Synchronize with noise

Synchronization in a mean-field model of interacting oscillators in a random environment

Dutch title: *Synchronisatieprocessen in een model voor gekoppelde oscillatoren in een willekeurige omgeving*

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Abstract

A variation on the random Kuramoto model was analyzed, a mean-field model which describes the behavior of coupled oscillators in a random environment. The analytic results are based on the behavior of the system in the infinite volume limit. In the analysis critical values were found for the parameters in the model which can be used to determine when the oscillators synchronize. Numerical simulations were used to verify the analytic results.

Contents

1 Introduction

Synchronization processes are present in a wide range of fields of science, for example in biology, chemistry but also in social sciences. A common example in biology is the behavior of fireflies. Initially, when a group of fireflies flash their lights this will be incoherent, but as time passes they will synchronize the flashing of their lights. An example in sociology can be observed after a concert, when people clap after a show they will begin clapping in their own rhythm. As time passes there is apparently a will for coherence, because eventually people will usually synchronize, they will clap in the same rhythm. A successful mathematical model to describe these processes as well as many others was introduced by Kuramoto (1975), which is known by the Kuramoto model. This model describes a system of coupled oscillators where the interaction between the oscillators is a mean-field interaction. This means that there is no geometry, the position of the oscillators with respect to each other does not matter. The Kuramoto model and many variations and applications have already been studied, for an extensive overview one could read the review article [2].

A variation on the Kuromoto model (which is deterministic) includes random noise in the form of Brownian motion. In this thesis a variation to the random Kuramoto model is analyzed. The main goal is to analyze for which values of the parameters in the model a phase transition occurs. In other terms: for which values of the parameters do the coupled oscillators synchronize? The model that will be introduced and analyzed was left as an open problem in [1]. The analysis of critical fluctuations for this model which was also suggested will not be covered however.

First, the microscopic model is introduced. This describes the behavior of each oscillator. Then an order parameter will be defined which will be used to determine if the oscillators behave coherent (synchronized) or incoherent. When considering a system of N oscillators and then taking the limit $N \to \infty$ it is possible to determine what happens on a macroscopic scale. The model will then be described by an ordinary differential equation (McKean-Vlasov equation), because of this the behavior of the model on a macroscopic scale is deterministic. After finding the stationary solutions to this differential equation it is possible to determine when phase transitions occur. Finally, the analytic results were validated by numerically simulating the process.

2 The microscopic model

As mentioned earlier the model describes the behavior of N coupled oscillators. This behavior can be described using a Hamiltonian which is also known as the energy function. A Hamiltonian is often used in physics for describing systems of interacting particles.
Let $\underline{\eta} = (\eta_j)_{j=1}^N$ be a sequence of i.i.d., random variables which are uniformly distributed on $[0, 2\pi)^N$ and denote their law by μ. Given a configuration $\underline{x} = (x_j)_{j=1}^N \in [0, 2\pi)^N$ the Hamiltonian $H_N(\underline{x}, \underline{\eta}) : [0, 2\pi)^N \times [0, 2\pi)^N \to \mathbb{R}$ can be defined as

$$H_N(\underline{x}, \underline{\eta}) = -\frac{\theta}{2N} \sum_{j,k=1}^{N} \cos(x_k - x_j) - h \sum_{j=1}^{N} \cos(x_j + \eta_j) \tag{1}$$

The x_j's are the phases of the oscillators. The parameter $\theta > 0$ is called the coupling parameter, the greater θ is the more it will cause the oscillators to synchronize. $\underline{\eta}$ is the random influence of the external field, its influence depends on $h > 0$ which is the strength of the external field. If h is sufficiently large the oscillators will not synchronize but instead will stay incoherent. Now a linear generator will be introduced to describe the change of the x_j's in continuous time, which is a Markov process. Therefore the x_j's depend on time: $\underline{x}(t) = (x_j(t))_{j=1}^N$ where $0 \leq t \leq T$ for some fixed T. Note that each path of an oscillator: $x_j[0, T]$ lies in $\mathcal{C}[0, T]$, this space contains all continuous functions from $[0, T]$ to $[0, 2\pi)$.

Define the generator L_N acting on $\mathcal{C}^2$ (twice continuously differentiable) functions $f : [0, 2\pi)^N \to \mathbb{R}$ as:

$$\begin{aligned} L_N f(\underline{x}) &= \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^{N} \left(-\frac{\partial H_N}{\partial x_j}\right) \frac{\partial f}{\partial x_j}(\underline{x}) \\ &= \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^{N} \left[\frac{\theta}{N}\sum_{k=1}^{N} \sin(x_k - x_j) - h\sin(x_j + \eta_j)\right] \frac{\partial f}{\partial x_j}(\underline{x}) \end{aligned} \tag{2}$$

Define the parameters $0 \leq r_N \leq 1$ and Ψ_N as follows:

$$r_N e^{i\Psi_N} = \frac{1}{N}\sum_{j=1}^{N} e^{ix_j} \tag{3}$$

r_N can be interpreted as a measure of coherence, when $r_N = 0$ the oscillators are incoherent, when $r_N = 1$ the oscillators are fully synchronized. Ψ_N is the average phase of all the oscillators. The generator L_N can be expressed in terms of r_N and Ψ_N:

$$L_N f(\underline{x}) = \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2 f}{\partial x_j^2}(\underline{x}) + \sum_{j=1}^{N} \left[\theta r_N \sin(\Psi_N - x_j) - h\sin(x_j + \eta_j)\right] \frac{\partial f}{\partial x_j}(\underline{x}) \tag{4}$$

It is assumed that that the initial condition $\underline{x}(0)$ satisfies the following property: $(x_j(0), \eta_j)_{j=1}^N$ are independent and identically distributed with a law λ of the form

$$\lambda(\mathrm{d}x, \mathrm{d}\eta) = q_0(x, \eta)\mu(\mathrm{d}\eta)\mathrm{d}x \tag{5}$$

Where $\int_0^{2\pi} q_0(x, \eta)\mathrm{d}x = 1$ μ-almost surely.

Note that the stochastic process defined by (4) can also be described by a stochastic differential equation:

$$\mathrm{d}X_j(t) = \left(h\sin(X_j(t) + \eta_j) + \frac{\theta}{N}\sum_{k=1}^{N} \sin(X_k(t) - X_j(t))\right)\mathrm{d}t + \mathrm{d}B_j(t) \tag{6}$$

Where $(B_j(t))_{j,t}$ is a sequence of independent Brownian motions. This representation of the stochastic process will not be used for the main analysis in this thesis. However, it will be used for validating the analytic results with numerical simulations in section 5.

3 Behavior of the process on macroscopic scale

We are interested in the behavior of the process defined by the generator (4) as $N \to \infty$. This can be studied by looking at empirical measures. For some function $f : [0, 2\pi) \times [0, 2\pi) \to \mathbb{C}$ consider the empirical averages of the following form:

$$\int f \mathrm{d}\rho_N(t) = \frac{1}{N} \sum_{j=1}^{N} f(x_j(t), \eta_j)$$

Where $(\rho_N(t))_{t \in [0,T]}$ is the flow of empirical measures.

$$\rho_N(t) = \frac{1}{N} \sum_{j=1}^{N} \delta_{(x_j(t), \eta_j)}$$

We choose $f(x, \eta) = e^{ix}$ such that we obtain the same expression as in (3):

$$\int f \mathrm{d}\rho_N(t) = \frac{1}{N} \sum_{j=1}^{N} e^{ix_j} = r_N e^{i\Psi_N}$$

For some function $q : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}$ we define the linear operator $\mathcal{L}_q$ acting on a $\mathcal{C}^2$ function $f : [0, 2\pi) \times [0, 2\pi) \to \mathbb{R}$ by:

$$\mathcal{L}_q f(x, \eta) = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x, \eta) - \frac{\partial}{\partial x} [(\theta r_q \sin(\Psi_q - x) - h \sin(x + \eta)) f(x, \eta)] \tag{7}$$

Where:

$$r_q e^{i\Psi_q} := \int_0^{2\pi} \int_0^{2\pi} e^{ix} q(x, \eta) \mathrm{d}x \mu(\mathrm{d}\eta)$$

Some more notations have to be introduced, for a given environment $\underline{\eta} \in [0, 2\pi)^N$ denote the distribution of the stochastic process defined by the generator in (4) with initial condition (5) by $\mathbb{P}_N^{\underline{\eta}}$. Then denote the joint law of the process and the environment by

$$\mathbb{P}_N(\mathrm{d}\underline{x}[0, T], \mathrm{d}\underline{\eta}) := \mathbb{P}_N^{\underline{\eta}}(\mathrm{d}\underline{x}[0, T]) \mu^{\otimes N}(\mathrm{d}\underline{\eta})$$

The next theorem, which was proven in [3] will make clear why one would introduce the linear operator $\mathcal{L}_q$ and the notation for the distribution $\mathbb{P}_N$.

Theorem 1. *The nonlinear McKean-Vlasov equation*

$$\begin{cases} \dfrac{\partial q_t}{\partial t}(x, \eta) & = \mathcal{L}_{q_t} q_t(x, \eta) \\ q_0(x, \eta) & \text{as given in (5)} \end{cases} \tag{8}$$

admits a unique solution in $\mathcal{C}^1[[0, T], L^1(\mathrm{d}x \otimes \mu)]$ and $q_t(\cdot, \eta)$ is a probability density on $[0, 2\pi)$, for μ-almost every η and every $t > 0$. Moreover, for every $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that

$$\mathbb{P}_N \left(\sup_{t \in [0,T]} \mathrm{d}_p(\rho_N(t), q_t) > \varepsilon \right) \leq e^{-C(\varepsilon) N}$$

for N sufficiently large, where, by abuse of notations, we identify q_t with the probability $q_t(x, \eta)\mu(\mathrm{d}\eta)\mathrm{d}x$ on $[0, 2\pi) \times [0, 2\pi)$.

