Delft University of Technology
Faculty of Electrical Engineering, Mathematics and Computer Science
Delft Institute of Applied Mathematics

Nonparametric Calibration of Inhomogeneous Lévy Processes using Fourier Techniques

A thesis submitted to the
Delft Institute of Applied Mathematics
in partial fulfillment of the requirements

for the degree

MASTER OF SCIENCE
in
APPLIED MATHEMATICS

by

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Delft, the Netherlands
September 2018

MSc Thesis APPLIED MATHEMATICS

"Nonparametric Calibration of Inhomogeneous Lévy Processes using Fourier Techniques"

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September 2018

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Abstract

In this report, inhomogeneous Lévy processes are studied in a discrete observational model based on derivatives of the process. First, homogeneous Lévy models are defined and an already known nonparametric method, using Fourier techniques and call and put option prices, for estimating the parameters of the model is described based on Belomestny and Reiß (2006a). Previous research suggests that there is a need for an extension of this concept since option prices with different maturities produce significantly different results. After all, the assumption that the parameters of the model are the same for any time window is not realistic and better results could be achieved once this premise is rejected.

That is why inhomogeneous Lévy processes are introduced and studied in this report. The estimation method for the homogeneous model from Belomestny and Reiß (2006a) is extended to fit into the inhomogeneous framework. Next, asymptotic normality of the estimators is proven for these processes in this setting and confidence intervals are constructed using the finite sample variance method. Asymptotic normality has already been shown and confidence intervals have been constructed in the homogeneous framework in the continuous observational model by Söhl (2014). Finally, data is simulated from an inhomogeneous Merton model to test the performance of the method and options from the S&P 500 index are used as a real-world application.

Preface

This report was written to conclude my master in Applied Mathematics. In the past 5 years, I have gained a lot of knowledge in almost all areas of mathematics and the most interesting one for me was (and still is) the world of statistics. That is also why I am going to spend the next 4 years to do a Ph. D. in statistics.

Maybe except for my first year, I always thought that one should do mathematics with the purpose of being useful. I did like to think about the uttermost abstract algebraic concepts but, in the end, it was always disappointing to me if there were little to no real-world applications. That is the most important reason why I love to do statistics. The balance between theory and practice can be really emphasized in this field. On one hand, you need to prove that the arguments work in theory and on the other hand, you should make sure they also work in practice.

In the past 8 months, I had the privilege to do research on a statistical topic with exactly this emphasis on the balance between theory and practice. Although it was great to find satisfying theoretical results, it felt better when the method started working in practice.

I would like to thank Jakob Söhl for supervising me during this project and helping me at points where I did not know what to do. Moreover, I would like to thank the thesis committee for reading this report and for judging me at my thesis defense. Last but not least, I want to thank my friends and family who motivated me for the last few months to complete this thesis while I was most of the time complaining about it.

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September 2018, Delft

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Chapter 1

Introduction

In the financial world, mathematics plays a huge role. Of course, when one wants to gamble on Apple or Bitcoin there is no need to worry about mathematics at all. However, when you are selling products related to Apple or Bitcoin shares, it is important that you are not selling products that introduce arbitrage, a risk-free strategy with non-negative profit. Thus, it is necessary to price these products as well as possible and that is where mathematics comes in. This even traces back to ancient Greek history.

Indeed, Lucretius, a Roman poet, described approximately in 60 BC that there exists something like a random walk process that we now know as the Brownian Motion (BM). In 1827, the BM was described extensively by Robert Brown whilst investigating pollen and 73 years later, the relation between stock prices and BM was made by Louis Bachelier in his Ph. D. thesis. Paul Samuelson noted that using Brownian motion to describe the movements of a stock price can be done better. Merely, because BM does not account for the non-negativity of stock prices. Therefore, he introduced a version which does not become negative and that allows for a certain drift, essentially he looked at the exponential BM defined by $\exp(rt + BM)$. This model comes close to reality, is easy to use and easy to interpret. Therefore, it is still used a lot in practice. For example, quantities that are used by investors such as the implied volatility, also known as the vega, are calculated under this particular assumption.

The biggest problem with this model is that it is an oversimplification for estimating the complex market and, for example, heavy-tails are not modeled well using BM. Approximately 40 years later, Paul Lévy introduced the more general Lévy processes and, as it might come as no surprise, the derived exponential Lévy processes, are now one of the better models one can use to estimate the market. As is shown in Cont and Tankov (2004a), Lévy processes model, for example, typical market characteristics as jumps, volatility smiles, and heavy tails well. Other applications of Lévy models in finance can be found in Schoutens (2003).

Lévy processes are already studied extensively, for example, in Bertoin (1996), Sato and Ken-Iti (1999) and Applebaum (2009). Moreover, different estimation methods for the parameters of certain Lévy processes are developed. For example, Gugushvili (2009) constructed an estimation method based on direct discrete observations from the Lévy process. An interesting case is when no direct data of the underlying process is available but only derived data. For example, options or futures related to a certain stock or index. In short, a call option is the right to buy and a put option is the right to sell a certain product for a given predetermined price at a given time. It happens to be the case that these option prices are related to the underlying Lévy process under the risk-neutral measure.

Cont and Tankov (2004b) and Cont and Tankov (2006) developed an estimation procedure of the Lévy triplet which minimizes the relative entropy with respect to a prior exponential Lévy model. Moreover, Qin and Todorov (2017) study the behavior of a nonparametric estimation method for the Lévy density in an Itô semimartingale model under the condition that the maturity decreases to 0.

One of the applications of knowing the underlying Lévy process under the risk-neutral measure is the arbitrage-free pricing of exotic derivatives, for example, Asian options, where the payoff depends on the average value of the share price, and other path-dependent options, see Albrecher and Predota (2004) or Shreve (2004). Moreover, this method can be used to find whether or not these exotic options are priced correctly.

As of now, nonparametric parameter estimation is only possible for homogeneous Lévy processes, i.e., the process on $[0, T]$ is estimated under the condition that the parameters are assumed to be constant over that time period. In Belomestny and Reiß (2006a), this method is introduced using a finite data sample and (asymptotic) confidence sets of the continuous variant of this model are derived in Söhl (2014).

When the maturity T is relatively large, it is not realistic to make a homogeneous assumption. However, derivatives with maturity T still need to be priced. In Belomestny and Reiß (2006b), the real data example shows that for different maturities parameters are estimated differently implying that this is indeed might not be a realistic assumption. Hence, there is a need for inhomogeneous Lévy processes. They are given a short introduction in Cont and Tankov (2004a) and in this report, inhomogeneous Lévy processes as an extension to the regular ones are studied. The estimation procedure used in Belomestny and Reiß (2006a) is extended to fit into the inhomogeneous framework and asymptotic confidence intervals are constructed for the parameters using the finite sample variance. Moreover, the performance of the implied estimators is tested as well as performance of the confidence sets.

In Chapter 2, the class of inhomogeneous Lévy processes is properly defined. In Chapter 3, the estimation procedure and the underlying model are explained in detail in both the homogeneous and inhomogeneous case. Moreover, one can find the formulas for the estimators of the parameters here. Theoretical results concerning the convergence of the estimators and the optimal rates of convergence are stated in Chapter 4. A significant part of the proof of this theorem is relocated to Chapter 10 and Appendix B-C. Chapter 5 is devoted to the explicit construction of confidence intervals using the finite sample variance method. In Chapters 6 and 7 simulations and applications are discussed. Discussion about particular aspects of this study and interesting further research topics are given in Chapter 8. The conclusion of this thesis can be found in Chapter 9.

Chapter 2

Lévy processes

In this chapter, both homogeneous and inhomogeneous Lévy processes are defined properly. Moreover, a few examples of particular Lévy processes are given.

2.1 Homogeneous Lévy processes

Let $(\Omega, \mathcal{F}, \mathrm{F}, P)$ be a filtered probability space, where $\mathcal{F} = \mathcal{F}_T$ and the filtration $\mathrm{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ satisfy the usual conditions. Let $T \in [0, \infty)$ denote the time horizon.

Definition 1. *A càdlàg, adapted, real valued stochastic process $(X_t)_{0\leq t\leq T}$ with $X_0 = 0$ a.s. is called a (homogeneous) Lévy process if the following conditions are met:*

1. *X has independent increments, i.e. $X_t - X_s$ is independent of $\mathcal{F}_s$ for any $0 \leq s < t \leq T$.*
2. *X has stationary increments, i.e. for any $0 \leq s \leq t \leq T$ the distribution of $X_t - X_s$ does only depend on $t - s$.*
3. *X is stochastically continuous, i.e. for every $0 \leq t \leq T$ and $\varepsilon > 0$: $\lim_{s\to t} P(|X_t - X_s| > \varepsilon) = 0$.*

In comparison to the definition of a Brownian motion, it is not assumed that X needs to have continuous paths with probability 1. This is representative in the financial markets since errors and imperfections of the market sometimes cause instantaneous jumps. In this report, only Lévy processes with a jump component of finite intensity and absolutely continuous jump distribution are under consideration.

Moreover, the assumption that $X_t - X_s$ should be Gaussian is dropped from the definition of Brownian motion to get an even more general stochastic process. Under these general assumptions, the formula for the characteristic function simplifies to

$$\varphi_T(u) := \mathbb{E}\left\{e^{iuX_T}\right\} = \exp\left(T\left(-\frac{\sigma^2 u^2}{2} + i\gamma u + \int_{-\infty}^{\infty}\left(e^{iux} - 1\right)\nu(x)\,\mathrm{d}x\right)\right). \tag{2.1}$$

$\sigma \geq 0$ is called the volatility and is essentially the Brownian motion component of the Lévy process which describes its variability. $\gamma \in \mathbb{R}$ will be referred to as the drift and the non-negative function $\nu \in L^1(\mathbb{R})$ is the Lévy density with intensity $\lambda = \|\nu\|_{L^1(\mathbb{R})}$. This representation of the characteristic function is called the Lévy-Khintchine representation and, as is a known fact for characteristic functions, it characterizes the complete Lévy process. Hence, the triplets (σ^2, γ, ν) correspond one-to-one with Lévy processes and it will therefore be called the Lévy triplet. This triplet will be the object of investigation.

Examples of Lévy processes:

1. The compound Poisson process. This process is locally constant and at exponentially distributed times, a jump in the path occurs. The size of the jump is allowed to follow any distribution.

2. The Brownian motion with drift. This is the only non-trivial Lévy process which has continuous paths with probability 1.

3. The Merton model. This is essentially a Brownian motion with normally distributed jumps which occur according to a Poisson process.

2.2 Inhomogeneous Lévy processes

As historical data suggested in Belomestny and Reiß (2006b) and Söhl and Trabs (2014), there is a need for an extension of homogeneous Lévy processes. In particular, it is desired to develop an inhomogeneous version where the parameters are time-dependent. In this report, these functions will be assumed to be locally constant for simplicity and interpretability purposes. Moreover, smooth parameter functions are not identifiable from options with a finite number of maturities only.

An inhomogeneous Lévy process is defined as

Definition 2. *A càdlàg, adapted, real valued stochastic process* $(X_t)_{0\leq t\leq T}$ *with* $X_t = 0$ *a.s. is called an inhomogeneous Lévy process if for* t_j, $j = 1, 2, \ldots, n$ *with* $t_1 = 0$ *and* $t_n = T$, *it holds true that for* $j = 1, \ldots, n-1$, $(X_t - X_{t_j})_{t_j\leq t<t_{j+1}}$ *behave like independent Lévy processes with Lévy triplets* $(\sigma_j^2, \gamma_j, \nu_j)$ *and for any* $j = 2, \ldots, n-1$ *and* $\varepsilon > 0$ *we have*

$$\lim_{h\downarrow 0} P(|X_{t_j-h} - X_{t_j+h}| > \varepsilon) = 0$$

According to the model,

$$\varphi_{T_j}(u) = \mathbb{E}\left\{e^{iuX_{T_j}}\right\} = \mathbb{E}\left\{\mathrm{e}^{iu\left(X_{T_j}-X_{T_{j-1}}\right)}\cdot \mathrm{e}^{iuX_{T_{j-1}}}\right\} = \mathbb{E}\left\{\mathrm{e}^{iu\left(X_{T_j}-X_{T_{j-1}}\right)}\right\}\cdot \varphi_{T_{j-1}}(u).$$

Moreover, the first term on the right-hand side can be simplified using the Lévy-Khintchine representation (2.1) which yields

$$\frac{\varphi_{T_j}(u)}{\varphi_{T_{j-1}}(u)} = \exp\left((T_j - T_{j-1})\left(-\frac{\sigma_j^2 u^2}{2} + i\gamma_j u + \int_{-\infty}^{\infty}\left(\mathrm{e}^{iux} - 1\right)\nu_j(x)\,\mathrm{d}x\right)\right). \tag{2.2}$$

In turn, the left-hand side of 2.2 will be estimated using option prices. From there it is possible to construct estimators for the Lévy triplets $(\sigma_j^2, \gamma_j, \nu_j)_{j=1}^n$ as will be described in detail in the next chapter.

Chapter 3

The model

From the Lévy-Khintchine representation (2.1), Belomestny and Reiß (2006a) developed an estimation procedure for the (homogeneous) Lévy triplet. This method will be explained in detail in the next section. In the second section of this chapter, this method will be generalized to inhomogeneous Lévy processes. It is possible for the reader to skip the homogeneous part and continue with the more general inhomogeneous part since that section is self-contained. However, for an interested reader, it is good to see where the differences in generalizing the concept come from, and an inexperienced reader in this topic might find the generalized case overwhelming in terms of indexes and notation. Therefore, it is advised not to skip the homogeneous case and to look at the inhomogeneous case how the idea and all concepts are generalized.

3.1 The homogeneous model

This section describes the model on which the estimation procedure of the Lévy triplet (σ^2, γ, ν) between 0 and T is based. To that end, suppose that option data maturity T is available. Assume that the Lévy triplet that is associated with the process X_t belongs to the class $\mathcal{G}_s(R, \sigma_{\max})$, which is defined below, for a known R and $\sigma_{\max}$.

Definition 3. *For $s \in \mathbb{N}$ and $R, \sigma_{\max} > 0$ let $\mathcal{G}_s(R, \sigma_{\max})$ denote the set of all Lévy triplets $\mathcal{T} = (\sigma^2, \gamma, \nu)$ satisfying*

$$\forall t \geq 0 : \; \mathbb{E}\left\{\exp(X_t)\right\} = 1 \quad \Leftrightarrow \quad \frac{\sigma^2}{2} + \gamma + \int_{-\infty}^{\infty} (\exp(x) - 1)\, \nu(x)\, \mathrm{d}x = 0 \tag{3.1}$$

and

$$C := \mathbb{E}\left\{\exp(2X_T)\right\} < \infty.$$

Moreover, $C \leq R$, μ is s-times (weakly) differentiable where $s \in \mathbb{Z}_{\geq 2}$ and

$$\sigma \in (0, \sigma_{\max}), \quad |\gamma|, \lambda \in [0, R], \quad \max_{0 \leq k \leq s} \left\| \mu^{(k)} \right\|_{L^2} \leq R, \quad \left\| \mu^{(s)} \right\|_{L^\infty} \leq R.$$

This definition invokes making a lot of different assumptions. Some may seem logical while that may not be the case for others. Therefore, the intuition and reasoning behind making these assumptions will be given before the remaining part of the model is explained.

The condition from equation (3.1), that is imposed on the parameters, is also called the martingale condition. This condition ensures that the exponential Lévy process induced by the Lévy triplet is a martingale. The second assumption, i.e., $C < \infty$, is a technical assumption which guarantees that the variance (or second moment) of the underlying stock price is finite. The R and $\sigma_{\max}$ are also introduced for theoretical purposes since it may be the case that either one of $C, \sigma, |\gamma|, \lambda, \mu^{(k)}$ becomes 'too large'. If one or more of these parameters become too big, it is impossible to construct a statistically stable estimation method. In that case, there would be no guarantee that the argument we evaluate the logarithm at is bounded from below by a positive constant. This causes either a statistically unstable method or it introduces an extra bias term. Both are undesirable.

Let S_t denote the underlying asset price. It will be assumed that S_t comes from an exponential inhomogeneous Lévy process, i.e.,

$$S_t = S_0 \exp(\overline{r}t + X_t),$$

with X_t a homogeneous Lévy process and $\overline{r}$ the risk-free interest rate. Note that it is thus assumed that the discounted asset price is a martingale under the risk-neutral measure. Define K_m as the strike price of the mth option with maturity T.

The risk-neutral option price for a European call option is given by (Shreve, 2004, p. 218)

$$C(K_m, T) = \mathrm{e}^{-\overline{r}T} \mathbb{E}\left\{(S_T - K_m)^+\right\}$$

with $z^+ := \max(0, z)$. Notice that the expectation is taken under the risk-neutral measure. Chapter A. is devoted to explaining briefly what the risk-neutral measure is and why the risk-neutral measure is used. For more about the risk-neutral measure, one can read Shreve (2012) and Shreve (2004). In the remaining part of this section and the upcoming chapters, the risk-neutral measure will always be used if not stated otherwise. It is common to define the log-forward moneyness x_m to replace the strike price K_m

$$x_m := \log(K_m/S_0) - \overline{r}T.$$

We reparametrize the function C to

$$C(K_m, T) = S_0 \mathbb{E}\left\{\left(\mathrm{e}^{X_T} - \mathrm{e}^{x_m}\right)^+\right\} =: \mathcal{C}(x_m, T)$$

For put options we have

$$\mathcal{P}(x_m, T) = S_0 \mathbb{E}\left\{\left(\mathrm{e}^{x_m} - \mathrm{e}^{X_T}\right)^+\right\}$$

(Shreve, 2004, p. 163). Note that $\mathcal{C}(x, T)$, $\mathcal{P}(x, T)$ do not converge to 0 when x tends to $-\infty$ and ∞, respectively. Hence, it makes no sense to take the Fourier transform of $\mathcal{C}$ or $\mathcal{P}$. However, $\lim_{x\to\infty} \mathcal{C}(x, T) = 0 = \lim_{x\to-\infty} \mathcal{P}(x, T)$. Therefore, the following function is introduced

$$\mathcal{O}(x) := \begin{cases} S_0^{-1}\mathcal{C}(x, T), & x \geq 0 \\ S_0^{-1}\mathcal{P}(x, T), & x < 0. \end{cases}$$

In Belomestny and Reiß (2005) and Carr and Madan (1999), the following properties of $\mathcal{O}$ and φ_T, defined in (2.1), are proven

Proposition 1. *The following properties hold.*

- *For all* $x \in \mathbb{R}$, $\mathcal{O}(x) = S_0^{-1}\mathcal{C}_j(x, T) - (1 - e^x)^+$.
- *For all* $x \in \mathbb{R}$, $\mathcal{O}(x) \in [0, 1 \wedge e^x]$.
- *If* $C_\alpha := \mathbb{E}\left\{e^{\alpha X_T}\right\}$ *is finite for some* $\alpha \geq 1$, *then for all* $x \geq 0$, $\mathcal{O}(x) \leq C_\alpha e^{(1-\alpha)x}$.
- *At any* $x \in \mathbb{R}\backslash\{0\}$, *the function* $\mathcal{O}$ *is twice differentiable with* $\|\mathcal{O}''\|_{L^1(\mathbb{R})} \leq 3$ *and the first derivative* $\mathcal{O}'$ *has a jump of height* -1 *at* 0.

- *The Fourier transform of $\mathcal{O}$ satisfies for all $v \in \mathbb{C}$ with* $\operatorname{Im}(v) \in [0,1]$,

$$\mathcal{F}\mathcal{O}(v) := \int_{-\infty}^{\infty} \mathcal{O}(x) e^{ivx} \, \mathrm{d}x = \frac{1 - \varphi_T(v-i)}{v(v-i)}.$$

The last relation, which is the most important, was first stated by Carr and Madan (1999) and this also explains why it is chosen to use call and put options in calibrating Lévy processes. For theoretical purposes and easier notations, define the following function

$$\psi(v) := T^{-1} \log\left(1 + iv(1+iv)\mathcal{F}\mathcal{O}(v)\right) = T^{-1} \log\left(\varphi_T(v-i)\right).$$

Under the assumptions of the model and (2.1) ψ simplifies to

$$\begin{aligned}
\psi(v) = \frac{\log\left(\varphi_T(v-i)\right)}{T} &= -\frac{\sigma^2 (v-i)^2}{2} + i\gamma(v-i) + \int_{-\infty}^{\infty} \left(\mathrm{e}^{i(v-i)x} - 1\right) \nu(x) \, \mathrm{d}x \\
&= -\frac{\sigma^2 v^2}{2} + i(\sigma^2 + \gamma)v + (\sigma^2/2 + \gamma - \lambda) + \int_{-\infty}^{\infty} \mathrm{e}^{ivx} \cdot (\mathrm{e}^x \nu(x)) \, \mathrm{d}x \\
&= -\frac{\sigma^2 v^2}{2} + i(\sigma^2 + \gamma)v + (\sigma^2/2 + \gamma - \lambda) + \mathcal{F}\mu(v).
\end{aligned}$$

Concluding, ψ is a quadratic polynomial plus the Fourier transform of μ. So, given an exact formula for the option function, it is not too hard to find the exact Lévy triplet. However, the option function is not given in practical situations and only option data is available. This data is likely to be corrupted by some noise and it is impossible to guarantee that always enough data would be available such that the ψ_j function can be estimated properly since only a fixed amount of data is available.

3.1.1 The observations and the estimators

In practice, the true prices of options are not observed due to imperfections of the market and the bid-ask spread. In theory, it is therefore assumed that the observed prices come from the following model

$$Y_m = C(K_m, T) + \varsigma_m \varepsilon_m,$$

where ε_m is assumed to follow a sub-Gaussian distribution, i.e., the tail of the distribution ε_m is dominated by a Gaussian tail. Define $\delta_m := S_0^{-1} \varsigma_m$, then

$$O_m = S_0^{-1} C(K_m, T) + \delta_m \varepsilon_m = \mathcal{O}(x_m) + \delta_m \varepsilon_m.$$

The first theoretical assumption that is made, is that $\delta_m = \delta(x_m)$ for a certain L^η function δ with $\eta > 2$. The assumption will be made intuitive in Chapter 4.

Another assumption that is made, is that the grid on logarithmic scale, i.e., $\{x_m : \ m = 1, \dots N\}$, is an equidistant grid. This assumption makes the calculations a lot easier without seriously affecting the model. An estimator for $\mathcal{O}$ is now given by

$$\tilde{\mathcal{O}}(x) := \beta_0(x) + \sum_{m=1}^{N} O_m b_m(x)$$

with $b_m(x) := \Lambda((x - x_m)/\Delta)$ with $\Delta := |x_{m+1} - x_m| = |x_m - x_{m-1}|$ and $\Lambda(x)$ the triangular function which is 0 at -1 and 1 and 1 at 0. It should be noted that for all limiting results, $\Delta \to 0$. Moreover, $A := \min(x_N, -x_0) \to \infty$.

Whilst estimating ψ, it could be that one has to evaluate the logarithm at an argument close to

0. A small error might then induce large statistical errors. Therefore in the estimation procedure, the logarithm will be trimmed around 0. To be precise, define

$$\kappa(v) := \frac{1}{2}\min\left\{\exp\left(-\frac{T\sigma_{\max}^2 v^2}{2} - 2RT\right), 1\right\}. \tag{3.2}$$

Define now the estimator for $\psi(v)$ by

$$\tilde{\psi}(v) := T^{-1}\log_{\geq\kappa(v)}\left(\tilde{\varphi}_T(v-i)\right). \tag{3.3}$$

The trimmed logarithm around $\kappa > 0$ is defined as

$$\log_{\geq\kappa}(z) := \begin{cases} \log(z), & \text{if } |z| \geq \kappa \\ \log(z/|z|), & \text{if } |z| \leq \kappa, \end{cases}$$

where the logarithm is taken such that $\tilde{\psi}$ is continuous with $\log(1) = 0$. This estimator is asymptotically well-defined and a proof can be found in section 10.1.

Now, define $\mu(x) := \mathrm{e}^x \nu(x)$ for notational purposes and note that μ is positive. Moreover, recall the martingale condition (3.1). Given this information, it is not difficult to see that the trimmed value does not affect the bias of the estimator for ψ. Indeed,

$$\begin{aligned} |\varphi_T(v-i)| &= |\exp(T\psi(v))| \\ &= \left|\exp\left(T\left(-\frac{\sigma^2 v^2}{2} + i(\sigma^2+\gamma)v + \left(\frac{\sigma^2}{2} + \gamma - \lambda\right) + \mathcal{F}\mu(v)\right)\right)\right| \\ &= \exp\left(-\frac{T\sigma^2 v^2}{2} - T\mathcal{F}\mu(0) + T\,\mathrm{Re}(\mathcal{F}\mu(v))\right) \\ &= \exp\left(-\frac{T\sigma^2 v^2}{2} - T\|\mu\|_{L^1} + T\,\mathrm{Re}(\mathcal{F}\mu(v))\right) \\ &\geq \exp\left(-\frac{T\sigma^2 v^2}{2} - 2T\|\mu\|_{L^1}\right) \geq \exp\left(-\frac{T\sigma^2 v^2}{2} - 2TR\right) \\ &\geq \exp\left(-\frac{T\sigma_{\max}^2 v^2}{2} - 2TR\right) \geq 2\kappa(v) \end{aligned} \tag{3.4}$$

is at least twice as large as the trimmed value.

Based on the theory above, the method given in Belomestny and Reiß (2006a) gives us estimates of $(\sigma^2, \gamma, \lambda)$ by

$$\begin{aligned} \hat{\sigma}^2 &:= \int_{-U}^{U} \mathrm{Re}\left(\tilde{\psi}(u)\right) w_\sigma^U(u)\,\mathrm{d}u, \\ \hat{\gamma} &:= -\hat{\sigma}^2 + \int_{-U}^{U} \mathrm{Im}\left(\tilde{\psi}(u)\right) w_\gamma^U(u)\,\mathrm{d}u, \\ \hat{\lambda} &:= \frac{\hat{\sigma}^2}{2} + \hat{\gamma} - \int_{-U}^{U} \mathrm{Re}\left(\tilde{\psi}(u)\right) w_\lambda^U(u)\,\mathrm{d}u, \end{aligned}$$

with the weight functions $w_\sigma^U(u)$, w_γ^U and w_λ^U such that certain terms cancel in integrating ψ from $-U$ to U. To be precise, the conditions

$$\int_{-1}^{1} u^2 w_\sigma^1(u)\,\mathrm{d}u = -2, \qquad \int_{-1}^{1} w_\sigma^1(u)\,\mathrm{d}u = 0,$$

$$\int_{-1}^{1} u w_\gamma^1(u)\,\mathrm{d}u = 1,$$

$$\int_{-1}^{1} u^2 w_\lambda^1(u)\, \mathrm{d}u = 0, \qquad \int_{-1}^{1} w_\lambda^1(u)\, \mathrm{d}u = 1,$$

are imposed, where w_ξ^1 will be defined as an even function for $\xi \in \{\sigma, \lambda\}$ and as an odd function for $\xi = \gamma$. Moreover, it is necessary to define

$$w_\sigma^U(u) = U^{-3} w_\sigma^1(u/U), \quad w_\gamma^U(u) = U^{-2} w_\gamma^1(u/U), \quad w_\lambda^U(u) = U^{-1} w_\lambda^1(u/U).$$

Note that these conditions on the weight functions are necessary to make. For example, the weight function for σ_j needs to be multiplied by u^2 and the integral of this product over an interval of length $2U$ should be a constant different from 0. Since $\int_{-U}^{U} u^2\, \mathrm{d}u$ is of the order U^3, the weight function should cancel this which thus implies it should be of the order U^{-3}.

Furthermore, it is assumed that

$$\mathcal{F}\left(w_\sigma^1(u)/u^s\right),\ \mathcal{F}\left(w_\gamma^1(u)/u^s\right),\ \mathcal{F}\left(w_\lambda^1(u)/u^s\right) \in L^1(\mathbb{R}).$$

which implies that the weight function divided by u^s are continuous and bounded. This can be translated back to introduce the inequalities

$$\left|w_\sigma^U(u)\right| \lesssim U^{-(s+3)}|u|^s, \quad \left|w_\gamma^U(u)\right| \lesssim U^{-(s+2)}|u|^s, \quad \left|w_\lambda^U(u)\right| \lesssim U^{-(s+1)}|u|^s,$$

where $a(u) \lesssim b(u)$ means there exists some constant $C > 0$ such that $a(u) \le Cb(u)$ for all u. Next to the estimators for the one dimensional parameters, an estimator for μ is implied by the definition of ψ.

$$\hat{\mu}(x) := \mathcal{F}^{-1}\left[\left(\tilde{\psi}(\bullet) + \frac{\hat{\sigma}^2}{2}(\bullet - i)^2 - i\hat{\gamma}(\bullet - i) + \hat{\lambda}\right) w_\mu^U(\bullet)\right](x)$$

and an estimator for $\nu(x)$ will be defined by

$$\hat{\nu}(x) := \mathcal{F}^{-1}\left[\left(\tilde{\psi}(\bullet + i) + \frac{\hat{\sigma}^2}{2}(\bullet)^2 - i\hat{\gamma}(\bullet) + \hat{\lambda}\right) w_\nu^U(\bullet)\right](x).$$

This estimator for the Lévy density at x is preferred over $\hat{\mu}(x) \cdot \exp(-x)$ since it provides a direct estimate for $\nu(x)$ which leads to more stable results.

3.2 The inhomogeneous model

This section describes the model on which the estimation procedure of the Lévy triplet $(\sigma_j^2, \gamma_j, \nu_j)$ between two consecutive time points T_j and T_{j+1} is based. To that end, suppose that option data with n maturities $T_1, \ldots, T_n$ is available and let $T_0 = 0$ and assume that the set of Lévy triplets that is associated with the process X_t belongs to the class $\mathcal{G}_s^n(R, \sigma_{\max}, T_n)$, which is defined below, for a known R and $\sigma_{\max}$.

Definition 4. *For $s \in \mathbb{N}$ and $R, \sigma_{\max} > 0$ let $\mathcal{G}_s^n(R, \sigma_{\max}, T_n)$ denote the set of all sets of size n containing all Lévy triplets $\mathcal{T} = (\sigma_i^2, \gamma_i, \nu_i)_{i=1}^n$ satisfying*

$$\forall t \ge 0:\ \mathbb{E}\left\{\exp(X_t)\right\} = 1 \quad \Leftrightarrow \quad \frac{\sigma_i^2}{2} + \gamma_i + \int_{-\infty}^{\infty} (\exp(x) - 1)\, \nu_i(x)\, \mathrm{d}x = 0 \tag{3.5}$$

and

$$C_i := \mathbb{E}\left\{\exp(2X_{T_i})\right\} < \infty.$$

Moreover, $C_i \leq R$, μ_i is s-times (weakly) differentiable where $s \in \mathbb{Z}_{\geq 2}$, $T_i \leq T_n$ and

$$\sigma_i \in (0, \sigma_{\max}), \quad |\gamma_i|, \lambda_i \in [0, R], \quad \max_{0 \leq k \leq s} \left\| \mu_i^{(k)} \right\|_{L^2} \leq R, \quad \left\| \mu_i^{(s)} \right\|_{L^\infty} \leq R.$$

This definition invokes making a lot of different assumptions. Some may seem logical while that may not be the case for others. Therefore, the intuition and reasoning behind making these assumptions will be given before the remaining part of the model is explained.

The condition from equation (3.5), that is imposed on the parameters, is also called the martingale condition. This condition ensures that the exponential Lévy process induced by the Lévy triplet is a martingale. The second assumption, i.e., $C_i < \infty$, is a technical assumption which guarantees that the variance (or second moment) of the underlying stock price is finite. The R and $\sigma_{\max}$ are also introduced for theoretical purposes since it may be the case that either one of $C_i, \sigma_i, |\gamma_i|, \lambda_i, \mu_i^{(k)}$ becomes 'too large'. If one or more of these parameters become too big, it is impossible to construct a statistically stable estimation method. In that case, there would be no guarantee that the argument we evaluate the logarithm at is bounded from below by a positive constant. This causes either a statistically unstable method or it introduces an extra bias term. Both are undesirable.

Also, note that it is possible to assume that μ_i is s_i times (weakly) differentiable. This, however, does not impact the model much in the sense that it only affects the convergence rate of the bias. Moreover, it does not matter for the convergence rate how smooth the previous μ_j for $j = 1, \ldots, i-1$ are. However, it should be noted that s plays a big role in the performance of the estimation method. It will later be shown that s is the parameter that determines the optimal choice of the cutoff value U and thus the optimal rate of the convergence in the estimation procedure. For practical purposes, if no further information about the smoothness of μ_i is given, one can choose $s = 2$.

Let S_t denote the underlying asset price. It will be assumed that S_t comes from an exponential inhomogeneous Lévy process, i.e.,

$$S_t = S_0 \exp(\overline{r}t + X_t),$$

with X_t an inhomogeneous Lévy process and $\overline{r}$ the risk-free interest rate. Note that it is thus assumed that the discounted asset price is a martingale under the risk-neutral measure. Define $K_{j,m}$ as the strike price of the mth option with maturity T_j. Section 3.1 gives an estimator for the triplet $(\sigma_1^2, \gamma_1, \nu_1)$ using only option data with maturity T_1. Belomestny and Reiß (2006a) mention that whenever more data on different maturities is available, it is straightforward to extend the model. It is indeed straightforward to modify the model to get better estimates for the parameters when different maturities are used. However, they are not extending it to inhomogeneous processes but they suggest to use a weighted average of the multiple estimates. Therefore, it remains to estimate the next $(n-1)$ triplets by modifying the framework introduced in section 3.1.

