Travelling waves for the spatially discretized bistable Allen-Cahn equation

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Abstract

We analyze the spatially discretized version of the Allen-Cahn partial differential equation. The second order derivative is numerically approximated by a weighted infinite sum. The coefficients of this sum as well as the function f in the differential equation have got freedom inside determined restrictions. For this spatially discretized variation of the Allen-Cahn partial differential equation, we prove the existence of a travelling wave solution.

Lekensamenvatting

In dit verslag bekijken we een speciale versie van de Allen-Cahn vergelijking. De Allen-Cahn vergelijking wordt gebruikt om verschillende processen uit de natuur te beschrijven, zodat we deze beter kunnen begrijpen. Een voorbeeld van zo'n proces is de transportatie van een impuls door een zenuw. We gaan onderzoeken of de Allen-Cahn vergelijking oplossingen heeft die zich gedragen als een golf. Dit kunnen we doen in het continue geval of in het discrete geval. In de continue wereld beweegt alles vloeiend, terwijl in de discrete wereld alles stapsgewijs beweegt. Een voorbeeld van een golf in een continue ruimte is een golf zoals we die zien in de zee, want deze beweegt zich vloeiend voort. Een voorbeeld van een golf in een discrete ruimte is een wave in het stadion. Deze beweegt zich namelijk stapsgewijs voort. Wij gaan ons verdiepen in het discrete geval en ons doel luidt dan ook: 'Het bewijzen van het bestaan van een golfoplossing voor de gediscretiseerde Allen-Cahn vergelijking'.

Summary

In this thesis we prove that the spatially discretized Allen-Cahn partial differential equation has a travelling wave solution. We start by giving some background information in the Introduction. The Allen-Cahn equation is introduced and some of its applications are mentioned. Furthermore, the spatial discretization used is discussed and motivated. The body of the report consists of two sections, namely the Problem Setup and the Main Proof.

In the first part, the problem for which we want to prove a solution exists is laid out. Several assumptions are made regarding the discretization, the function f in the Allen-Cahn equation and the fact that we seek a travelling wave solution. After applying these assumptions it is shown that the Allen-Cahn equation can be written as

$$c_\varepsilon u_\varepsilon' - \Delta_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0, \qquad u_\varepsilon(\pm\infty) = \pm 1.$$

Here

$$\Delta_\varepsilon u(x) = \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \left[u(x - k\varepsilon) + u(x + k\varepsilon) - 2u(x) \right].$$

This is the problem for which, mainly in the second part, it will be proven a solution exists. But first, the operators $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi$, $\mathcal{R}(c,\phi)$ and $\mathbf{N}(u_0,\phi)$ are introduced to rewrite the equation above in a form suitable for the proof in the second part, namely as $\mathcal{L}^{+}_{\varepsilon,\delta}\phi_\varepsilon = \mathcal{R}(c_\varepsilon, \phi_\varepsilon)$. Here $\mathcal{L}^{+}_{\varepsilon,\delta}\phi_\varepsilon$ is the linearization of the discrete Allen-Cahn equation around the travelling wave solution u_0 of the Allen-Cahn partial differential equation, while operator $\mathcal{R}(c_\varepsilon, \phi_\varepsilon)$ contains the difference between the discrete and the continuous equation. The nonlinear part of this difference is denoted as $\mathbf{N}(u_0,\phi)$. The first of the two main sections ends with proves of some properties of Δ_ε.

The second part consists almost exclusively of theorems and proofs. The entire section functions as a setup to be able to apply Banach's Fixed Point Theorem on the mapping $\mathbf{T}\phi = (\mathcal{L}^{+}_{\varepsilon,\delta})^{-1}\mathcal{R}(c_\varepsilon(\phi),\phi)$ in the end. From this it immediately follows that there exists a travelling wave solution and thus the goal is achieved. Most of the work is in showing that $\mathbf{T}$ satisfies the requirements in order to apply Banach's Fixed Point Theorem. This is done by making estimates involving the operators $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi$ and $\mathcal{R}(c_\varepsilon(\phi),\phi)$. To estimate expressions containing $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi$, there are first made similar estimations with the related operator $\mathcal{L}^{\pm}_{0}\phi$.

Finally, the result that has been proven in the report is discussed in the Conclusion. Furthermore, ideas for further research concerning this subject are shared.

Contents

1 Introduction

In this report we discuss a partial differential equation (PDE) that is related to the Allen-Cahn equation. The one dimensional Allen-Cahn equation is given by

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + f(u) = 0. \tag{1.1}$$

It is a reaction-diffusion equation used to describe numerous processes in nature. An example of such a process is the propagation phenomenon of nerve excitation. If the equation is used to describe this, it is referred to as the Nagumo equation.

The structure of the nonlinearity f has a major impact on the behaviour of the solutions to the PDE. We will examine this equation for a particular kind of functions f called bistable functions. These functions will include the function

$$f(u) = (u-1)(u+1)(u-q), \text{ with } q \in (-1, 1). \tag{1.2}$$

Then the diffusion-free equation of (1.1) has two stable equilibria $u = -1, 1$ and an unstable equilibrium $u = q$. Such a system is named a bistable system and (1.1) is then called a bistable reaction-diffusion equation.

An example of a situation that can be modelled by a bistable system is the competition between two species for dominance in a certain area[1]. Then the two stable equilibria each represent full dominance of one of the two species. Another example is the phase transitions of materials. Here the stable equilibria represent two material phases.

The Allen-Cahn equation has been used to understand a variety of concepts in dynamical systems theory. In many of these concepts, travelling wave type solutions play an important role. These can be written as

$$u(x,t) = \phi(x + ct), \qquad \phi(\pm\infty) = \pm 1. \tag{1.3}$$

These boundary conditions represent the stable equilibria, allowing that one of the stable equilibria dominates the total solution. To satisfy PDE (1.1), the pair (c, ϕ) has to satisfy

$$c\phi' = \phi'' - f(\phi), \tag{1.4}$$

known as the travelling wave equation.

We will study these wave type solutions, but in slightly different circumstances. We want to examine whether there exist wave type solutions with the presence of spatial discretization. Usually we discretize in space to avoid having to solve a PDE analytically. Famous methods for solving PDE's that make use of this idea are the finite difference method, the finite element method and the finite volume method. But in this case we can actually find solutions to (1.4) using analytic methods. These solutions, analyzed by Fife and McLeod[7], are given by

$$\phi(\xi) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\sqrt{2}}{4}\xi\right), \qquad c = \frac{\sqrt{2}}{2}(1-2q). \tag{1.5}$$

So why would we discretize in space? The answer is that for some phenomena in nature a discretized space fits them better than a continuous space. An example is the Josephson effect, which is a phenomenon that occurs when two superconductors are placed in proximity, with a small barrier of non-conducting material between them. Such a device is called a Josephson junction. The Josephson effect produces a current that flows across the Josephson junction. When we consider an array of Josephson junctions we can already see that this array has strong connections to a discretized grid. Other examples are chains of coupled diode resonators, coupled chemical or biochemical reactors, myelinated nerve fibers, neuronal networks, and patchy ecosystems. In all of these situations it makes more sense to use a discretized space, rather than a continuous space.

Now we specify how the spatial discretization will be executed. We discretize in space by setting the spatial coordinate $x = \varepsilon n$ in the travelling wave assumption (1.3). Here ε is small and represents the discretization step size and n is some integer denoting the amount of steps taken. So instead of (1.3), we use a discrete travelling wave assumption that is given by

$$u_n(t) = u(\varepsilon n + ct), \qquad u(\pm\infty) = \pm 1. \tag{1.6}$$

This assumption gives us a different equation compared to (1.4). What will make the problem we are going to study a really difficult one, is that the second derivative in space will be approximated by an infinite sum. In order to keep the analysis general, we don't impose too many restrictions on the infinite sum. So we consider a collection of approximations for the second spatial derivative. This collection contains well known numerical approximations such as the second order central difference. But a more complicated numerical approximation such as

$$\frac{\partial^2}{\partial x^2} u(x,t) \quad \to \quad \frac{1}{\varepsilon^2} \sum_{k>0} \left[u_{j+k}(t) + u_{j-k}(t) - 2u_j(t)\right] e^{-k} \tag{1.7}$$

is also included. The idea of the central difference approximation is being extended. The coefficients e^{-k} ensure the further away from x we are, the less impact the value of u has on the approximation.

For our research we combine all the ideas mentioned in the previous paragraphs to modify the Allen-Cahn equation. So we seek travelling wave solutions for a spatially discretized and bistable Allen-Cahn equation, where the second space derivative is approximated by an infinite sum. The existence of such a travelling wave solution was first proven by Bates, Chen and Chmaj[2], but their work omitted several calculations, proofs of claims and subtleties. The main contribution of this thesis is to fill in these gaps, by providing fully worked out proofs.

Likely, our work will make it easier to generalize these results. For example, consider a version of equation (1.1) where the second derivative is replaced by a convolution kernel of the form

$$\frac{\partial^2}{\partial x^2} u(x,t) \quad \to \quad \int_{-\infty}^{\infty} \mathcal{K}(y) u_j(y+t)\, dy. \tag{1.8}$$

These type of discretizations incorporate both continuous and discrete parts and have been studied, for instance, in [5] and [6].

Mathematically, the problem we will study is given by

$$\dot{u}_n(t) = \frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k u_{n-k}(t) - f(u_n), \quad n \in \mathbb{Z}, \tag{1.9}$$

with $u_n(t)$ as in (1.6). This differential equation originates from Ising models. These are models used to describe magnets, among other things. The magnet is discretized into a lattice where every site has its own magnetic moment, called spin. Each spin has a direction, which is either up or down. This idea of having two states, namely up and down, makes the Ising models also very fitting for describing all sorts of behaviour related to phase transitions. When an Ising model is used to describe a phase transition between liquid and gas, then liquid and gas behave as the two states.

2 Problem Setup

This section is mainly focused on introducing the problem for which we will prove a solution exists. The Allen-Cahn equation will be modified using some made assumptions and thereafter, by introducing a few operators, will be written in a convenient way such that we can later prove a solution exists.

Let us first state the partial differential equation that plays a central role throughout this thesis. It is called the Allen-Cahn or Nagumo partial differential equation (PDE) and is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f(u). \tag{2.1}$$

As mentioned we will make some assumptions and we start with an assumption regarding the function $f(u)$.

Assumption (A1). *$f \in C^2(\mathbb{R})$ is a function with exactly 3 zeroes at $-1, q \in (-1,1)$ and 1, with $f'(\pm 1) > 0$. Furthermore $\int_{-1}^{1} f(y)\,dy \neq 0$.*

An example of such a function is

$$f(u) = (u-1)(u+1)(u-q), \quad \text{with } q \in (-1,1). \tag{2.2}$$

Thus we could keep in mind the third order polynomial for f if we would like, but in general we will assume f is any function satisfying (A1).

We want to examine whether there exist wave type solutions to equation (2.1). So the next assumption we make is that $u(x,t) = \Phi(x+ct)$, which represents a travelling wave with wave speed c. We want to insert this travelling wave into equation (2.1). In order to do this we first compute the derivatives of $u(x,t)$ using our travelling wave assumption. We find

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Phi'(x+ct)\frac{\partial}{\partial t}\Big[x+ct\Big] = c\Phi'(x+ct), \\ \frac{\partial u}{\partial x} &= \Phi'(x+ct)\frac{\partial}{\partial x}\Big[x+ct\Big] = \Phi'(x+ct), \\ \frac{\partial^2 u}{\partial x^2} &= \Phi''(x+ct)\frac{\partial}{\partial x}\Big[x+ct\Big] = \Phi''(x+ct). \end{aligned} \tag{2.3}$$

After substitution, equation (2.1) is transformed into

$$c\Phi'(x+ct) = \Phi''(x+ct) - f(\Phi). \tag{2.4}$$

For better readability we introduce the variable $\xi = x + ct$, so that the equation can now be written as

$$c\Phi'(\xi) = \Phi''(\xi) - f(\Phi). \tag{2.5}$$

We will be examining a spatially discretized version of this equation. Namely, the problem

$$u_n'(t) = \frac{1}{\varepsilon^2}\sum_{k=-\infty}^{\infty} \alpha_k u_{n-k}(t) - f(u_n), \quad n \in \mathbb{Z}, \tag{2.6}$$

where $\varepsilon > 0$ and where $u_n(t)$ is a travelling wave solution of the form

$$u_n(t) = u(\varepsilon n + ct) \quad \text{satisfying} \;\; u(\pm\infty) = \pm 1. \tag{2.7}$$

Furthermore, an assumption is made for the coefficients $\{\alpha_k\}$.

Assumption (A2). *The coefficients* $\{\alpha_k\}$ *satisfy* $\sum_k \alpha_k = 0$, $\alpha_k = \alpha_{-k}$, $\sum_{k>0} \alpha_k k^2 = 1$ *and* $\sum_{k>0} |\alpha_k| k^2 < \infty$. *Furthermore, we have an extra condition when* $k = 0$, *namely* $\alpha_0 < 0$. *As a final condition we have that* $A(z) = \sum_{k>0} \alpha_k (1 - \cos(kz)) \geq 0$ *for all* $z \in [0, 2\pi]$.

Equation (2.6) seems like a completely different equation compared to (2.5), but we get the original equation (2.5) back after letting ε go to 0. We will first show this is true in the case of an example where the coefficients have certain given values. Later we will prove this for the general case in which the coefficients $\{\alpha_k\}$ satisfy (A2). But first we'll explain how we get to this spatially discretized equation.

To explain this we need to go back to equation (2.5). Here we discretize in space by writing the spatial coordinate as $x = \varepsilon n$ for $n \in \mathbb{Z}$. This makes ξ depend on n and thus we write $\xi_n = \varepsilon n + ct$. Applying this to (2.5), we get that $c\Phi'(\varepsilon n + ct) = \Phi''(\varepsilon n + ct) - f(\Phi)$ for each $n \in \mathbb{Z}$. Note that $\frac{\partial \Phi}{\partial t} = c\Phi'(\varepsilon n + ct)$ and replace every Φ with a u just for notation reasons. Then we have $\frac{\partial u}{\partial t} = u''(\varepsilon n + ct) - f(u)$. As a final step we can write $u(\varepsilon n + ct) = u_n(t)$ which is the definition for $u_n(t)$. This transforms the equation into $u_n'(t) = u''(\varepsilon n + ct) - f(u_n)$. Now the only difference with equation (2.6) is the second derivative term. But we will later see that if ε is approximating zero, the term including the summation in equation (2.6) approximates this second derivative with the help of the assumptions we made in (A2). So this term can be considered as a numerical approximation for the second derivative. Note that for this reason it makes sense see ε as the discretization step size and n as the amount of steps taken.

As an example we choose the coefficients as follows: $\alpha_1 = \alpha_{-1} = 1$, $\alpha_0 = -2$ and $\alpha_k = 0$ for all other values of k. Note that the coefficients in this example satisfy (A2). This choice of coefficients corresponds to the so-called nearest neighbour discretization. However, we also allow cases where infinitely many coefficients are nonzero. As mentioned before we want to show that we get equation (2.5) back when letting ε go to 0.

To be able to show this we first need a few derivations. We denote $x = \varepsilon n + ct$. With this notation we first derive expressions for the u terms in our equation which are needed later on. We have

$$u_n'(t) = u'(x)\frac{\partial}{\partial t}\Big[\varepsilon n + ct\Big] = cu'(x), \tag{2.8}$$

$$u_{n-k}(t) = u(\varepsilon(n-k) + ct) = u(\varepsilon n + ct - k\varepsilon) = u(x - k\varepsilon). \tag{2.9}$$

Furthermore, we write α_0 in a clever way. We can do this by using some of the made assumptions (A2). We obtain

$$\alpha_0 = \alpha_0 - \sum_k \alpha_k = -\sum_{k\neq 0} \alpha_k = -\sum_{k<0} \alpha_k - \sum_{k>0} \alpha_k = -2\sum_{k>0} \alpha_k. \tag{2.10}$$

Now everything is ready to show the claim that letting ε to 0 gives us back the original equation. We move all the terms in equation (2.6) to the left-hand side and then use (A2) and the expressions that just have been derived. This gives

$$
\begin{aligned}
0 &= u_n'(t) - \frac{1}{\varepsilon^2}\sum_{k=-\infty}^{\infty}\alpha_k u_{n-k} + f(u_n) \\
&= cu'(x) - \frac{1}{\varepsilon^2}\sum_{k=-\infty}^{\infty}\alpha_k u(x-k\varepsilon) + f(u) \\
&= cu'(x) - \frac{1}{\varepsilon^2}\sum_{k>0}\alpha_k u(x-k\varepsilon) - \frac{1}{\varepsilon^2}\sum_{k<0}\alpha_k u(x-k\varepsilon) - \frac{1}{\varepsilon^2}\alpha_0 u(x) + f(u) \\
&= cu'(x) - \frac{1}{\varepsilon^2}\sum_{k>0}\alpha_k u(x-k\varepsilon) - \frac{1}{\varepsilon^2}\sum_{k>0}\alpha_k u(x+k\varepsilon) - \frac{1}{\varepsilon^2}\alpha_0 u(x) + f(u) \qquad (2.11) \\
&= cu'(x) - \frac{1}{\varepsilon^2}\sum_{k>0}\alpha_k u(x-k\varepsilon) - \frac{1}{\varepsilon^2}\sum_{k>0}\alpha_k u(x+k\varepsilon) + \frac{2}{\varepsilon^2}\sum_{k>0}\alpha_k u(x) + f(u) \\
&= cu'(x) - \frac{1}{\varepsilon^2}\sum_{k>0}\alpha_k\left[u(x-k\varepsilon) + u(x+k\varepsilon) - 2u(x)\right] + f(u) \\
&= cu'(x) - \sum_{k>0}\alpha_k k^2\frac{u(x-k\varepsilon) + u(x+k\varepsilon) - 2u(x)}{(k\varepsilon)^2} + f(u).
\end{aligned}
$$

We now use the prescribed values of the coefficients from the earlier given example by inserting these into the final expression. Then we find that

$$0 = cu'(x) - \frac{u(x-\varepsilon) + u(x+\varepsilon) - 2u(x)}{\varepsilon^2} - f(u). \qquad (2.12)$$

To get to equation (2.5) there is one step left to do, which is letting ε go to 0. It is well known from numerical mathematics that the fraction $\frac{u(x-\varepsilon)+u(x+\varepsilon)-2u(x)}{\varepsilon^2}$ is converging to the second derivative of $u(x)$ if $\varepsilon \to 0$. So after letting $\varepsilon \to 0$ we end up with

$$cu'(x) - u''(x) + f(u) = 0, \qquad u(\pm\infty) = \pm 1. \qquad (2.13)$$

Note that this is exactly the same equation as (2.5). So we can conclude that (2.6) is indeed a discretized version of (2.5). But we only showed this in the case of the example coefficient values. It will later be shown in Corollary 1 that this is also valid in the general case.

We'll now rewrite equation (2.6) a little by introducing some new notation. Note that during the derivations in (2.11) we found out that (2.6) can be written as

$$cu'(x) - \frac{1}{\varepsilon^2}\sum_{k>0}\alpha_k\left[u(x-k\varepsilon) + u(x+k\varepsilon) - 2u(x)\right] + f(u) = 0, \qquad u(\pm\infty) = \pm 1. \quad (2.14)$$

To make this equation look a bit neater we introduce the operator Δ_ε that is denoted as

$$\Delta_\varepsilon u(x) = \frac{1}{\varepsilon^2}\sum_{k>0}\alpha_k\left[u(x-k\varepsilon) + u(x+k\varepsilon) - 2u(x)\right]. \qquad (2.15)$$

So now we can write (2.14) as

$$c_\varepsilon u_\varepsilon' - \Delta_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0, \qquad u_\varepsilon(\pm\infty) = \pm 1. \qquad (2.16)$$

Here $(c_\varepsilon, u_\varepsilon)$ is the solution pair to equation (2.14) with a certain value for ε that is given in the subscripts. The goal of this thesis is to show that this problem has a solution. To be able to prove this we have to further rewrite this problem into a form suitable for the proof. But first we'll discuss function spaces.

2.1 Function spaces

In the process of solving this problem we will often take functions that live in different function spaces. In this small section these function spaces will be discussed.

First a remark about notation. The inner product will be denoted as $\langle \cdot, \cdot \rangle$ and the norm will be denoted as $\|\cdot\|$. We now discuss the first type of function spaces called L^p spaces. For our problem we are only interested in $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$. For $L^2(\mathbb{R})$ we will give the corresponding definition, inner product and norm. These are given by

$$\begin{aligned} L^2(\mathbb{R}) &= \left\{ f : \mathbb{R} \to \mathbb{R} \,\middle|\, \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty \right\}, \\ \langle f, g \rangle_{L^2} &= \int_{-\infty}^{\infty} f(x) g(x) dx, \\ \|f\|_{L^2} &= \left(\int_{-\infty}^{\infty} |f(x)|^2 dx \right)^{1/2}. \end{aligned} \tag{2.17}$$

In this report we will omit writing L^2 in $\langle f, g \rangle_{L^2}$, since we will almost always use the inner product corresponding to $L^2(\mathbb{R})$. So this means that $\langle f, g \rangle$ denotes the L^2 inner product. If we use an inner product corresponding to another function space, then this will be specified.

Now we'll introduce $L^\infty(\mathbb{R})$. $L^\infty(\mathbb{R})$ contains measurable functions that are bounded. Furthermore the corresponding norm is given by

$$\|f\|_{L^\infty} = \sup_{x \in \mathbb{R}} |f(x)|. \tag{2.18}$$

The second type of function spaces we discuss are Sobolev spaces. These type of function spaces are denoted by $H^k(\mathbb{R})$. Here we are only interested in the cases $k = 1$ and $k = 2$. For both $H^1(\mathbb{R})$ and $H^2(\mathbb{R})$ we state the definition and the norm. These are given by

$$\begin{aligned} H^1(\mathbb{R}) &= \left\{ f : \mathbb{R} \to \mathbb{R} \mid f \in L^2(\mathbb{R}),\ f' \in L^2(\mathbb{R}) \right\}, \\ \|f\|_{H^1} &= \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2 \right)^{1/2}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} H^2(\mathbb{R}) &= \left\{ f : \mathbb{R} \to \mathbb{R} \mid f \in L^2(\mathbb{R}),\ f' \in L^2(\mathbb{R}),\ f'' \in L^2(\mathbb{R}) \right\}, \\ \|f\|_{H^2} &= \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2 + \|f''\|_{L^2}^2 \right)^{1/2}. \end{aligned} \tag{2.20}$$

Finally we also give the definition of function space $C_0^\infty(\mathbb{R})$. This function space won't occur too often. It's definition is given by

$$C_0^\infty(\mathbb{R}) = \left\{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is infinitely differentiable and } \lim_{x \to \pm\infty} f(x) = 0 \right\}. \tag{2.21}$$

All the function spaces we need have now been introduced. There are some useful properties for certain inner products. These are stated in the next lemma.

Lemma 1. *For ϕ in certain function spaces some useful identities for inner products follow. We have*

(i) for any $\phi \in H^1(\mathbb{R})$, $\langle \phi' , \phi \rangle = 0$;

(ii) for any $\phi \in H^2(\mathbb{R})$, $\langle \phi'' , \phi \rangle \leq 0$.

Proof. *(i)* Let $\phi \in H^1(\mathbb{R})$ arbitrary. Then applying integration by parts gives

$$\langle \phi' , \phi \rangle = \int_{\mathbb{R}} \phi'(x)\phi(x)\, dx = [\phi(x)\phi(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} \phi'(x)\phi(x)\, dx. \tag{2.22}$$

Since $\phi \in H^1(\mathbb{R})$ we have that $\phi(x) \to 0$ if we let $x \to \pm\infty$. This implies that $[\phi(x)\phi(x)]_{-\infty}^{\infty} = 0$. So we find

$$\langle \phi' , \phi \rangle = -\int_{\mathbb{R}} \phi'(x)\phi(x)\, dx = -\langle \phi' , \phi \rangle . \tag{2.23}$$

From this it follows that $\langle \phi' , \phi \rangle = 0$.

(ii) Let $\phi \in H^2(\mathbb{R})$ arbitrary. Then applying integration by parts gives

$$\langle \phi'' , \phi \rangle = \int_{\mathbb{R}} \phi''(x)\phi(x)\, dx = [\phi'(x)\phi(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} \phi'(x)\phi'(x)\, dx. \tag{2.24}$$

Since $\phi \in H^2(\mathbb{R})$ we have that $\phi(x) \to 0$ and $\phi'(x) \to 0$ if we let $x \to \pm\infty$. From this it follows that $[\phi'(x)\phi(x)]_{-\infty}^{\infty} = 0$. So we find

$$\langle \phi'' , \phi \rangle = -\int_{\mathbb{R}} \phi'(x)\phi'(x)\, dx = -\left\| \phi'(x) \right\|_{L^2}^2 \leq 0. \tag{2.25}$$

□

2.2 Rewriting the problem

We now return to the problem we want to solve. To be able to show that equation (2.16) has a solution the problem has to be rewritten. This will be done in this section.

If we study equation (2.16), we observe that for one value of ε we already know the solution to this equation. This is the case for ε equal to 0. We have seen this gives us equation (2.13) for which the solution is known. We denote this solution by (c_0, u_0). Thus we have got the identity

$$c_0 u_0' - u_0'' + f(u_0) = 0, \qquad u_0(\pm\infty) = \pm 1. \tag{2.26}$$

But there is more known about the solution (c_0, u_0), which is stated in a theorem and a lemma.

Theorem 1 ([8, §1])**.** *Consider the equation $c_0 u_0'(x) - u_0''(x) + f(u_0(x)) = 0$ with the boundary conditions $u_0(\pm\infty) = \pm 1$. Then there exists a solution $u_0(x)$. Furthermore $u_0'(x) > 0$ for all $x \in \mathbb{R}$ and $u_0(x)$ converges exponentially to ± 1 as $x \to \pm\infty$.*

Lemma 2. *Assume f is a function satisfying (A1) and let (c_0, u_0) be defined as the solution to* (2.26)*. Then $c_0 \neq 0$*

Proof. We first take equation (2.26) and multiply it on both sides by $u_0'(x)$. This gives

$$c_0 \left[u_0'(x)\right]^2 - u_0''(x)u_0'(x) + f(u_0(x))u_0'(x) = 0. \tag{2.27}$$

Now we integrate from $-\infty$ to ∞ on both sides of the equation, after which we get

$$c_0 \int_{-\infty}^{\infty} \left[u_0'(x)\right]^2 \, dx - \int_{-\infty}^{\infty} u_0''(x)u_0'(x) \, dx + \int_{-\infty}^{\infty} f(u_0(x))u_0'(x) \, dx = 0. \tag{2.28}$$

Let us now analyze the latter two integrals in the equation above. Using the substitution $y = u_0(x)$ we can write

$$\int_{-\infty}^{\infty} f(u_0(x))u_0'(x) \, dx = \int_{-1}^{1} f(y) \, dy. \tag{2.29}$$

It follows from Theorem 1 that $u_0' \in L^2(\mathbb{R})$. So we can apply Lemma 1*(i)* to u_0', which gives that

$$\int_{-\infty}^{\infty} u_0''(x)u_0'(x) \, dx = \langle u_0'' \, , \, u_0' \rangle = 0. \tag{2.30}$$

Using both of these observation we can rewrite (2.28) to

$$c_0 = \frac{-\int_{-1}^{1} f(y) \, dy}{\int_{-\infty}^{\infty} \left[u_0'(x)\right]^2 \, dx}. \tag{2.31}$$

(A1) tells us that $\int_{-1}^{1} f(y) \, dy \neq 0$. Furthermore $\int_{-\infty}^{\infty} \left[u_0'(x)\right]^2 \, dx > 0$. So we can conclude that $c_0 \neq 0$. ☐

To rewrite our problem we introduce the operators $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi$, $\mathcal{R}(c, \phi)$ and $\mathbf{N}(u_0, \phi)$. These are given by

$$\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi = \left\{\pm c_0 \tfrac{d}{dx} - \Delta_\varepsilon + f_u(u_0) + \delta\right\} \phi, \tag{2.32}$$

$$\mathcal{R}(c, \phi) = (c_0 - c)(u_0' + \phi') + (\Delta_\varepsilon - \tfrac{d^2}{dx^2})u_0 + \delta\phi - \mathrm{N}(u_0, \phi), \tag{2.33}$$

$$\mathbf{N}(u_0, \phi) = f(u_0 + \phi) - f(u_0) - f_u(u_0)\phi. \tag{2.34}$$

Here $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi$ is the linearization of the spatially discretized Allen-Cahn equation (2.16) around the travelling wave solution (c_0, u_0). Furthermore, $\mathcal{R}(c, \phi)$ contains the difference between the continuous and the discrete equation, where the nonlinear parts of this difference are contained in $\mathbf{N}(u_0, \phi)$. Now we are able to rewrite the problem with the help of a lemma.

Lemma 3. *Let (c_0, u_0) be the solution to equation* (2.13)*. Write $u_\varepsilon = u_0 + \phi_\varepsilon$, where $\phi_\varepsilon \in H^1(\mathbb{R})$ and let $\delta > 0$ be a small number. Then the following 2 problems are equivalent:*

1. $c_\varepsilon u_\varepsilon' - \Delta_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0$.
2. $\mathcal{L}^{+}_{\varepsilon,\delta}\phi_\varepsilon = \mathcal{R}(c_\varepsilon, \phi_\varepsilon)$.

Proof. We will prove this by showing $\mathcal{L}^+_{\varepsilon,\delta}\phi_\varepsilon - \mathcal{R}(c_\varepsilon, \phi_\varepsilon) = c_\varepsilon u'_\varepsilon - \Delta_\varepsilon u_\varepsilon + f(u_\varepsilon)$. We compute

$$\begin{aligned}
\mathcal{L}^+_{\varepsilon,\delta}\phi_\varepsilon - \mathcal{R}(c_\varepsilon, \phi_\varepsilon) &= c_0 \tfrac{d}{dx}\phi_\varepsilon - \Delta_\varepsilon\phi_\varepsilon + f_u(u_0)\phi_\varepsilon + \delta\phi_\varepsilon - (c_0 - c_\varepsilon)(u'_0 + \phi'_\varepsilon) - \Delta_\varepsilon u_0 \\
&\qquad + \tfrac{d^2}{dx^2}u_0 - \delta\phi_\varepsilon + f(u_0 + \phi_\varepsilon) - f(u_0) - f_u(u_0)\phi_\varepsilon \\
&= c_0\phi'_\varepsilon - \Delta_\varepsilon\phi_\varepsilon - c_0u'_0 - c_0\phi'_\varepsilon + c_\varepsilon u'_0 + c_\varepsilon\phi'_\varepsilon \\
&\qquad - \Delta_\varepsilon u_0 + u''_0 + f(u_0 + \phi_\varepsilon) - f(u_0) \\
&= c_\varepsilon(u'_0 + \phi'_\varepsilon) - \Delta_\varepsilon(u_0 + \phi_\varepsilon) + f(u_0 + \phi_\varepsilon) - (c_0u'_0 - u''_0 + f(u_0)) \\
&= c_\varepsilon u'_\varepsilon - \Delta_\varepsilon u_\varepsilon + f(u_\varepsilon).
\end{aligned} \tag{2.35}$$

Note that, in the final step, we used that $u_\varepsilon = u_0 + \phi_\varepsilon$ and that (c_0, u_0) is the solution to equation (2.13), since this implies that $c_0u'_0 - u''_0 + f(u_0) = 0$. Furthermore we remark that we can also do all steps in reverse. Thus it follows that *1.* and *2.* are equivalent. □

From now on we will work with the rewritten equation from Lemma 3. Before we start with the main part of this report where we prove this problem has a solution, we first treat two lemmas with some useful results for later on.