Here $\mathrm{d}_p(\cdot,\cdot)$ denotes the Prokhorov metric. Theorem 1 can be interpreted as follows, for every $t \in [0,T]$ we have the following weak convergence: $\rho_N(t) \overset{N\to\infty}{\longrightarrow} \rho_t$. Where ρ_t has the density $q_t(x,\eta)\mu(\mathrm{d}\eta)\mathrm{d}x$. As such the linear operator $\mathcal{L}_q$ defines the nonlinear McKean-Vlasov equation (8) where $q_t(\cdot,\eta)$ is a probability on $[0,2\pi)$. q satisfies the following conditions:

$$\begin{cases} \int_0^{2\pi} q(x,\eta)\mathrm{d}x = 1 & \text{for all } \eta \in [0,2\pi) \\ q(x,\eta) \text{ is } 2\pi\text{-periodic in } x & \text{for all } \eta \in [0,2\pi) \end{cases}$$

Now we will determine the equilibrium probability density which satisfies these conditions and which is a stationary solution to the ordinary differential equation in (8).

$$\frac{\partial q_t}{\partial t}(x,\eta) = 0 \Leftrightarrow \frac{1}{2}\frac{\partial^2 q}{\partial x^2}(x,\eta) = \frac{\partial}{\partial x}[(\theta r_q \sin(\Psi_q - x) - h\sin(x+\eta))q(x,\eta) \tag{9}$$

Integrating equation (9) with respect to x yields:

$$\frac{\partial q}{\partial x}(x,\eta) = 2[(\theta r_q \sin(\Psi_q - x) - h\sin(x+\eta))q(x,\eta) + c(\eta)$$

Where $c(\eta) \in \mathbb{R}$ is an integration constant. The general solution to this differential equation is given by:

$$\begin{aligned} q(x,\eta) &= \exp\left(\int 2\theta r_q \sin(\Psi_q - x) - 2h\sin(x+\eta)\mathrm{d}x\right)\cdot \\ &\quad \left(\int c(\eta)\exp\left(\int -2\theta r_q \sin(\Psi_q - x) + 2h\sin(x+\eta)\mathrm{d}x\right) + k(\eta)\right) \\ &= \exp\left(2\theta r_q \cos(\Psi_q - x) + 2h\cos(x+\eta)\right)\cdot \\ &\quad \left(\int c(\eta)\exp\left(-2\theta r_q \cos(\Psi_q - x) - 2h\cos(x+\eta)\right) + k(\eta)\right) \end{aligned} \tag{10}$$

Where $k(\eta) \in \mathbb{R}$ is another integration constant. The condition: $q(0,\eta) = q(2\pi,\eta)$ for all η, can now be used to show that $c(\eta) = 0$. When looking at equation (10), it is clear that the part outside of the integral satisfies this periodic condition:

$$\exp(2\theta r_q \cos(\Psi_q) + 2h\cos(\eta)) = \exp(2\theta r_q \cos(\Psi_q - 2\pi) + 2h\cos(2\pi + \eta))$$

Therefore, it is clear that if $\eta \in [0,2\pi)$ then

$$\begin{aligned} q(0,\eta) = q(2\pi,\eta) \quad &\Leftrightarrow \quad \left[\int c(\eta)\exp\left(-2\theta r_q \cos(\Psi_q - x) - 2h\cos(x+\eta)\right) + k(\eta)\right]_{x=0}^{x=2\pi} = 0 \\ &\Leftrightarrow \quad \int_0^{2\pi} c(\eta)\exp(-2\theta r_q \cos(\Psi_q - x) - 2h\cos(x+\eta))\mathrm{d}x = 0 \end{aligned}$$

Since $\exp(-2\theta r_q \cos(\Psi_q - x) - 2h\cos(x+\eta))$ is a positive function, the integral is greater than zero:

$$\int_0^{2\pi} \exp(-2\theta r_q \cos(\Psi_q - x) - 2h\cos(x+\eta))\mathrm{d}x > 0 \Rightarrow c(\eta) = 0$$

This yields the final expression for $q(x,\eta)$:

$$q(x,\eta) = k(\eta)\exp(2\theta r_q \cos(\Psi_q - x) + 2h\cos(x+\eta)) \tag{11}$$

Where q also satisfies the self-consistency relation:

$$r_q e^{i\Psi_q} = \int_0^{2\pi}\int_0^{2\pi} e^{ix} q(x,\eta)\mathrm{d}x\mu(\mathrm{d}\eta)$$

and where

$$k(\eta) = \left(\int_0^{2\pi} \exp(2\theta r_q \cos(\Psi_q - x) + 2h\cos(x+\eta))\mathrm{d}x \right)^{-1}$$

Because for this $k(\eta)$ we will have that $\int_0^{2\pi} q(x,\eta)\mathrm{d}x = 1$ for all $\eta \in [0, 2\pi)$. Note that $k(\eta)$ can be expressed in terms of a modified Bessel function of the first kind. A modified Bessel function of the first kind of order v can be represented by the following integral:

$$I_v(x) = \frac{1}{2\pi} \int_0^{2\pi} \cos(vy)\exp(x\cos(y))\mathrm{d}y$$

The following fact can be used to express $k(\eta)$ in terms of a modified Bessel function of the first kind: a linear combination of a sine and a cosine is again a cosine. For $A, B \in \mathbb{R}$ we have for some $\phi \in [0, 2\pi)$:

$$A\sin(x) + B\cos(x) = \sqrt{A^2 + B^2}\cos(x+\phi) \tag{12}$$

$$\begin{aligned}
k^{-1}(\eta) &= \int_0^{2\pi} \exp(2\theta r_q \cos(\Psi_q - x) + 2h\cos(x+\eta))\mathrm{d}x \\
&= \int_0^{2\pi} \exp(\cos(x)\left[2\theta r_q \cos(\Psi_q) + 2h\cos(\eta)\right] + \sin(x)\left[2\theta r_q \sin(\Psi_q) - 2h\sin(\eta)\right])\mathrm{d}x \\
&\overset{(12)}{=} \int_0^{2\pi} \exp\left(\sqrt{\left[2\theta r_q \cos(\Psi_q) + 2h\cos(\eta)\right]^2 + \left[2\theta r_q \sin(\Psi_q) - 2h\sin(\eta)\right]^2}\cos(x+\phi) \right)\mathrm{d}x \\
&= \int_0^{2\pi} \exp\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q + \eta)}\cos(x+\phi) \right)\mathrm{d}x \\
&\overset{y=x+\phi}{=} \int_\phi^{2\pi+\phi} \exp\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q + \eta)}\cos(y) \right)\mathrm{d}y \\
&\overset{(*)}{=} \int_0^{2\pi} \exp\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q + \eta)}\cos(y) \right)\mathrm{d}y \\
&= 2\pi I_0\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q + \eta)} \right) \qquad (13)
\end{aligned}$$

Note that $(*)$ holds since the function in the integrand is 2π-periodic. As long as the length of the integration interval is 2π, shifting the interval by any value of ϕ does not impact the result of the integral. This argument has been used quite often when simplifying integrals in the appendices.

3.1 Linearizing the operator $\mathcal{L}_q$

In this section the operator $\mathcal{L}_q$ will be linearized about the equilibrium solution q_*^0, which will be introduced in a moment. This has been done because the kernel of this linearized operator is related to critical values of the parameters θ and h which can be used to determine phase transitions. For $\varepsilon > 0$ the linearization of $\mathcal{L}_q$ is given by:

$$\mathscr{L}q(x,\eta) = \lim_{\varepsilon\to 0} \frac{\partial}{\partial x}\mathcal{L}_{\varepsilon q+q_*^0}(\varepsilon q(x,\eta) + q_*^0(x,\eta))$$

Where q_*^0 is the solution to equation (9) such that $r_{q_*^0} = 0$ and should therefore satisfy:

$$r_{q_*^0} e^{i\Psi_{q_*^0}} = \int_0^{2\pi}\int_0^{2\pi} e^{iy} q_*^0(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) = 0 \tag{14}$$

This is indeed the case, since we have:

$$\int_0^{2\pi}\int_0^{2\pi} \cos(y-\Psi_{q_*^0})q_*^0(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) \overset{(35)}{=} \int_0^{2\pi} \cos(\Psi_{q_*^0}+\eta)\frac{I_1(2h)}{I_0(2h)}\mu(\mathrm{d}\eta) = 0$$
$$\int_0^{2\pi}\int_0^{2\pi} \sin(y-\Psi_{q_*^0})q_*^0(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) \overset{(36)}{=} \int_0^{2\pi} -\sin(\Psi_{q_*^0}+\eta)\frac{I_1(2h)}{I_0(2h)}\mu(\mathrm{d}\eta) = 0$$

By multiplying both sides of equation (14) with e^{-ix} we obtain:

$$\begin{aligned} r_{q_*^0} e^{i(\Psi_{q_*^0}-x)} &= \int_0^{2\pi}\int_0^{2\pi} e^{i(y-x)} q_*^0(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) = 0 \\ \Rightarrow \quad r_{q_*^0}\sin(\Psi_{q_*^0}-x) &= \int_0^{2\pi}\int_0^{2\pi} \sin(y-x) q_*^0(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) = 0 \end{aligned} \tag{15}$$

