TU Delft

Master thesis Applied Mathematics

The Grothendieck property of Weak L^p spaces and Marcinkiewicz spaces

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Introduction

In this master thesis the proof of Lotz in [40] that Weak L^p spaces have the Grothendieck property is studied. The proof is *slightly modified to be more explicit and easier to comprehend* by introducing lemma's to better separate different parts of the proof that more clearly reveal its structure. Furthermore, the more general *Marcinkiewicz spaces are shown to sometimes have the Grothendieck property*, using the sufficient conditions for a Banach lattice to have the Grothendieck property that Lotz derived in [40] to prove the Grothendieck property of Weak L^p spaces. For most of these conditions that together are sufficient, proving that Marcinkiewicz spaces satisfy them is done in a way very similar to the case of Weak L^p spaces. However, the proof of the (necessary) condition that the dual sometimes has order continuous norm does not allow for such a simple generalization and requires more work. Finally, by using some more recent results [19] about the existence of symmetric functionals in the dual, *the conditions that are given for the Grothendieck property of Marcinkiewicz spaces are shown to be necessary*, and we thereby obtain a characterization of the Marcinkiewicz spaces that have the Grothendieck property.

Chapter 1 briefly summarizes well-known theory about Banach spaces required in the subsequent chapters: the weak topology, the Grothendieck property and characterizations thereof. In chapter 2 Banach lattices are introduced and Lotz' sufficient conditions for the Grothendieck property of a Banach lattice are derived. In chapter 3 the Weak L^p spaces and Marcinkiewicz spaces are introduced and after that in chapter 4 and 5 the Grothendieck property of these spaces on intervals with Lebesgue measure is studied, and finally in chapter 6 the Grothendieck property is studied on arbitrary measure spaces.

Contents

1 Grothendieck Banach spaces

The central mathematical structure in Section 3 will be a Banach lattice. However, since most of the results also hold in the more general case of a Banach space, that is, without the extra structure from the order, the introduction of a Banach lattice will be postponed to next section. First general topologies are considered, then weak topologies in Banach spaces will be introduced, reflexivity will be discussed, and at the end of the chapter the Grothendieck property will be defined and some equivalences will be derived.

1.1 Banach space preliminaries

All vector spaces considered are assumed to be real. Since primarily Banach lattices will be considered afterwards, this will turn out not te be a restrictive assumption.

Because dual spaces can also be defined for ordered vector spaces introduced in the next section, it is important to distinguish this dual from the continuous dual defined for normed vector spaces, especially because some results in the next section hold in the more general case of ordered vector spaces.

Definition 1.1. *The* algebraic dual *of a vector space E is denoted by $E^{\#}$ and consists of all linear functionals on E to $\mathbb{R}$ with a vector space structure given by $\big(u^{\#} + v^{\#}\big)(u) = u^{\#}(u) + v^{\#}(u)$ and $\big(\alpha u^{\#}\big)(u) = \alpha u^{\#}(u)$ for $u \in E$, $u^{\#} \in E^{\#}$, $v^{\#} \in E^{\#}$ and $\alpha \in \mathbb{R}$.*

If E is a vector space, then the evaluation of an element $u^{\#}$ of the algebraic dual $E^{\#}$ to an element u of the original space E, will also be denoted using the *natural pairing* $\langle \bullet\,, \bullet \rangle$, defined by $u^{\#}(u) = \big\langle u, u^{\#} \big\rangle$.

Definition 1.2. *The* continuous dual *of a normed vector space E is denoted by E' and consists of all $u' \in E^{\#}$ continuous in the topology induced by the norm on E, equipped with the norm $\|u'\| = \sup\{|\langle u, u' \rangle| : u \in \mathrm{ball}(E)\}$, where $\mathrm{ball}(E) = \{u \in E : \|u\| \leq 1\}$ is the closed unit ball of E.*

Definition 1.3. *A* Banach space *E is a complete normed vector space.*

The continuous dual is always complete.

Definition 1.4. *The continuous linear map j_E from a Banach space E into its bidual E'' defined by $(j_E(u))(u') = u'(u)$ for $u \in E$ and $u' \in E'$ will be called the* evaluation map *or* natural embedding of E into E''.

The evaluation map is an isometry, by the Hahn-Banach theorem.

1.2 Topological preliminaries

Before introducing the weak topology on Banach spaces, first some general topological definitions and remarks will be recalled.

Definition 1.5. *The* initial topology *on a set E, with respect to a collection H of real-valued functions on E, is the coarsest topology such that each $h \in H$ is continuous and is denoted by $\sigma(E, H)$.*

Lemma 1.6. *(See [13, Proposition 3.4 and 3.5]) Let E be a vector space and $H \subseteq E^{\#}$.*

(i) The topology $\sigma(E, H)$ is generated by the sets $h^{-1}(O)$ with $h \in H$ and O open in $\mathbb{R}$, that is, the following is a subbase's:

$$\left\{h^{-1}(O) : h \in H, O \subseteq \mathbb{R} \,\text{open}\right\}.$$

(ii) A neighbourhood basis of 0 of the topology $\sigma(E, H)$ is the collection of sets of the form

$$\{u \in E : \forall n \in \{1, \ldots, N\} : |\langle u, h_n\rangle| < \epsilon\}$$

with $N \in \mathbb{N}$, $h_1, \ldots, h_N \in H$ and $\epsilon > 0$.

(iii) the set E with the topology $\sigma(E, H)$ is a topological vector space and in particular $\sigma(E, H)$ is translation invariant

(iv) $\sigma(E, H)$ is Hausdorff if and only if H separates the points on E.

(v) A net (or sequence) $(u_t)_{t\in\mathbb{T}}$ in E converges to u in E in the topology $\sigma(E, H)$ if and only if $(h(u_t)_t)_{t\in\mathbb{T}}$ converges to $h(u)$ in the topology of $\mathbb{R}$ for every $h \in H$.

Definition 1.7. *A subset A of a topological Hausdorff space X is called*

1. compact *if and only if every open cover of A has a finite subcover, or, equivalently, if and only if every net in A has a subnet with limit in A.*
2. relatively compact *if and only if the closure $\bar{A}$ is compact, or, equivalently, if and only if every net in A has a subnet with limit in X.*
3. sequentially compact *if and only if every sequence in A has a subsequence with limit in A.*
4. sequentially relatively compact *if and only if every sequence in A has a subsequence with limit in X.*

In an arbitrary topology all nets must have convergent subnets in order to be compact, but the following result shows that for metrizable topologies this is equivalent to the sequential case, that is, the notions of (relative) compactness and (relative) sequential compactness coincide in metric spaces.

Lemma 1.8. *(See [26, Theorem 9.2 and Theorem 9.5]) In a metric space, (relative) compactness and (relative) sequential compactness coincide.*

The following very simple observation will become useful in the proof of Lemma 1.30.

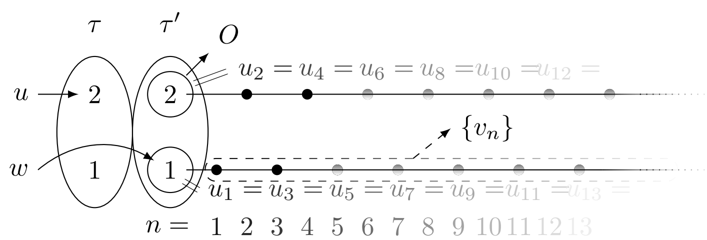
Lemma 1.9. *A subset A of a topological space X is relatively sequentially compact if and only if all sequences in A are relatively sequentially compact.*

Proof. The condition is trivially necessary. To show it is also sufficient, assume all sequences in A to be relatively sequentially compact and let a sequence $(u_n)_{n=1}^{\infty}$ in A be given. To show A is relatively sequentially compact, we need to show that $(u_n)_{n=1}^{\infty}$ has a subsequence with limit in X. Since $(u_n)_{n=1}^{\infty}$ is relatively sequentially compact, every subsequence has a subsequence with limit in X. Since the sequence $(u_n)_{n=1}^{\infty}$ is a subsequence of itself, relative sequential compactness of A has been proven. □

Lemma 1.10. *Let X be a set equipped with two Hausdorff topologies τ and τ'. Let τ' be a stronger topology than τ. If a sequence $(u_n)_{n=1}^{\infty}$ converges in τ and is relatively compact in τ', it converges also with respect to τ'.*

Proof. Assume by contradiction that $(u_n)_{n=1}^{\infty}$ converges to u in the Hausdorff topology τ, but it does not converge to u in the stronger Hausdorff topology τ', while it is relatively compact in τ'. Since $(u_n)_{n=1}^{\infty}$ does not converge in τ', there exists a subsequence $(v_n)_{n=1}^{\infty}$ and a τ'-open neighborhood O of u such that $v_n \notin O$ for all n.

Since $\{u_n\}_{n=1}^{\infty}$ is τ' relatively compact, the subsequence $(v_n)_{n=1}^{\infty}$ has an accumulation point w with respect to τ'. Clearly $w \neq u$. Since the topology τ is weaker than the topology τ', w is also an accumulation point of $(v_n)_{n=1}^{\infty}$ with respect to τ. Since τ is Hausdorff it follows that $w = u$, which is a contradiction. □



Definition 1.11. *If X is a topological space, then X will be called* extremally disconnected *if closures of open sets are open and X will is called* Stonian *if additionally it is compact and Hausdorff.*

1.3 Weak topologies in Banach spaces

Before introducing the Grothendieck property as a generalization of reflexivity, an alternative characterization for reflexivity will be derived concerning weak topologies.

Definition 1.12. *For a Banach space E, the topologies $\sigma(E, E')$ on E and $\sigma(E', j_E(E))$ on E', are called the weak and weak* topology, respectively.*

Definition 1.13. *A linear operator T from a Banach space E to a Banach space F is called* weakly compact *if and only if images of bounded sets in E are relatively weakly compact in F.*

It is easy to see that the embedding of a Banach space into its bidual is weak-weak* homeomorphic. The following lemma summarizes some of the properties of the weak and weak* topologies and their relationship with separability.

Lemma 1.14. *(See [44, Theorem 3.11 and p. 123 and p. 124]) The weak* topology on a dual is weaker than the weak topology on this dual, which in turn is weaker than the norm topology on the dual. The weak and weak* topologies on E and E' turn their corresponding spaces into Hausdorff locally convex topological vector spaces. For infinite dimensional spaces, the weak and weak* topologies are not complete and not metrizable. A Banach space is separable if and only if the unit ball of its dual is metrizable for the weak* topology, and similarly the unit ball is metrizable for the weak topology if and only if the dual is separable.*

The weak topology of an infinite dimensional space is not metrizable, but quite remarkably Lemma 1.8 also holds for the weak topology. One direction originates from [56] and the other from [20].

Theorem 1.15 (Eberlein-Šmulian theorem)**.** *(Originally from [56] and [20], also in [18, Chapter III]) For a Banach space E, (relative) weak compactness and (relative) sequential weak compactness in E coincide.*

This theorem implies that in order to show that an operator is weakly compact, Definition 1.13, it suffices to check that images of bounded sequences have weakly convergent subsequences.

The following fundamental result about the weak* topology was proved in [7, p. 123] for the separable case and later in [2, Theorem 1.3] for the general case.

Theorem 1.16 (Alaoglu-Bourbaki)**.** *(Originally from [2, Theorem 1.3], also in [18, p. 13]) The closed unit ball of the dual of a Banach space is weak* compact.*

Corollary 1.17. *The closed unit ball of the dual of a separable Banach space is weak* sequentially compact*

Proof. By Alaoglu's theorem, Theorem 1.16, the closed unit ball of the dual is always weak* compact. By Lemma 1.14 the weak* topology is metrizable on the closed unit ball of the dual of a separable Banach space. Since in metric spaces compactness and sequential compactness coincide, by Lemma 1.8, the closed unit ball of the dual is weak* sequentially compact. □

Gantmacher has shown in [23] that in order to check whether a continuous linear operator is weakly compact, one may also consider the so-called adjoint.

Definition 1.18. *For any continuous linear operator T from a Banach space E into a Banach space F, the* adjoint *$T' : F' \to E'$ is defined by $\langle Tu, v\rangle = \langle u, T'v\rangle$ for all $u \in E$ and $v \in F'$.*

For a continuous linear operator $T : E \to F$, the adjoint T' is a continuous linear operator with $\|T\| = \|T'\|$ which is weak*-to-weak* continuous, that is, continuous from $\sigma(F', F)$ to $\sigma(E', E)$.

Theorem 1.19 (Gantmacher-Nakamura)**.** *(Originally from [23, Satz 5] and [45, Theorem 1], also in [43, Corollary 3.5.5]) A continuous linear operator T from a Banach space E into a Banach space F is weakly compact if and only if its adjoint is weakly compact. Another equivalent condition is that the adjoint weak*-to-weak continuous*

Lemma 1.20. *(See [53, Exercise 6, p. 106] or [16, Solution 14.19]) Let E and F be Banach spaces. A continuous linear operator $T : F' \to E'$ is weak*-to-weak* continuous if and only if $T = S'$ for a continuous linear operator $S : E \to F$.*

1.4 Reflexivity

Definition 1.21. *A Banach space E will be called* reflexive *if and only if $j_E(E) = E''$.*

By definition of the weak and weak* topologies, the evaluation map is a homeomorphism from the weak topology to the the weak* topology.

Theorem 1.22 (Goldstine's Theorem)**.** *(Originally from [24, Theorem III], also in [18, p. 13]) The closed unit ball of a Banach space E is dense in the closed unit ball of the bidual E'' with respect to the weak* topology $\sigma(E'', E')$, where E is identified with a subspace of E'' via the natural embedding j_E.*

Theorem 1.23. *The following are equivalent for a Banach space E: (i) E is reflexive, (ii) the unit ball of E is weakly compact, (iii) the weak and weak* topologies on the dual coincide.*

Proof. If E is reflexive, by definition $j_E(E) = E''$, so $\sigma(E', E'') = \sigma(E', j_E(E))$ follows trivially, which proves that (i) implies (iii).

To prove (i) to (ii), first recall that the evaluation map is weak-weak* homeomorphic. By Theorem 1.16 the unit ball in the bidual E'' is weak* compact, and because continuous functions preserve compactness, the preimage of the unit ball in the bidual E'' is weakly compact. This preimage is the unit ball of E, because the evaluation map is assumed to be onto.

Now suppose (ii) holds. Using again that the evaluation map is homeomorphic from the unit ball of E in the weak topology to the unit ball of the bidual E'' in the weak* topology, it follows that the unit ball of E is weak* compact in the bidual E'', thus weak* closed, because the weak* topology is Hausdorff, by Lemma 1.14. Since this image is also dense in the weak* topology by Theorem 1.22, it is the entire unit ball of the bidual E'', so j_E is onto.

Finally suppose (iii) holds. By the Banach–Alaoglu theorem the unit ball in the dual E' is weak* compact, thus weakly compact, so E' is reflexive, by the implication (ii) to (i). It now follows that E is reflexive, since a Banach space is reflexive if and only if its dual is reflexive. □

1.5 Grothendieck spaces: characterizations and necessary conditions

It follows from Theorem 1.23 that the following definition is a generalization of reflexivity.

Definition 1.24. *A Banach space for which all weak* convergent (null) sequences in the dual are also weakly convergent is called a* Grothendieck space *(and the space is said to have the* Grothendieck property*).*

The restriction to null sequences in this definition is allowed by translation invariance of the topologies involved, (iii) of Lemma 1.6. We will see that in the context of Banach lattices, another simplification can be made by considering only sequences of positive pairwise disjoint elements, Theorem 2.21.

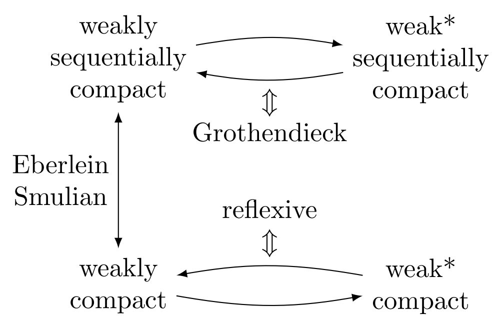
In this section several well-known results about the Grothendieck property will be discussed. Certain classes of Grothendieck spaces, among which the separable and weakly compactly generated, will be shown to be reflexive, which makes them uninteresting Grothendieck spaces. Then a few of the numerous characterizations for the Grothendieck property we be discussed, because they will be used later on in Lotz' proof of the Grothendieck property of Weak L^p spaces. The section ends with the most well-known non-trivial class of Grothendieck spaces due to Grothendieck himself.

The following equivalent condition for the Grothendieck property can be compared with the following: a Banach space is reflexive if and only if every weak* compact subset of the dual is also weakly compact.

Lemma 1.25. *A Banach space has the Grothendieck property if and only if every weak* sequentially (relatively) compact subset of the dual is also weakly sequentially (relatively) compact.*

Proof. First assume the Banach space E to have the Grothendieck property and let $C \subseteq E'$ be weak* sequentially relatively compact (the weak* sequentially compact case is similar). Let $(u_n')_{n=1}^\infty$ now be a given sequence of elements in C. Because C is assumed to be weak* sequentially relatively compact, there exists a weak* convergent subsequence $\left(u_{n_m}'\right)_{m=1}^\infty$. Because E is Grothendieck this subsequence is weakly convergent. This shows that C is weakly sequentially relatively compact. By Eberlein-Šmulian, Theorem 1.15, it is weakly relatively compact.

Now assume that every weak* sequentially relatively compact subset of the dual is weakly relatively compact. Let a weak* convergent sequence $(u_n')_{n=1}^\infty$ in the dual E' be given. Such a sequence is weak* sequentially relatively compact as a set. By (iv) of Lemma 1.6, by Lemma 1.14, and by Lemma 1.10, it follows that $(u_n')_{n=1}^\infty$ is also weakly convergent. □



An interesting consequence of Theorem 1.23 and Lemma 1.25 is that the dual of a non-reflexive Banach space with the Grothendieck property never has a weak* sequentially compact unit ball, though unit balls of duals are always weak* compact, by Alaoglu's theorem, Theorem 1.16. This implies that non-reflexive Banach spaces with the Grothendieck property cannot be weakly compactly generated, by the Amir-Lindenstrauss theorem [44, Theorem 4.8]. Since separable spaces are weakly compactly generated, examples of separable spaces with the Grothendieck property are not very interesting, since they have to be reflexive. A direct argument of this follows from Lemma 1.14 and Lemma 1.25.

A Banach space E is weakly sequentially complete if sequences $(u_n)_{n=1}^\infty$ in E are weakly convergent if $(\langle u_n, u' \rangle_n)_{n=1}^\infty$ converges for any $u' \in E'$.

Lemma 1.26. *For any Grothendieck space E its dual E' is weakly sequentially complete.*

Proof. Let $(u_n')_{n=1}^\infty$ be such that $(\langle u_n', u'' \rangle_n)_{n=1}^\infty$ converges for any $u'' \in E''$. In particular $\langle u, u_n' \rangle$ converges for $u \in E$, to say $\phi(u)$. By the Banach-Steinhaus theorem, $(u_n')_{n=1}^\infty$ is norm bounded, so $\phi \in E'$. Thus, $(u_n')_{n=1}^\infty$ is weak* convergent, and by the Grothendieck property is is weakly convergent. □

Theorem 1.27. *(See [43, Proposition 5.3.10]) The following are equivalent for a Banach space E:*

(i) E is a Grothendieck space

(ii) any continuous linear operator from E into an arbitrary separable Banach space is weakly compact.