Lemma 4. *Let Δ_ε be defined as in (2.15) and let the coefficients $\{\alpha_k\}$ satisfy (A2). Then*

(i) for any $\phi \in L^\infty(\mathbb{R})$ with $\phi'' \in L^2(\mathbb{R})$ and with $\lim_{x\to\pm\infty}\phi(x)$ and $\lim_{x\to\pm\infty}\phi'(x)$ existing, we have

$$\|\Delta_\varepsilon\phi - \phi''\|_{L^2} \to 0 \text{ as } \varepsilon \downarrow 0;$$

(ii) for any $\phi \in H^1(\mathbb{R})$, $\langle\Delta_\varepsilon\phi, \phi'\rangle = 0$;

(iii) for any $\phi, \psi \in L^2(\mathbb{R})$, $\langle\Delta_\varepsilon\phi, \psi\rangle = \langle\phi, \Delta_\varepsilon\psi\rangle$;

(iv) for any $\phi \in L^2(\mathbb{R})$, $\langle\Delta_\varepsilon\phi, \phi\rangle \leq 0$.

Proof. *(i)* First notice that, since $\lim_{x\to\pm\infty}\phi(x)$ and $\lim_{x\to\pm\infty}\phi'(x)$ exist, we have

$$\begin{aligned}
\lim_{x\to\pm\infty}\Delta_\varepsilon\phi(x) &= \frac{1}{\varepsilon^2}\sum_{k>0}\lim_{x\to\pm\infty}\alpha_k\left[\phi(x - k\varepsilon) + \phi(x + k\varepsilon) - 2\phi(x)\right] \\
&= \frac{1}{\varepsilon^2}\sum_{k>0}\lim_{x\to\pm\infty}\alpha_k\left[\phi(x) + \phi(x) - 2\phi(x)\right] \\
&= 0.
\end{aligned} \tag{2.36}$$

Because $[\Delta_\varepsilon\phi(x)]' = \Delta_\varepsilon\phi'(x)$, it follows in exactly the same way that $\lim_{x\to\pm\infty}\Delta_\varepsilon\phi'(x) = 0$ as well. We will use $\mathcal{F}[\phi](\xi)$ as notation for the Fourier transform of ϕ. So

$$\mathcal{F}[\phi](\xi) = \int_{-\infty}^{\infty}\phi(x)e^{i\xi x}\,dx. \tag{2.37}$$

Now integration by parts will be applied twice to rewrite the Fourier transform of $\Delta_\varepsilon \phi$. This gives

$$\begin{aligned}
\mathcal{F}\left[\Delta_\varepsilon \phi\right](\xi) &= \int_{-\infty}^{\infty} \Delta_\varepsilon \phi(x) e^{i\xi x}\, dx \\
&= \frac{1}{i\xi}\left[\Delta_\varepsilon \phi(x) e^{i\xi x}\right]_{-\infty}^{\infty} - \frac{1}{i\xi}\int_{-\infty}^{\infty} \Delta_\varepsilon \phi'(x) e^{i\xi x}\, dx \\
&= -\frac{1}{i\xi}\int_{-\infty}^{\infty} \Delta_\varepsilon \phi'(x) e^{i\xi x}\, dx \\
&= \frac{1}{\xi^2}\left[\Delta_\varepsilon \phi'(x) e^{i\xi x}\right]_{-\infty}^{\infty} - \frac{1}{\xi^2}\int_{-\infty}^{\infty} \Delta_\varepsilon \phi''(x) e^{i\xi x}\, dx \\
&= -\frac{1}{\xi^2}\int_{-\infty}^{\infty} \Delta_\varepsilon \phi''(x) e^{i\xi x}\, dx \\
&= -\frac{1}{\xi^2}\mathcal{F}\left[\Delta_\varepsilon \phi''\right](\xi).
\end{aligned} \tag{2.38}$$

Furthermore, using translations, we can write

$$\begin{aligned}
\mathcal{F}\left[\Delta_\varepsilon \phi''\right](\xi) &= \int_{-\infty}^{\infty} \Delta_\varepsilon \phi''(x) e^{i\xi x}\, dx \\
&= \frac{1}{\varepsilon^2}\int_{-\infty}^{\infty} \sum_{k>0} \alpha_k \left[\phi''(x-k\varepsilon) + \phi''(x+k\varepsilon) - 2\phi''(x)\right] e^{i\xi x}\, dx \\
&= \frac{1}{\varepsilon^2}\sum_{k>0} \alpha_k \left(\int_{-\infty}^{\infty} \phi''(x-k\varepsilon) e^{i\xi x} + \int_{-\infty}^{\infty} \phi''(x+k\varepsilon) e^{i\xi x} - 2\int_{-\infty}^{\infty} \phi''(x) e^{i\xi x}\, dx\right) \\
&= \frac{1}{\varepsilon^2}\sum_{k>0} \alpha_k \left(\int_{-\infty}^{\infty} \phi''(x) e^{i\xi(x+k\varepsilon)} + \int_{-\infty}^{\infty} \phi''(x) e^{i\xi(x-k\varepsilon)} - 2\int_{-\infty}^{\infty} \phi''(x) e^{i\xi x}\, dx\right) \\
&= \frac{1}{\varepsilon^2}\sum_{k>0} \alpha_k \left(e^{i\xi k\varepsilon} + e^{-i\xi k\varepsilon} - 2\right) \mathcal{F}[\phi''](\xi).
\end{aligned} \tag{2.39}$$

Using Euler's formula we find that $e^{i\xi k\varepsilon} + e^{-i\xi k\varepsilon} = 2\cos(\xi k\varepsilon)$. Combining this with the just derived (2.38) and (2.39) gives that

$$\begin{aligned}
\mathcal{F}\left[\Delta_\varepsilon \phi\right](\xi) &= -\frac{1}{\xi^2\varepsilon^2}\sum_{k>0} \alpha_k \left(2\cos(\xi k\varepsilon) - 2\right) \mathcal{F}[\phi''](\xi) \\
&= \frac{2}{\xi^2\varepsilon^2}\sum_{k>0} \alpha_k \left(1 - \cos(\xi k\varepsilon)\right) \mathcal{F}[\phi''](\xi),
\end{aligned} \tag{2.40}$$

and thus

$$\begin{aligned}
\mathcal{F}[\Delta_\varepsilon \phi - \phi''](\xi) &= \left(\frac{2\sum_{k>0} \alpha_k \left(1 - \cos(\xi k\varepsilon)\right)}{\xi^2\varepsilon^2} - 1\right) \mathcal{F}[\phi''](\xi) \\
&= \left(\frac{2\sum_{k>0} \alpha_k \left(1 - \cos(\xi k\varepsilon)\right) - \xi^2\varepsilon^2}{\xi^2\varepsilon^2}\right) \mathcal{F}[\phi''](\xi).
\end{aligned} \tag{2.41}$$

We denote

$$q_\varepsilon(\xi) = \frac{2\sum_{k>0} \alpha_k \left(1 - \cos(\xi k \varepsilon)\right) - \xi^2\varepsilon^2}{\xi^2\varepsilon^2}. \tag{2.42}$$

We want to estimate $q_\varepsilon(\xi)$. We use that $\sum_{k>0} \alpha_k \left(1 - \cos(\xi k\varepsilon)\right) \geq 0$, which follows from (A2), to find a lower bound. We estimate

$$q_\varepsilon(\xi) = \frac{2\sum_{k>0} \alpha_k \left(1 - \cos(\xi k \varepsilon)\right) - \xi^2\varepsilon^2}{\xi^2\varepsilon^2} \geq \frac{-\xi^2\varepsilon^2}{\xi^2\varepsilon^2} = -1. \tag{2.43}$$

Of course we also want to have an upper bound. To find one, we need the known inequality $2 - 2\cos(x) \leq x^2$. Furthermore, we need another assumption from (A2), namely $\sum_{k>0} \alpha_k k^2 = 1$. Applying both of these we find

$$\begin{aligned} q_\varepsilon(\xi) &= \frac{2\sum_{k>0} \alpha_k \left(1 - \cos(\xi k \varepsilon)\right) - \xi^2\varepsilon^2}{\xi^2\varepsilon^2} \\ &\leq \frac{\left(\sum_{k>0} \alpha_k k^2\right)\xi^2\varepsilon^2 - \xi^2\varepsilon^2}{\xi^2\varepsilon^2} = \frac{\xi^2\varepsilon^2 - \xi^2\varepsilon^2}{\xi^2\varepsilon^2} = 0. \end{aligned} \tag{2.44}$$

So we have that $q_\varepsilon(\xi) \in [-1, 0]$. Now we let $\delta > 0$ arbitrary. Notice that since $\phi'' \in L^2(\mathbb{R})$, Plancherel's identity implies that also $\mathcal{F}[\phi''] \in L^2(\mathbb{R})$. So this means we can pick $N > 0$ such that

$$\int\limits_{(\infty,-N]\cup[N,\infty)} \left|\mathcal{F}[\phi''](\xi)\right|^2 \, d\xi < \frac{\delta}{2}. \tag{2.45}$$

This implies that

$$\int\limits_{(\infty,-N]\cup[N,\infty)} \left|q_\varepsilon(\xi)\mathcal{F}[\phi''](\xi)\right|^2 d\xi < \frac{\delta}{2}, \tag{2.46}$$

since $|q_\varepsilon(\xi)| \leq 1$.

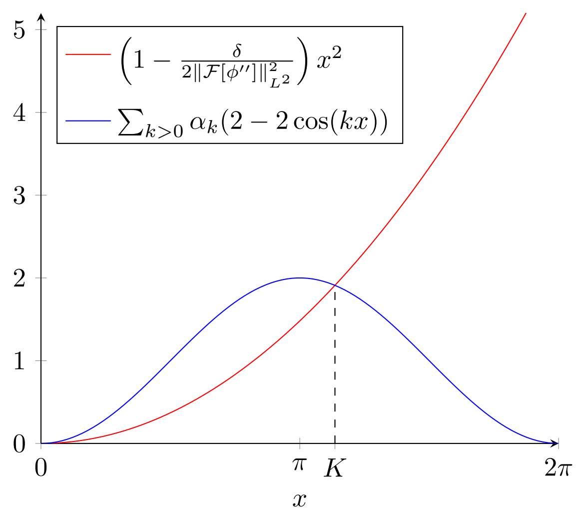
We claim that we can choose $\varepsilon_0 > 0$ such that $|q_\varepsilon(\xi)| \leq \frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2}$ for all $0 < \varepsilon < \varepsilon_0$ and all $-N \leq \xi \leq N$. To show this claim is true we first analyze the equation

$$\sum_{k>0} \alpha_k(2 - 2\cos(kx)) = \left(1 - \frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2}\right) x^2 \tag{2.47}$$

for positive x. Writing $\cos(kx)$ as its Taylor expansion around 0 allows us to write

$$\begin{aligned} \sum_{k>0} \alpha_k(2 - 2\cos(kx)) &= \sum_{k>0} \alpha_k \left(2 - 2\left(1 - \frac{k^2x^2}{2} + \sum_{n\geq 2} \frac{(-1)^n (kx)^{2n}}{(2n)!}\right)\right) \\ &= \sum_{k>0} \alpha_k k^2 x^2 - 2\sum_{k>0} \alpha_k \sum_{n\geq 2} \frac{(-1)^n (kx)^{2n}}{(2n)!} \\ &= x^2 + \mathcal{O}(x^4). \end{aligned} \tag{2.48}$$

Notice that, in the final step, we used that $\sum_{k>0} a_k k^2 = 1$, which is assumed in (A2). We can assume without loss of generality that $0 < 1 - \frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2} < 1$. This means that if x is small enough, then $\sum_{k>0} \alpha_k(2-2\cos(kx)) > \left(1 - \frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2}\right) x^2$. Furthermore we observe that at $x = 2\pi$ we have $\sum_{k>0} \alpha_k(2-2\cos(kx)) = 0$. So we get a similar situation to the one in the graph below.



From the graph it follows that equation (2.47) definitely has a solution. We denote this solution by $K > 0$. Now we are ready to show that the earlier made claim holds. We have

$$\begin{aligned} |q_\varepsilon(\xi)| &\leq \frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2} \\ \iff -q_\varepsilon(\xi) &\leq \frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2} \\ \iff \sum_{k>0} \alpha_k\left(2\cos(\xi k\varepsilon) - 2\right) + \xi^2\varepsilon^2 &\leq \frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2}\xi^2\varepsilon^2 \\ \iff \sum_{k>0} \alpha_k\left(2\cos(\xi k\varepsilon) - 2\right) &\leq \left(\frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2} - 1\right)\xi^2\varepsilon^2 \\ \iff \sum_{k>0} \alpha_k\left(2 - 2\cos(\xi k\varepsilon)\right) &\geq \left(1 - \frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2}\right)\xi^2\varepsilon^2. \end{aligned} \tag{2.49}$$

In the graph we can see that this inequality holds when $|\xi\varepsilon| \leq K$. Thus it also holds if $|\xi| \leq \frac{K}{\varepsilon}$. So choosing ε_0 such that $N = \frac{K}{\varepsilon_0}$ proves the claim. Now using the claim by letting $0 < \varepsilon < \varepsilon_0$, we can estimate

$$\int_{-N}^{N} \left|q_\varepsilon(\xi)\mathcal{F}[\phi''](\xi)\right|^2 \, d\xi \leq \frac{\delta}{2\|\mathcal{F}[\phi'']\|_{L^2}^2} \int_{-N}^{N} \left|\mathcal{F}[\phi''](\xi)\right|^2 \, d\xi \leq \frac{\delta}{2}. \tag{2.50}$$

From this together with (2.46) it follows that

$$\|q_\varepsilon \mathcal{F}[\phi'']\|_{L^2}^2 = \int_{(\infty,-N]\cup[N,\infty)} |q_\varepsilon(\xi)\mathcal{F}[\phi''](\xi)|^2\, d\xi + \int_{-N}^{N} |q_\varepsilon(\xi)\mathcal{F}[\phi''](\xi)|^2 \, d\xi$$
$$< \frac{\delta}{2} + \frac{\delta}{2} = \delta, \tag{2.51}$$

whenever $0 < \varepsilon < \varepsilon_0$. So we have now shown that

$$\lim_{\varepsilon\downarrow 0} \|\mathcal{F}[\Delta_\varepsilon \phi - \phi'']\|_{L^2}^2 = \lim_{\varepsilon\downarrow 0} \|q_\varepsilon \mathcal{F}[\phi'']\|_{L^2}^2 = 0. \tag{2.52}$$

Plancherel's identity implies that $\|\mathcal{F}[\Delta_\varepsilon \phi - \phi'']\|_{L^2} = \|\Delta_\varepsilon \phi - \phi''\|_{L^2}$. So it follows that

$$\lim_{\varepsilon\downarrow 0} \|\Delta_\varepsilon \phi - \phi''\|_{L^2}^2 = 0 \tag{2.53}$$

and thus also

$$\lim_{\varepsilon\downarrow 0} \|\Delta_\varepsilon \phi - \phi''\|_{L^2} = 0, \tag{2.54}$$

which concludes the proof.

(ii) We first claim that for any integer k,

$$\int_{\mathbb{R}} \phi'(x) \left[\phi(x+\varepsilon k) + \phi(x-\varepsilon k)\right] \, dx = 0. \tag{2.55}$$

We'll show now that this actually holds. We apply integration by parts to obtain

$$\int_{\mathbb{R}} \phi(x+\varepsilon k)\phi'(x)\, dx = [\phi(x+\varepsilon k)\phi(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} \phi'(x+\varepsilon k)\phi(x)\, dx. \tag{2.56}$$

We observe that $[\phi(x+\varepsilon k)\phi(x)]_{-\infty}^{\infty}$ vanishes. This is a consequence of $\phi(x)$ being a function that is part of function space $H^1(\mathbb{R})$. Because this tells us that if we let x go to $\pm\infty$, $\phi(x)$ will converge to 0. From this it follows that

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \phi(x+\varepsilon k)\phi'(x)\, dx + \int_{\mathbb{R}} \phi'(x+\varepsilon k)\phi(x)\, dx \\ &= \int_{\mathbb{R}} \phi(x+\varepsilon k)\phi'(x)\, dx + \int_{\mathbb{R}} \phi'(x)\phi(x-\varepsilon k)\, dx \\ &= \int_{\mathbb{R}} \phi'(x) \left[\phi(x+\varepsilon k) + \phi(x-\varepsilon k)\right] \, dx. \end{aligned} \tag{2.57}$$

Note that we applied a translation of εk to the second integral. Thus indeed we find that our claim holds. If we take $k = 0$, it follows from our claim that

$$2\int_{\mathbb{R}} \phi(x)\phi'(x)\, dx = 0. \tag{2.58}$$

Now we are ready to prove the original statement with the help of our claim and also its special case for when $k = 0$. We have

$$\begin{aligned}
\langle \Delta_\varepsilon \phi, \phi' \rangle &= \int_{\mathbb{R}} [\Delta_\varepsilon \phi(x)] \phi'(x)\, dx \\
&= \frac{1}{\varepsilon^2} \int_{\mathbb{R}} \sum_{k>0} \alpha_k \left[\phi(x+\varepsilon k) + \phi(x-\varepsilon k) - 2\phi(x)\right] \phi'(x)\, dx \\
&= \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \int_{\mathbb{R}} \left[\phi(x+\varepsilon k) + \phi(x-\varepsilon k) - 2\phi(x)\right] \phi'(x)\, dx \\
&= \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \left(\int_{\mathbb{R}} \phi'(x) \left[\phi(x+\varepsilon k) + \phi(x-\varepsilon k)\right]\, dx - 2 \int_{\mathbb{R}} \phi(x)\phi'(x)\, dx \right) \\
&= 0.
\end{aligned} \tag{2.59}$$

(iii)

$$\begin{aligned}
\langle \Delta_\varepsilon \phi, \psi \rangle &= \int_{\mathbb{R}} [\Delta_\varepsilon \phi(x)] \psi(x)\, dx \\
&= \int_{\mathbb{R}} \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \left[\phi(x+\varepsilon k) + \phi(x-\varepsilon k) - 2\phi(x)\right] \psi(x)\, dx \\
&= \frac{1}{\varepsilon^2} \left\{ \int_{\mathbb{R}} \sum_{k>0} \alpha_k \phi(x+\varepsilon k)\psi(x)\, dx + \int_{\mathbb{R}} \sum_{k>0} \alpha_k \phi(x-\varepsilon k)\psi(x)\, dx \right. \\
&\quad \left. -2 \int_{\mathbb{R}} \sum_{k>0} \alpha_k \phi(x)\psi(x)\, dx \right\},
\end{aligned} \tag{2.60}$$

where after translations with $w = x \pm \varepsilon k$ in the first 2 terms we get

$$\begin{aligned}
\langle \Delta_\varepsilon \phi, \psi \rangle &= \frac{1}{\varepsilon^2} \left\{ \int_{\mathbb{R}} \sum_{k>0} \alpha_k \phi(w)\psi(w-\varepsilon k)\, dw + \int_{\mathbb{R}} \sum_{k>0} \alpha_k \phi(w)\psi(w+\varepsilon k)\, dw \right. \\
&\quad \left. -2 \int_{\mathbb{R}} \sum_{k>0} \alpha_k \phi(w)\psi(w)\, dw \right\} \\
&= \int_{\mathbb{R}} \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \left[\psi(w+\varepsilon k) + \psi(w-\varepsilon k) - 2\psi(w)\right] \phi(w)\, dw \\
&= \langle \phi, \Delta_\varepsilon \psi \rangle.
\end{aligned} \tag{2.61}$$

(iv) To be able to show this we will need Plancherel's identity. Here it implies that $\langle \Delta_\varepsilon \phi\,,\, \phi \rangle = \langle \mathcal{F}[\Delta_\varepsilon \phi]\,,\, \mathcal{F}[\phi] \rangle$, where we use $\mathcal{F}[\phi]$ as notation for the Fourier transform of ϕ as defined in (2.37).

Before analyzing $\langle \mathcal{F}[\Delta_\varepsilon \phi]\,,\, \mathcal{F}[\phi] \rangle$, we will first rewrite $\mathcal{F}[\Delta_\varepsilon \phi](\xi)$ in terms of $\mathcal{F}[\phi](\xi)$.

This gives

$$
\begin{aligned}
\mathcal{F}[\Delta_\varepsilon \phi](\xi) &= \int_{-\infty}^{\infty} \Delta_\varepsilon \phi(x) e^{i\xi x}\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \big(\phi(x-k\varepsilon)+\phi(x+k\varepsilon)-2\phi(x)\big) e^{i\xi x}\, dx \\
&= \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \int_{-\infty}^{\infty} \big(\phi(x-k\varepsilon)+\phi(x+k\varepsilon)-2\phi(x)\big) e^{i\xi x}\, dx \\
&= \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \left(\int_{-\infty}^{\infty} \phi(x-k\varepsilon) e^{i\xi x}\, dx + \int_{-\infty}^{\infty} \phi(x+k\varepsilon) e^{i\xi x}\, dx - 2\int_{-\infty}^{\infty} \phi(x) e^{i\xi x}\, dx \right),
\end{aligned} \tag{2.62}
$$

where after translations with $k\varepsilon$ in the first two integrals we get

$$
\begin{aligned}
\mathcal{F}[\Delta_\varepsilon \phi](\xi) &= \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \left(\int_{-\infty}^{\infty} \phi(x) e^{i\xi(x+k\varepsilon)}\, dx + \int_{-\infty}^{\infty} \phi(x) e^{i\xi(x-k\varepsilon)}\, dx - 2\mathcal{F}[\phi](\xi) \right) \\
&= \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \left(e^{i\xi k\varepsilon} + e^{-i\xi k\varepsilon} - 2 \right) \mathcal{F}[\phi](\xi) \\
&= \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \left(2\cos(\xi k\varepsilon) - 2 \right) \mathcal{F}[\phi](\xi) \\
&= -\frac{2}{\varepsilon^2} A(\xi\varepsilon) \mathcal{F}[\phi](\xi).
\end{aligned} \tag{2.63}
$$

Note that in the last step we have used notation $A(z)$ which was introduced in assumption (A2). Now we can analyze $\langle \mathcal{F}[\Delta_\varepsilon \phi]\,,\, \mathcal{F}[\phi] \rangle$. Since $[\mathcal{F}[\phi](\xi)]^2 \geq 0$ and $A(\xi\varepsilon) \geq 0$ for all $\xi \in \mathbb{R}$, it follows that

$$
\langle \mathcal{F}[\Delta_\varepsilon \phi]\,,\, \mathcal{F}[\phi] \rangle = \frac{2}{\varepsilon^2} \int_{\mathbb{R}} [\mathcal{F}[\phi](\xi)]^2 A(\xi\varepsilon)\, dx \leq 0. \tag{2.64}
$$

So by Plancherel's identity we can conclude that $\langle \Delta_\varepsilon \phi\,,\, \phi \rangle = \langle \mathcal{F}[\Delta_\varepsilon \phi]\,,\, \mathcal{F}[\phi] \rangle \leq 0$. □

Corollary 1. *Let Δ_ε be defined as in* (2.15) *and let the coefficients $\{\alpha_k\}$ satisfy (A2). If we then let $\varepsilon \downarrow 0$ in the spatially discretized Allen-Cahn equation* (2.6)*, it converges to its continuous version* (2.5)*.*

Proof. During the derivations in (2.11) we found that

$$
\begin{aligned}
0 &= u_n'(t) - \frac{1}{\varepsilon^2} \sum_{k=-\infty}^{\infty} \alpha_k u_{n-k} + f(u_n) \\
&= cu'(x) - \frac{1}{\varepsilon^2} \sum_{k>0} \alpha_k \left[u(x-k\varepsilon) + u(x+k\varepsilon) - 2u(x) \right] + f(u) \\
&= cu'(x) - \Delta_\varepsilon u(x) + f(u).
\end{aligned} \tag{2.65}
$$

Now it follows from Lemma 4*(i)* that if we let $\varepsilon \downarrow 0$ on both sides, then we get

$$cu'(x) - u''(x) + f(u) = 0. \tag{2.66}$$

Furthermore, the boundary condition $u(\pm\infty) = \pm 1$ is still valid. So we see that we get back problem (2.5) if we let ε go to 0 in (2.6). Thus we can conclude that (2.6) is indeed a discretized version of (2.5). □

Lemma 5. *Assume f is a function satisfying (A1) and let (c_0, u_0) be defined as the solution to (2.26). Then $f_u(u_0)$ is bounded.*

Proof. To show $f_u(u_0)$ is bounded we will examine what happens if $x \to \pm\infty$. If we let this happen, we know that $u_0(x) \to \pm 1$ from (2.26). This causes $f_u(u_0) \to f_u(\pm 1)$, since f' is continuous by the assumption $f \in C^2(\mathbb{R})$. Note that $f_u(\pm 1)$ are both finite values. Because of the values being finite and $f_u(u_0)$ being continuous it follows that $f_u(u_0)$ is bounded. □

The properties stated in these two lemmas will come in handy during the proofs of some main theorems in later sections. The setting up of the problem has been finished and thus we can start with the main part of this report, which contains the body of the proof.

3 Main Proof

The Allen-Cahn equation and its discretized version have been been discussed in the previous section. Also the problem has been set up in more detail by introducing a different way to state the problem and showing some interesting properties. This means we are finally ready to state the theorem which is essentially the theorem we want to prove in this thesis.

Theorem 2. *Assume f satisfies (A1) and assume the coefficients $\{\alpha_k\}$ satisfy (A2). Then there exists a constant ε^* such that for all $\varepsilon \in (0, \varepsilon^*)$, problem* (2.16) *has at least one solution $(c_\varepsilon, u_\varepsilon)$. This solution is locally unique in $H^1(\mathbb{R})$ up to translation and has the property*

$$\lim_{\varepsilon \downarrow 0} (c_\varepsilon, u_\varepsilon) = (c_0, u_0) \quad in \quad \mathbb{R} \times H^1(\mathbb{R}). \tag{3.1}$$

If we are able to prove this, then our problem is solved. But we don't have enough firepower yet to prove this result. We need a proposition which will help us prove Theorem 2. This is Proposition 1 and is stated below.

Proposition 1. *There exists a positive constant C_0 and a positive function $\varepsilon_0(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ such that for every $\delta > 0$ and every $\varepsilon \in (0, \varepsilon_0(\delta))$, $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is a homeomorphism, see Definition A3, from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$. Furthermore*

$$\left\| \left(\mathcal{L}^{\pm}_{\varepsilon,\delta}\right)^{-1} \psi \right\|_{H^1} \leq C_0 \left\{ \|\psi\|_{L^2} + \frac{1}{\delta} \left| \langle \psi \, , \, \phi_0^{\mp} \rangle \right| \right\} \tag{3.2}$$

where $\phi_0^{\mp} \in L^2(\mathbb{R})$ is as in (3.5) *below and $\psi \in L^2(\mathbb{R})$ arbitrary. If also $\psi \perp \phi_0^{\mp}$, then we have*

$$\left\| \left(\mathcal{L}^{\pm}_{\varepsilon,\delta}\right)^{-1} \psi \right\|_{H^1} \leq C_0 \|\psi\|_{L^2}. \tag{3.3}$$

Unfortunately this proposition is also not easy to prove and preparations are necessary. These preparations will be executed in the following two subsections. More information about the operator $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is required, on which the focus will lie in the second section. To get there we consider another operator $\mathcal{L}^{\pm}_0$ in the first subsection. This operator is related to $\mathcal{L}^{\pm}_{\varepsilon,\delta}$, since both operators are linearizations of the Allen-Cahn equation around the solution (c_0, u_0). The difference is that $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is a linearization in the discrete case, while $\mathcal{L}^{\pm}_0$ is a linearization in the slightly easier continuous case. So it makes sense to first consider $\mathcal{L}^{\pm}_0$ such that we can make the connection to $\mathcal{L}^{\pm}_{\varepsilon,\delta}$.

After these preparations we will be ready to show Proposition 1 indeed holds. Then, after some more estimating, we are finally ready to prove the main result stated as Theorem 2.

3.1 Linearization $\mathcal{L}^{\pm}_0$ of the continuous Allen-Cahn equation

In this section we treat the operator $\mathcal{L}^{\pm}_0$ and the function $\phi^{\pm}_0(x)$. These are given by

$$\mathcal{L}^{\pm}_0 \phi = \pm c_0 \phi' - \phi'' + f_u(u_0)\phi, \tag{3.4}$$

$$\phi_0^+(x) = \frac{u_0'(x)}{\|u_0'(x)\|_{L^2}}, \qquad \phi_0^-(x) = \frac{u_0'(x) e^{-c_0 x}}{\|u_0'(x) e^{-c_0 x}\|_{L^2}}. \tag{3.5}$$

Here $\mathcal{L}_0^{\pm}$ maps functions from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$. To understand the operators a bit better, we remark that $\mathcal{L}_0^{+}$ can be seen as the linearization of the system (2.13) around the solution (c_0, u_0). The observant reader will question if the taking of the L^2-norms above is allowed. But we have exponential convergence for u_0 by Theorem 1, which is a property that is preserved under differentiation. This justifies the taking of the L^2-norms. We now treat a lemma consisting of properties involving $\mathcal{L}_0^{\pm}$ and $\phi_0^{\pm}(x)$.

Lemma 6. *Let $\mathcal{L}_0^{\pm}$ and $\phi_0^{\pm}$ be as above in (3.4) and (3.5). Then the following statements hold.*

(i) $\|\phi_0^{\pm}\|_{L^2} = 1$.

(ii) $\mathcal{L}_0^{\pm}\phi_0^{\pm} = 0$.

(iii) $\langle \mathcal{L}_0^{\pm}\phi\, ,\, \psi \rangle = \langle \phi\, ,\, \mathcal{L}_0^{\mp}\psi \rangle\, ,$ *for any* $\phi, \psi \in H^2(\mathbb{R})$.

(iv) $\phi_0^{\pm} \in H^2(\mathbb{R})$.

(v) *There exists a positive constant C such that for all $x > 0$, we have*

$$\phi_0^{\pm}(x) \int_0^x \frac{1}{\phi_0^{\pm}(y)}\, dy \leq C \quad \text{and} \quad \frac{1}{\left[\phi_0^{\pm}(x)\right]^2} \int_x^{\infty} \left[\phi_0^{\pm}(y)\right]^2\, dy \leq C. \tag{3.6}$$

(vi) *For every $\psi \in L^2(\mathbb{R})$ the problem*

$$\mathcal{L}_0^{\pm}\phi = \psi \qquad \text{with } \phi \in H^2(\mathbb{R}) \text{ and } \phi \perp \phi_0^{\pm} \tag{3.7}$$

has a unique solution if and only if $\psi \perp \phi_0^{\pm}$. Furthermore, there exists a positive constant C_1, such that

$$\|\phi\|_{H^2} \leq C_1 \left\|\mathcal{L}_0^{\pm}\phi\right\|_{L^2} \qquad \text{for all } \phi \in H^2(\mathbb{R}) \text{ satisfying } \phi \perp \phi_0^{\pm}. \tag{3.8}$$

Proof. *(i)* This result is not so hard to show. It follows directly from the definition of $\phi_0^{\pm}$. Since we have

$$\|\phi_0^{+}\|_{L^2} = \left\| \frac{u_0'(x)}{\|u_0'(x)\|_{L^2}} \right\|_{L^2} = \frac{\|u_0'(x)\|_{L^2}}{\|u_0'(x)\|_{L^2}} = 1, \tag{3.9}$$

which proves the statement for ϕ_0^{+}. In exactly the same way we have $\|\phi_0^{-}\|_{L^2} = 1$.