The risk-neutral option price for a European call option is given by (Shreve, 2004, p. 218)

$$C(K_{j,m}, T_j) = \mathrm{e}^{-\overline{r}T_j} \mathbb{E}\left\{(S_{T_j} - K_{j,m})^+\right\}$$

with $z^+ := \max(0, z)$. Notice that the expectation is taken under the risk-neutral measure. Chapter A. is devoted to explaining briefly what the risk-neutral measure is and why the risk-neutral measure is used. For more about the risk-neutral measure, one can read Shreve (2012) and Shreve (2004). In the remaining part of this section and the upcoming chapters, the risk-neutral measure will always be used if not stated otherwise. It is common to define the log-forward moneyness $x_{j,m}$ to replace the strike price $K_{j,m}$

$$x_{j,m} := \log(K_{j,m}/S_0) - \overline{r}T_j.$$

We reparametrize the function C to

$$C(K_{j,m}, T_j) = S_0\mathbb{E}\left\{\left(\mathrm{e}^{X_{T_j}} - \mathrm{e}^{x_{j,m}}\right)^+\right\} =: \mathcal{C}(x_{j,m}, T_j)$$

For put options we have

$$\mathcal{P}(x_{j,m}, T_j) = S_0\mathbb{E}\left\{\left(\mathrm{e}^{x_{j,m}} - \mathrm{e}^{X_{T_j}}\right)^+\right\}$$

(Shreve, 2004, p. 163). Note that $\mathcal{C}(x, T_j)$, $\mathcal{P}(x, T_j)$ do not converge to 0 when x tends to $-\infty$ and ∞, respectively. Hence, it makes no sense to take the Fourier transform of $\mathcal{C}$ or $\mathcal{P}$. However, $\lim_{x\to\infty} \mathcal{C}(x, T_j) = 0 = \lim_{x\to-\infty} \mathcal{P}(x, T_j)$. Therefore, the following function for $j = 1, 2, \ldots, n$ is introduced

$$\mathcal{O}_j(x) := \begin{cases} S_0^{-1}\mathcal{C}(x, T_j), & x \geq 0 \\ S_0^{-1}\mathcal{P}(x, T_j), & x < 0. \end{cases}$$

In Belomestny and Reiß (2005) and Carr and Madan (1999), the following properties of $\mathcal{O}_j$ and φ_{T_j}, defined in (2.1), are proven

Proposition 2. *The following properties hold.*

- *For all* $x \in \mathbb{R}$, $\mathcal{O}_j(x) = S_0^{-1}\mathcal{C}_j(x, T_j) - (1 - e^x)^+$.
- *For all* $x \in \mathbb{R}$, $\mathcal{O}_j(x) \in [0, 1 \wedge e^x]$.
- *If* $C_\alpha := \mathbb{E}\left\{e^{\alpha X_{T_j}}\right\}$ *is finite for some* $\alpha \geq 1$, *then for all* $x \geq 0$, $\mathcal{O}_j(x) \leq C_\alpha e^{(1-\alpha)x}$.
- *At any* $x \in \mathbb{R}\backslash\{0\}$, *the function* $\mathcal{O}_j$ *is twice differentiable with* $\left\|\mathcal{O}_j''\right\|_{L^1(\mathbb{R})} \leq 3$ *and the first derivative* $\mathcal{O}_j'$ *has a jump of height* -1 *at* 0.
- *The Fourier transform of* $\mathcal{O}_j$ *satisfies for all* $v \in \mathbb{C}$ *with* $\mathrm{Im}(v) \in [0, 1]$,

$$\mathcal{F}\mathcal{O}_j(v) := \int_{-\infty}^{\infty} \mathcal{O}_j(x)e^{ivx}\,\mathrm{d}x = \frac{1 - \varphi_{T_j}(v - i)}{v(v - i)}.$$

The last relation, which is the most important, was first stated by Carr and Madan (1999) and this also explains why it is chosen to use option prices in calibrating Lévy processes. For theoretical purposes and easier notations, define the following functions for $j = 1, 2, \ldots, n$ and $k = 0, 1$

$$\begin{aligned}\psi_j^k(v) &:= (T_j - T_{j-1})^{-1} \log\left(1 + iv(1 + iv)\mathcal{F}\mathcal{O}_{j-k}(v)\right) \\ &= (T_j - T_{j-1})^{-1} \log\left(\varphi_{T_{j-k}}(v - i)\right)\end{aligned}$$

and

$$\psi_j(v) := \psi_j^0(v) - \psi_j^1(v).$$

Under the assumptions of the model and (2.2) ψ_j simplifies to

$$\begin{aligned}\psi_j(v) = \frac{1}{T_j - T_{j-1}} \log\left(\frac{\varphi_{T_j}(v - i)}{\varphi_{T_{j-1}}(v - i)}\right) &= -\frac{\sigma_j^2(v - i)^2}{2} + i\gamma_j(v - i) + \int_{-\infty}^{\infty} \left(\mathrm{e}^{i(v-i)x} - 1\right)\nu_j(x)\,\mathrm{d}x \\ &= -\frac{\sigma_j^2 v^2}{2} + i(\sigma_j^2 + \gamma_j)v + (\sigma_j^2/2 + \gamma_j - \lambda_j) + \int_{-\infty}^{\infty} \mathrm{e}^{ivx} \cdot (\mathrm{e}^x \nu_j(x))\,\mathrm{d}x \\ &= -\frac{\sigma_j^2 v^2}{2} + i(\sigma_j^2 + \gamma_j)v + (\sigma_j^2/2 + \gamma_j - \lambda_j) + \mathcal{F}\mu_j(v).\end{aligned}$$

Concluding, ψ_j is a quadratic polynomial plus the Fourier transform of μ_j. So, given an exact formula for the option function, it is not too hard to find the exact Lévy triplet. However, the option function is not given in practical situations and only option data is available. This data is likely to be corrupted by some noise and it is impossible to guarantee that always enough data would be available such that the ψ_j function can be estimated properly since only a fixed amount of data is available.

3.2.1 The observations and the estimators

In practice, the true prices of options are not observed due to imperfections of the market and the bid-ask spread. In theory, it is therefore assumed that the observed prices come from the following model

$$Y_{j,m} = C(K_{j,m}, T_j) + \varsigma_{j,m}\varepsilon_{j,m},$$

where $\varepsilon_{j,m}$ is assumed to follow a sub-Gaussian distribution, i.e., the tail of the distribution $\varepsilon_{j,m}$ is dominated by a Gaussian tail. Define $\delta_{j,m} := S_0^{-1}\varsigma_{j,m}$, then

$$O_{j,m} = S_0^{-1}C(K_{j,m}, T_j) + \delta_{j,m}\varepsilon_{j,m} = \mathcal{O}_j(x_{j,m}) + \delta_{j,m}\varepsilon_{j,m}.$$

The first theoretical assumption that is made, is that $\delta_{j,m} = \delta_j(x_{j,m})$ for a certain L^η function δ_j with $\eta > 2$. The assumption will be made intuitive in Chapter 4.

Another assumption that is made, is that the grid on logarithmic scale, i.e., $\{x_{j,m} : m = 1, \dots N\}$, is an equidistant grid. This assumption makes the calculations a lot easier without seriously affecting the model. An estimator for $\mathcal{O}_j$ is now given by

$$\tilde{\mathcal{O}}_j(x) := \beta_{0,j}(x) + \sum_{m=1}^{N} O_{j,m} b_{j,m}(x)$$

with $b_{j,m}(x) := \Lambda((x - x_{j,m})/\Delta_j)$ with $\Delta_j := |x_{j,m+1} - x_{j,m}| = |x_{j,m} - x_{j,m-1}|$ and $\Lambda(x)$ the triangular function which is 0 at -1 and 1 and 1 at 0. It should be noted that for all limiting results, $\Delta_j \to 0$. Moreover, $A_j := \min(x_{j,N}, -x_{j,0}) \to \infty$.

Whilst estimating ψ_j, it could be that one has to evaluate the logarithm at an argument close to 0. A small error might then induce large statistical errors. Therefore in the estimation procedure, the logarithm will be trimmed around 0. To be precise, define

$$K(T, \sigma, R, v) := \frac{1}{2}\min\left\{\exp\left(-\frac{T\sigma^2 v^2}{2} - 2RT\right), 1\right\} \quad \text{and} \quad \kappa(v, T) := K(T, \sigma_{\max}, R, v). \tag{3.6}$$

Note that κ is decreasing in its first argument and that

$$\begin{aligned}
\prod_{r=1}^{j} 2K(T_r - T_{r-1}, \sigma_r, R, v) &\geq \prod_{r=1}^{j} \exp\left(-\frac{(T_r - T_{r-1})\sigma_r^2 v^2}{2} - 2R(T_r - T_{r-1})\right) \\
&= \exp\left(-\frac{v^2}{2}\sum_{r=1}^{j}(T_r - T_{r-1})\sigma_r^2 - 2R\sum_{r=1}^{j}(T_r - T_{r-1})\right) \\
&\geq \exp\left(-\frac{T_j\sigma_{\max}^2 v^2}{2} - 2RT_j\right) = 2\kappa(v, T_j)
\end{aligned}$$

holds for all $j = 1, 2, \dots, n$. Define now the estimator for $\psi_j(v)$ by

$$\tilde{\psi}_j(v) := (T_j - T_{j-1})^{-1}\left[\log_{\geq\kappa(v,T_j)}\left(\tilde{\varphi}_{T_j}(v - i)\right) - \log_{\geq\kappa(v,T_{j-1})}\left(\tilde{\varphi}_{T_{j-1}}(v - i)\right)\right]. \tag{3.7}$$

The trimmed logarithm around $\kappa > 0$ is defined as

$$\log_{\geq\kappa}(z) := \begin{cases} \log(z), & \text{if } |z| \geq \kappa \\ \log(z/|z|), & \text{if } |z| \leq \kappa, \end{cases}$$

where the logarithm is taken such that $\tilde{\psi}_j$ is continuous with $\log(1) = 0$. This estimator is asymptotically well-defined and a proof can be found in section 10.1.

Now, define $\mu_k(x) := \mathrm{e}^x \nu_k(x)$ for notational purposes and note that μ_k is positive. Moreover, recall the martingale condition (3.5). Given this information, it is not difficult to see that the trimmed value does not affect the bias of the estimator for ψ_k. Indeed,

$$\begin{aligned} \left|\frac{\varphi_{T_k}(v-i)}{\varphi_{T_{k-1}}(v-i)}\right| &= |\exp((T_k - T_{k-1})\psi_k(v))| \\ &= \left|\exp\left((T_k - T_{k-1})\left(-\frac{\sigma_k^2 v^2}{2} + i(\sigma_k^2 + \gamma_k)v + \left(\frac{\sigma_k^2}{2} + \gamma_k - \lambda_k\right) + \mathcal{F}\mu_k(v)\right)\right)\right| \\ &= \exp\left(-\frac{(T_k - T_{k-1})\sigma_k^2 v^2}{2} - (T_k - T_{k-1})\mathcal{F}\mu_k(0) + (T_k - T_{k-1})\operatorname{Re}(\mathcal{F}\mu_k(v))\right) \\ &= \exp\left(-\frac{(T_k - T_{k-1})\sigma_k^2 v^2}{2} - (T_k - T_{k-1})\|\mu_k\|_{L^1} + (T_k - T_{k-1})\operatorname{Re}(\mathcal{F}\mu_k(v))\right) \\ &\geq \exp\left(-\frac{(T_k - T_{k-1})\sigma_k^2 v^2}{2} - 2(T_k - T_{k-1})\|\mu_k\|_{L^1}\right) \\ &\geq \exp\left(-\frac{(T_k - T_{k-1})\sigma_k^2 v^2}{2} - 2(T_k - T_{k-1})R\right) \geq 2K(T_k - T_{k-1}, \sigma_k, R, v). \end{aligned} \tag{3.8}$$

It follows that

$$|\varphi_{T_k}(v-i)| = \prod_{m=1}^{k}\left|\frac{\varphi_{T_m}(v-i)}{\varphi_{T_{m-1}}(v-i)}\right| \geq \prod_{m=1}^{k}(2K(T_m - T_{m-1}, \sigma_m, R, v) \geq 2\cdot\kappa(v, T_k). \tag{3.9}$$

is at least twice as large as the trimmed value.

Based on the theory above, we adapt the method by Belomestny and Reiß (2006a) to estimate $\left(\sigma_j^2, \gamma_j, \lambda_j\right)$ by

$$\begin{aligned} \hat{\sigma}_j^2 &:= \int_{-U_j}^{U_j} \operatorname{Re}\left(\tilde{\psi}_j(u)\right) w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u, \\ \hat{\gamma}_j &:= -\hat{\sigma}_j^2 + \int_{-U_j}^{U_j} \operatorname{Im}\left(\tilde{\psi}_j(u)\right) w_{\gamma_j}^{U_j}(u)\,\mathrm{d}u, \\ \hat{\lambda}_j &:= \frac{\hat{\sigma}_j^2}{2} + \hat{\gamma}_j - \int_{-U_j}^{U_j} \operatorname{Re}\left(\tilde{\psi}_j(u)\right) w_{\lambda_j}^{U_j}(u)\,\mathrm{d}u, \end{aligned}$$

with the weight functions $w_{\sigma_j}^{U_j}(u)$, $w_{\gamma_j}^{U_j}$ and $w_{\lambda_j}^{U_j}$ such that certain terms cancel in integrating ψ_j from $-U_j$ to U_j. To be precise, the conditions

$$\begin{aligned} &\int_{-1}^{1} u^2 w_{\sigma_j}^1(u)\,\mathrm{d}u = -2, && \int_{-1}^{1} w_{\sigma_j}^1(u)\,\mathrm{d}u = 0, \\ &\int_{-1}^{1} u w_{\gamma_j}^1(u)\,\mathrm{d}u = 1, && \\ &\int_{-1}^{1} u^2 w_{\lambda_j}^1(u)\,\mathrm{d}u = 0, && \int_{-1}^{1} w_{\lambda_j}^1(u)\,\mathrm{d}u = 1, \end{aligned} \tag{3.10}$$

are imposed, where w_ξ^1 will be defined as an even function for $\xi \in \{\sigma_j, \lambda_j\}$ and as an odd function for

$\xi = \gamma_j$. Moreover, it is necessary to define

$$w_{\sigma_j}^{U_j}(u) = U_j^{-3} w_{\sigma_j}^1(u/U_j), \quad w_{\gamma_j}^{U_j}(u) = U_j^{-2} w_{\gamma_j}^1(u/U_j), \quad w_{\lambda_j}^{U_j}(u) = U_j^{-1} w_{\lambda_j}^1(u/U_j).$$

For example, the weight function for σ_j needs to be multiplied by u^2 and the integral of this product over an interval of length $2U_j$ should be a constant different from 0. Since $\int_{-U_j}^{U_j} u^2 \, du$ is of the order U_j^3, the weight function should cancel this which thus implies it should be of the order U_j^{-3}.

Furthermore, it is assumed that

$$\mathcal{F}\left(w_{\sigma_j}^1(u)/u^s\right), \ \mathcal{F}\left(w_{\gamma_j}^1(u)/u^s\right), \ \mathcal{F}\left(w_{\lambda_j}^1(u)/u^s\right) \in L^1(\mathbb{R}).$$

which implies that the weight function divided by u^s are continuous and bounded. This can be translated back to introduce the inequalities

$$\left|w_{\sigma_j}^{U_j}(u)\right| \lesssim U_j^{-(s+3)}|u|^s, \quad \left|w_{\gamma_j}^{U_j}(u)\right| \lesssim U^{-(s+2)}|u|^s, \quad \left|w_{\lambda_j}^{U_j}(u)\right| \lesssim U^{-(s+1)}|u|^s,$$

where $a(u) \lesssim b(u)$ means there exists some constant $C > 0$ such that $a(u) \leq Cb(u)$ for all u.

Next to the estimators for the one dimensional parameters, an estimator for μ_j is implied by the definition of ψ_j. However, it is preferred to directly estimate $\nu_j(x)$ instead of using $\exp(-x)\hat{\mu}_j(x)$ as an estimator, because it leads to more stable results. Hence, by a simple manipulation of the above, the estimator for $\nu_j(x)$ will be defined by

$$\hat{\nu}_j(x) := \mathcal{F}^{-1}\left[\left(\tilde{\psi}_j(\bullet + i) + \frac{\hat{\sigma}_j^2}{2}(\bullet)^2 - i\hat{\gamma}_j(\bullet) + \hat{\lambda}_j\right) w_{\nu_j}^{U_j}(\bullet)\right](x),$$

where $w_{\nu_j}^{U_j}(u) = w_{\nu_j}^1(u/U_j)$. Moreover, $w_{\nu_j}^{U_j}(u) = w_{\nu_j}^{U_j}(-u)$ for all $u \in \mathbb{R}$ and $w_{\nu_j}^{U_j}$ has support $[-U_j, U_j]$.

In particular, in the estimation procedure it is chosen to use the following weight functions, similar to Söhl and Trabs (2014),

$$\begin{aligned} w_{\sigma_j}^1(u) := c_1 \cdot \big((2s+1)u^{2s} - (8s+12)u^{2s+2} + (12s+30)u^{2s+4} \\ - (8s+32)u^{2s+6} + (2s+9)u^{2s+8}\big) \cdot \mathbb{1}\{|u| \leq 1\}, \qquad (3.11) \end{aligned}$$

$$w_{\gamma_j}^1(u) := c_2 \cdot \big(u^{2s+1} - 3u^{2s+3} + 3u^{2s+5} - u^{2s+7}\big) \cdot \mathbb{1}\{|u| \leq 1\}, \qquad (3.12)$$

$$\begin{aligned} w_{\lambda_j}^1(u) := c_3 \cdot \big((2s+3)u^{2s} - (8s+20)u^{2s+2} + (12s+42)u^{2s+4} \\ - (8s+36)u^{2s+6} + (2s+11)u^{2s+8}\big) \cdot \mathbb{1}\{|u| \leq 1\}, \qquad (3.13) \end{aligned}$$

$$w_{\nu_j}^1(u) := \begin{cases} 1, & |u| \leq 0.05, \\ \exp\left(-\frac{\exp(-(|u|-0.05)^{-2})}{(|u|-1)^2}\right), & 0.05 < |u| < 1, \\ 0, & |u| \geq 1 \end{cases} \qquad (3.14)$$

where the constants c_1, c_2, c_3 are chosen such that the conditions on the weight functions given in (3.10) are satisfied.

Note that the exact choice of the weight functions does not have a big influence on the performance of the estimation method since this function has the same role as a kernel in kernel density estimation, where it is more important to pick the optimal bandwidth than choosing the best kernel.

Chapter 4

Asymptotic normality

This chapter is centered around the main theoretical result of this research. Namely, that the estimators for σ_j^2, γ_j, λ_j and $\nu_j(x)$ are asymptotically Gaussian. From these results, it is clear that this particular ordering of the parameters is also the ordering in terms of estimator performance which makes sense when one investigates the definition of the estimators. Moreover, the optimal convergence rates are given.

Theorem 1. *Let $\delta_j \in L^\eta(\mathbb{R}) \cap C^\infty(\mathbb{R})$ with $\eta > 2$. Moreover, let $\Delta_j \|\delta_j\|_{l^2}^2 \lesssim \|\delta_j\|_\infty^2$, $e^{-A_j} \lesssim \Delta_j$ and assume that the Lévy triplets of the process belong to $\mathcal{G}_s^n(R, \sigma_{\max}, T_n)$. Define*

$$d_{j,j-k} = \frac{2\|\delta_{j-k}\|_{L^2}^2}{(T_j - T_{j-1})^2} \left(\sum_{r=1}^{j-k} (T_r - T_{r-1})\sigma_r^2 \right)^{-2} \exp\left(-2\left(\sum_{r=1}^{j-k} (T_r - T_{r-1}) \left(\frac{\sigma_r^2}{2} + \gamma_r - \lambda_r \right) \right) \right)$$

with $d_{j,0} := 0$ and

$$C_j := \frac{\exp\left(-U_j^2 \cdot \sum_{i=1}^j (T_i - T_{i-1})\sigma_i^2/2\right)}{\sqrt{d_{j,j}\Delta_j + d_{j,j-1}\Delta_{j-1}\exp\left(-U_j^2 \cdot (T_j - T_{j-1})\sigma_j^2\right)}}.$$

If U_j is chosen such that for all $j = 1, 2, \ldots, n$

$$\Delta_j U_j^4 \log U_j \exp\left(U_j^2 \left(\sum_{i=1}^j (T_i - T_{i-1})\sigma_i^2 \right) \right) \to 0, \tag{4.1}$$

for all $j = 2, 3, \ldots, n$

$$\frac{\Delta_{j-1}^2}{\Delta_j} U_j^4 \exp\left(U_j^2 \left(\sum_{i=1}^{j-1} (T_i - T_{i-1})\sigma_i^2 - (T_j - T_{j-1})\sigma_j^2 \right) \right) \to 0 \tag{4.2}$$

and for $s > 2$

$$U_j^{2(s+1)} \cdot \left(\Delta_j \exp\left(U_j^2 \left(\sum_{i=1}^j (T_i - T_{i-1})\sigma_i^2 \right) \right) + \Delta_{j-1} \exp\left(U_j^2 \left(\sum_{i=1}^{j-1} (T_i - T_{i-1})\sigma_i^2 \right) \right) \right) \to \infty, \tag{4.3}$$

then

$$\begin{aligned}
\Sigma &:= U_j^2 \cdot C_j \cdot \left(\hat{\sigma}_j^2 - \sigma_j^2\right) && \xrightarrow{\mathcal{D}} |w^1_{\sigma_j}(1)| \cdot Z_1 \\
\Gamma &:= U_j \cdot C_j \cdot (\hat{\gamma}_j - \gamma_j) && \xrightarrow{\mathcal{D}} |w^1_{\gamma_j}(1)| \cdot Z_2 \\
\Lambda &:= C_j \cdot \left(\hat{\lambda}_j - \lambda_j\right) && \xrightarrow{\mathcal{D}} |w^1_{\lambda_j}(1)| \cdot Z_3 \\
N(x) &:= U_j^{-1} \cdot C_j \cdot (\hat{\nu}_j(x) - \nu_j(x)) && \xrightarrow{\mathcal{D}} (2\pi)^{-1}|w^1_{\nu_j}(1)| \cdot Z_4
\end{aligned}$$

for all $j = 1, 2, \dots, n$ *where* $Z_i \sim \mathcal{N}(0, 1)$ *for* $i = 1, 2, 3, 4$.

First of all, it should be noted that the weight functions should not equal 0 at the cutoff value 1 for the asymptotic normality result to make sense. Thus, other weight functions than (3.11)-(3.14) should be used and one can use for example the proposed weight functions in Belomestny and Reiß (2006b). However, this result tells us that if it is chosen to use a weight function which equals 0 at the cutoff value 1, then the convergence rate of the estimators would be much faster and that is why the functions as in (3.11)-(3.14) are used in estimating the parameters.

This theorem invokes taking a lot of assumptions. First, it is assumed that $\delta \in L^\eta$ for $\eta > 2$. $\eta \geq 2$ is necessary for the L^2 norm of δ to exist, which is necessary for the asymptotic variance to exist. The assumption that η needs to be strictly bigger than 2 is necessary in showing asymptotic normality. The assumption stated in (4.1) is made to ensure that the estimator for ψ_j is asymptotically well-defined. Moreover, a slightly weaker version, without $\log U_j$, of equation (4.1) and (4.2) guarantee that the remainder term in a linearization of the stochastic error converges to 0 in probability. Moreover, (4.3) makes sure that the bias terms of the estimators converge to 0.

Since the full proof of this theorem is elaborate, it will be given in detail for σ_j, whereas the differences in the proofs with γ_j and λ_j will be noticed at the relevant points. For $\nu_j(x)$, the most interesting part, the computation of the variance, can be found in the Appendix. In this chapter, only a brief outline of the proof will be given and for the exact details, the reader is referred to Chapter 10.

First, Σ is split up into the bias and the stochastic error. The bias term is estimated as tight as possible and equation (4.3) makes sure that the bias tends to 0 as U_j tends to infinity. The stochastic part of Σ will then be split up into a linearized error term and a remainder. It is proven that under the conditions of the theorem, the remainder converges to 0 in probability, which essentially implies that it does not play a role in the limit distribution of Σ. Finally, it is shown that the linear error term converges to a normal distribution. To that end, the asymptotic variance is calculated and the Lyapunov condition of the Lindeberg-Feller central limit theorem is verified.

From the above, one could construct $(100 - \alpha)\%$ asymptotic confidence intervals for the parameters using Slutsky's theorem. However, it was found that using the asymptotic confidence intervals does not lead to satisfying coverage probabilities. This is due to the estimation of the finite sample by its limit. That is also the reason why this is not included in the report. A different approach in constructing $(100 - \alpha)\%$ confidence intervals is based on the finite sample variance and it turns out that this method works particularly well in terms of coverage probabilities. This will be explained in the next chapter.

4.1 The optimal convergence rates

Since it is impossible to ensure that condition (4.3) holds for all triplets in $\mathcal{G}_s(R, \sigma_{\max})$, the bias will dominate the stochastic error in the estimation process. Therefore the convergence rates are determined by the convergence rates of the bias. Proposition 5 states thus the rate of convergence for σ_j if the cutoff value U_j is replaced by the optimal cutoff value for all triplets in $\mathcal{G}_s(R, \sigma_{\max})$. It turns out that the following theorem holds

Theorem 2. *Let Δ_{j-1}/Δ_j be bounded as the cutoff tends to infinity. Define*

$$U_{\sigma_{\max}} := \sigma_{\max}^{-1} \sqrt{2 \log \Delta_j^{-1/2} / T_j}, \tag{4.4}$$

then $U_{\sigma_{\max}}$ is the optimal choice of the cutoff value in terms of the convergence rates of the estimators for all Lévy triplets $(\sigma_j^2, \gamma_j, \nu_j)$ in $\mathcal{G}_s(R, \sigma_{\max})$. Moreover, the convergence rates are determined by the bias and are given by

$$\begin{aligned}
\mathbb{E}\left|\hat{\sigma}_j^2 - \sigma_j^2\right| &\lesssim |\log \Delta_j|^{-(s+3)/2}, \\
\mathbb{E}\left|\hat{\gamma}_j - \gamma_j\right| &\lesssim |\log \Delta_j|^{-(s+2)/2}, \\
\mathbb{E}\left|\hat{\lambda}_j - \lambda_j\right| &\lesssim |\log \Delta_j|^{-(s+1)/2}, \\
\mathbb{E}\left|\hat{\nu}_j(x_0) - \nu_j(x_0)\right| &\lesssim |\log \Delta_j|^{-s/2}.
\end{aligned}$$

The proof of this theorem is relatively easy given the propositions from Chapter 10. and that is why it is chosen to give the proof here.

Proof. (4.4) guarantees that (4.1) holds. Indeed,

$$\Delta_j \exp\left(U_{\sigma_{\max}}^2 \left(\sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2\right)\right) = \Delta_j^{1 - \frac{\sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2}{T_j \sigma_{\max}^2}} = \Delta_j^{\frac{\sum_{i=1}^{j} (T_i - T_{i-1}) \left(\sigma_{\max}^2 - \sigma_i^2\right)}{T_j \sigma_{\max}^2}}.$$

Since, $\sigma_{\max} > \sigma_i$ for all i, this term converges to 0 with a polynomial rate in Δ_j. Thus, whenever this is multiplied by $U_{\sigma_{\max}}^4 \log U_{\sigma_{\max}}$, this still converges to 0 since this particular term converges to infinity only at a logarithmic rate in Δ_j. Moreover, for (4.2) to hold, it is sufficient if

$$\frac{\Delta_{j-1}^2}{\Delta_j^2} \cdot \Delta_j^{1 - \frac{\sum_{i=1}^{j-1} (T_i - T_{i-1}) \sigma_i^2 - (T_j - T_{j-1}) \sigma_j^2}{T_j \sigma_{\max}^2}} = \frac{\Delta_{j-1}^2}{\Delta_j^2} \cdot \Delta_j^{\frac{\sum_{i=1}^{j-1} (T_i - T_{i-1}) \left(\sigma_{\max}^2 - \sigma_i^2\right) + (T_j - T_{j-1}) \left(\sigma_{\max}^2 + \sigma_j^2\right)}{T_j \sigma_{\max}^2}} \to 0.$$

Note that for this to be true no particular strong assumption need to be made. The assumption $\Delta_j / \Delta_{j-1} \leq C$ is sufficient for this limit relation to hold.

Note that it is now guaranteed that the scaled remainder term converges to 0 by Proposition 6. Moreover, the scaled bias term converges to ∞ since assumption (4.3) doesn't hold. That is the reason why the bias the stochastic error dominates.

(4.4), Proposition 5 and the conclusion that the bias the stochastic error dominates provide now the convergence rate of σ_j^2. Indeed,

$$\mathbb{E}\left|\hat{\sigma}_j^2 - \sigma_j^2\right| \lesssim U_{\sigma_{\max}}^{-(s+3)} = \left(\sigma_{\max}^{-1} \sqrt{2 \log \Delta_j^{-1/2} / T_j}\right)^{-(s+3)} \lesssim |\log \Delta_j|^{-(s+3)/2}.$$

Similar to the statement below Proposition 5, the argument for γ_j and λ_j are similar. The convergence rate of the bias of $\nu_j(x_0)$ is given in equation (C.1). Plugging in (4.4) yields the result for $\nu_j(x_0)$.

Finally, the choice $U_{\sigma_{\max}}$ in (4.4) is optimal since if it is chosen slightly bigger, it will not be guaranteed that (4.1) holds, and if it is chosen slightly smaller, the bound on the bias term is bigger which makes it less optimal. □

Chapter 5

The finite sample variance

In this chapter, the method of constructing confidence intervals via the finite sample variance method is used. In Söhl and Trabs (2014), this has been done for the continuous case. Although both derivations are similar, it is more straightforward in the discrete case.

The idea of the method is to estimate the variance of the stochastic error. This is done similarly to the asymptotic analysis before with respect to the fact that this method also depends on the variance of the linear stochastic error term. The bias and the remainder term are thus neglected in this method. In the next chapter, it is argued that if the Lévy density has a sharp peak, there is a significant negative bias around this peak for the estimator of ν_j. This method doesn't take that into account because it is very difficult to track the bias explicitly. This could, however, be an interesting further research topic.

So, because of the similarities with the asymptotic analysis earlier, the method starts in a similar fashion. However, the linear term will not be estimated in the limit but rather in the finite case. First, rewrite σ_j^2 and compare to (10.2)

$$\hat{\sigma}_j^2 - \sigma_j^2 \approx \int_{-U_j}^{U_j} \mathrm{Re}\left(\tilde{\psi}_j(u) - \psi_j(u)\right) w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u.$$

As before, the idea is to linearize the logarithm $\tilde{\psi}_j(u) - \psi_j(u)$ as

$$\tilde{\psi}_j(u) - \psi_j(u) \approx \mathcal{L}_j^0(u) - \mathcal{L}_j^1(u)$$

with

$$\mathcal{L}_j^k(u) = \frac{1}{T_j - T_{j-1}} \cdot \frac{\varphi_{T_{j-k}}(u-i) - \tilde{\varphi}_{T_{j-k}}(u-i)}{\varphi_{T_{j-k}}(u-i)}$$

Recall $\tilde{\mathcal{O}}_j(x) = \sum_{r=1}^N (\mathcal{O}_j(x_{j,r}) + \varepsilon_{j,r})\delta_{j,r} b_{j,r}(x)$ and thus

$$\mathcal{F}\tilde{\mathcal{O}}_j(x) - \mathcal{F}\mathcal{O}_j(x) \approx \sum_{r=1}^N \delta_{j,r}\mathcal{F}b_{j,r}(x)\varepsilon_{j,r}.$$

The linearization error of $\mathcal{O}_j(x)$ is again left out of the equation, because it was found to be negligible according to Proposition 4. Moreover, recall

$$\mathcal{F}b_{j,r}(u) = \int_{\mathbb{R}} \exp(iux) b_{j,r}(x)\,\mathrm{d}x \approx (x_{j,r+1} - x_{j,r-1})/2 \cdot \exp(iux_{j,r}) = \Delta_j \cdot \exp(iux_{j,r}).$$

Hence, we approximate

$$\tilde{\psi}_j^0(u) - \psi_j^0(u) \approx \frac{1}{T_j - T_{j-1}} \cdot \frac{iu(1+iu)}{\varphi_{T_j}(u-i)} \sum_{r=1}^{N} \delta_{j,r} \Delta_j \exp(iux_{j,r}) \varepsilon_{j,r},$$

$$\tilde{\psi}_j^1(u) - \psi_j^1(u) \approx \frac{1}{T_j - T_{j-1}} \cdot \frac{iu(1+iu)}{\varphi_{T_{j-1}}(u-i)} \sum_{r=1}^{N} \delta_{j-1,r} \Delta_{j-1} \exp(x_{j-1,r}) \varepsilon_{j-1,r}.$$

The following convenient function is introduced

$$f_{\sigma_j}^k(u) := w_{\sigma_j}^{U_j}(u) iu(1+iu)/((T_j - T_{j-1})\varphi_{T_{j-k}}(u-i)) \tag{5.1}$$

such that we obtain

$$\begin{aligned}\int_{-U_j}^{U_j} \operatorname{Re}\left(\mathcal{L}_j^0(u)\right) w_{\sigma_j}^{U_j}(u) \,\mathrm{d}u &\approx \int_{-U_j}^{U_j} \operatorname{Re}\left(\frac{1}{T_j - T_{j-1}} \cdot \frac{iu(1+iu)}{\varphi_{T_j}(u-i)} \sum_{r=1}^{N} \delta_{j,r} \Delta_j \exp(iux_{j,r}) \varepsilon_{j,r}\right) w_{\sigma_j}^{U_j}(u) \,\mathrm{d}u \\ &= \int_{-U_j}^{U_j} \operatorname{Re}\left(f_{\sigma_j}^0(u) \cdot \sum_{r=1}^{N} \delta_{j,r} \exp(iux_{j,r}) \Delta_j \varepsilon_{j,r}\right) \mathrm{d}u \\ &= 2\pi \sum_{r=1}^{N} \delta_{j,r} \varepsilon_{j,r} \operatorname{Re}\left(\mathcal{F}^{-1} f_{\sigma_j}^0(-x_{j,r})\right) \Delta_j.\end{aligned}$$

Similarly,

$$\int_{-U_j}^{U_j} \operatorname{Re}\left(\mathcal{L}_j^1(u)\right) w_{\sigma_j}^{U_j}(u) \,\mathrm{d}u \approx 2\pi \sum_{r=1}^{N} \delta_{j-1}(x_{j-1,r}) \varepsilon_{j-1,r} \operatorname{Re}\left(\mathcal{F}^{-1} f_{\sigma_j}^1(-x_{j-1,r})\right) \Delta_{j-1}.$$

Thus, an estimation for the variance of the estimator of the square of the volatility σ_j^2 is given by

$$s_{\sigma_j^2}^2 := 4\pi^2 \sum_{r=1}^{N} \left(\delta_j(x_{j,r}) \operatorname{Re}\left(\mathcal{F}^{-1} f_{\sigma_j}^0(-x_{j,r})\right) \Delta_j\right)^2 + \left(\delta_{j-1}(x_{j-1,r}) \operatorname{Re}\left(\mathcal{F}^{-1} f_{\sigma_j}^1(-x_{j-1,r})\right) \Delta_{j-1}\right)^2.$$

The $(100-\alpha)\%$ confidence intervals are now defined by $\left(\hat{\sigma}_j^2 + z_{\alpha/2} \cdot \hat{s}_{\sigma_j^2}, \hat{\sigma}_j^2 + z_{100-\alpha/2} \cdot \hat{s}_{\sigma_j^2}\right)$ where z_p is the pth quantile of a standard normal distribution and where $\tilde{\varphi}_{T_j}$ is used in $f_{\sigma_j}^k(u)$ instead of φ_{T_j}.

