$ with $|\sigma| \geq 1$, which clearly implies that $\int_0^1 |f| \, d\lambda < \infty$. This establishes that $L^1_w(m) \subseteq L^1([0,1])$, which implies (27). Since all four spaces involved are Banach function spaces, it follows that all norms are equivalent. □

Lemmas 5.1 and 5.2 and Proposition 5.3 yield an integral representation of $T : L^1([0,1]) \to E$, namely

$$Tf = \int_{[0,1]} f \, dm, \quad f \in L^1([0,1]).$$

REMARK 5.4. Of course, the norms in the spaces $L^1(|m|)$ and $L^1([0,1])$ are actually equal. It is readily verified that $|\langle m, x^*\rangle| \leq |m|$ for all $x^* \in E^*$ with $\|x^*\|_{E^*} \leq 1$. This implies that $\|f\|_{L^1(m)} \leq \|f\|_{L^1(|m|)} = \|f\|_1$ for all $f \in L^1(m) = L^1([0,1])$. It is not difficult to show that $\|f\|_1 \leq 3\|f\|_{L^1(m)}$, for $f \in L^1([0,1])$.

REMARK 5.5. The vector measure $m : \Sigma \to E$ is countably additive, has finite variation and satisfies $m \ll \lambda$ (cf. (17) or Lemma 5.1 and the discussion prior to it). However, m *cannot* possess an E-valued Bochner density with respect to λ; see Example 2.7 (c) and Theorem 4.3.

REFERENCES

1. C.D. ALIPRANTIS AND O. BURKINSHAW, *Positive Operators*, Academic Press, Orlando/San Diego/New York, 1985.

2. L. DREWNOWSKI AND W. WNUK, On the modulus of indefinite vector integrals with values in Banach lattices, *Atti Sem. Mat. Fis. Iniv. Modena* **47** (1999), 221–233.

3. ———————, On the modulus of measures with values in topological Riesz spaces, *Rev. Mat. Complutense* **15** (2002), 357–400.

4. M. Elliott, The Riesz-Kantorovich formulae, *Positivity* **23** (2019), 1245–1259.

5. G. Groenewegen, *On Spaces of Banach Lattice Valued Functions and Measures*, Ph.D. Thesis, Univ. Nijmegen, The Netherlands, 1982.

6. L.V. Kantorovich, Concerning the general theory of operations in partially ordered spaces, *Dokl. Akad., Nauk SSSR* **1** (1936), 283–286 (in Russian).

7. W.A.J. Luxemburg and A.C. Zaanen, *Riesz Spaces I,* North-Holland, Amsterdam/London, 1971.

8. P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, Berlin/Heidelberg/New York, 1991.

9. S. Okada, W.J. Ricker, and E.A. Sánchez Pérez, *Optimal Domains and Integral Extensions of Operators Acting in Function Spaces*, Operator Theory Advances Applications, Vol. 180, Birkhäuser, Basel/Berlin, 2008.

10. H.H. Schaefer and X.D. Zhang, A note on order bounded vector measures, *Arch. Math.* **63** (1994), 152–157.

11. K.D. Schmidt, *Jordan Decompositions of Generalized Vector Measures*, Longman Scientific & Technical, Harlow, 1989.

12. A.C. Zaanen, *Riesz Spaces II*, North-Holland, Amsterdam, 1983.

Received 9 November, 2022.