Similarly we also have

$$r_q \sin(\Psi_q - x) = \int_0^{2\pi}\int_0^{2\pi} \sin(y-x)q(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) \tag{16}$$

This can be used to simplify the expression for $\mathcal{L}_{\varepsilon q+q_*^0}(\varepsilon q + q_*^0)$.

$$\begin{aligned} \mathcal{L}_{\varepsilon q+q_*^0}(\varepsilon q+q_*^0) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}(\varepsilon q+q_*^0) - \frac{\partial}{\partial x}\Bigg[\Bigg(\theta\int_0^{2\pi}\int_0^{2\pi}\sin(y-x)(\varepsilon q(y,\eta)+q_*^0(y,\eta))\mathrm{d}y\mu(\mathrm{d}\eta) \\ &\quad - h\sin(x+\eta)\Bigg)(\varepsilon q+q_*^0)\Bigg] \\ &\overset{(15)}{=} \frac{1}{2}\frac{\partial^2}{\partial x^2}(\varepsilon q+q_*^0) - \frac{\partial}{\partial x}\Bigg[\Bigg(\theta\varepsilon\int_0^{2\pi}\int_0^{2\pi}\sin(y-x)q(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) \\ &\quad - h\sin(x+\eta)\Bigg)(\varepsilon q+q_*^0)\Bigg] \\ &= \frac{1}{2}\frac{\partial^2}{\partial x^2}(\varepsilon q+q_*^0) - \frac{\partial}{\partial x}\Bigg[\theta\varepsilon(\varepsilon q+q_*^0)\int_0^{2\pi}\int_0^{2\pi}\sin(y-x)q(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) \\ &\quad - h\sin(x+\eta)(\varepsilon q+q_*^0)\Bigg] \end{aligned}$$

Taking the derivative with respect to ε:

$$\begin{aligned} \frac{\partial}{\partial\varepsilon}\mathcal{L}_{\varepsilon q+q_*^0}(\varepsilon q+q_*^0) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}q(x,\eta) - \frac{\partial}{\partial x}\Bigg[\theta(2\varepsilon q+q_*^0)\int_0^{2\pi}\int_0^{2\pi}\sin(y-x)q(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) \\ &\quad - h\sin(x+\eta)q(x,\eta)\Bigg] \end{aligned}$$

Taking the limit of $\varepsilon \to 0$ yields the final expression for the linearization of $\mathcal{L}_q$:

$$\begin{aligned}\mathscr{L}q(x,\eta) &= \frac{1}{2}\frac{\partial^2}{\partial x^2}q(x,\eta) - \frac{\partial}{\partial x}\left[\theta q_*^0(x,\eta)\int_0^{2\pi}\int_0^{2\pi}\sin(y-x)q(y,\eta)\mathrm{d}y\mu(\mathrm{d}\eta) - h\sin(x+\eta)q(x,\eta)\right] \\ &\overset{(16)}{=} \frac{1}{2}\frac{\partial^2}{\partial x^2}q(x,\eta) - \frac{\partial}{\partial x}\left[\theta q_*^0(x,\eta)r_q\sin(\Psi_q - x) - h\sin(x+\eta)q(x,\eta)\right] \end{aligned} \tag{17}$$

3.2 The kernel of the linearized operator $\mathscr{L}$

The kernel of the operator $\mathscr{L}$ consists of the functions g such that $\mathscr{L}g = 0$, $\int_0^{2\pi} g(x,\eta)\mathrm{d}x = 0$ for all $\eta \in [0, 2\pi)$ and $g(0,\eta) = g(2\pi,\eta)$ for all $\eta \in [0, 2\pi)$. This results in the following differential equation:

$$\mathscr{L}g = 0 \Leftrightarrow \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(x,\eta) = \frac{\partial}{\partial x}[(\theta r_g \sin(\Psi_g - x)q_*^0(x,\eta) - h\sin(x+\eta))g(x,\eta)] \tag{18}$$

Integrating both sides with respect to x results in

$$\frac{\partial g}{\partial x}(x,\eta) = 2[(\theta r_g \sin(\Psi_g - x)q_*^0(x,\eta) - h\sin(x+\eta))g(x,\eta)] + c_1$$

Where $c_1 \in \mathbb{R}$ is an integration constant. The solution to this nonhomogenous linear differential equation is as follows: (this is proven in appendix A)

$$g(x,\eta) = 2\theta r_g q_*^0(x,\eta)\left(\cos(\Psi_g - x) - \int_0^{2\pi} q_*^0(x,\eta)\cos(\Psi_g - x)\mathrm{d}x\right) \tag{19}$$

Where r_g and Ψ_g have to satisfy the self-consistency relation

$$r_g e^{i\Psi_g} = \int_0^{2\pi}\int_0^{2\pi} e^{ix} g(x,\eta)\mathrm{d}x\mu(\mathrm{d}\eta) \Leftrightarrow r_g = \int_0^{2\pi}\int_0^{2\pi} e^{i(x-\Psi_g)} g(x,\eta)\mathrm{d}x\mu(\mathrm{d}\eta)$$

Which is clearly equivalent to the following two conditions

$$\begin{aligned} r_g &= \int_0^{2\pi}\int_0^{2\pi} \cos(x-\Psi_g)g(x,\eta)\mathrm{d}x\mu(\mathrm{d}\eta) \\ 0 &= \int_0^{2\pi}\int_0^{2\pi} \sin(x-\Psi_g)g(x,\eta)\mathrm{d}x\mu(\mathrm{d}\eta) \end{aligned} \tag{20}$$

Plugging equation (19) into (20) yields the following conditions

$$\begin{aligned} r_g &= 2\theta r_g \int_0^{2\pi}\left[\int_0^{2\pi} \cos^2(\Psi_g - x)q_*^0(x,\eta)\mathrm{d}x - \left(\int_0^{2\pi}\cos(\Psi_g - x)q_*^0(x,\eta)\mathrm{d}x\right)^2\right]\mu(\mathrm{d}\eta) =: 2\theta r_g A(\Psi_g) \\ 0 &= 2\theta r_g \int_0^{2\pi}\left[\int_0^{2\pi} \cos^2(\Psi_g - x)\sin(x-\Psi_g)q_*^0(x,\eta)\mathrm{d}x \right. \\ &\quad \left. - \left(\int_0^{2\pi}\sin(x-\Psi_g)q_*^0(x,\eta)\mathrm{d}x\right)\left(\int_0^{2\pi}\cos(x-\Psi_g)q_*^0(x,\eta)\mathrm{d}x\right)\right]\mu(\mathrm{d}\eta) =: 2\theta r_g B(\Psi_g) \end{aligned} \tag{21}$$

$A(\Psi_g)$ and $B(\Psi_g)$ can be simplified, which is shown in appendix B. There it is shown that

$$A(\Psi_g) = \frac{1}{2} - \frac{1}{2}\left(\frac{I_1(2h)}{I_0(2h)}\right)^2 \quad \text{and} \quad B(\Psi_g) \equiv 0$$

Therefore the conditions in equation (21) are satisfied for $r_g = 0$ and general Ψ_g. For $r_g > 0$ and general Ψ_g the conditions can be reduced to the following condition for θ:

$$\theta(h) = \frac{1}{2}(A(\Psi_g))^{-1} = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}\left(\frac{I_1(2h)}{I_0(2h)}\right)^2\right)^{-1} = \left(1 - \left(\frac{I_1(2h)}{I_0(2h)}\right)^2\right)^{-1} \tag{22}$$

4 Phase transitions for the mean field limit

The expression for $\theta(h)$ that was found in the previous section can be seen as a critical value. The value of θ in a general stationary solution of the form as shown in (11) affects whether or not synchronization will occur in the mean field limit. Let q be such a general stationary solution. Recall that r_q and Ψ_q have to satisfy the following conditions:

$$r_q = \int_0^{2\pi} \int_0^{2\pi} \cos(x - \Psi_q) q(x, \eta) \mathrm{d}x \mu(\mathrm{d}\eta) \tag{23}$$

$$0 = \int_0^{2\pi} \int_0^{2\pi} \sin(x - \Psi_q) q(x, \eta) \mathrm{d}x \mu(\mathrm{d}\eta) \tag{24}$$

In this section the solutions (r_q, Ψ_q) to these conditions will be determined, which will depend on θ and h. In order to simplify the integrals in these conditions the following lemma is useful.

Lemma 4.1. *For $A, B \in \mathbb{R}$ where $A \neq 0$ or $B \neq 0$ the following equality holds:*

$$\int_0^{2\pi} \cos(x) \exp(2A\cos(x) + 2B\sin(x)) \mathrm{d}x = 2\pi \frac{A}{\sqrt{A^2 + B^2}} I_1(2\sqrt{A^2 + B^2}) \tag{25}$$

Where $I_v(.)$ denotes a modified Bessel function of the first kind of order v.