We define $f_{\gamma_j}^k$ and $f_{\lambda_j}^k$ by replacing the weight function $w_{\sigma_j}^{U_j}(u)$ in (5.1) by $w_{\gamma_j}^{U_j}(u)$ and $w_{\lambda_j}^{U_j}(u)$, repectively. This gives the following results for γ_j and λ_j where the derivation is similar to the one for σ_j and thus will be left to the reader.

$$\begin{aligned} s_{\gamma_j}^2 := &4\pi^2 \sum_{r=1}^{N} \left(-\operatorname{Re}\left(\mathcal{F}^{-1} f_{\sigma_j}^0(-x_{j,r})\right) + \operatorname{Im}\left(\mathcal{F}^{-1} f_{\gamma_j}^0(-x_{j,r})\right)\right)^2 \delta_{j,r}^2 \Delta_j^2 \\ &+ 4\pi^2 \sum_{r=1}^{N} \left(-\operatorname{Re}\left(\mathcal{F}^{-1} f_{\sigma_j}^1(-x_{j-1,r})\right) + \operatorname{Im}\left(\mathcal{F}^{-1} f_{\gamma_j}^0(-x_{j-1,r})\right)\right)^2 \delta_{j-1,r}^2 \Delta_{j-1}^2 \end{aligned}$$

and

$$\begin{aligned} s_{\lambda_j}^2 := &4\pi^2 \sum_{i=1}^{N} \left(-\operatorname{Re}\left(\frac{1}{2}\mathcal{F}^{-1} f_{\sigma_j}^0(-x_{j,r}) + \mathcal{F}^{-1} f_{\lambda_j}^0(-x_{j,r})\right) + \operatorname{Im}\left(\mathcal{F}^{-1} f_{\gamma_j}^0(-x_{j,r})\right)\right)^2 \delta_{j,r}^2 \Delta_j^2 \\ &+ 4\pi^2 \sum_{i=1}^{N} \left(-\operatorname{Re}\left(\frac{1}{2}\mathcal{F}^{-1} f_{\sigma_j}^1(-x_{j-1,r}) + \mathcal{F}^{-1} f_{\lambda_j}^1(-x_{j-1,r})\right) + \operatorname{Im}\left(\mathcal{F}^{-1} f_{\gamma_j}^1(-x_{j-1,r})\right)\right)^2 \delta_{j-1,r}^2 \Delta_{j-1}^2. \end{aligned}$$

The computation of the variance of the estimator for $\nu_j(x_0)$ works a little bit different and the expressions become longer. That is why $g_{U_j}^{(k)}$ is introduced for $k = 0, 1, 2$

$$g_{U_j}^{(k)}(x_0) := \mathcal{F}^{-1}\left[u^k \cdot w_{\nu_j}^{U_j}(u)\right](x_0).$$

This function will be approached numerically since it is impossible to find $g_{U_j}^{(k)}(x_0)$ in closed form if $w_{\nu_j}^{U_j}(u)$ is chosen as in (3.14). Now, define

$$f_{\nu_j}^k(u) := -\frac{w_{\nu_j}^{U_j}(u)u(u+i)}{(T_j - T_{j-1})\varphi_{T_{j-k}}(u)}$$

such that the expression for the variance of the estimator of ν_j at x_0 can be written as

$$\begin{aligned}
s_{\nu_j(x_0)}^2 := 4\pi^2 \sum_{r=1}^{N} \Bigg(& \frac{e^{-x_{j,r}}}{2\pi}\mathcal{F}^{-1} f_{\nu_j}^0(x_0 - x_{j,r}) \\
& + \operatorname{Re}\left(\mathcal{F}^{-1} f_{\sigma_j}^0(-x_{j,r})\right) \cdot \left(\frac{1}{2} g_{U_j}^{(2)}(x_0) + i g_{U_j}^{(1)}(x_0) - \frac{1}{2} g_U^{(0)}(x_0)\right) \\
& + \operatorname{Im}\left(\mathcal{F}^{-1} f_{\gamma_j}^0(-x_{j,r})\right)\left(-i g_{U_j}^{(1)}(x_0) + g_{U_j}^{(0)}(x_0)\right) \\
& - \operatorname{Re}\left(\mathcal{F}^{-1} f_{\lambda_j}^0(-x_{j,r})\right) g_{U_j}^{(0)}(x_0)\Bigg)^2 \delta_{j,r}^2 \Delta_j^2 \\
+ \frac{4\pi^2}{N} \sum_{r=1}^{N} \Bigg(& \frac{e^{-x_{j-1,r}}}{2\pi}\mathcal{F}^{-1} f_{\nu_j}^1(x_0 - x_{j-1,r}) \\
& + \operatorname{Re}\left(\mathcal{F}^{-1} f_{\sigma_j}^1(-x_{j-1,r})\right) \cdot \left(\frac{1}{2} g_{U_j}^{(2)}(x_0) + i g_{U_j}^{(1)}(x_0) - \frac{1}{2} g_U^{(0)}(x_0)\right) \\
& + \operatorname{Im}\left(\mathcal{F}^{-1} f_{\gamma_j}^1(-x_{j-1,r})\right)\left(-i g_{U_j}^{(1)}(x_0) + g_{U_j}^{(0)}(x_0)\right) \\
& - \operatorname{Re}\left(\mathcal{F}^{-1} f_{\lambda_j}^1(-x_{j-1,r})\right) g_{U_j}^{(0)}(x_0)\Bigg)^2 \delta_{j-1,r}^2 \Delta_{j-1}^2.
\end{aligned}$$

The derivation is similar to Söhl and Trabs (2014) and is therefore left to the reader.

Chapter 6

Simulations

In this chapter, the model is tested for simulated data. For the purpose of showing the properties and the performance of the inhomogeneous model, data is simulated from the Merton model with option data available at times T_1 and T_2. In this model, the Lévy triplet is given by (σ^2, γ, ν) where $\nu \sim \lambda \cdot \mathcal{N}(\mu, \delta^2)$ The objective of the simulation will be to evaluate the performance of the model in estimating the Lévy triplet between T_1 and T_2 in comparison to the performance in estimating the Lévy triplet between 0 and T_1. First, however, the model parameters should be chosen adequately.

These parameters include:

1. The amount of available option data
2. The noise level in the option prices
3. The time to maturity, the interest rate, the smoothness parameter s, and the Merton model parameters
4. The choice of the weight functions $w^1_{\sigma_j}(u)$, $w^1_{\gamma_j}(u)$ and $w^1_{\gamma_j}(u)$

To represent a realistic model, the amount of available option data is chosen to be 100 because it is realistic to have 100 different call and put options at the same maturity. Moreover, the noise level in the option prices is set to 5% relative to the value of the option function.

The times to maturity for the options are chosen to be $T_1 = 1$ month and $T_2 = 2$ months, the interest rate $\bar{r}$ is chosen to be 6%, the smoothness parameter s will be chosen to be 2, and the Merton model parameters of the time period $[0, T_1]$ are fixed and chosen to be $\sigma_1 = 0.2$, $\lambda_1 = 5$, $\nu_1 = \mathcal{N}\left(-0.3, 0.3^2\right)$, and $\gamma_1 = 1.11$ where the latter is implied by the martingale condition. Moreover, the parameters of the time period $[T_1, T_2]$ will vary according to $\sigma_2 \in \{0.1, 0.2, 0.3\}$, $\lambda_2 \in \{3, 5, 7\}$, $\operatorname{Var} \nu_2 \in \left\{0.2^2, 0.3^2, 0.5^2\right\}$, and γ_2 is again implied by the martingale condition (3.5). Not every combination is tested because that would mean 27 different results would have to be included. The default of the second parameter will be given by the second value in the set as given above and a simulation includes varying at most one parameter such that only 7 different types of simulations are performed. Note, that these parameters make sure that the trimmed value κ does not become too small. In particular, huge statistical errors are avoided.

The weight functions that are used in the simulations will be given by the formulas as in (3.11)-(3.14). As argued before, the exact choice of the weight functions does not influence the estimation much. However, as was also argued before, the value at the cutoff matters, as well as some smoothness assumptions imposed on the weight functions. These weight functions are chosen such that the value and its first two derivatives at the cutoff are equal to 0, making it a smooth function and therefore more optimal than, for example, the weight functions proposed by Belomestny and Reiß (2006b) which aren't continuous.

Finally, an up-to-now undiscussed topic should be addressed, namely, how the cutoff U should be chosen. It should be noted that the main focus of this research was not to find a good method to choose U. However, to see whether the simulations give good results, it is necessary to invoke a method to choose the cutoff U. Three methods described below are implemented to find a value for the cutoff U. The oracle method is used in the simulations.

6.1 The oracle method

The oracle method is one of the best methods when one starts doing simulations. The idea of the method is to choose the U that gives the best parameter estimates in terms of the L^1-distance between the estimated parameters and the real parameters. In the simulations, we used the following function for the oracle method

$$U_O = \underset{U>0}{\operatorname{argmin}} \left\{ |\hat{\sigma}(U) - \sigma| + |\hat{\gamma}(U) - \gamma| + \left|\hat{\lambda}(U) - \lambda\right| \right\}.$$

The major drawback of this method is that it is not applicable to real-world applications. However, it does provide in some sense a lower bound of what we could expect in an ideal situation.

6.2 The flat method

The flat method is a method that tries to find the flattest region for all estimates of the parameters. In theory, every value for the cutoff U should give a good estimation for the parameters. Therefore, it makes sense that whenever a few values of U in a row give similar parameter estimates the estimate is better than whenever a few values of U in a row that do not give similar values. Moreover, Bauer and Reiß (2008) give arguments why this method works well in practice. We used the following function for the flat method

$$U_F = \underset{U>0}{\operatorname{argmin}} \sum_{i=1}^{m} \{ |\hat{\sigma}(U + (i - m/2)x) - \hat{\sigma}(U + (i - m/2 - 1)x)| \}$$

for some $x > 0$. This x will be chosen as small as possible as the simulations allow and m is heuristically chosen to be 5.

6.3 The PLS method

The PLS, partial least squares, method is a method that finds the best Lévy triplet such that its implied option function $\hat{\mathcal{O}}_U$ is close to the estimated option function $\tilde{\mathcal{O}}$. We defined it to be equal to the L^2-distance

$$U_P = \underset{U>0}{\operatorname{argmin}} \int_{-\infty}^{\infty} \left(\hat{\mathcal{O}}_U(x) - \tilde{\mathcal{O}}(x) \right)^2 \mathrm{d}x.$$

6.4 Results

To assess the performance of the model in the simulations, the oracle method is chosen to determine the cutoff value U. In that way, the error is minimized and it is shown how well the model behaves in the best case scenario. In Figure 6.1 the option functions $\mathcal{O}_1$ and $\mathcal{O}_2$ under the Merton model defined by the default parameters, as described above, are plotted against the log-forward moneyness. Moreover, the simulated noisy option prices are plotted such that a 1% noise level is visualized. Based on these observations, the model is estimated and the results of the simulations are plotted in Figures 6.2-6.15.

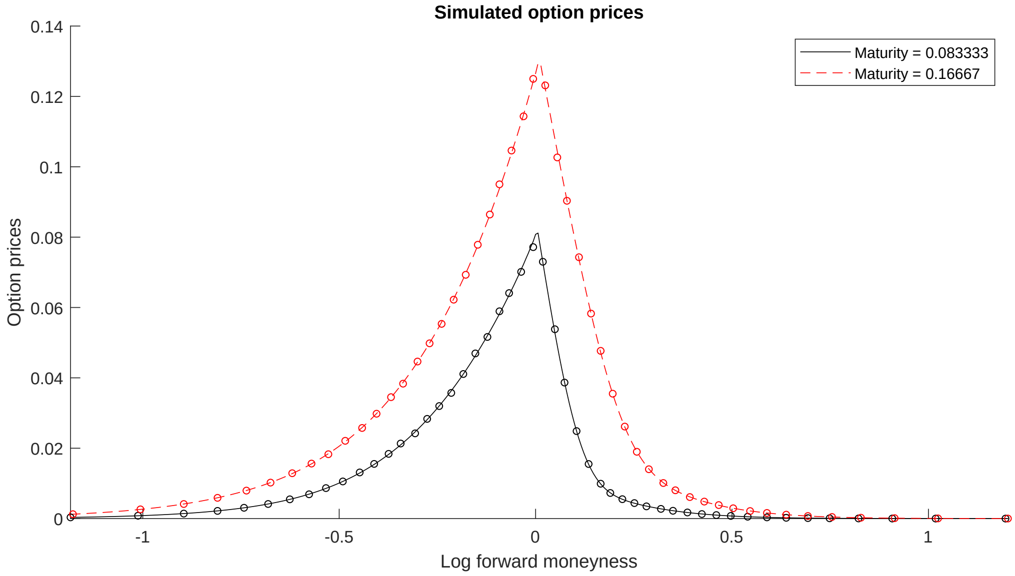

Figure 6.1: The option functions $\mathcal{O}_1$ and $\mathcal{O}_2$ and the simulated option function values under the Merton model defined by the default parameters plotted for maturity $T_1 = 1$ month and for maturity $T_2 = 2$ months.

From the simulations, the following conclusions are drawn. First of all, $\psi_2(v)$ is estimated quite well for $|v| \leq 20$ and that it is not a good idea to choose the cutoff value much higher than 20 in these kinds of models. This is explained by the definition of ψ_2 which involves taking a logarithm of a characteristic function. The characteristic function converges to 0 as $|u|$ tends to infinity such that small errors lead to big ones. Since it is desirable to choose the value of U as high as possible, it is therefore suggested that a cutoff value of approximately 20 is optimal.

Secondly, the estimate of ν_2 is generally better when the variance is higher because the estimate of ν_2 is generally a smooth version of ν_2. This makes it particularly difficult to estimate steep peaks well which can be viewed in almost all figures where the variance of the jump distribution is assumed to be 0.3^2 or less. In particular, Figure 6.7, where the variance has been set to 0.2^2, the method completely misses the peak of the distribution due to the smoothing.

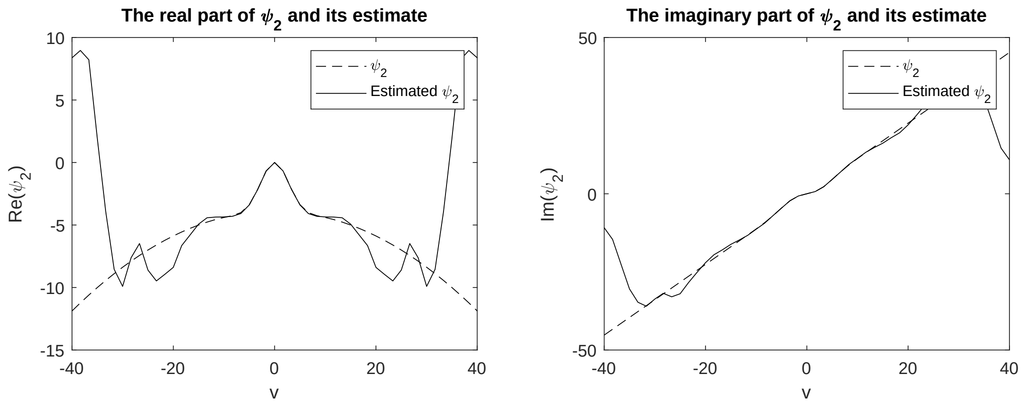

Figure 6.2: Estimate of $\psi_2(v)$ as a function of v of one simulation under the Merton model implied by $\sigma_2 = 0.1$, $\lambda_2 = 5$ and $\operatorname{Var} \nu_2 = 0.3^2$.

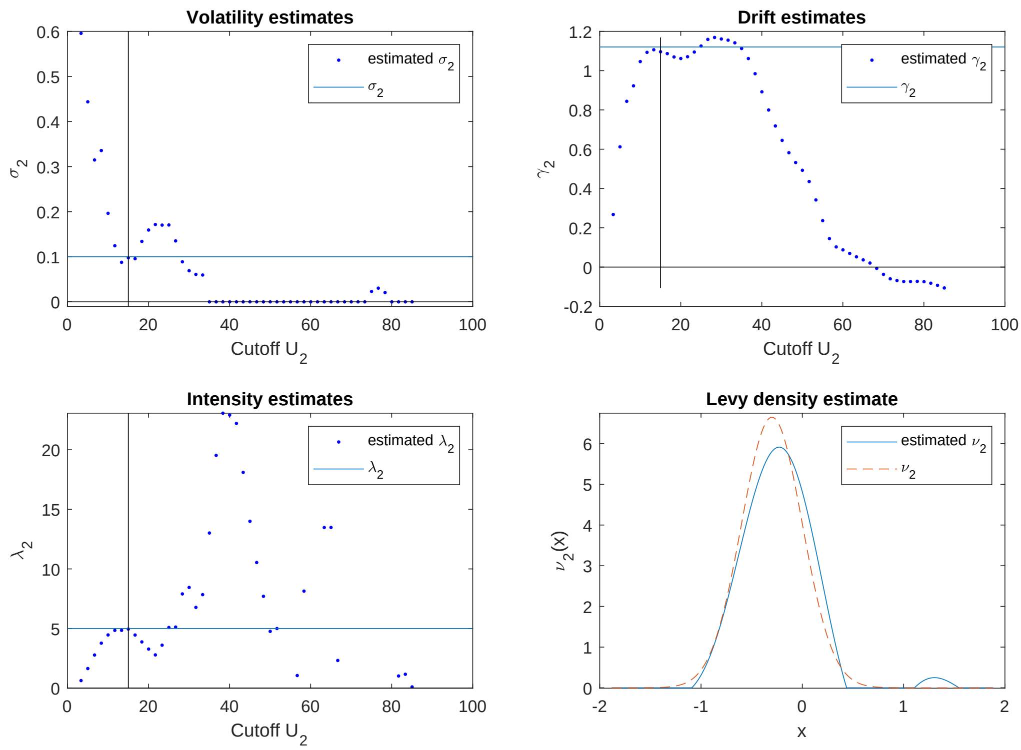

Figure 6.3: Estimates of the parameters of one simulation under the Merton model implied by $\sigma_2 = 0.1$, $\lambda_2 = 5$ and $\operatorname{Var} \nu_2 = 0.3^2$. The optimal cutoff value U_2 according to the oracle method is plotted as vertical line.

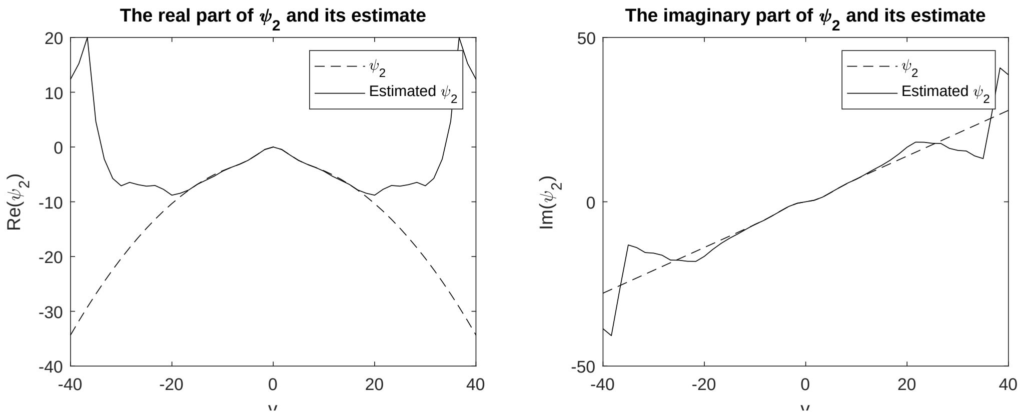

Figure 6.4: Estimate of $\psi_2(v)$ as a function of v of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 3$ and $\operatorname{Var} \nu_2 = 0.3^2$.

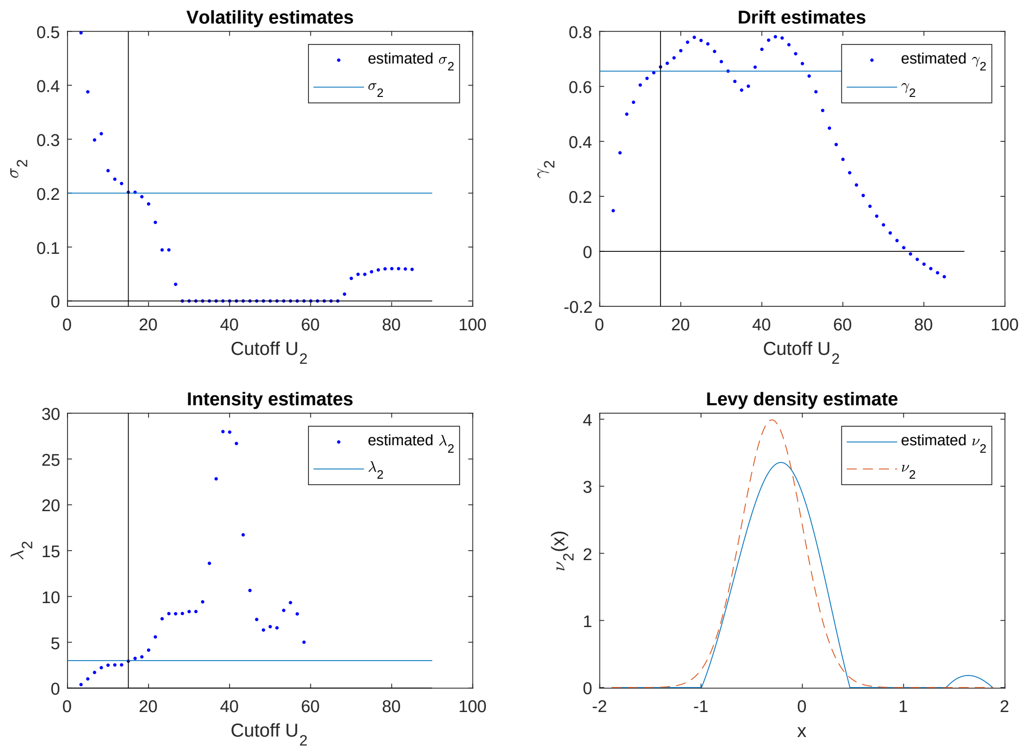

Figure 6.5: Estimates of the parameters of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 3$ and $\operatorname{Var} \nu_2 = 0.3^2$. The optimal cutoff value U_2 according to the oracle method is plotted as vertical line.

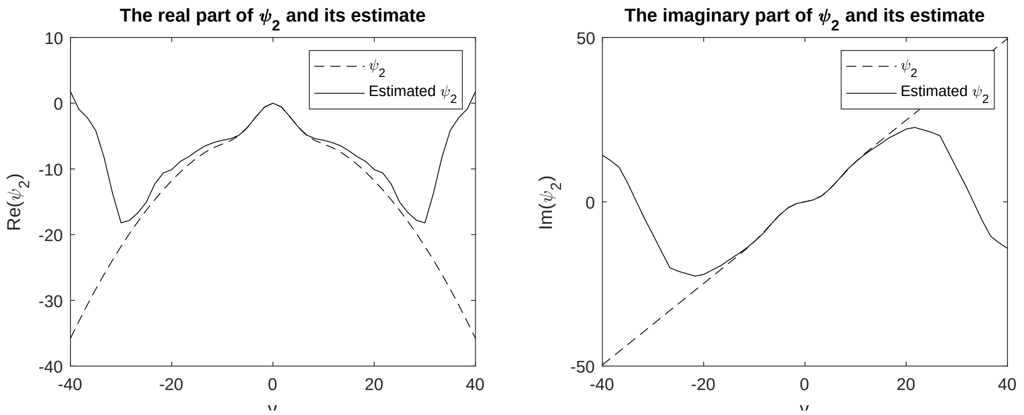

Figure 6.6: Estimate of $\psi_2(v)$ as a function of v of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 5$ and $\operatorname{Var}\nu_2 = 0.2^2$.

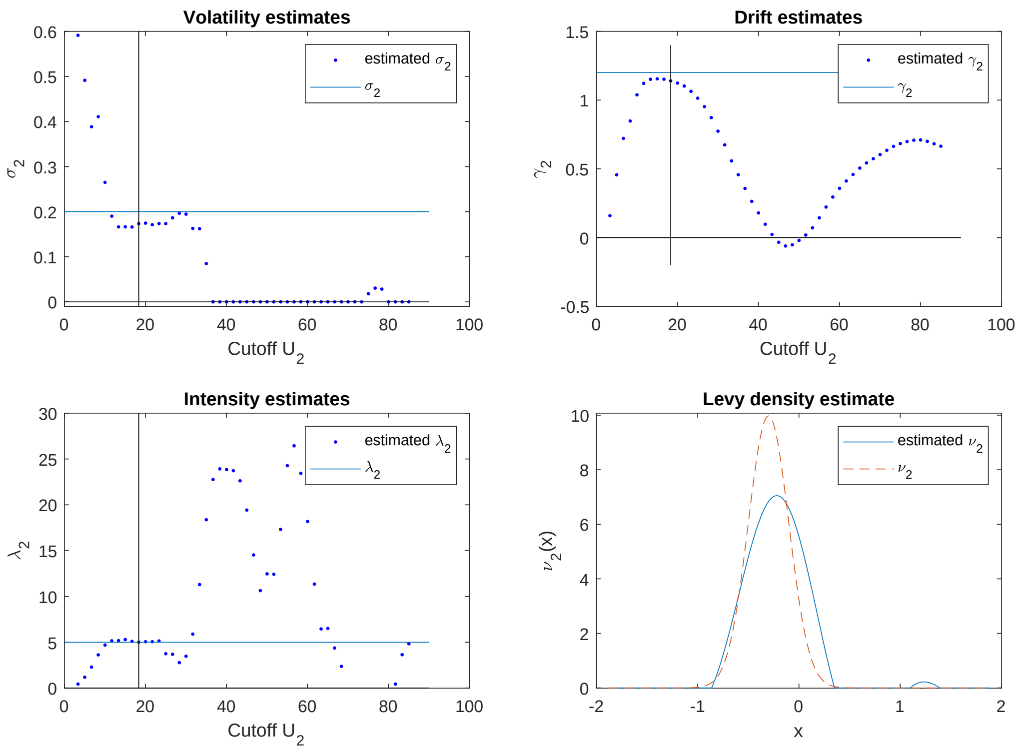

Figure 6.7: Estimates of the parameters of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 5$ and $\operatorname{Var}\nu_2 = 0.2^2$. The optimal cutoff value U_2 according to the oracle method is plotted as vertical line.

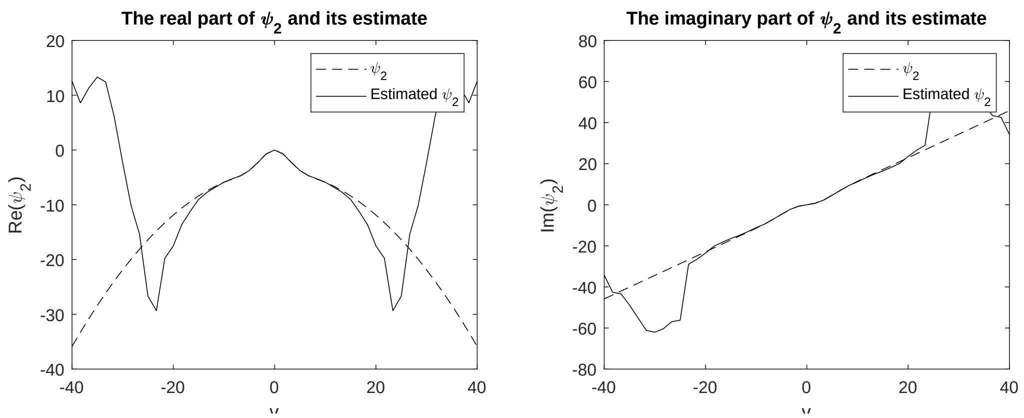

Figure 6.8: Estimate of $\psi_2(v)$ as a function of v of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 5$ and $\operatorname{Var} \nu_2 = 0.3^2$.

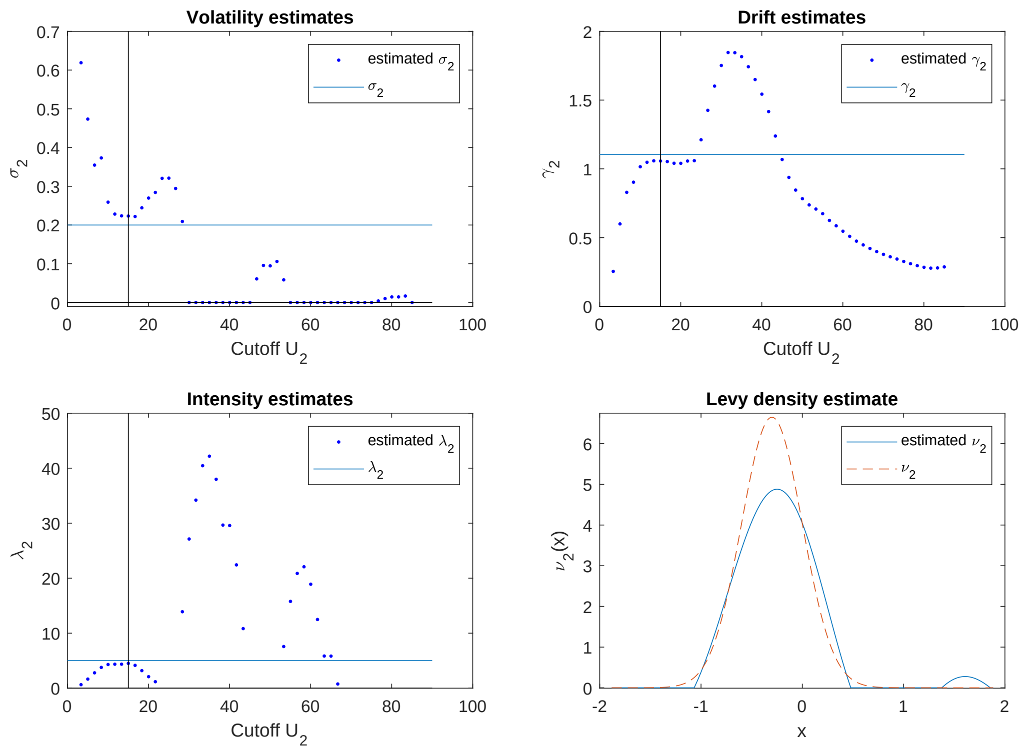

Figure 6.9: Estimates of the parameters of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 5$ and $\operatorname{Var} \nu_2 = 0.3^2$. The optimal cutoff value U_2 according to the oracle method is plotted as vertical line.

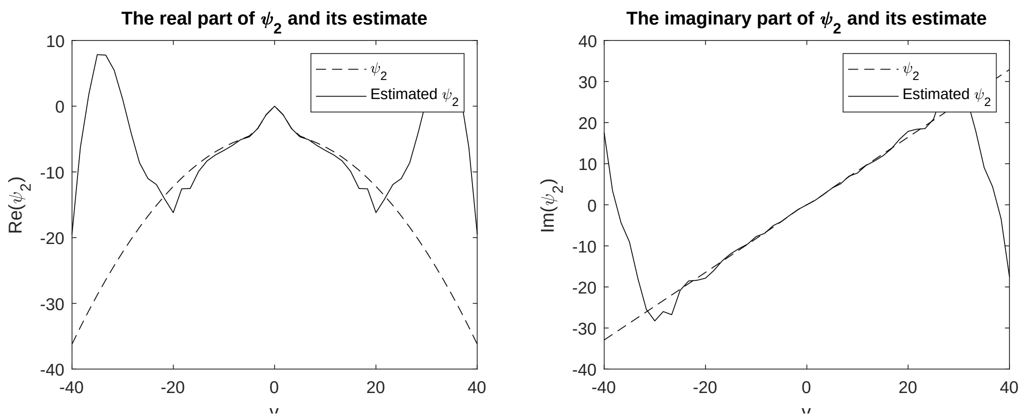

Figure 6.10: Estimate of $\psi_2(v)$ as a function of v of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 5$ and $\operatorname{Var} \nu_2 = 0.5^2$.

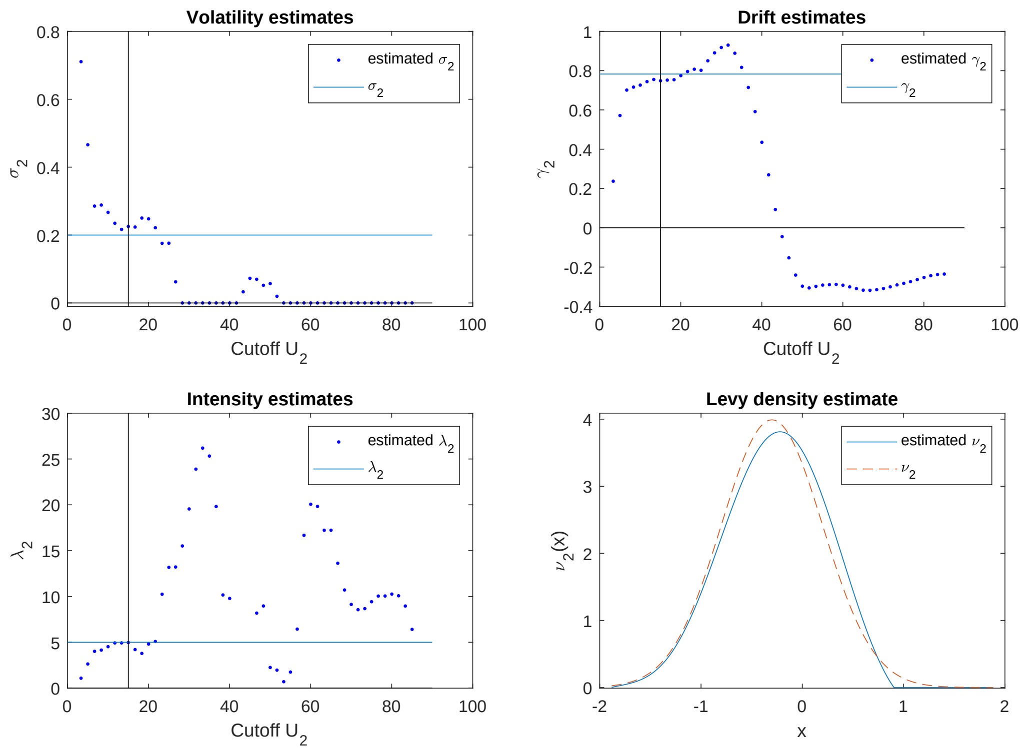

Figure 6.11: Estimates of the parameters of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 5$ and $\operatorname{Var} \nu_2 = 0.5^2$. The optimal cutoff value U_2 according to the oracle method is plotted as vertical line.