(ii) We want to show $\mathcal{L}_0^{\pm}\phi_0^{\pm} = 0$. First we take identity (2.26) and differentiate this on both sides. This gives

$$c_0 u_0'' - u_0''' + f_u(u_0)u_0' = 0, \tag{3.10}$$

which we will use later on. Now using (3.10) it follows almost immediately that $\mathcal{L}_0^{+}\phi_0^{+} = 0$. Since we get

$$\mathcal{L}_0^{+}\phi_0^{+} = \frac{1}{\|u_0'(x)\|_{L^2}} \left(c_0 u_0'' - u_0''' + f_u(u_0)u_0' \right) = 0. \tag{3.11}$$

To show $\mathcal{L}_0^- \phi_0^- = 0$ we have to put in some more effort. First, $(\phi_0^-)'$ and $(\phi_0^-)''$ have to be computed. We get

$$(\phi_0^-)' = \frac{1}{\|u_0'(x)e^{-c_0x}\|_{L^2}}\left(u_0''e^{-c_0x} - c_0u_0'e^{-c_0x}\right) = \frac{e^{-c_0x}}{\|u_0'(x)e^{-c_0x}\|_{L^2}}\left(u_0'' - c_0u_0'\right), \tag{3.12}$$

$$(\phi_0^-)'' = \frac{e^{-c_0x}}{\|u_0'(x)e^{-c_0x}\|_{L^2}}\left(u_0''' - 2c_0u_0'' + c_0^2u_0'\right). \tag{3.13}$$

From this it follows that

$$\begin{aligned}\mathcal{L}_0^- \phi_0^- &= \frac{e^{-c_0x}}{\|u_0'(x)e^{-c_0x}\|_{L^2}}\left(c_0^2u_0' - c_0u_0'' - u_0''' + 2c_0u_0'' - c_0^2u_0' + f_u(u_0)u_0'\right)\\ &= \frac{e^{-c_0x}}{\|u_0'(x)e^{-c_0x}\|_{L^2}}\left(c_0u_0'' - u_0''' + f_u(u_0)u_0'\right)\\ &= 0.\end{aligned} \tag{3.14}$$

Here we again used (3.10). So we indeed see that $\mathcal{L}_0^{\pm} \phi_0^{\pm} = 0$.

(iii) Let $\phi, \psi \in H^2(\mathbb{R})$ arbitrary. We obtain, by applying integration by parts, that

$$\langle \pm c_0\phi' \,,\, \psi\rangle = \pm c_0 \int_{\mathbb{R}} \phi'\psi \, dx = \pm c_0 \left[\phi\, \psi\right]_{-\infty}^{\infty} \mp c_0 \int_{\mathbb{R}} \phi\psi' \, dx = \langle \phi \,,\, \mp c_0\psi'\rangle \,. \tag{3.15}$$

Notice that $[\phi\, \psi]_{-\infty}^{\infty}$ vanishes here because both $\phi, \psi \in H^2(\mathbb{R})$. By using repetitive integration by parts and using that $\phi, \psi \in H^2(\mathbb{R})$, we get the following result in a similar way as done above. We find

$$\langle \phi'' \,,\, \psi\rangle = \langle \phi \,,\, \psi''\rangle \,. \tag{3.16}$$

Furthermore, we have the more trivial

$$\langle f_u(u_0)\phi \,,\, \psi\rangle = \langle \phi \,,\, f_u(u_0)\psi\rangle \,. \tag{3.17}$$

Using these three results we find

$$\begin{aligned}\left\langle \mathcal{L}_0^{\pm}\phi \,,\, \psi\right\rangle &= \langle \pm c_0\phi' - \phi'' + f_u(u_0)\phi \,,\, \psi\rangle\\ &= \langle \pm c_0\phi' \,,\, \psi\rangle - \langle \phi'' \,,\, \psi\rangle + \langle f_u(u_0)\phi \,,\, \psi\rangle\\ &= \langle \phi \,,\, \mp c_0\psi'\rangle - \langle \phi \,,\, \psi''\rangle + \langle \phi \,,\, f_u(u_0)\psi\rangle\\ &= \langle \phi \,,\, \mp c_0\psi' - \psi'' + f_u(u_0)\psi\rangle\\ &= \left\langle \phi \,,\, \mathcal{L}_0^{\mp}\psi\right\rangle .\end{aligned} \tag{3.18}$$

(iv) Differentiating (2.26) gives the differential equation $c_0u_0'' - u_0''' + f_u(u_0)u_0' = 0$ with the boundary condition $u_0(\pm\infty) = \pm 1$. Note that $u_0 = u_0(x)$. So if we let $x \to \pm\infty$ our differential equation will be very similar to

$$c_0\tilde{u_0}'' - \tilde{u_0}''' + f_u(\pm 1)\tilde{u_0}' = 0. \tag{3.19}$$

We will solve this equation for $\tilde{u_0}'$. This way we get to know how u_0 roughly behaves at $\pm\infty$ and thus also understand $\phi_0^{\pm}$ a lot better. To solve this equation we write $\tilde{u_0}'$ in the form $\tilde{u_0}'(x) = e^{rx}$, where r is a constant, and substitute this into differential equation (3.19). This gives

$$e^{rx}\left(c_0 r - r^2 + f_u(\pm 1)\right) = 0. \tag{3.20}$$

Because e^{rx} is never equal to 0 we get

$$r^2 - c_0 r - f_u(\pm 1) = 0. \tag{3.21}$$

Then completing the square gives

$$\left(r - \frac{c_0}{2}\right)^2 = \frac{c_0^2}{4} + f_u(\pm 1) \tag{3.22}$$

so that

$$r = \frac{c_0}{2} \pm \sqrt{\frac{c_0^2}{4} + f_u(\pm 1)}. \tag{3.23}$$

Hence

$$\tilde{u_0}'(x) = \alpha_\pm e^{\left(\frac{c_0}{2}+\sqrt{\frac{c_0^2}{4}+f_u(\pm 1)}\right)x} + \beta_\pm e^{\left(\frac{c_0}{2}-\sqrt{\frac{c_0^2}{4}+f_u(\pm 1)}\right)x} \quad \text{as } x \to \pm\infty. \tag{3.24}$$

So we have that

$$u_0'(x) \sim \alpha_\pm e^{\left(\frac{c_0}{2}+\sqrt{\frac{c_0^2}{4}+f_u(\pm 1)}\right)x} + \beta_\pm e^{\left(\frac{c_0}{2}-\sqrt{\frac{c_0^2}{4}+f_u(\pm 1)}\right)x} \quad \text{as } x \to \pm\infty. \tag{3.25}$$

Boundary condition $u_0(\pm\infty) = \pm 1$ implies that $u_0'(x)$ must be bounded. We can use this to be more specific about $u_0'(x)$. First we make an observation. We know from (A1) that $f_u(\pm 1) > 0$. So we get

$$\frac{c_0}{2} + \sqrt{\frac{c_0^2}{4} + f_u(\pm 1)} \ > \ 0 \qquad \text{and} \qquad \frac{c_0}{2} - \sqrt{\frac{c_0^2}{4} + f_u(\pm 1)} \ < \ 0. \tag{3.26}$$

We consider the case when $x \to \infty$. Using the two inequalities above, we see that

$$e^{\left(\frac{c_0}{2}+\sqrt{\frac{c_0^2}{4}+f_u(\pm 1)}\right)x} \longrightarrow \infty \qquad \text{and} \qquad e^{\left(\frac{c_0}{2}-\sqrt{\frac{c_0^2}{4}+f_u(\pm 1)}\right)x} \longrightarrow 0. \tag{3.27}$$

Since $\tilde{u_0}'(x)$ has to be bounded, it follows that $\alpha_+ = 0$. We can apply similar reasoning in the case that $x \to -\infty$. Using the two inequalities from (3.26) again, we find that

$$e^{\left(\frac{c_0}{2}+\sqrt{\frac{c_0^2}{4}+f_u(\pm 1)}\right)x} \longrightarrow 0 \qquad \text{and} \qquad e^{\left(\frac{c_0}{2}-\sqrt{\frac{c_0^2}{4}+f_u(\pm 1)}\right)x} \longrightarrow \infty. \tag{3.28}$$

By the boundedness of $u_0'(x)$, it now follows that $\beta_- = 0$. Using that $\alpha_+ = \beta_- = 0$ we can rewrite (3.25). We define the constant $\gamma_\pm$, where $\gamma_+ = \beta_+$ and $\gamma_- = \alpha_-$. This allows us to write (3.25) as

$$u_0'(x) \sim \gamma_\pm e^{\left(\frac{c_0}{2} \mp \sqrt{\frac{c_0^2}{4} + f_u(\pm 1)}\right)x} \quad \text{as } x \to \pm\infty. \tag{3.29}$$

From this we observe that $u_0'(x)$ converges exponentially to 0 $x \to \pm\infty$. Exponential convergence is a property that is maintained while taking derivatives. Hence $u_0''(x)$ converges exponentially to 0 as $x \to \pm\infty$. Thus both $u_0'(x), u_0''(x) \in L^2(\mathbb{R})$. So, from this it follows that $\phi_0^\pm \in H^2(\mathbb{R})$.

(v) First of all note that $\phi_0^\pm(x) > 0$, because $u_0'(x) > 0$ by Theorem 1. We keep this in mind during the proof. In the proof of Lemma 6*(iv)* we have found that $u_0'(x)$ converges exponentially to 0 when $x \to \infty$. It follows from (3.29) there exists a $r > 0$, such that for every $x \geq r$, we have

$$u_0'(x) \sim \gamma_+ e^{\left(\frac{c_0}{2} - \sqrt{\frac{c_0^2}{4} + f_u(\pm 1)}\right)x}. \tag{3.30}$$

For notation reasons we define the positive constant $A = -\left(\frac{c_0}{2} - \sqrt{\frac{c_0^2}{4} + f_u(\pm 1)}\right)$. So for every $x \geq r$, we now have

$$u_0'(x) \sim \gamma_+ e^{-Ax}. \tag{3.31}$$

Furthermore we define

$$m = \min_{x \in (0,r]} \phi_0^\pm(x) \qquad \text{and} \qquad M = \max_{x \in (0,r]} \phi_0^\pm(x). \tag{3.32}$$

We start with showing that the first of the two inequalities holds. Let $x > 0$ be arbitrary. We'll consider the cases $x \geq r$ and $x \leq r$ separately. First assume $x \leq r$. Then

$$\phi_0^\pm(x) \int_0^x \frac{1}{\phi_0^\pm(y)}\, dy \leq M \int_0^x \frac{1}{m}\, dy = \frac{Mx}{m} \leq \frac{Mr}{m}. \tag{3.33}$$

Now we consider the other case where $x \geq r$. We find that

$$\begin{aligned} \phi_0^+(x) \int_r^x \frac{1}{\phi_0^+(y)}\, dy &= \frac{u_0'(x)}{\|u_0'(x)\|_{L^2}} \int_r^x \frac{\|u_0'(x)\|_{L^2}}{u_0'(y)}\, dy \\ &= \gamma_+ e^{-Ax} \int_r^x \frac{e^{Ay}}{\gamma_+}\, dy \\ &= \frac{e^{-Ax}}{A} \left[e^{Ay}\right]_r^x \\ &= \frac{e^{-Ax}}{A} \left(e^{Ax} - e^{Ar}\right) = \frac{1}{A}\left(1 - e^{A(r-x)}\right) \leq \frac{1}{A}. \end{aligned} \tag{3.34}$$

Using the previous estimations (3.33) and (3.34) we obtain

$$\phi_0^+(x) \int_0^x \frac{1}{\phi_0^+(y)}\, dy = \phi_0^+(x) \left(\int_0^r \frac{1}{\phi_0^+(y)}\, dy + \int_r^x \frac{1}{\phi_0^+(y)}\, dy\right) \leq \frac{Mr}{m} + \frac{1}{A}. \tag{3.35}$$

Note that we have only estimated for ϕ_0^+. If we take ϕ_0^- instead of ϕ_0^+, the estimations above can be done in exactly the same way. The only difference is that A has to be replaced by $A + c_0$, which is also a positive constant. So, after we define $A^+ = A$ and $A^- = A + c_0$, we have

$$\phi_0^\pm(x) \int_0^x \frac{1}{\phi_0^\pm(y)}\, dy \le \frac{Mr}{m} + \frac{1}{A^\pm} \qquad \text{for all } x > 0. \tag{3.36}$$

Now we move on to the second inequality. Let $x > 0$ be arbitrary and again we split the cases $x \ge r$ and $x \le r$. First we condider $x \ge r$. Then

$$\begin{aligned}
\frac{1}{\left[\phi_0^+(x)\right]^2} \int_x^\infty \left[\phi_0^+(y)\right]^2\, dy &= \frac{\|u_0'(x)\|_{L^2}^2}{\left[u_0'(x)\right]^2} \int_x^\infty \frac{\left[u_0'(y)\right]^2}{\|u_0'(x)\|_{L^2}^2}\, dy \\
&= \frac{e^{2Ax}}{\gamma_+^2} \int_x^\infty \gamma_+^2 e^{-2Ay}\, dy \\
&= -\frac{e^{2Ax}}{2A} \left[e^{-2Ay}\right]_x^\infty = -\frac{e^{2Ax}}{2A} \left(0 - e^{-2Ax}\right) = \frac{1}{2A}.
\end{aligned} \tag{3.37}$$

Notice that this estimation is almost the same when ϕ_0^+ is replaced by ϕ_0^-. We only have to replace A, by $A + c_0$ as mentioned before. So, to be precise, we have

$$\frac{1}{\left[\phi_0^\pm(x)\right]^2} \int_x^\infty \left[\phi_0^\pm(y)\right]^2\, dy \le \frac{1}{2A^\pm} \qquad \text{for all } x \ge r. \tag{3.38}$$

Now we consider the case where $x \le r$. Then we obtain

$$\frac{1}{\left[\phi_0^\pm(x)\right]^2} \int_x^r \left[\phi_0^\pm(y)\right]^2\, dy \le \frac{1}{m^2} \int_0^r M^2\, dy = \frac{M^2 r}{m^2}. \tag{3.39}$$

Combining (3.38) and (3.39) we find that

$$\begin{aligned}
&\frac{1}{\left[\phi_0^\pm(x)\right]^2} \int_x^\infty \left[\phi_0^\pm(y)\right]^2\, dy \\
&\quad = \frac{1}{\left[\phi_0^\pm(x)\right]^2} \left(\int_x^r \left[\phi_0^\pm(y)\right]^2\, dy + \int_r^\infty \left[\phi_0^\pm(y)\right]^2\, dy \right) \le \frac{M^2 r}{m^2} + \frac{1}{2A^\pm}.
\end{aligned} \tag{3.40}$$

So we have that

$$\frac{1}{\left[\phi_0^\pm(x)\right]^2} \int_x^\infty \left[\phi_0^\pm(y)\right]^2\, dy \le \frac{M^2 r}{m^2} + \frac{1}{2A^\pm} \qquad \text{for all } x > 0. \tag{3.41}$$

Thus if we take $C = \max\left\{ \frac{Mr}{m} + \frac{1}{A^\pm}, \frac{M^2 r}{m^2} + \frac{1}{2A^\pm} \right\}$, we see that both of the inequalities indeed hold.

(vi) Let's first prove the if and only if statement in the forward direction. Let $\psi \in L^2(\mathbb{R})$ arbitrary. We assume that ϕ is the unique solution to

$$\mathcal{L}_0^\pm \phi = \psi \qquad \text{with } \phi \in H^2(\mathbb{R}) \text{ and } \phi \perp \phi_0^\pm. \tag{3.42}$$

So we have that $\phi \in H^2(\mathbb{R})$ and, by Lemma 6*(iv)*, also $\phi_0^\mp \in H^2(\mathbb{R})$. Hence we can apply Lemma 6*(iii)*. It implies that $\langle \psi \,, \, \phi_0^\mp \rangle = \langle \mathcal{L}_0^\pm \phi \,, \, \phi_0^\mp \rangle = \langle \phi \,, \, \mathcal{L}_0^\mp \phi_0^\mp \rangle$. Lemma 6*(ii)* states that $\mathcal{L}_0^\mp \phi_0^\mp = 0$ and thus it follows that $\langle \psi \,, \, \phi_0^\mp \rangle = 0$.

Now we want to prove the backward direction. We find a special solution $\phi_{sp}(x)$ to the equation $\mathcal{L}_0^\pm \phi = \psi$ if we solve this by variation of parameters. This special solution $\phi_{sp}(x)$ is given by

$$\phi_{sp}(x) = \phi_0^\pm(x) \int_0^x \frac{1}{\phi_0^+(y)\phi_0^-(y)} \int_y^\infty \phi_0^\mp(z)\psi(z)\, dz\, dy. \tag{3.43}$$

We will show that $\phi_{sp}(x)$ indeed satisfies the equation $\mathcal{L}_0^\pm \phi = \psi$. To do this we first introduce some new notation

$$\eta(y) = \int_y^\infty \phi_0^\mp(z)\psi(z)\, dz, \tag{3.44}$$

so that we now can write

$$\phi_{sp}(x) = \phi_0^\pm(x) \int_0^x \frac{\eta(y)}{\phi_0^+(y)\phi_0^-(y)}\, dy. \tag{3.45}$$

To be able to elaborate $\mathcal{L}_0^\pm \phi_{sp}$ we need to know the derivatives $\eta'(y), \phi_{sp}'(x)$ and $\phi_{sp}''(x)$. $\phi_{sp}''(x)$ can be derived using $\eta'(y)$ and $\phi_{sp}'(x)$ and these are given by

$$\eta'(y) = -\phi_0^\mp(y)\psi(y), \tag{3.46}$$

$$\phi_{sp}'(x) = \left(\phi_0^\pm(x)\right)' \int_0^x \frac{\eta(y)}{\phi_0^+(y)\phi_0^-(y)}\, dy + \frac{\eta(x)}{\phi_0^\mp(x)}. \tag{3.47}$$

With the help of these two derivatives we find

$$\begin{aligned}
\phi_{sp}''(x) &= \left(\phi_0^\pm(x)\right)'' \int_0^x \frac{\eta(y)}{\phi_0^+(y)\phi_0^-(y)}\, dy + \frac{\left(\phi_0^\pm(x)\right)' \eta(x)}{\phi_0^+(x)\phi_0^-(x)} \\
&\quad - \frac{\phi_0^\mp(x)\psi(x)\phi_0^\mp(x) + \left(\phi_0^\mp(x)\right)' \eta(x)}{\left[\phi_0^\mp(x)\right]^2} \\
&= \left(\phi_0^\pm(x)\right)'' \int_0^x \frac{\eta(y)}{\phi_0^+(y)\phi_0^-(y)}\, dy + \frac{\left(\phi_0^\pm(x)\right)' \eta(x)}{\phi_0^+(x)\phi_0^-(x)} \\
&\quad - \psi(x) - \frac{\left(\phi_0^\mp(x)\right)' \eta(x)}{\left[\phi_0^\mp(x)\right]^2}.
\end{aligned} \tag{3.48}$$

Now we are ready to write out $\mathcal{L}_0^\pm \phi_{sp}$ which should equal $\psi(x)$. This gives

$$
\begin{aligned}
\mathcal{L}_0^{\pm}\phi_{sp} = &\pm c_0 \left(\phi_0^{\pm}(x)\right)' \int_0^x \frac{\eta(y)}{\phi_0^+(y)\phi_0^-(y)}\,dy \pm c_0 \frac{\eta(x)}{\phi_0^{\mp}(x)} - \left(\phi_0^{\pm}(x)\right)'' \int_0^x \frac{\eta(y)}{\phi_0^+(y)\phi_0^-(y)}\,dy \\
&- \frac{\left(\phi_0^{\pm}(x)\right)'\eta(x)}{\phi_0^+(x)\phi_0^-(x)} + \psi(x) + \frac{\left(\phi_0^{\mp}(x)\right)'\eta(x)}{\left[\phi_0^{\mp}(x)\right]^2} + f_u(u_0)\phi_0^{\pm}(x)\int_0^x \frac{\eta(y)}{\phi_0^+(y)\phi_0^-(y)}\,dy \\
= &\int_0^x \frac{\eta(y)}{\phi_0^+(y)\phi_0^-(y)}\,dy \Big(\pm c_0 \left(\phi_0^{\pm}(x)\right)' - \left(\phi_0^{\pm}(x)\right)'' + f_u(u_0)\phi_0^{\pm}(x) \Big) \\
&+ \frac{\eta(x)}{\left[\phi_0^{\mp}(x)\right]^2} \left(\pm c_0 \phi_0^{\mp}(x) - \frac{\left(\phi_0^{\pm}(x)\right)'\phi_0^{\mp}(x)}{\phi_0^{\pm}(x)} + \left(\phi_0^{\mp}(x)\right)' \right) + \psi(x).
\end{aligned} \tag{3.49}
$$

Note that the expression in brackets after the integral is just $\mathcal{L}_0^{\pm}\phi_0^{\pm}$ and by Lemma 6*(ii)* $\mathcal{L}_0^{\pm}\phi_0^{\pm}=0$. So

$$
\mathcal{L}_0^{\pm}\phi_{sp} = \frac{\eta(x)}{\left[\phi_0^{\mp}(x)\right]^2} \left(\pm c_0 \phi_0^{\mp}(x) - \frac{\left(\phi_0^{\pm}(x)\right)'\phi_0^{\mp}(x)}{\phi_0^{\pm}(x)} + \left(\phi_0^{\mp}(x)\right)' \right) + \psi(x). \tag{3.50}
$$

Remember $\phi_{sp}(x)$ is a solution to the PDE if $\mathcal{L}_0^{\pm}\phi_{sp} = \psi$. So if the first term of (3.50) equals 0, then $\phi_{sp}(x)$ is a solution to the PDE. We claim that the expression between the round brackets equals 0. We will use the definitions of $\phi_0^+(x)$ and $\phi_0^-(x)$, stated in (3.5), to evaluate this expression for $\phi_0^+(x)$ and $\phi_0^-(x)$ separately. We find that

$$
\begin{aligned}
c_0\phi_0^-(x) - &\frac{\left(\phi_0^+(x)\right)'\phi_0^-(x)}{\phi_0^+(x)} + \left(\phi_0^-(x)\right)' \\
&= c_0 \frac{u_0'(x)e^{-c_0x}}{\left\|u_0'(x)e^{-c_0x}\right\|_{L^2}} - \frac{u_0''(x)\,u_0'(x)e^{-c_0x}\,\|u_0'(x)\|_{L^2}}{\|u_0'(x)\|_{L^2}\left\|u_0'(x)e^{-c_0x}\right\|_{L^2}u_0'(x)} \\
&\quad + \frac{u_0''(x)e^{-c_0x} - c_0u_0'(x)e^{-c_0x}}{\left\|u_0'(x)e^{-c_0x}\right\|_{L^2}} \\
&= \frac{c_0u_0'(x)e^{-c_0x} - u_0''(x)e^{-c_0x} + u_0''(x)e^{-c_0x} - c_0u_0'(x)e^{-c_0x}}{\left\|u_0'(x)e^{-c_0x}\right\|_{L^2}} \\
&= 0,
\end{aligned} \tag{3.51}
$$

$$
\begin{aligned}
-c_0\phi_0^+(x) - &\frac{\left(\phi_0^-(x)\right)'\phi_0^+(x)}{\phi_0^-(x)} + \left(\phi_0^+(x)\right)' \\
&= -c_0\frac{u_0'(x)}{\|u_0'(x)\|_{L^2}} \\
&\quad - \frac{\Big(u_0''(x)e^{-c_0x} - c_0u_0'(x)e^{-c_0x}\Big)u_0'(x)\,\|u_0'(x)e^{-c_0x}\|_{L^2}}{\|u_0'(x)e^{-c_0x}\|_{L^2}\,\|u_0'(x)\|_{L^2}\,u_0'(x)e^{-c_0x}} \\
&\quad + \frac{u_0''(x)}{\|u_0'(x)\|_{L^2}} \\
&= \frac{-c_0u_0'(x) - u_0''(x) + c_0u_0'(x) + u_0''(x)}{\|u_0'(x)\|_{L^2}} \\
&= 0.
\end{aligned}
\tag{3.52}
$$

So we can conclude that

$$
\pm c_0\phi_0^\mp(x) - \frac{\left(\phi_0^\pm(x)\right)'\phi_0^\mp(x)}{\phi_0^\pm(x)} + \left(\phi_0^\mp(x)\right)' = 0 \tag{3.53}
$$

and thus we find that $\mathcal{L}_0^\pm\phi_{sp}(x) = \psi(x)$, which means $\phi_{sp}(x)$ is indeed a solution to $\mathcal{L}_0^\pm\phi = \psi$.

Now we'll derive an upper estimate for $\eta(y)$ which will be useful to obtain an upper estimate for $\phi_{sp}(x)$. To find such an estimate we use Cauchy-Schwarz and the second inequality of (3.6) stated in Lemma 6*(v)*. It follows that for every $y > 0$ we have

$$
\begin{aligned}
|\eta(y)| = \left|\int_y^\infty \phi_0^\mp(z)\psi(z)\,dz\right| &\le \sqrt{\int_y^\infty \left[\phi_0^\mp(z)\right]^2\,dz\int_y^\infty \psi(z)^2\,dz} \\
&\le \sqrt{C\left[\phi_0^\mp(y)\right]^2\int_y^\infty \psi(z)^2\,dz} \\
&= \sqrt{C}\left|\phi_0^\mp(y)\right|\sqrt{\int_y^\infty \psi(z)^2\,dz}.
\end{aligned}
\tag{3.54}
$$

Using the derived inequality above, we obtain that for every $x > 0$

$$
\begin{aligned}
|\phi_{sp}(x)| &\le \left|\phi_0^\pm(x)\right|\int_0^x \left|\frac{1}{\phi_0^+(y)\phi_0^-(y)}\right|\,|\eta(y)|\,dy \\
&\le \sqrt{C}\left|\phi_0^\pm(x)\right|\int_0^x\left(\left|\frac{1}{\phi_0^+(y)\phi_0^-(y)}\right|\,\left|\phi_0^\mp(y)\right|\sqrt{\int_y^\infty \psi(z)^2\,dz}\right)dy \\
&= \sqrt{C}\left|\phi_0^\pm(x)\right|\int_0^x\left(\frac{1}{\left|\phi_0^\pm(y)\right|}\sqrt{\int_y^\infty \psi(z)^2\,dz}\right)dy.
\end{aligned}
\tag{3.55}
$$

To further bound $|\phi_{sp}(x)|$ we need some observations. Note that $\phi_0^{\pm}(x) > 0$, because $u_0'(x) > 0$ by Theorem 1. Together with the first inequality of (3.6) from Lemma 6*(v)* this implies that

$$\left|\phi_0^{\pm}(x)\right| \int_0^x \frac{1}{\left|\phi_0^{\pm}(y)\right|} dy = \phi_0^{\pm}(x) \int_0^x \frac{1}{\phi_0^{\pm}(y)} dy \leq C. \tag{3.56}$$

Furthermore we notice that $\sqrt{\int_y^\infty \psi(z)^2\, dz} \leq \sqrt{\int_{-\infty}^\infty \psi(z)^2\, dz} = \|\psi\|_{L^2}$. Using these two derived inequalities we find that

$$\sqrt{C}\left|\phi_0^{\pm}(x)\right| \int_0^x \left(\frac{1}{\left|\phi_0^{\pm}(y)\right|} \sqrt{\int_y^\infty \psi(z)^2\, dz} \right) dy \leq C\sqrt{C}\, \|\psi\|_{L^2}\,. \tag{3.57}$$

So if we denote $A_1 = C\sqrt{C}$, it follows that

$$|\phi_{sp}(x)| \leq A_1 \|\psi\|_{L^2} \quad \text{for } x > 0. \tag{3.58}$$

We can also write this as $\|\phi_{sp}\|_{L^\infty(0,\infty)} \leq A_1 \|\psi\|_{L^2}$. Here the norm $\|\phi_{sp}\|_{L^\infty(0,\infty)}$ takes the maximum value of $|\phi_{sp}(x)|$ for $x \in (0, \infty)$. Furthermore, by applying l'Hôpital's rule, it follows that $\lim_{x\to\infty} \phi_{sp}(x) = 0$.

Now we want to show there exists another constant A_2 such that a similar statement holds for $x \in (-\infty, 0)$, namely $\|\phi_{sp}\|_{L^\infty(-\infty,0)} \leq A_2 \|\psi\|_{L^2}$. To show this we assume $\psi \perp \phi_0^{\pm}$, then for every $y \in \mathbb{R}$ we have

$$\begin{aligned} 0 &= \int_{-\infty}^\infty \psi(x)\phi_0^{\pm}(x)\, dx \\ &= \int_{-\infty}^y \psi(x)\phi_0^{\pm}(x)\, dx + \int_y^\infty \psi(x)\phi_0^{\pm}(x)\, dx \end{aligned} \tag{3.59}$$

so that

$$\int_y^\infty \psi(x)\phi_0^{\pm}(x)\, dx = \int_y^{-\infty} \psi(x)\phi_0^{\pm}(x)\, dx. \tag{3.60}$$

Bounds as in Lemma 6*(v)* can also be obtained for $x < 0$. These are found in the same way as for $x > 0$. In a manner similar to the case when x is positive it can be shown, using (3.60), that $\|\phi_{sp}\|_{L^\infty(-\infty,0)} \leq A_2 \|\psi\|_{L^2}$ for some positive constant A_2 and that $\lim_{x\to-\infty} \phi_{sp}(x) = 0$. So we find that ϕ_{sp} is bounded by ψ and that $\lim_{x\to\pm\infty} \phi_{sp}(x) = 0$. It is claimed in [2, Lemma 5(2)] that this is enough to show there exists a positive constant A such that $\|\phi_{sp}\|_{H^2} \leq A \|\psi\|_{L^2}$[1].

We define $\phi = \phi_{sp} - \langle \phi_{sp}\,,\, \phi_0^{\pm} \rangle \phi_0^{\pm}$ and claim that ϕ is a unique solution to problem (3.7). This would finish the proof immediately. We'll show that this claim indeed holds.

[1]It is claimed this result follows using the differential equation $\mathcal{L}_0^{\pm}\phi_{sp} = \psi$ and an energy estimate.