Proof. The proof can be found in [1] in Appendix A.2. □

In appendix C it is shown that (24) holds true for all possible values of all the parameters. In the same appendix it is shown that:

$$\int_0^{2\pi} \cos(x-\Psi_q) q(x, \eta) \mathrm{d}x = \frac{\theta r_q + h\cos(\Psi_q + \eta)}{\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h \cos(\Psi_q + \eta)}} \frac{I_1\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h \cos(\Psi_q + \eta)}\right)}{I_0\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h \cos(\Psi_q + \eta)}\right)}$$

Define the function $F : [0, \infty) \to \mathbb{R}$ by:

$$\begin{aligned} F(r_q) &= \int_0^{2\pi} \int_0^{2\pi} \cos(x - \Psi_q) q(x, \eta) \mathrm{d}x \mu(\mathrm{d}\eta) \\ &= \int_0^{2\pi} \frac{\theta r_q + h\cos(\eta)}{\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h \cos(\eta)}} \frac{I_1\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h \cos(\eta)}\right)}{I_0\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h \cos(\eta)}\right)} \mu(\mathrm{d}\eta) \end{aligned} \tag{26}$$

Note that F does not depend on Ψ_q (this is shown in appendix D), therefore in most of the calculations involving the function F we chose $\Psi_q = 0$. It is clear that finding solutions (r_q, Ψ_q) to conditions (23), (24) comes down to finding values of r_q which satisfy $F(r_q) = r_q$. In appendix D it is shown that F has the following properties:

- $F(0) = 0$ and F is continuous
- $\lim_{r_q \to \infty} F(r_q) = 1$
- $F'(r_q) = 2\theta \mathbb{E}_\mu \left[\mathbb{V}\mathrm{ar}_{q(x,\eta)}(\cos(X))\right]$. Therefore F is increasing, since its first derivative can be expressed as a variance which is strictly positive.
- $F''(r_q) = 4\theta^2 \mathbb{E}_\mu \left[\mathbb{E}_{q(x,\eta)}\left(\left[\cos(X) - \mathbb{E}_{q(x,\eta)}(\cos(X))\right]^3\right)\right]$. This expression unfortunately does not reveal when F is convex or concave.

Note that $F'(0) = 2\theta\mathbb{E}_\mu\left[\mathbb{V}\text{ar}_{q_*^0(x,\eta)}(\cos(X))\right] = 2\theta A(\Psi_g)$ with $A(\Psi_g)$ as in (21) with $\Psi_g = 0$. It is clear that if $\theta = \theta(h)$ we have $F'(0) = 1$. Therefore we can conclude that if $\theta > \theta(h)$ then $F'(0) > 1$ which means there is at least one positive solution to $F(r_q) = r_q$ since F is continuous and $\lim_{r_q\to\infty} F(r_q) = 1$.

Since it is not known when F changes curvature, numerical methods were required to determine the number of positive solutions to $F(r_q) = r_q$ for different values of θ and h. The following conclusions are based on numerically finding the number of positive solutions to $F(r_q) = r_q$ for $(\theta, h) \in [0, 30] \times [0, 30]$ where the square $[0, 30] \times [0, 30]$ was discretized as a grid with step size 0.1 in both directions.

- When $\theta > \theta(h)$ there exists a $r > 0$ such that there are synchronized solutions of the form: (r, Ψ) where $\Psi \in [0, 2\pi)$.
- There exists a $\theta_*(h) \leq \theta(h)$ (which was found numerically as well) such that
  - When $\theta < \theta_*(h)$ there are no synchronized solutions.
  - When $\theta_*(h) < \theta < \theta(h)$ there exist $r_1, r_2 > 0$ such that there are synchronized solutions of the form (r_1, Ψ) and (r_2, Ψ) where $\Psi \in [0, 2\pi)$.

The amount of synchronized solutions for different values θ and h as described above can be visualized in a phase diagram which is shown below.

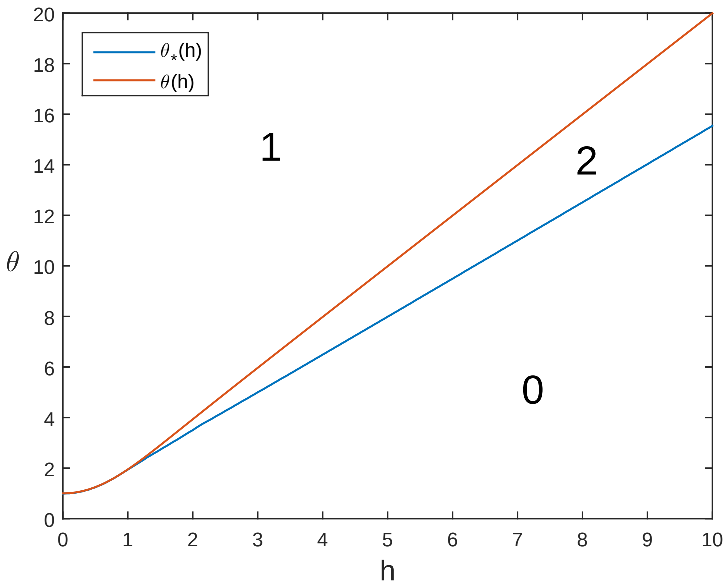

Figure 1: A phase diagram for the parameters (θ, h). $\theta(h)$ was derived analytically (as shown in equation (22)) and $\theta_*(h)$ was found numerically. The number in each of the regions is the number of positive solutions to $F(r) = r$. Each positive solution $r > 0$ results in synchronized solutions of the form (r, Ψ) where $\Psi \in [0, 2\pi)$.

5 Simulations

In order to verify the analytic results one can numerically simulating the process. This has been done by numerically integrating equation (6) using the Euler-Maruyama method. For each of the three regions in figure 1 a pair (θ, h) was chosen and then the process was simulated. The initial state of the oscillators was random, taken from a uniform distribution on $[0, 2\pi)^N$. The results are visualized in figure 2, 3 and 4.

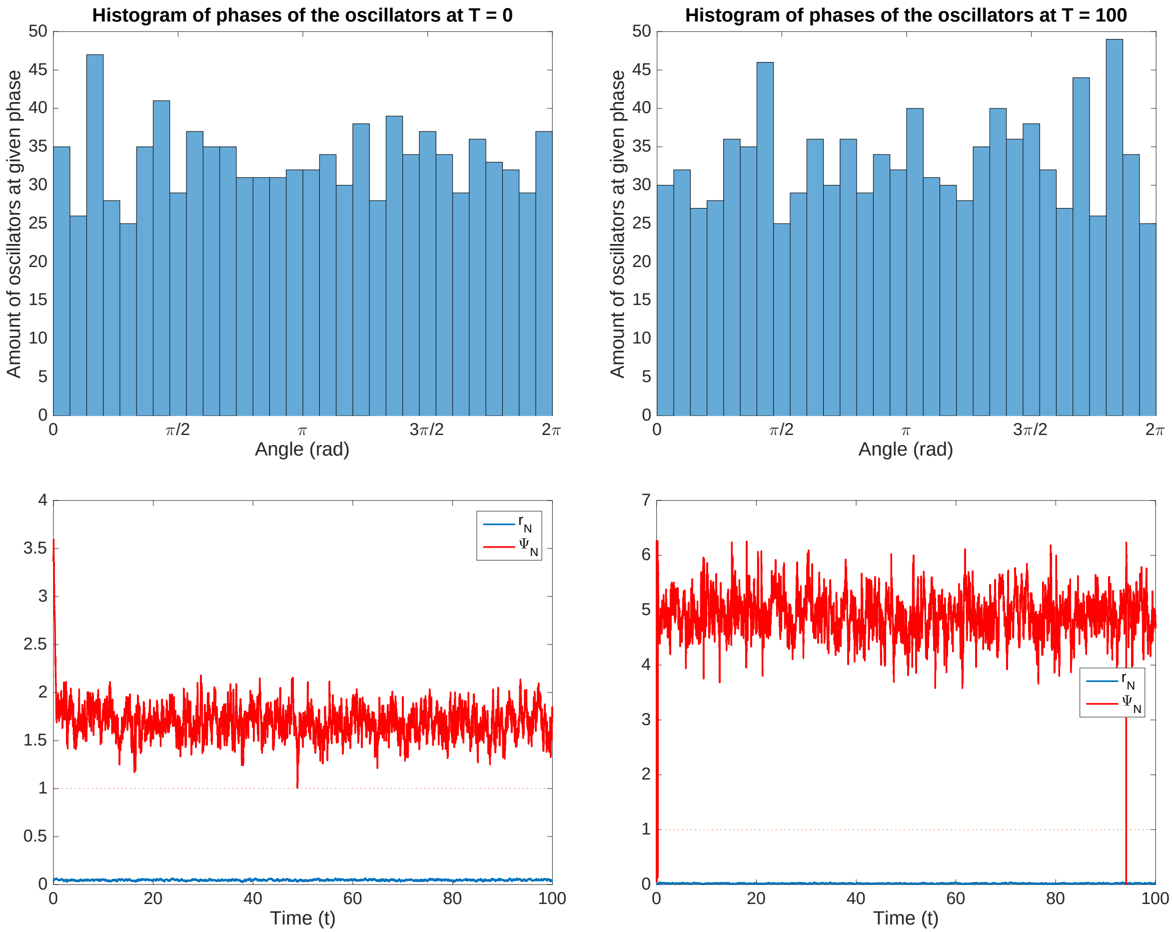

Figure 2: The result of numerically simulating the process with the following parameters: $N = 1000$, $t \in [0, 100]$, $\Delta t = 0.01$, $\theta = 4$, $h = 8$. This corresponds to region 0 in the phase diagram in figure 1. In the histograms on top one can see the amount of oscillators at each possible phase angle at a specified moment in time. This is the result of one simulation but it was repeated for 100 times and the results were very similar each time. In the plots on the bottom one can see the behavior of r_N and Ψ_N over time for two simulations. It is clear that r_N remains relatively stable over time, while Ψ_N fluctuates quite a bit. It was found that $r_N(100) \approx 0.033$ which means that the oscillators are incoherent. As for Ψ_N, during each simulation Ψ_N fluctuates around a different value, the two plots that are shown are an example of this behavior.