(iii) any continuous linear operator from E into c_0 is weakly compact

Proof. To prove (i) implies (ii), assume E is a Grothendieck space. By Gantmacher's theorem, Theorem 1.19, it suffices to show that, for any $T \in \mathcal{L}(E, F)$, with F a separable Banach space, T' is weakly compact, that is, for each sequence $(u'_n)_{n=1}^{\infty}$ in the closed unit ball of the dual of F, the image $(T'u'_n)_{n=1}^{\infty}$ has a weakly convergent subsequence. Let $(u'_n)_{n=1}^{\infty}$ be such a sequence. By separability of F and Corollary 1.17, the closed unit ball of the dual F' is weak* sequentially compact, so $(u'_n)_{n=1}^{\infty}$ has a weak* convergent subsequence, say $\left(u'_{n_m}\right)_{m=1}^{\infty}$. Since the adjoint of a continuous operator is always weak*-weak* continuous by Lemma 1.20, T' maps $\left(u'_{n_m}\right)_{m=1}^{\infty}$ to a weak* convergent sequence $\left(T'\left(u'_{n_m}\right)\right)_{m=1}^{\infty}$ in E', which is also weakly convergent by the assumption of the Grothendieck property of E.

It is evident that (ii) implies (iii).

To prove (iii) implies (i), let $(u'_n)_{n=1}^{\infty}$ be a weak* null sequence in the dual E'. Since the weak topology is stronger than the weak* topology and both are Hausdorff, weak relative compactness of the set $\{u'_n\}_{n=1}^{\infty}$ would imply weak convergence of the sequence $(u'_n)_{n=1}^{\infty}$, by Lemma 1.10. To show $\{u'_n\}_{n=1}^{\infty}$ is weakly relatively compact, consider the operator $T \in \mathcal{L}(E, c_0)$ defined by

$$(Tu)_n = \langle u, u'_n \rangle ,$$

for $u \in E$. Since weak* convergence implies norm boundedness (by the uniform boundedness principle), $\sup_n \|u'_n\|$ is finite, and hence T is bounded. By the assumption that T is weakly compact, and by Gantmacher's theorem, Theorem 1.19, T' is also weakly compact. Since the set of coordinate functionals of c_0, $(e'_n)_{n=1}^{\infty}$, is bounded, the image

$$T'\{e'_n\} = \{e'_n \circ T : n \in \mathbb{N}\} = \{u'_n\} ,$$

for $n \in \mathbb{N}$, is relatively weakly compact. ☐

That separable Grothendieck spaces are reflexive also follows from the preceding theorem in the following way. If a Grothendieck space is separable, the identity operator is weakly compact by Theorem 1.27, so, by definition, the image of the closed unit ball is relatively weakly compact. Since convex sets are weakly closed if and only if they are norm closed, by Mazur's theorem, the unit ball is weakly compact. By Theorem 1.23 this implies the space is reflexive.

Clearly the Grothendieck property is an isomorphic invariant of the class of Banach spaces. For the following theorem some notation will have to be introduced. Let F be a closed subspace of a Banach space E. The *quotient space*, denoted by E/F, is the quotient group of E by F with respect to the vector operation $+$. It is a Banach space with norm

$$\|[u]\|_{E/F} = \inf_{v \in [u]} \|v\|_E ,$$

where $u \in E$ and $[\,\cdot\,] : E \to E/F$ is the quotient map.

A closed subspace F is called *complemented* in a Banach space E if and only if there exists a closed subspace G of E such that $E = F \oplus G$. In this case G is isomorphic to E/F and E is isomorphic to $F \times G$ with norm given by

$$\|(u, v)\| = \|u\| + \|v\|$$

for $u, v \in E$.

Lemma 1.28. *If E is a Grothendieck space and $T : E \to F$ is a surjective continuous linear map onto a Banach space F, then F is also a Grothendieck space.*

Proof. By Theorem 1.27 it suffices to show that a given linear continuous map $S : F \to c_0$ is weakly compact. Since E is a Grothendieck space, $S \circ T$ is weakly compact. Since T is a surjective continuous linear map, it is an open map by the open mapping theorem ([53, Theorem 2.11]), so there exists a $\delta > 0$ such that $\delta\,\mathrm{ball}(F) \subseteq T(\mathrm{ball}(E))$, so $S(\mathrm{ball}(F)) \subseteq \frac{1}{\delta}(S \circ T)(\mathrm{ball}(E))$. It follows from weak compactness of $S \circ T$ that $\frac{1}{\delta}(S \circ T)(\mathrm{ball}(E))$ is relatively weakly compact, so $S(\mathrm{ball}(F))$ is relatively weakly compact as well. Therefore, S is weakly compact. ☐

The following two results on the Grothendieck property will be used in Section 6.

Corollary 1.29. *Let E be a Grothendieck space. If the closed subspaces E and F are such that $E = F \oplus G$, then F and G are also Grothendieck spaces. If F is a closed subspace, then E/F is also a Grothendieck space.*

Proof. Apply Lemma 1.28 to the projection map to see that F is Grothendieck whenever a Banach space E is the direct sum of two closed subspaces F and G. Apply Lemma 1.28 to the the quotient map to see that the Grothendieck property is also preserved by quotients. ☐

Lemma 1.30. *Let E be a Banach space. If for every sequence $(u_n)_{n=1}^{\infty}$ in E there exists a closed Grothendieck subspace F such that $\{u_n\} \subseteq F$, then E is a Grothendieck space.*

Proof. The characterization of Grothendieck spaces in Theorem 1.27 is used. Let a continuous linear map $T : E \to c_0$ be given. To show that it is weakly compact, let a bounded sequence $(u_n)_{n=1}^{\infty}$ in E also be given. By assumption there exists a closed Grothendieck subspace F such that $\{u_n\}_{n=1}^{\infty} \subseteq F$. By the Grothendieck property of F, the restriction $T|_F$ is weakly compact, so $(Tu)_{n=1}^{\infty}$ has a weakly convergent subsequence. It follows that E is a Grothendieck space. ☐

Grothendieck spaces are named after Alexander Grothendieck, because he discovered one of the first important non-trivial class of Grothendieck spaces.

Theorem 1.31. *(Originally from [25]) The Banach space of continuous functions $C(K)$ on a Stonian space K is a Grothendieck space.*

This result was improved upon first in [6] and then in [55] requiring K to be an F-space. See the remark after [44, Definition 4.5] for a chronological summary or the introduction of [49].

Note that the space of continuous functions on $[0, 1]$ is not Grothendieck (since it is separable but not reflexive).

1.6 Other generalizations of reflexivity

The Grothendieck property is not the only generalization of reflexivity. Quasi-reflexivity, introduced in [15], generalizes the non-reflexive space isomorphic to its bidual given in [29], with the property that $E''/j_E(E)$ is finite dimensional. Almost reflexivity, defined in [28] and [17] for example, also known as weak conditional compactness, is shown to be equivalent to having no isomorphic copy of ℓ_1 in [51]. Another generalization, closely related to Banach spaces with the Grothendieck property, is the class of weakly compactly generated Banach spaces, [38], [30], [21, Chapter 13], which equal a closed linear span of a weakly compact subset. A quantative Grothendieck property has been introduced in [9] and investigated in [37]. So-called c-Grothendieck spaces are Grothendieck for $c \geq 1$, but there are Grothendieck spaces not c-Grothendieck for any $c \geq 1$.

2 Grothendieck Banach lattices

2.1 Preliminaries on Banach lattices

Banach lattice theory lies in the intersection of functional analysis and order theory. Like the algebraic structure has to be compatible with the norm structure in a normed vector space, the order structure in a Banach lattice has to be compatible with the Banach space.

The terminology concerning Banach lattices is in accordance with the terminology of Meyer-Nieberg [43].

A real vector space E is called an *ordered vector space* if a partial order $\leq$ is defined with $\alpha u + w \leq \alpha v + w$ if $u \leq v$ for $u, v, w \in E$ and $\alpha \geq 0$. An ordered vector space is called a *vector lattice* (or *Riesz space*) if any two elements $u, v \in E$ have a least upper bound $u \vee v$ and greatest lower bound $u \wedge v$. For each element $u \in E$ we define the *positive part* $u^+ = u \wedge 0$, the *negative part* $u^- = (-u) \vee 0$, and the absolute value $|u| = u^+ + u^- = u \vee (-u)$. It follows that $u = u^+ - u^-$. We write E_+ for the set of all $u \in E$ with $u \geq 0$ and we call $[u, v] = \{w \in E : u \leq w \leq v\}$ for u and v in E an *order interval*. A subset of E is called *order bounded* if and only if it is contained in an order interval. Two elements $u, v \in E$ will be called *disjoint* if and only if $|u| \wedge |v| = 0$. This is equivalent with $|u| + |v| = |u| \vee |v|$ and a necessary condition is $|u| + |v| = |u + v|$. A sequence $(u_n)_{n=1}^\infty$ of E will be called disjoint only if the elements in $\{u_n\}_{n=1}^\infty$ are mutually disjoint. Furthermore, we will write $u_\tau \downarrow 0$ for a net $(u_\tau)_{\tau \in \mathbb{T}}$ in a vector lattice with $\inf_{\tau \in \mathbb{T}} u_\tau = 0$ that is decreasing, that is, such that $u_\tau \leq u_\sigma$ for all $\tau, \sigma \in \mathbb{T}$ with $\sigma \leq \tau$.

A vector lattice E has the *countable interpolation property* if for all sequences $(u_n)_{n=1}^\infty$ and $(v_n)_{n=1}^\infty$ in E with $u_n \leq u_{n+1} \leq v_{n+1} \leq v_n$ for all n, there exists a $w \in E$ such that $u_n \leq w \leq v_n$ for all n. A vector lattice E satisfies a stronger condition called *(σ-)Dedekind completeness* if every order bounded (sequence) set in E has a supremum and an infimum in E.

Definition 2.1. *A Banach space E which is also a vector lattice with respect to the same linear structure, is called a* Banach lattice *if $|u| \leq |v|$ implies $\|u\| \leq \|v\|$ for all $u, v \in E$.*

This is not the only way to combine an ordered vector space with a normed vector space. For example, a more general class of Banach spaces with an order structure is the class of *ordered Banach spaces*, see [8].

All vector spaces considered are assumed to be real. The term *complex Banach lattice* sounds contradictory, but it is possible to extend the absolute value on a real Banach lattice to the complexification of the real vector space. See [39], [46], and [48, page 157].

Definition 2.2. *The* order dual *of an ordered vector space is denoted by $E^\sim$ and consists of all $\tilde{u} \in E^\#$ that map order bounded sets into bounded subsets of $\mathbb{R}$. The order given by $\tilde{u} \leq \tilde{v}$ if and only if $(\tilde{u} - \tilde{v})E_+ \subseteq F_+$ for $\tilde{u}, \tilde{v} \in E^\sim$, turns the order dual into a vector lattice.*

The following formulae can be derived for the least upper bound and greatest lower bound of two elements $\tilde{u}, \tilde{v} \in E^\sim$ of the dual:

$$\langle w, \tilde{u} \vee \tilde{v} \rangle = \sup\{\langle u, \tilde{u} \rangle + \langle v, \tilde{v} \rangle : u, v \in E_+ \text{ and } u + v = w\},$$

$$\langle w, \tilde{u} \wedge \tilde{v} \rangle = \inf\{\langle u, \tilde{u} \rangle + \langle v, \tilde{v} \rangle : u, v \in E_+ \text{ and } u + v = w\},$$

for $w \in E_+$. The following *Riesz-Kantorovich* formulae can be derived for the positive part, negative part, and absolute value of an element $\tilde{u} \in E^\sim$ of the order dual:

$$\left\langle u, \tilde{u}^+ \right\rangle = \sup_{0 \leq v \leq u} \langle v, \tilde{u} \rangle, \qquad \left\langle u, \tilde{u}^- \right\rangle = \inf_{0 \leq v \leq u} \langle v, \tilde{u} \rangle, \qquad \langle u, |\tilde{u}| \rangle = \sup_{|v| \leq u} |\langle v, \tilde{u} \rangle|,$$

for $u \in E_+$. From the rightmost expression it immediately follows that for $\tilde{u} \in E^\sim$ and $u \in E$:

$$|\langle u, \tilde{u} \rangle| \leq \langle |u|, |\tilde{u}| \rangle.$$

To see that the order dual $E^\sim$ is Dedekind complete ([4, Theorem 1.18]), let A be an upwards directed order bounded subset of the order dual $E^\sim$ (this is sufficient, since any order bounded set is contained in an upwards directed set with the same upper bounds, by [42, Theorem 15.11]). The infimum and supremum of A can be shown to be $\langle u, \tilde{a} \rangle = \inf_{\tilde{v} \in A} \langle u, \tilde{v} \rangle$ and $\left\langle u, \tilde{b} \right\rangle = \sup_{\tilde{v} \in A} \langle u, \tilde{v} \rangle$ for $u \in E_+^\sim$. Since $\tilde{a}$ and $\tilde{b}$ are linear, they can be extended to the whole of $E^\sim$ by setting $\langle u, \tilde{a} \rangle = \langle u^+, \tilde{a} \rangle - \langle u^-, \tilde{a} \rangle$ and $\left\langle u, \tilde{b} \right\rangle = \left\langle u^+, \tilde{b} \right\rangle - \left\langle u^-, \tilde{b} \right\rangle$ for $u \in E$.

For Banach lattices the order dual coincides with the continuous dual, which is in contrast with the fact that the algebraic dual and the continuous dual coincide if and only if the space is finite dimensional.

Definition 2.3. *Let E and F be Banach lattices. A continuous linear operator $T : E \to F$ is called a* lattice homomorphism *if it $Tu \vee Tv = T(u \vee v)$ and $Tu \wedge Tv = T(u \wedge v)$ for $u, v \in E$.*

The norm of a Banach lattice is called *order continuous* whenever for any net $(u_\tau)_{\tau \in \mathbb{T}}$ with $u_t \downarrow 0$ we have $\|u_t\| \downarrow 0$. For Dedekind complete Banach lattices it suffices to check this for sequences only.

The following is sometimes called Dini's theorem.

Lemma 2.4. *(See [43, Proposition 1.4.1]) Decreasing weakly null sequences are norm null.*

Proof. Let $(u_n)_{n=1}^\infty$ be a decreasing weakly null sequence in a Banach lattice E. Since 0 is clearly contained in the closure of $\{u_n\}_{n=1}^\infty$, certainly 0 is contained in the weak closure of the convex set generated by the elements in the sequence. Since the weak closure of a convex set equals its norm closure, by Mazur's theorem, it follows that 0 is also contained in its norm closure.

Now let $\epsilon > 0$ be given. Then there exists an M, and M positive real numbers $(\alpha_m)_{m=1}^M$ with $\sum_{m=1}^M \alpha_m = 1$, and M different positive natural numbers $(n_m)_{m=1}^M$, such that

$$\left\| \sum_{m=1}^M \alpha_m u_{n_m} \right\| \leq \epsilon.$$

Since $(u_n)_{n=1}^{\infty}$ is decreasing

$$u_{n_M} = \sum_{m=1}^{M} \alpha_m u_{n_M} \leq \sum_{m=1}^{M} \alpha_m u_{n_m}$$

and because the norm preserves order, it follows that $\|u_{n_M}\| \leq \epsilon$. Since $(u_n)_{n=1}^{\infty}$ is decreasing and the norm preserves order, also $\|u_n\| \leq \epsilon$ for any $n \geq n_M$, so $(u_n)_{n=1}^{\infty}$ is a null sequence. □

Lemma 2.5. *Disjoint order bounded sequences in Banach lattices are weakly null.*

Proof. Let u be an element of a Banach lattice E and let $(u_n)_{n=1}^{\infty} \subseteq [0, u]$ be a disjoint sequence. It suffices to show that $\langle u_n, u' \rangle \to 0$ as $n \to 0$ for all positive u' in the dual of E. Let such a u' be given. Then for any $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} \langle u_n, u' \rangle = \left\langle \sum_{n=1}^{N} u_n, u' \right\rangle = \left\langle \bigvee_{n=1}^{N} u_n, u' \right\rangle \leq \langle u, u' \rangle \,,$$

so $\sum_{n=1}^{\infty} \langle u_n, u' \rangle$ is finite, and we have $\langle u_n, u' \rangle \to 0$ as $n \to 0$. □

The following is a sliding hump argument.

Lemma 2.6. *Let $(u_m)_{m=1}^{\infty}$ be a weakly null sequence in a Banach space E. If $(u'_k)_{k=1}^{\infty}$ is a sequence in the dual E' and $\epsilon > 0$ are such that*

$$\sup_{k \in \mathbb{N}} |\langle u_m, u'_k \rangle| > \epsilon \tag{1}$$

for all $m \in \mathbb{N}$, then there exists subsequences $(u_{m_n})_{n=1}^{\infty}$ and $\left(u'_{k_n}\right)_{n=1}^{\infty}$ such that for all $n \in \mathbb{N}$:

$$\left|\left\langle u_{m_n}, u'_{k_n} \right\rangle\right| > \epsilon \,.$$

Proof. Take $m_1 = 1$. By the supremum in Equation (1) for $m = 1$, we can let k_1 be such that $\left|\left\langle x_1, x'_{k_1} \right\rangle\right| > \epsilon$. Suppose that $m_1 < \cdots < m_N$ and $k_1 < \cdots < k_N$ have been constructed such that $\left|\left\langle u_{m_n}, u'_{k_n} \right\rangle\right| > \epsilon$ for $n = 1, \cdots, N$. Since u_m is weakly null, there exists $m_{N+1} > m_N$ such that $\left|\left\langle u_{m_{N+1}}, u'_k \right\rangle\right| < \epsilon$ for all $1 \leq k \leq k_N$. It follows by the supremum in Equation (1) for m_{N+1} that there exists a $k_{N+1} \geq k_N$ such that $\left|\left\langle u_{m_{N+1}}, u'_{k_{N+1}} \right\rangle\right| > \epsilon$. This completes the proof (by induction). □

The following theorem contains just two of the many different equivalent conditions for the dual to have order continuous norm.

Theorem 2.7. *(See [43, Theorem 2.4.14] and [4, Theorem 4.69]) For a Banach lattice E the following statements are equivalent:*

(i) the dual of E has order continuous norm;

(ii) any disjoint bounded sequence in E is weakly null;

(iii) ℓ^1 is not lattice-embeddable in E;

(iv) ℓ^∞ is not lattice-embeddable in the dual of E.

Lemma 2.8. *If E is a Banach lattice for which there exists a constant $0 < \alpha < 1$ such that for any two mutually disjoint positive elements u and v in the unit sphere of E we have $\left\| \frac{u+v}{2} \right\|_E \leq \alpha$ (that is, E is* quasi-uniformly convex *[36, Definition 3.31]), then the norm in the dual of E is order continuous.*

Proof. First we observe what the condition in the statement of the theorem implies for arbitrary mutually disjoint positive non-zero elements u and v. We always have

$$0 \leq u + v \leq (\|u\| \vee \|v\|) \left(\frac{|u|}{\|u\|} + \frac{|v|}{\|v\|} \right),$$

where the last step follows from disjointness. Taking norms and applying the quasi-uniformly convex property yields:

$$\|u + v\| \leq 2\alpha(\|u\| \vee \|v\|) \,. \tag{2}$$

The proof is by contradiction. Suppose the dual E' does not have order continuous norm. It follows from Theorem 2.7 that there exists a disjoint sequence $(u_n)_{n=1}^{\infty}$ in E that is not weakly null. So there exists a norm one ϕ in the dual of E such that $(\langle u_n, \phi \rangle_n)_{n=1}^{\infty}$ is not a null sequence. Since $|\langle u_n, \phi\rangle| \leq \langle |u_n|, |\phi| \rangle$ for all n, by replacing ϕ by $|\phi|$ and u_n by $|u_n|$, it may be assumed that ϕ and all u_n's are positive. Furthermore, by passing, if necessary, to a subsequence of $(u_n)_{n=1}^{\infty}$, it may be assumed also that there exists a positive ϵ such that $\langle u_n, \phi \rangle > \epsilon$ for all n. Since $\langle u_n / \|u_n\|, \phi \rangle \geq \langle u_n, \phi \rangle$ for all n, replacing u_n by $\frac{u_n}{\|u_n\|}$, it may be assumed that $(u_n)_{n=1}^{\infty}$ is normalized.