Recall that $\mathcal{L}_0^{\pm}\phi_{sp} = \psi$. Furthermore we have $\mathcal{L}_0^{\pm}\phi_0^{\pm} = 0$ by Lemma 6*(ii)*. Using both of these observations, we find

$$\mathcal{L}_0^{\pm}\phi = \mathcal{L}_0^{\pm}\left[\phi_{sp} - \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle\phi_0^{\pm}\right] = \mathcal{L}_0^{\pm}\phi_{sp} - \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle\mathcal{L}_0^{\pm}\phi_0^{\pm} = \mathcal{L}_0^{\pm}\phi_{sp} = \psi. \quad (3.61)$$

Now we show that $\phi \perp \phi_0^{\pm}$. Since $\|\phi_0^{\pm}\|_{L^2} = 1$ by Lemma 6*(i)*, we have

$$\begin{aligned}\langle \phi\,,\, \phi_0^{\pm}\rangle = \left\langle \phi_{sp} - \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle\phi_0^{\pm}\,,\, \phi_0^{\pm}\right\rangle &= \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle - \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle\langle \phi_0^{\pm}\,,\, \phi_0^{\pm}\rangle \\ &= \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle - \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle\|\phi_0^{\pm}\|_{L^2}^2 \\ &= \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle - \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle \\ &= 0.\end{aligned} \quad (3.62)$$

It is left to show that ϕ is a unique solution. First consider the problem $\mathcal{L}_0^{\pm}\hat{\phi} = 0$. It follows from Lemma 6*(ii)* that $\phi_0^{\pm}$ is a solution. Another solution to this problem is $\phi_0^{\pm}\int_0^x \frac{1}{\phi_0^{+}(y)\phi_0^{-}(y)}\,dy$, which can be found by following the same procedure as during the evaluation of $\mathcal{L}_0^{\pm}\phi_{sp}$, but with setting $\eta(y) = 1$. These two solutions are clearly linearly independent. Since $\mathcal{L}_0^{\pm}\hat{\phi} = 0$ is a second order ordinary differential equation, all of its solutions must be linear combinations of these two solutions.

Now let $\tilde{\phi}$ be another solution to problem (3.7). Then

$$\mathcal{L}_0^{\pm}\left[\tilde{\phi} - \phi\right] = \mathcal{L}_0^{\pm}\tilde{\phi} - \mathcal{L}_0^{\pm}\phi = \psi - \psi = 0. \quad (3.63)$$

As we have just observed, $\tilde{\phi} - \phi$ must be a linear combination of $\phi_0^{\pm}\int_0^x \frac{1}{\phi_0^{+}(y)\phi_0^{-}(y)}\,dy$ and $\phi_0^{\pm}$. But $\phi_0^{\pm}\int_0^x \frac{1}{\phi_0^{+}(y)\phi_0^{-}(y)}\,dy$ is unbounded and since $\tilde{\phi} - \phi \in H^2(\mathbb{R})$, it can only be a multiple of $\phi_0^{\pm}$. So there exists a constant d such that $\tilde{\phi} - \phi = d\phi_0^{\pm}$. Using this together with the fact that both $\tilde{\phi} \perp \phi_0^{\pm}$ and $\tilde{\phi} \perp \phi_0^{\pm}$, we find

$$0 = \langle \tilde{\phi} - \phi\,,\, \phi_0^{\pm}\rangle = d\langle \phi_0^{\pm}\,,\, \phi_0^{\pm}\rangle = d\|\phi_0^{\pm}\|_{L^2}^2. \quad (3.64)$$

But Lemma 6*(i)* states that $\|\phi_0^{\pm}\|_{L^2} = 1$ and thus it follows that $d = 0$. Because $\tilde{\phi} - \phi = d\phi_0^{\pm}$, we find that $\tilde{\phi} = \phi$ and we can conclude that ϕ is a unique solution to (3.7) proving the claim.

Furthermore, we can estimate the H^2-norm of ϕ by applying the Cauchy-Schwarz inequality. We find that

$$\begin{aligned}\|\phi\|_{H^2} = \left\|\phi_{sp} - \langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle\phi_0^{\pm}\right\|_{H^2} &\le \|\phi_{sp}\|_{H^2} + \left\|\langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle\phi_0^{\pm}\right\|_{H^2} \\ &= \|\phi_{sp}\|_{H^2} + \left|\langle \phi_{sp}\,,\, \phi_0^{\pm}\rangle\right|\,\|\phi_0^{\pm}\|_{H^2} \\ &\le \|\phi_{sp}\|_{H^2} + \|\phi_{sp}\|_{L^2}\,\|\phi_0^{\pm}\|_{L^2}\|\phi_0^{\pm}\|_{H^2} \\ &\le \|\phi_{sp}\|_{H^2}\left(1 + \|\phi_0^{\pm}\|_{L^2}\|\phi_0^{\pm}\|_{H^2}\right).\end{aligned} \quad (3.65)$$

Using that $\|\phi_0^{\pm}\|_{L^2} = 1$, see Lemma 6*(i)*, and using the earlier obtained bound $\|\phi_{sp}\|_{H^2} \le A\,\|\psi\|_{L^2}$ we can further estimate this expression. This gives

$$\|\phi\|_{H^2} \le A\left(1 + \|\phi_0^{\pm}\|_{H^2}\right)\|\psi\|_{L^2}\,. \quad (3.66)$$

So defining $C_1 = A\left(1 + \|\phi_0^{\pm}\|_{H^2}\right)$ confirms that also (3.8) holds.

□

This lemma is mainly needed to prove the next lemma, where we find an estimate in which a function $\phi \in H^2(\mathbb{R})$ is compared to $(\mathcal{L}_0^{\pm} + \delta)\phi$. Later, we will see that a similar estimate can be obtained where the operator $\mathcal{L}_0^{\pm} + \delta$ is replaced by $\mathcal{L}_{\varepsilon,\delta}^{\pm}$. So, the continuous case is then being connected to the discrete case. But this will happen in the next section 3.2, where the operator $\mathcal{L}_{\varepsilon,\delta}^{\pm}$ is treated.

Lemma 7. *Let $\mathcal{L}_0^{\pm}$ and $\phi_0^{\pm}$ be as in* (3.4) *and* (3.5)*. Then there exists a positive constant C_2, such that for every $\delta > 0$ and for all $\phi \in H^2(\mathbb{R})$ we have*

$$\|\phi\|_{H^2} \leq C_2 \left\{ \frac{1}{\delta} \left| \langle \psi \, , \, \phi_0^{\mp} \rangle \right| + \|\psi\|_{L^2} \right\} \qquad \textit{where } \psi = \mathcal{L}_0^{\pm}\phi + \delta\phi. \tag{3.67}$$

Proof. We will consider 3 cases separately to prove this statement. Namely δ is (a)large, (b)small and (c)intermediate.

(a) We start with δ being large. We denote $\delta_1 = 1 + \|f_u(u_0)\|_{L^\infty}$ and we assume $\delta \geq \delta_1$. Let $\phi \in H^2(\mathbb{R})$ be arbitrary and set $\psi = \mathcal{L}_0^{\pm}\phi + \delta\phi$. Our goal is to show that

$$\|\phi\|_{H^2}^2 = \|\phi\|_{L^2}^2 + \|\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 \leq C \, \|\psi\|_{L^2}^2 \, , \quad \text{where } C \text{ is a constant.}$$

If we can verify this, then (3.67) will easily follow. We will start by bounding $\|\phi\|_{L^2}^2$. The Cauchy-Schwarz inequality implies that

$$\|\psi\|_{L^2} \|\phi\|_{L^2} \geq \langle \psi \, , \, \phi \rangle \, . \tag{3.68}$$

By Lemma 1*(i)*, Lemma 1*(ii)* and the definition of ψ we obtain

$$\begin{aligned} \langle \psi \, , \, \phi \rangle &= \langle \pm c_0 \phi' - \phi'' + f_u(u_0)\phi + \delta\phi \, , \, \phi \rangle \\ &= \pm c_0 \langle \phi' \, , \, \phi \rangle - \langle \phi'' \, , \, \phi \rangle + \langle f_u(u_0)\phi \, , \, \phi \rangle + \delta \langle \phi \, , \, \phi \rangle \\ &\geq \langle f_u(u_0)\phi \, , \, \phi \rangle + \delta \langle \phi \, , \, \phi \rangle \, . \end{aligned} \tag{3.69}$$

Since for all $x \in \mathbb{R}$ we have $f_u(u_0(x)) \geq - \|f_u(u_0)\|_{L^\infty}$, we find

$$\begin{aligned} \langle f_u(u_0)\phi \, , \, \phi \rangle &= \int_{\mathbb{R}} f_u(u_0(x))\phi(x)^2 \, dx \\ &\geq - \|f_u(u_0)\|_{L^\infty} \int_{\mathbb{R}} \phi(x)^2 \, dx = - \|f_u(u_0)\|_{L^\infty} \|\phi\|_{L^2}^2 \, . \end{aligned} \tag{3.70}$$

Furthermore we of course have $\langle \phi \, , \, \phi \rangle = \|\phi\|_{L^2}^2$. From these two observations it follows that

$$\langle f_u(u_0)\phi \, , \, \phi \rangle + \delta \langle \phi \, , \, \phi \rangle \geq (\delta - \|f_u(u_0)\|_{L^\infty}) \, \|\phi\|_{L^2}^2 \, . \tag{3.71}$$

Connecting (3.68), (3.69) and (3.71) gives

$$\|\psi\|_{L^2} \|\phi\|_{L^2} \geq (\delta - \|f_u(u_0)\|_{L^\infty}) \, \|\phi\|_{L^2}^2 \, , \tag{3.72}$$

which after dividing both sides by $\|\phi\|_{L^2}$ is equivalent to

$$(\delta - \|f_u(u_0)\|_{L^\infty}) \, \|\phi\|_{L^2} \leq \|\psi\|_{L^2} \, . \tag{3.73}$$

This together with the assumption $\delta \geq \delta_1$ implies that

$$\begin{aligned} \|\phi\|_{L^2} &= (\delta_1 - \|f_u(u_0)\|_{L^\infty}) \|\phi\|_{L^2} \\ &\leq (\delta - \|f_u(u_0)\|_{L^\infty}) \|\phi\|_{L^2} \\ &\leq \|\psi\|_{L^2} . \end{aligned} \tag{3.74}$$

After squaring both sides we obtain $\|\phi\|_{L^2}^2 \leq \|\psi\|_{L^2}^2$. So we have found a bound for $\|\phi\|_{L^2}^2$.

Now we also want to bound $\|\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2$ by $B_1 \|\psi\|_{L^2}^2$ for some constant B_1. If $|c_0| > 1$, we find

$$\|\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 \leq \|c_0\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 , \tag{3.75}$$

while if $|c_0| \leq 1$ we can estimate

$$\begin{aligned} \|\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 &= \frac{1}{c_0^2} \left(\|c_0\phi'\|_{L^2}^2 + \|c_0\phi''\|_{L^2}^2 \right) \\ &\leq \frac{1}{c_0^2} \left(\|c_0\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 \right). \end{aligned} \tag{3.76}$$

So we can conclude that

$$\|\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 \leq \max\left\{1, \frac{1}{c_0^2}\right\} \left(\|c_0\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 \right). \tag{3.77}$$

Note that by Lemma 1*(i)* we have that $\langle \phi'' , \phi' \rangle = 0$. We'll use this to rewrite $\|c_0\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2$. This gives

$$\begin{aligned} \|c_0\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 &= \int_{\mathbb{R}} \left[c_0^2\phi'(x)^2 + \phi''(x)^2\right] dx \\ &= \int_{\mathbb{R}} \left[\pm c_0\phi'(x) - \phi''(x)\right]^2 dx \pm 2c_0 \int_{\mathbb{R}} \phi''(x)\phi'(x)\, dx \\ &= \int_{\mathbb{R}} \left[\pm c_0\phi'(x) - \phi''(x)\right]^2 dx \pm 2c_0 \langle \phi'' , \phi' \rangle \\ &= \left\| \pm c_0\phi'(x) - \phi''(x) \right\|_{L^2}^2. \end{aligned} \tag{3.78}$$

By the definition of ψ we obtain $\pm c_0\phi' - \phi'' = \psi - (f_u(u_0) + \delta)\,\phi$. So we have

$$\begin{aligned} \|c_0\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 &\leq \left\|\psi - (f_u(u_0) + \delta)\,\phi\right\|_{L^2}^2 \\ &\leq \left(\|\psi\|_{L^2} + \left\|(f_u(u_0) + \delta)\,\phi\right\|_{L^2}\right)^2 . \end{aligned} \tag{3.79}$$

Since $f_u(u_0(x)) \leq \|f_u(u_0)\|_{L^\infty}$ for all $x \in \mathbb{R}$, using the same reasoning as in (3.70), we have that $\|(f_u(u_0) + \delta)\,\phi\|_{L^2} \leq (\|f_u(u_0)\|_{L^\infty} + \delta) \|\phi\|_{L^2}$. Using this together with

(3.73) and (3.74) we find that

$$\begin{aligned}\|c_0\phi'\|_{L^2}^2+\|\phi''\|_{L^2}^2 &\leq \Big(\|\psi\|_{L^2}+(\|f_u(u_0)\|_{L^\infty}+\delta)\,\|\phi\|_{L^2}\Big)^2\\ &= \Big(\|\psi\|_{L^2}+(\delta-\|f_u(u_0)\|_{L^\infty})\,\|\phi\|_{L^2}+2\,\|f_u(u_0)\|_{L^\infty}\,\|\phi\|_{L^2}\Big)^2\\ &\leq \Big(2\,\|f_u(u_0)\|_{L^\infty}+2\Big)^2\,\|\psi\|_{L^2}^2\end{aligned} \tag{3.80}$$

If we now choose $B_1=\max\left\{1,\frac{1}{c_0^2}\right\}\left(2\,\|f_u(u_0)\|_{L^\infty}+2\right)^2$, then it follows from (3.77) and (3.80) that

$$\|\phi'\|_{L^2}^2+\|\phi''\|_{L^2}^2\leq B_1\,\|\psi\|_{L^2}^2\,. \tag{3.81}$$

If we now combine this with the estimation $\|\phi\|_{L^2}^2\leq\|\psi\|_{L^2}^2$ obtained earlier, then we have

$$\|\phi\|_{H^2}^2=\|\phi\|_{L^2}^2+\|\phi'\|_{L^2}^2+\|\phi''\|_{L^2}^2\leq(B_1+1)\,\|\psi\|_{L^2}^2\,. \tag{3.82}$$

Taking the square root on both sides gives

$$\|\phi\|_{H^2}\leq\sqrt{B_1+1}\,\|\psi\|_{L^2}\leq\sqrt{B_1+1}\left\{\frac{1}{\delta}\left|\left\langle\psi\,,\,\phi_0^{\mp}\right\rangle\right|+\|\psi\|_{L^2}\right\}. \tag{3.83}$$

The final step is taking $C_2=\sqrt{B_1+1}$, after which we end up with the result we were seeking for.

(b) We consider the case where δ is small. We assume $\delta\in(0,\delta_0]$ where δ_0 will be defined later. Again we set $\psi=\mathcal{L}_0^{\pm}\phi+\delta\phi$ and we let $\phi\in H^2(\mathbb{R})$ arbitrary.

First we decompose ϕ. We do this by choosing $\phi^{\perp}$ such that $\phi=\left\langle\phi\,,\,\phi_0^{\pm}\right\rangle\phi_0^{\pm}+\phi^{\perp}$. Therefore we can now write $\phi^{\perp}=\phi-\left\langle\phi\,,\,\phi_0^{\pm}\right\rangle\phi_0^{\pm}$. Now Lemma 6*(i)*, which states that $\|\phi_0^{\pm}\|_{L^2}=1$, implies that

$$\begin{aligned}\left\langle\phi^{\perp}\,,\,\phi_0^{\pm}\right\rangle &= \left\langle\phi\,,\,\phi_0^{\pm}\right\rangle-\left\langle\phi\,,\,\phi_0^{\pm}\right\rangle\left\langle\phi_0^{\pm}\,,\,\phi_0^{\pm}\right\rangle\\ &= \left\langle\phi\,,\,\phi_0^{\pm}\right\rangle-\left\langle\phi\,,\,\phi_0^{\pm}\right\rangle\left\|\phi_0^{\pm}\right\|_{L^2}^2\\ &= 0.\end{aligned} \tag{3.84}$$

So we have found that $\phi^{\perp}\perp\phi_0^{\pm}$. Furthermore Lemma 6*(ii)*, which states that $\mathcal{L}_0^{\pm}\phi_0^{\pm}=0$, helps us finding

$$\mathcal{L}_0^{\pm}\phi^{\perp}=\mathcal{L}_0^{\pm}\phi-\left\langle\phi\,,\,\phi_0^{\pm}\right\rangle\mathcal{L}_0^{\pm}\phi_0^{\pm}=\psi-\delta\phi. \tag{3.85}$$

Now applying the second part of Lemma 6*(vi)* to $\phi^{\perp}$ gives

$$\|\phi^{\perp}\|_{H^2}\leq C_1\left\|\mathcal{L}_0^{\pm}\phi^{\perp}\right\|_{L^2}=C_1\,\|\psi-\delta\phi\|_{L^2}\leq C_1\left\{\|\psi\|_{L^2}+\delta\,\|\phi\|_{L^2}\right\}. \tag{3.86}$$

This inequality will be used later in the proof. In the proof of Lemma 6*(vi)* we have seen that $\mathcal{L}_0^{\pm}\phi\perp\phi_0^{\mp}$ for $\phi\in H^2(\mathbb{R})$. Since $\mathcal{L}_0^{\pm}\phi=\psi-\delta\phi$, we also have $\psi-\delta\phi\perp\phi_0^{\mp}$ and thus

$$\left\langle\psi-\delta\phi\,,\,\phi_0^{\mp}\right\rangle=\left\langle\psi\,,\,\phi_0^{\mp}\right\rangle-\delta\left\langle\phi\,,\,\phi_0^{\mp}\right\rangle=0. \tag{3.87}$$

Now bringing $\delta \left\langle \phi \, , \, \phi_0^{\mp} \right\rangle$ to the other side and replacing ϕ by its decomposition gives

$$\left\langle \psi \, , \, \phi_0^{\mp} \right\rangle = \delta \left\langle \phi \, , \, \phi_0^{\mp} \right\rangle = \delta \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle \left\langle \phi_0^{+} \, , \, \phi_0^{-} \right\rangle + \delta \left\langle \phi^{\perp} \, , \, \phi_0^{\mp} \right\rangle . \tag{3.88}$$

We denote

$$\sigma = \left\langle \phi_0^{+} \, , \, \phi_0^{-} \right\rangle = \int_{\mathbb{R}} \frac{\left[u_0'(x)\right]^2 e^{-c_0 x}}{\|u_0'(x)\|_{L^2} \, \|u_0'(x) e^{-c_0 x}\|} \, dx. \tag{3.89}$$

Notice that $\sigma > 0$. Furthermore $\sigma \leq 1$ by the Cauchy-Schwarz inequality in combination with Lemma 6*(i)*. Consider equality (3.88). We divide it by δ and rewrite it with our definition of σ. This gives

$$\frac{1}{\delta} \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle = \sigma \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle + \left\langle \phi^{\perp} \, , \, \phi_0^{\mp} \right\rangle . \tag{3.90}$$

We can now bound $\sigma |\langle \phi \, , \, \phi_0^{\pm} \rangle|$ using Lemma 6*(i)*. We get

$$\begin{aligned}
\sigma \left| \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle \right| &= \left| \sigma \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle \right| \\
&= \left| \frac{1}{\delta} \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle - \left\langle \phi^{\perp} \, , \, \phi_0^{\mp} \right\rangle \right| \\
&\leq \frac{1}{\delta} \left| \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle \right| + \left| \left\langle \phi^{\perp} \, , \, \phi_0^{\mp} \right\rangle \right| \\
&\leq \frac{1}{\delta} \left| \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle \right| + \|\phi^{\perp}\|_{L^2} \|\phi_0^{\pm}\|_{L^2} = \frac{1}{\delta} \left| \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle \right| + \|\phi^{\perp}\|_{L^2}.
\end{aligned} \tag{3.91}$$

From this together with the earlier obtained inequality for $\|\phi^{\perp}\|_{H^2}$ it follows that

$$\begin{aligned}
\sigma \left| \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle \right| + \|\phi^{\perp}\|_{H^2} &\leq \frac{1}{\delta} \left| \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle \right| + \|\phi^{\perp}\|_{L^2} + \|\phi^{\perp}\|_{H^2} \\
&\leq \frac{1}{\delta} \left| \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle \right| + 2\|\phi^{\perp}\|_{H^2} \\
&\leq \frac{1}{\delta} \left| \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle \right| + 2C_1 \, \|\psi\|_{L^2} + 2\delta C_1 \, \|\phi\|_{L^2} \, .
\end{aligned} \tag{3.92}$$

After rewriting this is equivalent to

$$\frac{1}{\delta} \left| \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle \right| + 2C_1 \, \|\psi\|_{L^2} \geq \sigma \left| \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle \right| + \|\phi^{\perp}\|_{H^2} - 2\delta C_1 \, \|\phi\|_{L^2} \, . \tag{3.93}$$

We want to find a lower bound for the left side of this expression that is of the form $A(\delta) \, \|\phi\|_{L^2}$. To find such a lower bound one more inequality is required. This inequality is obtained via the definition of ϕ in the following way. We have

$$\begin{aligned}
\|\phi\|_{L^2} &= \left\| \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle \phi_0^{\pm} + \phi^{\perp} \right\|_{L^2} \\
&\leq \left\| \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle \phi_0^{\pm} \right\|_{L^2} + \|\phi^{\perp}\|_{L^2} \\
&= \left| \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle \right| \|\phi_0^{\pm}\|_{L^2} + \|\phi^{\perp}\|_{L^2} \\
&= \left| \left\langle \phi \, , \, \phi_0^{\pm} \right\rangle \right| + \|\phi^{\perp}\|_{L^2}.
\end{aligned} \tag{3.94}$$

Now we are ready to find the lower bound for $\sigma|\langle\phi\,,\,\phi_0^\pm\rangle| + \|\phi^\perp\|_{H^2} - 2\delta C_1\,\|\phi\|_{L^2}$. Besides the just derived inequality we'll also use that $\sigma \leq 1$ during our derivation. We find

$$\begin{aligned}\sigma\left|\left\langle\phi\,,\,\phi_0^\pm\right\rangle\right| + \|\phi^\perp\|_{H^2} - 2\delta C_1\,\|\phi\|_{L^2} &\geq \sigma\left|\left\langle\phi\,,\,\phi_0^\pm\right\rangle\right| + \|\phi^\perp\|_{L^2} - 2\delta C_1\,\|\phi\|_{L^2} \\ &\geq \sigma\left\{\,\left|\left\langle\phi\,,\,\phi_0^\pm\right\rangle\right| + \|\phi^\perp\|_{L^2}\right\} - 2\delta C_1\,\|\phi\|_{L^2} \\ &\geq \sigma\,\|\phi\|_{L^2} - 2\delta C_1\,\|\phi\|_{L^2} \\ &= (\sigma - 2\delta C_1)\,\|\phi\|_{L^2}\,.\end{aligned} \tag{3.95}$$

So we also have got

$$\frac{1}{\delta}\left|\left\langle\psi\,,\,\phi_0^\mp\right\rangle\right| + 2C_1\,\|\psi\|_{L^2} \geq (\sigma - 2\delta C_1)\,\|\phi\|_{L^2}\,, \tag{3.96}$$

which is equivalent to

$$\|\phi\|_{L^2} \leq \frac{1}{\sigma - 2\delta C_1}\left\{\frac{1}{\delta}\left|\left\langle\psi\,,\,\phi_0^\mp\right\rangle\right| + 2C_1\,\|\psi\|_{L^2}\right\}. \tag{3.97}$$

We will now finally set the value of δ_0. We let $\delta_0 = \sigma/4C_1$ such that $\delta \leq \sigma/4C_1$. This implies

$$\|\phi\|_{L^2} \leq \frac{2}{\sigma}\left\{\frac{1}{\delta}\left|\left\langle\psi\,,\,\phi_0^\mp\right\rangle\right| + 2C_1\,\|\psi\|_{L^2}\right\} = \frac{4C_1}{\sigma}\left\{\frac{1}{2C_1\delta}\left|\left\langle\psi\,,\,\phi_0^\mp\right\rangle\right| + \|\psi\|_{L^2}\right\}. \tag{3.98}$$

Note that we can choose C_1 as big as we like. We take $C_1 \geq 1/2$. If we now set $B_2 = 4C_1/\sigma$ we get

$$\|\phi\|_{L^2} \leq B_2\left\{\frac{1}{\delta}\left|\left\langle\psi\,,\,\phi_0^\mp\right\rangle\right| + \|\psi\|_{L^2}\right\}. \tag{3.99}$$

In a very similar way as in (a) we can find a bound for $\|\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2$. The main difference is that we use (3.99) instead of (3.74) when bounding $\|c_0\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2$ as in (3.80). This leads to a slightly different bound of course. We find that

$$\|c_0\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 \leq B_3\left\{\frac{1}{\delta}\left|\left\langle\psi\,,\,\phi_0^\mp\right\rangle\right| + \|\psi\|_{L^2}\right\}^2, \tag{3.100}$$

where $B_3 = \left(2B_2\,\|f_u(u_0)\|_{L^\infty} + 2\right)^2$. We now use (3.77) which implies

$$\|\phi'\|_{L^2}^2 + \|\phi''\|_{L^2}^2 \leq B_3\max\left\{1, \frac{1}{c_0^2}\right\}\left\{\frac{1}{\delta}\left|\left\langle\psi\,,\,\phi_0^\mp\right\rangle\right| + \|\psi\|_{L^2}\right\}^2. \tag{3.101}$$

Just like in (a), we square inequality (3.99) and add inequality (3.101) to it. Thereafter we take the square root on both sides of the obtained inequality to get the final result. Namely

$$\|\phi\|_{H^2} \leq C_2\left\{\frac{1}{\delta}\left|\left\langle\psi\,,\,\phi_0^\mp\right\rangle\right| + \|\psi\|_{L^2}\right\}, \tag{3.102}$$

with $C_2 = \sqrt{B_2^2 + B_3\max\left\{1, \frac{1}{c_0^2}\right\}}$.

(c) The final case we consider is for $\delta \in [\delta_0, \delta_1]$, where we assume that $\delta_0 \leq \delta_1$. If this is not the case, then parts (a) and (b) already complete the full proof. We define

$$\tilde{\Lambda}^{\pm}(\delta) = \inf_{\|\phi\|_{H^2}=1} \left\| \mathcal{L}_0^{\pm}\phi + \delta\phi \right\|_{L^2}, \tag{3.103}$$

$$\hat{\Lambda}^{\pm} = \inf_{\delta \in [\delta_0, \delta_1]} \tilde{\Lambda}^{\pm}(\delta). \tag{3.104}$$

Let $\{\phi_j\}_{j=0}^{\infty}$ and $\{\delta_j\}_{j=0}^{\infty}$ be sequences in respectively $H^2(\mathbb{R})$ and $\mathbb{R}$, with $\|\phi_j\|_{H^2} = 1$ and $\delta_j \in [\delta_0, \delta_1]$, such that

$$\lim_{j\to\infty} \left\| \mathcal{L}_0^{\pm}\phi_j + \delta_j\phi_j \right\|_{L^2} = \hat{\Lambda}^{\pm}. \tag{3.105}$$

Define $\psi_j = \mathcal{L}_0^{\pm}\phi_j + \delta_j\phi_j$. Then $\lim_{j\to\infty} \|\psi_j\|_{L^2} = \hat{\Lambda}^{\pm}$ and therefore $\{\psi_j\}_{j=0}^{\infty}$ is a bounded sequence in $L^2(\mathbb{R})$. Since $\|\phi_j\|_{H^2} = 1$ for all j, we have that $\{\phi_j\}_{j=0}^{\infty}$ is a bounded sequence in $H^2(\mathbb{R})$. So it follows from Theorem A1 that, by taking subsequences if necessary, there exist $\phi \in H^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R})$ such that

$$\psi_j \to \psi \text{ in } L^2(\mathbb{R}) \text{ weakly, as } j \to \infty, \tag{3.106}$$

$$\phi_j \to \phi \text{ in } H^2(\mathbb{R}) \text{ weakly, as } j \to \infty. \tag{3.107}$$

Furthermore Theorem A2 implies that, by taking another subsequence if necessary, we have

$$\phi_j \to \phi \text{ in } L^2_{loc}(\mathbb{R}), \text{ as } j \to \infty. \tag{3.108}$$

For each j we have that $\delta_j \leq \delta_1$. So $\{\delta_j\}_{j=0}^{\infty}$ is a bounded sequence. Now the Bolzano-Weierstrass Theorem tells us there exists a $\delta \in [\delta_0, \delta_1]$ such that, by taking a subsequence if necessary, $\delta_j \to \delta$ as $j \to \infty$.

Now we take a test function $\zeta \in C_0^{\infty}(\mathbb{R}) \cap H^2(\mathbb{R})$. Then

$$\begin{aligned} \langle \psi_j \,, \zeta \rangle &= \left\langle \mathcal{L}_0^{\pm}\phi_j + \delta_j\phi_j \,, \zeta \right\rangle \\ &= \left\langle \mathcal{L}_0^{\pm}\phi_j \,, \zeta \right\rangle + \langle \delta_j\phi_j \,, \zeta \rangle = \left\langle \phi_j \,, \mathcal{L}_0^{\mp}\zeta \right\rangle + \delta_j \langle \phi_j \,, \zeta \rangle \,. \end{aligned} \tag{3.109}$$

We want to examine what happens if we let $j \to \infty$ on both sides. Since $\psi_j \to \psi$ weakly in $L^2(\mathbb{R})$, we know that $\langle \psi_j \,, \zeta \rangle$ converges to $\langle \psi \,, \zeta \rangle$. Since $\phi_j \to \phi$ weakly in $H^2(\mathbb{R})$, it follows that $\left\langle \phi_j \,, \mathcal{L}_0^{\mp}\zeta \right\rangle \to \left\langle \phi \,, \mathcal{L}_0^{\mp}\zeta \right\rangle$ and $\langle \phi_j \,, \zeta \rangle \to \langle \phi \,, \zeta \rangle$. We also observed earlier that $\delta_j \to \delta$. Using all of these observations, we see that letting $j \to \infty$ on both sides of (3.109) gives

$$\begin{aligned} \langle \psi \,, \zeta \rangle &= \left\langle \phi \,, \mathcal{L}_0^{\mp}\zeta \right\rangle + \delta \langle \phi \,, \zeta \rangle \\ &= \left\langle \mathcal{L}_0^{\pm}\phi \,, \zeta \right\rangle + \langle \delta\phi \,, \zeta \rangle = \left\langle \mathcal{L}_0^{\pm}\phi + \delta\phi \,, \zeta \right\rangle \,. \end{aligned} \tag{3.110}$$

Because this holds for every $\zeta \in C_0^{\infty}(\mathbb{R}) \cap H^2(\mathbb{R})$, we find that $\psi = \mathcal{L}_0^{\pm}\phi + \delta\phi$. We claim that $\hat{\Lambda}^{\pm} > 0$. We will show this by contradiction and therefore we assume from now on that $\hat{\Lambda}^{\pm} = 0$. It follows from Theorem A3 that

$$\|\psi\|_{L^2} \leq \liminf_{j\to\infty} \|\psi_j\|_{L^2} = \hat{\Lambda}^{\pm}. \tag{3.111}$$

This implies that $\psi = 0$ and thus it follows that $\mathcal{L}_0^{\pm}\phi + \delta\phi = \psi = 0$. In [2, Lemma 5(3)], the authors claim that the positivity of the functions $\phi_0^{\pm}$ and the fact that $\mathcal{L}_0^{\pm}\phi_0^{\pm} = 0$ allows one to use Liouville's theorem to conclude that the equation $(\mathcal{L}_0^{\pm} + \delta)\phi = 0$ does not have any nontrivial bounded solution[2]. So we find that $\phi = 0$.