During the simulations it was found that if synchronization occurs, that is if r_N stabilizes at some $r > 0$, then that happened (with a notable margin) in the interval $t \in [0, 30]$. The plots suggest that if r_N stabilizes, simulating the process for more iterations will not impact the results. Therefore we introduce the notation $r_N(\infty) := \lim_{t\to\infty} r_N(t) \approx r_N(30)$. When keeping h fixed while varying θ one can plot $r_N(\infty)$ depending on θ, which has been done in figure 5.

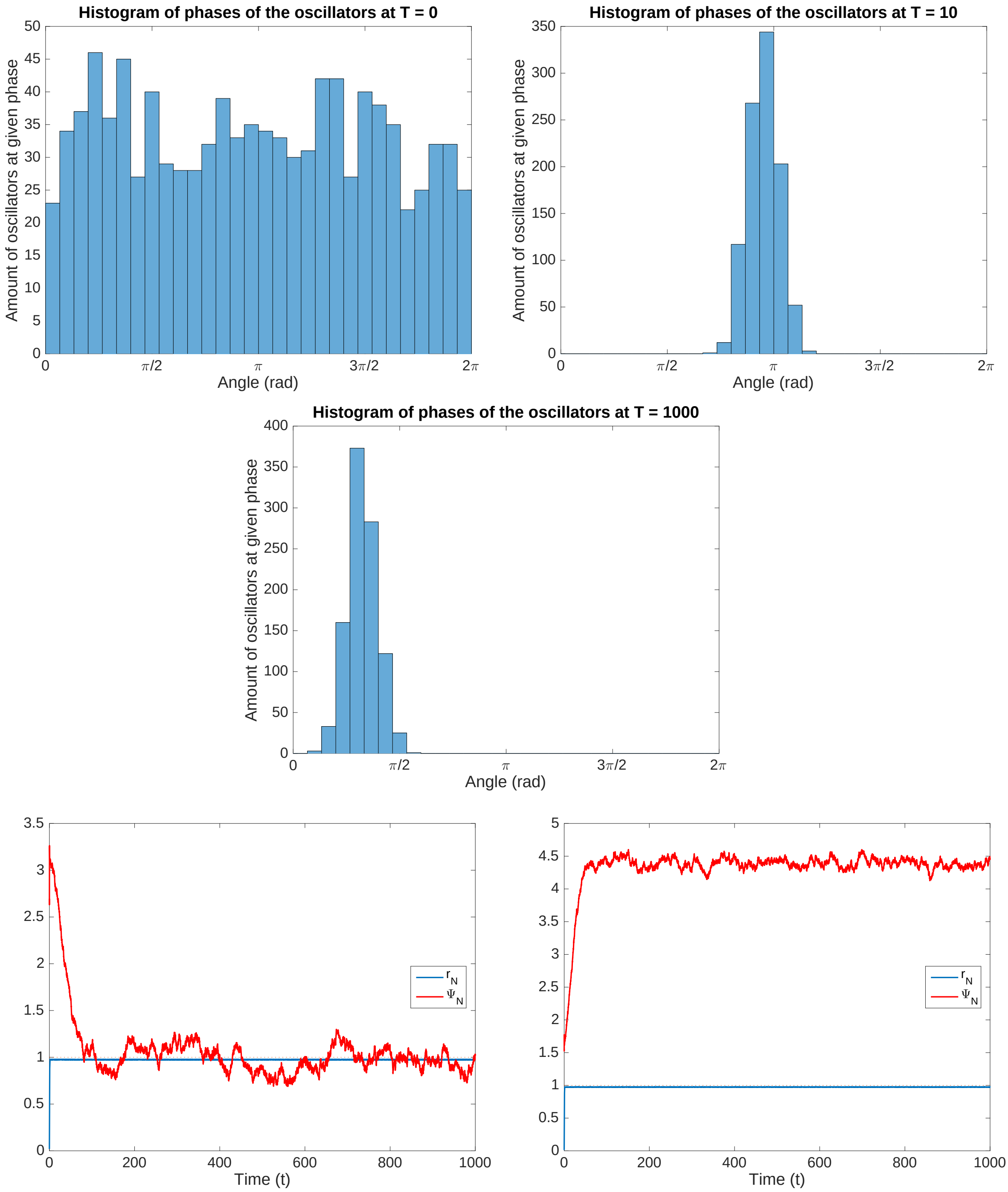

Figure 3: The result of numerically simulating the process with the following parameters: $N = 1000$, $t \in [0, 1000]$, $\Delta t = 0.01$, $\theta = 14$, $h = 2$. This corresponds to region 1 in the phase diagram in figure 1. In the histograms on top one can see the amount of oscillators at each possible phase angle at a specified moment in time. This is the result of one simulation but it was repeated for 100 times and the results were very similar each time. In the plots on the bottom one can see the behavior of r_N and Ψ_N over time for two simulations. It is clear that r_N stabilizes very quickly over time. As for Ψ_N one can see that its behavior is not as stable as that of r_N but during all simulations it was observed that Ψ_N fluctuates around a specific value. Again, each simulation resulted in Ψ_N 'stabilizing' at a different value. It was found that $r_N(1000) \approx 0.97$ which means that the oscillators are almost fully synchronized.

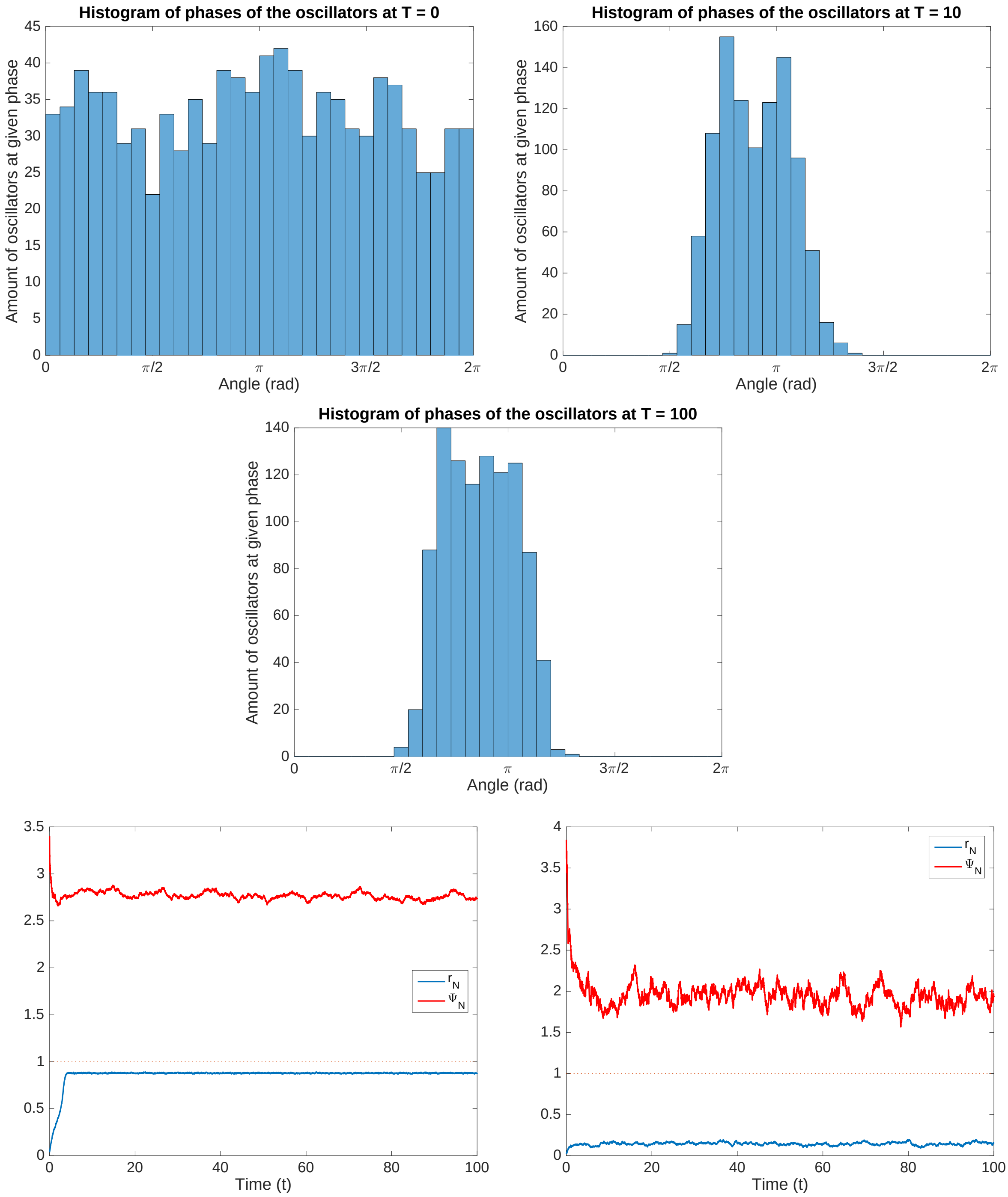

Figure 4: The result of numerically simulating the process with the following parameters: $N = 1000$, $t \in [0, 100]$, $\Delta t = 0.01$, $\theta = 16.5$, $h = 9$. This corresponds to region 2 in the phase diagram in figure 1. In the histograms on top one can see the amount of oscillators at each possible phase angle at a specified moment in time. This is the result of one simulation but it was repeated for 100 times, the behavior of Ψ_N was very consistent in all simulation but this was not the case for r_N. In the plots on the bottom one can see the behavior of r_N and Ψ_N over time for two simulations. It is clear that r_N stabilizes over time, but there was some variation in the value it stabilized at. In many simulations it was very clear that the oscillators were synchronized as was expected, but in some cases this was not the case. Again, it was observed that Ψ_N does fluctuate quite a bit around a certain value, which was different for each simulation.