Define the sequence $\left(v_n^k\right)_{n=1}^{\infty}$ in the convex span of $\{u_n\}_{n=1}^{\infty}$ recursively by $v_n^0 = u_n$ and $v_n^{k+1} = \frac{v_{2n-1}^k + v_{2n}^k}{2}$ for all n. From $\langle u_n, \phi \rangle > \epsilon$ for all n follows $\left\langle v_n^k, \phi \right\rangle > \epsilon$ for all n and all k. From Equation (2), we see that $\left\|v_n^k\right\| \leq \alpha^k$. Combining this we get for all n and all k:

$$\epsilon < \left\langle v_n^k, \phi \right\rangle \leq \|\phi\| \left\|v_n^k\right\| \leq \alpha^k \,,$$

but the latter can be made arbitrary small by choosing k large enough. This is a contradiction and the norm of the dual is thus continuous. □

2.2 L-spaces and M-spaces and representation theorems

Several representation theorems will become convenient in the proofs of this chapter, but in order to be able to present them, first some terminology will be introduced.

First of all, it will become worthwhile to abstract L^1 spaces preserving the property of additivity of the norm. Furthermore a dual notion will be introduced, abstracting from L^∞ spaces, and typical examples of both will be defined.

Definition 2.9. *A Banach lattice E is called an* M-space *if $\|u \vee v\| = \|u\| \vee \|v\|$ for u and v in the positive cone of E.*

Definition 2.10. *A Banach lattice E is called an* L-space *if $\|u + v\| = \|u\| + \|v\|$ for u and v in the positive cone of E.*

Infinite dimensional M-spaces and L-spaces are not reflexive. An element u of a Banach lattice E is called an *order unit* if for all $v \in E$ there exists an $\alpha > 0$ such that $|v| \leq \alpha u$.

Definition 2.11. *For any Banach lattice E, the* principal ideal E_u *of a positive u in E is the set of all $v \in E$ such that $|v| \leq \alpha u$ for some $\alpha > 0$.*

Lemma 2.12. *(See [43, Proposition 1.2.13] or [54, Proposition 7.2]) If the ideal E_u of a Banach lattice E is normed by*

$$\|v\|_u = \inf\{\alpha > 0 : |v| \leq \alpha u\}\,,$$

then E_u becomes an M-space with order unit u.

Proof. Let v and w be arbitrary positive elements of the principal ideal E_u of some Banach lattice E. By definition of the norm on E_u we have $v \leq \|v\|_u u$ and $w \leq \|w\|_u u$, so also $v \vee w \leq (\|v\|_u \vee \|w\|_u)u$, which by definition means $\|v \vee w\|_u \leq \|v\|_u \vee \|w\|_u$. The other inequality is trivial.

From the definition of the norm on E_u it immediately follows that the inclusion of E_u in E is continuous with $\|v\|_E \leq \|u\|_E \|v\|_u$ for $v \in E_u$. A given Cauchy sequence $(v_n)_{n=1}^{\infty}$ in E_u is thus Cauchy in E, and therefore convergent, to say $v \in E$. To show completeness of E_u it thus remains to show that $(v_n)_{n=1}^{\infty}$ converges to v in E_u, and that $v \in E_u$. Let $\epsilon > 0$ be given. Since $(v_n)_{n=1}^{\infty}$ is Cauchy in E_u, there exists an N, such that for any $n, m \geq N$, $|v_n - v_m| \leq \epsilon u$. Since the absolute value is continuous, it follows by taking the limit of m to infinity that $|v_n - v| \leq \epsilon u$ for any $n \geq N$. Firstly this means that $|v| \leq |v_n| + \epsilon u$, so $v \in E_u$. Secondly, by taking norms on both sides, we obtain $\|v_n - v\|_u \leq \epsilon$, so v_n converges to v in E_u. ☐

Theorem 2.13. *(Originally from [32], also in [43, Proposition 1.4.7])*

1. *The dual of an L-space is an M-space with order unit.*
2. *The dual of an M-space is an L-space.*

The following two theorems allow M-spaces and L-spaces to be concretely represented as certain function lattices.

Theorem 2.14. *(Originally from [31, Theorem 2], also in [4, Theorem 4.21 and 4.29]) If E is an M-space with order unit u, there exists a compact Hausdorff topological space K and an isometric and lattice isomorphic map $S : E \to C(K)$ into $C(K)$ such that $S(u) = 1$.*

Theorem 2.15 (Kakutani-Bohnenblust-Nakano)**.** *(Originally from [32], also in [4, Theorem 4.27] or [43, Theorem 2.7.1]) An L-space is isometric and lattice isomorphic to a concrete $L^1(X, \mu)$ space over a locally compact space (X, μ).*

Before the dual of $C(K)$, for compact Hausdorff K, can be described more precisely using the representation theorem Theorem 2.19, a specific L-space will be introduced, consisting of certain signed measures.

Definition 2.16. *A* signed measure *μ is a real-valued function defined on a σ-algebra Σ of subsets of a nonempty set X, such that $\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$ whenever $(A_n)_{n=1}^{\infty}$ are mutually disjoint in Σ.*

The set of signed measures, called ca(Σ) from "countably additive functions on Σ", is given the partial order defined by $\mu \leq \nu$ whenever $\mu(A) \leq \nu(A)$ for all $A \in \Sigma$, for μ and ν in ca(Σ). With this partial order ca(Σ) is a vector lattice with the following expression for the supremum and infimum ([5, page 335])

$$(\mu \vee \nu)(A) = \sup_{A \supseteq B \in \Sigma} (\mu(B) + \nu(A \setminus B)) \quad (\mu \wedge \nu)(A) = \inf_{A \supseteq B \in \Sigma} (\mu(B) + \nu(A \setminus B))\,,$$

for μ and ν in $\mathrm{ca}(\Sigma)$ and $A \in \Sigma$. Furthermore, the following expression for the absolute value of an element μ of $\mathrm{ca}(\Sigma)$, also known as the *variation* of μ, can be derived

$$|\mu|(A) = \sup_{A \supseteq B \in \Sigma} (\mu(B) - \mu(A \setminus B)) = \sup_{\pi \in \Pi(\Sigma)} \sum_{A \in \pi} |\mu(A)| \, ,$$

for $A \in \Sigma$, where $\Pi(\Sigma)$ is the set of all partitions of Σ.

Lemma 2.17. *For $\mu \in \mathrm{ca}(\Sigma)$, define $\|\mu\| = |\mu|(X)$. Then $\mathrm{ca}(\Sigma)$ is an L-space.*

Proof. A proof that $\mathrm{ca}(\Sigma)$ is complete with respect to the given norm can be found in [12, Theorem 4.6.1]. It is trivial that the norm is additive on the positive cone. □

Whenever Σ is the σ-algebra of the Borel sets of a compact Hausdorff space K, we denote $\mathrm{ca}(\Sigma)$ by $\mathrm{ca}(K)$.

Definition 2.18. *If K is a compact Hausdorff space with a σ-algebra of Borel sets Σ, a measure $\mu \in \mathrm{ca}(K)$ is called* regular *if*

$$\sup_{\substack{A \supseteq C \in \Sigma \\ C \text{ compact}}} |\mu|(C) = |\mu|(A) = \inf_{\substack{A \subseteq O \in \Sigma \\ O \text{ open}}} |\mu|(O) \, ,$$

for all $A \in \Sigma$. The set of all regular measures on Σ is denoted by $\mathrm{rca}(K)$, which is a closed ideal of $\mathrm{ca}(K)$ ([3, Theorem 12.12]).

This space $\mathrm{rca}(K)$ is particularly interesting because it is the dual of the space of continuous functions on a compact Hausdorff space K. The duality pairing is given by the integral: $\phi_\mu(u) = \langle u, \mu \rangle = \int_K u \, \mathrm{d}\, \mu$ for $\mu \in \mathrm{rca}(K)$ and $u \in C(K)$.

Theorem 2.19 (Riesz–Markov–Kakutani)**.** *(Originally from [31, Theorem 10]) For any compact Hausdorff space K, the mapping $\mu \mapsto \phi_\mu$ is an isometry and a lattice isomorphism from $\mathrm{rca}(K)$ onto the dual of $C(K)$.*

Combining these representation theorems, Theorems 2.14, 2.15 and 2.19, we get the following corollary:

Corollary 2.20. *The dual of any principal ideal E_u in a Banach lattice E is isometric lattice isomorphic to both $\mathrm{rca}(K)$, for some compact Hausdorff K, and some concrete $L^1(\mu)$.*

2.3 Characterization Grothendieck Banach lattices

The following important result is due to Kühn.

Theorem 2.21. *(Originally from [35, Proposition 1], also in [43, Theorem 5.3.13]) A Banach lattice with the interpolation property whose dual has order continuous norm, is Grothendieck if and only if any weak* null sequence of positive pairwise disjoint elements of the dual converges weakly.*

Proposition 2.22. *The dual of a Banach lattice with the Grothendieck property has an order continuous norm.*

Proof. Since the dual is Dedekind complete, it suffices to consider only sequences $(u'_n)_{n=1}^{\infty}$ of positive elements of the dual E' satisfying $u' \downarrow 0$. It follows directly from the definition of the order on the dual E', see the beginning of Section 2.1, that $(u'_n)_{n=1}^{\infty}$ is a weak* null sequence, that is, the weak* topology is an *order continuous topology.* By the Grothendieck property, such a sequence is thus weakly null. By Lemma 2.4 it follows that such a sequence must be norm null.

□

2.3.1 Lotz' characterization

The following is a small lemma that will be used to rephrase an assertion about open disjoints sets using disjoint functions instead.

Lemma 2.23. *Let X be a compact Hausdorff space and $\mu \in \mathrm{rca}(X)$. Let $\epsilon > 0$ be given.*

1. *If O is an open subset of X then there exists a norm one, $\|f\|_\infty = 1$, continuous function f on X whose support is contained in O, with $\left|\int_X f \, \mathrm{d}\,\mu\right| > |\mu(O)| - \epsilon$.*
2. *If f is a norm one continuous function on X, then there exists an open O subset contained in the support of f, with $|\mu(O)| > \frac{1}{2}\left|\int_X f \, \mathrm{d}\,\mu\right| - \epsilon$.*

Proof. 1. Since $|\mu|$ is regular, there exists a compact $C \subseteq O$ such that $|\mu|(O \setminus C) < \frac{1}{2}\epsilon$. By the triangle inequality we thus have $|\mu(C)| > |\mu(O)| - \frac{1}{2}\epsilon$. By Urysohn's lemma, there exists a continuous function f on X whose support is contained in O such that $0 \le f \le 1$ and $f = 1$ on C. It follows that

$$\left|\int_X f \, \mathrm{d}\,\mu\right| = \left|\int_{O \setminus C} f \, \mathrm{d}\,\mu + \int_C f \, \mathrm{d}\,\mu\right| \ge |\mu(C)| - \left|\int_{O\setminus C} f \, \mathrm{d}\,\mu\right|$$
$$\ge |\mu(C)| - \int_{O \setminus C} |f| \, \mathrm{d}|\mu| \ge |\mu(C)| - |\mu|(O \setminus C) \ge |\mu(C)| - \tfrac{1}{2}\epsilon \ge |\mu(O)| - \epsilon\,.$$

2. Let a norm one continuous function f on X be given. Define $A = \{f \neq 0\}$, then

$$\left|\int_X f \, \mathrm{d}\,\mu\right| \le \int_X |f| \, \mathrm{d}|\mu| \le |\mu|(A)\,.$$

If (P, N) is the Hahn decomposition ([22, 231E]) of μ, then

$$|\mu|(A) = \mu(A \cap P) - \mu(A \cap N)\,.$$

Without loss of generality, assume that $|\mu(A \cap P)| \ge \frac{1}{2}|\mu|(A)$. By regularity of the measure μ, we can pick a compact set $C \subseteq A \cap N$ such that $|\mu((A \cap N) \setminus C)| < \epsilon$. Define $O = A \setminus C$, then by the reverse triangle inequality,

$$\begin{aligned}|\mu(O)| \ge& |\mu(O \cap P)| - |\mu(O \cap N)| \\ =& |\mu(A \cap P)| - |\mu((A \cap N) \setminus C)| \ge \tfrac{1}{2}|\mu(A)| - \epsilon \ge \tfrac{1}{2}\left|\int_X f \, \mathrm{d}\,\mu\right| - \epsilon\end{aligned}$$

□

The following characterization of relatively weakly compact sets of $\mathrm{rca}(X)$, for compact Hausdorff X was used by Grothendieck to prove Theorem 1.31, and was also used by Lotz to prove the Grothendieck property of Weak L^p spaces.

Theorem 2.24 (Grothendieck's weak compactness criterion)**.** *(Originally from [25, Théoremè 2], also in [18, Theorem 14 of chapter VII] and [43, Theorem 2.5.5] for M-spaces) For any bounded subset A of $\mathrm{rca}(K)$ with K a compact Hausdorff space the following are equivalent*

(i) A is relatively weakly compact,

(ii) if $(O_n)_{n=1}^{\infty}$ is a sequence of disjoint open sets in K, then $(|\mu(O_n)|_n)_{n=1}^{\infty}$ converges to zero uniformly in $\mu \in A$,

(iii) if $(f_n)_{n=1}^{\infty}$ is a sequence of disjoint functions in $C(K)$ of norm one, then $\left(\left|\int_K f_n \,\mathrm{d}\,\mu\right|_n\right)_{n=1}^{\infty}$ converges to zero uniformly in $\mu \in A$.

Partial proof. For the equivalences of (i) and (ii), see [18, page 98, Theorem VII. 14] or [10, Theorem 6.4.2]. The last equivalence follows from Lemma 2.23. □

Some facts about Banach lattices are required for the main result of this section.

Definition 2.25. *A subset A of a Banach lattice E is called* almost (or approximately) order bounded *if and only if for any positive ϵ there exists a positive u in E such that $A \subseteq [-u, u] + \epsilon\,\mathrm{ball}(E)$.*

The following theorem describes the relatively weakly compact sets of L^1.

Theorem 2.26 (Dunford-Pettis)**.** *(See [43, Theorem 2.5.4]) A non-empty bounded subset A of an L-space is relatively weakly compact if and only if A is almost order bounded.*

Lemma 2.27. *If E is an L-space, then the number of mutually disjoint norm one elements in $F = [-u, u] + \epsilon\,\mathrm{ball}(E)$, with $u \in E_+$ and $0 < \epsilon < 1$, is bounded by $\frac{1}{1-\epsilon}\|u\|$.*

Proof. It is first established that the absolute value of each element $v \in F$ can be written as

$$|v| = v_1 + v_2 \text{ with } v_1 = u \wedge |v| \in [-u, u] \text{ and } v_2 = (|v| - u)^+ \in \epsilon\,\mathrm{ball}(E)\,.$$

Only the claim that the second term lies in $\epsilon\,\mathrm{ball}(E)$ requires an explanation. Write $v = v_1' + v_2'$ with $-u \le v_1' \le u$ and $v_2' \in \epsilon\,\mathrm{ball}(E)$. By the Riesz decomposition property, see [43, Theorem 1.1.1 (viii)], it follows from $|v| \le |v_1'| + |v_2'|$ that we can choose $0 \le v_1'' \le |v_1'|$ and $0 \le v_2'' \le |v_2'|$ such that $|v| = v_1'' + v_2''$. Using

$$(|v| - u)^+ = (v_1'' + v_2'' - u)^+ \le (u + v_2'' - u)^+ = v_2'' \le |v_2'|. \tag{3}$$

we then find $\left\|(|v| - u)^+\right\| \le \epsilon$.

The number of elements in a set of mutually disjoint norm one elements $G \subseteq F$ is

$$|G| = \sum_{v\in G} \|v\| \le \sum_{v\in G} \|v_1\| + \sum_{v\in G} \|v_2\|\,,$$

where v_1 and v_2 are defined as above. The second term can be estimated by $\epsilon|G|$ and the first term is estimated first using additivity of the norm on E_+ and then using the mutual disjointness:

$$\sum_{v\in G} \|u \wedge |v|\| \overset{\text{AL}}{=\!=} \left\|\sum_{v\in G} u \wedge |v|\right\| \overset{\text{disj}}{=\!=} \left\|u \wedge \sum_{v\in G} |v|\right\| \le \|u\|\,. \tag{4}$$

It follows that $|G| \le \|u\| + \epsilon|G|$, so $|G| \le \frac{1}{1-\epsilon}\|u\|$. □

It will be convenient to introduce the following terminology, not defined in common literature.

Definition 2.28. *A sequence* $(u_n)_{n=1}^{\infty}$ *in a Banach lattice* E *will be called* finitely disjoint *if for each* $N \in \mathbb{N}$ *there exists* N *elements in* $\{u_n\}_{n=1}^{\infty}$ *that are mutually disjoint.*

The following corollary follows immediately from Theorem 2.26 and Lemma 2.27.

Corollary 2.29. *If* $(u_n)_{n=1}^{\infty}$ *is a normalized finitely disjoint sequence in an L-space, then* $\{u_n\}_{n=1}^{\infty}$ *is not relatively weakly compact.*

Definition 2.30. *Let* E *be a Banach space. A sequence* $(u_n)_{n=1}^{\infty}$ *in* E *is called an* ℓ^1*-basic sequence if there exist positive constants* c *and* C *such that*

$$c\sum_{n=1}^{N}|\alpha_n| \leq \left\|\sum_{n=1}^{N}\alpha_n u_n\right\|_E \leq C\sum_{n=1}^{N}|\alpha_n|$$

for all $\alpha_1, ..., \alpha_N \in \mathbb{R}$ *and all* $N \in \mathbb{N}$.

A sequence $(u_n)_{n=1}^{\infty}$ of a Banach space E is an ℓ^1 basic sequence if and only if $(u_n)_{n=1}^{\infty}$ is a Schauder basis in $\overline{\text{span}}\{u_n : n \in \mathbb{N}\}$ and the map T from $\overline{\text{span}}\{u_n : n \in \mathbb{N}\}$ into ℓ^1, defined by $T(\sum_{n=1}^{\infty}\alpha_n u_n) = \sum_{n=1}^{\infty}\alpha_n e_n$ is a linear isomorphism, with $(e_n)_{n=1}^{\infty}$ the standard basis of ℓ^1.

This implies, in particular, that an ℓ^1-basic sequence is not weakly convergent, since the standard basis $(e_n)_{n=1}^{\infty}$ in ℓ^1 is not weakly convergent in ℓ^1. The following dichotomy for norm bounded disjoint sequences will avoid us from relying on the involved Rosenthal's ℓ^1 theorem.