Our goal is to also show that $\phi \neq 0$ and thus obtain a contradiction. We start by deriving two inequalities that will come in handy later in the proof.

Let's derive the first of the two inequalities. We start by observing that

$$\begin{aligned} \left\langle f_u(u_0)\phi_j \,,\, \phi_j'' \right\rangle - \|\phi_j''\|_{L^2}^2 &= \left\langle f_u(u_0)\phi_j \,,\, \phi_j'' \right\rangle - \left\langle \phi_j'' \,,\, \phi_j'' \right\rangle \\ &= \left\langle f_u(u_0)\phi_j - \phi_j'' \,,\, \phi_j'' \right\rangle \\ &= \left\langle \mathcal{L}_0^{\pm}\phi_j \mp c_0\phi_j' \,,\, \phi_j'' \right\rangle \\ &= \left\langle \mathcal{L}_0^{\pm}\phi_j + \delta_j\phi_j \,,\, \phi_j'' \right\rangle + \left\langle \mp c_0\phi_j' - \delta_j\phi_j \,,\, \phi_j'' \right\rangle . \end{aligned} \tag{3.112}$$

Now it follows from Lemma 1*(i)* and Lemma 1*(ii)* that

$$\left\langle \mp c_0\phi_j' - \delta_j\phi_j \,,\, \phi_j'' \right\rangle = \mp c_0 \left\langle \phi_j' \,,\, \phi_j'' \right\rangle - \delta_j \left\langle \phi_j \,,\, \phi_j'' \right\rangle \geq 0 \tag{3.113}$$

and thus

$$\left\langle f_u(u_0)\phi_j \,,\, \phi_j'' \right\rangle - \|\phi_j''\|_{L^2}^2 \geq \left\langle \mathcal{L}_0^{\pm}\phi_j + \delta_j\phi_j \,,\, \phi_j'' \right\rangle = \left\langle \psi_j \,,\, \phi_j'' \right\rangle . \tag{3.114}$$

By using the Cauchy-Schwarz inequality we find that

$$\left\langle f_u(u_0)\phi_j \,,\, \phi_j'' \right\rangle \leq \|f_u(u_0)\|_{L^\infty} \left\langle \phi_j \,,\, \phi_j'' \right\rangle \leq \|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} \|\phi_j''\|_{L^2}, \tag{3.115}$$

$$\left\langle \psi_j \,,\, \phi_j'' \right\rangle \geq -\|\psi_j\|_{L^2} \|\phi_j''\|_{L^2}. \tag{3.116}$$

Applying these estimates to both sides of (3.114) gives

$$\|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} \|\phi_j''\|_{L^2} - \|\phi_j''\|_{L^2}^2 \geq -\|\psi_j\|_{L^2} \|\phi_j''\|_{L^2} \tag{3.117}$$

and thus

$$\|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} \|\phi_j''\|_{L^2} + \|\psi_j\|_{L^2} \|\phi_j''\|_{L^2} \geq \|\phi_j''\|_{L^2}^2. \tag{3.118}$$

After diving this inequality by $\|\phi_j''\|_{L^2}$ we get

$$\|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} + \|\psi_j\|_{L^2} \geq \|\phi_j''\|_{L^2}. \tag{3.119}$$

[2] We could not find which theorem the authors refer to as Liouville's theorem. Likely, they mean a theorem that provides a link between the number of zeroes of an eigenfunction and an ordering of the eigenvalues. In particular, the eigenfuction $\phi_0^{\pm}$ having no zeroes would imply that 0 is the smallest eigenvalue, which means that $-\delta$ cannot be an eigenvalue. This is similar to classical Sturm-Liouville theory. However, the operator $\mathcal{L}_0^{\pm}$ does not correspond to a Sturm-Liouville boundary value problem, so this theory cannot be applied here directly.

Squaring both sides gives

$$\left(\|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} + \|\psi_j\|_{L^2}\right)^2 \geq \|\phi_j''\|_{L^2}^2 \tag{3.120}$$

We now further estimate the left hand side using the well known inequality $(x+y)^2 \leq 2x^2 + 2y^2$. So then

$$2\|f_u(u_0)\|_{L^\infty}^2 \|\phi_j\|_{L^2}^2 + 2\|\psi_j\|_{L^2}^2 \geq \|\phi_j''\|_{L^2}^2 \tag{3.121}$$

and we end up with

$$0 \geq \|\phi_j''\|_{L^2}^2 - 2\|f_u(u_0)\|_{L^\infty}^2 \|\phi_j\|_{L^2}^2 - 2\|\psi_j\|_{L^2}^2 . \tag{3.122}$$

This is the first of two inequalities we will need later on in the proof. Now we begin with deriving the second inequality. From the definition of $\mathcal{L}_0^\pm$ it follows that $\pm c_0\phi_j' = \mathcal{L}_0^\pm \phi_j + \phi_j'' - f_u(u_0)\phi_j$. Using this as well as Lemma 1*(i)*, we observe that

$$\begin{aligned}
\pm c_0\|\phi_j'\|_{L^2}^2 &= \left\langle \pm c_0\phi_j' \, , \, \phi_j' \right\rangle \\
&= \left\langle \mathcal{L}_0^\pm \phi_j + (\delta_j\phi_j - \delta_j\phi_j) + \phi_j'' - f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle \\
&= \left\langle \mathcal{L}_0^\pm \phi_j + \delta_j\phi_j \, , \, \phi_j' \right\rangle - \delta_j \left\langle \phi_j \, , \, \phi_j' \right\rangle + \left\langle \phi_j'' \, , \, \phi_j' \right\rangle - \left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle \\
&= \left\langle \psi_j \, , \, \phi_j' \right\rangle - \left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle .
\end{aligned} \tag{3.123}$$

Multiplying this on both sides by $\pm\,\mathrm{sign}(c_0)$ gives

$$|c_0|\,\|\phi_j'\|_{L^2}^2 = \pm\,\mathrm{sign}(c_0)\left\langle \psi_j \, , \, \phi_j' \right\rangle \mp \mathrm{sign}(c_0)\left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle \tag{3.124}$$

and thus

$$|c_0|\,\|\phi_j'\|_{L^2}^2 \mp \mathrm{sign}(c_0)\left\langle \psi_j \, , \, \phi_j' \right\rangle = \mp\,\mathrm{sign}(c_0)\left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle . \tag{3.125}$$

We can estimate both sides of this equality by Cauchy-Schwarz. For estimating the right hand side we also use that $\|f_u(u_0)\phi_j\|_{L^2} \leq \|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2}$, which follows from the fact that $f_u(u_0(x)) \leq \|f_u(u_0)\|_{L^\infty}$ for all $x \in \mathbb{R}$. We find

$$|c_0|\,\|\phi_j'\|_{L^2}^2 \mp \mathrm{sign}(c_0)\left\langle \psi_j \, , \, \phi_j' \right\rangle \geq |c_0|\,\|\phi_j'\|_{L^2}^2 - \|\psi_j\|_{L^2}\|\phi_j'\|_{L^2}, \tag{3.126}$$

$$\mp\,\mathrm{sign}(c_0)\left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle \leq \|f_u(u_0)\|_{L^\infty}\|\phi_j\|_{L^2}\|\phi_j'\|_{L^2}. \tag{3.127}$$

Applying both of these estimates to (3.125) gives

$$|c_0|\,\|\phi_j'\|_{L^2}^2 - \|\psi_j\|_{L^2}\|\phi_j'\|_{L^2} \leq \|f_u(u_0)\|_{L^\infty}\|\phi_j\|_{L^2}\|\phi_j'\|_{L^2}, \tag{3.128}$$

which after dividing by $\|\phi_j'\|$ is equivalent to

$$\begin{aligned}
&|c_0|\,\|\phi_j'\|_{L^2} - \|\psi_j\|_{L^2} \leq \|f_u(u_0)\|_{L^\infty}\|\phi_j\|_{L^2} \\
\iff\ &|c_0|\,\|\phi_j'\|_{L^2} \leq \|f_u(u_0)\|_{L^\infty}\|\phi_j\|_{L^2} + \|\psi_j\|_{L^2} .
\end{aligned} \tag{3.129}$$

Now we square both sides of the inequality and thereafter use the earlier seen inequality $(x+y)^2 \leq 2x^2 + 2y^2$. This gives

$$\begin{aligned}
c_0^2\|\phi_j'\|_{L^2}^2 &\leq \left(\|f_u(u_0)\|_{L^\infty}\|\phi_j\|_{L^2} + \|\psi_j\|_{L^2}\right)^2 \\
&\leq 2\|f_u(u_0)\|_{L^\infty}^2\|\phi_j\|_{L^2}^2 + 2\|\psi_j\|_{L^2}^2
\end{aligned} \tag{3.130}$$

and thus

$$0 \geq c_0^2 \|\phi_j'\|_{L^2}^2 - 2\, \|f_u(u_0)\|_{L^\infty}^2 \, \|\phi_j\|_{L^2}^2 - 2\, \|\psi_j\|_{L^2}^2 \,. \tag{3.131}$$

This is the second inequality that will come in handy later on in proof. Now we introduce the positive constants a and m. We first define

$$a = \frac{1}{2} \min\{f_u(1), f_u(-1)\}\,. \tag{3.132}$$

Note that (A1) implies that $a > 0$. Now we take a positive constant m satisfying

$$a = \min_{|x|\geq m} \{f_u(u_0(x))\}\,. \tag{3.133}$$

This is possible because $f_u(u_0(x))$ converges to $f_u(\pm 1)$ as $x \to \pm\infty$ and is taking values smaller then $\frac{1}{2}\min\{f_u(1), f_u(-1)\}$ for certain values of x. In particular there exist values of x such that $f_u(u_0(x)) = 0$, which follows from $u_0(x)$ taking values in $(-1, 1)$ in combination with (A1).

Recall that our goal is to show that $\phi \neq 0$. Therefore we are seeking a positive lower bound for $\int_{|x|\leq m} \phi^2(x)\, dx$. To obtain such a bound a lot of estimating has to be done.

Using Lemma 1*(i)*, Lemma 1*(ii)* and the fact that $f_u(u_0(x)) \geq -\|f_u(u_0)\|_{L^\infty}$ for all $x \in \mathbb{R}$, we obtain

$$\begin{aligned}
\langle \psi_j\,,\, \phi_j\rangle &= \langle \mathcal{L}_0^{\pm}\phi_j + \delta_j\phi_j\,,\, \phi_j\rangle \\
&= \left\langle \pm c_0\phi_j' - \phi_j'' + f_u(u_0)\phi_j + \delta_j\phi_j\,,\, \phi_j\right\rangle \\
&= \pm c_0 \left\langle \phi_j'\,,\, \phi_j\right\rangle - \left\langle \phi_j''\,,\, \phi_j\right\rangle + \langle f_u(u_0)\phi_j\,,\, \phi_j\rangle + \delta_j \langle \phi_j\,,\, \phi_j\rangle \\
&= -\left\langle \phi_j''\,,\, \phi_j\right\rangle + \langle f_u(u_0)\phi_j\,,\, \phi_j\rangle + \delta_j \|\phi_j\|_{L^2}^2 \\
&\geq \langle f_u(u_0)\phi_j\,,\, \phi_j\rangle \\
&= \int_{\mathbb{R}} f_u(u_0(x))\phi_j^2(x)\, dx \\
&= \int_{|x|\geq m} f_u(u_0(x))\phi_j^2(x)\, dx + \int_{|x|\leq m} f_u(u_0(x))\phi_j^2(x)\, dx \\
&\geq \min_{|x|\geq m}\{f_u(u_0(x))\} \int_{|x|\geq m} \phi_j^2(x)\, dx - \|f_u(u_0)\|_{L^\infty} \int_{|x|\leq m} \phi_j^2(x)\, dx \\
&= a\left(\int_{\mathbb{R}} \phi_j^2(x)\, dx - \int_{|x|\leq m} \phi_j^2(x)\, dx\right) - \|f_u(u_0)\|_{L^\infty} \int_{|x|\leq m} \phi_j^2(x)\, dx \\
&= a\, \|\phi_j\|_{L^2}^2 - (a + \|f_u(u_0)\|_{L^\infty}) \int_{|x|\leq m} \phi_j^2(x)\, dx.
\end{aligned} \tag{3.134}$$

We now make another estimate using both Cauchy-Schwarz and the inequality $-xy \geq -\frac{1}{2}x^2 - \frac{1}{2}y^2$. This inequality follows from the well known inequality $x^2 + y^2 \geq 2xy$. We find

$$\begin{aligned} a \|\phi_j\|_{L^2}^2 - \langle \psi_j \, , \, \phi_j \rangle &\geq a \|\phi_j\|_{L^2}^2 - \|\psi_j\|_{L^2} \|\phi_j\|_{L^2} \\ &= a \|\phi_j\|_{L^2}^2 - \left(\frac{1}{\sqrt{a}} \|\psi_j\|_{L^2} \right) \left(\sqrt{a} \|\phi_j\|_{L^2} \right) \\ &\geq a \|\phi_j\|_{L^2}^2 - \frac{1}{2a} \|\psi_j\|_{L^2}^2 - \frac{a}{2} \|\phi_j\|_{L^2}^2 \\ &= \frac{a}{2} \|\phi_j\|_{L^2}^2 - \frac{1}{2a} \|\psi_j\|_{L^2}^2 \, . \end{aligned} \tag{3.135}$$

We combine the two previous obtained inequalities by first rewriting (3.134) and then applying (3.135) to it. This gives

$$\begin{aligned} (a + \|f_u(u_0)\|_{L^\infty}) \int_{|x| \leq m} \phi_j^2(x) \, dx &\geq a \|\phi_j\|_{L^2}^2 - \langle \psi_j \, , \, \phi_j \rangle \\ &\geq \frac{a}{2} \|\phi_j\|_{L^2}^2 - \frac{1}{2a} \|\psi_j\|_{L^2}^2 \, . \end{aligned} \tag{3.136}$$

It is time to use the two inequalities we derived at the start of the proof. We denote the positive constant $B = c_0^2 + 2(c_0^2+1) \|f_u(u_0)\|_{L^\infty}^2$. We now multiply (3.122) by $\frac{ac_0^2}{2B}$ and (3.131) by $\frac{a}{2B}$. So we get

$$0 \geq \frac{ac_0^2}{2B} \|\phi_j''\|_{L^2}^2 - \frac{ac_0^2 \|f_u(u_0)\|_{L^\infty}^2}{B} \|\phi_j\|_{L^2}^2 - \frac{ac_0^2}{B} \|\psi_j\|_{L^2}^2 \tag{3.137}$$

and

$$0 \geq \frac{ac_0^2}{2B} \|\phi_j'\|_{L^2}^2 - \frac{a \|f_u(u_0)\|_{L^\infty}^2}{B} \|\phi_j\|_{L^2}^2 - \frac{a}{B} \|\psi_j\|_{L^2}^2 . \tag{3.138}$$

Adding both of these to (3.136) gives

$$\begin{aligned} (a + \|f_u(u_0)\|_{L^\infty}) \int_{|x| \leq m} \phi_j^2(x) \, dx \geq \frac{ac_0^2}{2B} &\left(\|\phi_j''\|_{L^2}^2 + \|\phi_j'\|_{L^2}^2 \right) \\ &+ \left(\frac{a}{2} - \frac{a \|f_u(u_0)\|_{L^\infty}^2 (c_0^2+1)}{B} \right) \|\phi_j\|_{L^2}^2 \\ &- \left(\frac{1}{2a} + \frac{a(c_0^2+1)}{B} \right) \|\psi_j\|_{L^2}^2 . \end{aligned} \tag{3.139}$$

Furthermore, we notice that

$$\begin{aligned} \frac{a}{2} - &\frac{a \|f_u(u_0)\|_{L^\infty}^2 (c_0^2+1)}{B} \\ &= \frac{a\big(c_0^2 + 2(c_0^2+1) \|f_u(u_0)\|_{L^\infty}^2\big)}{2B} - \frac{2a \|f_u(u_0)\|_{L^\infty}^2 (c_0^2+1)}{2B} = \frac{ac_0^2}{2B} . \end{aligned} \tag{3.140}$$

We can substitute this into (3.139). To further simplify inequality (3.139) we introduce the positive constants B_4 and B_5. These are defined as

$$B_4 = \frac{ac_0^2}{2B\left(a + \|f_u(u_0)\|_{L^\infty}\right)}, \tag{3.141}$$

$$B_5 = \left(\frac{1}{2a} + \frac{a(c_0^2+1)}{B}\right)\frac{1}{\left(a + \|f_u(u_0)\|_{L^\infty}\right)}. \tag{3.142}$$

Note that B_4 is indeed positive, since $c_0 \neq 0$, shown in Lemma 2. We can now rewrite (3.139) using these constants. We first divide the inequality by $(a + \|f_u(u_0)\|_{L^\infty})$ and thereafter insert the constants B_4 and B_5 where possible. This way we find that

$$\begin{aligned}\int_{|x|\leq m} \phi_j^2(x)\,dx &\geq B_4\Big(\|\phi_j''\|_{L^2}^2 + \|\phi_j'\|_{L^2}^2 + \|\phi_j\|_{L^2}^2\Big) - B_5\|\psi_j\|_{L^2}^2 \\ &= B_4\|\phi_j\|_{H^2}^2 - B_5\|\psi_j\|_{L^2}^2 \\ &= B_4 - B_5\|\psi_j\|_{L^2}^2.\end{aligned} \tag{3.143}$$

Here we used that $\|\phi_j\|_{H^2} = 1$, which we assumed in the beginning of the proof. Recall that $\lim_{j\to\infty}\|\psi_j\|_{L^2} = \hat{\Lambda}^\pm$ and that $\phi_j \to \phi$ in L^2_{loc}. So letting $j \to \infty$ on both sides of the inequality gives

$$\int_{|x|\leq m} \phi^2(x)\,dx \geq B_4 - B_5(\hat{\Lambda}^\pm)^2 = B_4, \tag{3.144}$$

since we assumed that $\hat{\Lambda}^\pm = 0$. Because B_4 is a positive constant it follows that $\phi \neq 0$. But this gives a contradiction since we earlier obtained that $\phi = 0$. Thus we must have that $\hat{\Lambda}^\pm > 0$.

So we know there exists a positive constant C_2 such that $\hat{\Lambda}^\pm \geq \frac{1}{C_2}$. We now let $\delta \in [\delta_0, \delta_1]$ and $\phi \in H^2(\mathbb{R})$ be arbitrary and we denote $\psi = \mathcal{L}_0^\pm\phi + \delta\phi$. Since we have $\|\frac{\phi}{\|\phi\|_{H^2}}\|_{H^2} = 1$, it follows from the definition of $\hat{\Lambda}^\pm$ that

$$\hat{\Lambda}^\pm \leq \left\|\mathcal{L}_0^\pm\frac{\phi}{\|\phi\|_{H^2}} + \delta\frac{\phi}{\|\phi\|_{H^2}}\right\|_{L^2} = \frac{\left\|\mathcal{L}_0^\pm\phi + \delta\phi\right\|_{L^2}}{\|\phi\|_{H^2}} = \frac{\|\psi\|_{L^2}}{\|\phi\|_{H^2}}. \tag{3.145}$$

Dividing by $\hat{\Lambda}^\pm$ and multiplying by $\|\phi\|_{H^2}$ gives

$$\|\phi\|_{H^2} \leq \frac{1}{\hat{\Lambda}^\pm}\|\psi\|_{L^2}. \tag{3.146}$$

After applying $\hat{\Lambda}^\pm \geq \frac{1}{C_2}$, we obtain

$$\|\phi\|_{H^2} \leq C_2\|\psi\|_{L^2} \tag{3.147}$$

and thus also

$$\|\phi\|_{H^2} \leq C_2\left\{\frac{1}{\delta}\left|\langle\psi\,,\,\phi_0^\mp\rangle\right| + \|\psi\|_{L^2}\right\} \tag{3.148}$$

concluding the proof.

□

3.2 Linearization $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ of the discretized Allen-Cahn equation

Now we'll study the operator $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ as defined in (2.32). We do this by also using the information we gained about $\mathcal{L}^{\pm}_{0}$ in the previous section.

Let us first introduce two new quantities. We define $\Lambda^{\pm}(\varepsilon,\delta)$ and $\Lambda^{\pm}(\delta)$ for every $\varepsilon > 0$ and for every $\delta > 0$. These are given by

$$\Lambda^{\pm}(\varepsilon,\delta) = \inf_{\|\phi\|_{H^1}=1} \left\{ \left\| \mathcal{L}^{\pm}_{\varepsilon,\delta}\phi \right\|_{L^2} + \frac{1}{\delta} \left| \left\langle \mathcal{L}^{\pm}_{\varepsilon,\delta}\phi \, , \, \phi_0^{\mp} \right\rangle \right| \right\}, \tag{3.149}$$

$$\Lambda^{\pm}(\delta) = \liminf_{\varepsilon \downarrow 0} \Lambda^{\pm}(\varepsilon,\delta). \tag{3.150}$$

Before stating the main lemma of this section we treat some key properties of $\mathcal{L}^{\pm}_{\varepsilon,\delta}$. These will help us proving Lemma 9 right after.

Lemma 8. *Let $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ be as in* (2.32)*. Then*

(i) $\mathcal{L}^{\pm}_{\varepsilon,\delta} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ *is a bounded operator;*

(ii) for any $\phi \in L^2(\mathbb{R})$ and any $\zeta \in C_0^{\infty}(\mathbb{R}) \cap H^1(\mathbb{R})$, we have $\left\langle \mathcal{L}^{\pm}_{\varepsilon,\delta}\phi \, , \, \zeta \right\rangle = \left\langle \phi \, , \, \mathcal{L}^{\mp}_{\varepsilon,\delta}\zeta \right\rangle$.

Proof. *(i)* Let $\phi \in H^1(\mathbb{R})$. Since we want to show boundedness, we need an upper bound for $\|\mathcal{L}_{\varepsilon,\delta}\phi\|_{L^2}$ of the form $A\|\phi\|_{H^1}$, with A being a positive constant. To find such an upper bound we estimate

$$\begin{aligned} \|\mathcal{L}_{\varepsilon,\delta}\phi\|_{L^2} &= \|\pm c_0\phi' - \Delta_\varepsilon\phi + f_u(u_0)\phi + \delta\phi\|_{L^2} \\ &\leq \|\pm c_0\phi'\|_{L^2} + \|\Delta_\varepsilon\phi\|_{L^2} + \|f_u(u_0)\phi\|_{L^2} + \|\delta\phi\|_{L^2} \end{aligned} \tag{3.151}$$

Now since $f_u(u_0(x)) \leq \|f_u(u_0)\|_{L^\infty}$ for all $x \in \mathbb{R}$, we get

$$\begin{aligned} \|\mathcal{L}_{\varepsilon,\delta}\phi\|_{L^2} &\leq |c_0|\,\|\phi'\|_{L^2} + \|\Delta_\varepsilon\phi\|_{L^2} + (\|f_u(u_0)\|_{L^\infty} + \delta)\,\|\phi\|_{L^2} \\ &\leq \|\Delta_\varepsilon\phi\|_{L^2} + (|c_0| + \|f_u(u_0)\|_{L^\infty} + \delta)\,\|\phi\|_{H^1} \end{aligned} \tag{3.152}$$

The upper bound is almost in the right form. Only the term $\|\Delta_\varepsilon\phi\|_{L^2}$ is still a problem. Estimating $\|\Delta_\varepsilon\phi\|_{L^2}$ gives

$$\begin{aligned} \|\Delta_\varepsilon\phi\|_{L^2} &= \left\| \frac{1}{\varepsilon^2}\sum_{k>0} \alpha_k \left[\phi(x-k\varepsilon) + \phi(x-k\varepsilon) - 2\phi(x)\right] \right\|_{L^2} \\ &\leq \frac{1}{\varepsilon^2}\sum_{k>0} |\alpha_k| \left(\|\phi(x-k\varepsilon)\|_{L^2} + \|\phi(x-k\varepsilon)\|_{L^2} + 2\,\|\phi(x)\|_{L^2} \right) \\ &= \frac{4}{\varepsilon^2}\sum_{k>0} |\alpha_k|\,\|\phi\|_{L^2} \\ &\leq \frac{4}{\varepsilon^2}\sum_{k>0} |\alpha_k|\,\|\phi\|_{H^1}. \end{aligned} \tag{3.153}$$

Since $\sum_{k>0} |\alpha_k| < \infty$ by (A2), we can choose

$$A = \frac{4}{\varepsilon^2} \sum_{k>0} |\alpha_k| + |c_0| + \|f_u(u_0)\|_{L^\infty} + \delta, \tag{3.154}$$

satisfying $\|\mathcal{L}_{\varepsilon,\delta}\phi\|_{L^2} \le A \|\phi\|_{H^1}$. So we can conclude that $\mathcal{L}_{\varepsilon,\delta}$ is a bounded operator from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$.

(ii) Let $\phi \in L^2(\mathbb{R})$ and $\zeta \in C_0^\infty(\mathbb{R}) \cap H^1(\mathbb{R})$. Then

$$\begin{aligned} \left\langle \mathcal{L}^{\pm}_{\varepsilon,\delta}\phi \,,\, \zeta \right\rangle &= \langle \pm c_0 \phi' - \Delta_\varepsilon \phi + f_u(u_0)\phi + \delta\phi \,,\, \zeta \rangle \\ &= \pm c_0 \langle \phi' \,,\, \zeta \rangle - \langle \Delta_\varepsilon \phi \,,\, \zeta \rangle + \langle f_u(u_0)\phi \,,\, \zeta \rangle + \langle \delta\phi \,,\, \zeta \rangle \,. \end{aligned} \tag{3.155}$$

It is easy to see that $\langle f_u(u_0)\phi \,,\, \zeta \rangle = \langle \phi \,,\, f_u(u_0)\zeta \rangle$ and that $\langle \delta\phi \,,\, \zeta \rangle = \langle \phi \,,\, \delta\zeta \rangle$. By Lemma 4*(iii)* we have that $\langle \Delta_\varepsilon \phi \,,\, \zeta \rangle = \langle \phi \,,\, \Delta_\varepsilon \zeta \rangle$. Using integrating by parts we find

$$\begin{aligned} \langle \phi' \,,\, \zeta \rangle = \int_{\mathbb{R}} \phi'(x)\zeta(x)\,dx &= [\phi(x)\zeta(x)]_{-\infty}^{\infty} - \int_{\mathbb{R}} \phi(x)\zeta'(x)\,dx \\ &= [\phi(x)\zeta(x)]_{-\infty}^{\infty} - \langle \phi \,,\, \zeta' \rangle \,. \end{aligned} \tag{3.156}$$

Since $\zeta \in C_0^\infty(\mathbb{R})$, we have $\lim_{x\to\pm\infty} \zeta(x) = 0$. This causes $[\phi(x)\zeta(x)]_{-\infty}^{\infty}$ to vanish. Thus we get that $\langle \phi' \,,\, \zeta \rangle = -\langle \phi \,,\, \zeta' \rangle$. Now we can rewrite the final expression in (3.155). This gives

$$\begin{aligned} \left\langle \mathcal{L}^{\pm}_{\varepsilon,\delta}\phi \,,\, \zeta \right\rangle &= \mp c_0 \langle \phi \,,\, \zeta' \rangle - \langle \phi \,,\, \Delta_\varepsilon \zeta \rangle + \langle \phi \,,\, f_u(u_0)\zeta \rangle + \langle \phi \,,\, \delta\zeta \rangle \\ &= \langle \phi \,,\, \mp c_0 \zeta' - \Delta_\varepsilon \zeta + f_u(u_0)\zeta + \delta\zeta \rangle \\ &= \left\langle \phi \,,\, \mathcal{L}^{\mp}_{\varepsilon,\delta}\zeta \right\rangle . \end{aligned} \tag{3.157}$$

□

Lemma 9. *There exists a positive constant C_0 such that $\Lambda^{\pm}(\delta) \ge \frac{1}{C_0}$ for all $\delta > 0$.*

Proof. First of all we remark that the proof of this statement will be very similar to part (c) of the proof of Lemma 7. But, for completeness, we will yet show the proof.