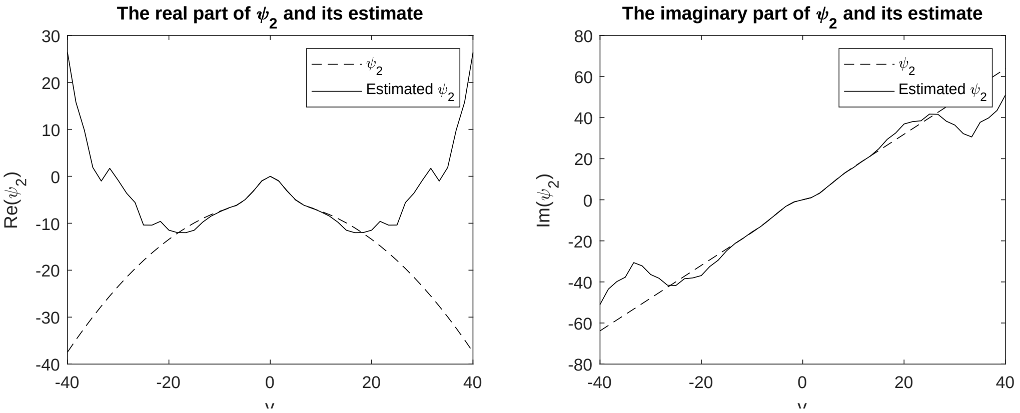

Figure 6.12: Estimate of $\psi_2(v)$ as a function of v of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 7$ and $\operatorname{Var}\nu_2 = 0.3^2$.

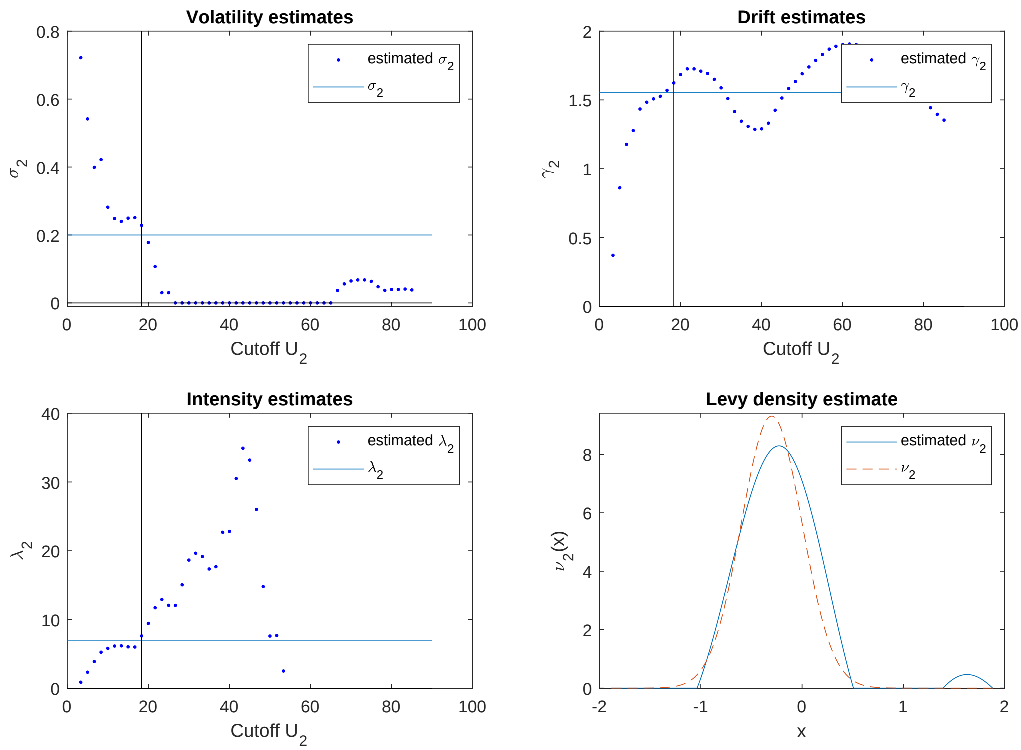

Figure 6.13: Estimates of the parameters of one simulation under the Merton model implied by $\sigma_2 = 0.2$, $\lambda_2 = 7$ and $\operatorname{Var}\nu_2 = 0.3^2$. The optimal cutoff value U_2 according to the oracle method is plotted as vertical line.

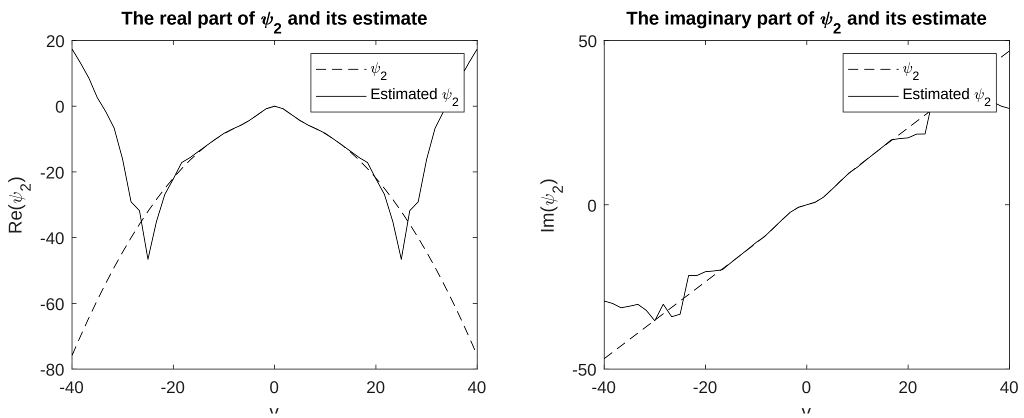

Figure 6.14: Estimate of $\psi_2(v)$ as a function of v of one simulation under the Merton model implied by $\sigma_2 = 0.3$, $\lambda_2 = 5$ and $\mathrm{Var}\, \nu_2 = 0.3^2$.

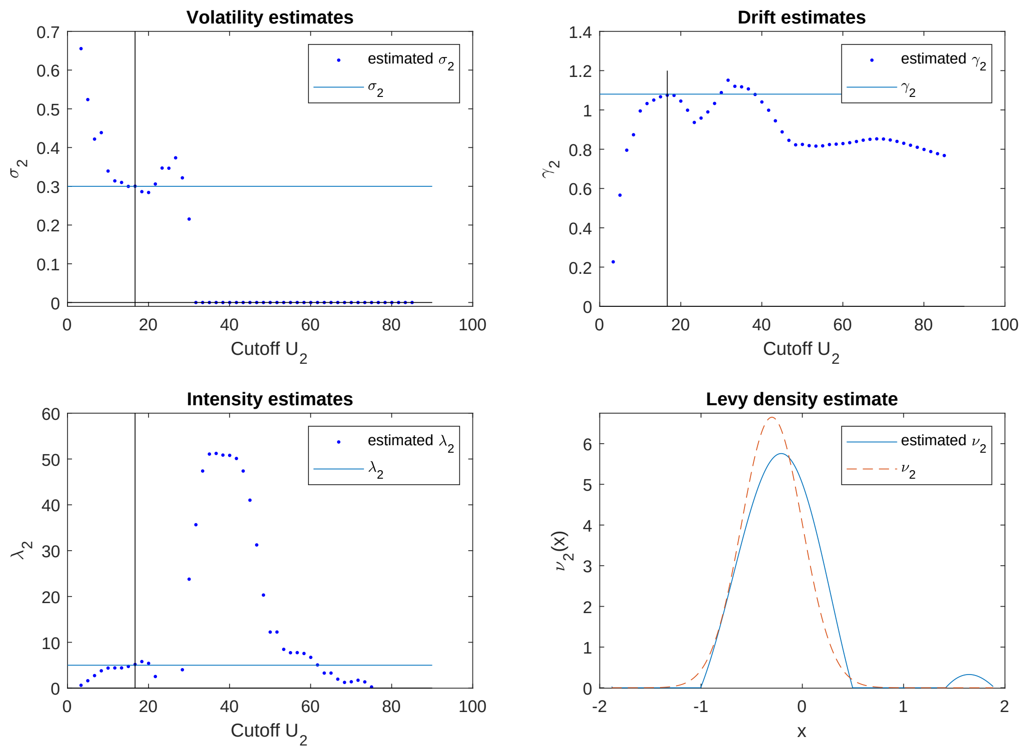

Figure 6.15: Estimates of the parameters of one simulation under the Merton model implied by $\sigma_2 = 0.3$, $\lambda_2 = 5$ and $\mathrm{Var}\, \nu_2 = 0.3^2$. The optimal cutoff value U_2 according to the oracle method is plotted as vertical line.

To assess the performance of the finite sample variance method in practice, it is not allowed to pick the cutoff value U according to the oracle method since that will influence the results heavily. Moreover, that is not how that particular model was built. In this case, it is thus necessary to pick a fixed U. Based on the simulations, it is not an unreasonable choice to let U be equal to 20. In Table 6.1, one can find the coverage probabilities of the confidence intervals which are constructed in section 5 in estimating the Merton model implied by the default parameters defined above.

Moreover, in Figures 6.16-6.17 the 95% pointwise confidence intervals for ν_2 are plotted where the cutoff value U is chosen to be approximately 10. The first figure suggests that the confidence interval method does not work very good since it is significantly wrong around the peak. Moreover, t also just misses to cover the curve between $(-0.9, -0.7) \cup (0.1, 0.2) \cup (1.4, 1.8)$. The negative bias around the mode is due to the smoothing in the estimation process and cannot be avoided. Moreover, it is very hard to track this error. The bump around 1.6 comes from the estimation procedure. The estimate for the density is the inverse of a Fourier transform and thus it will oscillate around 0 which sometimes cause these kinds of bumps. The other errors are explained by other flaws in the model.

In the other figure, the peak of the density is less sharp, which makes the estimator perform better, at least around the mode. Unfortunately, this pointwise confidence interval also doesn't cover the complete density. However, it only misses approximately 5% of the density over the area $(-2, 2)$ which is reasonable since one should note that this is not a uniform confidence interval.

It turns out that, when the peak of the Lévy density is too sharp, the bias can influence the model significantly and the confidence intervals may become useless. In theory, one can find an exact upper bound for the convergence rate of the bias since the measure μ_j is known. So, it might be theoretically possible to incorporate the bias if the goal is to cover the complete function. However, this is impossible for any application since the infinity norm of the sth derivative of the measure μ_j is unknown. Therefore incorporating the bias in confidence intervals will not be considered.

	σ_2^2	γ_2	λ_2
50%	60.8%	56.8%	57.5%
95%	99.1%	98.8%	98.8%

Table 6.1: Coverage probabilities of 50% and 95% confidence intervals constructed using the finite sample variance method. The probabilities are calculated via a Monte Carlo simulation study using 1000 Monte Carlo iterations.

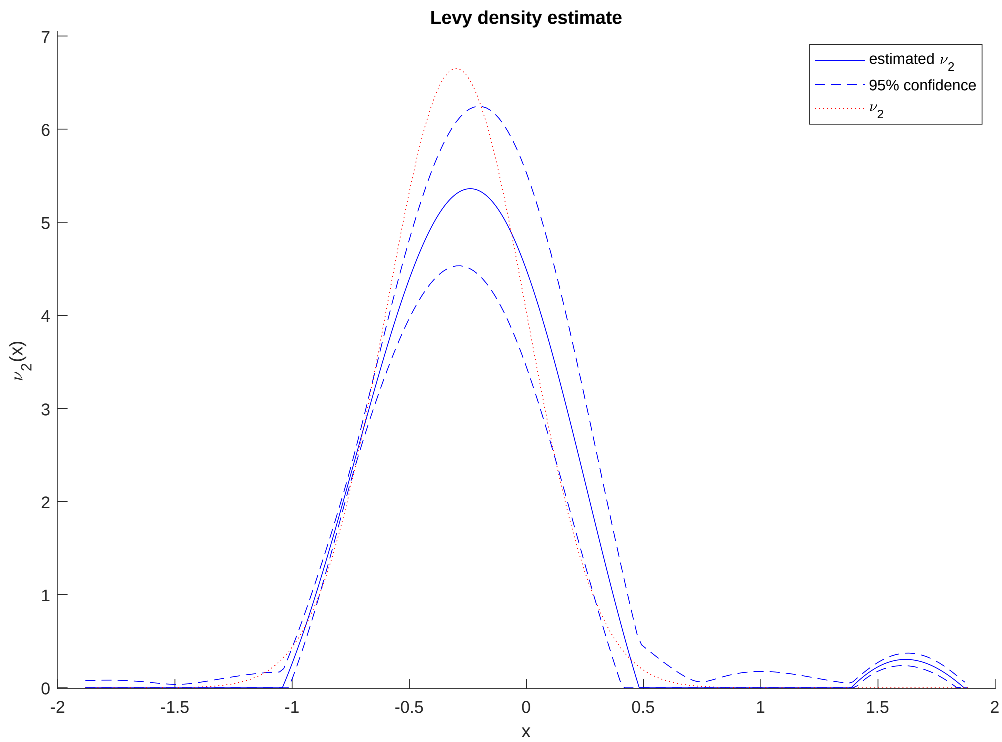

Figure 6.16: The estimate of ν_2 is plotted with its 95% pointwise confidence intervals. Moreover, the true ν_2 is also plotted in this figure to show the performance of the confidence intervals.

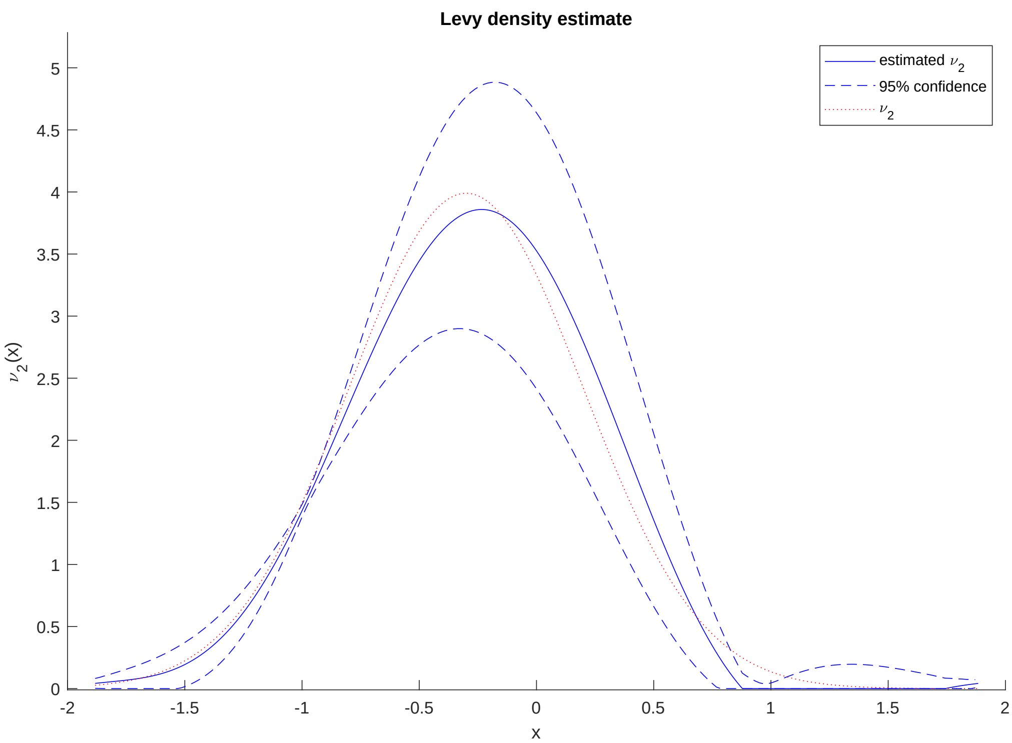

Figure 6.17: The estimate of ν_2 is plotted with its 95% pointwise confidence intervals. Moreover, the true ν_2 is also plotted in this figure to show the performance of the confidence intervals.

Chapter 7

Applications

Option data from the S&P 500 index is collected. The date from which the data is collected is January 5, 2016 and the data can be found at `https://www.historicaloptiondata.com/content/sample-files-0`. 3 different maturities of the options are investigated, 15 January 2016, 19 February 2016, and 18 March 2016. These maturities will be referred to with T_1, T_2, and T_3, respectively. Moreover, T_0 will be defined as January 5, 2016, the date of collection. To construct the confidence intervals, it is assumed that $\delta_j(x) = 0.01\mathcal{O}_j(x)$, similar to (Cont and Tankov, 2004a, p. 439) and the smoothness parameter s is set equal to 2.

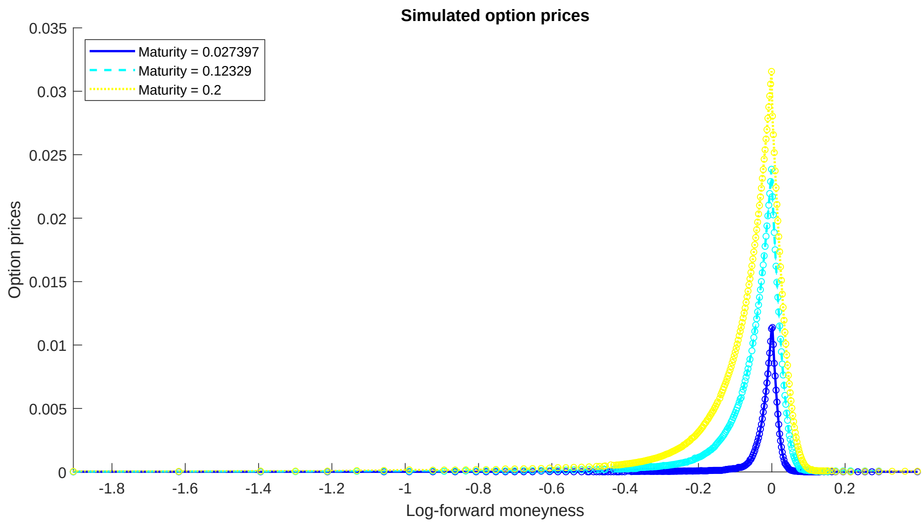

Figure 7.1: The option function values of the S&P 500 index.

From Figure 7.1, it appears that a lot more put options with a relatively small strike price are sold than call option with a relatively high strike price. This is due to the risk-averseness of investors. If the index drops a lot, the put options with low strike prices provide an insurance to some extent to investors.

To compare the homogeneous model to the inhomogeneous model, the results of the homogeneous model with the third maturity, i.e., 18 March 2016, are presented below and below those results, the results of the inhomogeneous model are presented. The figure structure of the plots of the estimates of the Lévy parameters of the inhomogeneous Lévy process, i.e., Figures 7.4-7.7, corresponding to the maturities is given by the matrix

$$\begin{pmatrix} T_1 & T_2 \\ T_3 & \end{pmatrix}.$$

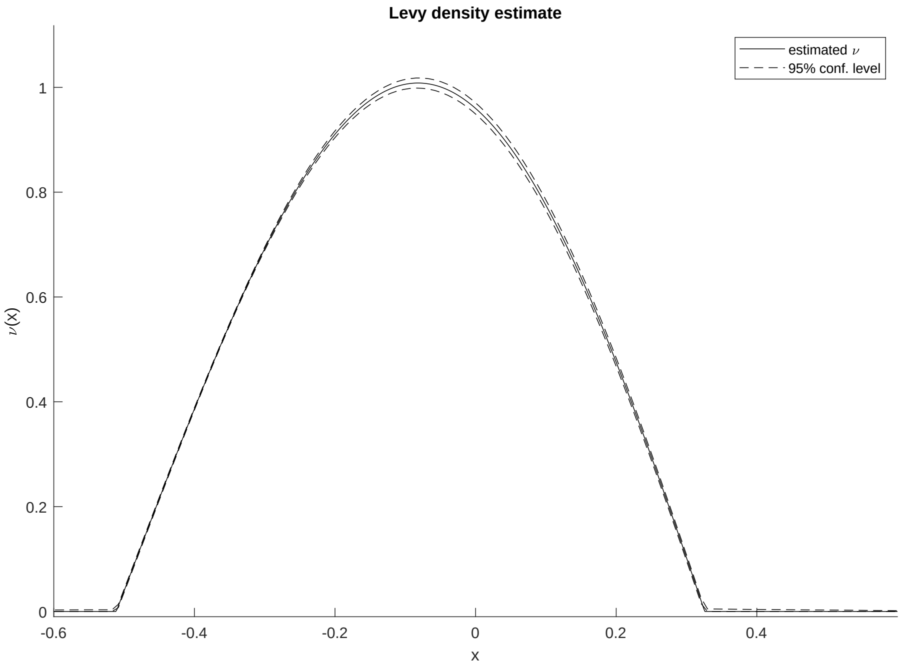

Figure 7.2: Estimates of the Lévy density under the assumption of homogeneity on $[T_0, T_3]$ including a 95% pointwise confidence level of the S&P 500 index where the cutoff U is chosen using the flat method.

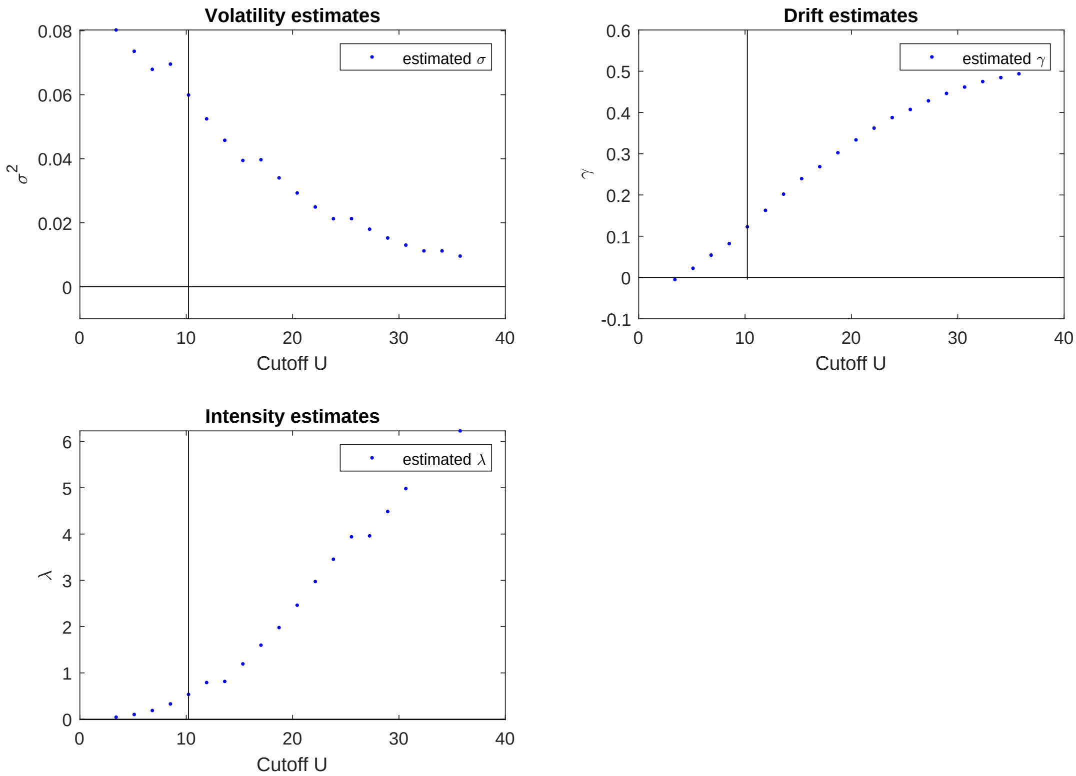

Figure 7.3: Estimates of the parameters of the Lévy process under the assumption of homogeneity on $[T_0, T_3]$ of the S&P 500 index where the cutoff U is chosen using the flat method.

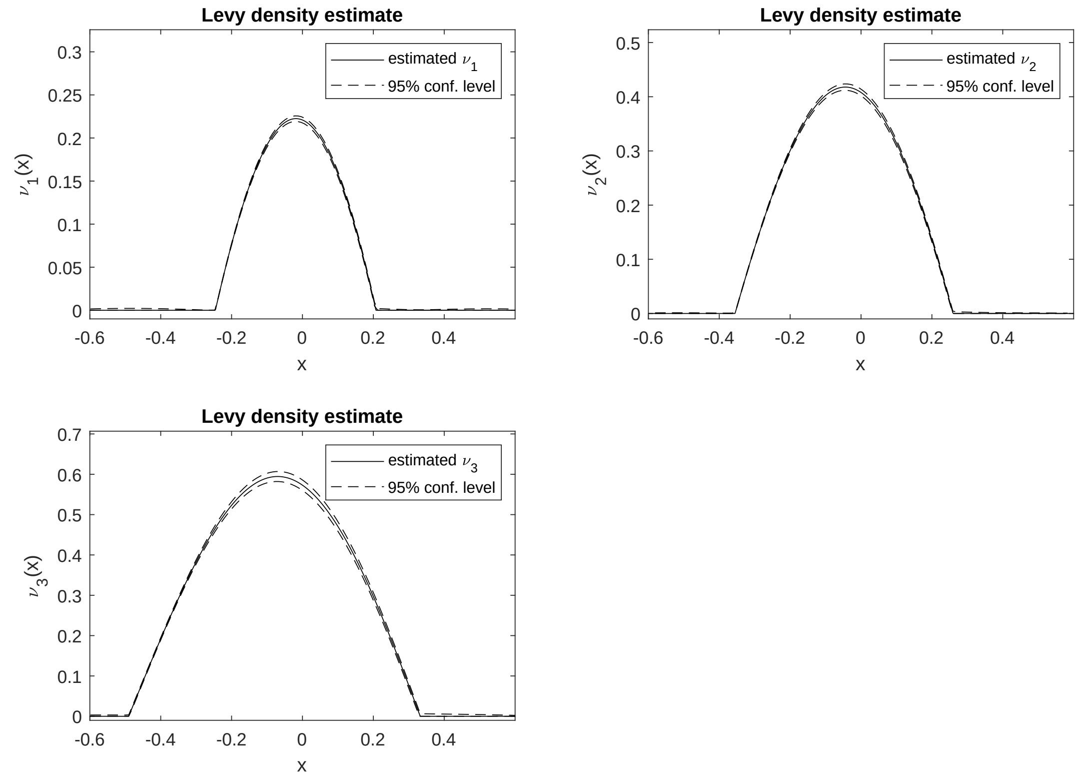

Figure 7.4: Estimates of the Lévy density under the assumption of homogeneity on $[T_0, T_1]$, $[T_1, T_2]$ and $[T_2, T_3]$ including a 95% pointwise confidence level of the S&P 500 index where the cutoff U is chosen using the flat method.

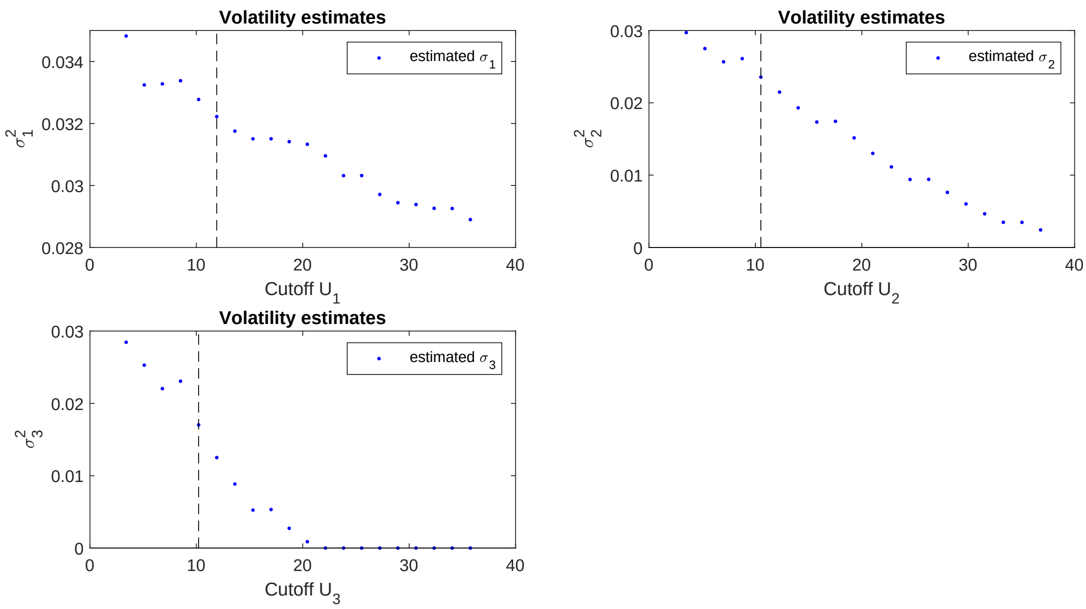

Figure 7.5: Estimates of the volatility parameter of the Lévy model under the assumption of homogeneity on $[T_0, T_1]$, $[T_1, T_2]$ and $[T_2, T_3]$ of the S&P 500 index where the cutoff U is chosen using the flat method.

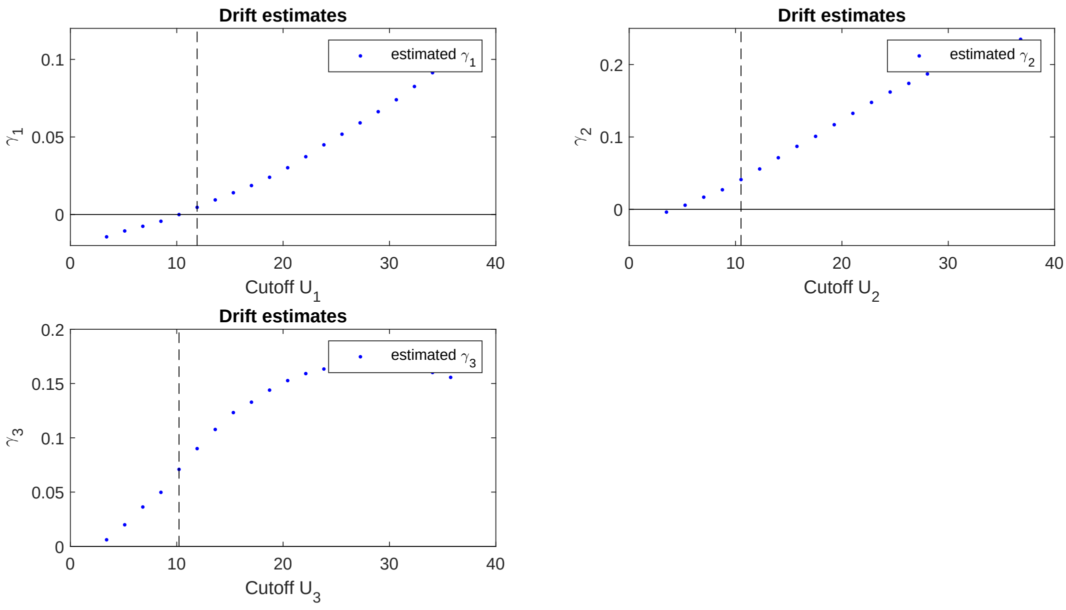

Figure 7.6: Estimates of the drift parameter of the Lévy model under the assumption of homogeneity on $[T_0, T_1]$, $[T_1, T_2]$ and $[T_2, T_3]$ of the S&P 500 index where the cutoff U is chosen using the flat method.

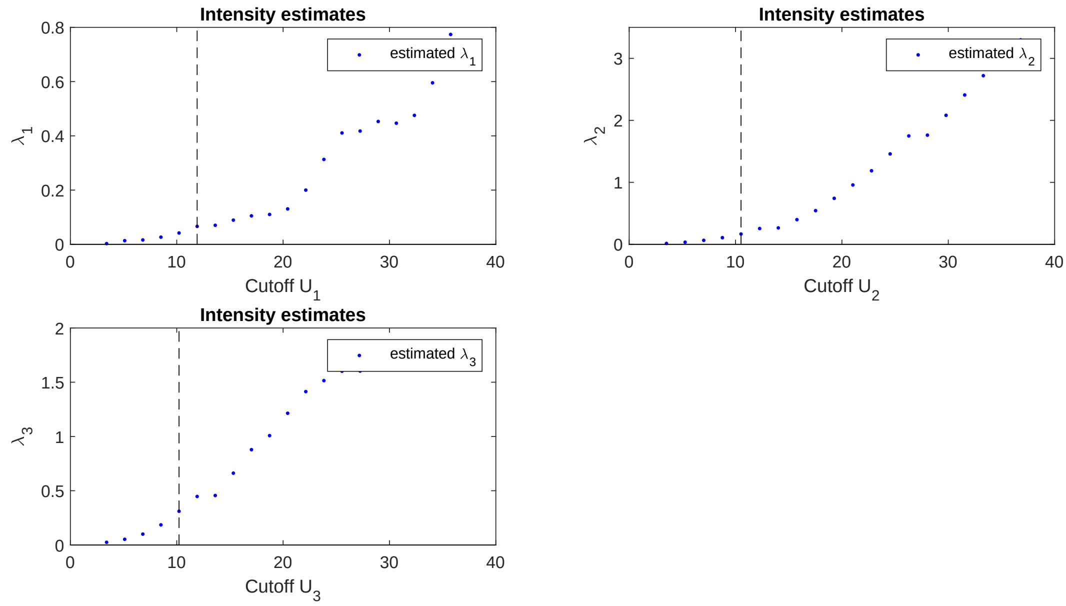

Figure 7.7: Estimates of the intensity of the Lévy model under the assumption of homogeneity on $[T_0, T_1]$, $[T_1, T_2]$ and $[T_2, T_3]$ of the S&P 500 index where the cutoff U is chosen using the flat method.

From the figures above and the tables displayed below, it is clear that the inhomogeneous Lévy process gives significantly different results when compared with the homogeneous Lévy process. This shows that assuming homogeneity might indeed be wrong. For example, the intersection of the 95% confidence intervals is disjoint when comparing the estimators of the parameters from different time intervals. Also when comparing the parameters of the inhomogeneous Lévy model with the homogeneous model, there is no overlap in the confidence intervals. Moreover, the Lévy densities for $[0, T_1]$, $[T_1, T_2]$ and $[T_2, T_3]$ are also significantly different from each other and from the estimated Lévy density under the assumption

that the process is homogeneous.
This does not necessarily mean that we are allowed to reject the homogeneity premise but it does give verification to some extent that this premise is wrong. However, to conclude that such a statement is wrong, one should of course perform a certain hypothesis test. A possible approach is explained in more detail in the next chapter.