Lemma 2.31. *A norm bounded disjoint sequence in a Banach lattice* E *is either weakly convergent, or has an* ℓ^1*-basic subsequence.*

Proof. Let $(u_n)_{n=1}^{\infty}$ be a norm bounded disjoint sequence in the Banach lattice E which is not weakly convergent, thus certainly not a weakly null sequence. Then there exists an element u' in the dual of E such that $(\langle u_n, u'\rangle)_{n=1}^{\infty}$ does not converge to zero. There exists a positive ϵ and a subsequence $(v_n)_{n=1}^{\infty}$ of $(u_n)_{n=1}^{\infty}$, such that $|\langle v_n, u'\rangle| > \epsilon$ for all $n \in \mathbb{N}$. We claim that this subsequence is an ℓ^1-basic sequence. Let an $N \in \mathbb{N}$ and $\alpha_1, ..., \alpha_N \in \mathbb{R}$ be given. Then by norm-boundedness of $(v_n)_{n=1}^{\infty}$,

$$\left\|\sum_{n=1}^{N}\alpha_n v_n\right\| \leq \sum_{n=1}^{N}|\alpha_n|\|v_n\| \leq \left(\sup_{n\in\mathbb{N}}\|v_n\|\right)\sum_{n=1}^{N}|\alpha_n|.$$

On the other hand, using the disjointness of $(v_n)_{n=1}^{\infty}$,

$$\begin{aligned}\left\|\sum_{n=1}^{N}\alpha_n v_n\right\|\|v'\| = \left\|\sum_{n=1}^{N}|\alpha_n v_n|\right\|\|v'\| &\geq \left\langle\sum_{n=1}^{N}|\alpha_n v_n|, |v'|\right\rangle \\ &= \sum_{n=1}^{N}|\alpha_n|\langle|v_n|, |v'|\rangle \geq \sum_{n=1}^{N}|\alpha_n||\langle v_n, v'\rangle| \geq \epsilon\sum_{n=1}^{N}|\alpha_n|\,,\end{aligned}$$

It follows that

$$\left\|\sum_{n=1}^{N}\alpha_n v_n\right\| \geq \frac{\epsilon}{\|v'\|}\sum_{n=1}^{N}|\alpha_n|.$$

We may conclude that $(v_n)_{n=1}^{\infty}$ is an ℓ^1-basic sequence, so $(u_n)_{n=1}^{\infty}$ has an ℓ^1-basic sequence. □

Lemma 2.32. *Let E be a Banach space and let $(u'_n)_{n=1}^{\infty}$ be an ℓ^1-basic sequence in E'. Then there exist a $\delta > 0$ such that for each finite subset $A \subseteq \mathbb{N}$ there exists $u_A \in E$ satisfying $\|u_A\| = 1$ and $\langle u_A, u'_n \rangle > \delta$ for $n \in A$.*

Proof. Let $T : \overline{\operatorname{span}}\{u'_n\} \to \ell^1$ be the isomorphism $T(\sum_n \alpha_n u'_n) = \sum \alpha_n e_n$, with $(e_n)_{n=1}^{\infty}$ the standard basis in ℓ^1. Then:

$$\frac{1}{\|T\|} \sum_{n=1}^{\infty} |\alpha_n| \leq \left\| \sum_{n=1}^{\infty} \alpha_n u'_n \right\| \leq \|T^{-1}\| \sum_{n=1}^{\infty} |\alpha_n| \,, \tag{5}$$

for all $(\alpha_n)_{n=1}^{\infty} \in \ell^1$. Define the linear functional $v'' : \overline{\operatorname{span}}\{u'_n\}_{n=1}^{\infty} \to \mathbb{R}$ by

$$\left\langle \sum_{n=1}^{\infty} \alpha_n u'_n, v'' \right\rangle = \sum_{n=1}^{\infty} \alpha_n \,.$$

Now $\|v''\| \leq \|T\|$, by (5), and by construction $\langle u'_n, v'' \rangle = 1$ for $n \in \mathbb{N}$. By the Hahn-Banach theorem, this functional v'' can be extended to the whole of E' preserving its norm. Call this extension v'' again. By Goldstine's theorem, there exists a net $(v_t)_{t \in T}$ of elements in E such that $j_E(v_t)$ converges to v'' in the weak* topology of the bidual E'', and such that $\|v_t\| \leq \|v''\| \leq \|T\|$ for all $t \in T$. By definition of the weak* topology and our choice of v'', $\lim_{t \in T} \langle v_t, u'_n \rangle = 1$ for all $n \in \mathbb{N}$.

Let a finite set $A \subseteq \mathbb{N}$ be given. Choose t such that $\langle v_t, u'_n \rangle > \frac{1}{2}$ for all $n \in A$. By the latter inequality and the bound on the norms of v_t, the element $u_A = \frac{v_t}{\|v_t\|}$ satisfies $\|u_A\| = 1$ and $\langle u_A, u'_n \rangle > \frac{1}{2}\|T\|^{-1}$ for all $n \in A$. Take $\delta = \frac{1}{2}\|T\|^{-1}$. □

Proposition 2.33. *Let E be a Banach lattice and suppose that $(u'_n)_{n=1}^{\infty}$ is a finitely disjoint sequence of positive elements in the dual of E. If there exists a positive u_0 in E and a $\delta > 0$ such that $\langle u_0, u'_n \rangle \geq \delta$ for all n, then there exists a subsequence $\left(u'_{n_m}\right)_{m=1}^{\infty}$, a disjoint sequence $(v_m)_{m=1}^{\infty}$ in $[0, u_0]$ and an $\epsilon > 0$ such that $\left\langle v_m, u'_{n_m} \right\rangle \geq \epsilon$ for all $m \in \mathbb{N}$.*

Proof. For $n \in \mathbb{N}$, define the functionals $w'_n \in E'_+$ by

$$w'_n = \frac{u'_n}{\langle u_0, u'_n \rangle} \,.$$

Let E_{u_0} be the principal ideal in E equipped with the order unit norm $\|\cdot\|_{E_{u_0}}$ and let ϕ_n be the restriction of w'_n to E_{u_0}. Since $\phi_n \geq 0$ and $\langle u_0, \phi_n \rangle = 1$, it follows that $\|\phi_n\|_{E'_{u_0}} = 1$ for all n. Furthermore, the restriction map $w' \mapsto w'|_{E_{u_0}}$ is a lattice homomorphism from the dual of E into the dual of E_{u_0} and so, the sequence $(\phi_n)_{n=1}^{\infty}$ is also finitely disjoint. Since the dual of E_{u_0} is an L-space, by Lemma 2.12 and Theorem 2.13, it follows from Corollary 2.29 that $\{\phi_n\}_{n=1}^{\infty}$ is not relatively weakly compact in the dual of E_{u_0}. Using that E_{u_0} is isometrically lattice isomorphic to $C(K)$ for some compact Hausdorff space K, by Theorem 2.14, it follows from Grothendieck's weak compactness criterion Theorem 2.24 that there exists a disjoint sequence $(f_m)_{m=1}^{\infty}$ in E_{u_0} such that for all m we have $\|f_m\|_{E_{u_0}} = 1$ and

$$\sup_{n \in \mathbb{N}} |\langle f_m, \phi_n \rangle| > \delta \,,$$

for some γ. Since $\phi_n \geq 0$, we have $|\langle f_m, \phi_n \rangle| \leq \langle |f_m|, |\phi_n| \rangle$ and so, replacing f_m by $|f_m|$ if necessary, it may be assumed that $0 \leq f_m \leq u_0$ for all m. Hence, setting $\epsilon = \delta\gamma$, it follows that for all $m \in \mathbb{N}$

$$\sup_{n \in \mathbb{N}} \langle f_m, u'_n \rangle = \sup_{n \in \mathbb{N}} \langle u_0, u'_n \rangle \langle f_m, w'_n \rangle > \epsilon > 0 \,,$$

By Lemma 2.5, $f_m \to 0$ weakly as $m \to \infty$ and so, it follows from Lemma 2.6 that there exists a subsequence $(f_{m_k})_{k=1}^\infty$ of $(f_m)_{m=1}^\infty$ and a subsequence $\left(u'_{n_k}\right)_{k=1}^\infty$ of $(u'_n)_{n=1}^\infty$ such that $\left\langle f_{m_k}, u'_{n_k} \right\rangle > \epsilon$ for all k. Setting $v_k = f_{m_k}$ finishes the proof. □

Theorem 2.34. *(Originally from [40, Theorem 1]) Let E be a Banach lattice with the countable interpolation property whose dual has order continuous norm. Then E is a Grothendieck space if there exists a collection $\mathcal{G}$ of positive linear operators on E with lattice homomorphic adjoints, and there exists a positive element u_0 in E such that:*

1. *for every $u \in E$ there exists a $T \in \mathcal{G}$ such that $|u| \leq \|u\| T u_0$,*
2. *for every sequence $(u_n)_{n=1}^\infty$ of mutually disjoint elements in $[0, u_0]$ and every sequence $(T_n)_{n=1}^\infty$ in $\mathcal{G}$, there exists a non-negative v in E such that $T_n u_n \leq v$ for all n.*

Proof. By Theorem 2.21 it suffices to show that weak* null sequences of positive pairwise disjoint elements of the dual converge weakly. Any weak* convergent sequence is necessarily norm-bounded and by Lemma 2.31 a norm-bounded disjoint sequence either converges weakly or contains a basic ℓ^1 subsequence. Suppose by contradiction that $(u'_n)_{n=1}^\infty$ is a weak* null sequence of positive pairwise disjoint elements in the dual of E that is not weakly null. By passing to a subsequence we may thus assume $(u'_n)_{n=1}^\infty$ to be an ℓ^1-basic sequence.

Take an arbitrary sequence of mutually disjoint subsets of $\{u'_n\}_{n=1}^\infty$ such that the number of elements tends to infinity. To be explicit, define for each $N \in \mathbb{N}$,

$$F_N = \left\{ n \in \mathbb{N} : \frac{(N-1)N}{2} < n \leq \frac{N(N+1)}{2} \right\} .$$

Lemma 2.32 gives a $\delta > 0$ and a sequence $(u_N)_{N=1}^\infty$ of normalized elements of E such that $\langle u_N, u'_n \rangle \geq \delta$ whenever $n \in F_N$. The same holds for $(|u_N|)_{N=1}^\infty$, since the u'_n's are positive.

By the first requirement on $\mathcal{G}$ and u_0, there exists a sequence $(S_N)_{N=1}^\infty$ in $\mathcal{G}$ such that $|u_N| \leq S_N u_0$. Define the map $\sigma : \mathbb{N} \to \mathbb{N}$ by setting $\sigma(n) = N$ whenever $n \in F_N$ and let $T_n = S_{\sigma(n)}$ for $n \in \mathbb{N}$. Observe that, if $n \in \mathbb{N}$, then

$$\langle T_n u_0, u'_n \rangle = \langle S_N u_0, u'_n \rangle \geq \langle |u_N|, u'_n \rangle \geq \delta \,,$$

where N is such that $n \in F_N$.

Furthermore, for each $N \in \mathbb{N}$, the system

$$\{T'_n u'_n : n \in F_N\} = \{S'_N u'_n : n \in F_N\}$$

is disjoint, as S'_N is a lattice homomorphism, by the hypothesis on $\mathcal{G}$. Therefore, the sequence $(T'_n u'_n)_{n=1}^\infty$ is finitely disjoint and for all $n \in \mathbb{N}$,

$$\langle u_0, T'_n u'_n \rangle = \langle T_n u_0, u'_n \rangle \geq \delta \,.$$

Hence, by Proposition 2.33, there exists a disjoint sequence $(v_m)_{m=1}^{\infty}$ in $[0, u_0]$, a subsequence $\left(T'_{n_m} u'_{n_m}\right)_{m=1}^{\infty}$ of $(T'_n u'_n)_{n=1}^{\infty}$ and an $\epsilon > 0$ such that for $m \in \mathbb{N}$

$$\left\langle v_m, T'_{n_m} u'_{n_m} \right\rangle > \epsilon\,.$$

By hypothesis, there exists a positive $v \in E$ such that $T_{n_m} v_m \leq v$ for all $m \in \mathbb{N}$. This implies that for all $m \in \mathbb{N}$,

$$\left\langle v, u'_{n_m} \right\rangle \geq \left\langle T_{n_m} v, u'_{n_m} \right\rangle = \left\langle v_m, T'_{n_m} u'_{n_m} \right\rangle > \epsilon$$

which contradicts the assumption that $\left(u'_{n_m}\right)_{m=1}^{\infty}$ is weak* null. This suffices for a proof of the theorem.

□

3 Banach function spaces

For a given σ-finite measure space (X, Σ, μ), let $\mathcal{M}(\mu)$ denote the collection of all measurable functions from X to the extended real numbers $\mathbb{R}^*$, with the usual identification of μ-almost everywhere equal functions. The *vector space* structure, can only be naturally defined on the subset $\mathcal{M}_0(\mu)$ of $\mathcal{M}(\mu)$ consisting of all $f \in \mathcal{M}(\mu)$ which are finite μ-almost everywhere, and is obtained by considering pointwise operations, that is, by defining $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = cf(x)$ for all $c \in \mathbb{R}$, all $f, g \in \mathcal{M}_0(\mu)$, and for μ-almost all $x \in X$. The *lattice* structure on $\mathcal{M}(\mu)$ is pointwise inherited from the natural order on $\mathbb{R}^*$: $f \leq g$ if and only if $f(x) \leq g(x)$ for μ-almost all $x \in X$.

Definition 3.1. *A function $\rho : \mathcal{M}(\mu)_+ \to [0, \infty]$ is called a* function norm *if and only if it satisfies the following three conditions:*

(i) ρ has trivial kernel, that is, $\rho(f) = 0$ implies $f = 0$ for $f \in \mathcal{M}(\mu)_+$.

(ii) ρ is sublinear, that is, $\rho(f+g) \leq \rho(f) + \rho(g)$ for $f, g \in \mathcal{M}(\mu)_+$ and $\rho(\alpha f) = \alpha\rho(f)$ for $f \in \mathcal{M}(\mu)_+$ and $\alpha \geq 0$. We set $0\infty = 0$.

(iii) ρ is order preserving, that is, $f \leq g$ implies $\rho(f) \leq \rho(g)$ for f and $g \in \mathcal{M}(\mu)_+$.

Furthermore, ρ is called a function quasi-norm *if sublinearity is relaxed to these two conditions: there exists a finite constant C such that $\rho(f+g) \leq C(\rho(f) + \rho(g))$ for $f, g \in \mathcal{M}(\mu)_+$, and $\rho(\alpha f) = \alpha\rho(f)$ for $f \in \mathcal{M}(\mu)_+$ and $\alpha \geq 0$.*

Whenever ρ is a function (quasi-)norm, the subset $\mathcal{M}_\rho(\mu) = \{f \in \mathcal{M}(\mu) : \rho(|f|) < \infty\}$ is defined to be its corresponding (quasi-)normed function space *and $\rho(|f|)$ is denoted as $\|f\|_\rho$, or $\|f\|$, if no ambiguity is possible.*

A complete normed function space is called a Banach function space*.*

Lemma 3.2. *If μ is a measure and ρ is a function norm on $\mathcal{M}_\rho(\mu)_+$, then $\mathcal{M}_\rho(\mu) \subseteq \mathcal{M}_0(\mu)$ is a vector subspace and $\|\cdot\|_\rho$ is a norm on $\mathcal{M}_\rho(\mu)$.*

Proof. To show $\mathcal{M}_\rho(\mu) \subseteq \mathcal{M}_0(\mu)$, let $f \in \mathcal{M}_\rho(\mu)$. Clearly $c\mathbf{1}_{\{|f|=\infty\}} \leq |f|$ for all $c > 0$, and so by by property (ii) of Definition 3.1 it follows that $c\rho\big(\mathbf{1}_{\{|f|=\infty\}}\big) \leq \rho(|f|)$. Hence if $\rho(|f|) < \infty$, then $\rho\big(\mathbf{1}_{\{f=\infty\}}\big) = 0$ and so, $\mathbf{1}_{\{|f|=\infty\}} = 0$ μ-almost everywhere, that is, $f \in \mathcal{M}_0(\mu)$. Trivially $\mathcal{M}_\rho(\mu)$ is closed under scalar multiplication and closed under addition, by property (ii). By properties (i) and (ii) ρ is a norm on $\mathcal{M}_\rho(\mu)$. □

Definition 3.3. *A function norm $\|\cdot\|_\rho$ on $\mathcal{M}(\mu)_+$ is said to have the* Fatou property *if $f_n \uparrow f$ in $\mathcal{M}(\mu)_+$ implies $\rho(f_n) \uparrow \rho(f)$.*

Lemma 3.4. *The Fatou property of a function norm ρ implies completeness of its corresponding normed function space.*

Proof. The Fatou property implies the so-called Riesz Fisher property ([59, §65 Theorem 1]), which implies completeness ([59, §30 Theorem 2]). □

3.1 Weak L^p spaces

The terminology concerning Weak L^p spaces is in accordance with the terminology of Bennett and Sharpley [11].

3.1.1 Definition from Chebyshev's inequality

The necessity to consider something like Weak L^p spaces originates from a part of functional analysis called interpolation theory of which the the Marcinkiewicz's interpolation theorem, and the Riesz-Thorin interpolation theorem, [50] and [58], lie at its foundation. However, the definition of Weak L^p spaces will be presented here without such a justification as to why such a generalization of L^p might be interesting to consider.

The following property of functions in L^p will become the defining property of Weak L^p spaces.

Theorem 3.5 (Chebyshev inequality)**.** *(Originally from [57]) For all $f \in L^p(\mu)$, with $0 < p < \infty$:*

$$\forall y > 0 : \mu(\{|f| > y\}) \leq \left(\frac{\|f\|_{L^p}}{y} \right)^p . \tag{6}$$

where $\{x \in X : |f(x)| > y\}$ is denoted as $\{|f| > y\}$.

Proof. For any μ-measurable function f, monotonicity of the integral gives

$$y^p \mu(\{|f|^p > y^p\}) = \int_{\{|f|^p > y^p\}} y^p \,\mathrm{d}\,\mu \leq \int_{\{|f|^p > y^p\}} |f|^p \,\mathrm{d}\,\mu \leq \int_X |f|^p \,\mathrm{d}\,\mu = \||f|^p\|_{L^1(\mu)} = \|f\|_{L^p(\mu)} .$$

□

Definition 3.6. *For $1 < p < \infty$, the Weak L^p spaces, denoted $L^{p,\infty}(\mu)$, are the vector subspaces of $\mathcal{M}_0(\mu)$ of all μ-measurable functions f with*

$$\|f\|'_{p,\infty} = \inf\{C > 0 : \forall y > 0 : \mu(\{|f| > y\}) \leq (C/y)^p\} < \infty \,, \tag{7}$$

with the convention that the infimum of the empty set is infinity.

In Lemma 3.14 the quantity defined in Equation (7) will be shown to be a quasi-norm, which justifies the notation $\|\cdot\|'_{p,\infty}$. The apostrophe distinguishes the quasi-norm from the equivalent norm defined in Equation (12), proven to be a norm in Lemma 3.20. The name of Weak L^p spaces is justified by the inclusion $L^p \subseteq L^{p,\infty}$, which follows from Equation (6). On $\mathbb{R}$ with Lebesgue measure λ the function $f \in L^{p,\infty}(\lambda) \setminus L^p(\lambda)$ defined by $f(x) = |x|^{-1/p}$ shows that in general $L^p \subsetneq L^{p,\infty}$.

The definition of Weak L^p spaces presented here does not easily extend to the even more general Lorentz space $L^{p,q}(\mu)$, and does not yield the most convenient expression for the quasi-norm, but Equation (7) can be cast into such a form. The first step is to take a closer look at the expression $\mu(\{|f| > y\})$.

3.1.2 The distribution function

The function that assigns to each positive y the left hand side of the inequality in Equation (6) is given a name in the following definition.

Definition 3.7. *For a μ-measurable function f defined on a measure space (X, Σ, μ) its* distribution function *d_f^μ, or d_f whenever μ is fixed, is defined by $d_f^\mu(y) = \mu(\{|f| > y\})$ for positive $y \geq 0$. A ν-measurable g defined on a measure space (Y, Ξ, ν) is called* equimeasurable *with f if $d_f^\mu = d_g^\nu$.*

Notice that if there exists some positive y_0 such that $d_f(y_0)$ is finite, then $d_f(y) \to \mu(\{|f| = \infty\})$ as $y \to \infty$ (in particular, $d_f(y) \to 0$ as $y \to 0$ if $f \in \mathcal{M}_0(\mu)$). Note that in general it may occur that d_f is identically infinity (take for example $f(x) = x$ on $[0, \infty)$).