Let $\delta > 0$ be any positive fixed constant. We take two sequences with certain properties, namely $\{\varepsilon_j\}_{j=0}^{\infty}$ and $\{\phi_j\}_{j=0}^{\infty}$. We let $\{\varepsilon_j\}_{j=0}^{\infty}$ be a sequence with $\varepsilon_j \in (0,1)$ for all j, where $\lim_{j\to\infty} \varepsilon_j = 0$. For our other sequence $\{\phi_j\}_{j=0}^{\infty}$, we let every $\phi_j \in H^1(\mathbb{R})$ such that $\|\phi_j\|_{H^1} = 1$. By the definition of $\Lambda^{\pm}(\delta)$ there exist such sequences $\{\varepsilon_j\}_{j=0}^{\infty}$ and $\{\phi_j\}_{j=0}^{\infty}$ that satisfy

$$\lim_{j\to\infty} \left\{ \|\psi_j\|_{L^2} + \frac{1}{\delta} \left| \langle \psi_j \,,\, \phi_0^{\mp} \rangle \right| \right\} = \Lambda^{\pm}(\delta), \quad \text{where } \psi_j = \mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j. \tag{3.158}$$

From this it follows that $\limsup_{j\to\infty} \|\psi_j\|_{L^2} \le \Lambda^{\pm}(\delta)$. Thus $\{\psi_j\}_{j=0}^{\infty}$ is a bounded sequence in $L^2(\mathbb{R})$. Since $\|\phi_j\|_{H^1} = 1$ for all j, we have that $\{\phi_j\}_{j=0}^{\infty}$ is also a bounded sequence, but in $H^1(\mathbb{R})$. Now Theorem A1 can be applied to both sequences. It tells us that, by taking subsequences if necessary, there exist functions $\psi \in L^2(\mathbb{R})$ and $\phi \in H^1(\mathbb{R})$ such that

$$\psi_j \to \psi \text{ in } L^2(\mathbb{R}) \text{ weakly, as } j \to \infty, \tag{3.159}$$

$$\phi_j \to \phi \text{ in } H^1(\mathbb{R}) \text{ weakly, as } j \to \infty. \tag{3.160}$$

Furthermore Theorem A2 implies that, by taking another subsequence if necessary, we have

$$\phi_j \to \phi \text{ in } L^2_{loc}(\mathbb{R}), \text{ as } j \to \infty. \tag{3.161}$$

By the definition of weak convergence, see Definition A1, and by the continuity of the absolute value we have

$$\frac{1}{\delta}\left|\left\langle \psi\, ,\, \phi_0^{\mp}\right\rangle\right| = \lim_{j\to\infty}\frac{1}{\delta}\left|\left\langle \psi_j\, ,\, \phi_0^{\mp}\right\rangle\right|. \tag{3.162}$$

Together with Theorem A3, this gives that

$$\|\psi\|_{L^2} + \frac{1}{\delta}\left|\left\langle \psi\, ,\, \phi_0^{\mp}\right\rangle\right| \le \liminf_{j\to\infty}\left\{\|\psi_j\|_{L^2} + \frac{1}{\delta}\left|\left\langle \psi_j\, ,\, \phi_0^{\mp}\right\rangle\right|\right\} = \Lambda^{\pm}(\delta). \tag{3.163}$$

Now we take a test function $\zeta \in C_0^\infty(\mathbb{R}) \cap H^2(\mathbb{R})$. From Lemma 8*(ii)* it follows that

$$\langle \psi_j\, ,\, \zeta\rangle = \left\langle \mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j\, ,\, \zeta\right\rangle = \left\langle \phi_j\, ,\, \mathcal{L}^{\mp}_{\varepsilon_j,\delta}\zeta\right\rangle. \tag{3.164}$$

Now we let $j \to \infty$ on both sides of the equation. On the left hand side it's quite easy to see what happens when we let $j \to \infty$. Since $\psi_j \to \psi$ weakly in $L^2(\mathbb{R})$, we know that $\langle \psi_j\, ,\, \zeta\rangle$ converges to $\langle \psi\, ,\, \zeta\rangle$. On the right hand side things are a bit more difficult. Writing out gives

$$\begin{aligned}\left\langle \phi_j\, ,\, \mathcal{L}^{\mp}_{\varepsilon_j,\delta}\zeta\right\rangle &= \langle \phi_j\, ,\, \mp c_0\zeta' - \Delta_{\varepsilon_j}\zeta + f_u(u_0)\zeta + \delta\zeta\rangle \\ &= \mp c_0\langle \phi_j\, ,\, \zeta'\rangle - \left\langle \phi_j\, ,\, \Delta_{\varepsilon_j}\zeta\right\rangle + \langle \phi_j\, ,\, f_u(u_0)\zeta\rangle + \langle \phi_j\, ,\, \delta\zeta\rangle.\end{aligned} \tag{3.165}$$

Now we examine what happens with each of the terms if we send $j \to \infty$. Since ϕ_j converges to ϕ weakly in $H^1(\mathbb{R})$, it immediately follows that $\langle \phi_j\, ,\, \zeta'\rangle \to \langle \phi\, ,\, \zeta'\rangle$, $\langle \phi_j\, ,\, f_u(u_0)\zeta\rangle \to \langle \phi\, ,\, f_u(u_0)\zeta\rangle$ and $\langle \phi_j\, ,\, \delta\zeta\rangle \to \langle \phi\, ,\, \delta\zeta\rangle$. By Lemma 4*(i)* we have $\lim_{j\to\infty}\left\|\Delta_{\varepsilon_j}\zeta - \zeta''\right\|_{L^2} = 0$. Thus Theorem A4 implies that $\left\langle \phi_j\, ,\, \Delta_{\varepsilon_j}\zeta\right\rangle \to \langle \phi\, ,\, \zeta''\rangle$ as $j \to \infty$. So we find

$$\begin{aligned}\lim_{j\to\infty}\left\langle \phi_j\, ,\, \mathcal{L}^{\mp}_{\varepsilon_j,\delta}\zeta\right\rangle &= \mp c_0\langle \phi\, ,\, \zeta'\rangle - \langle \phi\, ,\, \zeta''\rangle + \langle \phi\, ,\, f_u(u_0)\zeta\rangle + \langle \phi\, ,\, \delta\zeta\rangle \\ &= \langle \phi\, ,\, \mp c_0\zeta' - \zeta'' + f_u(u_0)\zeta + \delta\zeta\rangle \\ &= \left\langle \phi\, ,\, \left(\mathcal{L}_0^{\mp} + \delta\right)\zeta\right\rangle.\end{aligned} \tag{3.166}$$

So we conclude that letting $j \to \infty$ on both sides of (3.164) gives

$$\langle \psi\, ,\, \zeta\rangle = \left\langle \phi\, ,\, \left(\mathcal{L}_0^{\mp} + \delta\right)\zeta\right\rangle = \left\langle \left(\mathcal{L}_0^{\pm} + \delta\right)\phi\, ,\, \zeta\right\rangle. \tag{3.167}$$

Since this holds for every $\zeta \in C_0^\infty(\mathbb{R}) \cap H^2(\mathbb{R})$, we have $\psi = \left(\mathcal{L}_0^{\pm} + \delta\right)\phi = \mathcal{L}_0^{\pm}\phi + \delta\phi$. This is one of the requirements to be able to apply Lemma 7 to ϕ. Now writing out $\mathcal{L}_0^{\pm}\phi$ in this expression enables us to write $\phi'' = \pm c_0\phi' + f_u(u_0)\phi + \delta\phi - \psi$. Each term on the right hand side is part of $L^2(\mathbb{R})$. So $\phi'' \in L^2(\mathbb{R})$. Thus also $\phi \in H^2(\mathbb{R})$, as we assumed earlier that

$\phi \in H^1(\mathbb{R})$. We have shown that all requirements to be able to apply Lemma 7 on ϕ hold. Using Lemma 7 and (3.163) we get

$$\|\phi\|_{H^2} \leq C_2 \left\{ \frac{1}{\delta} \left| \left\langle \psi \, , \, \phi_0^{\mp} \right\rangle \right| + \|\psi\|_{L^2} \right\} \leq C_2 \Lambda^{\pm}(\delta). \tag{3.168}$$

It remains to find a lower bound for $\|\phi\|_{L^2}$. Directly from the definition of $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ we find that $\pm c_0 \phi_j' = \mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j + \Delta_{\varepsilon_j}\phi_j - f_u(u_0)\phi_j - \delta\phi_j$. Together with Lemma 1*(i)* and Lemma 4*(ii)* this implies

$$\begin{aligned} \pm c_0 \|\phi_j'\|_{L^2}^2 &= \left\langle \pm c_0 \phi_j' \, , \, \phi_j' \right\rangle \\ &= \left\langle \mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j + \Delta_{\varepsilon_j}\phi_j - f_u(u_0)\phi_j - \delta\phi_j \, , \, \phi_j' \right\rangle \\ &= \left\langle \mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j \, , \, \phi_j' \right\rangle + \left\langle \Delta_{\varepsilon_j}\phi_j \, , \, \phi_j' \right\rangle - \left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle - \delta \left\langle \phi_j \, , \, \phi_j' \right\rangle \\ &= \left\langle \psi_j \, , \, \phi_j' \right\rangle - \left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle . \end{aligned} \tag{3.169}$$

Multiplying this on both sides by $\pm\,\text{sign}(c_0)$ gives

$$|c_0| \, \|\phi_j'\|_{L^2}^2 = \pm\,\text{sign}(c_0) \left\langle \psi_j \, , \, \phi_j' \right\rangle \mp \text{sign}(c_0) \left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle \tag{3.170}$$

$$\Longleftrightarrow \; |c_0| \, \|\phi_j'\|_{L^2}^2 \mp \text{sign}(c_0) \left\langle \psi_j \, , \, \phi_j' \right\rangle = \mp\,\text{sign}(c_0) \left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle . \tag{3.171}$$

We can estimate both sides of this equality by Cauchy-Schwarz. For estimating the right hand side we again use $\|f_u(u_0)\phi_j\|_{L^2} \leq \|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2}$ as seen in multiple proofs before. This gives

$$|c_0| \, \|\phi_j'\|_{L^2}^2 \mp \text{sign}(c_0) \left\langle \psi_j \, , \, \phi_j' \right\rangle \geq |c_0| \, \|\phi_j'\|_{L^2}^2 - \|\psi_j\|_{L^2} \|\phi_j'\|_{L^2}, \tag{3.172}$$

$$\mp\,\text{sign}(c_0) \left\langle f_u(u_0)\phi_j \, , \, \phi_j' \right\rangle \leq \|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} \|\phi_j'\|_{L^2}. \tag{3.173}$$

If we apply both of these estimates to (3.171) we obtain

$$|c_0| \, \|\phi_j'\|_{L^2}^2 - \|\psi_j\|_{L^2} \|\phi_j'\|_{L^2} \leq \|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} \|\phi_j'\|_{L^2}, \tag{3.174}$$

which after dividing by $\|\phi_j'\|$ is equivalent to

$$\begin{aligned} |c_0| \, \|\phi_j'\|_{L^2} - \|\psi_j\|_{L^2} &\leq \|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} \\ \Longleftrightarrow \; |c_0| \, \|\phi_j'\|_{L^2} &\leq \|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} + \|\psi_j\|_{L^2} . \end{aligned} \tag{3.175}$$

Now squaring both sides of the inequality and thereafter applying the well known inequality $(x+y)^2 \leq 2x^2 + 2y^2$ gives

$$\begin{aligned} c_0^2 \|\phi_j'\|_{L^2}^2 &\leq \left(\|f_u(u_0)\|_{L^\infty} \|\phi_j\|_{L^2} + \|\psi_j\|_{L^2} \right)^2 \\ &\leq 2 \, \|f_u(u_0)\|_{L^\infty}^2 \|\phi_j\|_{L^2}^2 + 2 \, \|\psi_j\|_{L^2}^2 . \end{aligned} \tag{3.176}$$

We will need this inequality later in the proof. We define

$$a = \frac{1}{2} \min \left\{ f_u(1), f_u(-1) \right\} . \tag{3.177}$$

Then by (A1) we notice that $a > 0$. Now we take a positive constant m satisfying

$$a = \min_{|x|\geq m} \{f_u(u_0(x))\}. \tag{3.178}$$

This is possible because $f_u(u_0(x))$ converges to $f_u(\pm 1)$ as $x \to \pm\infty$ and is taking values smaller then $\frac{1}{2}\min\{f_u(1), f_u(-1)\}$ for certain values of x. In particular there exist values of x such that $f_u(u_0(x)) = 0$, which follows from $u_0(x)$ taking values in $(-1, 1)$ in combination with (A1). Using Lemma 1*(i)*, Lemma 4*(iv)* and the fact that $f_u(u_0(x)) \geq -\|f_u(u_0)\|_{L^\infty}$ for all $x \in \mathbb{R}$, we obtain

$$\begin{aligned}
\langle \psi_j\,,\,\phi_j\rangle &= \langle \mathcal{L}^{\pm}_{\varepsilon_j,\delta}\phi_j\,,\,\phi_j\rangle \\
&= \left\langle \pm c_0\phi_j' - \Delta_{\varepsilon_j}\phi_j + f_u(u_0)\phi_j + \delta\phi_j\,,\,\phi_j\right\rangle \\
&= \pm c_0\left\langle \phi_j'\,,\,\phi_j\right\rangle - \left\langle \Delta_{\varepsilon_j}\phi_j\,,\,\phi_j\right\rangle + \langle f_u(u_0)\phi_j\,,\,\phi_j\rangle + \delta\langle\phi_j\,,\,\phi_j\rangle \\
&= -\left\langle \Delta_{\varepsilon_j}\phi_j\,,\,\phi_j\right\rangle + \langle f_u(u_0)\phi_j\,,\,\phi_j\rangle + \delta\|\phi_j\|^2_{L^2} \\
&\geq \langle f_u(u_0)\phi_j\,,\,\phi_j\rangle \\
&= \int_{\mathbb{R}} f_u(u_0(x))\phi_j^2(x)\,dx \\
&= \int_{|x|\geq m} f_u(u_0(x))\phi_j^2(x)\,dx + \int_{|x|\leq m} f_u(u_0(x))\phi_j^2(x)\,dx \\
&\geq \min_{|x|\geq m}\{f_u(u_0(x))\}\int_{|x|\geq m}\phi_j^2(x)\,dx - \|f_u(u_0)\|_{L^\infty}\int_{|x|\leq m}\phi_j^2(x)\,dx \\
&= a\left(\int_{\mathbb{R}}\phi_j^2(x)\,dx - \int_{|x|\leq m}\phi_j^2(x)\,dx\right) - \|f_u(u_0)\|_{L^\infty}\int_{|x|\leq m}\phi_j^2(x)\,dx \\
&= a\|\phi_j\|^2_{L^2} - (a + \|f_u(u_0)\|_{L^\infty})\int_{|x|\leq m}\phi_j^2(x)\,dx.
\end{aligned} \tag{3.179}$$

We now make another estimate using Cauchy-Schwarz and the inequality $-xy \geq -\frac{1}{2}x^2 - \frac{1}{2}y^2$, which follows from the well known inequality $x^2 + y^2 \geq 2xy$. We find

$$\begin{aligned}
a\|\phi_j\|^2_{L^2} - \langle\psi_j\,,\,\phi_j\rangle &\geq a\|\phi_j\|^2_{L^2} - \|\psi_j\|_{L^2}\|\phi_j\|_{L^2} \\
&= a\|\phi_j\|^2_{L^2} - \left(\frac{1}{\sqrt{a}}\|\psi_j\|_{L^2}\right)\left(\sqrt{a}\|\phi_j\|_{L^2}\right) \\
&\geq a\|\phi_j\|^2_{L^2} - \frac{1}{2a}\|\psi_j\|^2_{L^2} - \frac{a}{2}\|\phi_j\|^2_{L^2} \\
&= \frac{a}{2}\|\phi_j\|^2_{L^2} - \frac{1}{2a}\|\psi_j\|^2_{L^2}.
\end{aligned} \tag{3.180}$$

We'll combine the two previous derived inequalities by rewriting (3.179) and applying (3.180) to this rewritten inequality. This gives

$$\begin{aligned}
(a + \|f_u(u_0)\|_{L^\infty})\int_{|x|\leq m}\phi_j^2(x)\,dx &\geq a\|\phi_j\|^2_{L^2} - \langle\psi_j\,,\,\phi_j\rangle \\
&\geq \frac{a}{2}\|\phi_j\|^2_{L^2} - \frac{1}{2a}\|\psi_j\|^2_{L^2}.
\end{aligned} \tag{3.181}$$

If we now multiply (3.176) by $\frac{a}{2(2\|f_u(u_0)\|_{L^\infty}^2+c_0^2)}$, we get

$$\frac{a\left\|f_u(u_0)\right\|_{L^\infty}^2\left\|\phi_j\right\|_{L^2}^2}{2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2}+\frac{a\left\|\psi_j\right\|_{L^2}^2}{2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2}\geq\frac{ac_0^2\|\phi_j'\|_{L^2}^2}{2\big(2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2\big)}. \tag{3.182}$$

So we have

$$0\geq\frac{ac_0^2\|\phi_j'\|_{L^2}^2}{2\big(2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2\big)}-\frac{a\left\|f_u(u_0)\right\|_{L^\infty}^2\left\|\phi_j\right\|_{L^2}^2}{2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2}-\frac{a\left\|\psi_j\right\|_{L^2}^2}{2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2}. \tag{3.183}$$

Adding this to (3.181) gives

$$\begin{aligned}(a+\|f_u(u_0)\|_{L^\infty})\int_{|x|\leq m}\phi_j^2(x)\,dx\geq\ &\frac{a}{2}\left(1-\frac{\|f_u(u_0)\|_{L^\infty}^2}{\|f_u(u_0)\|_{L^\infty}^2+c_0^2/2}\right)\|\phi_j\|_{L^2}^2\\&+\frac{ac_0^2}{2\big(2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2\big)}\|\phi_j'\|_{L^2}^2\\&-\left(\frac{1}{2a}+\frac{a}{2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2}\right)\|\psi_j\|_{L^2}^2\,.\end{aligned} \tag{3.184}$$

Furthermore, we notice that

$$\begin{aligned}\frac{a}{2}\left(1-\frac{\|f_u(u_0)\|_{L^\infty}^2}{\|f_u(u_0)\|_{L^\infty}^2+c_0^2/2}\right)&=\frac{a}{2}\left(\frac{\|f_u(u_0)\|_{L^\infty}^2+c_0^2/2-\|f_u(u_0)\|_{L^\infty}^2}{\|f_u(u_0)\|_{L^\infty}^2+c_0^2/2}\right)\\&=\frac{ac_0^2}{2\big(2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2\big)}.\end{aligned} \tag{3.185}$$

To simplify inequality (3.184) we introduce the positive constants C_3 and C_4. These are defined as

$$C_3=\frac{ac_0^2}{2\big(2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2\big)\left(a+\|f_u(u_0)\|_{L^\infty}\right)}\ , \tag{3.186}$$

$$C_4=\left(\frac{1}{2a}+\frac{a}{2\left\|f_u(u_0)\right\|_{L^\infty}^2+c_0^2}\right)\frac{1}{(a+\|f_u(u_0)\|_{L^\infty})}. \tag{3.187}$$

We are now able to rewrite (3.184) in a nice way. We first divide inequality (3.184) by $(a+\|f_u(u_0)\|_{L^\infty})$ and afterwards insert the just defined constants C_3 and C_4 where possible. After executing these operations we find

$$\begin{aligned}\int_{|x|\leq m}\phi_j^2(x)\,dx&\geq C_3\left(\|\phi_j\|_{L^2}^2+\|\phi_j'\|_{L^2}^2\right)-C_4\left\|\psi_j\right\|_{L^2}^2\\&=C_3\left\|\phi_j\right\|_{H^1}^2-C_4\left\|\psi_j\right\|_{L^2}^2\\&=C_3-C_4\left\|\psi_j\right\|_{L^2}^2\,.\end{aligned} \tag{3.188}$$

Here we used that $\|\phi_j\|_{H^1}=1$, which we assumed at the start of the proof. Recall that $\phi_j\to\phi$ in $L^2_{loc}(\mathbb{R})$. From the definition of convergence in $L^2_{loc}(\mathbb{R})$, see Definition A2, it follows that letting $j \to \infty$ on both sides of the inequality gives

$$\int_{|x|\le m} \phi^2(x)\, dx \ge C_3 - C_4 \lim_{j\to\infty} \|\psi_j\|_{L^2}^2 . \tag{3.189}$$

Because $\frac{1}{\delta}|\langle \psi \,,\, \phi_0^{\mp}\rangle|$ is nonnegative, it follows from (3.158) that $\lim_{j\to\infty} \|\psi_j\|_{L^2}^2 \le \Lambda^{\pm}(\delta)^2$. If we apply this, we get that

$$\int_{|x|\le m} \phi^2(x)\, dx \ge C_3 - C_4 \Lambda^{\pm}(\delta)^2. \tag{3.190}$$

From (3.168) it follows that $\|\phi\|_{H^2}^2 \le C_2^2 \Lambda^{\pm}(\delta)^2$. We can connect this to the inequality above. We find

$$C_2^2 \Lambda^{\pm}(\delta)^2 \ge \|\phi\|_{H^2}^2 \ge \|\phi\|_{L^2}^2 = \int_{\mathbb{R}} \phi^2(x)\, dx \ge \int_{|x|\le m} \phi^2(x)\, dx \ge C_3 - C_4 \Lambda^{\pm}(\delta)^2. \tag{3.191}$$

Rewriting this gives

$$\begin{aligned} \left(C_2^2 + C_4\right) \Lambda^{\pm}(\delta)^2 &\ge C_3 \\ \iff \qquad \Lambda^{\pm}(\delta)^2 &\ge \frac{C_3}{C_2^2 + C_4}. \end{aligned} \tag{3.192}$$

Now defining $\frac{1}{C_0} = \left(\frac{C_3}{C_2^2 + C_4}\right)^{1/2}$ gives that $\Lambda^{\pm}(\delta) \ge \frac{1}{C_0}$ concluding the proof. □

Remark. For this proof to make sense it is important that $c_0 \neq 0$, which has been proven in Lemma 2. Because in the case that $c_0 = 0$ we would only have shown that $\Lambda^{\pm}(\delta) \ge 0$, since in that case $C_3 = 0$. But this result immediately follows from the definition of $\Lambda^{\pm}(\delta)$ and clearly is too weak to advance in proving Theorem 2.

With this theorem we managed to use the knowledge from the continuous case and connect this to the discrete case. Now we are ready to prove Proposition 1 as stated on page 22.

3.3 Proof of Proposition 1

Proof. Let $\delta > 0$ be arbitrary. From Lemma 9 we know there exists a positive constant C_0 not depending on δ such that $\Lambda^{\pm}(\delta) \ge \frac{1}{C_0}$. By the definition of $\Lambda^{\pm}(\delta)$ there has to exist a positive constant $\varepsilon_0(\delta)$ such that for all $\varepsilon \in (0, \varepsilon_0(\delta))$ we have $\Lambda^{\pm}(\varepsilon, \delta) \ge \frac{1}{C_0}$. Now we consider the operator $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ for $\varepsilon \in (0, \varepsilon_0(\delta))$.

Before showing $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is a homeomorphism from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$, we will first prove that we have a homeomorphism when $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ maps from $H^1(\mathbb{R})$ to its image $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R}))$. Surjectivity is easily given, since we let $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ map to its image. Furthermore, since Lemma 8*(i)* implies that $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is bounded, it follows from Theorem A5 that $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is continuous.

For injectivity we show that $\ker(\mathcal{L}^{\pm}_{\varepsilon,\delta}) = \{0\}$. We first take $\phi \in H^1(\mathbb{R})$ with $\|\phi\|_{H^1} = 1$. Assuming $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi = 0$ then gives $\Lambda^{\pm}(\varepsilon, \delta) = 0$. This is impossible because we observed earlier that $\Lambda^{\pm}(\varepsilon, \delta) \ge \frac{1}{C_0}$. So we must have $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi \neq 0$ in the case of $\|\phi\|_{H^1} = 1$. Now we can go to the general case taking $\phi \in H^1(\mathbb{R})$ with $\|\phi\|_{H^1} = A > 0$. Using the linearity of $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ we get $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi = \mathcal{L}^{\pm}_{\varepsilon,\delta}(A\frac{\phi}{A}) = A\mathcal{L}^{\pm}_{\varepsilon,\delta}(\frac{\phi}{A})$. Since $\|\frac{\phi}{A}\|_{H^1} = 1$, we have $\mathcal{L}^{\pm}_{\varepsilon,\delta}(\frac{\phi}{A}) \neq 0$ and thus also $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi \neq 0$. So we indeed have $\ker(\mathcal{L}^{\pm}_{\varepsilon,\delta}) = \{0\}$ and thus $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is injective.

To show that the inverse of $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is also continuous we use Theorem A5. It tells us the inverse of $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is continuous if there exists a constant $K > 0$ such that

$$\sup_{\psi \in \mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R})),\, \|\psi\|_{L^2} \leq 1} \left\| \left(\mathcal{L}^{\pm}_{\varepsilon,\delta}\right)^{-1} \psi \right\|_{H^1} = K. \tag{3.193}$$

So our goal is to find a constant $K > 0$ such that this holds. First we let $\psi \in \mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R}))$ and assume $0 < \|\psi\|_{L^2} \leq 1$. For better readability we denote $\phi = (\mathcal{L}^{\pm}_{\varepsilon,\delta})^{-1}\psi$, from which we can derive $\psi = \mathcal{L}^{\pm}_{\varepsilon,\delta}\phi$. Since $\|\frac{\phi}{\|\phi\|_{H^1}}\|_{H^1} = 1$, we have, by the definition of $\Lambda^{\pm}(\varepsilon,\delta)$, that

$$\begin{aligned} \Lambda^{\pm}(\varepsilon,\delta) &\leq \left\| \mathcal{L}^{\pm}_{\varepsilon,\delta} \frac{\phi}{\|\phi\|_{H^1}} \right\|_{L^2} + \frac{1}{\delta} \left| \left\langle \mathcal{L}^{\pm}_{\varepsilon,\delta} \frac{\phi}{\|\phi\|_{H^1}} \,,\, \phi_0^{\mp} \right\rangle \right| \\ &= \frac{1}{\|\phi\|_{H^1}} \|\psi\|_{L^2} + \frac{1}{\delta \|\phi\|_{H^1}} \left| \langle \psi \,,\, \phi_0^{\mp} \rangle \right|. \end{aligned} \tag{3.194}$$

Dividing by $\Lambda^{\pm}(\varepsilon,\delta)$ and multiplying by $\|\phi\|_{H^1}$ gives

$$\|\phi\|_{H^1} \leq \frac{1}{\Lambda^{\pm}(\varepsilon,\delta)} \left(\|\psi\|_{L^2} + \frac{1}{\delta} \left| \langle \psi \,,\, \phi_0^{\mp} \rangle \right| \right). \tag{3.195}$$

Now applying the inequality $\Lambda^{\pm}(\varepsilon,\delta) \geq \frac{1}{C_0}$ we obtain

$$\|\phi\|_{H^1} \leq C_0 \left(\|\psi\|_{L^2} + \frac{1}{\delta} \left| \langle \psi \,,\, \phi_0^{\mp} \rangle \right| \right), \tag{3.196}$$

which is one of the results in Proposition 1 indicated by (3.2). From this we can also derive (3.3), because after assuming $\psi \perp \phi_0^{\mp}$ it immediately follows that

$$\|\phi\|_{H^1} \leq C_0 \|\psi\|_{L^2}. \tag{3.197}$$

After this little detour we now get back to showing that (3.193) holds for some $K > 0$. We apply the inequality $\|\psi\|_{L^2} \leq 1$, Lemma 6*(i)* and Cauchy-Schwarz to (3.196). This gives

$$\begin{aligned} \|\phi\|_{H^1} &\leq C_0 \left(\|\psi\|_{L^2} + \frac{1}{\delta} \|\psi\|_{L^2} \|\phi_0^{\mp}\|_{L^2} \right) \\ &\leq C_0 \left(1 + \frac{1}{\delta} \right). \end{aligned} \tag{3.198}$$

So we have found that $\|(\mathcal{L}^{\pm}_{\varepsilon,\delta})^{-1}\psi\|_{H^1} = \|\phi\|_{H^1}$ is bounded above by a constant when assuming $\psi \in \mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R}))$ and $\|\psi\|_{L^2} \leq 1$. So it follows that there has to exist a constant $K > 0$ such that (3.193) holds. Thus we can conclude that the inverse of $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is continuous.

So we have now shown that the operator $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ mapping functions from $H^1(\mathbb{R})$ to its image $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R}))$ satisfies all the properties of a homeomorphism. We are done if we can show that $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R})) = L^2(\mathbb{R})$. We claim that this is the case and we'll prove it by contradiction.

So we first assume $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R})) \neq L^2(\mathbb{R})$. We have just shown that $H^1(\mathbb{R})$ and $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R}))$ are homeomorphic. Because $H^1(\mathbb{R})$ is complete, it follows from Theorem A9 that $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R}))$ is complete as well. Lemma 8*(i)* tells us that $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R})) \subset L^2(\mathbb{R})$. So we can use Theorem A10 to conclude that $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R}))$ must be closed in $L^2(\mathbb{R})$.

Now Theorem A6 implies there exists a nontrivial $\psi \in L^2(\mathbb{R})$ being orthogonal to $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R}))$. So by Lemma 8*(ii)* we have $0 = \langle \mathcal{L}^{\pm}_{\varepsilon,\delta}\phi \,,\, \psi\rangle = \langle \phi \,,\, \mathcal{L}^{\mp}_{\varepsilon,\delta}\psi\rangle$ for all $\phi \in H^1(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$. Writing this out we find

$$\begin{aligned} 0 = \langle \phi \,,\, \mathcal{L}^{\mp}_{\varepsilon,\delta}\psi\rangle &= \langle \phi \,,\, \mp c_0\psi' - \Delta_\varepsilon\psi + f_u(u_0)\psi + \delta\psi\rangle \\ &= \mp c_0 \langle \phi \,,\, \psi'\rangle + \langle \phi \,,\, -\Delta_\varepsilon\psi + f_u(u_0)\psi + \delta\psi\rangle \,. \end{aligned} \tag{3.199}$$

Using integration by parts it can be found, in the same way as in the proof of Lemma 8*(ii)*, that $\langle \phi \,,\, \psi'\rangle = -\langle \phi' \,,\, \psi\rangle$. After applying this, we do some rewriting to find

$$\langle \phi' \,,\, \psi\rangle = -\Big\langle \phi, \; \frac{1}{\mp c_0}\left(\Delta_\varepsilon\psi - f_u(u_0)\psi - \delta\psi\right)\Big\rangle. \tag{3.200}$$

So we know the weak derivative of ψ, see Definition A4, and we will denote it as ψ_w. Hence we have $\psi_w = \frac{1}{\mp c_0}\left(\Delta_\varepsilon\psi - f_u(u_0)\psi - \delta\psi\right)$. In the proof of Lemma 8*(ii)* we showed that $\|f_u(u_0)\psi\|_{L^2} \leq \|f_u(u_0)\|_{L^\infty}\|\psi\|_{L^2}$ and that $\|\Delta_\varepsilon\psi\|_{L^2} \leq \frac{4}{\varepsilon^2}\sum_{k>0}|\alpha_k|\,\|\psi\|_{L^2}$. Using these we find that

$$\begin{aligned} \|\psi_w\|_{L^2} &= \frac{1}{\mp c_0}\|\Delta_\varepsilon\psi - f_u(u_0)\psi - \delta\psi\|_{L^2} \\ &\leq \frac{1}{\mp c_0}\left(\|\Delta_\varepsilon\psi\|_{L^2} + \|f_u(u_0)\psi\|_{L^2} + \|\delta\psi\|_{L^2}\right) \\ &\leq \frac{1}{\mp c_0}\left(\frac{4}{\varepsilon^2}\sum_{k>0}|\alpha_k| + \|f_u(u_0)\|_{L^\infty} + \delta\right)\|\psi\|_{L^2} \,. \end{aligned} \tag{3.201}$$

Notice that $\sum_{k>0}|\alpha_k| < \infty$ by (A2). Since we also have that $\psi \in L^2(\mathbb{R})$, it follows that $\|\psi_w\|_{L^2} < \infty$ and thus $\psi_w \in L^2(\mathbb{R})$. A more formal definition for $H^1(\mathbb{R})$ states that a function f lies in $H^1(\mathbb{R})$ if both f and its weak derivative lie in $L^2(\mathbb{R})$. Because this is the case for ψ, we can conclude that $\psi \in H^1(\mathbb{R})$ and therefore $\mathcal{L}^{\pm}_{\varepsilon,\delta}\psi \in L^2(\mathbb{R})$.

Now we take $\zeta \in L^2(\mathbb{R})$ arbitrary. Since $H^1(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exists a sequence $\{\zeta_n\}_{n=0}^\infty$ in $H^1(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$ such that $\zeta_n \to \zeta$ as $n \to \infty$. Using this together with the earlier obtained fact that $\langle \phi \,,\, \mathcal{L}^{\mp}_{\varepsilon,\delta}\psi\rangle = 0$ for all $\phi \in H^1(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$, we find that

$$\langle \zeta \,,\, \mathcal{L}^{\mp}_{\varepsilon,\delta}\psi\rangle = \lim_{n\to\infty}\langle \zeta_n \,,\, \mathcal{L}^{\mp}_{\varepsilon,\delta}\psi\rangle = \lim_{n\to\infty} 0 = 0. \tag{3.202}$$

This tells us that $\mathcal{L}^{\mp}_{\varepsilon,\delta}\psi$ is orthogonal to every function in $L^2(\mathbb{R})$. Since $\mathcal{L}^{\mp}_{\varepsilon,\delta}\psi \in L^2(\mathbb{R})$, it must be that $\mathcal{L}^{\mp}_{\varepsilon,\delta}\psi = 0$. From the injectivity of $\mathcal{L}^{\mp}_{\varepsilon,\delta}$ it follows that $\psi = 0$. So ψ is indeed trivial and we have a contradiction. Thus we must have $\mathcal{L}^{\pm}_{\varepsilon,\delta}(H^1(\mathbb{R})) = L^2(\mathbb{R})$. We can now conclude that $\mathcal{L}^{\pm}_{\varepsilon,\delta}$ is a homeomorphism from $H^1(\mathbb{R})$ to $L^2(\mathbb{R})$. □

3.4 Proof of Theorem 2

Before proving Theorem 2 we do some last preparations. During these preparations we want to find bounds for some norms involving operator $\mathcal{R}(c,\phi)$, where $\mathcal{R}(c,\phi)$ is as in (2.33). These bounds will turn out to be very helpful when proving Theorem 2. To obtain these, a handful of lemmas is treated. But first we introduce a few things.