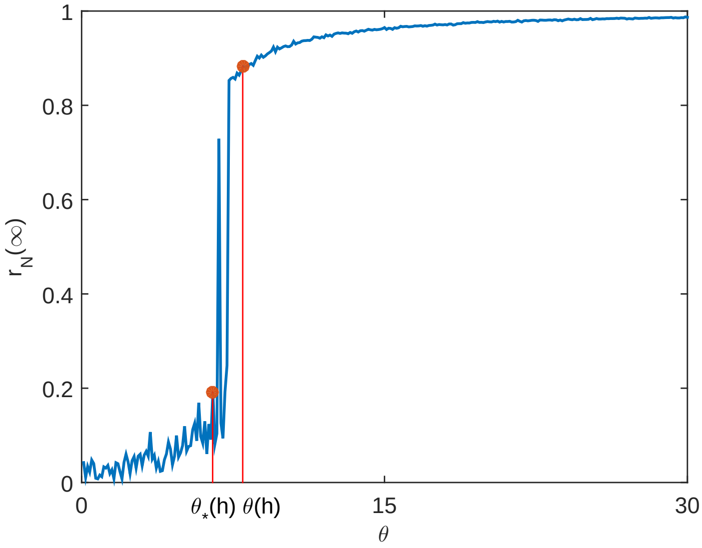

Figure 5: A plot of $r_N(\infty)$ depending on θ, h was fixed at $h = 4$. The two vertical red lines are $\theta_*(h)$ and $\theta(h)$. The following parameters were used: $N = 1000$, $\Delta t = 0.005$, $t \in [0, 30]$.

Note that in figure 3, it was decided to use T = 1000 instead of T = 100 (which was chosen for the plots for the other regions in the phase diagram). This interval was chosen since otherwise the behavior of Ψ_N could not be properly observed, in the simulations for figure 2 and 4 it was sufficient to choose a smaller time interval.

As for the the plots in figure 2 and 3 it is clear that the results are in agreement with the phase diagram in figure 1. The plots in figure 4 are not as conclusive, one should however take into account that there is an error caused by numerically simulating the process and because of finite size effects (One cannot simulate N coupled oscillators where $N \to \infty$). Besides these two reasons one should also consider that in comparison to the other two regions in the phase diagram, region 2 is relatively small which causes the parameters h and θ to be quite close to the critical values $\theta(h)$ and $\theta_*(h)$. The plot in figure 5 also shows that when simulating for $\theta_*(h) < \theta < \theta(h)$ there is some fluctuation in the results, but overall it seems fair to conclude that the numerical results agree with the analytic results.

6 Conclusions

The critical values for θ (the coupling parameter) and h (the strength of the external field) that were found have shown when phase transitions can be expected. The numerical simulations have shown that synchronization does indeed occur when the parameters h and θ satisfy the conditions for synchronization as determined in the analysis. The numerical results have shown that Ψ_N (the average phase of the oscillators) will be different for each simulation, while r_N (the measure of synchronization) shows very consistent behavior in the simulations (except for the case $\theta_*(h) < \theta < \theta(h)$). The notion that Ψ_N is different for each simulation agrees with the analytic results which said that if there exists a synchronized solution $r > 0$ for conditions (23), (24) then every pair (r, Ψ) for $\Psi \in [0, 2\pi)$ is a synchronized solution for those same conditions.

7 Acknowledgements

I would like to thank Wioletta Ruszel for all the help that she has given during this project. Wioletta has provided guidance by explaining the major steps that were necessary in the analysis of the model, as well as by discussing the results and making the subject more intuitively understandable.

8 References

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A Appendix: Solution to differential equation 18

The general solution to the nonhomogenous linear differential equation in (18) is

$$g(x,\eta) = \exp\left(2h\cos(x+\eta)\right)\left(\int [2\theta r_g \sin(\Psi_g - x) q_*^0(x,\eta) + c_1] \exp\left(-2h\cos(x+\eta)\right) \mathrm{d}x + c_2\right)$$

Where $c_2 \in \mathbb{R}$ is another integration constant. Note that $q_*^0(x,\eta) = k_*^0(\eta)\exp(2h\cos(x+\eta))$

$$\begin{aligned}
g(x,\eta) &= \exp\left(2h\cos(x+\eta)\right)\left(\int 2\theta r_g \sin(\Psi_g - x) k_*^0(\eta) + c_1 \exp\left(-2h\cos(x+\eta)\right) \mathrm{d}x + c_2\right) \\
&= \exp\left(2h\cos(x+\eta)\right) k_*^0(\eta) \int 2\theta r_g \sin(\Psi_g - x)\mathrm{d}x \\
&\quad + \exp\left(2h\cos(x+\eta)\right)\left(c_1 \int \exp\left(-2h\cos(x+\eta)\right) \mathrm{d}x + c_2\right) \\
&= q_*^0(x,\eta) 2\theta r_g \cos(\Psi_g - x) + \exp\left(2h\cos(x+\eta)\right)\left(c_1 \int \exp\left(-2h\cos(x+\eta)\right) \mathrm{d}x + c_2\right)
\end{aligned}$$

The periodic condition $g(0,\eta) = g(2\pi,\eta)$ can now be used to determine c_1. Since

$$\left[q_*^0(x,\eta) 2\theta r_g \cos(\Psi_g - x) + \exp\left(2h\cos(x+\eta)\right)\left(c_1 \int \exp\left(-2h\cos(x+\eta)\right) \mathrm{d}x + c_2\right)\right]_{x=0}^{x=2\pi} = 0$$

We clearly must have that $c_1 \int_0^{2\pi} \exp\left(-2h\cos(x+\eta)\right) \mathrm{d}x = 0$. Because $\exp\left(-2h\cos(x+\eta)\right)$ is a positive function, the integral is greater than zero:

$$\int_0^{2\pi} \exp\left(-2h\cos(x+\eta)\right) \mathrm{d}x > 0 \Rightarrow c_1 = 0$$

This results in the following expression for $g(x,\eta)$

$$g(x,\eta) = q_*^0(x,\eta) 2\theta r_g \cos(\Psi_g - x) + c_2 \exp(2h\cos(x+\eta)) \tag{27}$$

The condition $\int_0^{2\pi} g(x,\eta)\mathrm{d}x = 0$ can be used to solve for c_2.

$$\int_0^{2\pi} g(x,\eta)\mathrm{d}x = \int_0^{2\pi} q_*^0(x,\eta) 2\theta r_g \cos(\Psi_g - x)\mathrm{d}x + c_2 \int_0^{2\pi} \exp(2h\cos(x+\eta))\mathrm{d}x = 0$$

Using the following identity

$$\int_0^{2\pi} q_*^0(x,\eta)\mathrm{d}x = \int_0^{2\pi} k_*^0(\eta)\exp(2h\cos(x+\eta))\mathrm{d}x = 1 \Leftrightarrow \int_0^{2\pi} \exp(2h\cos(x+\eta))\mathrm{d}x = \frac{1}{k_*^0(\eta)}$$

Gives us the following expression for c_2:

$$c_2 = -k_*^0(\eta) \int_0^{2\pi} q_*^0(x,\eta) 2\theta r_g \cos(\Psi_g - x)\mathrm{d}x$$

Plugging this back into equation (27) yields the final expression for $g(x,\eta)$:

$$\begin{aligned}
g(x,\eta) &= q_*^0(x,\eta) 2\theta r_g \cos(\Psi_g - x) - \exp(2h\cos(x+\eta)) k_*^0(\eta) \int_0^{2\pi} q_*^0(x,\eta) 2\theta r_g \cos(\Psi_g - x)\mathrm{d}x \\
&= q_*^0(x,\eta) 2\theta r_g \cos(\Psi_g - x) - q_*^0(x,\eta) \int_0^{2\pi} q_*^0(x,\eta) 2\theta r_g \cos(\Psi_g - x)\mathrm{d}x \\
&= 2\theta r_g q_*^0(x,\eta)\left(\cos(\Psi_g - x) - \int_0^{2\pi} q_*^0(x,\eta)\cos(\Psi_g - x)\mathrm{d}x\right)
\end{aligned} \tag{28}$$

B Appendix: Simplifying $A(\Psi_g)$ and $B(\Psi_g)$

Simplifying the integrals in the definitions of $A(\Psi_g)$ and $B(\Psi_g)$ as shown below will show that $A(\Psi_g) = \frac{1}{2} - \frac{1}{2}\left(\frac{I_1(2h)}{I_0(2h)}\right)^2$ and $B(\Psi_g) \equiv 0$.

$$A(\Psi_g) := \int_0^{2\pi} \left[\int_0^{2\pi} \cos^2(\Psi_g - x) q_*^0(x,\eta) \mathrm{d}x - \left(\int_0^{2\pi} \cos^2(\Psi_g - x) q_*^0(x,\eta) \mathrm{d}x \right)^2 \right] \mu(\mathrm{d}\eta) \quad (29)$$

$$\begin{aligned} B(\Psi_g) := & \int_0^{2\pi} \Big[\int_0^{2\pi} \cos(\Psi_g - x) \sin(x - \Psi_g) q_*^0(x,\eta) \mathrm{d}x \\ & - \left(\int_0^{2\pi} \sin(x - \Psi_g) q_*^0(x,\eta) \mathrm{d}x \right) \left(\int_0^{2\pi} \cos(x - \Psi_g) q_*^0(x,\eta) \mathrm{d}x \right) \Big] \mu(\mathrm{d}\eta) \end{aligned} \quad (30)$$