Lemma 3.8. *The distribution function is decreasing and right-continuous.*

Proof. For a μ-measurable function f, its distribution function d_f is decreasing, since it is the composition of a decreasing $y \mapsto \{|f| > y\}$ and an increasing μ (with the partial order $\subseteq$ on X).

Let y be positive. To show right-continuity of the distribution function, we define $A_n = \{|f| > y + 1/n\}$. Since d_f is decreasing, the limit $\lim_{z \downarrow y} d_f(z)$ exists if the limit $\lim_{n\to\infty} f\big(y + \frac{1}{n}\big)$ exists. Since $A_n \uparrow \{|f| > y\}$ implies $\mu(A_n) \uparrow \mu(\{|f| > y\})$, it follows that $\lim_{n\to\infty} f\big(y + \frac{1}{n}\big) = f(y)$. ☐

Simplifying the expression for $\|\cdot\|'_{p,\infty}$ and substituting this definition yields the following alternative expression (which will be shown to be a quasi-norm in Lemma 3.14.

Lemma 3.9. *For* $1 < p < \infty$ *and any* μ*-measurable function* f

$$\|f\|'_{p,\infty} = \sup_{y>0} y(d_f(y))^{1/p} . \tag{8}$$

Proof. Substituting Definition 3.7 in Equation (7) and simplifying gives the desired result:

$$\begin{aligned} \|f\|'_{p,\infty} &= \inf\Big\{C \geq 0 : \forall y > 0 : y(d_f(y))^{1/p} \leq C\Big\} \\ &= \inf\Big\{C \geq 0 : \quad \sup_{y>0} y(d_f(y))^{1/p} \leq C\Big\} = \sup_{y>0} y(d_f(y))^{1/p} \end{aligned}$$

for $f \in \mathcal{M}(\mu)$. ☐

An intuitive way to interpret the distribution function is that it somewhat resembles the inverse function: if X is the strictly positive real axis $(0, \infty)$ with Lebesgue measure, and $f > 0$ is strictly decreasing, then $d_f = f^{-1}$.

3.1.3 The decreasing rearrangement function

Definition 3.10. *For any* μ*-measurable function* f *the* decreasing rearrangement function *of* f*, denoted* $f^* : [0, \infty) \to [0, \infty]$*, is defined by*

$$f^*(t) = \inf\{0 < y : d_f(y) \leq t\} .$$

Notice that if the distribution function of an almost everywhere finite measurable function f is not everywhere infinite, then $f^*(t)$ is finite for all positive t. The collections of such functions will be denoted by $S(\mu)$, so

$$\begin{aligned} S(\mu) &= \{f \in \mathcal{M}_0(\mu) : \exists y_0 > 0 : d_f(y_0) < \infty\} \\ &= \{f \in \mathcal{M}_0(\mu) : \forall t > 0 : f^*(t) < \infty\} . \end{aligned}$$

Lemma 3.11. *(See [14, Theorem 4.5]) For any* μ*-measurable function* f*,* f^* *is decreasing and right-continuous,* $f^* = d_{d_f}$*, and* $d_f = d_{f^*}$*, that is,* f *and* f^* *are equimeasurable.*

Proof. For any partition of $(0, \infty)$ of nonempty sets A and B with $a \leq b$ for all $a \in A$ and all $b \in B$, $\inf B = \sup A$. By Lemma 3.8, $A_t = \{0 < y : d_f(y) > t\}$ and $B_t = \{0 < y : d_f(y) \leq t\}$ satisfy these requirements if they are not empty, so

$$f^*(t) = \inf B_t = \sup A_t = \lambda(A_t) = d_{d_f}(t) ,$$

follows in this case. When A_t or B_t is empty, the assertion follows trivially.

We first show that that $f^*(t) > y$ if and only if $d_f(y) > t$, for t and y positive. From $\inf B_t = f^*(t) > y$ it follows that $y \notin B_t$, so $d_f(y) > t$. For the converse, first note that any $y_n \downarrow f^*(t)$ satisfies $d_f(y_n) \leq t$ and by right continuity of the distribution function (Lemma 3.8) $d_f(f^*(t)) \leq t$. In view of this observation $f^*(t) < y$ and $d_f(y) > t$ are contradictory, since the latter inequality implies $d_f(d_f(y)) < d_f(t)$, since the decreasing rearrangement function is a decreasing function.

Let a positive y now be given. The derived equivalence gives the following equality:

$$\{t > 0 : f^*(t) > y\} = \{t > 0 : d_f(y) > t\} = (0, d_f(y)) \,,$$

and by taking Lebesgue measures of both sides we obtain $d_{f^*}(y) = d_f(y)$.

□

Lemma 3.12. *For any μ-measurable function f,*

$$\|f\|'_{p,\infty} = \sup_{t>0} t^{1/p} f^*(t) \,. \tag{9}$$

Proof. By Equation (8) it suffices to show that $\sup_{y\geq 0} y(d_f(y))^{1/p} = \sup_{t\geq 0} t^{1/p} f^*(t)$. Intuitively we would like to set $d_f(y) = t$ and $y = d_f^{-1}(t) = f^*(t)$, which is correct if f^* is strictly decreasing. Equation (9) is proven in two parts.

For the inequality $\geq$ the case $\|f\|'_{p,\infty} = \infty$ is clear, so assume $\|f\|'_{p,\infty} < \infty$. Then by definition $d_f(y) \leq \left(\|f\|'_{p,\infty}\right)^p y^{-p}$ for all $y \geq 0$, and so,

$$f^*(t) = \inf\{y > 0 : d_f(y) \leq t\} \leq \inf\Big\{y > 0 : \left(\|f\|'_{p,\infty}\right)^p y^{-p} \leq t\Big\} = \|f\|'_{p,\infty} t^{-1/p}.$$

This shows that $\sup_{t\geq 0} t^{1/p} f^*(t) \leq \|f\|'_{p,\infty}$.

For the inequality $\leq$ the case $\sup_{y\geq 0} y(d_f(y))^{1/p} = 0$ is clear, so fix a $0 < \lambda < \sup_{y\geq 0} y(d_f(y))^{1/p}$. Now there exists a $y > 0$ such that $\lambda < y d_f(y)^{1/p}$, that is, $d_f(y) > \frac{\lambda^p}{y^p}$. Defining $t = \frac{\lambda^p}{y^p}$ it follows that

$$f^*(t) = \inf\left\{z \geq 0 : d_f(z) \leq \frac{\lambda^p}{y^p}\right\} \geq y = \lambda t^{-1/p} \,,$$

that is, $t^{1/p} f^*(t) \geq \lambda$. This shows that $\sup_{t\geq 0} t^{1/p} f^*(t) \geq \lambda$. This holds for all $0 < \lambda < \sup_{y\geq 0} y(d_f(y))^{1/p}$, which shows that $\sup_{t\geq 0} t^{1/p} f^*(t) \geq \sup_{y\geq 0} y(d_f(y))^{1/p}$. □

3.1.4 Normability

First it will be shown in Lemma 3.14 that $\|\cdot\|'_{p,\infty}$ is in fact a quasi-norm and then an equivalent norm will be defined.

Lemma 3.13. *For μ-measurable functions f and g and $s, t \in [0, \infty)$,*

$$d_{f+g}(t+s) \leq d_f(t) + d_g(s) \qquad \textit{and} \qquad (f+g)^*(t+s) \leq f^*(t) + g^*(s) \,.$$

Proof. The statement of this lemma can be generalized by replacing each + on the left hand sides of both inequalities by an arbitrary binary operator on $[0, \infty)$ increasing in both variables, so the proof only relies on this fact.

From $\{|f + g| > t + s\} \subseteq \{|f| > t\} \cup \{|g| > s\}$ and monotonicity and subadditivity of μ follows

$$d_{f+g}(t + s) = \mu(\{|f + g| > t + s\}) \le \mu(\{|f| > t\}) + \mu(\{|g| > s\}) = d_f(t) + d_g(s)\,.$$

A similar inequality holds for decreasing rearrangement functions. Since

$$\{d_f \le t\} + \{d_g \le s\} \subseteq \{d_{f+g} \le t + s\}$$

it follows that

$$(f + g)^*(t + s) = \inf\{d_{f+g} \le t + s\} \le \inf\{d_f \le t\} + \inf\{d_g \le s\} = f^*(t) + g^*(s)\,.$$

□

Lemma 3.14. *The Weak L^p spaces are quasi-normed by $\|\cdot\|'_{p,\infty}$.*

Proof. Only the quasi-triangle inequality requires justification. This is an immediate consequence of Lemma 3.13 by setting $a = b = x/2$. For f and g in $L^{p,\infty}(\mu)$:

$$\|f + g\|'_{p,\infty} \le \sup_{x\ge 0} x^{1/p}(f^*(x/2) + g^*(x/2)) \tag{10}$$

$$\le \sup_{x\ge 0}(2x)^{1/p} f^*(x) + \sup_{x\ge 0}(2x)^{1/p} f^*(x) = 2^{1/p}\Big(\|f\|'_{p,\infty} + \|g\|'_{p,\infty}\Big)\,. \tag{11}$$

□

While $f \mapsto f^*$ does not satisfy the subadditivity required for $\|\cdot\|'_{p,\infty}$ to be a norm, replacing the decreasing rearrangement function in the expression for $\|\cdot\|'_{p,\infty}$ with the so-called maximal function f^{**} turns Equation (7) into an equivalent norm. Lemma 3.20 will show that this is actually a norm.

Definition 3.15. *The* maximal function *of a μ-measurable function f is defined for $x > 0$ as*

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(y)\,\mathrm{d}\,y\,.$$

For a ν-measurable g, we write $f \prec\prec g$ whenever $f^{**} \le g^{**}$. In this case f is said to be submajorized *by g (in [47] f and g are said to be in Hardy-Littlewood-Pólya relation).*

Several properties of the maximal function are summarized in the following lemma, the proof of which is from [11, Chapter 2 Proposition 3.2].

Lemma 3.16. *The maximal function is a decreasing non-negative continuous function that dominates the rearrangement function. It is either finite or constant ∞.*

Proof. Let a μ-measurable function f be given. Nonnegativity follows from non-negativity of f^*. If $f \ne S(\mu)$ then f^{**} is identically infinity, otherwise f^{**} is finite μ-almost everywhere. Since the antiderivative of a decreasing function is continuous, the function $\frac{1}{x}$ is continuous, and the product of continuous functions is continuous, f^{**} is as well. By Lemma 3.8 and Lemma 3.11, f^* is decreasing, so $f^*(\alpha) \leq f^*\left(\alpha\frac{x}{y}\right)$ for $0 < x \leq y$ and $0 < \alpha$, so

$$f^{**}(y) = \frac{1}{y}\int_0^y f^*(y')\,\mathrm{d}\,y' \leq \frac{1}{y}\int_0^y f^*(y'x/y)\,\mathrm{d}\,y' = \frac{1}{x}\int_0^x f^*(x')\,\mathrm{d}\,x' = f^{**}(x)\,,$$

which shows the maximal function is decreasing. It dominates the rearrangement function by

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(x')\,\mathrm{d}\,x' \geq \frac{1}{x}f^*(x)\int_0^x \mathrm{d}\,x' = f^*(x)\,.$$

□

Lemma 3.17. *(See [11, Theorem 3.4 of Section 2]) The maximal function is subadditive, that is, for two μ-measurable functions f and g we have $(f+g)^{**} \leq f^{**} + g^{**}$.*

Sketch of proof. The proof is not very difficult for non-atomic or completely atomic measure spaces, the so-called *resonant* measure spaces. An arbitrary totally σ-finite measure space can then be embedded in a non-atomic measure space, a procedure called the *method of retracts*, to obtain the same result for this much larger class of measure spaces. □

Lemma 3.18. *For all sequences $(f_n)_{n=1}^\infty$ of non-negative measurable functions and all non-negative measurable functions f, $f_n \uparrow f$ implies $d_{f_n} \uparrow d_f$, $f_n^* \uparrow f^*$, and $f_n^{**} \uparrow f^{**}$.*

Proof. By $f_n \leq f_{n+1}$, $\{|f_n| > t\} \subseteq \{|f_{n+1}| > t\}$ almost everywhere, for all n. By $\lim_n f_n = f$ also $\cup_n\{|f_n| > t\} = \{|f| > t\}$. Just like in the proof of right continuity, Lemma 3.8,

$$\lim_{n\to\infty} d_{f_n}(t) = \lim_{n\to\infty} \mu(\{|f_n| > t\}) = \mu(\{|f| > t\}) = d_f(t)\,,$$

so $d_{f_n} \uparrow d_f$. By Lemma 3.11 also $f_n^* \uparrow f^*$. By the monotone convergence theorem also $f_n^{**} \uparrow f^{**}$. □

Definition 3.19. *Let $1 < p < \infty$. For any μ-measurable function f let*

$$\|f\|_{p,\infty} = \sup_{x>0} x^{1/p} f^{**}(x)\,. \tag{12}$$

Lemma 3.20. *The quasi-norms $\|\cdot\|'_{p,\infty}$ and $\|\cdot\|_{p,\infty}$ are equivalent, for measurable functions f,*

$$\|f\|'_{p,\infty} \leq \|f\|_{p,\infty} \leq p'\|f\|'_{p,\infty},$$

with p' the conjugate exponent of p, that is, satisfying $\frac{1}{p} + \frac{1}{p'} = 1$. The quasi-norm $\|\cdot\|_{p,\infty}$ is a function norm with the Fatou property (hence $\left(L^{p,\infty}, \|\cdot\|_{p,\infty}\right)$ is a Banach function space).

Proof. One side of the equivalence follows trivially, since f^* is dominated by f^{**}, by Lemma 3.16. The other direction of the equivalence of norms follows from Equation (9) and

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}\,s = \frac{1}{t}\int_0^t s^{-1/p}s^{1/p}f^*(s)\,\mathrm{d}\,s \leq \frac{1}{t}\left(\int_0^t s^{-1/p}\,\mathrm{d}\,s\right)\sup_{s>0} s^{1/p}f^*(s) = p't^{-1/p}\|f\|'_{p,\infty}\,.$$

That the quasi-norm $\|\cdot\|_{p,\infty}$ is actually a norm follows from Lemma 3.17.

By Lemma 3.4 only the Fatou property has to be proven to show completeness. Since by Lemma 3.18 $(f_n)_{n=1}^\infty \uparrow f$ implies $f_n^{**} \uparrow f^{**}$ for $(f_n)_{n=1}^\infty$ a sequence of non-negative elements in $L^{p,\infty}$, the Fatou property follows from the definition in Equation (12) of the norm $\|\cdot\|_{p,\infty}$. □

Definition 3.21. *A Banach function space E on (X, Σ, μ) is called* rearrangement invariant *if for equimeasurable f and g in $\mathcal{M}(\mu)$, $f \in E$ implies $g \in E$ and $\|f\| = \|g\|$. The norm is called* fully-symmetric *if it follows from $f \in \mathcal{M}(\mu)$, $g \in E$ and $f \prec\prec g$ that $f \in E$ and $\|f\|_E \leq \|g\|_E$ (in [47, Theorem 7.7.1] this property is called the Hardy–Littlewood–Pólya principle.*

Clearly the spaces $L^{p,\infty}$ are rearrangement invariant fully-symmetric.

Lemma 3.22. *Let $1 < p < \infty$ and μ be a σ-finite measure. The dual of $L^{p,\infty}(\mu)$ has an order continuous norm.*

Proof. By Theorem 2.7 it suffices to to show that the dual cannot contain ℓ^∞ isomorphically. A sufficient condition is that there exists a $q \geq 1$ such that for every disjoint order bounded sequence $(u_n)_{n=1}^\infty$ of positive elements of the dual, we have $(\|u_n\|)_{n=1}^\infty \in \ell^q$. By [43, Corollary 2.8.10], this is equivalent with the dual having a lower q-estimate, that is, for any finite set A of pairwise disjoint elements in the dual of $L^{p,\infty}(\mu)$ we must have an M such that

$$\left\| \sum_{u \in A} u \right\| \geq M \left(\sum_{u \in A} \|u\|^q \right)^{1/q}$$

which is equivalent to $L^{p,\infty}(\mu)$ satisfying an upper p-estimate, with $p = \frac{1}{1-\frac{1}{q}}$, by [43, Corollary 2.8.5]:

$$\left\| \sum_{u \in A} u \right\| \leq M \left(\sum_{u \in A} \|u\|^p \right)^{1/p} .$$

Take any finite set of disjoints elements $\{f_n\}_{n=1}^N$ in $L^{p,\infty}(\mu)$. Then for all y,

$$d_{\sum f_n}(y) = \sum d_{f_n}(y) \leq y^{-p} \sum \left(\|f_n\|'_{p,\infty} \right)^p \leq y^{-p} \sum \left(\|f_n\|_{p,\infty} \right)^p$$

where the first step follows since for disjoint elements, the distribution function of the sum of functions is the sum of distribution functions, and the last inequality follows from equivalence of the norms, Lemma 3.20. Taking the y^{-p} to the other side and taking the $\left(\frac{1}{p}\right)^{\text{th}}$ power, gives

$$\left\| \sum f_n \right\|'_{p,\infty} \leq \left(\sum \|f_n\|^p_{p,\infty} \right)^{1/p}$$

Using now the other direction of equivalence of the norms, we get

$$\left\| \sum f_n \right\|_{p,\infty} \leq p' \left\| \sum f_n \right\|'_{p,\infty} \leq p' \left(\sum \|f_n\|^p_{p,\infty} \right)^{1/p} ,$$

and we obtain an upper p-estimate, which implies the dual to have an order continuous norm. $\square$

Lemma 3.23. *For all sequences $(f_n)_{n=1}^\infty$ of positive measurable functions*

$$d_{\sup_n f_n} \leq \sum_n d_{f_n} .$$

Proof. By σ-additivity of μ, for any such sequence $(f_n)_{n=1}^{\infty}$ and any $y \geq 0$,

$$d_{\sup_n f}(y) = \mu\left(\left\{x \in X : \left(\sup_n f\right)(x) > y\right\}\right) = \mu(\{x \in X : \exists n : f_n(x) > y\})$$
$$= \mu(\cup_n \{x \in X : f_n(x) > y\}) \leq \sum_n \mu(\{x \in X : f_n(x) > y\}) = \sum_n d_{f_n}(y).$$

□

Proposition 3.24. *(See [11, Proposition 2.3.3]) Let (μ, Σ, X) be a σ-finite non-atomic measure and $t \leq \mu(X)$. For any μ-a.e. finite measurable function f,*

$$\int_0^t f^*(s) \,\mathrm{d}\, s = \sup\left\{\int_A f \,\mathrm{d}\, \mu : A \in \Sigma, \mu(A) = t\right\}.$$

An alternative expression for Equation (12) will be derived.

Theorem 3.25. *For any μ-measurable function f,*

$$\|f\|_{p,\infty} = \sup_{\substack{A \subseteq X \\ \mu(A) < \infty}} (\mu(A))^{-1/p'} \int_A |f| \,\mathrm{d}\, \mu \,. \tag{13}$$

Proof. Let f be a non-negative μ-measurable function. Let A now be a measurable set of positive finite measure. The Hardy–Littlewood inequality ([11, Theorem 2.2 of Section 2]) gives the second inequality of

$$\|f\|_{p,\infty} \geq (\mu(A))^{-1/p'} \int_0^{\mu(A)} f^*(s) \,\mathrm{d}\, s = (\mu(A))^{-1/p'} \int_0^{\infty} \chi_A^*(s) f^*(s) \,\mathrm{d}\, s$$
$$\geq (\mu(A))^{-1/p'} \int_0^{\infty} (\chi_A f)^*(s) \,\mathrm{d}\, s = (\mu(A))^{-1/p'} \int_X |\chi_A f| \,\mathrm{d}\, \mu = (\mu(A))^{-1/p'} \int_A |f| \,\mathrm{d}\, \mu \,,$$

which proves the inequality $\geq$ of Equation (13).