We define

$$\mathbf{X}_\eta = \left\{\phi \in H^1(\mathbb{R}) \mid \ \|\phi\|_{H^1} \leq \eta\right\}. \tag{3.203}$$

For every $\phi \in \mathbf{X}_\eta$, we choose $c_\varepsilon = c_\varepsilon(\phi)$ such that $\mathcal{R}(c_\varepsilon,\phi) \perp \phi_0^-$. So we have

$$\begin{aligned}\langle \mathcal{R}(c_\varepsilon,\phi)\,,\,\phi_0^-\rangle = \ & c_0\langle u_0' + \phi'\,,\,\phi_0^-\rangle - c_\varepsilon\langle u_0'+\phi'\,,\,\phi_0^-\rangle \\ & + \langle \Delta_\varepsilon u_0 - u_0''\,,\,\phi_0^-\rangle + \delta\langle\phi\,,\,\phi_0^-\rangle - \langle \mathbf{N}(u_0,\phi)\,,\,\phi_0^-\rangle = 0.\end{aligned} \tag{3.204}$$

After rewriting this we find that

$$c_\varepsilon(\phi) = c_0 + \frac{\langle \Delta_\varepsilon u_0 - u_0''\,,\,\phi_0^-\rangle + \delta\langle\phi\,,\,\phi_0^-\rangle - \langle \mathbf{N}(u_0,\phi)\,,\,\phi_0^-\rangle}{\langle u_0'+\phi'\,,\,\phi_0^-\rangle}. \tag{3.205}$$

Furthermore we let $\hat\sigma$ be defined as $\hat\sigma = \frac{1}{2}\left\langle u_0'\,,\,\phi_0^-\right\rangle$. Writing this out gives

$$\hat\sigma = \frac{1}{2}\left\langle u_0'\,,\,\phi_0^-\right\rangle = \int_{\mathbb{R}} \frac{[u_0'(x)]^2\, e^{-c_0x}}{2\,\|u_0'(x)e^{-c_0x}\|_{L^2}}\,dx = \frac{\int_{\mathbb{R}} [u_0'(x)]^2\, e^{-c_0x}\,dx}{2\left(\int_{\mathbb{R}} [u_0'(x)]^2\, e^{-2c_0x}\right)^{1/2} dx} > 0. \tag{3.206}$$

So $\hat\sigma$ is a positive constant. Using Cauchy-Schwarz and Lemma 6*(i)* we obtain

$$|\langle\phi'\,,\,\phi_0^-\rangle| \leq \|\phi'\|_{L^2}\,\|\phi_0^-\|_{L^2} \leq \|\phi\|_{H^1} \leq \eta \quad \text{for every } \phi \in \mathbf{X}_\eta. \tag{3.207}$$

We use this and from now on require $\eta \leq \hat\sigma$. It follows that

$$\langle u_0'+\phi'\,,\,\phi_0^-\rangle = \langle u_0'\,,\,\phi_0^-\rangle + \langle\phi'\,,\,\phi_0^-\rangle = 2\hat\sigma + \langle\phi'\,,\,\phi_0^-\rangle \geq 2\hat\sigma - \eta \geq \hat\sigma \quad \text{for every } \phi \in \mathbf{X}_\eta. \tag{3.208}$$

This property of $\hat\sigma$ will be helpful when deriving some of the estimates in the following lemmas.

Lemma 10. *Let* $\mathbf{N}(u_0,\phi)$ *be defined as in* (2.34) *and require* $\eta \leq \hat\sigma$. *Then there exists a positive constant* M *such that*

$$|\mathbf{N}(u_0,\phi)| \leq M\eta\,|\phi| \quad \textit{and} \quad |\mathbf{N}(u_0,\phi_1) - \mathbf{N}(u_0,\phi_2)| \leq M\eta\,|\phi_1-\phi_2| \tag{3.209}$$

pointwise for all $\phi,\phi_1,\phi_2 \in \mathbf{X}_\eta$.

Proof. We let $\phi,\phi_1,\phi_2 \in \mathbf{X}_\eta$ arbitrary and we require $\eta \leq \hat\sigma$. From Theorem A7 it follows that there exists a positive constant a such that $\|\phi\|_{L^\infty} \leq a\,\|\phi\|_{H^1}$, $\|\phi_1\|_{L^\infty} \leq a\,\|\phi_1\|_{H^1}$ and $\|\phi_2\|_{L^\infty} \leq a\,\|\phi_2\|_{H^1}$. These inequalities will be used quite frequently in this proof. Furthermore we can, without loss of generality, assume that $a \geq 2$. This will turn out to be quite helpful.

Before we estimate the nonlinear term $\mathbf{N}(u_0,\phi)$, we first rewrite $f(u_0+\phi)$ pointwise using Taylor's theorem. It tells us that

$$f(u_0+\phi) = f(u_0) + f_u(u_0)\phi + \frac{1}{2}f_{uu}(t)\phi^2 \quad \text{with } t \text{ lying between } u_0 \text{ and } \phi. \tag{3.210}$$

So $|t|$ can be estimated by $|t| \leq |u_0| + |\phi|$. Theorem 1 implies that $\|u_0\|_{L^\infty} \leq 1$. This allows us to further bound $|t|$. We get

$$|t| \leq |u_0| + |\phi| \leq \|u_0\|_{L^\infty} + \|\phi\|_{L^\infty} \leq 1 + a\,\|\phi\|_{H^1} \leq 1 + a\eta \leq 1 + a\hat{\sigma}. \tag{3.211}$$

We define $\widetilde{M} = \sup_{|s|\leq 1+3a\hat{\sigma}} |f_{uu}(s)|$ and set $M = a\widetilde{M}$. Notice that $|f_{uu}(t)| \leq \widetilde{M}$. We are now ready to estimate $\mathbf{N}(u_0, \phi)$. Using (3.210) we find that

$$\begin{aligned} |\mathbf{N}(u_0,\phi)| &= |f(u_0+\phi) - f(u_0) - f_u(u_0)\phi| = \frac{1}{2}\left|f_{uu}(t)\phi^2\right| \leq \left|f_{uu}(t)\phi^2\right| \\ &\leq |f_{uu}(t)|\,\|\phi\|_{L^\infty}\,|\phi| \leq a\,|f_{uu}(t)|\,\|\phi\|_{H^1}\,|\phi| \leq a\widetilde{M}\eta\,|\phi| = M\eta\,|\phi|\,. \end{aligned} \tag{3.212}$$

It remains to estimate the distance between two nonlinear terms. Using Taylor's theorem we can write that

$$f_u(u_0+\phi_2) = f_u(u_0) + f_{uu}(t_2)\phi_2 \quad \text{pointwise with } t_2 \text{ lying between } u_0 \text{ and } \phi_2. \tag{3.213}$$

We also rewrite $f(u_0 + \phi_1)$ pointwise using Taylor's theorem. It states that there exists a constant t_1 lying between $u_0 + \phi_2$ and $\phi_1 - \phi_2$ such that

$$f(u_0+\phi_1) = f(u_0+\phi_2) + f_u(u_0+\phi_2)(\phi_1-\phi_2) + \frac{1}{2}f_{uu}(t_1)(\phi_1-\phi_2)^2. \tag{3.214}$$

Replacing $f_u(u_0 + \phi_2)$ by (3.213) gives

$$f(u_0+\phi_1) = f(u_0+\phi_2) + (f_u(u_0) + f_{uu}(t_2)\phi_2)(\phi_1-\phi_2) + \frac{1}{2}f_{uu}(t_1)(\phi_1-\phi_2)^2. \tag{3.215}$$

We estimate the difference of the nonlinear terms by replacing $f(u_0+\phi_1)$ with (3.215). This gives

$$\begin{aligned} |\mathbf{N}(u_0,\phi_1) - \mathbf{N}(u_0,\phi_2)| &= |f(u_0+\phi_1) - f(u_0) - f_u(u_0)\phi_1 - f(u_0+\phi_2) + f(u_0) + f_u(u_0)\phi_2| \\ &= |f_{uu}(t_2)\phi_2(\phi_1-\phi_2) + \frac{1}{2}f_{uu}(t_1)(\phi_1-\phi_2)^2| \\ &\leq |f_{uu}(t_2)||\phi_2||\phi_1-\phi_2| + \frac{1}{2}|f_{uu}(t_1)||\phi_1-\phi_2|^2. \end{aligned} \tag{3.216}$$

This bound is not the one we are after and thus we want to bound it even further. Since $|t_2|$ can be bounded in the same way as $|t|$ in (3.211), it follows that $|f_{uu}(t_2)| \leq \widetilde{M}$. $|t_1|$ can be estimated in a similar way. We have

$$\begin{aligned} |t_1| &\leq |u_0+\phi_2| + |\phi_1-\phi_2| \leq 1 + |\phi_1| + 2\,|\phi_2| \leq 1 + \|\phi_1\|_{L^\infty} + 2\,\|\phi_2\|_{L^\infty} \\ &\leq 1 + a\,\|\phi_1\|_{H^1} + 2a\,\|\phi_2\|_{H^1} \leq 1 + 3a\eta \leq 1 + 3a\hat{\sigma}. \end{aligned} \tag{3.217}$$

So we also have $|f_{uu}(t_1)| \leq \widetilde{M}$. As estimated many times before in this proof we have $|\phi_1| \leq \eta$ and $|\phi_2| \leq \eta$. Using these we can bound $|\phi_1 - \phi_2|^2$. We get

$$\begin{aligned} |\phi_1-\phi_2|^2 &= |(\phi_1-\phi_2)^2| = |\phi_1^2 - \phi_1\phi_2 + \phi_2^2 - \phi_1\phi_2| \\ &\leq |\phi_1^2 - \phi_1\phi_2| + |\phi_2^2 - \phi_1\phi_2| = |\phi_1||\phi_1-\phi_2| + |\phi_2||\phi_2-\phi_1| \leq 2\eta|\phi_1-\phi_2|. \end{aligned} \tag{3.218}$$

Combining all these observations we can further bound (3.216). We also use the assumption $a \geq 2$ that was made in the beginning of the proof. We find that

$$|\mathbf{N}(u_0, \phi_1) - \mathbf{N}(u_0, \phi_2)| \leq \widetilde{M}\eta|\phi_1 - \phi_2| + \widetilde{M}\eta|\phi_1 - \phi_2| \leq a\widetilde{M}\eta|\phi_1 - \phi_2| = M\eta|\phi_1 - \phi_2| \tag{3.219}$$

concluding the proof. □

Corollary 2. *Let $\mathbf{N}(u_0, \phi)$ be defined as in (2.34) and require $\eta \leq \hat{\sigma}$. Then there exists a positive constant M such that*

$$\|\mathbf{N}(u_0, \phi)\|_{L^2} \leq M\eta^2 \quad \text{and} \quad \|\mathbf{N}(u_0, \phi_2) - \mathbf{N}(u_0, \phi_1)\|_{L^2} \leq M\eta \|\phi_1 - \phi_2\|_{H^1} \tag{3.220}$$

for all $\phi, \phi_1, \phi_2 \in \mathbf{X}_\eta$.

Proof. Let M be the positive constant from Lemma 10 and let $\phi, \phi_1, \phi_2 \in \mathbf{X}_\eta$ be arbitrary. Both results follow almost immediately from Lemma 10. It implies that

$$\begin{aligned} \|\mathbf{N}(u_0, \phi)\|_{L^2} &= \left(\int_{\mathbb{R}} |\mathbf{N}(u_0, \phi)|^2 \, dx\right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} M^2\eta^2\phi(x)^2 \, dx\right)^{1/2} = M\eta \|\phi\|_{L^2} \leq M\eta^2 \end{aligned} \tag{3.221}$$

and that

$$\begin{aligned} \|\mathbf{N}(u_0, \phi_2) - \mathbf{N}(u_0, \phi_1)\|_{L^2} &= \left(\int_{\mathbb{R}} |\mathbf{N}(u_0, \phi_2) - \mathbf{N}(u_0, \phi_1)|^2 \, dx\right)^{1/2} \\ &\leq \left(\int_{\mathbb{R}} M^2\eta^2 |\phi_1(x) - \phi_2(x)|^2 \, dx\right)^{1/2} \\ &= M\eta \|\phi_1 - \phi_2\|_{L^2} \\ &\leq M\eta \|\phi_1 - \phi_2\|_{H^1}. \end{aligned} \tag{3.222}$$

□

Lemma 11. *Let $c_\varepsilon(\phi)$ be defined as in (3.205) and require $\eta \leq \hat{\sigma}$. Then there exists a positive constant M such that the inequalities*

$$|c_\varepsilon(\phi) - c_0| \leq \frac{1}{\hat{\sigma}}\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\delta + M\eta)\eta\Big) \tag{3.223}$$

and

$$|c_\varepsilon(\phi_1) - c_\varepsilon(\phi_2)| \leq \|\phi_1 - \phi_2\|_{H^1} \frac{2}{\hat{\sigma}^2}\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\hat{\sigma} + \eta)(\delta + M\eta)\Big) \tag{3.224}$$

hold for all $\phi, \phi_1, \phi_2 \in \mathbf{X}_\eta$.

Proof. Let M be the positive constant from Lemma 10 and let $\phi, \phi_1, \phi_2 \in \mathbf{X}_\eta$ be arbitrary. From (3.208) and Lemma 6*(i)* it follows that

$$\begin{aligned}
|c_\varepsilon(\phi) - c_0| &= \left| \frac{\langle \Delta_\varepsilon u_0 - u_0'' \, , \, \phi_0^- \rangle + \delta \langle \phi \, , \, \phi_0^- \rangle - \langle \mathbf{N}(u_0, \phi) \, , \, \phi_0^- \rangle}{\langle u_0' + \phi' \, , \, \phi_0^- \rangle} \right| \\
&\leq \frac{1}{\hat{\sigma}} \Big(\left| \langle \Delta_\varepsilon u_0 - u_0'' \, , \, \phi_0^- \rangle \right| + \delta \left| \langle \phi \, , \, \phi_0^- \rangle \right| + \left| \langle \mathbf{N}(u_0, \phi) \, , \, \phi_0^- \rangle \right| \Big) \\
&\leq \frac{1}{\hat{\sigma}} \Big(\| \Delta_\varepsilon u_0 - u_0'' \|_{L^2} \| \phi_0^- \|_{L^2} + \delta \| \phi \|_{L^2} \| \phi_0^- \|_{L^2} + \| \mathbf{N}(u_0, \phi) \|_{L^2} \| \phi_0^- \|_{L^2} \Big) \\
&\leq \frac{1}{\hat{\sigma}} \Big(\| \Delta_\varepsilon u_0 - u_0'' \|_{L^2} + \delta \eta + \| \mathbf{N}(u_0, \phi) \|_{L^2} \Big).
\end{aligned} \tag{3.225}$$

Using the estimate of $\| \mathbf{N}(u_0, \phi) \|_{L^2}$ from Corollary 2 we find that

$$|c_\varepsilon(\phi) - c_0| \leq \frac{1}{\hat{\sigma}} \Big(\| \Delta_\varepsilon u_0 - u_0'' \|_{L^2} + (\delta + M\eta)\eta \Big). \tag{3.226}$$

The first part of the proof is now completed. So we can start with the second part, in which we estimate the difference between $c_\varepsilon(\phi_1)$ and $c_\varepsilon(\phi_2)$. For readability reasons we introduce $\mathcal{D}(\phi)$ as

$$\mathcal{D}(\phi) = \langle \Delta_\varepsilon u_0 - u_0'' \, , \, \phi_0^- \rangle + \delta \langle \phi \, , \, \phi_0^- \rangle - \langle \mathbf{N}(u_0, \phi) \, , \, \phi_0^- \rangle. \tag{3.227}$$

Rewriting the difference between $c_\varepsilon(\phi_1)$ and $c_\varepsilon(\phi_2)$ as one fraction gives

$$\begin{aligned}
|c_\varepsilon(\phi_1) - c_\varepsilon(\phi_2)| &= \left| \frac{\mathcal{D}(\phi_1)}{\langle u_0' + \phi_1' \, , \, \phi_0^- \rangle} - \frac{\mathcal{D}(\phi_2)}{\langle u_0' + \phi_2' \, , \, \phi_0^- \rangle} \right| \\
&= \left| \frac{\langle u_0' + \phi_2' \, , \, \phi_0^- \rangle \mathcal{D}(\phi_1) - \langle u_0' + \phi_1' \, , \, \phi_0^- \rangle \mathcal{D}(\phi_2)}{\langle u_0' + \phi_1' \, , \, \phi_0^- \rangle \langle u_0' + \phi_2' \, , \, \phi_0^- \rangle} \right|.
\end{aligned} \tag{3.228}$$

Now it follows from (3.208) that

$$|c_\varepsilon(\phi_1) - c_\varepsilon(\phi_2)| \leq \frac{1}{\hat{\sigma}^2} \left| \langle u_0' + \phi_2' \, , \, \phi_0^- \rangle \mathcal{D}(\phi_1) - \langle u_0' + \phi_1' \, , \, \phi_0^- \rangle \mathcal{D}(\phi_2) \right|. \tag{3.229}$$

Now we take $\left|\langle u_0' + \phi_2' , \phi_0^- \rangle \mathcal{D}(\phi_1) - \langle u_0' + \phi_1' , \phi_0^- \rangle \mathcal{D}(\phi_2)\right|$ and write it out in a handy form such that we can estimate it. We compute

$$
\begin{aligned}
&\left|\langle u_0' + \phi_2' , \phi_0^- \rangle \mathcal{D}(\phi_1) - \langle u_0' + \phi_1' , \phi_0^- \rangle \mathcal{D}(\phi_2)\right| \\
&\quad= \Big| \left[\langle u_0' , \phi_0^- \rangle + \langle \phi_2' , \phi_0^- \rangle\right] \left[\langle \Delta_\varepsilon u_0 - u_0'' , \phi_0^- \rangle + \delta \langle \phi_1 , \phi_0^- \rangle - \langle \mathbf{N}(u_0, \phi_1) , \phi_0^- \rangle\right] \\
&\qquad - \left[\langle u_0' , \phi_0^- \rangle + \langle \phi_1' , \phi_0^- \rangle\right] \left[\langle \Delta_\varepsilon u_0 - u_0'' , \phi_0^- \rangle + \delta \langle \phi_2 , \phi_0^- \rangle - \langle \mathbf{N}(u_0, \phi_2) , \phi_0^- \rangle\right] \Big| \\
&\quad= \Big| \langle u_0' , \phi_0^- \rangle \Big(\delta \langle \phi_1 - \phi_2 , \phi_0^- \rangle + \big\langle \mathbf{N}(u_0, \phi_2) - \mathbf{N}(u_0, \phi_1) , \phi_0^- \big\rangle \Big) \\
&\qquad + \langle \Delta_\varepsilon u_0 - u_0'' , \phi_0^- \rangle \langle \phi_2' - \phi_1' , \phi_0^- \rangle \\
&\qquad + \delta \Big(\langle \phi_2' , \phi_0^- \rangle \langle \phi_1 , \phi_0^- \rangle - \langle \phi_1' , \phi_0^- \rangle \langle \phi_2 , \phi_0^- \rangle \Big) \\
&\qquad - \langle \phi_2' , \phi_0^- \rangle \langle \mathbf{N}(u_0, \phi_1) , \phi_0^- \rangle + \langle \phi_1' , \phi_0^- \rangle \langle \mathbf{N}(u_0, \phi_2) , \phi_0^- \rangle \Big| \\
&\quad\le \Big| \langle u_0' , \phi_0^- \rangle \Big(\delta \langle \phi_1 - \phi_2 , \phi_0^- \rangle + \big\langle \mathbf{N}(u_0, \phi_2) - \mathbf{N}(u_0, \phi_1) , \phi_0^- \big\rangle \Big) \Big| \\
&\qquad + \left| \langle \Delta_\varepsilon u_0 - u_0'' , \phi_0^- \rangle \langle \phi_2' - \phi_1' , \phi_0^- \rangle \right| \\
&\qquad + \Big| \delta \Big(\langle \phi_2' , \phi_0^- \rangle \langle \phi_1 , \phi_0^- \rangle - \langle \phi_1' , \phi_0^- \rangle \langle \phi_2 , \phi_0^- \rangle \Big) \Big| \\
&\qquad + \left| \langle \phi_1' , \phi_0^- \rangle \langle \mathbf{N}(u_0, \phi_2) , \phi_0^- \rangle - \langle \phi_2' , \phi_0^- \rangle \langle \mathbf{N}(u_0, \phi_1) , \phi_0^- \rangle \right| .
\end{aligned}
\tag{3.230}
$$

We want to bound this further. This will be done by treating each term in absolute value separately. Lemma 6*(i)*, implying that $\|\phi_0^-\|_{L^2} = 1$, will be used a lot of times in this part of the proof.

To estimate the first term we use the estimate of $\|\mathbf{N}(u_0, \phi_2) - \mathbf{N}(u_0, \phi_1)\|_{L^2}$ from Corollary 2. Furthermore we recall the definition of $\hat{\sigma}$, which implies $\langle u_0' , \phi_0^- \rangle = 2\hat{\sigma}$. We find that

$$
\begin{aligned}
&\Big| \langle u_0' , \phi_0^- \rangle \Big(\delta \langle \phi_1 - \phi_2 , \phi_0^- \rangle + \big\langle \mathbf{N}(u_0, \phi_2) - \mathbf{N}(u_0, \phi_1) , \phi_0^- \big\rangle \Big) \Big| \\
&\quad\le \delta \left| \langle u_0' , \phi_0^- \rangle \right| \left| \langle \phi_1 - \phi_2 , \phi_0^- \rangle \right| + \left| \langle u_0' , \phi_0^- \rangle \right| \left| \big\langle \mathbf{N}(u_0, \phi_2) - \mathbf{N}(u_0, \phi_1) , \phi_0^- \big\rangle \right| \\
&\quad\le 2\delta\hat{\sigma} \|\phi_1 - \phi_2\|_{L^2} \|\phi_0^-\|_{L^2} + 2\hat{\sigma} \|\mathbf{N}(u_0, \phi_2) - \mathbf{N}(u_0, \phi_1)\|_{L^2} \|\phi_0^-\|_{L^2} \\
&\quad\le 2\delta\hat{\sigma} \|\phi_1 - \phi_2\|_{H^1} + 2\hat{\sigma} M \eta \|\phi_1 - \phi_2\|_{H^1} \\
&\quad= 2\hat{\sigma} \|\phi_1 - \phi_2\|_{H^1} \big(\delta + M\eta\big).
\end{aligned}
\tag{3.231}
$$

The first term has been estimated, so we move on to the second term. We estimate

$$
\begin{aligned}
\left| \langle \Delta_\varepsilon u_0 - u_0'' , \phi_0^- \rangle \langle \phi_2' - \phi_1' , \phi_0^- \rangle \right| &\le \|\Delta_\varepsilon u_0 - u_0''\|_{L^2} \|\phi_0^-\|_{L^2} \|\phi_2' - \phi_1'\|_{L^2} \|\phi_0^-\|_{L^2} \\
&\le \|\Delta_\varepsilon u_0 - u_0''\|_{L^2} \|\phi_2 - \phi_1\|_{H^1} \\
&\le 2 \|\Delta_\varepsilon u_0 - u_0''\|_{L^2} \|\phi_2 - \phi_1\|_{H^1} .
\end{aligned}
\tag{3.232}
$$

In the final step the expression is doubled. This isn't necessarily needed, but it will turn out that it fits in very nicely with the estimations of the other terms. Now we estimate the third term. This gives

$$
\begin{aligned}
&\Big|\delta\Big(\langle\phi_2',\,\phi_0^-\rangle\,\langle\phi_1,\,\phi_0^-\rangle-\langle\phi_1',\,\phi_0^-\rangle\langle\phi_2,\,\phi_0^-\rangle\Big)\Big|\\
&\qquad=\delta\Big|\langle\phi_2',\,\phi_0^-\rangle\langle\phi_1,\,\phi_0^-\rangle-\langle\phi_2',\,\phi_0^-\rangle\langle\phi_2,\,\phi_0^-\rangle+\langle\phi_2',\,\phi_0^-\rangle\langle\phi_2,\,\phi_0^-\rangle-\langle\phi_1',\,\phi_0^-\rangle\langle\phi_2,\,\phi_0^-\rangle\Big|\\
&\qquad=\delta\Big|\langle\phi_2',\,\phi_0^-\rangle\langle\phi_1-\phi_2,\,\phi_0^-\rangle+\langle\phi_2,\,\phi_0^-\rangle\langle\phi_2'-\phi_1',\,\phi_0^-\rangle\Big|\\
&\qquad\le\delta\left|\langle\phi_2',\,\phi_0^-\rangle\langle\phi_1-\phi_2,\,\phi_0^-\rangle\right|+\delta\left|\langle\phi_2,\,\phi_0^-\rangle\langle\phi_2'-\phi_1',\,\phi_0^-\rangle\right|\\
&\qquad\le\delta\Big(\|\phi_2'\|_{L^2}\,\|\phi_0^-\|_{L^2}\,\|\phi_1-\phi_2\|_{L^2}\,\|\phi_0^-\|_{L^2}+\|\phi_2\|_{L^2}\,\|\phi_0^-\|_{L^2}\,\|\phi_1'-\phi_2'\|_{L^2}\,\|\phi_0^-\|_{L^2}\Big)\\
&\qquad\le\delta\Big(\|\phi_2\|_{H^1}\,\|\phi_1-\phi_2\|_{H^1}+\|\phi_2\|_{H^1}\,\|\phi_1-\phi_2\|_{H^1}\Big)\\
&\qquad=2\delta\,\|\phi_2\|_{H^1}\,\|\phi_1-\phi_2\|_{H^1}\\
&\qquad\le2\delta\eta\,\|\phi_1-\phi_2\|_{H^1}\,.
\end{aligned}
\tag{3.233}
$$

Only the final term is yet to be estimated. For this we use the estimates from Corollary 2. We find that

$$
\begin{aligned}
&\Big|\langle\phi_1',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_2)\,,\,\phi_0^-\rangle-\langle\phi_2',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_1)\,,\,\phi_0^-\rangle\Big|\\
&\quad=\Big|\langle\phi_1',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_2)\,,\,\phi_0^-\rangle-\langle\phi_1',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_1)\,,\,\phi_0^-\rangle\\
&\qquad+\langle\phi_1',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_1)\,,\,\phi_0^-\rangle-\langle\phi_2',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_1)\,,\,\phi_0^-\rangle\Big|\\
&\quad=\Big|\langle\phi_1',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_2)-\mathbf{N}(u_0,\phi_1)\,,\,\phi_0^-\rangle+\langle\phi_1'-\phi_2',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_2)\,,\,\phi_0^-\rangle\Big|\\
&\quad\le\left|\langle\phi_1',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_2)-\mathbf{N}(u_0,\phi_1)\,,\,\phi_0^-\rangle\right|+\left|\langle\phi_1'-\phi_2',\,\phi_0^-\rangle\langle\mathbf{N}(u_0,\phi_2)\,,\,\phi_0^-\rangle\right|\\
&\quad\le\|\phi_1'\|_{L^2}\,\|\phi_0^-\|_{L^2}\,\|\mathbf{N}(u_0,\phi_2)-\mathbf{N}(u_0,\phi_1)\|_{L^2}\,\|\phi_0^-\|_{L^2}+\|\phi_1'-\phi_2'\|_{L^2}\,\|\phi_0^-\|_{L^2}\,\|\mathbf{N}(u_0,\phi_2)\|_{L^2}\,\|\phi_0^-\|_{L^2}\\
&\quad\le\|\phi_1\|_{H^1}\,M\eta\,\|\phi_1-\phi_2\|_{H^1}+\|\phi_1-\phi_2\|_{H^1}\,M\eta^2\\
&\quad\le2M\eta^2\,\|\phi_1-\phi_2\|_{H^1}\,.
\end{aligned}
\tag{3.234}
$$

We have found an estimation for each of the four terms. Thus we are now able to further bound the expression we were left with at (3.230). Using the four estimations we find that

$$
\begin{aligned}
&\Big|\langle u_0'+\phi_2',\,\phi_0^-\rangle\mathcal{D}(\phi_1)-\langle u_0'+\phi_1',\,\phi_0^-\rangle\mathcal{D}(\phi_2)\Big|\\
&\qquad\le2\hat{\sigma}\,\|\phi_1-\phi_2\|_{H^1}\,\big(\delta+M\eta\big)+2\,\|\Delta_\varepsilon u_0-u_0''\|_{L^2}\,\|\phi_2-\phi_1\|_{H^1}\\
&\qquad\quad+2\delta\eta\,\|\phi_1-\phi_2\|_{H^1}+2M\eta^2\,\|\phi_1-\phi_2\|_{H^1} \qquad (3.235)\\
&\qquad=2\,\|\phi_1-\phi_2\|_{H^1}\,\Big(\|\Delta_\varepsilon u_0-u_0''\|_{L^2}+\hat{\sigma}\,(\delta+M\eta)+\delta\eta+M\eta^2\Big)\\
&\qquad=2\,\|\phi_1-\phi_2\|_{H^1}\,\Big(\|\Delta_\varepsilon u_0-u_0''\|_{L^2}+(\hat{\sigma}+\eta)\,(\delta+M\eta)\Big).
\end{aligned}
$$

So it now follows from (3.229) that

$$|c_\varepsilon(\phi_1) - c_\varepsilon(\phi_2)| \le \|\phi_1 - \phi_2\|_{H^1} \frac{2}{\hat{\sigma}^2}\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\hat{\sigma} + \eta)\,(\delta + M\eta)\Big) \tag{3.236}$$

completing the proof. □

Lemma 12. *Let $\mathcal{R}(c,\phi)$ be defined as in* (2.33) *and let $c_\varepsilon(\phi)$ be as in* (3.205)*. Furthermore require $\eta \le \hat{\sigma}$. Then there exists a positive constant M such that the inequalities*

$$\|\mathcal{R}(c_\varepsilon(\phi), \phi)\|_{L^2} \le \left(\frac{1}{\hat{\sigma}}\big\{\|u_0'\|_{L^2} + \eta\big\} + 1\right)\left(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\delta + M\eta)\eta\right) \tag{3.237}$$

and

$$\begin{aligned}&\|\mathcal{R}(\,c_\varepsilon(\phi_1), \phi_1) - \mathcal{R}\;(c_\varepsilon(\phi_2), \phi_2)\|_{L^2} \\ &\qquad \le \|\phi_1 - \phi_2\|_{H^1} \frac{2}{\hat{\sigma}^2}\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\hat{\sigma} + \eta)(\delta + M\eta)\Big)\Big(\|u_0'\|_{L^2} + \eta + \hat{\sigma}/2\Big)\end{aligned} \tag{3.238}$$

hold for all $\phi, \phi_1, \phi_2 \in \mathbf{X}_\eta$.