	σ^2	γ	λ
$p_{0.025}$	0.0592	0.1222	0.5291
$p_{0.975}$	0.0606	0.1236	0.5416

Table 7.1: 95% confidence intervals constructed using the finite sample variance method for the parameters of the underlying process of the S&P 500 index under the homogeneous Lévy assumption. The confidence intervals for the parameters are given by $(p_{0.025}, p_{0.975})$.

	σ_1^2	σ_2^2	σ_3^2	γ_1	γ_2	γ_3	λ_1	λ_2	λ_3
$p_{0.025}$	0.0319	0.0232	0.0162	0.0044	0.0408	0.0701	0.0651	0.1632	0.3033
$p_{0.975}$	0.0326	0.0239	0.0179	0.0049	0.0414	0.0717	0.0673	0.1698	0.3182

Table 7.2: 95% confidence intervals constructed using the finite sample variance method for the parameters of the underlying process of the S&P 500 index under the inhomogeneous Lévy assumption. The confidence intervals for the parameters are given by $(p_{0.025}, p_{0.975})$.

Chapter 8

Discussion and further research

In this chapter, possible extensions of this research are discussed.

In the simulations only the inhomogeneous Merton model is used for testing the performance of the method. Of course, there are a lot more interesting Lévy models that could be the subject of study. For example, the double exponential jump-diffusion model described by Kou (2002) is interesting since the Lévy density of this model has a very sharp peak. Moreover, it is interesting to study the performance of the model when the Lévy process follows a mixture of different Lévy processes, for example, the Lévy process could follow a Merton model on $[0, T_1]$ and a double exponential jump-diffusional model on $[T_1, T_2]$.

In the construction of the confidence intervals using the finite sample variance method, sharp peaks of the Lévy density are not estimated well because of the intermediate smoothing in the estimation procedure which smooths the peak of the Lévy density. It is interesting to investigate if the sharp peaks can be captured by different confidence intervals if, for example, the bias term is taken into account.

Moreover, as a rule of thumb the option function error δ_j has been set equal to 1% of $\mathcal{O}_j$, similar to (Cont and Tankov, 2004a, p. 439). It might give better confidence estimates if one estimates δ_j using regression techniques. Another possibility is to assume that δ_j is proportional to $\mathcal{O}_j$ and estimate the L^2-norm of δ_j by the test statistic of the PLS method, i.e.,

$$\begin{aligned}\int_{-U_j}^{U_j} \left(\tilde{\mathcal{O}}_j(x) - \mathcal{O}_j(x)\right)^2 \mathrm{d}x &\approx \int_{-U_j}^{U_j} \left(\tilde{\mathcal{O}}_j(x) - \mathcal{O}_j^l(x)\right)^2 \mathrm{d}x = \int_{-U_j}^{U_j} \left(\sum_{r=1}^{N} \varepsilon_{j,r}\delta_j(x_{j,r})b_{j,r}(x)\right)^2 \mathrm{d}x \\ &\approx \int_{-U_j}^{U_j} \sum_{r=1}^{N} \delta_j(x_{j,r})^2 b_{j,r}(x)^2 \,\mathrm{d}x = \sum_{r=1}^{N} \delta_j(x_{j,r})^2 \int_{x_{j-1,r}}^{x_{j+1,r}} b_{j,r}(x)^2 \,\mathrm{d}x \\ &= \sum_{r=1}^{N} \delta_j(x_{j,r})^2 \cdot \frac{2\Delta_j}{3} \approx \frac{2}{3} \|\delta_j\|_{L^2(\mathbb{R})}^2\end{aligned}$$

where in step one, we estimated the option function $\mathcal{O}_j$ by its linearization. In the second approximation, the random variable is estimated by its expectation. In the end, the summation is approximated by its limit as U_j tends to ∞. This also provides a direct estimate of the function δ_j. It should be note, however, that the approximation steps should be dealt with more rigorously if one wants to use this approximation.

One different topic that could be studied in further research is the construction of hypothesis tests for testing whether or not the process is inhomogeneous on $[T_j, T_{j+2}]$.

Of course, one could estimate the Lévy triplets on $[T_j, T_{j+1}]$ and $[T_{j+1}, T_{j+2}]$ separately without worrying about possible homogeneity. On one hand, it gives better results in general when inhomogeneity is assumed. However, when in a perfect world, the process actually is homogeneous on $[T_j, T_{j+2}]$, better estimation results are achieved when all three (or more) maturities T_j, T_{j+1}, T_{j+2} are used in the estimation process of the corresponding Lévy triplet.

In the application of the model, both the inhomogeneous and the homogeneous model are fitted in the same application. Parameter estimates turned out to be completely different, however, it might be the case that both models produce similar results in different applications of the model and that the homogeneous model produces more stable results, for example. To test this, one could calculate the implied option functions and compare the difference of the implied option functions with the estimated option function from the data. From there, one could perform a bootstrap to determine p-values. Another possibility would be to use different derivatives and investigate the differences with these derivatives.

Chapter 9

Conclusion

In this report, estimators for inhomogeneous Lévy processes are developed and extensively studied. It has been shown that the estimators are asymptotically normal, optimal convergence rates are determined and theoretic confidence intervals are constructed. Moreover, the model is applied to simulations from an inhomogeneous Merton model and to the S&P 500 index.

It turned out from the simulations that the asymptotic normality result shouldn't be applied to determine confidence intervals. However, the result isn't useless since from the statement the optimal convergence rates of the estimators are determined. Moreover, it wasn't necessary to get the confidence intervals from the asymptotic normality result since a different approach based on the finite sample variance turned out to work pretty well in terms of coverage probabilities. This worked better since it was based on the direct estimation of the variance of the estimators instead of using a limiting result.

From the simulations of the inhomogeneous Merton model, it was clear that the cutoff value shouldn't be chosen to high due to large stochastic and numerical errors as was expected from the model.

Moreover, it was found that the method had a lot of difficulty in estimating sharp peaks of the Lévy density. This was explained by the Fourier techniques in the method which essentially smoothens the density causing a (significant) negative bias around the mode of the density. This negative bias also sometimes invalidates the pointwise confidence intervals around this area when the actual peak is too sharp. It is suggested that this problem might lead to an interesting further research topic.

Finally, the application of the model to the S&P 500 stock agrees with the questionability of the homogeneity assumption. Indeed, significantly different results are found for different time frames. However, to actually reject the premise, it is necessary to perform a hypothesis test or something similar. This would also be a very interesting new research topic that is induced by the introduction of inhomogeneous Lévy processes.

Chapter 10

Proofs

10.1 The estimator $\tilde{\psi}_j$ for the function ψ_j

In order to ensure that the estimation procedure works, it is necessary to check whether the estimator for ψ_j is well-defined, i.e., that $\mathbb{P}(\tilde{\varphi}_{T_j}(u-i)=0)=0$. This turns out to be asymptotically true under the assumption that the error distribution is sub-Gaussian and if U_j does not converge to infinity too quickly. One should recall the definition of $K(T,\sigma,R,v)$ which can be found in (3.6).

In the same spirit of Söhl (2010), the following results hold.

Proposition 3. *Let $j \in \{1,2,\dots,n\}$, let the error distribution be sub-Gaussian and let U_j be such that*

$$\Delta_j U_j^4 \log U_j \exp\left(U_j^2 \cdot \sum_{m=1}^{j} (T_m - T_{m-1})\sigma_m^2\right) \to 0.$$

Moreover, if there exists a $p>1$ such that

$$\lim_{U_j\to\infty} \sum_{j=1}^{N} \delta_j^2 \left(1+|x_j|^p\right) \Delta_j < \infty,$$

then

$$\lim_{U_j\to\infty} \mathbb{P}\left(\sup_{u\in[0,U_j]} |\tilde{\varphi}_{T_j}(u-i)-\varphi_{T_j}(u-i)| > 2^{j-1} \inf_{u\in[0,U_j]} \prod_{m=1}^{j} K(T_m-T_{m-1},\sigma_m,R,u)\right) = 0.$$

Moreover, the estimator $\tilde{\psi}_j$ defined in (3.7) is asymptotically well-defined.

Corollary 1. *Under the assumptions of Proposition 3.,*

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{u\in[0,U_j]} |\arg\tilde{\varphi}_{T_j}(u-i) - \arg\varphi_{T_j}(u-i)| > \pi\right) = 0 \qquad (10.1)$$

holds.

Note that if the limit relation holds, well-definedness is immediate by Lemma 2, $|\varphi_{T_j}(u-i)| > 2\kappa(u,T_j)$ and

$$2^{j-1} \inf_{u\in[0,U_j]} \prod_{m=1}^{j} K(T_m-T_{m-1},\sigma_m,R,u) \geq \inf_{u\in[0,U_j]} \kappa(u,T_j).$$

Proof (Proposition 3). To prove the limit relation, Markov's inequality will be applied, where the main difficulty is found in estimating the expectation of the supremum of the difference

$$|\tilde{\varphi}_{T_j}(u-i) - \varphi_{T_j}(u-i)|^2 = \left| iu(1+iu) \cdot \sum_{r=1}^{N} \delta_{j,r} \mathcal{F} b_{j,r}(u) \varepsilon_{j,r} \right|^2 = \Delta_j \left(u^4 + u^2\right) \cdot |G_u|^2$$

with

$$G_u := \frac{1}{\sqrt{\Delta_j}} \sum_{r=1}^{N} \delta_{j,r} \mathcal{F} b_{j,r}(u) \varepsilon_{j,r}.$$

G_u will be bounded using an entropy argument. To that end, we bound the difference $|G_u - G_v|$ in terms of $|u-v|$. Let $q \in [0,1]$, then

$$\begin{aligned}
\Delta_j \mathbb{E}\left\{|G_u - G_v|^2\right\} &= \mathbb{E}\left\{ \left| \sum_{r=1}^{N} \delta_{j,r} \varepsilon_{j,r} \int_{-\infty}^{\infty} b_{j,r}(x) \left(\mathrm{e}^{ixu} - \mathrm{e}^{ixv}\right) \mathrm{d}x \right|^2 \right\} \\
&= \sum_{r=1}^{N} \delta_{j,r}^2 \left| \int_{-\infty}^{\infty} b_{j,r}(x) \left(\mathrm{e}^{ixu} - \mathrm{e}^{ixv}\right) \mathrm{d}x \right|^2 \\
&\leq \sum_{r=1}^{N} \delta_{j,r}^2 \left(\int_{-\infty}^{\infty} b_{j,r}(x) \left|\mathrm{e}^{ixu} - \mathrm{e}^{ixv}\right| \mathrm{d}x \right)^2 \\
&\leq \sum_{r=1}^{N} \delta_{j,r}^2 \left(\int_{-\infty}^{\infty} b_{j,r}(x) \cdot \min(2, |u-v||x|) \, \mathrm{d}x \right)^2 \\
&= \sum_{r=1}^{N} \delta_{j,r}^2 \left(\int_{|x|>2/|u-v|} 2 b_{j,r}(x) \, \mathrm{d}x + \int_{|x|\leq 2/|u-v|} b_{j,r}(x) \cdot |u-v||x| \, \mathrm{d}x \right)^2 \\
&\leq \sum_{r=1}^{N} \delta_{j,r}^2 \left(\int_{|x|>2/|u-v|} 2 b_{j,r}(x) \cdot \left(\frac{|x||u-v|}{2}\right)^q \mathrm{d}x \right. \\
&\qquad \left. + \int_{|x|\leq 2/|u-v|} b_{j,r}(x) \cdot |u-v||x| \cdot \left(\frac{2}{|x||u-v|}\right)^{1-q} \mathrm{d}x \right)^2 \\
&= \sum_{r=1}^{N} \delta_{j,r}^2 \left(\int_{-\infty}^{\infty} 2^{1-q} b_{j,r}(x) \cdot |x|^q \cdot |u-v|^q \, \mathrm{d}x \right)^2 \\
&= |u-v|^{2q} \cdot \sum_{r=1}^{N} \delta_{j,r}^2 \left(\int_{-\infty}^{\infty} 2^{1-q} b_{j,r}(x) \cdot |x|^q \, \mathrm{d}x \right)^2 \\
&\leq |u-v|^{2q} \cdot 2^{2-2q} \cdot \sum_{r=1}^{N} \delta_{j,r}^2 \left(\int_{x_{r-1}}^{x_{r+1}} b_{j,r}(x) \cdot |x|^q \, \mathrm{d}x \right)^2 \\
&\leq |u-v|^{2q} \cdot 2^{2-2q} \cdot \sum_{r=1}^{N} \delta_{j,r}^2 \cdot (x_{r+1} - x_{r-1})^2 \cdot |\max\{x_{r+1}, -x_{r-1}\}|^{2q} \\
&\leq |u-v|^{2q} \cdot 2^{4-2q} \Delta_j^2 \sum_{r=1}^{N} \delta_{j,r}^2 (|x_r| + \Delta_j)^{2q}
\end{aligned}$$

From Lemma 3. we find

$$(|x_r| + \Delta_j)^{2q} \leq \max\left(2^{2q-1}, 1\right) \left(|x_r|^{2q} + \Delta_j^{2q}\right) \leq 2 \left(|x_r|^{2q} + \Delta_j^{2q}\right).$$

Hence, the supremum of the difference $|G_u - G_v|$ is bounded as $N \to \infty$. Indeed,

$$\Delta_j \mathbb{E}\left\{|G_u - G_v|^2\right\} \leq |u-v|^{2q} \cdot 2^{4-2q+1}\Delta_j^2 \sum_{r=1}^{N} \delta_{j,r}^2 \left(|x_r|^{2q} + \Delta_j^{2q}\right)$$

is bounded as $N \to \infty$ by the second assumption in the proposition. This leads to the choice of $q = \min(p/2, 1)$ where the minimum is taken with 1 such that $q \in [0,1]$, a necessary condition for the above calculation to hold.

Concluding, there exist a constant $c > 0$ such that

$$d(u,v) := \sqrt{\mathbb{E}\left\{|G_u - G_v|^2\right\}} \leq c|u-s|^H =: \rho(u,v)$$

with $H = q = \min(p/2, 1)$. Hence, $B_\rho(x,r) \subset B_d(x,r)$ for all $x \in \mathbb{R}$ and $r > 0$ and thus $N_d(X,r) \leq N_\rho(X,r)$ for all sets X and all $r > 0$.

Before Dudley's theorem is applied, the metric entropy will be estimated. First of all, it is noted that there exists a $D < \infty$ such that $d(u,v) \leq D$ for all $u, v \in \mathbb{R}$. This is immediate when one applies the inequality $|\mathrm{e}^{ix} - \mathrm{e}^{iy}| \leq 2$ for $x, y \in \mathbb{R}$.

The covering number of $[0, U_j]$ of the metric ρ given a radius r is equal to

$$N_\rho([0,U_j], r) = \lceil U_j (c/r)^{1/H}/2 \rceil .$$

Moreover, we assume U_j to be large enough such that $U_j \geq (eD/c)^{1/H}$, then

$$N_\rho([0,U_j], r) \leq U_j (c/r)^{1/H}.$$

The metric entropy is now estimated as follows

$$\begin{aligned}
J([0,U_j], d) &:= \int_0^\infty \sqrt{\log(N_d([0,U_j],r))}\,\mathrm{d}r = \int_0^D \sqrt{\log(N_d([0,U_j],r))}\,\mathrm{d}r \\
&\leq \int_0^D \sqrt{\log(N_\rho([0,U_j],r))}\,\mathrm{d}r \leq \int_0^D \sqrt{\log\left(U_j (c/r)^{1/H}\right)}\,\mathrm{d}r \\
&= H^{-1/2} \int_0^D \sqrt{\log\left(U_j^H (c/r)\right)}\,\mathrm{d}r \\
&= cH^{-1/2} U_j^H \int_0^{D/(U_j^H c)} \sqrt{\log(1/s)}\,\mathrm{d}s \\
&\leq cH^{-1/2} U_j^H \cdot D/(U_j^H c) \sqrt{\log\left((U_j^H c)/D\right)} \\
&= \sqrt{\log(U_j) + \log\left(c^{1/H}/D^{1/H}\right)} \\
&\lesssim \sqrt{\log U_j}
\end{aligned}$$

where the one but last inequality needs a little verification. Define

$$x := D/(U_j^H c) \leq e^{-1}$$

then

$$\log x^{-1} \geq 1.$$

The integral is now solved and estimated as follows

$$\int_0^x \sqrt{\log(1/s)}\,\mathrm{d}s = \frac{\sqrt{\pi}}{2}\cdot\left(1-\mathrm{Erf}\left(\sqrt{\log x^{-1}}\right)\right) + x\sqrt{\log x^{-1}}$$

with

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-s^2}\,\mathrm{d}s.$$

Moreover, we have

$$1-\mathrm{Erf}\left(\sqrt{\log x^{-1}}\right) \leq \exp\left(-\log x^{-1}\right)/\left(\sqrt{\pi}\sqrt{\log x^{-1}}\right) = \frac{x}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{\log x^{-1}}}$$

Thus

$$\int_0^x \sqrt{\log(1/s)}\,\mathrm{d}s \leq \frac{x}{2}\cdot\left(\frac{1}{\sqrt{\log x^{-1}}}+\sqrt{\log x^{-1}}\right) \leq x\sqrt{\log x^{-1}}$$

where$\log x^{-1} \geq 1$ is used.

Dudley's theorem now states that for all $U>0$ we have a version of the process which is almost surely continuous on $[0,U]$ with respect to the metric d. Moreover, it provides the following bound (van der Vaart and Wellner, 1997, Corollary 2.2.8) for all $a \geq 1$

$$\mathbb{E}\left\{\sup_{u\in[0,U_j]}|G_u|^a\right\} \lesssim (\log U_j)^{a/2}$$

Moreover,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{u\in[0,U_j]}|\tilde{\varphi}_{T_j}(u-i)-\varphi_{T_j}(u-i)| > 2^{j-1}\inf_{u\in[0,U_j]}\prod_{m=1}^{j}K(T_m-T_{m-1},\sigma_m,R,u)\right)\\
&\quad\leq \mathbb{E}\left\{\sup_{u\in[0,U_j]}|\tilde{\varphi}_{T_j}(u-i)-\varphi_{T_j}(u-i)|^2\right\}\cdot\left(2^{j-1}\inf_{u\in[0,U_j]}\prod_{m=1}^{j}K(T_m-T_{m-1},\sigma_m,R,u)\right)^{-2}\\
&\quad\leq \Delta_j\left(U_j^4+U_j^2\right)\mathbb{E}\left\{\sup_{u\in[0,U_j]}|G_u|^2\right\}\cdot 2^{2-2j}\prod_{m=1}^{j}\left(K(T_m-T_{m-1},\sigma_m,R,U_j)\right)^{-2}\\
&\quad\leq \Delta_j\left(U_j^4+U_j^2\right)\mathbb{E}\left\{\sup_{u\in[0,U_j]}|G_u|^2\right\}\cdot 4\exp\left(U_j^2\cdot\sum_{m=1}^{j}(T_m-T_{m-1})\sigma_m^2-4R\sum_{m=1}^{j}(T_m-T_{m-1})\right)\\
&\quad\lesssim \Delta_j U_j^4\log(U_j)\cdot\exp\left(U_j^2\cdot\sum_{m=1}^{j}(T_m-T_{m-1})\sigma_m^2\right)
\end{aligned}$$

The latter converges to 0 by assumption.

Now, we can conclude that the process is asymptotically well-defined since

$$\left|\varphi_{T_j}(u-i)\right| \geq 2\cdot\left(2^{j-1}\prod_{m=1}^{j}K(T_m-T_{m-1},\sigma_m,R,u)\right)$$

for all $u\in\mathbb{R}$. Moreover, the argument difference is uniformly bounded as stated in Corollary 1. For the details of the latter, it is referred to Lemma 2.

□

10.2 Proof of Theorem 1

Before it is possible to prove Theorem 1, it is necessary to invoke the following result from Belomestny and Reiß (2006a). It states how the estimator for $\mathcal{O}_j$ behaves.

Proposition 4. *Under the assumptions of Theorem 1, we obtain*

$$\sup_{u\in\mathbb{R}}\left|\mathbb{E}\left\{\mathcal{F}\tilde{\mathcal{O}}_j(u)-\mathcal{F}\mathcal{O}_j(u)\right\}\right| = \sup_{u\in\mathbb{R}}\left|\mathcal{F}\mathcal{O}^l_j-\mathcal{F}\mathcal{O}_j(u)\right| \lesssim \Delta_j^2$$

where $\mathcal{O}^l_j(u)$ is the linear approximation of $\mathcal{O}_j(u)$ such that at the given data points $x_{j,r}$, one has $\mathcal{O}^l_j(x_{j,r}) = \mathcal{O}_j(x_{j,r})$.

To prove the theorem, we start by rewriting $\sigma_j^2 - \hat{\sigma}_j^2$,

$$\begin{aligned}
\hat{\sigma}_j^2-\sigma_j^2 &= \int_{-U_j}^{U_j}\operatorname{Re}\left(\mathcal{F}\mu_j(u)\right)w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u + \int_{-U_j}^{U_j}\operatorname{Re}\left(\tilde{\psi}_j(u)-\psi_j(u)\right)w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u \\
&= \int_{-U_j}^{U_j}\operatorname{Re}\left(\mathcal{F}\mu_j(u)\right)w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u + \int_{-U_j}^{U_j}\operatorname{Re}\left(\tilde{\psi}^0_j(u)-\psi^0_j(u)\right)w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u \\
&\qquad - \int_{-U_j}^{U_j}\operatorname{Re}\left(\tilde{\psi}^1_j(u)-\psi^1_j(u)\right)w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u \\
&= \int_{-U_j}^{U_j}\operatorname{Re}(\mathcal{F}\mu_j(u))w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u + \int_{-U_j}^{U_j}\operatorname{Re}(\mathcal{L}^0_j(u))w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u + \int_{-U_j}^{U_j}\operatorname{Re}(\mathcal{R}^0_j(u))w_{\sigma_j}^{U_j}(u),\mathrm{d}u \\
&\qquad - \int_{-U_j}^{U_j}\operatorname{Re}(\mathcal{L}^1_j(u))w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u - \int_{-U}^{U}\operatorname{Re}(\mathcal{R}^1_j(u))w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u \\
&=: B_{\sigma^2} + L^{(0)}_{\sigma^2} + R^{(0)}_{\sigma^2} - L^{(1)}_{\sigma^2} - R^{(1)}_{\sigma^2}
\end{aligned} \tag{10.2}$$

with $\mathcal{L}^k_j$ the linear error term en $\mathcal{R}^k_j$ the remainder error term defined as

$$\mathcal{L}^k_j(u) = \frac{1}{T_j - T_{j-1}}\cdot\frac{\tilde{\varphi}_{T_{j-k}}(u-i)-\varphi_{T_{j-k}}(u-i)}{\varphi_{T_{j-k}}(u-i)} \quad\text{and}\quad \mathcal{R}^k_j(u) = \tilde{\psi}^k_j(u) - \psi^k_j(u) - \mathcal{L}^k_j(u)$$

for $k=0$ and $k=1$.

For readability, the proof of Theorem 1. is split up into the following 4 parts, all having their own subsection. In the first and second subsection, it is shown that the bias and the remainder error term, respectively, converge to 0. In the third subsection, the asymptotic variance of the linear error term is calculated and in the fourth, the last, subsection it is proven that the linear error term is asymptotically normally distributed.

10.2.1 The bias B_{σ^2}

Proposition 5.

$$|B_{\sigma^2}| \le U_j^{-(s+3)}\left\|\mu_j^{(s)}\right\|_\infty \left\|\mathcal{F}\left(\frac{w^1_{\sigma_j}(u)}{u^{s_j}}\right)\right\|_{L_1}$$

The difference with γ_j and λ_j will be the order of convergence. Instead of $U^{-(s_j+3)}$, they will have the terms $U^{-(s_j+2)}$ and, respectively, $U^{-(s_j+1)}$. The proof changes only at the point where the definition of $w^U_\xi(u)$ in terms of $w^1_\xi(u)$ is inserted for $\xi\in\{\sigma,\gamma,\lambda\}$.

Proof. We will use $(iu)^{s_j}\mathcal{F}\mu_j(u) = \mathcal{F}\mu_j^{(s_j)}(u)$ and Plancherel's isometry which states

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,\mathrm{d}\xi,$$

where

$$\hat{\hat{f}}(\xi) := \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi i\xi x}\,\mathrm{d}x$$

to find a bound on the first term. We use, however,

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{ivx}\,\mathrm{d}x$$

Thus, Plancherel's isometry changes to

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\,\mathrm{d}x = \int_{-\infty}^{\infty} \hat{\hat{f}}(\xi)\overline{\hat{\hat{g}}(\xi)}\,\mathrm{d}\xi = \int_{-\infty}^{\infty} \hat{f}(-2\pi\xi)\overline{\hat{g}(-2\pi\xi)}\,\mathrm{d}\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\zeta)\overline{\hat{g}(\zeta)}\,\mathrm{d}\zeta$$

We get now for the first integral

$$\begin{aligned}
\left|\int_{-U_j}^{U_j} \mathcal{F}\mu_j(u)w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u\right| &= \left|\int_{-\infty}^{\infty} \mathcal{F}\mu_j^{(s)}(u)\cdot\frac{w_{\sigma_j}^{U_j}(u)}{(iu)^s}\,\mathrm{d}u\right| \\
&= \left|\int_{-\infty}^{\infty} \mathcal{F}\mu_j^{(s)}(u)\cdot\overline{\left(\frac{w_{\sigma_j}^{U_j}(u)}{(-iu)^s}\right)}\,\mathrm{d}u\right| \\
&= 2\pi\left|\int_{-\infty}^{\infty} \mu_j^{(s)}(x)\overline{\mathcal{F}^{-1}\left(\frac{w_{\sigma_j}^{U_j}(u)}{(-iu)^s}\right)(x)}\,\mathrm{d}x\right| \\
&\le 2\pi\int_{-\infty}^{\infty}\left|\mu_j^{(s)}(x)\right|\left|\overline{\mathcal{F}^{-1}\left(\frac{w_{\sigma_j}^{U_j}(u)}{(-iu)^s}\right)(x)}\right|\mathrm{d}x \\
&\le 2\pi\left\|\mu_j^{(s)}\right\|_\infty\int_{-\infty}^{\infty}\left|\mathcal{F}^{-1}\left(\frac{w_{\sigma_j}^{U_j}(u)}{(-iu)^s}\right)(x)\right|\mathrm{d}x \\
&\le 2\pi\left\|\mu_j^{(s)}\right\|_\infty\left\|\frac{1}{2\pi}\cdot\mathcal{F}\left(\frac{w_{\sigma_j}^{U_j}(u)}{(-iu)^s}\right)(x)\right\|_{L^1} \\
&= \left\|\mu_j^{(s)}\right\|_\infty\left\|\mathcal{F}\left(\frac{w_{\sigma_j}^{U_j}(u)}{(-iu)^s}\right)(x)\right\|_{L^1} \\
&= \left\|\mu_j^{(s)}\right\|_\infty\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{w_{\sigma_j}^{U_j}(u)}{(-iu)^s}\cdot\mathrm{e}^{i\xi u}\,\mathrm{d}u\right|\mathrm{d}\xi \\
&= \left\|\mu_j^{(s)}\right\|_\infty\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{U_j^{-3}w_{\sigma_j}^{1}(u/U_j)}{(-iu)^s}\cdot\mathrm{e}^{i\xi u}\,\mathrm{d}u\right|\mathrm{d}\xi \\
&= U_j^{-3}\cdot\left\|\mu_j^{(s_j)}\right\|_\infty\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{w_{\sigma_j}^{1}(\tilde{u})}{(\tilde{u}U_j)^s}\cdot\mathrm{e}^{i\xi\tilde{u}U_j}\cdot U_j\,\mathrm{d}\tilde{u}\right|\mathrm{d}\xi \\
&= U_j^{-2-s}\cdot\left\|\mu_j^{(s)}\right\|_\infty\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{w_{\sigma_j}^{1}(\tilde{u})}{\tilde{u}^s}\cdot\mathrm{e}^{i\xi\tilde{u}U_j}\,\mathrm{d}\tilde{u}\right|\mathrm{d}\xi \\
&= U_j^{-2-s}\cdot\left\|\mu_j^{(s)}\right\|_\infty\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty}\frac{w_{\sigma_j}^{1}(\tilde{u})}{\tilde{u}^s}\cdot\mathrm{e}^{i\tilde{\xi}\tilde{u}}\,\mathrm{d}\tilde{u}\right|\cdot\frac{1}{U_j}\,\mathrm{d}\tilde{\xi} \\
&= U_j^{-(s+3)}\cdot\left\|\mu_j^{(s)}\right\|_\infty\int_{-\infty}^{\infty}\left|\mathcal{F}\left(\frac{w_{\sigma_j}^{1}(\tilde{u})}{\tilde{u}^{s_j}}\right)(\tilde{\xi})\right|\mathrm{d}\tilde{\xi} \\
&= U_j^{-(s+3)}\left\|\mu_j^{(s)}\right\|_\infty\left\|\mathcal{F}\left(\frac{w_{\sigma_j}^{1}(u)}{u^s}\right)\right\|_{L_1}
\end{aligned}$$

□

10.2.2 The remainder $R^{(k)}_{\sigma^2}$

Proposition 6. *Let*

$$\Delta_j \cdot U_j^4 \exp\left(U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2\right) \to 0$$

and

$$\frac{\Delta_{j-1}^2}{\Delta_j} \cdot U_j^4 \exp\left(U_j^2 \left(\left(\sum_{i=1}^{j-1} (T_i - T_{i-1})\sigma_i^2\right) - (T_j - T_{j-1})\sigma_j^2\right)\right) \to 0.$$

Then,

$$\frac{U_j^2 \exp\left(-U_j^2 \cdot \sum_{i=1}^{j}(T_i - T_{i-1})\sigma_i^2/2\right) \cdot R^{(k)}_{\sigma^2}}{d_{j,j}\sqrt{\Delta_j} + d_{j,j-1}\sqrt{\Delta_{j-1}} \exp\left(-U_j^2 \cdot (T_j - T_{j-1})\sigma_j^2/2\right)} \xrightarrow{\mathcal{P}} 0$$

We have shown in (3.9) that

$$\left|\varphi_{T_j}(u - i)\right| \geq \prod_{m=1}^{j} (2K(T_m - T_{m-1}, \sigma_m, R, u) =: 2K_j(u).$$

For the remainder term the following lemma is proven first.

Lemma 1. *For all $u \in \mathbb{R}$ the remainder term satisfies with probability tending to* 1

$$|\mathcal{R}_j^k(u)| \leq \frac{1}{2}(T_j - T_{j-1})^{-1} K_{j-k}(u)^{-2} \left(u^4 + u^2\right) \left|\mathcal{F}(\mathcal{O}_{j-k} - \tilde{\mathcal{O}}_{j-k})(u)\right|^2$$

Proof (Lemma). Recall the definition of $\tilde{\psi}_j^k$

$$\tilde{\psi}_j^k(u) := \frac{1}{T_j - T_{j-1}} \log_{\geq \kappa(u, T_{j-k})} \left(\tilde{\varphi}_{T_{j-k}}(u - i)\right)$$

with $C_j(u) = 2^{j-1}\kappa^j(u)$ and note that

$$(T_j - T_{j-1})\tilde{\psi}_j^k(u) = \log\left(\mathrm{e}^{(T_j - T_{j-1})\tilde{\psi}_j^k(u)}\right).$$

Hence, we can write the remainder term as follows, where the idea is to write it such that we can use a second order Taylor expansion of the logarithm.

$$\begin{aligned}
(T_j - T_{j-1})|\mathcal{R}_j^k(u)| &= \left|(T_j - T_{j-1})\tilde{\psi}_j^k(u) - (T_j - T_{j-1})\psi_j^k(u) - (T_j - T_{j-1})\mathcal{L}_j^k(u)\right| \\
&= \left|\log\left(\mathrm{e}^{(T_j - T_{j-1})\tilde{\psi}_j^k(u)}\right) - \log\left(\mathrm{e}^{(T_j - T_{j-1})\psi_j^k(u)}\right) - \frac{\tilde{\varphi}_{T_{j-k}}(u-i) - \varphi_{T_{j-k}}(u-i)}{\varphi_{T_{j-k}}(u-i)}\right| \\
&\leq \left|\log\left(\mathrm{e}^{(T_j - T_{j-1})\tilde{\psi}_j^k(u)}\right) - \log\left(\mathrm{e}^{(T_j - T_{j-1})\psi_j^k(u)}\right) - \frac{\mathrm{e}^{(T_j - T_{j-1})\tilde{\psi}_j^k(u)} - \varphi_{T_{j-k}}(u-i)}{\varphi_{T_{j-k}}(u-i)}\right| \\
&\quad + \left|\frac{\tilde{\varphi}_{T_{j-k}}(u-i) - \mathrm{e}^{(T_j - T_{j-1})\tilde{\psi}_j^k(u)}}{\varphi_{T_{j-k}}(u-i)}\right| \\
&= \left|\log\left(\mathrm{e}^{(T_j - T_{j-1})\tilde{\psi}_j^k(u)}\right) - \log\left(\mathrm{e}^{(T_j - T_{j-1})\psi_j^k(u)}\right) - \frac{\mathrm{e}^{(T_j - T_{j-1})\tilde{\psi}_j^k(u)} - \mathrm{e}^{(T_j - T_{j-1})\psi_j^k(u)}}{\mathrm{e}^{(T_j - T_{j-1})\psi_j^k(u)}}\right| \\
&\quad + \left|\varphi_{T_{j-k}}(u-i)^{-1}\left(\tilde{\varphi}_{T_{j-k}}(u-i) - \mathrm{e}^{(T_j - T_{j-1})\tilde{\psi}_j^k(u)}\right)\right|
\end{aligned}$$

A direct consequence of Proposition 3. is

$$\lim_{U \to \infty} \mathbb{P}\left(\left|\tilde{\varphi}_{T_{j-k}}(u-i)\right| \geq K_{j-k}(u)\right) = 1 \tag{10.3}$$

This implies that

$$\lim_{U\to\infty} \mathbb{P}\left(\mathrm{e}^{(T_j-T_{j-1})\tilde{\psi}_j^k(u)} = \tilde{\varphi}_{T_{j-k}}(u-i)\right) = 1 \tag{10.4}$$

since $K_j(u) \geq C_j(u)$. Thus, the second term will be zero with probability tending to one.