To prove the inequality $\leq$, first assume that the measure μ is non-atomic, that is, there does not exists a measurable set A of positive measure such that there does not exist a measurable subset of A of smaller nonzero measure. It follows from Proposition 3.24 that

$$\sup_{0 < t \leq \mu(X)} t^{-1/p'} \int_0^t f^*(s) \,\mathrm{d}\, s = \sup_{0 < t \leq \mu(X)} \sup_{\mu(A) = t} t^{-1/p'} \int_A f \,\mathrm{d}\, \mu$$
$$= \sup_{0 < t \leq \mu(X)} \sup_{\mu(A) = t} (\mu(A))^{-1/p'} \int_A f \,\mathrm{d}\, \mu$$
$$= \sup_{\substack{A \subseteq X \\ \mu(A) < \infty}} (\mu(A))^{-1/p'} \int_A |f| \,\mathrm{d}\, \mu \,,$$

If $\mu(X) = \infty$ this proves equality of Equation (13), otherwise including $t > \mu(X)$ does not increase the supremum, since in this case $f^*(t) = 0$, so

$$t^{-1/p'} \int_0^t f^*(s) \,\mathrm{d}\, s = t^{-1/p'} \int_0^{\mu(X)} f^*(s) \,\mathrm{d}\, s \leq (\mu(X))^{1/p'} \int_0^{\mu(X)} f^*(s) \,\mathrm{d}\, s$$

for all $t > \mu(X)$. This proves the inequality $\leq$ of Equation (13) if μ is non-atomic.

Let (X, Σ, μ) now be an arbitrary σ-finite measure space. Let $I = [0, 1]$ be equipped with Lebesgue measure λ. We define $\Omega = X \times I$, equipped with the product measure $\mu \times \lambda$. It should be observed that the measure $\mu \times \lambda$ is non-atomic. For a given non-negative μ-measurable function f we define the $(\mu \times \lambda)$-measurable function $\tilde{f}$ by $\tilde{f}(x, y) = f(x)$ for all $(x, y) \in \Omega$. It is easily checked that $\tilde{f}$ and f are equimeasurable, and therefore their norms agree. It thus remains to show that

$$\sup_{\substack{A \subseteq X \times I \\ (\mu \times \lambda)(A) < \infty}} ((\mu \times \lambda)(A))^{-1/p'} \int_{A \times I} \tilde{f} \, \mathrm{d}\, \mu \leq \sup_{\substack{A \subseteq X \\ \mu(A) < \infty}} (\mu(A))^{-1/p'} \int_A f \, \mathrm{d}\, \mu \, . \tag{14}$$

For a given $(\mu \times \lambda)$-measurable set A of nonzero finite measure, define $A_y = \{x \in X : (x, y) \in A\}$. Fubini's theorem implies that

$$\int_A \tilde{f} \, \mathrm{d}(\mu \times \lambda) = \int_I \int_{A_y} f \, \mathrm{d}\, \mu \, \mathrm{d}\, \lambda(y) \, . \tag{15}$$

By the supremum we have

$$\int_I \int_{A_y} f \, \mathrm{d}\, \mu \, \mathrm{d}\, \lambda(y) \leq \int_I (\mu(A_y))^{1/p'} \, \mathrm{d}\, \lambda(y) \sup_{\substack{A \subseteq X \\ \mu(A) < \infty}} (\mu(A))^{-1/p'} \int_A f \, \mathrm{d}\, \mu \, . \tag{16}$$

By Fubini's theorem and Hölder's inequality

$$\int_I (\mu(A_y))^{1/p'} \, \mathrm{d}\, \lambda(y) \leq \left(\int_I \mu(A_y) \, \mathrm{d}\, \lambda(y) \right)^{1/p'} = ((\mu \times \lambda)(A))^{1/p'} \, . \tag{17}$$

Combining Equation (15), Equation (16), and Equation (17), we obtain Equation (14). ☐

We end this section with some lemmata that will be needed later.

Lemma 3.26. *If $u, v \in S(0, \infty)^+$, then*

$$\int_0^{t_1} u^*(s) \, \mathrm{d}\, s + \int_0^{t_2} v^*(s) \, \mathrm{d}\, s \leq \int_0^{t_1 + t_2} u + v^*(s) \, \mathrm{d}\, s$$

for all $t_1, t_2 > 0$.

Proof. We may assume that $\int_0^{t_1} u^*(s) \, \mathrm{d}\, s$ and $\int_0^{t_2} v^*(s) \, \mathrm{d}\, s$ are finite. Given $\epsilon > 0$, it follows from Proposition 3.24 that there exists measurable sets $A, B \subseteq (0, \infty)$ with $\lambda(A) \leq t_1$ and $\lambda(B) \leq t_2$, such that

$$\int_0^{t_1} u^*(s) \, \mathrm{d}\, s \leq \int_A u \, \mathrm{d}\, \lambda + \epsilon \text{ and } \int_0^{t_2} v^*(s) \, \mathrm{d}\, s \leq \int_B v \, \mathrm{d}\, \lambda + \epsilon \, .$$

This implies that

$$\begin{aligned}\int_0^{t_1} u^*(s)\,\mathrm{d}\,s + \int_0^{t_2} v^*(s)\,\mathrm{d}\,s &\le \int_A u\,\mathrm{d}\,\lambda + \int_B v\,\mathrm{d}\,\lambda + 2\epsilon \\ &\le \int_{A\cup B} (u+v)\,\mathrm{d}\,\lambda + 2\epsilon \\ &\le \int_0^{\lambda(A\cup B)} (u+v)^*(s)\,\mathrm{d}\,s + 2\epsilon \\ &\le \int_0^{t_1+t_2} (u+v)^*(s)\,\mathrm{d}\,s + 2\epsilon\,.\end{aligned}$$

Since this holds for arbitrary positive ϵ, the result follows. ☐

Lemma 3.27. *If $g_1, g_2 \in S(0,\infty)^+$ are disjoint and if $f \in S(0,\infty)^+$ is such that $g_1 \prec\prec f$ and $g_2 \prec\prec f$, then*

$$g_1 + g_2 \prec\prec D_2 f\,,$$

where $D_\alpha f$ is the dilation of f defined by $(D_\alpha f)(x) = f(x/\alpha)$, for $\alpha > 0$.

Proof. Pick disjoint $f_1, f_2 \in S(0,\infty)$ equimeasurable with f. Note that such functions exist:

$$\begin{aligned}f_1(x) &= \sum_{n=0}^{\infty} f((x-n)\vee 0)\chi_{(2n,2n+1]}(x)\,, \\ f_2(x) &= \sum_{n=0}^{\infty} f((x-n-1)\vee 0)\chi_{(2n+1,2n+2]}(x)\,.\end{aligned}$$

Given positive ϵ and t, it follows from Proposition 3.24 that there exists a measurable set $A \subseteq (0,\infty)$ with $\lambda(A) \le t$ such that

$$\int_0^t (g_1+g_2)^*(s)\,\mathrm{d}\,s \le \int_A (g_1+g_2)\,\mathrm{d}\,\lambda + \epsilon\,.$$

Define $A_1 = A \cap \{g_1 > 0\}$ and $A_2 = A \cap \{g_2 > 0\}$. Since g_1 and g_2 are disjoint, it follows that A_1 and A_2 are disjoint, and so $\lambda(A_1) + \lambda(A_2) \le \lambda(A)$. Using that $g_1 \prec\prec f_1$ and $g_2 \prec\prec f_2$, in combination with Lemma 3.26, we find that

$$\begin{aligned}\int_A (g_1+g_2)\,\mathrm{d}\,\lambda &= \int_{A_1} g_1\,\mathrm{d}\,\lambda + \int_{A_2} g_2\,\mathrm{d}\,\lambda \\ &\le \int_0^{\lambda(A_1)} g_1^*(s)\,\mathrm{d}\,s + \int_0^{\lambda(A_2)} g_2^*(s)\,\mathrm{d}\,s \\ &\le \int_0^{\lambda(A_1)} f_1^*(s)\,\mathrm{d}\,s + \int_0^{\lambda(A_2)} f_2^*(s)\,\mathrm{d}\,s \\ &\le \int_0^{\lambda(A_1)+\lambda(A_2)} (f_1^*+f_2^*)(s)\,\mathrm{d}\,s \\ &\le \int_0^{t} (f_1^*+f_2^*)(s)\,\mathrm{d}\,s\end{aligned}$$

Consequently,

$$\int_0^t (g_1+g_2)^*(s)\,\mathrm{d}\,s \le \int_0^t (f_1+f_2)^*(s)\,\mathrm{d}\,s + \epsilon\,,$$

for all $\epsilon > 0$ and so, for $t > 0$,

$$\int_0^t (g_1+g_2)^*(s)\,\mathrm{d}\,s \le \int_0^t (f_1+f_2)^*(s)\,\mathrm{d}\,s\,.$$

Since f_1 and f_2 are disjoint, it follows that, for $\lambda > 0$,

$$d_{f_1+f_2}(\lambda) = d_{f_1}(\lambda) + d_{f_2}(\lambda) = 2d_f(\lambda)\,,$$

and hence, for $s > 0$,

$$(f_1+f_2)^*(s) = f^*(s/2) = D_2 f^*(s)\,.$$

□

3.2 Marcinkiewicz spaces

Marcinkiewicz spaces are a class of rearrangement invariant Banach function spaces, indexed by a function ψ, that contains the Weak L^p spaces studied in the previous section as the special case $\psi(t) = t^{1/p}$.

Throughout this section, let $\gamma \in (0,\infty]$ and let $\psi : [0,\gamma) \to [0,\infty)$ be a concave function, which is necessarily increasing, continuous except possibly at 0, absolutely continuous on finite intervals, and differentiable except possibly on a countable set [52, Theorem 5.17]. Then the fundamental theorem of Lebesgue integral calculus [52, Theorem 5.14] gives

$$\psi(t) = \lim_{t\downarrow 0}\psi(t) + \int_0^t \psi'\,\mathrm{d}\,\lambda\,,$$

for all $t > 0$, where ψ' is decreasing and non-negative and can be defined on the entire real axis by setting $\psi'(t) = \lim_{t\downarrow 0}\psi'(t)$. It will be assumed that ψ is not the zero function.

Definition 3.28. *Let (X,μ) be a σ-finite measure space, with $\mu(X) = \gamma$. The* Marcinkiewicz space *$M_\psi = M_\psi(\mu)$ on (X,μ) is defined to be the fully symmetric Banach function space of all $f \in S(\mu)$ for which the following is finite:*

$$\|f\|_{M_\psi} = \sup_{0<t<\gamma} \frac{1}{\psi(t)} \int_0^t f^*\,\mathrm{d}\,\lambda\,. \tag{18}$$

Define $\phi : [0,\gamma) \to [0,\infty)$ by $\phi(t) = \frac{t}{\psi(t)}$ and

$$\|f\|'_{M_\psi} = \sup_{0<t<\gamma} \phi(t) f^*(t)\,, \tag{19}$$

for $f \in S(\mu)$. Also define $\mathbb{M}_\psi(s,t) = \frac{\psi(st)}{\psi(t)}$.

To see that the Marcinkiewicz space M_ψ is indeed a fully symmetric Banach function space with the Fatou property, see [47, Proposition 7.10.2] and [34, Theorem II.5.2]. To find when $\|\cdot\|'_{M_\psi}$ is equivalent to the norm $\|\cdot\|_{M_\psi}$, it will become necessary to determine when the function $\frac{1}{\phi}$ is an element of M_ψ. The resulting condition will depend on $\mathbb{M}$.

Lemma 3.29. *(See [34, Lemma II.1.3]) For all $s > 1$ and all $t > 0$, $\mathbb{M}_\psi(s,t) \geq 1$. If $\gamma = \infty$ and $\inf_{t>0} \mathbb{M}_\psi(s,t) = 1$ for some $s > 1$, then $\inf_{t>0} \mathbb{M}_\psi(s',t) = 1$ for all $s' > 1$.*

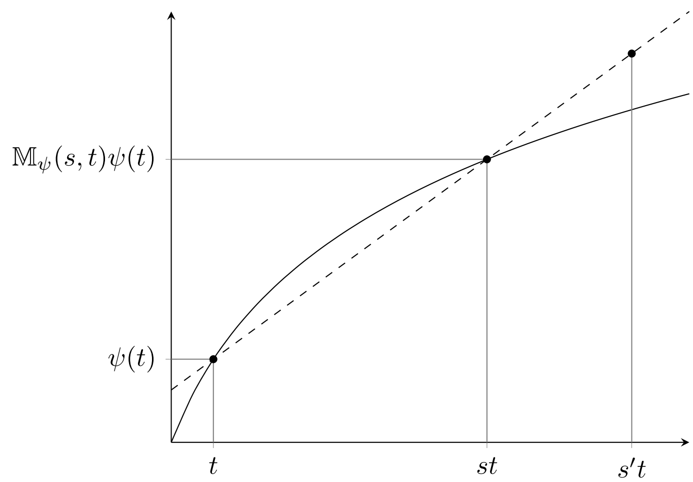
Proof. The first assertion follows directly, since ψ is non-decreasing. Let s satisfy the assumption and choose $s' > 1$ arbitrary. If $s' < s$ then the assertion follows trivially since ψ is non-decreasing. Assume now that $s' > s$, and choose $t > 0$ arbitary. By concavity of ψ, the line through $(t, \psi(t))$ and $(st, \psi(st))$, given by

$$t' \mapsto \frac{\psi(t)(\mathbb{M}_\psi(s,t) - 1)}{t(s-1)}(t' - t) + \psi(t)\,,$$

lies above ψ at $s't$, that is:

$$\mathbb{M}_\psi(s',t) \leq (\mathbb{M}_\psi(s,t) - 1)\frac{s'-1}{s-1} + 1\,.$$

Since t was arbitrary, this implies that $\inf_{t>0} \mathbb{M}_\psi(s',t) = 1$.



□

Lemma 3.30. *If*

$$\inf_t \mathbb{M}_\psi(s,t) > 1 \text{ for some } s > 1\,, \tag{20}$$

then $\frac{1}{\phi} \in M_\psi$. This condition is necessary if and only if $\gamma = \infty$.

Proof. To show sufficiency, let $\alpha > 1$ and $s > 0$ be such that $\psi(st) \geq \alpha\psi(t)$ for all $t > 0$. Let $t > 0$ be arbitrary. By applying this inequality n times, we get $\psi(s^{-n}t) \leq \alpha^{-n}\psi(t)$. We now wish to replace the discrete n by a continuous x at the cost of one extra α. To see why this is allowed, notice that since ψ is non-decreasing, $\psi(s^{-x}t) \leq \alpha^{-n}\psi(t)$ for any real $x \geq n$. Therefore, $\psi(s^{-x}t) \leq \alpha^{-\lfloor x \rfloor}\psi(t) \leq \alpha^{-x+1}\psi(t)$, where $\lfloor x \rfloor$ is the largest integer below x. Replace $s^{-x}t$ by t' and obtain: for all $t' \leq t$

$$\psi(t') \leq \alpha^{-\log_s \frac{t}{t'} + 1}\psi(t) = \alpha\left(\frac{t}{t'}\right)^{-\log_s \alpha}\psi(t)\,.$$

We can now bound the following integral:

$$\int_0^t \frac{\psi(t')}{t'}dt' \leq \frac{\alpha}{\log_s \alpha}\psi(t)$$

And we obtain:

$$\left\|\frac{1}{\phi}\right\|_{M_\psi} = \sup_{0<t<\gamma} \frac{1}{\psi(t)}\int_0^t \frac{\psi(t')}{t'}dt' \leq \frac{\alpha}{\log_s \alpha} < \infty$$

The proof of necessity if $\gamma = \infty$, is contained in [34, Lemma 5.3]. To see that this condition is not sufficient if $\gamma < \infty$, consider a an arbitrary ψ that does satisfy Equation (20). Choose $0 < \gamma_2 < \gamma$ arbitrary and define ψ_2 to equal ψ on $(0, \gamma_2)$ and define ψ_2 to be constant $\psi(\gamma_2)$ otherwise. In this case for $s = \gamma/\gamma_2$ the condition in Equation (20) is satisfied, but $\frac{1}{\phi} \in M_\psi$. □

Lemma 3.31. *(See [34, Theorem 5.3]) The quasi-norm* $\|\cdot\|'_{M_\psi}$ *is equivalent to the norm* $\|\cdot\|_{M_\psi}$ *if Equation* (20) *holds. This is necessary if* $\gamma = \infty$.

Proof. Note that trivially one side of the equivalence of the norms always holds:

$$f^* \text{ decreasing} \Rightarrow f^*(t) \leq \frac{1}{t}\int_0^t f^* \Rightarrow \frac{t}{\psi(t)}f^*(t) \leq \frac{1}{\psi(t)}\int_0^t f^* \Rightarrow \|f\|' \leq \|f\|$$

For sufficiency of the other side of the equivalence of the norms, we use Hölder's inequality,

$$\int_0^t f^*(s)ds \overset{\text{Hölder}}{\leq} \int_0^t \frac{\psi(s)}{s}ds \cdot \sup_{s>0}\frac{s}{\psi(s)}f^*(s) \leq \psi(t)\left\|\frac{\psi(s)}{s}\right\|\|f\|',$$

we obtain after division by $\psi(t)$ and after taking the supremum, that $\|f\| \leq \left\|\frac{1}{\phi}\right\|\|f\|'$. Lemma 3.30 shows that this is bounded.

We now prove necessity, assuming $\gamma = \infty$. Since $\frac{1}{\phi}$ is a decreasing function, $\left(\frac{1}{\phi}\right)^* = \frac{1}{\phi}$, so $\left\|\frac{1}{\phi}\right\|' = 1$. By equivalence of the norms, $\left\|\frac{1}{\phi}\right\| < \infty$, which implies by Lemma 3.30 that condition the Equation (20) holds. □

Lemma 3.32. *The condition in Equation* (20) *is satisfied if and only if:*

$$\liminf_{t\downarrow 0} \mathbb{M}(s,t) > 1 \textit{ and } \liminf_{t\uparrow\gamma} \mathbb{M}(s,t) > 1 \qquad \textit{for some } s > 1\,. \tag{21}$$

Proof. The 'if' part is trivial.

Assume that there exists a sequence $(t_n)_{n=1}^\infty$ such that $\mathbb{M}(s, t_n)$ converges to 1. Then there exists a subsequence $(s_n)_{n=1}^\infty$ that converges to $0 \leq t' \leq \gamma$ with $\mathbb{M}(s, s_n) \to 1$. By $\liminf_{t\downarrow 0} \mathbb{M}(s,t) > 1$ and $\liminf_{t\downarrow\infty} \mathbb{M}(s,t) > 1$, both $t' \neq 0$ and $t' \neq \gamma$. By continuity of $\mathbb{M}$, $\mathbb{M}(s,t') = 1$, that is, $\psi(t') = \psi(st')$. By concavity and positivity of ψ this implies that $\psi(t) = \psi(t')$ for all $t \geq t'$, which contradicts $\liminf_{t\to\infty} \mathbb{M}(s,t) > 1$. □

An example of a concave $\psi : (0,\gamma) \to (0,\infty)$ that does not satisfy the condition in Lemma 3.32 at ∞ is $\psi(t) = \ln(1+t)$ and similarly an example that does not satisfy the condition at 0 is $\psi(t) = \frac{1}{\ln(1/t)}$. An example that satisfies neither conditions is $\psi(t) = \frac{\ln(t+2)}{t+2}\frac{1}{\mathrm{W}(1/t)}$, where W is the *Lambert W* function defined by $W(t)e^{W(t)} = t$ for $t > 0$. Note also that the condition at 0 is satisfied if there exists an n such that the n^{th} derivative of ψ at $t = 0$ exists and is nonzero.