Proof. Let M be the positive constant from Lemma 10 and let $\phi, \phi_1, \phi_2 \in \mathbf{X}_\eta$ be arbitrary. Since $c_0 - c_\varepsilon(\phi)$ is just a constant we have that

$$\begin{aligned}\|\mathcal{R}(c_\varepsilon(\phi), \phi)\|_{L^2} &= \|(c_0 - c_\varepsilon(\phi))(u_0' + \phi') + (\Delta_\varepsilon u_0 - u_0'') + \delta\phi + \mathbf{N}(u_0, \phi)\|_{L^2} \\ &\le |c_0 - c_\varepsilon(\phi)|\,\|u_0' + \phi'\|_{L^2} + \|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + \delta\,\|\phi\|_{L^2} + \|\mathbf{N}(u_0, \phi)\|_{L^2} \\ &\le |c_0 - c_\varepsilon(\phi)|\left(\|u_0'\|_{L^2} + \|\phi'\|_{L^2}\right) + \|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + \delta\,\|\phi\|_{L^2} + \|\mathbf{N}(u_0, \phi)\|_{L^2}\,.\end{aligned} \tag{3.239}$$

We can bound this further using Lemma 11 and Corollary 2. We get

$$\begin{aligned}\|\mathcal{R}(c_\varepsilon(\phi), \phi)\|_{L^2} &\le \frac{1}{\hat{\sigma}}\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\delta + M\eta)\eta\Big)\Big(\|u_0'\|_{L^2} + \|\phi\|_{H^1}\Big) \\ &\quad + \|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + \delta\,\|\phi\|_{H^1} + M\eta^2 \\ &\le \frac{1}{\hat{\sigma}}\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\delta + M\eta)\eta\Big)\Big(\|u_0'\|_{L^2} + \eta\Big) \\ &\quad + \|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\delta + M\eta)\eta \\ &= \left(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\delta + M\eta)\eta\right)\left(\frac{1}{\hat{\sigma}}\big\{\|u_0'\|_{L^2} + \eta\big\} + 1\right).\end{aligned} \tag{3.240}$$

So the first of the two inequalities has now been proved. It remains to show that the second inequality holds as well. We get

$$\begin{aligned}
&\|\mathcal{R}(c_\varepsilon(\phi_1),\phi_1)-\mathcal{R}(c_\varepsilon(\phi_2),\phi_2)\|_{L^2}\\
&\quad=\big\|(c_0-c_\varepsilon(\phi_1))(u_0'+\phi_1')+\Delta_\varepsilon u_0-u_0''+\delta\phi_1-\mathbf{N}(u_0,\phi_1)\\
&\qquad-(c_0-c_\varepsilon(\phi_2))(u_0'+\phi_2')-(\Delta_\varepsilon u_0-u_0'')-\delta\phi_2+\mathbf{N}(u_0,\phi_2)\big\|_{L^2}\\
&\quad=\big\|(c_\varepsilon(\phi_2)-c_\varepsilon(\phi_1))u_0'+(c_0-c_\varepsilon(\phi_1))\phi_1'\\
&\qquad-(c_0-c_\varepsilon(\phi_2))\phi_2'+\delta(\phi_1-\phi_2)+\mathbf{N}(u_0,\phi_2)-\mathbf{N}(u_0,\phi_1)\big\|_{L^2}\\
&\quad\le|c_\varepsilon(\phi_2)-c_\varepsilon(\phi_1)|\,\|u_0'\|_{L^2}+\|(c_0-c_\varepsilon(\phi_1))\phi_1'-(c_0-c_\varepsilon(\phi_2))\phi_2'\|_{L^2}\\
&\qquad+\delta\,\|\phi_1-\phi_2\|_{L^2}+\|\mathbf{N}(u_0,\phi_2)-\mathbf{N}(u_0,\phi_1)\|_{L^2}\,.
\end{aligned}\tag{3.241}$$

The term $\|(c_0-c_\varepsilon(\phi_1))\phi_1'-(c_0-c_\varepsilon(\phi_2))\phi_2'\|_{L^2}$ is the most difficult to estimate. Therefore we will first estimate it seperately. We find that

$$\begin{aligned}
&\|(c_0-c_\varepsilon(\phi_1))\phi_1'-(c_0-c_\varepsilon(\phi_2))\phi_2'\|_{L^2}\\
&\quad=\|(c_0-c_\varepsilon(\phi_1))\phi_1'-(c_0-c_\varepsilon(\phi_2))\phi_2'+(c_0-c_\varepsilon(\phi_1))\phi_2'-(c_0-c_\varepsilon(\phi_1))\phi_2'\|_{L^2}\\
&\quad=\|(c_0-c_\varepsilon(\phi_1))(\phi_1'-\phi_2')+(c_\varepsilon(\phi_2)-c_\varepsilon(\phi_1))\phi_2'\|_{L^2}\\
&\quad\le|c_0-c_\varepsilon(\phi_1)|\,\|\phi_1'-\phi_2'\|_{L^2}+|c_\varepsilon(\phi_2)-c_\varepsilon(\phi_1)|\,\|\phi_2'\|_{L^2}\\
&\quad\le|c_0-c_\varepsilon(\phi_1)|\,\|\phi_1-\phi_2\|_{H^1}+|c_\varepsilon(\phi_2)-c_\varepsilon(\phi_1)|\,\|\phi_2\|_{H^1}\\
&\quad\le|c_0-c_\varepsilon(\phi_1)|\,\|\phi_1-\phi_2\|_{H^1}+|c_\varepsilon(\phi_2)-c_\varepsilon(\phi_1)|\,\eta.
\end{aligned}\tag{3.242}$$

Furthermore we have that $\delta\,\|\phi_1-\phi_2\|_{L^2}\le\delta\,\|\phi_1-\phi_2\|_{H^1}$. So now bounding (3.241) further gives

$$\begin{aligned}
&\|\mathcal{R}(c_\varepsilon(\phi_1),\phi_1)-\mathcal{R}(c_\varepsilon(\phi_2),\phi_2)\|_{L^2}\\
&\quad\le|c_\varepsilon(\phi_2)-c_\varepsilon(\phi_1)|\left(\|u_0'\|_{L^2}+\eta\right)+|c_0-c_\varepsilon(\phi_1)|\,\|\phi_1-\phi_2\|_{H^1}\\
&\qquad+\delta\,\|\phi_1-\phi_2\|_{H^1}+\|\mathbf{N}(u_0,\phi_2)-\mathbf{N}(u_0,\phi_1)\|_{L^2}\,.
\end{aligned}\tag{3.243}$$

Now we apply the inequalities from Lemma 11 and Corollary 2 to this expression. It follows that

$$\begin{aligned}
&\|\mathcal{R}(c_\varepsilon(\phi_1),\phi_1)-\mathcal{R}(c_\varepsilon(\phi_2),\phi_2)\|_{L^2}\\
&\quad\le\|\phi_1-\phi_2\|_{H^1}\,\frac{2}{\hat\sigma^2}\Big(\|\Delta_\varepsilon u_0-u_0''\|_{L^2}+(\hat\sigma+\eta)\,(\delta+M\eta)\Big)\Big(\|u_0'\|_{L^2}+\eta\Big)\\
&\qquad+\|\phi_1-\phi_2\|_{H^1}\,\frac{1}{\hat\sigma}\Big(\|\Delta_\varepsilon u_0-u_0''\|_{L^2}+(\delta+M\eta)\eta\Big)\\
&\qquad+\|\phi_1-\phi_2\|_{H^1}\,(\delta+M\eta)\\
&\quad=\|\phi_1-\phi_2\|_{H^1}\,\frac{2}{\hat\sigma^2}\Big(\|\Delta_\varepsilon u_0-u_0''\|_{L^2}+(\hat\sigma+\eta)\,(\delta+M\eta)\Big)\Big(\|u_0'\|_{L^2}+\eta\Big)\\
&\qquad+\|\phi_1-\phi_2\|_{H^1}\,\frac{1}{\hat\sigma}\Big(\|\Delta_\varepsilon u_0-u_0''\|_{L^2}+(\hat\sigma+\eta)(\delta+M\eta)\Big)\\
&\quad=\|\phi_1-\phi_2\|_{H^1}\,\frac{2}{\hat\sigma^2}\Big(\|\Delta_\varepsilon u_0-u_0''\|_{L^2}+(\hat\sigma+\eta)\,(\delta+M\eta)\Big)\Big(\|u_0'\|_{L^2}+\eta+\hat\sigma/2\Big).
\end{aligned}\tag{3.244}$$

□

Consequence 1. *Since $\|\mathcal{R}(c_\varepsilon(\phi),\phi)\|_{L^2}$ is bounded, it follows that $\mathcal{R}(c_\varepsilon(\phi),\phi) \in L^2(\mathbb{R})$.*

We have acquired enough information about the operators $\mathcal{L}^{\pm}_{\varepsilon,\delta}\phi$ and $\mathcal{R}(c,\phi)$ to be able to prove Theorem 2, see page 22, which is the main goal of this thesis.

Proof of Theorem 2

Proof. We start by letting δ and η be small positive constants. The exact values will be determined later, but we already require $\eta \leq \hat{\sigma}$. Furthermore we require $\varepsilon < \varepsilon_0(\delta)$, where $\varepsilon_0(\cdot)$ is the function as defined in Proposition 1. We define $\mathbf{T} : \mathbf{X}_\eta \subset H^1(\mathbb{R}) \to H^1(\mathbb{R})$ by

$$\mathbf{T}\phi = (\mathcal{L}^{+}_{\varepsilon,\delta})^{-1}\mathcal{R}(c_\varepsilon(\phi),\phi). \tag{3.245}$$

Consequence 1 states that $\mathcal{R}(c_\varepsilon(\phi),\phi) \in L^2(\mathbb{R})$ for any $\phi \in \mathbf{X}_\eta$. Since $(\mathcal{L}^{+}_{\varepsilon,\delta})^{-1}$ maps function from $L^2(\mathbb{R})$ to $H^1(\mathbb{R})$, it is clear that $\mathbf{T}$ indeed maps functions from $\mathbf{X}_\eta$ to $H^1(\mathbb{R})$.

The goal is to show that this mapping $\mathbf{T}$ has got a fixed point. If this is the case, it is not too hard to show that equation (2.16) has a solution. We can show that $\mathbf{T}$ has a fixed point by using Theorem A8, known as Banach's Fixed Point Theorem. We are allowed to use this theorem if $\mathbf{T}$ satisfies two properties. The first one is that $\mathbf{T}$ has to be a mapping to itself, so from $\mathbf{X}_\eta$ to $\mathbf{X}_\eta$. The second one is that $\mathbf{T}$ has to be a contraction mapping, see Definition A5.

We will proceed as follows. First we find bounds for $\|\mathbf{T}\phi\|_{H^1}$ and $\|\mathbf{T}\phi_1 - \mathbf{T}\phi_2\|_{H^1}$. Thereafter we determine the values of δ and η. This will be done in such a way that both required properties for $\mathbf{T}$ follow.

Let us first find a bound for $\|\mathbf{T}\phi\|_{H^1}$ with $\phi \in \mathbf{X}_\eta$ arbitrary. Since $\mathcal{R}(c_\varepsilon,\phi) \perp \phi_0^-$, Proposition 1 implies that we have

$$\|\mathbf{T}\phi\|_{H^1} = \left\|(\mathcal{L}^{+}_{\varepsilon,\delta})^{-1}\mathcal{R}(c_\varepsilon(\phi),\phi)\right\|_{H^1} \leq C_0 \left\|\mathcal{R}(c_\varepsilon(\phi),\phi)\right\|_{L^2}. \tag{3.246}$$

We can further bound this using Lemma 12. This gives

$$\|\mathbf{T}\phi\|_{H^1} \leq C_0\left(\frac{1}{\hat{\sigma}}\left\{\|u_0'\|_{L^2} + \eta\right\} + 1\right)\left(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\delta + M\eta)\eta\right). \tag{3.247}$$

We now introduce $E = \max\{1, 2\hat{\sigma}\}$. Then $E \geq 2\hat{\sigma}$, from which it also follows that $\frac{C_0 E}{\hat{\sigma}} \geq 2C_0 \geq C_0$. Remember we also did require $\eta \leq \hat{\sigma}$. So we can estimate

$$C_0\left(\frac{1}{\hat{\sigma}}\left\{\|u_0'\|_{L^2} + \eta\right\} + 1\right) \leq \frac{C_0 E}{\hat{\sigma}}\left(\frac{2}{\hat{\sigma}}\left\{\|u_0'\|_{L^2} + \hat{\sigma}\right\} + 1\right). \tag{3.248}$$

We will denote the constant on the right hand side by A. So

$$A = \frac{C_0 E}{\hat{\sigma}}\left(\frac{2}{\hat{\sigma}}\left\{\|u_0'\|_{L^2} + \hat{\sigma}\right\} + 1\right). \tag{3.249}$$

This constant does not depend on δ, η or ε. This will be an important detail when determining the values of δ and η. Now we apply (3.248) to (3.247). We find

$$\|\mathbf{T}\phi\|_{H^1} \leq A\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\delta + M\eta)\eta\Big). \tag{3.250}$$

Later in the proof it will turn out that this bound is precisely the one we need. Now we move on and try to find a bound for $\|\mathbf{T}\phi_1 - \mathbf{T}\phi_2\|_{H^1}$ where we let $\phi_1, \phi_2 \in \mathbf{X}_\eta$ arbitrary. Note that $(\mathcal{L}^+_{\varepsilon,\delta})^{-1}$ is linear since it's an inverse of a linear bijective mapping. We use this and again apply Proposition 1 to find that

$$\begin{aligned}\|\mathbf{T}\phi_1 - \mathbf{T}\phi_2\|_{H^1} &= \Big\|(\mathcal{L}^+_{\varepsilon,\delta})^{-1}\mathcal{R}(c_\varepsilon(\phi_1),\phi_1) - (\mathcal{L}^+_{\varepsilon,\delta})^{-1}\mathcal{R}(c_\varepsilon(\phi_2),\phi_2)\Big\|_{H^1} \\ &= \Big\|(\mathcal{L}^+_{\varepsilon,\delta})^{-1}\Big(\mathcal{R}(c_\varepsilon(\phi_1),\phi_1) - \mathcal{R}(c_\varepsilon(\phi_2),\phi_2)\Big)\Big\|_{H^1} \\ &\leq C_0 \|\mathcal{R}(c_\varepsilon(\phi_1),\phi_1) - \mathcal{R}(c_\varepsilon(\phi_2),\phi_2)\|_{L^2}.\end{aligned} \tag{3.251}$$

Now it follows from Lemma 12 and the definition of E that

$$\begin{aligned}&\|\mathbf{T}\phi_1 - \mathbf{T}\phi_2\|_{H^1} \\ &\quad\leq C_0\|\phi_1-\phi_2\|_{H^1}\frac{2}{\hat\sigma^2}\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\hat\sigma+\eta)(\delta+M\eta)\Big)\Big(\|u_0'\|_{L^2} + \eta + \hat\sigma/2\Big) \\ &\quad\leq C_0\|\phi_1-\phi_2\|_{H^1}\frac{2}{\hat\sigma^2}\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + 2\hat\sigma(\delta+M\eta)\Big)\Big(\|u_0'\|_{L^2} + \hat\sigma + \hat\sigma/2\Big) \\ &\quad\leq \|\phi_1-\phi_2\|_{H^1}\frac{2C_0E}{\hat\sigma^2}\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + \delta + M\eta\Big)\Big(\|u_0'\|_{L^2} + \hat\sigma + \hat\sigma/2\Big) \\ &\quad= \|\phi_1-\phi_2\|_{H^1}\frac{C_0E}{\hat\sigma}\cdot\frac{2}{\hat\sigma}\Big(\|u_0'\|_{L^2} + \hat\sigma + \hat\sigma/2\Big)\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + \delta + M\eta\Big) \\ &\quad= \|\phi_1-\phi_2\|_{H^1}\frac{C_0E}{\hat\sigma}\Big(\frac{2}{\hat\sigma}\big\{\|u_0'\|_{L^2} + \hat\sigma\big\} + 1\Big)\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + \delta + M\eta\Big) \\ &\quad= A\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + \delta + M\eta\Big)\|\phi_1-\phi_2\|_{H^1}.\end{aligned} \tag{3.252}$$

We have now also got a bound for $\|\mathbf{T}\phi_1 - \mathbf{T}\phi_2\|_{H^1}$, which means we are ready to determine δ and η. We choose

$$\delta = \frac{1}{4A} \qquad \text{and} \qquad \eta = \min\Big\{\hat\sigma, \frac{1}{4MA}\Big\}. \tag{3.253}$$

From Theorem 1 it follows that $u_0(x)$ satisfies the conditions of Lemma 4*(i)*. So we have that

$$\lim_{\varepsilon\downarrow 0}\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} = 0. \tag{3.254}$$

Therefore we can choose a small positive constant $\varepsilon^* \leq \varepsilon_0(\delta)$ such that

$$\sup_{\varepsilon\in(0,\varepsilon^*)}\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} \leq \frac{\min\{1,\eta\}}{4A}. \tag{3.255}$$

Keep in mind that A doesn't depend on δ, η or ε. From now on we assume $\varepsilon \in (0,\varepsilon^*)$.

Using the bound above and the given values to δ and η we can further bound (3.250). We estimate

$$\begin{aligned}\|\mathbf{T}\phi\|_{H^1} &\leq A\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + (\delta + M\eta)\eta\Big) \\ &\leq A\left(\frac{\min\{1,\eta\}}{4A} + \delta\eta + M\eta^2\right) \\ &\leq A\left(\frac{\eta}{4A} + \frac{\eta}{4A} + M\eta\min\left\{\hat{\sigma}, \frac{1}{4MA}\right\}\right) \\ &\leq A\left(\frac{\eta}{4A} + \frac{\eta}{4A} + \frac{\eta}{4A}\right) \\ &= \frac{3\eta}{4} \\ &\leq \eta.\end{aligned} \tag{3.256}$$

Hence $\mathbf{T}\phi \in \mathbf{X}_\eta$. So we can conclude that $\mathbf{T}$ is a mapping to itself, namely from $\mathbf{X}_\eta$ to $\mathbf{X}_\eta$. We still have to show that $\mathbf{T}$ is a contraction. We again use (3.253) and (3.255), but now to further bound (3.252). We estimate

$$\begin{aligned}\|\mathbf{T}\phi_1 - \mathbf{T}\phi_2\|_{H^1} &\leq C_5\Big(\|\Delta_\varepsilon u_0 - u_0''\|_{L^2} + \delta + M\eta\Big)\|\phi_1 - \phi_2\|_{H^1} \\ &\leq C_5\left(\frac{\min\{1,\eta\}}{4C_5} + \frac{1}{4C_5} + M\min\left\{\hat{\sigma}, \frac{1}{4MC_5}\right\}\right)\|\phi_1 - \phi_2\|_{H^1} \\ &\leq C_5\left(\frac{1}{4C_5} + \frac{1}{4C_5} + \frac{1}{4C_5}\right)\|\phi_1 - \phi_2\|_{H^1} \\ &= \frac{3}{4}\|\phi_1 - \phi_2\|_{H^1}.\end{aligned} \tag{3.257}$$

So we see that $\mathbf{T}$ is indeed a contraction mapping. Thus we can use Theorem A8, which states that $\mathbf{T}$ must have a unique fixed point. So there exists a $\phi_\varepsilon \in \mathbf{X}_\eta$ such that $\mathbf{T}\phi_\varepsilon = \phi_\varepsilon$. Hence $(\mathcal{L}^+_{\varepsilon,\delta})^{-1}\mathcal{R}(c_\varepsilon(\phi_\varepsilon), \phi_\varepsilon) = \phi_\varepsilon$ by the definition of $\mathbf{T}$. Now we let $\mathcal{L}^+_{\varepsilon,\delta}$ operate on both sides of this equation to obtain

$$\mathcal{R}(c_\varepsilon(\phi_\varepsilon), \phi_\varepsilon) = \mathcal{L}^+_{\varepsilon,\delta}\,\phi_\varepsilon. \tag{3.258}$$

Now it follows from Lemma 3 that $u_\varepsilon = u_0 + \phi_\varepsilon$ must satisfy $c_\varepsilon u_\varepsilon' - \Delta_\varepsilon u_\varepsilon + f(u_\varepsilon) = 0$, where we use the shorthand notation $c_\varepsilon = c_\varepsilon(\phi_\varepsilon)$. So $(c_\varepsilon, u_\varepsilon)$ is a solution to equation (2.16). Since we have let $\varepsilon \in (0, \varepsilon^*)$ arbitrarily, there exist such a solution pair $(c_\varepsilon, u_\varepsilon)$ for every $\varepsilon \in (0, \varepsilon^*)$. Uniqueness follows from the uniqueness of the fixed point ϕ_ε.

It is only left to show that $\lim_{\varepsilon\downarrow 0}(c_\varepsilon, u_\varepsilon) = (c_0, u_0)$. If we let $\varepsilon \downarrow 0$, then the left hand side of (3.255) will become very small. So this means we can pick η very small without violating (3.255). Thus if we let $\varepsilon \downarrow 0$, we can let $\eta \downarrow 0$ as well. Since $\phi_\varepsilon \in \mathbf{X}_\eta$, ϕ_ε has to converge to 0. So $\lim_{\varepsilon\downarrow 0} u_\varepsilon = u_0 + \lim_{\varepsilon\downarrow 0}\phi_\varepsilon = u_0$.

We can use this reasoning again to derive that c_ε converges to c_0 if we let $\varepsilon \downarrow 0$. From the definition of $c_\varepsilon(\phi)$, see (3.205), it follows that

$$c_\varepsilon = c_\varepsilon(\phi_\varepsilon) = c_0 + \frac{\langle \Delta_\varepsilon u_0 - u_0'' \,,\, \phi_0^- \rangle + \delta\langle \phi_\varepsilon \,,\, \phi_0^- \rangle - \langle \mathbf{N}(u_0, \phi_\varepsilon) \,,\, \phi_0^- \rangle}{\langle u_0' + \phi_\varepsilon' \,,\, \phi_0^- \rangle} \tag{3.259}$$

We'll show that all of the terms in the numerator converge to 0 as $\varepsilon \downarrow 0$. Lemma 4*(i)* tells us that $\Delta_\varepsilon u_0 - u_0''$ converges to 0 in L^2-norm as we let $\varepsilon \downarrow 0$. So applying Theorem A4 gives that $\lim_{\varepsilon\downarrow 0}\langle \Delta_\varepsilon u_0 - u_0'', \phi_0^- \rangle = 0$.

Recall that letting $\varepsilon \downarrow 0$ allows us to let $\eta \downarrow 0$. Then Lemma 10 implies that

$$\lim_{\varepsilon\downarrow 0} \|\mathbf{N}(u_0, \phi_\varepsilon)\|_{L^2} \le \lim_{\varepsilon\downarrow 0} M\eta^2 = \lim_{\eta\downarrow 0} M\eta^2 = 0. \tag{3.260}$$

So $\mathbf{N}(u_0, \phi_\varepsilon)$ converges to 0 in L^2-norm and we can again apply Theorem A4. This gives that $\lim_{\varepsilon\downarrow 0}\langle \mathbf{N}(u_0, \phi_\varepsilon), \phi_0^- \rangle = 0$.

Since ϕ_ε converges to 0 as $\varepsilon \downarrow 0$ it follows that $\lim_{\varepsilon\downarrow 0} \delta\langle \phi_\varepsilon, \phi_0^- \rangle = 0$. Combining these observations we find that

$$\lim_{\varepsilon\downarrow 0} c_\varepsilon(\phi_\varepsilon) = c_0 + \lim_{\varepsilon\downarrow 0} \frac{\langle \Delta_\varepsilon u_0 - u_0'', \phi_0^- \rangle + \delta\langle \phi_\varepsilon, \phi_0^- \rangle - \langle \mathbf{N}(u_0, \phi_\varepsilon), \phi_0^- \rangle}{\langle u_0' + \phi_\varepsilon', \phi_0^- \rangle} = c_0. \tag{3.261}$$

Thus indeed $\lim_{\varepsilon\downarrow 0}(c_\varepsilon, u_\varepsilon) = (c_0, u_0)$, which concludes the proof. □

4 Conclusion

In this thesis we have shown that discretized partial differential equation (2.16) has a solution. The equation depends on the value of ε and thus we get different solutions when we vary the discretization step size. So if we regard a real process in nature that can be described by (2.16), then the space we work in can really impact the behaviour of such a process. But what is still unclear, is how big this impact is. It would be very interesting to examine what the impact of choosing the discretization step size has on the solution. One could also wonder for what values of the discretization step this model is suitable.

Furthermore, we have only looked into travelling waves, so waves with a nonzero wave speed. But what if the wave speed is equal to zero? Can we then still find a solution to our problem? We could also replace the infinite sum we used to approach the second order spatial derivative with some other approximation, for example in the form of an integral. Does the equation then still have solutions? And if it has solutions, how does it differ from the solution to the equation in this report?

Thus we see that this study can still be extended in various ways and perhaps improved. But the proof shown in this study could be helpful in future research regarding this subject. An example is when the second order spatial derivative is replaced by a convolution kernel such as

$$\frac{\partial^2}{\partial x^2} u(x,t) \quad \rightarrow \quad \int_{-\infty}^{\infty} \mathcal{K}(y) u_j(y+t)\, dy.$$

This research can also be helpful when considering the discretized Allen-Cahn equation in more than one dimension. So there are plenty of opportunities for further research.

Appendix

Definition A1. *Let H be a Hilbert space and let $\{\phi_j\}_{j=0}^{\infty}$ be a sequence in H. Then we say ϕ_j converges weakly to ϕ in H if*

$$\langle \phi_j \, , \, \psi \rangle_H \to \langle \phi \, , \, \psi \rangle_H \text{ as } j \to \infty, \text{ for all } \psi \in H.$$

Definition A2. *Let $\{\phi_j\}_{j=0}^{\infty}$ be a sequence in $L^2(\mathbb{R})$. Then we say ϕ_j converges to ϕ in $L^2_{loc}(\mathbb{R})$ if for every compact set $K \subseteq \mathbb{R}$*

$$\int_K (\phi_j - \phi)^2 \to 0.$$

Definition A3. *Let X and Y be normed vector spaces and let $T : X \to Y$ be a mapping. Then T is called a homeomorphism if it is a bijective and continuous linear mapping that has an inverse mapping which is continuous. If such a mapping exists, we call X and Y homeomorphic.*

Definition A4. *Let $\psi \in L^2(\mathbb{R})$. We say that $\psi_w \in L^2(\mathbb{R})$ is the weak derivative of ψ if*

$$\langle \phi' \, , \, \psi \rangle = - \langle \phi \, , \, \psi_w \rangle$$

for all $\phi \in C_0^{\infty}(\mathbb{R})$.

Definition A5. *Let X be a subset of a normed vector space. Then $T : X \to X$ is called a contraction mapping if there exists some constant α with $0 \leq \alpha < 1$ such that*

$$\|Tx - Ty\| \leq \alpha \|x - y\|$$

for all $x, y \in X$.

Theorem A1 ([9, Theorem 5.73])**.** *Let $\{\psi_j\}_{j=0}^{j=\infty}$ be a sequence in either $L^2(\mathbb{R}), H^1(\mathbb{R})$ or $H^2(\mathbb{R})$. If $\{\psi_j\}_{j=0}^{j=\infty}$ is a bounded sequence, then $\{\psi_j\}_{j=0}^{j=\infty}$ has a subsequence that weakly converges in the corresponding function space.*

Theorem A2. *Let $\{\psi_j\}_{j=0}^{j=\infty}$ be a bounded sequence in $H^1(\mathbb{R})$. Then $\{\psi_j\}_{j=0}^{j=\infty}$ has a subsequence that converges in $L^2_{loc}(\mathbb{R})$.*

Theorem A3 ([9, Exercise 5.29])**.** *Let H be a Hilbert space. Assume that $\psi_j \to \psi$ weakly in H as $j \to \infty$. Then*

$$\|\psi\|_H \leq \liminf_{j \to \infty} \|\psi_j\|_H \, .$$

Theorem A4. *Let H be a Hilbert space. Assume $\{\phi_j\}_{j=0}^{\infty}$ is a bounded sequence in H. Furthermore assume $\phi_j \to \phi$ weakly in H and $\psi_j \to \psi$ in H-norm as $j \to \infty$. Then*

$$\langle \phi_j \, , \, \psi_j \rangle_H \to \langle \phi \, , \, \psi \rangle_H \quad \text{as } j \to \infty.$$

Proof. Let $\{\phi_j\}_{j=0}^{\infty}$ be a bounded sequence in H that converges weakly to ϕ in H as $j \to \infty$. Let $\{\psi_j\}_{j=0}^{\infty}$ be a sequence in H that converges to ψ in H-norm as $j \to \infty$. Then

$$\lim_{j \to \infty} \langle \phi_j \, , \, \psi \rangle_H = \langle \phi \, , \, \psi \rangle_H$$

and

$$\lim_{j\to\infty} |\langle \phi_j \,,\, \psi_j - \psi \rangle_H| \leq \lim_{j\to\infty} \|\phi_j\|_H \, \|\psi_j - \psi\|_H = 0.$$

So $\lim_{j\to\infty} \langle \phi_j \,,\, \psi_j - \psi \rangle_H = 0$ and it follows that

$$\lim_{j\to\infty} \langle \phi_j \,,\, \psi_j \rangle_H = \lim_{j\to\infty} \langle \phi_j \,,\, \psi \rangle_H + \lim_{j\to\infty} \langle \phi_j \,,\, \psi_j - \psi \rangle_H = \langle \phi \,,\, \psi \rangle_H.$$

□

Theorem A5 ([9, Lemma 4.1]). *Let X and Y be normed vector spaces and let $T : X \to Y$ be a linear mapping. Then the following are equivalent:*

(i) T is continuous.

(ii) T is continuous in 0.

(iii) There exists a constant $K > 0$ such that

$$\sup_{x\in X,\, \|x\|\leq 1} \|Tx\| = K.$$

(iv) There exists a constant $K > 0$ such that

$$\|Tx\| \leq K \, \|x\| \text{ for all } x \in X.$$

Theorem A6 ([9, Exercise 3.19]). *Let Y be a closed linear subspace of a Hilbert space $\mathcal{H}$. If $Y \neq \mathcal{H}$, then $Y^{\perp} \neq \{0\}$.*

Theorem A7 ([3, Theorem 8.8]). *There exists a constant $a > 0$ such that*

$$\|\phi\|_{L^\infty} \leq a \, \|\phi\|_{H^1} \quad \text{for all } \phi \in H^1(\mathbb{R}).$$

Theorem A8 ([4, Theorem 7.13]). *Let X be a complete metric space, and let $T : X \to X$ be a contraction mapping, see Definition A5. Then T has a unique fixed point x_0 in X(i.e. $Tx_0 = x_0$).*

Theorem A9 ([9, Lemma 4.38(c)]). *Let X, Y be normed linear spaces that are homeomorphic, see Definition A3. Then X is complete if and only if Y is complete.*

Theorem A10 ([4, Theorem 7.9]). *Let M be a complete metric space and let A be a subset of M. Then A is complete if and only if A is closed in M*

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