In (3.8) it is shown that $|\exp((T_j - T_{j-1})\psi_j^k(u))| \geq 2K_{j-k}(u)$ and from (10.3) and (10.4) we find that with probability tending to one, $|\exp((T_j - T_{j-1})\tilde{\psi}_j^k(u))| = |\tilde{\varphi}_{T_{j-k}}(u-i)| \geq K_{j-k}(u)$. Hence, we can apply Lemma 5 with probability tending to one to bound the first term and complete the proof

$$\begin{aligned}
&\left|\log\left(\mathrm{e}^{(T_j-T_{j-1})\tilde{\psi}_j^k(u)}\right) - \log\left(\mathrm{e}^{(T_j-T_{j-1})\psi_j^k(u)}\right) - \mathrm{e}^{-(T_j-T_{j-1})\psi_j^k(u)}\left(\mathrm{e}^{(T_j-T_{j-1})\tilde{\psi}_j^k(u)} - \mathrm{e}^{(T_j-T_{j-1})\psi_j^k(u)}\right)\right| \\
&\qquad \leq \frac{1}{2}K_{j-k}(u)^{-2}\left|\mathrm{e}^{(T_j-T_{j-1})\tilde{\psi}_j^k(u)} - \mathrm{e}^{(T_j-T_{j-1})\psi_j^k(u)}\right|^2 \\
&\qquad = \frac{1}{2}K_{j-k}(u)^{-2}\left(u^4+u^2\right)\left|\mathcal{F}(\mathcal{O}_{j-k} - \tilde{\mathcal{O}}_{j-k})(u)\right|^2.
\end{aligned}$$

□

Proof (Proposition). From the Lemma, we find

$$\begin{aligned}
\mathbb{E}\left\{\left|\int_{-U_j}^{U_j} \mathcal{R}_j^k(u) w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u\right|^2\right\} &\leq \mathbb{E}\left\{\left(\int_{-U_j}^{U_j} |\mathcal{R}_j^k(u)||w_{\sigma_j}^{U_j}(u)|\,\mathrm{d}u\right)^2\right\} \\
&\lesssim \mathbb{E}\left\{\left(\int_{-U_j}^{U_j} K_{j-k}(u)^{-2}\left(u^4+u^2\right)\left|\mathcal{F}(\tilde{\mathcal{O}}_{j-k} - \mathcal{O}_{j-k})(u)\right|^2 |w_{\sigma_j}^{U_j}(u)|\,\mathrm{d}u\right)^2\right\} \\
&= \mathbb{E}\bigg\{\int_{-U_j}^{U_j}\int_{-U_j}^{U_j} K_{j-k}(u)^{-2}K_{j-k}(v)^{-2}\left(u^4+u^2\right)\left(v^4+v^2\right) \\
&\qquad \cdot \left|\mathcal{F}(\tilde{\mathcal{O}}_{j-k} - \mathcal{O}_{j-k})(u)\right|^2 \left|\mathcal{F}(\tilde{\mathcal{O}}_{j-k} - \mathcal{O}_{j-k})(v)\right|^2 |w_{\sigma_j}^{U_j}(u)||w_{\sigma_j}^{U_j}(v)|\,\mathrm{d}u\,\mathrm{d}v\bigg\} \\
&= \int_{-U_j}^{U_j}\int_{-U_j}^{U_j} K_{j-k}(u)^{-2}\kappa_j(v)^{-2}\left(u^4+u^2\right)\left(v^4+v^2\right) \\
&\qquad \cdot \mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_{j-k} - \mathcal{O}_{j-k})(u)\mathcal{F}(\tilde{\mathcal{O}}_{j-k} - \mathcal{O}_{j-k})(v)\right|^2\right\} |w_{\sigma_j}^{U_j}(u)||w_{\sigma_j}^{U_j}(v)|\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

Recall the definition $\mathcal{O}_{m,l}(x) := \mathbb{E}\{\tilde{\mathcal{O}}_m(x)\}$. We are going to further look into the expected value with $m = j-k$

$$\begin{aligned}
&\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_m)(u)\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_m)(v)\right|^2\right\} \\
&\quad = \mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l} + \mathcal{O}_{m,l} - \mathcal{O}_m)(u)\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l} + \mathcal{O}_{m,l} - \mathcal{O}_m)(v)\right|^2\right\} \\
&\quad \leq 4\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(u)\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(v)\right|^2\right\} + 4\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(u)\mathcal{F}(\mathcal{O}_{m,l} - \mathcal{O}_m)(v)\right|^2\right\} \\
&\qquad + 4\mathbb{E}\left\{\left|\mathcal{F}(\mathcal{O}_{m,l} - \mathcal{O}_m)(u)\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(v)\right|^2\right\} + 4\mathbb{E}\left\{|\mathcal{F}(\mathcal{O}_{m,l} - \mathcal{O}_m)(u)\mathcal{F}(\mathcal{O}_{m,l} - \mathcal{O}_m)(v)|^2\right\} \\
&\quad \leq 4\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(u)\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(v)\right|^2\right\} + 4\left|\mathcal{F}(\mathcal{O}_{m,l} - \mathcal{O}_m)(v)\right|^2 \mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(u)\right|^2\right\} \\
&\qquad + 4\left|\mathcal{F}(\mathcal{O}_{m,l} - \mathcal{O}_m)(u)\right|^2 \mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(v)\right|^2\right\} + 4\left|\mathcal{F}(\mathcal{O}_{m,l} - \mathcal{O}_m)(u)\mathcal{F}(\mathcal{O}_{m,l} - \mathcal{O}_m)(v)\right|^2 \\
&\quad \leq 4\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(u)\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(v)\right|^2\right\} + 4\left\|\mathcal{F}(\mathcal{O}_{m,l} - \mathcal{O}_m)\right\|_\infty^2 \mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m - \mathcal{O}_{m,l})(u)\right|^2\right\}
\end{aligned}$$

$$+ 4\left\|\mathcal{F}(\mathcal{O}_{m,l}-\mathcal{O}_m)\right\|_\infty^2 \mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(v)\right|^2\right\} + 4\left\|\mathcal{F}(\mathcal{O}_{m,l}-\mathcal{O}_m)\right\|_\infty^4$$

Hence, the second and third terms become a constant times the variance of $\mathcal{F}\tilde{\mathcal{O}}_m$. This, in particular, can also be estimated

$$\begin{aligned}
\operatorname{Var}\left[\mathcal{F}\tilde{\mathcal{O}}_m(u)\right] &= \operatorname{Var}\left[\mathcal{F}\left(\beta_{0,m}(x)+\sum_{r=1}^N O_{m,r}b_{m,r}(x)\right)(u)\right] = \operatorname{Var}\left[\mathcal{F}\beta_{0,m}(u)+\sum_{r=1}^N O_{m,r}\mathcal{F}b_{m,r}(u)\right]\\
&= \operatorname{Var}\left[\sum_{r=1}^N O_{m,r}\mathcal{F}b_{m,r}(u)\right] = \operatorname{Var}\left[\sum_{r=1}^N (\mathcal{O}_m(x_r)+\delta_{m,r}\varepsilon_{m,r})\mathcal{F}b_{jk}(u)\right]\\
&= \operatorname{Var}\left[\sum_{r=1}^N \delta_{m,r}\varepsilon_{m,r}\mathcal{F}b_{m,r}(u)\right] = \sum_{r=1}^N |\delta_{m,r}\mathcal{F}b_{m,r}(u)|^2 \operatorname{Var}\left[\varepsilon_{m,r}\right]\\
&= \sum_{r=1}^N |\delta_{m,r}\mathcal{F}b_{m,r}(u)|^2 \leq \sum_{r=1}^N |\delta_{m,r}|^2 \left\|\mathcal{F}b_{m,r}\right\|_\infty^2 \leq \sum_{r=1}^N |\delta_{m,r}|^2 \left\|b_{m,r}\right\|_{L^1}^2\\
&\leq \Delta_m^2 \sum_{k=1}^N |\delta_{m,r}|^2 = \Delta_m^2 \left\|\delta_m\right\|_{l^2}^2 \lesssim \Delta_m \left\|\delta_m\right\|_\infty^2
\end{aligned}$$

It follows that

$$\begin{aligned}
&\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_m)(u)\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_m)(v)\right|^2\right\}\\
&\qquad\lesssim \mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(u)\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(v)\right|^2\right\} + \left\|\delta_m\right\|_\infty^2 \left\|\mathcal{F}(\mathcal{O}_{m,l}-\mathcal{O}_m)\right\|_\infty^2\\
&\qquad\quad + \left\|\delta_m\right\|_\infty^2 \left\|\mathcal{F}(\mathcal{O}_{m,l}-\mathcal{O}_m)\right\|_\infty^2 + \left\|\mathcal{F}(\mathcal{O}_{m,l}-\mathcal{O}_m)\right\|_\infty^4\\
&\qquad\lesssim \mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(u)\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(v)\right|^2\right\} + \left\|\delta_m\right\|_\infty^2 \cdot \Delta_m^4 + \Delta_m^8.
\end{aligned}$$

Since the behavior of the integrand is interesting when U_j tends to infinity, the lower order u^2 and v^2 are left outside the equation. It could be taken into account with these terms. In the end, however, it will turn out that these terms are negligible. To avoid long negligible expressions, these terms are left out of the equation.

$$\begin{aligned}
\mathbb{E}\left\{\left|\int_{-U_j}^{U_j} \mathcal{R}_j^k(u) w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u\right|^2\right\} &\lesssim \int_{-U_j}^{U_j}\int_{-U_j}^{U_j} K_m(u)^{-2}K_m(v)^{-2}\left(u^4+u^2\right)\left(v^4+v^2\right)\\
&\qquad\cdot\left(\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(u)\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(v)\right|^2\right\}+\right.\\
&\qquad\left.\left\|\delta_m\right\|_\infty^2\cdot\Delta_m^4+\Delta_m^8\right)|w_{\sigma_j}^{U_j}(u)||w_{\sigma_j}^{U_j}(v)|\,\mathrm{d}u\,\mathrm{d}v\\
&\lesssim \int_{-U_j}^{U_j}\int_{-U_j}^{U_j} K_m(u)^{-2}K_m(v)^{-2}u^4v^4\\
&\qquad\cdot\left(\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(u)\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(v)\right|^2\right\}+\right.\\
&\qquad\left.\left\|\delta_m\right\|_\infty^2\cdot\Delta_m^4+\Delta_m^8\right)|w_{\sigma_j}^{U_j}(u)||w_{\sigma_j}^{U_j}(v)|\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

The next step is to simplify the expectation in the integral.

$$\begin{aligned}
&\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(u)\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(v)\right|^2\right\}\\
&\quad=\mathbb{E}\left\{\left|\mathcal{F}\left(\beta_{0,m}(x)+\sum_{r=1}^N O_{m,r}b_{m,r}(x)-\beta_{0,m}(x)-\sum_{r=1}^N \mathcal{O}_m(x_r)b_{m,r}(x)\right)(u)\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(v)\right|^2\right\}\\
&\quad=\mathbb{E}\left\{\left|\mathcal{F}\left(\sum_{r=1}^N(O_{m,r}-\mathcal{O}_m(x_{m,r}))b_{m,r}(x)\right)(u)\mathcal{F}\left(\sum_{s=1}^N(O_{m,s}-\mathcal{O}_m(x_{m,s}))b_{m,s}(x)\right)(v)\right|^2\right\}\\
&\quad=\mathbb{E}\left\{\left|\mathcal{F}\left(\sum_{r=1}^N\delta_{m,r}\varepsilon_{m,r}b_{m,r}(x)\right)(u)\mathcal{F}\left(\sum_{s=1}^N\delta_{m,s}\varepsilon_{m,s}b_{m,s}(x)\right)(v)\right|^2\right\}\\
&\quad=\mathbb{E}\left\{\left|\sum_{r=1}^N\delta_{m,r}\varepsilon_{m,r}\mathcal{F}b_{m,r}(u)\sum_{s=1}^N\delta_{m,s}\varepsilon_{m,s}\mathcal{F}b_{m,s}(v)\right|^2\right\}\\
&\quad=\mathbb{E}\left\{\left|\sum_{r=1}^N\sum_{s=1}^N\delta_{m,r}\delta_{m,s}\varepsilon_{m,r}\varepsilon_{m,s}\mathcal{F}b_{m,r}(u)\mathcal{F}b_{m,s}(v)\right|^2\right\}\\
&\quad=\mathbb{E}\left\{\sum_{r=1}^N\sum_{s=1}^N\delta_{m,r}^2\delta_{m,s}^2\varepsilon_{m,r}^2\varepsilon_{m,s}^2|\mathcal{F}b_{m,r}(u)|^2|\mathcal{F}b_{m,s}(v)|^2\right\}\\
&\quad=\sum_{r=1}^N\sum_{s=1}^N\delta_{m,r}^2\delta_{m,s}^2\mathbb{E}\left\{\varepsilon_{m,r}^2\varepsilon_{m,s}^2\right\}|\mathcal{F}b_{m,r}(u)|^2|\mathcal{F}b_{m,s}(v)|^2
\end{aligned}$$

which simplifies further to

$$\begin{aligned}
&\mathbb{E}\left\{\left|\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(u)\mathcal{F}(\tilde{\mathcal{O}}_m-\mathcal{O}_{m,l})(v)\right|^2\right\}\\
&\quad=\sum_{r=1}^N\delta_{m,r}^4\mathbb{E}\left\{\varepsilon_{m,r}^4\right\}|\mathcal{F}b_{m,r}(u)|^2|\mathcal{F}b_{m,r}(v)|^2\\
&\qquad\quad+\sum_{r=1}^N\sum_{s=1,s\neq r}^N\delta_{m,r}^2\delta_{m,s}^2\mathbb{E}\left\{\varepsilon_{m,r}^2\right\}\mathbb{E}\left\{\varepsilon_{m,s}^2\right\}|\mathcal{F}b_{m,r}(u)|^2|\mathcal{F}b_{m,s}(v)|^2\\
&\quad\lesssim\sum_{r=1}^N\delta_{m,r}^4|\mathcal{F}b_{m,r}(u)|^2|\mathcal{F}b_{m,r}(v)|^2+\sum_{r=1}^N\sum_{s=1,s\neq r}^N\delta_{m,r}^2\delta_{m,s}^2|\mathcal{F}b_{m,r}(u)|^2|\mathcal{F}b_{m,s}(v)|^2\\
&\quad=\sum_{r=1}^N\sum_{s=1}^N\delta_{m,r}^2\delta_{m,s}^2|\mathcal{F}b_{m,r}(u)|^2|\mathcal{F}b_{m,s}(v)|^2=\left(\sum_{r=1}^N\delta_{m,r}^2|\mathcal{F}b_{m,r}(u)|^2\right)\cdot\left(\sum_{r=1}^N\delta_{m,r}^2|\mathcal{F}b_{m,r}(v)|^2\right),
\end{aligned}$$

where we used the finiteness of the fourth moments of $\varepsilon_{m,r}$. We obtain that the integral is bounded by

$$\begin{aligned}
&\mathbb{E}\left\{\left|\int_{-U_j}^{U_j}\mathcal{R}_j^k(u)w_{\sigma_j}^{U_j}(u)\,\mathrm{d}u\right|^2\right\}\\
&\lesssim\int_{-U_j}^{U_j}\int_{-U_j}^{U_j}K_m(u)^{-2}K_m(v)^{-2}u^4v^4\cdot\left(\left(\sum_{r=1}^N\delta_{m,r}^2|\mathcal{F}b_{m,r}(u)|^2\right)\cdot\left(\sum_{r=1}^N\delta_{m,r}^2|\mathcal{F}b_{m,r}(v)|^2\right)\right.\\
&\qquad\qquad\left.+\|\delta_m\|_\infty^2\cdot\Delta_m^4+\Delta_m^8\right)|w_{\sigma_j}^{U_j}(u)||w_{\sigma_j}^{U_j}(v)|\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}$$

$$= \left(\int_{-U_j}^{U_j} K_m(u)^{-2} u^4 \sum_{r=1}^{N} \delta_{m,r}^2 |\mathcal{F} b_{m,r}(u)|^2 \cdot |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u \right)^2 + \left(\|\delta_m\|_\infty^2 \cdot \Delta_m^4 + \Delta_m^8 \right) \left(\int_{-U_j}^{U_j} K_m(u)^{-2} u^4 |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u \right)^2 .$$

Note that the first integral can be bounded in terms of the second one

$$\begin{aligned} &\int_{-U_j}^{U_j} K_m(u)^{-2} u^4 \sum_{r=1}^{N} \delta_{m,r}^2 |\mathcal{F} b_{m,r}(u)|^2 \cdot |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u \\ &\qquad \le \int_{-U_j}^{U_j} K_m(u)^{-2} u^4 \sum_{r=1}^{N} \delta_{m,r}^2 \, \|\mathcal{F} b_{m,r}\|_\infty^2 \cdot |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u \\ &\qquad \lesssim \Delta_m^2 \int_{-U_j}^{U_j} K_m(u)^{-2} u^4 \sum_{r=1}^{N} \delta_{m,r}^2 \cdot |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u \\ &\qquad \lesssim \Delta_m^2 \, \|\delta_m\|_{l^2}^2 \int_{-U_j}^{U_j} K_m(u)^{-2} u^4 |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u \\ &\qquad \lesssim \Delta_m \, \|\delta_m\|_\infty^2 \int_{-U_j}^{U_j} K_m(u)^{-2} u^4 |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u. \end{aligned}$$

Hence, we get with Lemma 6 the following bound for the remainder term

$$\begin{aligned} &\mathbb{E} \left\{ \left| \int_{-U_j}^{U^j} \mathcal{R}_j^k(u) w_{\sigma_j}^{U_j}(u) \, \mathrm{d}u \right|^2 \right\} \\ &\qquad \lesssim \left(\|\delta_m\|_\infty^4 \cdot \Delta_m^2 + \left(\|\delta_m\|_\infty^2 \cdot \Delta_m^4 + \Delta_m^8 \right) \right) \left(\int_{-U_j}^{U_j} K_m(u)^{-2} u^4 |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u \right)^2 \\ &\qquad \lesssim \left(\|\delta_m\|_\infty^4 \cdot \Delta_m^2 + \|\delta_m\|_\infty^2 \cdot \Delta_m^4 + \Delta_m^8 \right) \cdot \exp\left(2U_j^2 \cdot \sum_{i=1}^{m} (T_i - T_{i-1}) \sigma_i^2 \right). \end{aligned}$$

We can now prove convergence in probability of this term, i.e. for all $\varepsilon > 0$

$$\begin{aligned} &\mathbb{P} \left(\frac{U_j^2 \exp\left(-U_j^2 \cdot \sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2 / 2 \right)}{d_{j,j} \sqrt{\Delta_j} + d_{j,j-1} \sqrt{\Delta_{j-1}} \exp\left(-U_j^2 \cdot (T_j - T_{j-1}) \sigma_j^2 / 2 \right)} \cdot \left| \int_{-U_j}^{U_j} \mathrm{Re}\left(\mathcal{R}_j^k(u) \right) w_{\sigma_j}^{U_j}(u) \, \mathrm{d}u \right| > \varepsilon \right) \\ &\qquad \le \mathbb{P} \left(\frac{U_j^2 \exp\left(-U_j^2 \cdot \sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2 / 2 \right)}{d_{j,j} \sqrt{\Delta_j} + d_{j,j-1} \sqrt{\Delta_{j-1}} \exp\left(-U_j^2 \cdot (T_j - T_{j-1}) \sigma_j^2 / 2 \right)} \cdot \left| \int_{-U_j}^{U_j} \mathcal{R}_j^k(u) w_{\sigma_j}^{U_j}(u) \, \mathrm{d}u \right| > \varepsilon \right) \\ &\qquad \lesssim \frac{1}{\varepsilon^2} \cdot \frac{\|\delta_m\|_\infty^4 \cdot \Delta_m^2 \cdot \exp\left(2U_j^2 \sum_{i=1}^{m} (T_i - T_{i-1}) \sigma_i^2 \right) \cdot U_j^4 \exp\left(-U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2 \right)}{\left(d_{j,j} \sqrt{\Delta_j} + d_{j,j-1} \sqrt{\Delta_{j-1}} \exp\left(-U_j^2 \cdot (T_j - T_{j-1}) \sigma_j^2 / 2 \right) \right)^2} \\ &\qquad \le \frac{1}{\varepsilon^2} \cdot \frac{\|\delta_m\|_\infty^4 \cdot \Delta_m^2 \cdot \exp\left(2U_j^2 \sum_{i=1}^{m} (T_i - T_{i-1}) \sigma_i^2 \right) \cdot U_j^4 \exp\left(-U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2 \right)}{d_{j,j}^2 \Delta_j + d_{j,j-1}^2 \Delta_{j-1} \exp\left(-U_j^2 \cdot (T_j - T_{j-1}) \sigma_j^2 \right)} \\ &\qquad \le \frac{\|\delta_m\|_\infty^4}{\varepsilon^2 \cdot d_{j,j}^2} \cdot \frac{\Delta_m^2}{\Delta_j} \cdot U_j^4 \cdot \exp\left(2U_j^2 \sum_{i=1}^{m} (T_i - T_{i-1}) \sigma_i^2 - U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2 \right) =: p_{j,m}. \end{aligned}$$

For $k = 0$, i.e., $m = j$, we have

$$p_{j,j} = \frac{\|\delta_j\|_\infty^4}{\varepsilon^2 \cdot d_{j,j}^2} \cdot \Delta_j \cdot U_j^4 \cdot \exp\left(U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2 \right) \to 0$$

and for $k = 1$, i.e., $m = j - 1$, we have

$$p_{j,j-1} = \frac{\|\delta_{j-1}\|_\infty^4}{\varepsilon^2 \cdot d_{j,j}^2} \cdot \frac{\Delta_{j-1}^2}{\Delta_j} U_j^4 \exp\left(U_j^2 \left(\left(\sum_{i=1}^{j-1} (T_i - T_{i-1})\sigma_i^2 \right) - (T_j - T_{j-1})\sigma_j^2 \right) \right) \to 0$$

Proving convergence to 0 in probability of the remainder term. □

The differences in the proof with γ_j and λ_j occur again only at a single point. In applying Lemma 6 we get extra U_j terms, but in the normalizing constant exactly these terms are missing such that the product will still be the same. Moreover, note that we have constructed a proof for σ for the absolute value of the remainder. Hence, this proof works for both the real part of the remainder as well as the imaginary part of the remainder.

10.2.3 Asymptotic variance of the linear term $L_{\sigma^2}^{(k)}$

In this subsection, the asymptotic variance of $L_{\sigma^2}^{(k)}$ is calculated.

Proposition 7. *As U_j to infinity,*

$$\frac{\operatorname{Var}\left(L_{\sigma^2}^{(k)}\right)}{w_{\sigma_j}^1(1)^2} \sim U_j^{-4} \cdot \left(d_{j,j} \cdot \Delta_j \exp\left(\sum_{r=1}^{j} (T_r - T_{r-1})\sigma_r^2 U_j^2 \right) + d_{j,j-1} \cdot \Delta_{j-1} \exp\left(\sum_{r=1}^{j-1} (T_r - T_{r-1})\sigma_r^2 U_j^2 \right) \right)$$

where $a \sim b$ if $\lim ab^{-1} = 1$.

Proof. Recall

$$\mathcal{L}_j^k(u) = \frac{1}{T_j - T_{j-1}} \cdot \frac{\tilde{\varphi}_{T_{j-k}}(u-i) - \varphi_{T_{j-k}}(u-i)}{\varphi_{T_{j-k}}(u-i)}$$

and

$$L_{\sigma^2}^{(k)} = \int_{-U_j}^{U_j} \operatorname{Re}(\mathcal{L}_j^k(u)) w_{\sigma_j}^{U_j}(u)\, \mathrm{d}u.$$

Working this out yields

$$\begin{aligned}
L_{\sigma^2}^{(k)} &= \int_{-U_j}^{U_j} \operatorname{Re}(\mathcal{L}_j^k(u)) w_{\sigma_j}^{U_j}(u)\, \mathrm{d}u = U_j \int_{-1}^{1} \operatorname{Re}(\mathcal{L}_j^k(uU_j)) w_{\sigma_j}^{U_j}(uU_j)\, \mathrm{d}u = U_j^{-2} \int_{-1}^{1} \operatorname{Re}\left(\mathcal{L}_j^k(uU_j)\right) w_{\sigma_j}^1(u)\, \mathrm{d}u \\
&= U_j^{-2} \int_0^1 \operatorname{Re}\left(\mathcal{L}_j^k(uU_j)\right) w_{\sigma_j}^1(u)\, \mathrm{d}u + U_j^{-2} \int_0^1 \operatorname{Re}\left(\mathcal{L}_j^k(-uU_j)\right) w_{\sigma_j}^1(-u)\, \mathrm{d}u \\
&= U_j^{-2} \int_0^1 \operatorname{Re}\left(\mathcal{L}_j^k(uU_j)\right) w_{\sigma_j}^1(u)\, \mathrm{d}u + U_j^{-2} \int_0^1 \operatorname{Re}\left(\overline{\mathcal{L}_j^k(uU_j)}\right) w_{\sigma_j}^1(u)\, \mathrm{d}u \\
&= 2U_j^{-2} \int_0^1 \operatorname{Re}\left(\mathcal{L}_j^k(uU_j)\right) w_{\sigma_j}^1(u)\, \mathrm{d}u \\
&= 2U_j^{-2} \int_0^1 \operatorname{Re}\left(\frac{1}{T_j - T_{j-1}} \cdot \frac{\tilde{\varphi}_{T_{j-k}}(uU_j - i) - \varphi_{T_{j-k}}(uU_j - i)}{\varphi_{T_{j-k}}(uU_j - i)} \right) w_{\sigma_j}^1(u)\, \mathrm{d}u \\
&= \frac{2U_j^{-2}}{T_j - T_{j-1}} \int_0^1 \operatorname{Re}\left(\frac{iuU_j(1 + iuU_j)\mathcal{F}\left(\tilde{\mathcal{O}}_{j-k} - \mathcal{O}_{j-k}\right)(uU_j)}{\varphi_{T_{j-k}}(uU_j - i)} \right) w_{\sigma_j}^1(u)\, \mathrm{d}u.
\end{aligned}$$

We will now use the following formula with $m = j - k$

$$\begin{aligned}\varphi_{T_m}(u-i) = \exp\Bigg(&\frac{-u^2}{2}\cdot\left(\sum_{r=1}^{m}(T_r - T_{r-1})\sigma_r^2\right) + iu\cdot\left(\sum_{r=1}^{m}(T_r - T_{r-1})\left(\sigma_r^2+\gamma_r\right)\right) \\ &+\left(\sum_{r=1}^{m}(T_r - T_{r-1})\left(\frac{\sigma_r^2}{2}+\gamma_r-\lambda_r\right)\right) + \left(\sum_{r=1}^{m}(T_r - T_{r-1})\mathcal{F}\mu_r(u)\right)\Bigg) \\ =: \exp&\left(-\frac{u^2}{2}\cdot A_m + iu\cdot B_m + C_m + D_m(u)\right).\end{aligned}$$

Inserting this into the equation yields

$$\begin{aligned}L_{\sigma^2}^{(k)} &= \frac{2U_j^{-2}}{T_j - T_{j-1}}\operatorname{Re}\left(\int_0^1 \frac{iuU_j(1+iuU_j)\mathcal{F}\left(\tilde{\mathcal{O}}_m - \mathcal{O}_m\right)(uU_j)w_{\sigma_j}^1(u)}{\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right) \\ &= \frac{2U_j^{-2}}{T_j - T_{j-1}}\operatorname{Re}\left(\int_0^1 \frac{iuU_j(1+iuU_j)\left(\sum_{r=1}^{N}\delta_{m,r}\mathcal{F}b_{m,r}(uU_j)\varepsilon_{m,r}\right)w_{\sigma}^1(u)}{\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right) \\ &= \frac{2U_j^{-2}}{T_j - T_{j-1}}\sum_{r=1}^{N}\delta_{m,r}\varepsilon_{m,r}\operatorname{Re}\left(\int_0^1 \frac{iuU_j(1+iuU_j)\mathcal{F}b_{m,r}(uU_j)w_{\sigma_j}^1(u)}{\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right).\end{aligned} \tag{10.5}$$

We will now apply the Lindeberg-Feller central limit theorem. Verification of this theorem depends on the computation of the asymptotic variance. Thus, we will compute the variance $V_{\operatorname{Re}^2 z}^k$ of $L_{\sigma^2}^{(k)}$

$$V_{\operatorname{Re}^2 z}^k = \frac{4U_j^{-4}}{(T_j - T_{j-1})^2}\sum_{r=1}^{N}\delta_{m,r}^2\operatorname{Re}^2\left(\int_0^1 \frac{iuU_j(1+iuU_j)\mathcal{F}b_{m,r}(uU_j)w_{\sigma_j}^1(u)}{\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right)$$

Instead of computing the real part immediately, we will make use of the following identity

$$\operatorname{Re}^2 z = \left(\frac{z+\overline{z}}{2}\right)^2 = \frac{1}{4}\left(z^2 + 2z\overline{z} + \overline{z}^2\right)$$

and compute the three different parts instead, where we will start with z^2

$$V_{z^2}^k := \frac{4U_j^{-4}}{(T_j - T_{j-1})^2}\sum_{r=1}^{N}\delta_{m,r}^2\left(\int_0^1 \frac{iuU_j(1+iuU_j)\mathcal{F}b_{m,r}(uU_j)w_{\sigma_j}^1(u)}{\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right)^2$$

and we will study the behaviour of the integral inside the summation

$$\begin{aligned}I_{z^2} &:= \left(\int_0^1 \frac{iuU_j(1+iuU_j)\mathcal{F}b_{m,r}(uU_j)w_{\sigma_j}^1(u)}{\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right)^2 \\ &= \int_0^1\int_0^1 \frac{iuU_j(1+iuU_j)\mathcal{F}b_{m,r}(uU_j)w_{\sigma_j}^1(u)ivU_j(1+ivU_j)\mathcal{F}b_{m,r}(vU_j)w_{\sigma_j}^1(v)}{\exp\left(-\left(u^2+v^2\right)U_j^2A_m/2 + i(u+v)U_jB_m + 2C_m + D_m(uU_j) + D_m(vU_j)\right)}\,\mathrm{d}u\,\mathrm{d}v \\ &= -U_j^2\exp(-2C_m)\int_0^1\int_0^1 uv\exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right)\cdot(1+iuU_j)(1+ivU_j)\cdot g(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ &= -U_j^2\exp\left(-2C_m\right)\cdot\Bigg(\int_0^1\int_0^1 uv\exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right)\cdot g(u,v)\,\mathrm{d}u\,\mathrm{d}v \\ &\qquad + iU_j\int_0^1\int_0^1 uv\exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right)\cdot(u+v)g(u,v)\,\mathrm{d}u\,\mathrm{d}v\end{aligned}$$

$$- U_j^2 \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot uvg(u,v)\,\mathrm{d}u\,\mathrm{d}v\Bigg).$$

with g defined as

$$g(u,v) := \frac{\mathcal{F}b_{m,r}(uU_j)w_{\sigma_j}^1(u)\mathcal{F}b_{m,r}(vU_j)w_{\sigma_j}^1(v)}{\exp\left(i(u+v)U_jB_m + D_m(uU_j) + D_m(vU_j)\right)}$$

where the weight function $w_{\sigma_j}^1$ is symmetrically extended to $(-1,0)$, i.e., such that $w_{\sigma_j}^1(u) = w_{\sigma_j}^1(-u)$ for all $u \in (-1,0)$. Similarly,

$$\begin{aligned} I_{\overline{z}^2} = -U_j^2 \exp\left(-2C_m\right) \cdot \Bigg(&\int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot g(-u,-v)\,\mathrm{d}u\,\mathrm{d}v \\ &+ iU_j \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot (-u-v)g(-u,-v)\,\mathrm{d}u\,\mathrm{d}v \\ &- U_j^2 \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot uvg(-u,-v)\,\mathrm{d}u\,\mathrm{d}v\Bigg) \end{aligned}$$

and

$$\begin{aligned} I_{z\overline{z}} = U_j^2 \exp\left(-2C_m\right) \cdot \Bigg(&\int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot g(u,-v)\,\mathrm{d}u\,\mathrm{d}v \\ &+ iU_j \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot (u-v)g(u,-v)\,\mathrm{d}u\,\mathrm{d}v \\ &- U_j^2 \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot -uvg(u,-v)\,\mathrm{d}u\,\mathrm{d}v\Bigg) \end{aligned}$$

We will now define a function that is equal to $uv\exp(2iU_jB_m)\mathcal{F}b_{m,r}(U_j)^{-2}g(u,v)$, or in more detail

$$\tilde{g}_{U_j}(u,v) := uv \cdot \frac{\mathcal{F}b_{m,r}(uU_j)\mathcal{F}b_{m,r}(vU_j)}{\mathcal{F}b_{m,r}(U_j)^2} \cdot w_{\sigma_j}^1(u)w_{\sigma_j}^1(v) \cdot \frac{\exp\left(2iU_jB_m\right)}{\exp\left(i(u+v)U_jB_m\right)} \cdot \exp\left(-D_m(uU_j) - D_m(vU_j)\right)$$

and we will apply Lemma 7. The intuition behind this choice is that the limit for $\lim_{U_j\to\infty} \tilde{g}_{U_j}(1,1)$ exists because of these factors. Hence, we need to check the conditions on $\tilde{g}_{U_j}$ and we need to find functions f_{U_j} which converge to a Dirac delta function at $(1,1)$. Rescaling the other factors in the integrals we obtain

$$f_{U_j}(u,v) := A_m^2 U_j^4 \exp\left(-A_m U_j^2\right) uv \exp\left(A_m(u^2+v^2)U_j^2/2\right) =: F(u) \cdot F(v).$$

However, we still need to check the conditions of the lemma on the function f_{U_j}.

$$\begin{aligned} \int_{1-U_j^{-3/2}}^1 F(u)\,\mathrm{d}u &= \exp\left(-A_m U_j^2/2\right) \int_{1-U_j^{-3/2}}^1 uA_m U_j^2 \exp\left(A_m u^2 U_j^2/2\right)\,\mathrm{d}u \\ &= \exp\left(-A_m U_j^2/2\right) \cdot \left[\exp\left(A_m u^2 U_j^2/2\right)\right]_{u=1-U_j^{-3/2}}^1 \\ &= \exp\left(-A_m U_j^2/2\right) \cdot \left[\exp\left(A_m U_j^2/2\right) - \exp\left(A_m\left(1-U_j^{-3/2}\right)^2 U_j^2/2\right)\right] \\ &= 1 - \exp\left(A_m U_j^2\left[\left(1-2U_j^{-3/2}+U_j^{-3}\right)-1\right]/2\right) \\ &= 1 - \exp\left(-A_m U_j^{1/2} + A_m U_j^{-1}/2\right) \end{aligned}$$

Hence,

$$\lim_{U_j\to\infty}\int_{1-U_j^{-3/2}}^{1}\int_{1-U_j^{-3/2}}^{1} f_{U_j}(u,v)\,\mathrm{d}u\,\mathrm{d}v = \lim_{U_j\to\infty}\left(\int_{1-U_j^{-3/2}}^{1} F(u)\,\mathrm{d}u\right)^2$$
$$= \left(\lim_{U_j\to\infty} 1-\exp\left(-A_mU_j^{1/2}+A_mU_j^{-1}/2\right)\right)^2 = 1.$$

Checking the other condition on the function f is left to the reader. So, what remains is checking boundedness of $\tilde{g}_U$ on the unit square and we need to check that

$$\lim_{U_j\to\infty}\sup_{(u,v)\in[1-U^{-3/2},1]^2}|\tilde{g}_{U_j}(u,v)-\tilde{g}_{U_j}(1,1)| = 0$$

which can be found in Lemma 8.