4 The Weak L^p space $L^{p,\infty}(0,\gamma)$ is Grothendieck

Lotz' objective in [40] was to prove the Grothendieck property of Weak L^p spaces. Instead of applying Theorem 2.34 directly to $L^{p,\infty}(X)$ for arbitrary measure spaces X, first an interval of the real line with the Lebesgue measure is considered. Since in Lemma 3.22 order continuity of the dual is established, it remains to be shown that there exists a pair $(u_0, \mathcal{G})$ imitating the action of the identity on the constant-one function of $C(K)$. The general case follows in Section 6.

Definition 4.1. *A* measure preserving *map $\phi : X \to Y$ between the measure spaces (X,μ) and (Y,ν) is such that the preimages of measurable subsets $A \subseteq Y$ are measurable and equally measured: $\mu\big(\phi^{-1}(A)\big) = \nu(A)$. If its inverse exists and is measure preserving as well, ϕ is called a* measure preserving isomorphism.

Lemma 4.2. *For any strictly positive function in $S(0,\gamma)$, for $\gamma \in (0,\infty)$, and any $a > 1$, there exists a measure preserving isomorphism ϕ on $(0,\gamma)$ such that λ-almost everywhere $a^{-1} f \leq f^* \circ \phi \leq af$.*

Proof. For $i \in \mathbb{Z}$ define the partition $A_i = \big(a^i, a^{i+1}\big]$ of $(0,\infty)$ and the partitions

$$\Omega_i^f = f^{-1}[A_i] \text{ and } \Omega_i^{f^*} = f^{*-1}[A_i]\,,$$

of $(0,\gamma)$. By equimeasurability, $\lambda\Big(\Omega_i^f\Big) = \lambda\Big(\Omega_i^{f^*}\Big)$ for $i \in \mathbb{Z}$. Moreover, $\lambda\Big(\Omega_i^f\Big) = 0$ if and only if $\Omega_i^{f^*} = \varnothing$, since f^* is right continuous and decreasing, Lemma 3.11. Let $\Omega^f = \cup\Big\{\Omega_i^f : \lambda(A_i) > 0\Big\}$. If $\lambda\Big(\Omega_i^f\Big) > 0$ then there exists a measure preserving isomorphism ϕ_i from Ω_i^f onto $\Omega_i^{f^*}$, by [52, Theorem 12, Chapter 15 and Problem 18]. Hence there is a measure preserving isomorphism ψ from Ω^f onto $(0,\gamma)$ with $\psi|_{\Omega_i^f} = \phi_i$ if $\lambda\Big(\Omega_i^f\Big) > 0$. Let now $s \in \Omega^f$ be given and define i to satisfy $s \in \Omega_i^f$, then $f(s)$ and $f^*(\psi(s))$ are both in A_i, which implies that $a^{-1}f(s) \leq f^*(\psi(s)) \leq af(s)$ for $s \in \Omega^f$. Because the assertion requires a bijection, the image of the null set $(0,\gamma) \setminus \Omega^f$ has to be defined. Let $N \subseteq (0,\gamma)$ be an arbitrary uncountable null set. Let θ be a bijection from $C = \psi^{-1}(N) \cup \big((0,\gamma) \setminus \Omega^f\big)$ onto B. Let

$$\phi(s) = \begin{cases} \psi(s) & s \in (0,\gamma) \setminus N \\ \theta(s) & s \in N \end{cases},$$

then ϕ is a measure preserving isomorphism with the desired property. □

The isomorphisms ϕ of the previous lemma will now be considered as bounded linear operators T_ϕ on $L^{p,\infty}(0,\infty)$ with $(T_\phi f)^* = f^*$ by defining $T_\phi f = f \circ \phi$ for $f \in L^{p,\infty}(0,\infty)$. The adjoints of such T_ϕ are clearly lattice homomorphic, since T_ϕ is invertible and bipositive.

Theorem 4.3. *(Originally from [40, Theorem 2]) $L^{p,\infty}((0,\gamma))$ is Grothendieck, for all $\gamma \in (0,\infty]$.*

Proof. We wish to apply Theorem 2.34. Order continuity of the norm of the dual has been shown in Lemma 3.22. Define $u_0 \in L^{p,\infty}$ by $u_0(t) = 2t^{-1/p}$ and define $\mathcal{G}$ to be the set of all T_ϕ with ϕ a measure preserving isomorphism on $(0,\gamma)$.

In order to show the first requirement of Theorem 2.34, assume that $\|f\|_{p,\infty} \leq 1$, that is, by equivalence of the norms, Lemma 3.20, $\sup_t t^{1/p} f^*(t) \leq 1$, which implies $f^*(t) \leq t^{-1/p}$ for all t. By Lemma 4.2 for $a = 2$, there exists a measure preserving isomorphism ϕ such that,

$$|f|(t) \leq 2f^*(\phi(t)) \leq 2(\phi(t))^{-1/p} = (T_\phi u_0)(t)\,,$$

which implies $\|f\| \leq T_\phi u_0$. The first requirement of Theorem 2.34 is thus satisfied.

To check the second requirement in Theorem 2.34, let a disjoint sequence $(f_n)_{n=1}^\infty$ in $[0, u_0]$ and a sequence $(T_{\phi_n})_{n=1}^\infty$ in $\mathcal{G}$ be given. It suffices to show that $v \equiv \sup_n f_n \circ \phi_n \in L^{p,\infty}$. Its distribution function can be bounded by

$$d_{\sup_n f_n \circ \phi_n} \leq \sum d_{f_n \circ \phi_n} = \sum d_{f_n} \leq d_{u_0}\,,$$

where the first estimate follows from Lemma 3.23, the second equality follows from equimeasurability after composition with a measure preserving isomorphism, and the last estimate follows from disjointness. It follows that $v^* \leq {u_0}^*$, so $v \in L^{p,\infty}$. □

5 When is the Marcinkiewicz space $M_\psi(0,\gamma)$ Grothendieck?

Proving when the Marcinkiewicz spaces have the Grothendieck property is very similar to the proof that Weak L^p spaces are Grothendieck.

As before, let $\gamma \in (0, \infty]$ and let $\psi : [0, \gamma) \to [0, \infty)$ be a concave function. We first determine necessary and sufficient conditions for order continuity of the norm of the dual M_ψ in the case $\gamma = \infty$, and then consider the case $\gamma < \infty$. After these results, the necessary and sufficient conditions for the Grothendieck property follow easily in the same way as in the previous section.

The following result is already from 1974.

Theorem 5.1. *(Originally from [41, Теорема 4], also in [1, Theorem 8.6] on* $[0,1]$ *without proof) If* $\gamma = \infty$*, then the dual of the Marcinkiewicz space* M_ψ*, has order continuous norm if*

$$\inf_t \mathbb{M}_\psi(s,t) > 1 \text{ for some } s > 1\,. \qquad (20 \text{ again})$$

Proof. To prove order continuity of the norm of the dual, we use Lemma 2.8. Let u and v thus be two mutually disjoint positive elements in the unit ball of M_ψ. By definition of the norm we have

$$\sup_{x>0} \frac{1}{\psi(x)} \int_0^x u^* \,\mathrm{d}\,\lambda \leq 1\,,$$

and similarly for v, which means that $\int_0^x u^* \,\mathrm{d}\,\lambda \leq \int_0^x \psi' \,\mathrm{d}\,\lambda$ for all $x > 0$, that is, $u \prec\prec \psi'$ and $v \prec\prec \psi'$. By Lemma 3.27 $u+v \prec\prec D_2\psi'$. Since the norm is monotone with respect to submajorization, it follows that $\|u+v\|_{M_\psi} \leq \|D_2\psi'\|_{M_\psi}$ and it remains to show that $\|D_2\psi'\|_{M_\psi} < 2$. By the following change of variables

$$
\begin{aligned}
\|D_2\psi'\|_{M_\psi} &= \sup_{x>0} \frac{1}{\psi(x)} \int_0^x D_2\psi' \,\mathrm{d}\,\lambda \\
&= \sup_{x>0} \frac{1}{\psi(x)} \int_0^x \psi'\Big(\frac{y}{2}\Big) \,\mathrm{d}\,\lambda(y) \\
&= \sup_{x>0} \frac{1}{\psi(x)} \int_0^{x/2} \psi'(z) 2 \,\mathrm{d}\,\lambda(z) \\
&= 2 \sup_{x>0} \frac{\psi(x/2)}{\psi(x)} = 2 / \inf_{x>0} \frac{\psi(2x)}{\psi(x)}\,.
\end{aligned}
$$

Assuming Equation (20) and using Lemma 3.29 for $s' = 2$ finishes the proof. □

Lemma 5.4 will show that if γ is finite, the lim inf of $\mathbb{M}$ at $t = \gamma$ is not required for order continuity of the norm of the dual. However, a similar proof as in the case of $(0, \infty)$ does not apply, since the Marcinkiewicz space on a finite interval is not necessarily quasi-uniformly convex. The following example demonstrates this:

Example. *Consider $\gamma = 1$ and $\psi(t) = 2t \wedge \frac{1}{2}$, and define the disjoint elements $u, v \in M_\psi(0,1)$ by*

$$
u = \chi_{[0,\frac{1}{2}]} \qquad \text{and} \qquad v = \chi_{[\frac{1}{2},1]}\,.
$$

It is clear that $u^* = v^* = u$. To compute the norms, we evaluate Equation (19):

$$
\|u\|_{M_\psi} = \|v\|_{M_\psi} = \sup_{0<t<\gamma} \frac{1}{\psi(t)} \int_0^t u^* \,\mathrm{d}\,\lambda = \sup_{0<t<\gamma} \frac{1}{2t \wedge \frac{1}{2}} \int_0^t \chi_{[0,\frac{1}{2}]} \,\mathrm{d}\,\lambda = \sup_{0<t<\gamma} \frac{t \wedge \frac{1}{2}}{2t \wedge \frac{1}{2}} = 1
$$

Similarly we find that $\|u + v\|_{M_\psi} = 2$, so $M_\psi(0,1)$ is not quasi-uniformly convex, so Lemma 2.8 cannot be used to prove order continuity of the norm of the dual.

To prove the converse of the of these results, we start with the following simple observation:

Lemma 5.2. *If the dual of the Marcinkiewicz space M_ψ has order continuous norm, then we have $\lim_{t\downarrow 0} \psi(t) = 0$. If additionally $\gamma = \infty$, then $\lim_{t\to\infty} \psi(t) = \infty$.*

Proof. The proof is by contraposition. It will be shown that M_ψ contains ℓ^1, which suffices for a proof, since then the dual cannot have an order continuous norm, by Theorem 2.7.

First assume that $\gamma = \infty$. Assume thus that $\lim_{t\to\infty} \psi(t) = L < \infty$. Define the map $T : \ell^1 \to M_\psi(0,\infty)$ by $T(e_i) = L\chi_{(i-1,i)}$, where $(e_i)_{i=1}^\infty$ is the natural basis of ℓ^1. This map is clearly a lattice isomorphism. To show that it is also an isomorphism onto its range, we will calculate the image of an arbitrary element of ℓ_1: $\sum_{i=1}^\infty \alpha_i e_i$. Since the sup in the definition of the norm of f on M_ψ will never be attained at values for t for which there exists a $s > t$ such that $f(s) = f(t)$, there exists an $N \in \mathbb{N}$ such that for any $n \geq N$,

$$
\|T(\alpha_i)_{i=1}^\infty\|_{M_\psi} = \frac{1}{\psi(N)} \sum_{i=1}^N L\big|\alpha_{\sigma(i)}\big| \geq \frac{1}{\psi(n)} \sum_{i=1}^n L\big|\alpha_{\sigma(i)}\big|\,, \tag{22}
$$

where $\sigma : \mathbb{N} \to \mathbb{N}$ is such that $\left(\alpha_{\sigma(i)}\right)_{i=1}^{\infty}$ is non-increasing. Since this holds for any $n \geq N$, we can also take the limit $n \to \infty$ and obtain:

$$\|T(\alpha_i)_{i=1}^{\infty}\|_{M_\psi} \geq \frac{1}{L}\sum_{i=1}^{\infty} L\left|\alpha_{\sigma(i)}\right| = \|(\alpha_i)_{i=1}^{\infty}\|_{\ell^1}$$

One side of the equivalence of norms has been proven. Since for any N we bound the middle expression of Equation (22) by

$$\frac{L}{\psi(N)}\|(\alpha_i)_{i=1}^{\infty}\|_{\ell^1}$$

and since $\psi(N)$ is bounded from below by $\psi(1)$, the other side of the equivalence of norms has been proven as well.

Now assume that $\lim_{t\downarrow 0}\psi(t) = L > 0$. Choose $\delta = \gamma$ if γ is finite, and otherwise choose $\delta = 1$. Define $T : \ell^1 \to M_\psi(0,\infty)$ now by $T(e_i) = \frac{2^i}{\delta}\chi_{\left[\frac{\delta}{i+1},\frac{\delta}{i}\right]}$. The proof that this map is an isomorphism onto its range is similar to the case above.

□

Proposition 5.3. *Assume $\gamma = \infty$. If the dual of M_ψ has order continuous norm, then condition in Equation* (20) *is satisfied.*

Proof. Suppose that the condition in Equation (20) is satisfied. As observed in Lemma 5.2, $\psi(0+) = 0$ and $\psi(\infty) = \infty$. Therefore, it follows from [19, Theorem 3.4] that there exists a positive $u' \in M'_\psi$ which is *symmetric*, that is, if $0 \leq f, g \in M_\psi$ and $f \prec\prec g$, then $u'(f) \leq u'(g)$. In particular, if $0 \leq f, g \in M_\psi$ and $f^* = g^*$, then $u'(f) = u'(g)$.

Since $0 < u' \in M'_\psi$, there exists a norm one positive $u \in M_\psi$ such that $u'(u) > 0$. Let $(u_n)_{n=1}^{\infty}$ be a sequence of positive mutually disjoint elements in M_ψ, all equimeasurable with u. This implies that $\|u_n\|_{M_\psi} = \|u\|_{M_\psi} = 0$ and $u'(u_n) = u'(u) > 0$ for all n. In particular, $(u_n)_{n=1}^{\infty}$ does not converge to zero weakly. It follows from Theorem 2.7 that the norm in the dual is not order continuous, which is a contradiction. The proof is complete. □

The following lemma considers the case $\gamma < \infty$.

Lemma 5.4. *Assume $\gamma < \infty$. The dual of the Marcinkiewicz space $M_\psi(0,\gamma)$, has order continuous norm if and only if, for some $s > 1$*

$$\liminf_{t\downarrow 0}\mathbb{M}(s,t) > 1\,. \tag{23}$$

Proof. By the conditions on ψ, there exists a δ such that $\psi'(\delta) > 0$. Define $\psi_\delta \in \mathcal{M}(0,\infty)$ by

$$\psi_\delta(t) = \begin{cases}\psi(t) & \text{if } t \leq \delta\\ \psi(\delta) + (t-\delta)\psi'(\delta) & \text{if } t \geq \delta\end{cases},$$

for $t > 0$. It remains to be shown that $M_\psi(0,\gamma)$ and $M_{\psi_\delta}(0,\infty)\cap\mathcal{M}(0,\gamma)$ are isomorphic, since then Theorem 5.1 and Proposition 5.3 finish the proof.

Since clearly the norms of the two Marcinkiewicz spaces M_{ψ_1} and M_{ψ_2} on some finite interval $(0,\gamma)$ are equivalent if $\frac{\psi_1}{\psi_2}$ is bounded from above and below (in some neighbourhood of zero is sufficient), $M_\psi(0,\gamma)$ is isomorphic with $M_{\psi_\delta|_{(0,\gamma)}}(0,\gamma)$.

It remains to be shown that the canonical embedding $j : M_{\psi_\delta|_{(0,\gamma)}}(0,\gamma) \to M_{\psi_\delta}(0,\infty)$ is an isomorphism onto its range. It is trivial that $\|j(f)\|_{M_{\psi_\delta}(0,\infty)} \leq \|f\|_{M_{\psi_\delta|_{(0,\gamma)}}(0,\gamma)}$. We use the fact that the maximal function is decreasing, Lemma 3.16:

$$\frac{1}{\psi_\delta(t)} \int_0^t (j(f))^*(s)\,\mathrm{d}\,s = \frac{t}{\psi_\delta(t)} \frac{1}{t} \int_0^t (j(f))^*(s)\,\mathrm{d}\,s \leq \frac{t}{\psi_\delta(t)} \frac{1}{\gamma} \int_0^\gamma f^*(s)\,\mathrm{d}\,s\,,$$

for $t > \gamma$. Using $\psi_\delta(t) \geq \psi'_\delta(\gamma)t$ we find

$$\frac{t}{\psi_\delta(t)} \frac{1}{\gamma} \int_0^\gamma f^*(s)\,\mathrm{d}\,s \leq \frac{1}{\gamma\psi'_\delta(\gamma)} \int_0^\gamma f^*(s)\,\mathrm{d}\,s \leq \frac{\psi_\delta(\gamma)}{\gamma\psi'_\delta(\gamma)} \|f\|_{M_{\psi_\delta}(0,\gamma)}\,,$$

so j is indeed an isomorphism onto its range. □

Theorem 5.5. *Let $\psi : [0,\gamma) \to [0,\infty)$ be a concave function. Let $0 < \gamma \leq \infty$. The Marcinkiewicz space $M_\psi(0,\gamma)$ is a Grothendieck space if and only if either*

(i) $\gamma < \infty$ and $\liminf_{t\downarrow 0} \mathbb{M}(s,t) > 1$ for some (or equivalently, all) $s > 1$ (Equation (23)), or

(ii) $\gamma = \infty$ and $\liminf_{t\downarrow 0} \mathbb{M}(s,t) > 1$ and $\liminf_{t\to\infty} \mathbb{M}(s,t) > 1$ for some (or equivalently, all) $s > 1$ (Equation (21)).

Proof. By Proposition 5.3, lemma 5.4, and theorem 5.1, the condition is equivalent with order continuity of the dual M'_ψ. Since this is required for the Grothendieck property of M_ψ, it remains to prove the Grothendieck property using the condition. We will proceed similarly as in Theorem 4.3, now using $u_0(t) = 2\frac{\psi(t)}{t} = \frac{2}{\phi(t)}$ instead. We will apply Theorem 2.34 again with the same $\mathcal{G}$: the set of all T_ϕ with ϕ a measure preserving isomorphism on $(0,\gamma)$.

In order to show the first requirement in Theorem 2.34, assume that $\|f\| \leq 1$, that is, by equivalence of the norms, Lemma 3.31, $\sup_{t>0} \phi(t) f^*(t) \leq 1$, which implies $f^*(t) \leq \frac{1}{\phi(t)}$ for all t. By Lemma 4.2, there exists a measure preserving isomorphism Φ such that, given an $a > 1$, $|f|(t) \leq af^*(\Phi(t)) \leq \frac{a}{\phi(\Phi(t))}$. Choose $a = 2$ and the first requirement is satisfied.