Recall that

$$q_*^0(x,\eta) = k_*^0(\eta) \exp(2h \cos(x + \eta))$$

Note that the normalization constant can be expressed as a modified Bessel function of the first kind of order zero.

$$k_*^0(\eta) = \left(\int_0^{2\pi} \exp(2h \cos(x + \eta)) \mathrm{d}x \right)^{-1} = (2\pi I_0(2h))^{-1}$$

The following identities will be proven in the next subsections

$$\int_0^{2\pi} \sin(x) \exp(2h \cos(x)) \mathrm{d}x = 0 \quad (31)$$

$$\int_0^{2\pi} \sin(2x) \exp(2h \cos(x)) \mathrm{d}x = 0 \quad (32)$$

$$\int_0^{2\pi} \cos(x) q_*^0(x,\eta) \mathrm{d}x = \cos(\eta) \frac{I_1(2h)}{I_0(2h)} \quad (33)$$

$$\int_0^{2\pi} \sin(x) q_*^0(x,\eta) \mathrm{d}x = -\sin(\eta) \frac{I_1(2h)}{I_0(2h)} \quad (34)$$

$$\int_0^{2\pi} \cos(\Psi_g - x) q_*^0(x,\eta) \mathrm{d}x = \cos(\Psi_g + \eta) \frac{I_1(2h)}{I_0(2h)} \quad (35)$$

$$\int_0^{2\pi} \sin(x - \Psi_g) q_*^0(x,\eta) \mathrm{d}x = -\sin(\Psi_g + \eta) \frac{I_1(2h)}{I_0(2h)} \quad (36)$$

$$\int_0^{2\pi} \cos(\Psi_g - x) \sin(x - \Psi_g) q_*^0(x,\eta) \mathrm{d}x = -\frac{1}{2} \sin(2(\eta + \Psi_g)) \frac{I_2(2h)}{I_0(2h)} \quad (37)$$

$$\int_0^{2\pi} \cos^2(\Psi_g - x) q_*^0(x,\eta) \mathrm{d}x = \frac{1}{2} + \frac{1}{2} \left(\cos(2(\eta + \Psi_g)) \frac{I_2(2h)}{I_0(2h)} \right) \quad (38)$$

Equations (35) and (38) can be used to simplify the expression for $A(\Psi_g)$.

$$\begin{aligned} A(\Psi_g) &= \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \left(\cos(2(\eta + \Psi_g)) \frac{I_2(2h)}{I_0(2h)} \right) - \cos^2(\Psi_g + \eta) \left(\frac{I_1(2h)}{I_0(2h)} \right)^2 \right) \mu(\mathrm{d}\eta) \\ &= \frac{1}{2\pi} \Big[\pi + \int_0^{2\pi} \frac{1}{2} \left(\cos(2(\eta + \Psi_g)) \frac{I_2(2h)}{I_0(2h)} \right) \mathrm{d}\eta \\ &\quad - \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2} \cos(2(\Psi_g + \eta)) \right) \left(\frac{I_1(2h)}{I_0(2h)} \right)^2 \mathrm{d}\eta \Big] \\ &= \frac{1}{2} - \frac{1}{2} \left(\frac{I_1(2h)}{I_0(2h)} \right)^2 \end{aligned}$$

Equations (35), (36) and (37) can be used to simplify the expression for $B(\Psi_g)$

$$\begin{aligned}
B(\Psi_g) &= \int_0^{2\pi} \left(-\frac{1}{2}\sin(2(\eta+\Psi_g))\frac{I_2(2h)}{I_0(2h)} + \cos(\Psi_g+\eta)\sin(\Psi_g+\eta)\left(\frac{I_1(2h)}{I_0(2h)}\right)^2 \right)\mu(\mathrm{d}\eta) \\
&= \frac{1}{2\pi}\left[-\frac{1}{2}\frac{I_2(2h)}{I_0(2h)}\int_0^{2\pi}\sin(2(\eta+\Psi_g))\mathrm{d}\eta + \left(\frac{I_1(2h)}{I_0(2h)}\right)^2 \int_0^{2\pi}\frac{1}{2}\sin(2(\eta+\Psi_g))\mathrm{d}\eta \right] \\
&= 0
\end{aligned}$$

Proof of equations 31, 32

The proof of these identities is very simple since these functions have explicit antiderivatives.

$$\int_0^{2\pi}\sin(x)\exp(2h\cos(x))\mathrm{d}x = \left[-\frac{1}{2h}\exp(2h\cos(x))\right]_0^{2\pi} = 0$$

$$\int_0^{2\pi}\sin(2x)\exp(2h\cos(x))\mathrm{d}x = \left[-\frac{1}{2h^2}(2h\cos(x)\exp(2h\cos(x)) - \exp(2h\cos(x)))\right]_0^{2\pi} = 0$$

Proof of equations 33, 34

Evaluating the first integral

$$\begin{aligned}
\int_0^{2\pi}\cos(x)q_*^0(x,\eta)\mathrm{d}x &= \int_0^{2\pi}\cos(x)(2\pi I_0(2h))^{-1}\exp(2h\cos(x+\eta))\mathrm{d}x \\
&\overset{y=x+\eta}{=} \int_\eta^{2\pi+\eta}\cos(y-\eta)(2\pi I_0(2h))^{-1}\exp(2h\cos(y))\mathrm{d}y \\
&= \int_0^{2\pi}\cos(y-\eta)(2\pi I_0(2h))^{-1}\exp(2h\cos(y))\mathrm{d}y \\
&= (2\pi I_0(2h))^{-1}\left[\cos(\eta)\int_0^{2\pi}\cos(y)\exp(2h\cos(y))\mathrm{d}y \right. \\
&\qquad \left. + \sin(\eta)\int_0^{2\pi}\sin(y)\exp(2h\cos(y))\mathrm{d}y\right] \\
&\overset{(31)}{=} \cos(\eta)(2\pi I_0(2h))^{-1}\int_0^{2\pi}\cos(y)\exp(2h\cos(y))\mathrm{d}y \\
&= \cos(\eta)(2\pi I_0(2h))^{-1}2\pi I_1(2h) \\
&= \cos(\eta)\frac{I_1(2h)}{I_0(2h)}
\end{aligned}$$

The second integral can be handled in a similar fashion.

$$\begin{aligned}\int_0^{2\pi} \sin(x) q_*^0(x,\eta)\mathrm{d}x &\overset{y=x+\eta}{=} \int_\eta^{2\pi+\eta} \sin(y-\eta)(2\pi I_0(2h))^{-1}\exp(2h\cos(y))\mathrm{d}y \\ &= (2\pi I_0(2h))^{-1}\Big[\cos(\eta)\int_0^{2\pi}\sin(y)\exp(2h\cos(y))\mathrm{d}y \\ &\quad -\sin(\eta)\int_0^{2\pi}\cos(y)\exp(2h\cos(y))\mathrm{d}y\Big] \\ &\overset{(31)}{=} -\sin(\eta)(2\pi I_0(2h))^{-1}\int_0^{2\pi}\cos(y)\exp(2h\cos(y))\mathrm{d}y \\ &= -\sin(\eta)(2\pi I_0(2h))^{-1} 2\pi I_1(2h) \\ &= -\sin(\eta)\frac{I_1(2h)}{I_0(2h)}\end{aligned}$$

Proof of equations 35, 36

These integrals can be evaluated using equations (33), (34).

$$\begin{aligned}\int_0^{2\pi}\cos(\Psi_g - x)q_*^0(x,\eta)\mathrm{d}x &= \cos(\Psi_g)\int_0^{2\pi}\cos(x)q_*^0(x,\eta)\mathrm{d}x + \sin(\Psi_g)\int_0^{2\pi}\sin(x)q_*^0(x,\eta)\mathrm{d}x \\ &= (\cos(\Psi_g)\cos(\eta) - \sin(\Psi_g)\sin(\eta))\frac{I_1(2h)}{I_0(2h)} \\ &= \cos(\Psi_g+\eta)\frac{I_1(2h)}{I_0(2h)}\end{aligned}$$

And similarly,

$$\begin{aligned}\int_0^{2\pi}\sin(x-\Psi_g)q_*^0(x,\eta)\mathrm{d}x &= \cos(\Psi_g)\int_0^{2\pi}\sin(x)q_*^0(x,\eta)\mathrm{d}x - \sin(\Psi_g)\int_0^{2\pi}\cos(x)q_*^0(x,\eta)\mathrm{d}x \\ &= (-\cos(\Psi_g)\sin(\eta) - \sin(\Psi_g)\cos(\eta))\frac{I_1(2h)}{I_0(2h)} \\ &= -\sin(\Psi_g+\eta)\frac{I_1(2h)}{I_0(2h)} \qquad (39)\end{aligned}$$