From this, it appears that all the integrals in the final expressions for I_{z^2}, $I_{\overline{z}^2}$ and $I_{z\overline{z}}$ converge equally fast to 0. Hence, the dominating term is the last one by the U_j^4 factor in front of it. Henceforth, the first two integrals will be left out of the equation.

$$\begin{aligned}
&\lim_{U\to\infty} I_{z^2}\cdot A_m^2\exp(-A_mU_j^2)\mathcal{F}b_{m,r}(U_j)^{-2}\exp(2iU_jB_m)\\
&\quad= \lim_{U\to\infty}\exp(-2C_m)\int_0^1\int_0^1 f_{U_j}(u,v)\tilde{g}_{U_j}(u,v)\,\mathrm{d}u\,\mathrm{d}v\\
&\quad= \exp(-2C_m)\lim_{U\to\infty}\tilde{g}_{U_j}(1,1)\\
&\quad= \exp(-2C_m)w_{\sigma_j}^1(1)^2\lim_{U\to\infty}\exp\left(-2D_m(U_j)\right)\\
&\quad= \exp(-2C_m)w_{\sigma_j}^1(1)^2
\end{aligned}\tag{10.6}$$

Similarly,

$$\lim_{U\to\infty} I_{\overline{z}^2}\cdot A_m^2\exp(-A_mU_j^2)\overline{\mathcal{F}b_{m,r}(U_j)}^{-2}\exp(-2iU_jB_m)$$
$$= \exp(-2C_m)w_{\sigma_j}^1(1)^2$$

and

$$\lim_{U\to\infty} I_{z\overline{z}}\cdot A_m^2\exp(-A_mU_j^2)|\mathcal{F}b_{m,r}(U_j)|^{-2}$$
$$= \exp(-2C_m)w_{\sigma_j}^1(1)^2$$

We can now compute the asymptotic variance. Note that the summands are replaced by their respective asymptotic behavior. Since N grows as U_j grows, this replacement is not a trivial step. The details of the verification of this step can be found in Lemma 9.

$$\begin{aligned}
&\lim_{U_j\to\infty}\Delta_m^{-1}U_j^4A_m^2\exp(-A_mU_j^2)V_{z^2}^k\\
&\quad= \frac{4}{(T_j-T_{j-1})^2}\lim_{U_j\to\infty}\Delta_m^{-1}\sum_{r=1}^N\delta_{m,r}^2A_m^2\exp(-A_mU_j^2)I_{z^2}\\
&\quad= \frac{4}{(T_j-T_{j-1})^2}\lim_{U\to\infty}\Delta_m^{-1}\sum_{r=1}^N\delta_{m,r}^2\exp(-2C_m)w_{\sigma_j}^1(1)^2\mathcal{F}b_{m,r}(U_j)^2\exp\left(-2iB_mU_j\right)\\
&\quad= 4(T_j-T_{j-1})^{-2}\exp(-2C_m)w_{\sigma_j}^1(1)^2\lim_{U_j\to\infty}\exp\left(-2iB_mU_j\right)\Delta_m^{-1}\sum_{r=1}^N\delta_{m,r}^2\mathcal{F}b_{m,r}(U_j)^2
\end{aligned}$$

Similarly,

$$\lim_{U\to\infty} \Delta_m^{-1} U_j^4 A_m^2 \exp(-A_m U_j^2) V_{\bar{z}^2}^k$$
$$= 4(T_j - T_{j-1})^{-2} \exp(-2C_m) w_{\sigma_j}^1(1)^2 \lim_{U_j\to\infty} \exp(2iB_m U_j)\, \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 \mathcal{F} b_{m,r}(-U_j)^2$$

and

$$\lim_{U\to\infty} \Delta_m^{-1} U_j^4 A_m^2 \exp(-A_m U_j^2) V_{z\bar{z}}^k$$
$$= 4(T_j - T_{j-1})^{-2} \exp(-2C_m) w_{\sigma_j}^1(1)^2 \lim_{U_j\to\infty} \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 |\mathcal{F} b_{m,r}(U_j)|^2.$$

To compute the limit, it is necessary to actually calculate $\mathcal{F} b_{m,r}$. First, we know that the Fourier transform of the unit box function with limits $-1/2$ and $1/2$ is equal to the sinc $= \sin(\bullet)/\bullet$ function. Hence the Fourier transform of the convolution of two unit box functions is equal to the sinc^2 function, whereas the convolution is equal to the triangular function Λ with limits -1 and 1. We have

$$b_{m,r}(\Delta_m x + x_{m,r}) = \Lambda(x)$$

Then

$$\text{sinc}^2(u) = \int_{-1}^{1} \exp(iux)\Lambda(x)\,\mathrm{d}x = \Delta_m^{-1} \int_{x_{m,r-1}}^{x_{m,r+1}} \exp\left(iu \cdot \frac{y - x_{m,r}}{\Delta_m}\right) b_{m,r}(y)\,\mathrm{d}y$$
$$= \Delta_m^{-1} \exp\left(-iux_{m,r}\Delta_m^{-1}\right) \mathcal{F} b_{m,r}\left(u\Delta_m^{-1}\right)$$

Thus,

$$\mathcal{F} b_{m,r}(y) = \Delta_m \exp(iyx_{m,r})\, \text{sinc}^2(y\Delta_m). \tag{10.7}$$

Hence,

$$\lim_{U_j\to\infty} \exp(-2iB_m U_j) \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 \mathcal{F} b_{m,r}(U_j)^2$$
$$= \lim_{U_j\to\infty} \exp(-2iB_m U_j)\, \text{sinc}^4(U_j\Delta_m) \sum_{r=1}^{N} \delta_{m,r}^2 \exp(2iU_j x_{m,r})\, \Delta_m$$
$$= \lim_{U_j\to\infty} \exp(-2iB_m U_j)\, \text{sinc}^4(U_j\Delta_m) \int_{-\infty}^{\infty} \delta_m(x)^2 \exp(2iU_j x)\,\mathrm{d}x$$
$$= \lim_{U_j\to\infty} \exp(-2iB_m U_j)\, \text{sinc}^4(U_j\Delta_m)\, \mathcal{F}\delta_m^2(2U_j).$$

We have assumed δ to be an L^2 function, hence $\mathcal{F}\delta_m^2(2U_j) \to 0$ as $U_j \to \infty$. Moreover, $U_j\Delta_m \to 0$, thus $\text{sinc}^4(U_j\Delta_m) \to 1$ as $U_j \to \infty$ and the first term is bounded in norm with 1. So, we conclude that this sum is equal to 0 in the limit. Similarly,

$$\lim_{U\to\infty} \exp(2iB_m U_j)\, \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 \mathcal{F} b_{m,r}(-U_j)^2 = 0$$

and

$$\lim_{U_j\to\infty} \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 |\mathcal{F} b_{m,r}(U_j)|^2 = \lim_{U_j\to\infty} \text{sinc}^4(U_j\Delta_m) \sum_{r=1}^{N} \delta_{m,r}^2 \Delta_m$$
$$= \lim_{U_j\to\infty} \text{sinc}^4(U_j\Delta_m) \int_{-\infty}^{\infty} \delta_m(x)^2\,\mathrm{d}x$$
$$= \lim_{U_j\to\infty} \text{sinc}^4(U_j\Delta_m)\, \|\delta_m\|_{L^2}^2 = \|\delta_m\|_{L^2}^2$$

Concluding, the asymptotic variance of $L_{\sigma^2}^{(k)}$ is equal to

$$\begin{aligned} V^k_{\mathrm{Re}^2 z} &= \Delta_m U_j^{-4} A_m^{-2} \exp(A_m U^2) \cdot \frac{1}{4} \Big(0 + 2 \cdot 4(T_j - T_{j-1})^{-2} \exp(-2C_m) w^1_{\sigma_j}(1)^2 \left\|\delta_m\right\|^2_{L^2} + 0\Big) \\ &= 2 \left\|\delta_m\right\|^2_{L^2} (T_j - T_{j-1})^{-2} A_m^{-2} \exp(-2C_m) w^1_{\sigma_j}(1)^2 \cdot \Delta_m U_j^{-4} \exp(A_m U_j^2) \end{aligned} \tag{10.8}$$

If we define

$$d_{j,m} = 2 \left\|\delta_m\right\|^2_{L^2} (T_j - T_{j-1})^{-2} A_m^{-2} \exp(-2C_m),$$

we can write the asymptotic variance AV of $\hat{\sigma}_j^2 - \sigma_j^2$ as

$$AV = U_j^{-4} \cdot \Big(d_{j,j} \cdot \Delta_j \exp\left(A_j U_j^2\right) + d_{j,j-1} \cdot \Delta_{j-1} \exp\left(A_{j-1} U_j^2\right)\Big) \cdot w^1_{\sigma_j}(1)^2$$

□

Note that in this proof, the difference with γ_j and λ_j is only found in the order of U_j as the first term. In the other cases, we have U_j^{-2} and, respectively, 1 instead of U_j^{-4}.

10.2.4 Asymptotic normality of the linear term $L_{\sigma^2}^{(k)}$

To prove asymptotic normality, we will apply the Lindeberg-Feller version of the central limit theorem. A version of their theorem is formulated as

Theorem 3. *Let $X_1, X_2, \ldots$ be independent random variables such that $\mathbb{E}X_n = 0$ and $\operatorname{Var} X_n = \sigma_n^2 < \infty$. Define $T_n = \sum_{k=1}^n X_k$ and $s_n^2 = \operatorname{Var} T_n = \sum_{k=1}^n \sigma_k^2$. Then the following statement holds*

$$\left(\exists\, \eta > 2: \lim_{n\to\infty} s_n^{-\eta} \sum_{k=1}^{n} \mathbb{E}\left\{|X_k|^\eta\right\} = 0\right) \quad \Rightarrow \quad \left(\frac{T_n}{s_n} \xrightarrow{\mathcal{D}} N(0,1)\right).$$

The left-hand side will be referred to as the Lyapunov condition.

Before the next proposition can be proved, it is necessary to introduce sub-Gaussian distributions.

Definition 5. *The distribution of a random variable Z is called sub-Gaussian if there exist positive constants A and B such that for every $t > 0$*

$$\mathbb{P}(|Z| > t) \leq A e^{-Bt^2}.$$

In other words, Z is called sub-Gaussian if the tail of its distribution is dominated by a Gaussian tail. Moreover, one should note that any bounded random variable is sub-Gaussian. Since in practice everything is bounded, it will not be wrong to assume sub-Gaussianity. It is now possible to prove the next proposition, where in the case of γ_j and λ_j nothing really changes since the U_j terms will always cancel in the end.

Proposition 8. *If the distributions of $\varepsilon_{j,r}$ and $\varepsilon_{j-1,r}$ are sub-Gaussian for all r and if $\delta_j, \delta_{j-1} \in L^\eta$ with $\eta > 2$, then $L_\xi^{(k)}$ is asymptotically normal for $\xi \in \{\sigma^2, \gamma, \lambda, \nu(x)\}$*

Proof. The Lindeberg-Feller central limit theorem will be applied to conclude that $L_{\sigma^2}^{(k)}$ as given in (10.5) is asymptotically normal. In particular, the Lyapunov condition is proven to hold.

Note that we have already computed s_n^2, i.e.,

$$s_n^2 = U_j^{-4} \cdot \Big(d_{j,j} \cdot \Delta_j \exp\left(A_j U_j^2\right) + d_{j,j-1} \cdot \Delta_{j-1} \exp\left(A_{j-1} U_j^2\right)\Big) \geq d_{j,m} \cdot U_j^{-4} \Delta_m \exp\left(A_m U_j^2\right).$$

We can also bound $\sigma_{r,\eta} := \mathbb{E}|X_r|^\eta$, where we can make use of the sub-Gaussianity of $\varepsilon_{r,j}$ to estimate the expectation with a constant which may depend on the choice of η.

$$\begin{aligned}\sigma_{r,\eta} &= \mathbb{E}\left\{\left|\frac{2U_j^{-2}}{T_j - T_{j-1}}\delta_{m,r}\varepsilon_{m,r} \operatorname{Re}\left(\int_0^1 \frac{iuU_j(1+iuU_j)\mathcal{F}b_{m,r}(uU_j)w^1_{\sigma_j}(u)}{\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right)\right|^\eta\right\} \\ &\lesssim U_j^{-2\eta}|\delta_{m,r}|^\eta \left|\operatorname{Re}\left(\int_0^1 \frac{iuU_j(1+iuU_j)\mathcal{F}b_{m,r}(uU_j)w^1_{\sigma_j}(u)}{\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right)\right|^\eta \\ &\leq U_j^{-2\eta}|\delta_{m,r}|^\eta \left|\int_0^1 \frac{iuU_j(1+iuU_j)\mathcal{F}b_{m,r}(uU_j)w^1_{\sigma_j}(u)}{\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right|^\eta\end{aligned}$$

We note that the integral I looks similar to the ones that we have calculated in the previous section. To avoid repeating arguments, the integral behaves in the limit as

$$\lim_{U\to\infty} |I| \cdot A_m \exp(-A_mU_j^2/2)|\mathcal{F}b_{m,r}(U_j)|^{-1} = \exp(-C_m)w^1_{\sigma_j}(1).$$

Hence, in the limit we have

$$\begin{aligned}|I| &\lesssim A_m^{-1}\exp(A_mU_j^2/2)|\mathcal{F}b_{m,r}(U_j)|\exp(-C_m)w^1_{\sigma_j}(1) \\ &= A_m^{-1}\exp(A_mU_j^2/2)\Delta_m \operatorname{sinc}^2(U_j\Delta_m)\exp(-C_m)w^1_{\sigma_j}(1).\end{aligned}$$

Thus

$$\sigma_{r,\eta} \lesssim U_j^{-2\eta}|\delta_{m,r}|^\eta \exp\left(A_mU_j^2\cdot\eta/2\right)\Delta_m^\eta.$$

Hence,

$$\begin{aligned}\lim_{n\to\infty} s_n^{-\eta}\sum_{k=1}^n \mathbb{E}\left\{|X_k|^\eta\right\} &\lesssim \lim_{n\to\infty} U_j^{2\eta}\Delta_m^{-\eta/2}\exp\left(-A_mU_j^2\cdot\eta/2\right)\sum_{r=1}^N U_j^{-2\eta}|\delta_{m,r}|^\eta\exp\left(A_mU_j^2\cdot\eta/2\right)\Delta_m^\eta \\ &= \lim_{n\to\infty}\Delta_m^{\eta/2-1}\sum_{r=1}^n|\delta_{m,r}|^\eta\Delta_m = \lim_{n\to\infty}\Delta_m^{\eta/2-1}\int_{-\infty}^\infty|\delta_m(x)|^\eta\,\mathrm{d}x \\ &= \|\delta_m\|_{L^\eta}\cdot\lim_{n\to\infty}\Delta_m^{\eta/2-1} = 0\end{aligned}$$

Where in the last step, it is used that $\eta > 2$, $\delta_m \in L^\eta$ and δ_m Riemann integrable. □

Appendix A

The risk-neutral measure

A commonly used measure in finance is the risk-neutral measure. This is a measure that is not related to the real-world probability measure but it is related to how investors are willing to invest. This chapter is devoted to explaining what the risk-neutral measure is and why it is important. Moreover, it may become more clear why the Lévy process under the risk-neutral measure is estimated instead of under the real-world probability measure.

Definition 6. *The risk-neutral measure is a probability measure such that the discounted expectation of a trade is equal to its value.*

Consider the following example. Alice needs to have 10 euros right now otherwise Alice has a big problem. The problem is that Alice only has 7 euros available. Luckily, there is one possibility available. Bob offers to pay 3 euros if Alice guesses one unbiased coin toss correctly. However, if Alice doesn't guess correctly, she has to pay Bob 7 euros. This is not a winning game but since there are no other options available at the time, Alice is likely to play the game.

The real-world probability of Alice winning is 50% and the risk-neutral probability is 70% since in that case, the expectation of the game would be 0. It might be tempting to think that she has more chance of winning the game in the risk-neutral world. However, one should never think of it that way.

One of the ways, the risk-neutral probabilities are useful, is in pricing derivatives. Suppose that Charlie notices that Alice and Bob were playing this game, then he could offer Dave the opportunity to make money out of this game. He offers to pay 2 euros to Dave if Alice wins and 0 if Alice loses. How much would the price of this contract cost for Dave? The exact price of this contract should be $1.40(= 0.7 \cdot 2 + 0.3 \cdot 0)$ euros, which is the expectation of the contract under the risk-neutral measure. It looks like this contract costs too much if one considers the real-world probabilities, however, if Dave could buy it for 1.20 euros in real life, he should do this. Moreover, Dave should sell 20% of the game generating a direct income of 1.40 euros. At time 0, Dave would have $1.40 - 1.20 = 0.20$ euros. If Alice wins the game, Dave would get 2 euros from the contract with Charlie and he would have to pay Alice 2 euros for selling the game which leads to a zero net gain. Moreover, if Alice loses the game, Dave would get 0 from the contract with Charlie and he has to pay 0 from selling the game because it is not worth anything anymore. Either way, Dave has a net gain of 0 at time 1. Hence, he gains the 20 cents he received at the beginning without having to face any risk.

In the case described above, an arbitrage situation occurred. In the real world, Dave probably wouldn't get to sell 20% of the game because this stock is obviously overpriced. However, in a real-world situation, it is almost always possible to sell a stock for almost the exact price at which it is being sold. This will eventually imply that more people are willing to sell than to buy which induces a drop in price but that is not the point of this example. The point is that because of this phenomenon option pricing should be done very carefully and, instead of using the real-world probability measure, the risk-neutral probability measure should be used in pricing options such that arbitrage is excluded.

Thus since the risk-neutral probability measure is used in pricing options and not the real-world probability measure, it should come as no surprise that option prices don't imply anything about the real-world probability measure. Moreover, knowing the risk-neutral measure is crucial for pricing options such as the regular ones, like European calls, and the exotic ones, like Bermudan puts.

Appendix B

Additional lemmas

The following two lemmas are used in the proof for Proposition 3. The first lemma provides a technical proof of a fairly obvious result and the second one gives a precise construction for the constant C such that $(y+\Delta)^{2q}$ can be bounded by $\left(y^{2q}+\Delta^{2q}\right)$ for positive y.

Lemma 2. *Given two continuous functions $f, g : [0,T] \to \mathbb{C}$ for which holds $|f(t)| > C$ and $|f(t)-g(t)| \leq C$ for all $t \in [0,T]$. Moreover, $\arg f(0) = \arg g(0)$. Then, if $\arg$ is chosen to be such that $t \mapsto \arg(\gamma(t))$ is continuous for a continuous function γ, then*

$$\sup_{t\in[0,T]} |\arg g(t) - \arg f(t)| \leq \pi$$

Proof. Suppose $\sup_{t\in[0,T]} |\arg g(t) - \arg f(t)| > \pi$. Then, by continuity of $\arg g(t)$ and $\arg f(t)$, there exists a $t_0 \in [0,T]$ such that $|\arg g(t_0) - \arg f(t_0)| = \pi$, which is equivalent with $g(t_0) = -rf(t_0)$ with $r \in \mathbb{R}_{>0}$. However, then

$$|f(t_0) - g(t_0)| = |f(t_0) + rf(t_0)| = (1+r)|f(t_0)| > |f(t_0)| \geq C$$

which is a contradiction with the assumption $|f(t) - g(t)| \leq C$ for all $t \in [0,T]$. □

Lemma 3. *For any $q, x \in \mathbb{R}$ and $c > 0$, we have*

$$\left(x^2+c\right)^{2q} \leq \max\left(2^{2q-1}, 1\right) \cdot \left(\left(x^2\right)^{2q} + c^{2q}\right),$$

where in the case of $x, q = 0$, one should read $\left(x^2\right)^{2q} = 1$.

Proof. If $q = 0$, the result is obvious. So without loss of generality, we will assume $q \neq 0$. Moreover, we define $y = x^2$ and we consider $y \geq 0$. Define $f_c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as follows

$$f_c(y) = \frac{(y+c)^{2q}}{y^{2q}+c^{2q}}$$

and note that this function has no singularities and that it is differentiable with derivative

$$\begin{aligned}
\frac{\mathrm{d}f_c}{\mathrm{d}y}(y) &= \frac{\left(y^{2q}+c^{2q}\right) \cdot 2q(y+c)^{2q-1} - (y+c)^{2q} \cdot 2qy^{2q-1}}{\left(y^{2q}+c^{2q}\right)^2} \\
&= \frac{2q(y+c)^{2q-1} \cdot \left(\left(y^{2q}+c^{2q}\right) - (y+c)y^{2q-1}\right)}{\left(y^{2q}+c^{2q}\right)^2}
\end{aligned}$$

$$= \frac{2qc(y+c)^{2q-1} \cdot \left(c^{2q-1} - y^{2q-1}\right)}{\left(y^{2q} + c^{2q}\right)^2}$$

Moreover, we note that

$$\lim_{y\to\infty} f_c(y) = 1$$

and $f_c(0) = 1$. Thus either $|f_c(y)| \leq 1$ for all $y > 0$ or $|f_c(y)| \leq f_c(z)$ with z a zero of f_c'. From the equation above it is clear that f_c' has one zero on its domain which equals c. For this zero we have $f(c) = 2^{2q-1}$. It is now clear that for any $q, x \in \mathbb{R}$ and $c > 0$

$$|f_c(y)| \leq \max\left(2^{2q-1}, 1\right)$$

holds, which completes the proof. □

The next three lemmas are used in bounding the remainder error term.

Lemma 4. *Given* $|\tilde{z}| > C$ *and given* $|z| > 2C$*. Then*

$$|\log(\tilde{z}) - \log(z) - (\tilde{z} - z)z^{-1}| \leq \frac{|\tilde{z} - z|^2}{2C^2}$$

if $|\arg z - \arg \tilde{z}| \leq \pi$

Proof. First note that if we have proven the result for $C = 1$, the result for general C follows immediately by replacing z and $\tilde{z}$ with z/C and $\tilde{z}/C$, respectively.
Second observation to be made is that if the result is proven for positive z, the result for general z follows immediately by a rotational argument, i.e., replacing w with $we^{i\varphi}$ for $w = z$ and $\tilde{z}$. So, without loss of generality, we will take $C = 1$ and $z \in \mathbb{R}_{>0}$.

The proof is based on showing that $1/4$ bounds the following real-valued function

$$f : \left(\mathbb{R}^2 \setminus \{(x, y) : \ x^2 + y^2 < 1\}\right) \times [2, \infty) \to \mathbb{R}_{>0}$$

defined by

$$f(x, y, z) = \frac{|\log(x + iy) - \log(z) - (x + iy - z)z^{-1}|^2}{|x + iy - z|^4}.$$

Via the `fminsearch` function from `MATLAB`, it is quickly believed that the maximum of this function is attained at $z = 2$ and $x^2 + y^2 = 1$. Moreover, it is found, iteratively, that the maximum will be located at the point such that the difference in argument is maximized. In our case, this would mean $x = -1$ and $y = 0$, leading to a difference of π in the argument. This leads to the following upper bound of the function f

$$|f(z)| \leq \frac{(3/2 - \log(2))^2 + \pi^2}{3^4} \approx 0.1299 \leq \frac{1}{4},$$

which completes the proof. □

Lemma 5. *For the functions* ψ_j *and* $\tilde{\psi}_j$ *as defined before we have with* $T = T_j - T_{j-1}$

$$\left|\log\left(e^{(T_j-T_{j-1})\tilde{\psi}_j^k(u)}\right) - \log\left(e^{(T_j-T_{j-1})\psi_j^k(u)}\right) - e^{-(T_j-T_{j-1})\psi_j^k(u)}\left(e^{(T_j-T_{j-1})\tilde{\psi}_j^k(u)}\right) - e^{(T_j-T_{j-1})\psi_j^k(u)}\right|$$
$$\leq \frac{1}{2}K_j(u)^{-2}\left|e^{(T_j-T_{j-1})\tilde{\psi}_j^k(u)} - e^{(T_j-T_{j-1})\psi_j^k(u)}\right|^2$$

with probability tending to 1

Proof. This is an immediate result of combining Proposition 3, Corollary 1 and Lemma 4. □

Lemma 6. *For $j = 1, 2, \ldots, n$, we have*

$$\int_{-U_j}^{U_j} K_j(T, \sigma_{\max}, R, u)^{-2} u^4 |w_{\xi_1}^{U_j}(u)| \, \mathrm{d}u \lesssim \xi_2 \exp\left(U_j^2 \cdot \sum_{m=1}^{j} (T_m - T_{m-1}) \sigma_m^2 \right)$$

for $\xi_1 \in \{\sigma_j, \gamma_j, \lambda_j\}$ and, respectively, $\xi_2 \in \{1, U_j, U_j^2\}$.

Proof. The result will be proven in detail for $\xi = \sigma_j$. The proof of the two others option for ξ are similar. Define $A_j := \sum_{m=1}^{j} (T_m - T_{m-1}) \sigma_m^2$ and recall

$$K_j(u) := 2^{j-1} \exp\left(-\frac{u^2}{2} \cdot A_j + 2R \sum_{m=1}^{j} (T_m - T_{m-1}) \right)$$

Thus

$$K_j(u)^{-2} \lesssim \exp\left(A_j u^2\right).$$

Hence,

$$\begin{aligned}
\int_{-U_j}^{U_j} K_j(u)^{-2} u^4 |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u &\lesssim \int_{-U_j}^{U_j} \exp\left(A_j u^2\right) u^4 |w_{\sigma_j}^{U_j}(u)| \, \mathrm{d}u \\
&\lesssim \int_{-U_j}^{U_j} \exp\left(A_j u^2\right) u^4 U^{-(s+3)} |u|^s \, \mathrm{d}u \\
&\lesssim U_j^{-(s+3)} \int_0^{U_j} u^{3+s} \cdot 2A_j u \exp\left(A_j u^2\right) \, \mathrm{d}u \\
&\leq \int_0^{U_j} 2A_j u \exp\left(A_j u^2\right) \, \mathrm{d}u \\
&= \left[\exp\left(A_j u^2\right)\right]_{u=0}^{U_j} \lesssim \exp\left(A_j U_j^2\right)
\end{aligned}$$

□

The last two lemmas are used in computing the asymptotic variance of the linear error term.

Lemma 7. *Let $g_U(u, v)$ be bounded functions on the unit square such that $0 \leq |g_U(u, v)| \leq C$. Moreover, let $h(x) \downarrow 0$ as $x \to \infty$ and assume that the functions g_U satisfy the following condition for all $\eta > 0$*

$$\lim_{U \to \infty} \sup_{(u,v) \in [1-h(U), 1]^2} |g_U(u, v) - g_U(1, 1)| = 0.$$

Let $f_U(u, v)$ be a positive function such that

$$\lim_{U \to \infty} \int_0^1 \int_0^1 f_U(u, v) \, \mathrm{d}u \, \mathrm{d}v = 1$$

and

$$\lim_{U \to \infty} \int_{1-h(U)}^1 \int_{1-h(U)}^1 f_U(u, v) \, \mathrm{d}u \, \mathrm{d}v = 1.$$

Then

$$\lim_{U \to \infty} \int_0^1 \int_0^1 f_U(u, v) g_U(u, v) \, \mathrm{d}u \, \mathrm{d}v = \lim_{U \to \infty} g_U(1, 1).$$

Proof. By the monotonicity property of integrals, we have

$$\lim_{U\to\infty}\left|\int\int_{(0,1)^2\setminus[1-h(U)]^2} f_U(u,v)g_U(u,v)\,\mathrm{d}(u,v)\right| \le \lim_{U\to\infty}\int\int_{(0,1)^2\setminus[1-h(U)]^2} f_U(u,v)\,\mathrm{d}u\,\mathrm{d}v\cdot C$$
$$= 0$$

Thus,

$$\lim_{U\to\infty}\int_0^1\int_0^1 f_U(u,v)g_U(u,v)\,\mathrm{d}u\,\mathrm{d}v = \lim_{U\to\infty}\int_{1-h(U)}^1\int_{1-h(U)}^1 f_U(u,v)g_U(u,v)\,\mathrm{d}u\,\mathrm{d}v$$

Now, we will prove that this equals $g(1,1)$ by looking at the difference and rewriting it as follows

$$\begin{aligned}\lim_{U\to\infty}&\left|\int_{1-h(U)}^1\int_{1-h(U)}^1 f_U(u,v)(g_U(u,v)-g_U(1,1))\,\mathrm{d}u\,\mathrm{d}v\right|\\ &\le \lim_{U\to\infty}\int_{1-h(U)}^1\int_{1-h(U)}^1 f_U(u,v)\,\mathrm{d}u\,\mathrm{d}v\cdot\sup_{(u,v)\in[1-h(U)]^2}|g_U(u,v)-g_U(1,1)|\\ &= \lim_{U\to\infty}\sup_{(u,v)\in[1-h(U)]^2}|g_U(u,v)-g_U(1,1)| = 0\end{aligned}$$

Where the last equal sign is due to the assumption on g_U. Concluding,

$$\lim_{U\to\infty}\int_0^1\int_0^1 f_U(u,v)g_U(u,v)\,\mathrm{d}u\,\mathrm{d}v = \lim_{U\to\infty} g_U(1,1)$$

□

Lemma 8. *Define*

$$\tilde{g}_U(u,v) := uv\cdot\frac{\mathcal{F}b_{m,r}(uU_j)\mathcal{F}b_{m,r}(vU_j)}{\mathcal{F}b_{m,r}(U_j)^2}\cdot w^1_{\sigma_j}(u)w^1_{\sigma_j}(v)\cdot\frac{\exp(2iU_mB_m)}{\exp(i(u+v)U_mB_m)}\cdot\exp(-D_m(uU_m)-D_m(vU_m))$$

then $\tilde{g}$ is uniformly bounded on the unit square given $U_j, U_m > c$ for a certain $c > 0$. Moreover,

$$\sup_{(u,v)\in[1-U_m^{-3/2},1]^2}|\tilde{g}_U(u,v)-\tilde{g}_U(1,1)|\to 0.$$

Proof. For ease of notation, define

$$\begin{aligned}\tilde{g}_1(u,v) &= uv,\\ \tilde{g}_2(u,v) &= \mathcal{F}b_{m,r}(uU_j)\mathcal{F}b_{m,r}(vU_j)\mathcal{F}b_{m,r}(U_j)^{-2},\\ \tilde{g}_3(u,v) &= w^1_{\sigma_j}(u)w^1_{\sigma_j}(v),\\ \tilde{g}_4(u,v) &= \exp(i(2-u-v)U_mB_m)\ \text{ and}\\ \tilde{g}_5(u,v) &= \exp(-D_m(uU_m)-D_m(vU_m)).\end{aligned}$$

Note that $\tilde{g}_1$, $\tilde{g}_3$ and $\tilde{g}_4$ are uniformly bounded (in U) on the unit square. Also note that $\mathcal{F}\mu_j(x)\to 0$ for $x\to\infty$. Hence, D_m is a bounded function, which implies that $\tilde{g}_5$ is bounded uniformly in U. Proving boundedness of $\tilde{g}_2$, recall (C.2), which states that

$$\mathcal{F}b_{m,r}(u) = \Delta_m\exp(iux_r)\,\mathrm{sinc}^2(u\Delta_m).$$

Hence,

$$\tilde{g}_2(u,v) = \exp(i(u+v-2)U_jx_r)\cdot\frac{\mathrm{sinc}^2(uU_j\Delta_m)\,\mathrm{sinc}^2(vU_j\Delta_m)}{\mathrm{sinc}^2(U_j\Delta_m)} \tag{B.1}$$

Since $U_j\Delta_m \to 0$, we can find a $c > 0$, such that for all $U_j > c$, we have $\operatorname{sinc}^2(U_j\Delta_m) \geq 1/2$, which leads to the bound

$$|\tilde{g}_2(u,v)| = \left|\frac{\operatorname{sinc}^2(uU_j\Delta_m)\operatorname{sinc}^2(vU_j\Delta_m)}{\operatorname{sinc}^2(U_j\Delta_m)}\right| \leq \left|\frac{1}{\operatorname{sinc}^2(U_j\Delta_m)}\right| \leq 2.$$

Concluding, $\tilde{g}$ is bounded.