The verification of the second requirement in Theorem 2.34 is identical to that in Theorem 4.3. □

6 The Grothendieck property of BFS's on arbitrary measure spaces

The main result of this section is Proposition 6.7. The proof consists of three steps, each weakening the assumptions on the measure space under consideration. The first (weakest) assumption is that the measure space is 'separable', a property introduced in the following definition.

Definition 6.1. *The* Fréchet–Nikodym *pseudo-metric ρ is defined on the subset Σ_f of the finitely measured elements from a measure space (X, Σ, μ) by $\rho(A_1, A_2) = \mu(A_1 \triangle A_2)$ for $A_1, A_2 \in \Sigma_f$.*

An equivalence relation $\sim$ can then be defined on Σ_f, identifying sets that differ by a null set: $A \sim B$ if and only if $A \triangle B$ is a null set for $A, B \in \Sigma_f$. This measure algebra *(or* Boolean algebra*) $\Sigma_f/\sim$ will be denoted by Σ/μ, and the metric ρ on Σ/μ will be denoted as ρ_μ.*

A measure space (X, Σ, μ) is called separable *if its corresponding metric space $(\Sigma/\mu, \rho_\mu)$ is separable.*

The following lemma describes how separable measure spaces appear in the proof of Proposition 6.7.

Lemma 6.2. *(See [27, Theorem 40.B]) A countably generated σ-algebra with σ-finite measure is separable.*

The following definition of an isomorphism between measure spaces is a 'modulo null sets' generalization of an isomorphism between measure spaces.

Definition 6.3. *The measure algebras of two measure spaces (X, Σ, μ) and (Y, T, ν) are called* isomorphic *if a bijection $\phi : \Sigma/\mu \to T/\nu$ exists measure preserving, $\mu = \nu \circ \phi$, and homomorphic w.r.t. the set operations $\setminus$, $\cup$, and $\cap$. In this case ϕ is also called a* Boolean isomorphism*.*

The following is a non-trivial result.

Theorem 6.4. *(See [27, Theorem C and Exercise 6 of Section 41] or [12, 9.3.4. Theorem] for probability measures) Let (X, μ, Σ) be a σ-finite non-atomic measure space. If the corresponding measure algebra is separable, then it is isomorphic to the measure algebra of the interval $(0, \mu(X))$ with Lebesgue measure.*

The following lemma defines for a given Boolean isomorphism from a measure algebra Σ/μ to a measure algebra Ξ/ν, a natural mapping can be defined from the corresponding space of μ-almost everywhere finite measurable functions $\mathcal{M}_0(\mu)$ to $\mathcal{M}_0(\nu)$.

Lemma 6.5. *Let (X, Σ, μ) and (Y, Ξ, ν) be measure spaces and let ϕ be a Boolean isomorphism from the measure algebras Ξ/ν to the measure algebra Σ/μ. There exists a bijective lattice isomorphism $T_\phi : \mathcal{M}_0(\mu) \to \mathcal{M}_0(\nu)$ that satisfies, for $f \in \mathcal{M}_0(\mu)$,*

$$\{f(x) : x \in \phi(B)\} = \{(T_\phi f)(y) : y \in B\} \tag{24}$$

for all $B \in \Xi/\nu$, and thus $d_{T_\phi f} = d_f$ and $(T_\phi f)^* = f^*$.

Proof. We first define this operator T_ϕ for simple non-negative functions, and proceed like in the definition of the Lebesgue integral. Let $f = \sum_{n=1}^{N} \alpha_n \chi_{A_n}$ be given, where each α_i is a real positive constant and each $A_n \in \Sigma$ is a μ-measurable set. We define:

$$T_\phi f = \sum_{n=1}^{N} \alpha_i \chi_{\phi(A_n)} \, . \tag{25}$$

It follows from the properties of the Boolean isomorphism ϕ that this definition does not depend on the specific representation of f. Substituting the expression of f as a linear combination of indicator functions into the left side of Equation (24), and the expression in Equation (25) for $T_\phi f$ into the right side of Equation (24), immediately reveals equality for simple functions.

We now define T_ϕ for non-negative measurable functions f finite μ-almost everywhere. Let $(s_m)_{m=1}^\infty$ be a sequence of positive simple functions such that $s_m \uparrow f$. We define

$$T_\phi f = \sup_m T_\phi s_m \, . \tag{26}$$

This definition clearly coincides with Equation (25) for simple non-negative functions. To see that this does not depend on the specific representation of f, let $t_m \uparrow f$. Let $M \in \mathbb{N}$ be arbitrary and $0 < \rho < 1$. It suffices to show that

$$\sup_m T_\phi s_m \geq \rho T_\phi t_M \, , \tag{27}$$

since by symmetry we have the reverse inequality as well. Let $t_M = \sum_{n=1}^N \alpha_n \chi_{A_n}$ and define $X_m = \{s_m \geq \rho t_M\}$. Because $s_m(x) \uparrow f_m(x)$ for all $x \in X$, $X_m \uparrow \{f \geq \rho t_m\} = X$. Write $s_m = s_m \chi_{X_m} + s_m \chi_{X_m^C}$, then

$$T_\phi s_m \geq T_\phi(s_m \chi_{X_m}) \geq \rho T_\phi(t_M \chi_{X_m}) = \rho \sum_{n=1}^N \alpha_n \chi_{\phi(A_n \cap X_m)} = \rho \sum_{n=1}^N \alpha_n \chi_{\phi(A_n) \cap \phi(X_m)} \, , \tag{28}$$

where the last step follows since ϕ is homomorphic with respect to $\cap$. Since $X_m \uparrow X$ implies $\phi(X_m) \uparrow \phi(X) = Y$, monotone continuity from below of ν gives that taking limits in Equation (28) yields Equation (27). We now show that Equation (24) still holds for non-negative measurable functions f, with $(s_m)_{m=1}^\infty$ a sequence of positive simple functions such that $s_m \uparrow f$. Since for simple functions we have proven Equation (24) already, for any m,

$$\{s_m(x) : x \in \phi(B)\} = \{(T_\phi s_m)(x) : x \in B\} \, .$$

This clearly implies that (since $s_m(x)$ is non-decreasing for constant x)

$$\left\{\sup_m s_m(x) : x \in \phi(B)\right\} = \left\{\sup_m (T_\phi s_m)(x) : x \in B\right\} ,$$

which means

$$\left\{\left(\sup_m s_m\right)(x) : x \in \phi(B)\right\} = \left\{\left(\sup_m T_\phi s_m\right)(x) : x \in B\right\} ,$$

which means

$$\{f(x) : x \in \phi(B)\} = \{(T_\phi f)(x) : x \in B\} \, .$$

The last part of the definition of T_ϕ is for not non-negative measurable functions f finite μ-almost everywhere:

$$T_\phi f = T_\phi\big(f \chi_{\{f \geq 0\}}\big) + T_\phi\big(f \chi_{\{f < 0\}}\big) \, , \tag{29}$$

This definition is obviously compatible with the previous definitions, and the assertion still holds.

We now show that T_ϕ is bijective. We claim that $T_{\phi^{-1}}$ is its inverse. First let a simple non-negative function $f = \sum_{n=1}^{N} \alpha_i \chi_{A_n}$ be given. The representation of $T_\phi f$ as a simple non-negative function follows immediately from the definition of T_ϕ in Equation (25). We obtain

$$T_{\phi^{-1}} T_\phi f = \sum_{n=1}^{N} \alpha_i \chi_{\phi^{-1}(\phi(A_n))} = \sum_{n=1}^{N} \alpha_i \chi_{A_n} \overset{\text{a.e.}}{=} f\,.$$

Now let an arbitrary non-negative function f be given. Let $(s_m)_{m=1}^{\infty}$ be a sequence of positive simple functions such that $s_m \uparrow f$. It follows that $T_\phi s_m \uparrow T_\phi f$, so by definition, Equation (26), $T_{\phi^{-1}} T_\phi f = \sup_m T_{\phi^{-1}} T_\phi s_m = \sup_m s_m$. We have thus shown T_ϕ is bijective.

It is easy to see that T_ϕ is lattice isomorphic. $\square$

To reduce the general case of an arbitrary σ-finite measure space X to the non-atomic case, the method of retracts can be used, as in the proof of Lemma 3.17, but it can also be proven by considering the product measure space $X \times [0, 1]$. For this we then need the following lemma.

Lemma 6.6. *(See [33, Lemma 2] and [33, Proposition 1] for the converse) Let (X, Σ, μ) be a measure space and $\Lambda \subseteq \Sigma$ a σ-finite sub-σ-algebra of Σ. Let E be a fully symmetric rearrangement invariant Banach function space over X. The conditional expectation ('averaging operator') $\mathbb{E}_E(\cdot|\Lambda) : E \to E \cap \mathcal{M}(\mu_\Lambda)$ defined by*

$$\int_A \mathbb{E}_E(f|\Lambda)\,\mathrm{d}\,\mu = \int_A f\,\mathrm{d}\,\mu$$

for all $f \in E$ and all Λ-measurable A with μ-finite measure, is a well-defined contraction.

Proof. Since all rearrangement invariant Banach function spaces are continuously embedded in $L^1 + L^\infty$, by [11, Theorem 6.6 of chapter 2] or [34, Theorem 4.1 of chapter 4] (sometimes part of the definition of a rearrangement invariant space), it suffices to show that the conditional expectation is well-defined on L^1 and on L^∞.

The conditional expectation $\mathbb{E}_1(\cdot|\Lambda)$ defined on L^1 is a well-defined contraction on L^1 by the Radon Nikodym theorem (see [22, p. 233D] and [22, 242J]).

To see that the conditional expectation is also a well defined contraction as an operator $\mathbb{E}_\infty(\cdot|\Lambda) : L^\infty(\mu) \to L^\infty(\mu|_\Sigma)$, remark that $\mathbb{E}_\infty(\cdot|\Lambda)$ is the adjoint of the natural embedding $j : L^1(\mu|_\Sigma) \to L^1(\mu)$ ([22, 243 Notes and comments]). The defining property of the conditional expectation then follows by choosing $g = \chi_A \,\mathrm{d}\,\mu \in (L^\infty(\mu))'$ in the definition of the adjoint: $\langle jf, g\rangle = \langle f, j'g\rangle$ for $f \in L^\infty(\mu|_\Sigma)$.

Since the norm is assumed to be fully symmetric, to show $\mathbb{E}_E(\cdot|\Lambda)$ is a contraction, it suffices to show that $f \in E$ implies $\mathbb{E}_E(f|\Lambda) \prec\prec f$. By subadditivity of the maximal function, Lemma 3.16, and by linearity of the conditional expectation,

$$\int_0^t (\mathbb{E}_E(f|\Lambda))^*(s)\,\mathrm{d}\,s \le \|\mathbb{E}_1(f_1|\Lambda)\|_1 + t\|\mathbb{E}_\infty(f_1|\Lambda)\|_\infty$$

where $f_1 \in L^1$ and $f_\infty \in L^\infty$ are such that $f = f_1 + f_\infty$. Since both $\mathbb{E}_1(\cdot|\Lambda)$ and $\mathbb{E}_\infty(\cdot|\Lambda)$ are contractions, the right-hand side can be estimated by $\|f_1\|_1 + t\|f_\infty\|_\infty$. By [11, Theorem 2.6 of section 2], the following formula finishes the proof: for all μ-measurable f and all $t > 0$,

$$\int_0^t f^*(s)\,\mathrm{d}\,s = \inf_{f=f_1+f_\infty} \{\|f_1\|_1 + t\|f_\infty\|_\infty\}\,.$$

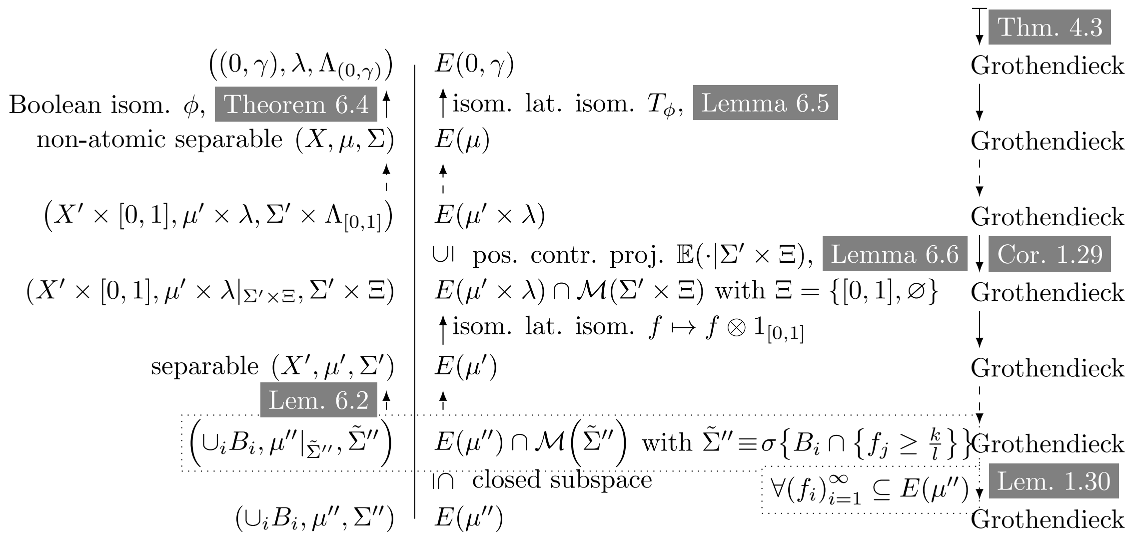
$\square$

Although a measure space is clearly σ-finite if it contains a sub-σ-algebra σ-finite with respect to the restriction of the measure, a sub-σ-algebra of a σ-finite σ-algebra is not necessarily σ-finite (consider the sub-σ-algebra $\{\varnothing, X\}$ if $\mu(X) > 0$).

We have now made all preparations to prove the main proposition of this section. However, to formulate the statement of the proposition, we first introduce for a fully symmetric Banach function space $E(0,\gamma)$ with the Fatou property defined on $(0,\gamma)$ with Lebesgue measure, a natural corresponding fully symmetric Banach function space $E(\mu)$ defined on X for a given arbitrary σ-finite measure space (X,μ) with $\mu(X)=\gamma$, by setting

$$E(\mu) = \{f \in S(\mu) : f^* \in E(0,\gamma)\} \text{ and } \|f\|_{E(\mu)} = \|f^*\|_{E(0,\gamma)}\,. \tag{30}$$

It is well-known and easy to prove that $E(\mu)$ is a fully symmetric Banach function space with the Fatou property (see [11, Theorem 4.9]).



Proposition 6.7. *If a Banach function space $E(0,\gamma)$ has the Grothendieck property, then $E(\mu)$ also has the Grothendieck property.*

Proof. Step 1: Suppose first that the measure space (X,Σ,μ) is separable and non-atomic. By Theorem 6.4 there exists a measure preserving Boolean isomorphism ϕ from the measure algebra Σ/μ of (X,Σ,μ) onto the measure algebra $\Lambda_{(0,\gamma)}/\lambda$ of $\big((0,\gamma),\Lambda_{(0,\gamma)},\lambda\big)$, with $\Lambda_{(0,\gamma)}$ the σ-algebra of Lebesgue measurable sets in $(0,\gamma)$. The isomorphism ϕ induces a bijective lattice isomorphism T_ϕ satisfying $(T_\phi f)^* = f^*$ for all $f \in \mathcal{M}_0(\mu)$, by Lemma 6.5. From the definition of $E(\mu)$ it follows that the restriction of T_ϕ to $E(\mu)$ is an isometrical isomorphism from $E(\mu)$ onto $E(0,\gamma)$. Consequently, $E(\mu)$ has the Grothendieck property.

Step 2: Suppose now that the measure space (X',Σ',μ') is separable (and possibly not non-atomic). It will be shown that $E(\mu')$ has the Grothendieck property by embedding it as a complemented subspace of the space $E(\mu'\times\lambda)$, where λ is the Lebesgue measure on $[0,1]$. It is easily verified that $\mu'\times\lambda$ is a separable measure and so, by Step 1, the space $E(\mu'\times\lambda)$ has the Grothendieck property. Corollary 1.29 then implies the Grothendieck property of $E(\mu')$.

To be explicit we will denote by

$$E_0 = \left\{ f \otimes \chi_{[0,1]} : f \in E(\mu') \right\}$$

the image of the norm-one embedding $f \mapsto f \otimes \chi_{[0,1]}$ from $E(\mu')$ into $E(\mu' \times \lambda)$. To see that this mapping is norm-one, notice that $\left(f \otimes \chi_{[0,1]}\right)^* = f^*$. An alternative description of E_0 is given by

$$E_0 = E(\Sigma \times \Lambda_0) = E(\mu' \times \lambda) \cap \mathcal{M}_0(\Sigma, \Lambda_0)$$

where $\Lambda_0 = \{\varnothing, [0,1]\}$ is the indiscrete σ-algebra. It remains to be shown that E_0 is a complemented subspace of $E(\mu' \times \lambda)$. By Lemma 6.6 the conditional expectation operator $\mathbb{E}_{E(\mu' \times \lambda)}(\cdot | \Sigma \times \Lambda_0)$ is a positive contractive projection from $E(\mu' \times \lambda)$ to E_0. This shows that E_0 is a complemented subspace of $E(\mu' \times \lambda)$, which finishes the proof.

Step 3: Suppose finally that (X'', μ'', Σ'') is an arbitrary σ-finite measure space. It suffices to show that for any sequence $(f_n)_{n=1}^\infty$ in $E(\mu'')$ there exists a closed subspace F of $E(\mu'')$ with the Grothendieck property such that $f_n \in F$ for all n, by Lemma 1.30.

To this end, suppose that $(f_n)_{n=1}^\infty$ is a sequence in $E(\mu'')$. By σ-finiteness of (X'', μ'', Σ'') there exists a disjoint sequence $(B_m)_{m=1}^\infty$ in Σ'' such that $\mu(B_m)$ is finite for all m and $X = \cup_{m=1}^\infty B_m$. Define $\tilde{\Sigma}''$ to be the σ-algebra generated by the sets

$$A_{n,m,k,l} = B_m \cap \{f_n > q\},$$

for $n, m, \in \mathbb{N}$ and $q \in \mathbb{Q}$, and let $\mu|_{\tilde{\Sigma}''}$ be the restriction of μ to $\tilde{\Sigma}''$. Since $X = \cup_{m,q} A_{1,m,q}$, it is clear that $\left(X, \tilde{\Sigma}'', \mu|_{\tilde{\Sigma}''}\right)$ is also σ-finite. It is also separable by Lemma 6.2 since $\tilde{\Sigma}''$ is generated by a countable collection. The Grothendieck property of $E(\mu'')$ now follows from Step 2. Since all of $\{f_n\}_{n=1}^\infty$ is contained in $F \equiv E(\mu'')$, Lemma 1.30 finishes the proof. □

The next follows from a combination of Proposition 6.7 and Theorem 5.5.

Theorem 6.8. *Let (X, Σ, μ) be a σ-finite measure space. Let $\psi : [0, \mu(X)) \to [0, \infty)$ be a concave function. The Marcinkiewicz space $M_\psi(X)$ is a Grothendieck space if*

$$\liminf_{t \downarrow 0} \frac{\psi(2t)}{\psi(t)} > 1 \text{ and } \left(\mu(X) < \infty \text{ or } \liminf_{t \to \infty} \frac{\psi(2t)}{\psi(t)} > 1 \right).$$

Let (X, Σ, μ) be a σ-finite measure space. Since the Weak L^p space $L^{p,\infty}(\mu)$ is the Marcinkiewicz space $M_\psi(\mu)$ for $\psi(t) = t^{1/p}$, the following is a simple corollary.

Corollary 6.9. *If (X, Σ, μ) be a σ-finite measure space, then $L^{p,\infty}(\mu)$ is a Grothendieck space.*

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