We note that $\tilde{g}_1 \cdot \tilde{g}_3$ is independent of U and continuous in $(1,1)$. Moreover, the second part of $\tilde{g}_2$ also behaves nicely. Thus these factor can be taken out of the equation. Moreover, $\tilde{g}_5$ converges uniformly to 1 for $U_m \to \infty$ because of the smoothness of $\mu_j(x)$. The only problems here occur thus in the first part of $\tilde{g}_2$ as expressed in (B.1) and in $\tilde{g}_4$.

$$\begin{aligned}\sup_{(u,v)\in[1-U_m^{-3/2},1]^2} |\tilde{g}_4(u,v)-1| &= \left|\exp\left(i(2-(1-U_m^{-3/2})-(1-U_m^{-3/2}))U_m\Sigma_2\right)-1\right| \\ &= \left|\exp\left(i\cdot U_m^{-1/2}\Sigma_2\right)-\exp(i\cdot 0)\right| \leq |U_m^{-1/2}\Sigma_2| \to 0.\end{aligned}$$

In a similar way, the first part of $\tilde{g}_2$ can be controlled. This completes the proof. □

Lemma 9. *Under the assumptions of Theorem 1*

$$\lim_{U_j\to\infty} \Delta_m^{-1}\sum_{r=1}^{N}\delta_{m,r}^2 A_m^2 e^{-A_m U_j^2} I_{z^2} = \lim_{U\to\infty}\Delta_m^{-1}\sum_{r=1}^{N}\delta_{m,r}^2 e^{-2C_m} w_{\sigma_j}^1(1)^2 \mathcal{F}b_{m,r}(U_j)^2 e^{-2iB_m U_j}.$$

Proof. It will be shown that the difference

$$\begin{aligned}&\lim_{U_j\to\infty}\Delta_m^{-1}\sum_{r=1}^{N}\delta_{m,r}^2\left(A_m^2 e^{-A_m U_j^2} I_{z^2} - e^{-2C_m} w_{\sigma_j}^1(1)^2\mathcal{F}b_{m,r}(U_j)^2 e^{-2iB_m U_j}\right) \\ &\qquad = \lim_{U_j\to\infty}\sum_{r=1}^{N}\delta_{m,r}^2\Delta_m\left(A_m^2 e^{-A_m U_j^2} I_{z^2}\Delta_m^{-2} - e^{-2C_m} w_{\sigma_j}^1(1)^2\mathcal{F}b_{m,r}(U_j)^2 e^{-2iB_m U_j}\Delta_m^{-2}\right)\end{aligned}$$

equals 0. Since $\lim\sum_{r=1}^N \delta_{m,r}^2\Delta_m = \|\delta_m\|_{L^2(\mathbb{R})}^2$ by the Riemann integrability of δ_m and the assumption that $\delta_m \in L^2(\mathbb{R})$, it is enough to show that the other part of the summation converges to 0 as $U_j \to \infty$. If that is the case, we can bound it by any $\varepsilon > 0$ provided that U_j is big enough. Letting ε to converge to 0 yields the result. Thus, it remains to show that

$$\lim_{U_j\to\infty}\left(A_m^2 e^{-A_m U_j^2} I_{z^2}\Delta_m^{-2} - e^{-2C_m} w_{\sigma_j}^1(1)^2\mathcal{F}b_{m,r}(U_j)^2 e^{-2iB_m U_j}\Delta_m^{-2}\right) = 0. \tag{B.2}$$

Since, we know that the result holds without the Δ_m^{-2} factor as is shown in (10.6), it is enough to show that the second term is bounded. Indeed, if $f_n/g_n \to 1$, $h_n \in \mathbb{C}$ and $|g_n h_n| \leq C$ we have

$$\lim_{n\to\infty}|f_n h_n - g_n h_n| = \lim_{n\to\infty}|g_n h_n|\cdot\left|\frac{f_n}{g_n}-1\right| \leq C\cdot\lim_{n\to\infty}\left|\frac{f_n}{g_n}-1\right| = 0.$$

Hence, we insert $\mathcal{F}b_{m,r}(U_j) = \Delta_m\exp(iU_j x_{m,r})\operatorname{sinc}^2(U_j\Delta_m)$. This gives that (B.2) equals

$$\lim_{U_j\to\infty}\left(A_m^2 e^{-A_m U_j^2} I_{z^2}\Delta_m^{-2} - e^{-2C_m} w_{\sigma_j}^1(1)^2 e^{2iU_j x_{m,r}}\operatorname{sinc}^4(U_j\Delta_m)\, e^{-2iB_m U_j}\right)$$

The absolute value of the second part converges to $e^{-2C_m} w_{\sigma_j}^1(1)^2$. In other words, it is bounded which concludes the proof. □

Appendix C

Additional details of the proof of Theorem 1 for $\nu_j(x)$.

In this chapter, one can find the relevant details of the proof of Theorem 1. for the estimator $\hat{\nu}_j(x)$ for $\nu_j(x)$. In contrast with the other estimators, we define

$$\varphi_{\nu,T_j}(v) = 1 - v(v+i) \cdot \mathcal{F}(\mathcal{O}_j(x - \overline{r}T_j))(v),$$

where we use the notation $\overline{r}$ for the risk-free interest rate, instead of r,

$$\begin{aligned}\psi^k_{\nu,j}(v) &:= (T_j - T_{j-1})^{-1} \log\left(1 - v(v+i)\mathcal{F}\mathcal{O}_{j-k}(\bullet - \overline{r}T_{j-k})(v)\right) \\ &= (T_j - T_{j-1})^{-1} \log\left(\varphi_{\nu,T_{j-k}}(v)\right)\end{aligned}$$

and

$$\psi_{\nu,j}(v) := \psi^0_{\nu,j}(v) - \psi^1_{\nu,j}(v).$$

These functions will be used in the estimation of $\nu_j(x)$ because under the assumptions of the model this function simplfies as follows

$$\begin{aligned}\psi_{\nu,j}(v) &= \frac{1}{T_j - T_{j-1}} \log\left(\frac{\varphi_{\nu,T_j}(v)}{\varphi_{\nu,T_{j-1}}(v)}\right) = -\frac{\sigma_j^2 v^2}{2} + i\gamma_j v + \int_{-\infty}^{\infty} \left(\mathrm{e}^{ivx} - 1\right) \nu_j(x)\,\mathrm{d}x \\ &= -\frac{\sigma_j^2 v^2}{2} + i\gamma_j v - \lambda_j + \mathcal{F}\nu_j(v)\end{aligned}$$

And $\hat{\nu}_j(x)$ is defined as

$$\hat{\nu}_j(x) := \mathcal{F}^{-1}\left[\left(\tilde{\psi}_{\nu,j}(\bullet) + \frac{\hat{\sigma}_j^2}{2}(\bullet)^2 - i\hat{\gamma}_j(\bullet) + \hat{\lambda}_j\right) w^{U_j}_{\nu_j}(\bullet)\right](x).$$

We start by rewriting $\hat{\nu}_j(x) - \nu_j(x)$,

$$\begin{aligned}\hat{\nu}_j(x) - \nu_j(x) = \mathcal{F}^{-1} &\left[\left(\tilde{\psi}_{\nu,j}(\bullet) + \frac{\hat{\sigma}_j^2}{2}(\bullet)^2 - i\hat{\gamma}_j(\bullet) + \hat{\lambda}_j\right) w^{U_j}_{\nu_j}(\bullet)\right](x) \\ &- \mathcal{F}^{-1}\left[\left(\psi_{\nu,j}(\bullet) + \frac{\sigma_j^2}{2}(\bullet)^2 - i\gamma_j(\bullet) + \lambda_j\right)\right](x)\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\tilde{\psi}_{\nu,j}(u)+\frac{\hat{\sigma}_j^2}{2}u^2 - i\hat{\gamma}_j u + \hat{\lambda}_j\right) w_{\nu_j}^{U_j}(u)\cdot\exp(-iux)\,\mathrm{d}u \\
&\quad - \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\psi_{\nu,j}(u)+\frac{\sigma_j^2}{2}u^2 - i\gamma_j u + \lambda_j\right)\cdot\exp(-iux)\,\mathrm{d}u \\
&= \frac{1}{2\pi}\Bigg[\int_{-U_j}^{U_j}\left(\tilde{\psi}_{\nu,j}(w)-\psi_{\nu,j}(w)\right) w_{\nu_j}^{U_j}(w)\cdot\exp(-iwx)\,\mathrm{d}w + \frac{\hat{\sigma}_j^2-\sigma_j^2}{2}\int_{-U_j}^{U_j} w^2 w_{\nu_j}^{U_j}(w)\cdot\exp(-iwx)\,\mathrm{d}w \\
&\quad - i(\hat{\gamma}_j-\gamma_j)\int_{-U_j}^{U_j} w w_{\nu_j}^{U_j}(w)\cdot\exp(-iwx)\,\mathrm{d}w + (\hat{\lambda}_j-\lambda_j)\int_{-U_j}^{U_j} w_{\nu_j}^{U_j}(w)\exp(-iwx)\,\mathrm{d}w \\
&\quad + \int_{\mathbb{R}\setminus[-U_j,U_j]}\left(\psi_{\nu,j}(w)+\frac{\sigma_j^2}{2}w^2 - i\gamma_j w+\lambda_j\right)\left(1-w_{\nu_j}^{U_j}(w)\right)\cdot\exp(-iwx)\,\mathrm{d}w\Bigg] \\
&=: \Psi+\Sigma+\Gamma+\Lambda+B.
\end{aligned}
$$

From Theorem 1. it follows that Σ, Γ and Λ converge all with the same rate to 0. The bias sqaured B^2 can be estimated according to the following line of reasoning

$$
\begin{aligned}
B^2 &\leq \int_{\mathbb{R}\setminus[-U_j,U_j]}\left|\left(\psi_{\nu,j}(u)+\frac{\sigma_j^2}{2}u^2 - i\gamma_j u+\lambda_j\right)\left(1-w_{\nu_j}^{U_j}(u)\right)\cdot\exp(-iux)\right|^2\,\mathrm{d}u \\
&= \int_{\mathbb{R}}\left|\mathcal{F}\nu_j(u)\left(1-w_{\nu_j}^{U_j}(u)\right)\right|^2\,\mathrm{d}u \leq \int_{\mathbb{R}}|\mathcal{F}\nu_j(u)|^2\cdot\frac{u^{2s}}{U_j^{2s}}\,\mathrm{d}u \\
&= U_j^{-2s}\int_{\mathbb{R}}|\mathcal{F}\nu_j(u)|^2\cdot u^{2s}\,\mathrm{d}u = U_j^{-2s}\int_{\mathbb{R}}\left|\frac{\mathcal{F}\nu_j^{(s)}(u)}{(-iu)^s}\right|^2\cdot u^{2s}\,\mathrm{d}u \\
&= U_j^{-2s}\int_{\mathbb{R}}\left|\mathcal{F}\nu_j^{(s)}(u)\right|^2\,\mathrm{d}u = U_j^{-s}\left\|\nu_j^{(s)}\right\|_{L^2(\mathbb{R})}^2
\end{aligned}
\tag{C.1}
$$

Thus the bias converges to 0 with the order U^{-s} and thus, similarly to Proposition 5, the scaled bias converges to 0. Hence, we are left with Ψ. We will split Ψ up into a linear term and a remainder term.

$$
\begin{aligned}
\Psi &= \frac{1}{2\pi}\int_{-U_j}^{U_j}\left(\tilde{\psi}_{\nu,j}(u)-\psi_{\nu,j}(u)\right) w_{\nu_j}^{U_j}(u)\cdot\exp(-iux)\,\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-U_j}^{U_j}\left(\log(\tilde{\varphi}_{\nu,T_j}(u))-\log(\varphi_{\nu,T_j}(u))\right) w_{\nu_j}^{U_j}(u)\cdot\exp(-iux)\,\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-U_j}^{U_j}\left(\mathcal{L}_j^0(u)-\mathcal{L}_j^1(u)-\mathcal{R}_j^0(u)+\mathcal{R}_j^1(u)\right) w_{\nu_j}^{U_j}(u)\cdot\exp(-iux)\,\mathrm{d}u
\end{aligned}
$$

with $\mathcal{L}_j^k$ the linear error term en $\mathcal{R}_j^k$ the remainder error term defined as

$$
\mathcal{L}_j^k(u) = \frac{1}{T_j-T_{j-1}}\cdot\frac{\tilde{\varphi}_{\nu,T_{j-k}}(u)-\varphi_{\nu,T_{j-k}}(u)}{\varphi_{\nu,T_{j-k}}(u)} \quad\text{and}\quad \mathcal{R}_j^k(u) = \tilde{\psi}_j^k(u)-\psi_j^k(u)-\mathcal{L}_j^k(u)
$$

for $k=0$ and $k=1$.

The remainder term can be bounded similarly to Proposition 6 and it remains to calculate the asymptotic variance of the linear term.

$$
\mathcal{L}_j^k(u) = \frac{1}{T_j-T_{j-1}}\cdot\frac{\tilde{\varphi}_{\nu,T_{j-k}}(u)-\varphi_{\nu,T_{j-k}}(u)}{\varphi_{\nu,T_{j-k}}(u)}
$$

and

$$L^{(k)}_{\sigma^2} = \frac{1}{2\pi}\int_{-U_j}^{U_j} \mathcal{L}^k_j(u) w^{U_j}_{\nu_j}(u)\exp(-iux)\,\mathrm{d}u.$$

This simplifies as follows

$$\begin{aligned}
L^{(k)}_{\sigma^2} &= \frac{1}{2\pi}\int_{-U_j}^{U_j} \mathcal{L}^k_j(u) w^{U_j}_{\nu_j}(u)\exp(-iux)\,\mathrm{d}w = \frac{U_j}{2\pi}\int_{-1}^{1} \mathcal{L}^k_j(uU_j) w^{1}_{\nu_j}(u)\exp(-iuU_jx)\,\mathrm{d}u \\
&= \frac{U_j}{2\pi}\int_{0}^{1} \mathcal{L}^k_j(uU_j) w^{1}_{\nu_j}(u)\exp(-iuU_jx)\,\mathrm{d}u + \frac{U_j}{2\pi}\int_{0}^{1} \mathcal{L}^k_j(-uU_j) w^{1}_{\nu_j}(-u)\exp(iuU_jx)\,\mathrm{d}u \\
&= \frac{U_j}{2\pi}\int_{0}^{1} \mathcal{L}^k_j(uU_j) w^{1}_{\nu_j}(u)\exp(-iuU_jx)\,\mathrm{d}u + \frac{U_j}{2\pi}\int_{0}^{1} \overline{\mathcal{L}^k_j(uU_j)} w^{1}_{\nu_j}(u)\exp(iuU_jx)\,\mathrm{d}u \\
&= \frac{U_j}{\pi}\int_{0}^{1} \mathrm{Re}\left[\mathcal{L}^k_j(uU_j) w^{1}_{\nu_j}(u)\cdot\exp(-iuU_jx)\right]\,\mathrm{d}u \\
&= \frac{U_j}{\pi}\int_{0}^{1} \mathrm{Re}\left(\frac{1}{T_j - T_{j-1}}\cdot\frac{\tilde{\varphi}_{\nu,T_{j-k}}(uU_j) - \varphi_{\nu,T_{j-k}}(uU_j)}{\varphi_{\nu,T_{j-k}}(uU_j)} w^{1}_{\nu_j}(u)\cdot\exp(-iuU_jx)\right)\,\mathrm{d}u \\
&= \frac{U_j}{\pi}\int_{0}^{1} \mathrm{Re}\left(\frac{-uU_j(uU_j+i)\cdot\left(\mathcal{F}(\tilde{\mathcal{O}}_{j-k}(x-\overline{r}T_{j-k}))(uU_j) - \mathcal{F}(\mathcal{O}_{j-k}(x-\overline{r}T_{j-k}))(uU_j)\right)}{(T_j - T_{j-1})\varphi_{\nu,T_{j-k}}(uU_j)\left(w^{1}_{\nu_j}(u)\right)^{-1} e^{iuU_jx}}\right)\,\mathrm{d}u
\end{aligned}$$

We will now use the following formula with $m = j - k$

$$\begin{aligned}
\varphi_{\nu,T_m}(u) = \exp\Bigg(&\frac{-u^2}{2}\cdot\left(\sum_{r=1}^{m}(T_r - T_{r-1})\sigma_r^2\right) + iu\cdot\left(\sum_{r=1}^{m}(T_r - T_{r-1})\gamma_r\right) \\
&- \left(\sum_{r=1}^{m}(T_r - T_{r-1})\lambda_r\right) + \left(\sum_{r=1}^{m}(T_r - T_{r-1})\mathcal{F}\nu_r(u)\right)\Bigg) \\
=:& \exp\left(-\frac{u^2}{2}\cdot A_m + iu\cdot B_m + C_m + D_m(u)\right).
\end{aligned}$$

Inserting this into the equation yields

$$\begin{aligned}
L^{(k)}_{\sigma^2} &= \frac{U_j}{\pi}\int_{0}^{1} \mathrm{Re}\left(\frac{-uU_j(uU_j+i)\cdot\left(\mathcal{F}(\tilde{\mathcal{O}}_{j-k}(x-\overline{r}T_{j-k}))(uU_j) - \mathcal{F}(\mathcal{O}_{j-k}(x-\overline{r}T_{j-k}))(uU_j)\right)e^{-iuU_jx}}{(T_j - T_{j-1})\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)\left(w^{1}_{\nu_j}(u)\right)^{-1}}\right)\,\mathrm{d}u \\
&= -\frac{U_j}{\pi}\mathrm{Re}\left(\int_{0}^{1}\frac{uU_j(uU_j+i)\cdot\left(\sum_{r=1}^{N}\delta_{m,r}\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)\varepsilon_{m,r}\right)\exp(-iuU_jx)}{(T_j - T_{j-1})\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)\left(w^{1}_{\nu_j}(u)\right)^{-1}}\,\mathrm{d}u\right) \\
&= -\frac{U_j}{\pi}\sum_{r=1}^{N}\delta_{m,r}\varepsilon_{m,r}\,\mathrm{Re}\left(\int_{0}^{1}\frac{uU_j(uU_j+i)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)w^{1}_{\nu_j}(u)\exp(-iuU_jx)}{(T_j - T_{j-1})\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right).
\end{aligned}$$

The variance V of $L^{(k)}_{\sigma^2}$ can now be calculated as

$$V = \frac{U_j^2}{\pi^2}\sum_{r=1}^{N}\delta_{m,r}^2\left(\mathrm{Re}\left(\int_{0}^{1}\frac{uU_j(uU_j+i)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)w^{1}_{\nu_j}(u)\exp(-iuU_jx)}{(T_j - T_{j-1})\exp\left(-u^2U_j^2A_m/2 + iuU_jB_m + C_m + D_m(uU_j)\right)}\,\mathrm{d}u\right)\right)^2$$

Applying the identity

$$(\mathrm{Re}\,z)^2 = \frac{1}{4}z^2 + \frac{1}{2}z\overline{z} + \frac{1}{4}\overline{z}^2$$

yields

$$
\begin{aligned}
V = \frac{U_j^2}{\pi^2} \sum_{r=1}^{N} \delta_{m,r}^2 \Bigg(& \frac{1}{4} \left(\int_0^1 \frac{uU_j(uU_j+i)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)w_{\nu_j}^1(u)\exp(-iuU_jx)}{(T_j-T_{j-1})\exp\left(-u^2U_j^2A_m/2+iuU_jB_m+C_m+D_m(uU_j)\right)}\,\mathrm{d}u \right)^2 \\
& + \frac{1}{2} \left| \int_0^1 \frac{uU_j(uU_j+i)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)w_{\nu_j}^1(u)\exp(-iuU_jx)}{(T_j-T_{j-1})\exp\left(-u^2U_j^2A_m/2+iuU_jB_m+C_m+D_m(uU_j)\right)}\,\mathrm{d}u \right|^2 \\
& + \frac{1}{4} \overline{\left(\int_0^1 \frac{uU_j(uU_j+i)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)w_{\nu_j}^1(u)\exp(-iuU_jx)}{(T_j-T_{j-1})\exp\left(-u^2U_j^2A_m/2+iuU_jB_m+C_m+D_m(uU_j)\right)}\,\mathrm{d}u \right)^2} \Bigg) \\
=: {} & V_1 + V_2 + V_3
\end{aligned}
$$

The three parts V_1, V_2, V_3 above will be computed separately, where it is chosen to start with V_1

$$
V_1 = \frac{U_j^2}{4\pi^2(T_j-T_{j-1})^2} \sum_{r=1}^{N} \delta_{m,r}^2 \left(\int_0^1 \frac{uU_j(uU_j+i)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)w_{\nu_j}^1(u)\exp(-iuU_jx)}{\exp\left(-u^2U_j^2A_m/2+iuU_jB_m+C_m+D_m(uU_j)\right)}\,\mathrm{d}u \right)^2 .
$$

The integral I_1 inside the summation is simplified according to the following line of reasoning

$$
\begin{aligned}
I_1 &:= \left(\int_0^1 \frac{uU_j(uU_j+i)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)w_{\nu_j}^1(u)\exp(-iuU_jx)}{\exp\left(-u^2U_j^2A_m/2+iuU_jB_m+C_m+D_m(uU_j)\right)}\,\mathrm{d}u \right)^2 \\
&= U_j^2 \exp(-2C_m) \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot (i+uU_j)(i+vU_j) \cdot g(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&= U_j^2 \exp\left(-2C_m\right) \cdot \Bigg(- \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot g(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&\qquad + iU_j \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot (u+v) g(u,v)\,\mathrm{d}u\,\mathrm{d}v \\
&\qquad + U_j^2 \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot uv g(u,v)\,\mathrm{d}u\,\mathrm{d}v \Bigg).
\end{aligned}
$$

with g defined as

$$
g(u,v) := \frac{\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)w_{\nu_j}^1(u)\exp(-iuU_jx)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(vU_j)w_{\nu_j}^1(v)\exp(-ivU_jx)}{\exp\left(i(u+v)U_jB_m+D_m(uU_j)+D_m(vU_j)\right)},
$$

where $\overline{r}$ denotes the risk-free interest rate. Similarly,

$$
\begin{aligned}
I_3 = U_j^2 \exp\left(-2C_m\right) \cdot \Bigg(& - \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot g(-u,-v)\,\mathrm{d}u\,\mathrm{d}v \\
& - iU_j \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot (u+v) g(-u,-v)\,\mathrm{d}u\,\mathrm{d}v \\
& + U_j^2 \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot uv g(-u,-v)\,\mathrm{d}u\,\mathrm{d}v \Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 = U_j^2 \exp\left(-2C_m\right) \cdot \Bigg(& \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot g(u,-v)\,\mathrm{d}u\,\mathrm{d}v \\
& + iU_j \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot (v-u) g(u,-v)\,\mathrm{d}u\,\mathrm{d}v
\end{aligned}
$$

$$+ U_j^2 \int_0^1 \int_0^1 uv \exp\left(A_m\left(u^2+v^2\right)U_j^2/2\right) \cdot uvg(u,-v)\,\mathrm{d}u\,\mathrm{d}v \Bigg)$$

To give a precise argument, it is necessary to define the following function that looks like g

$$\begin{aligned} \tilde{g}_{U_j}(u,v) := uv \cdot & \frac{\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(uU_j)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(vU_j)}{\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(U_j)^2} \cdot w_{\nu_j}^1(u)w_{\nu_j}^1(v) \\ & \cdot \frac{\exp\left(2iU_j(B_m+x)\right)}{\exp\left(i(u+v)U_j(B_m+x)\right)} \cdot \exp\left(-D_m(uU_j)-D_m(vU_j)\right) \end{aligned}$$

and Lemma 7. will be applied next. Please note that, Lemma 7. was adjusted to fit in the proof of Theorem 1., however, it is not difficult to see that it does not need much modification such that the result also holds in this case. Hence, we need to check the conditions on $\tilde{g}_{U_j}$ and we need to find a relevant function f_{U_j}. After normalizing $uv\exp(A_m(u^2+v^2)U_j^2/2)$, we obtain the delta Dirac sequence

$$f_{U_j}(u,v) := A_m^2 U_j^4 \exp\left(-A_m U_j^2\right) uv \exp\left(A_m(u^2+v^2)U_j^2/2\right) =: F(u)\cdot F(v).$$

It remains to check the conditions of the lemma on the function f_{U_j}.

$$\begin{aligned} \int_{1-U_j^{-3/2}}^1 F(u)\,\mathrm{d}u &= \exp\left(-A_m U_j^2/2\right) \int_{1-U_j^{-3/2}}^1 uA_m U_j^2 \exp\left(A_m u^2 U_j^2/2\right) \\ &= \exp\left(-A_m U_j^2/2\right) \cdot \left[\exp\left(A_m u^2 U_j^2/2\right)\right]_{u=1-U_j^{-3/2}}^1 \\ &= \exp\left(-A_m U_j^2/2\right) \cdot \left[\exp\left(A_m U_j^2/2\right) - \exp\left(A_m\left(1-U_j^{-3/2}\right)^2 U_j^2/2\right)\right] \\ &= 1 - \exp\left(A_m U_j^2\left[\left(1-2U_j^{-3/2}+U_j^{-3}\right)-1\right]/2\right) \\ &= 1 - \exp\left(-A_m U_j^{1/2} + A_m U_j^{-1}/2\right) \end{aligned}$$

Hence,

$$\begin{aligned} \lim_{U_j\to\infty} \int_{1-U_j^{-3/2}}^1 \int_{1-U_j^{-3/2}}^1 f_{U_j}(u,v)\,\mathrm{d}u\,\mathrm{d}v &= \lim_{U_j\to\infty}\left(\int_{1-U_j^{-3/2}}^1 F(u)\,\mathrm{d}u\right)^2 \\ &= \left(\lim_{U_j\to\infty} 1-\exp\left(-A_m U_j^{1/2}+A_m U_j^{-1}/2\right)\right)^2 = 1. \end{aligned}$$

Checking the other condition of the function f_{U_j} is left to the reader and the problem simplifies to checking boundedness of $\tilde{g}_U$ on the unit square and

$$\lim_{U_j\to\infty} \sup_{(u,v)\in[1-U^{-3/2},1]^2} |\tilde{g}_{U_j}(u,v) - \tilde{g}_{U_j}(1,1)| = 0.$$

The latter is a small modification to Lemma 8.

From this, it appears that all the integrals in the final expressions for I_1, I_2 and I_3 converge equally fast to 0. Hence, the dominating term is the last one by the U_j^4 factor in front of it. Henceforth, the first two integrals will be left out of the equation.

$$\begin{aligned} &\lim_{U_j\to\infty} I_1 \cdot A_m^2 \exp(-A_m U_j^2)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(U_j)^{-2}\exp(2iU_j(B_m+x)) \\ &\qquad = \lim_{U\to\infty} \exp(-2C_m) \int_0^1 \int_0^1 f_{U_j}(u,v)\tilde{g}_{U_j}(u,v)\,\mathrm{d}u\,\mathrm{d}v \end{aligned}$$

$$= \exp(-2C_m) \lim_{U\to\infty} \tilde{g}_{U_j}(1,1)$$
$$= \exp(-2C_m) \lim_{U\to\infty} \exp\left(-2D_m(U_j)\right) w^1_{\nu_j}(1)^2$$
$$= \exp(-2C_m) w^1_{\nu_j}(1)^2$$

Similarly,

$$\lim_{U_j\to\infty} I_3 \cdot A_m^2 \exp(-A_m U_j^2)\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(-U_j)^{-2}\exp(-2iU_j(B_m+x))$$
$$= \exp(-2C_m) w^1_{\nu_j}(1)^2$$

and

$$\lim_{U_j\to\infty} I_2 \cdot A_m^2 \exp(-A_m U_j^2)|\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(U_j)|^{-2}$$
$$= \exp(-2C_m) w^1_{\nu_j}(1)^2.$$

We can now compute the asymptotic variance where the swap of the limit and the sum is verified by Lemma 9.

$$\lim_{U_j\to\infty} \Delta_m^{-1} U_j^{-2} A_m^2 \exp(-A_m U_j^2) V_1$$
$$= \frac{1}{4\pi^2(T_j - T_{j-1})^2} \lim_{U_j\to\infty} \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 A_m^2 \exp(-A_m U_j^2) I_1$$
$$= \frac{1}{4\pi^2(T_j - T_{j-1})^2} \lim_{U_j\to\infty} \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 \exp(-2C_m) w^1_{\nu_j}(1)^2$$
$$\cdot \mathcal{F}(b_{m,r}(x-\overline{r}T_m))(U_j)^2 \exp\left(-2iU_j(B_m+x)\right)$$
$$= \frac{\exp(-2C_m) w^1_{\nu_j}(1)^2}{4\pi^2(T_j - T_{j-1})^2} \cdot \lim_{U_j\to\infty} \exp\left(-2iU_j(B_m+x)\right) \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 \mathcal{F}(b_{m,r}(x-\overline{r}T_m))(U_j)^2$$

Similarly,

$$\lim_{U\to\infty} \Delta_m^{-1} U_j^{-2} A_m^2 \exp(-A_m U_j^2) V_3$$
$$= \frac{\exp(-2C_m) w^1_{\nu_j}(1)^2}{4\pi^2(T_j - T_{j-1})^2} \lim_{U_j\to\infty} \exp\left(2iU_j(B_m+x)\right) \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 \mathcal{F}(b_{m,r}(x-\overline{r}T_m))(-U_j)^2$$

and

$$\lim_{U\to\infty} \Delta_m^{-1} U_j^{4} A_m^2 \exp(-A_m U_j^2) V_2$$
$$= \frac{\exp(-2C_m) w^1_{\nu_j}(1)^2}{2\pi^2(T_j - T_{j-1})^2} \lim_{U_j\to\infty} \Delta_m^{-1} \sum_{r=1}^{N} \delta_{m,r}^2 |\mathcal{F}(b_{m,r}(x-\overline{r}T_m))(U_j)|^2.$$

To compute the limit, it is necessary to find $\mathcal{F}(b_{m,r}(x-\overline{r}T_m))$. First, we know that the Fourier transform of the unit box function with limits $-1/2$ and $1/2$ is equal to the sinc $= \sin(\bullet)/\bullet$ function. Hence the Fourier transform of the convolution of two unit box functions is equal to sinc^2, whereas the convolution is equal to the triangular function Λ with limits -1 and 1. Define $c_{m,r}(x) = b_{m,r}(x - \overline{r}T_m)$, then $b_{m,r}(x) = c_{m,r}(x + \overline{r}T_m)$. Moreover,

$$b_{m,r}(\Delta_m x + x_r) = \Lambda\left(x\right).$$

Thus

$$c_{m,r}(\Delta_m x + x_r + \overline{r}T_m) = \Lambda(x).$$

Then, similarly as before (C.2)

$$\begin{aligned}\operatorname{sinc}^2(u) &= \int_{-1}^{1} \exp(iux)\Lambda(x)\,\mathrm{d}x = \Delta_m^{-1}\int_{x_{r-1}}^{x_{r+1}} \exp\left(iu\cdot\frac{y-x_r}{\Delta_m}\right) b_{m,r}(y)\,\mathrm{d}y \\ &= \Delta_m^{-1}\int_{x_{r-1}}^{x_{r+1}} \exp\left(iu\cdot\frac{y-x_r}{\Delta_m}\right) c_{m,r}(y+\overline{r}T_m)\,\mathrm{d}y \\ &= \Delta_m^{-1}\int_{x_{r-1}+\overline{r}T_m}^{x_{r+1}+\overline{r}T_m} \exp\left(iu\cdot\frac{x-\overline{r}T_m-x_r}{\Delta_m}\right) c_{m,r}(x)\,\mathrm{d}x \\ &= \Delta_m^{-1}\exp\left(-iu(x_r+\overline{r}T_m)\Delta_m^{-1}\right)\mathcal{F}c_{m,r}\left(u\Delta_m^{-1}\right).\end{aligned}$$

Thus,

$$\mathcal{F}c_{m,r}(y) = \Delta_m \exp\left(iy(x_r+\overline{r}T_m)\right)\operatorname{sinc}^2\left(y\Delta_m\right). \tag{C.2}$$

Hence,

$$\begin{aligned}&\lim_{U_j\to\infty} \exp\left(-2iU_j(B_m+x)\right)\Delta_m^{-1}\sum_{r=1}^{N}\delta_{m,r}^2\mathcal{F}c_{m,r}(U_j)^2 \\ &\quad= \lim_{U_j\to\infty} \exp\left(-2iU_j(B_m+x)\right)\operatorname{sinc}^4\left(U_j\Delta_m\right)\sum_{r=1}^{N}\delta_{m,r}^2\exp\left(2iU_j(x_r+\overline{r}T_m)\right)\Delta_m \\ &\quad= \lim_{U_j\to\infty} \exp\left(-2iU_j(B_m+x)\right)\operatorname{sinc}^4\left(U_j\Delta_m\right)\int_{-\infty}^{\infty}\delta_m(x)^2\exp\left(2iU_j(x+\overline{r}T_m)\right)\,\mathrm{d}x \\ &\quad= \lim_{U_j\to\infty} \exp\left(-2iU_j(B_m+x)\right)\operatorname{sinc}^4\left(U_j\Delta_m\right)\int_{-\infty}^{\infty}\delta_m(x-\overline{r}T_m)^2\exp\left(2iU_jx\right)\,\mathrm{d}x \\ &\quad= \lim_{U_j\to\infty} \exp\left(-2iU_j(B_m+x)\right)\operatorname{sinc}^4\left(U_j\Delta_m\right)\mathcal{F}\left(\delta_m^2(x-\overline{r}T_m)\right)(2U_j)\end{aligned}$$

We have assumed δ to be an L^2 function. Thus the shifted δ is still L^2 and hence $\mathcal{F}\left(\delta_m^2(x-\overline{r}T_m)\right)(2U_j)\to 0$ as $U_j\to\infty$ by the Riemann-Lebesgue lemma. Moreover, $\operatorname{sinc}^4(U_j\Delta_m)\to 1$, since $U_j\Delta_m\to 0$ as $U_j\to\infty$, and the first term is bounded in norm with 1. So, it is concluded that this sum is equal to 0 in the limit. Similarly,

$$\lim_{U\to\infty}\exp\left(2i(B_m+x)U_j\right)\Delta_m^{-1}\sum_{r=1}^{N}\delta_{m,r}^2\mathcal{F}c_{m,r}(-U_j)^2 = 0$$

and

$$\begin{aligned}\lim_{U_j\to\infty}\Delta_m^{-1}\sum_{r=1}^{N}\delta_{m,r}^2|\mathcal{F}c_{m,r}(U_j)|^2 &= \lim_{U_j\to\infty}\operatorname{sinc}^4\left(U_j\Delta_m\right)\sum_{r=1}^{N}\delta_{m,r}^2\Delta_m \\ &= \lim_{U_j\to\infty}\operatorname{sinc}^4\left(U_j\Delta_m\right)\int_{-\infty}^{\infty}\delta_m(x)^2\,\mathrm{d}x \\ &= \lim_{U_j\to\infty}\operatorname{sinc}^4(U_j\Delta_m)\left\|\delta_m\right\|_{L^2}^2 = \left\|\delta_m\right\|_{L^2}^2.\end{aligned}$$

Concluding, as before in (10.8), the asymptotic variance of $L_{\sigma^2}^{(k)}$ is equal to

$$\begin{aligned}V_{\mathrm{Re}^2 z}^k &= \Delta_m U_j^2 A_m^{-2}\exp(A_mU^2)\cdot\frac{\exp(-2C_m)}{2\pi^2(T_j-T_{j-1})^2}\left\|\delta_m\right\|_{L^2}^2 \\ &= 2^{-1}\pi^{-2}\left\|\delta_m\right\|_{L^2}^2(T_j-T_{j-1})^{-2}A_m^{-2}\exp(-2C_m)w_{\nu_j}^1(1)^2\cdot\Delta_mU_j^2\exp(A_mU_j^2)\end{aligned}$$

If we define

$$d_{j,m} = 2\left\|\delta_m\right\|_{L^2}^2(T_j-T_{j-1})^{-2}A_m^{-2}\exp(-2C_m),$$

we can write the asymptotic variance AV of $\hat{\sigma}_j^2 - \sigma_j^2$ as

$$AV = U_j^2 \cdot \left(\frac{d_{j,j}}{(2\pi)^2} \cdot \Delta_j \exp\left(A_j U_j^2\right) + \frac{d_{j,j-1}}{(2\pi)^2} \cdot \Delta_{j-1} \exp\left(A_{j-1} U_j^2\right) \right) w_{\nu_j}^1(1)^2.$$

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