Proof of equations 37, 38

$$\int_0^{2\pi} \cos(\Psi_g - x)\sin(x - \Psi_g) q_*^0(x,\eta)\mathrm{d}x =$$

$$\begin{aligned}
&= \int_0^{2\pi} \frac{1}{2}\sin(2(x-\Psi_g)) q_*^0(x,\eta)\mathrm{d}x \\
&= (2\pi I_0(2h))^{-1}\frac{1}{2}\int_0^{2\pi} \sin(2(x-\Psi_g))\exp(2h\cos(x+\eta))\mathrm{d}x \\
&\overset{y=x+\eta}{=} (2\pi I_0(2h))^{-1}\frac{1}{2}\int_0^{2\pi} \sin(2(y-(\Psi_g+\eta)))\exp(2h\cos(y))\mathrm{d}y \\
&= (2\pi I_0(2h))^{-1}\frac{1}{2}\Big[\cos(2(\Psi_g+\eta))\int_0^{2\pi}\sin(2y)\exp(2h\cos(y))\mathrm{d}y \\
&\quad - \sin(2(\Psi_g+\eta))\int_0^{2\pi}\cos(2y)\exp(2h\cos(y))\mathrm{d}y\Big] \\
&\overset{(32)}{=} -(2\pi I_0(2h))^{-1}\frac{1}{2}\cos(2(\Psi_g+\eta))\int_0^{2\pi}\cos(2y)\exp(2h\cos(y))\mathrm{d}y \\
&= -\frac{1}{2}\sin(2(\eta+\Psi_g))(2\pi I_0(2h))^{-1} I_2(2h) 2\pi \\
&= -\frac{1}{2}\sin(2(\eta+\Psi_g))\frac{I_2(2h)}{I_0(2h)}
\end{aligned}$$

$$\int_0^{2\pi} \cos^2(\Psi_g - x) q_*^0(x,\eta)\mathrm{d}x =$$

$$\begin{aligned}
&= \int_0^{2\pi} \left(\frac{1}{2} + \frac{1}{2}\cos(2(\Psi_g - x))\right) q_*^0(x,\eta)\mathrm{d}x \\
&= \frac{1}{2} + \frac{(2\pi I_0(2h))^{-1}}{2}\int_0^{2\pi}\cos(2(\Psi_g - x))\exp(2h\cos(x+\eta))\mathrm{d}x \\
&\overset{y=x+\eta}{=} \frac{1}{2} + \frac{(2\pi I_0(2h))^{-1}}{2}\int_0^{2\pi}\cos(2(y-(\eta+\Psi_g)))\exp(2h\cos(y))\mathrm{d}y \\
&= \frac{1}{2} + \frac{(2\pi I_0(2h))^{-1}}{2}\Big[\int_0^{2\pi}\cos(2y)\cos(2(\eta+\Psi_g))\exp(2h\cos(y))\mathrm{d}y \\
&\quad + \int_0^{2\pi}\sin(2y)\sin(2(\eta+\Psi_g))\exp(2h\cos(y))\mathrm{d}y\Big] \\
&\overset{(32)}{=} \frac{1}{2} + \frac{(2\pi I_0(2h))^{-1}}{2}\cos(2(\eta+\Psi_g))\int_0^{2\pi}\cos(2y)\exp(2h\cos(y))\mathrm{d}y \\
&= \frac{1}{2} + \frac{1}{2}\left(\cos(2(\eta+\Psi_g))\frac{I_2(2h)}{I_0(2h)}\right)
\end{aligned}$$

C Appendix: Integrals in the self-consistency relation

Evaluating the integral in equation (24).

$$\begin{aligned}
&\int_0^{2\pi} \sin(x-\Psi_q)q(x,\eta)\mathrm{d}x = \\
&\quad = \int_0^{2\pi} \cos\left(\frac{\pi}{2} - x + \Psi_q\right) q(x,\eta)\mathrm{d}x \\
&\quad = k(\eta) \int_0^{2\pi} \cos\left(\frac{\pi}{2} - x + \Psi_q\right) \exp(2\theta r_q \cos(x-\Psi_q) + 2h\cos(x+\eta))\mathrm{d}x \\
&\quad \overset{y=\frac{\pi}{2}-x+\Psi_q}{=} k(\eta) \int_{-\frac{3}{2}\pi+\Psi_q}^{\frac{\pi}{2}+\Psi_q} \cos(y) \exp\left(2\theta r_q \cos\left(\frac{\pi}{2} - y\right) + 2h\cos\left(\Psi_q + \frac{\pi}{2} + \eta - y\right)\right) \mathrm{d}y \\
&\quad = k(\eta) \int_0^{2\pi} \cos(y) \exp\Bigg(2\theta r_q \sin(y) + 2h\cos\left(\Psi_q + \frac{\pi}{2} + \eta\right)\cos(y) + \\
&\qquad + 2h\sin\left(\Psi_q + \frac{\pi}{2} + \eta\right)\sin(y)\Bigg)\mathrm{d}y \\
&\quad = k(\eta) \int_0^{2\pi} \cos(y) \exp\Bigg(\cos(y)2(-h\sin(\Psi_q+\eta)) + \sin(y)2(\theta r_q + h\cos(\Psi_q+\eta))\Bigg)\mathrm{d}y \\
&\quad \overset{\text{Lemma 4.1}}{=} \frac{-h\sin(\Psi_q+\eta)k(\eta)2\pi}{\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q+\eta)}} I_1\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q+\eta)}\right) \\
&\quad \overset{(13)}{=} \frac{-h\sin(\Psi_q+\eta)}{\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q+\eta)}} \frac{I_1\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q+\eta)}\right)}{I_0\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q+\eta)}\right)}
\end{aligned}$$

Using the following property

$$\frac{\mathrm{d}}{\mathrm{d}z} I_0(z) = I_1(z)$$

One can see that the integral in equation (24) can be evaluated

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} \sin(x-\Psi_q)q(x,\eta)\mathrm{d}x\mu(\mathrm{d}\eta) = \\
&\quad = \int_0^{2\pi} \frac{-h\sin(\Psi_q+\eta)}{\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q+\eta)}} \frac{I_1\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q+\eta)}\right)}{I_0\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q+\eta)}\right)} \mu(\mathrm{d}\eta) \\
&\quad = \frac{1}{2\pi}\left[\frac{1}{2\theta r_q} \ln\left(I_0\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\Psi_q+\eta)}\right)\right)\right]_{\eta=0}^{\eta=2\pi} \\
&\quad = 0
\end{aligned}$$

The integral in equation 23 can be handled similarly. However, it does not seem to have an explicit antiderivative.

$$\begin{aligned}
&\int_0^{2\pi} \cos(x-\Psi_q)q(x,\eta)\mathrm{d}x = \\
&\qquad = k(\eta)\int_0^{2\pi} \cos(x-\Psi_q)\exp(2\theta r_q\cos(x-\Psi_q)+2h\cos(x+\eta))\mathrm{d}x \\
&\qquad \overset{y=x-\Psi_q}{=} k(\eta)\int_{-\Psi_q}^{2\pi-\Psi_q} \cos(y)\exp\left(2\theta r_q\cos(y)+2h\cos\left(y+\Psi_q+\eta\right)\right)\mathrm{d}y \\
&\qquad = k(\eta)\int_0^{2\pi}\cos\left(y\right)\exp\Big(\cos(y)2(\theta r_q+h\cos(\Psi_q+\eta))+\sin(y)2(-h\sin(\Psi_q+\eta))\Big)\mathrm{d}y \\
&\qquad \overset{\text{Lemma 4.1}}{=} \frac{(\theta r_q+h\cos(\Psi_q+\eta))k(\eta)2\pi}{\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\Psi_q+\eta)}} I_1\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\Psi_q+\eta)}\right) \\
&\qquad \overset{(13)}{=} \frac{\theta r_q+h\cos(\Psi_q+\eta)}{\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\Psi_q+\eta)}} \frac{I_1\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\Psi_q+\eta)}\right)}{I_0\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\Psi_q+\eta)}\right)}
\end{aligned}$$

D Appendix: Properties of the function F

Using equation (35)

$$F(0) = \int_0^{2\pi} \cos(\Psi_q+\eta)\frac{I_1(2h)}{I_0(2h)}\mu(\mathrm{d}\eta) = 0$$

It turns out that F does not depend on Ψ_q because the integrand in the final integral is 2π-periodic in η.

$$\begin{aligned}
F(r_q) &= \int_0^{2\pi}\int_0^{2\pi}\cos(x-\Psi_q)q(x,\eta)\mathrm{d}x\mu(\mathrm{d}\eta) = \\
&= \int_0^{2\pi} \frac{\theta r_q+h\cos(\Psi_q+\eta)}{\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\Psi_q+\eta)}} \frac{I_1\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\Psi_q+\eta)}\right)}{I_0\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\Psi_q+\eta)}\right)}\mu(\mathrm{d}\eta) \\
&\overset{y=\eta+\Psi_q}{=} \frac{1}{2\pi}\int_{\Psi_q}^{2\pi+\Psi_q} \frac{\theta r_q+h\cos(y)}{\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(y)}} \frac{I_1\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(y)}\right)}{I_0\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(y)}\right)}\mathrm{d}y \\
&= \int_0^{2\pi} \frac{\theta r_q+h\cos(\eta)}{\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta)}} \frac{I_1\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta)}\right)}{I_0\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta)}\right)}\mu(\mathrm{d}\eta) \qquad (40)
\end{aligned}$$

Proof of $\lim_{r_q \to \infty} F(r_q) = 1$

$$\lim_{r_q \to \infty} F(r_q) = \lim_{r_q \to \infty} \int_0^{2\pi} \frac{\theta r_q + h\cos(\eta)}{\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\eta)}} \frac{I_1\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\eta)}\right)}{I_0\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\eta)}\right)} \mu(\mathrm{d}\eta)$$

Using the following two properties

$$I_1(z) = \frac{1}{2\pi}\int_0^{2\pi} \cos(x)\exp(z\cos(x))\mathrm{d}x \leq \frac{1}{2\pi}\int_0^{2\pi} \exp(z\cos(x))\mathrm{d}x = I_0(z)$$

$$\frac{\partial}{\partial z} I_v(z) = I_{v-1}(z) - \frac{v}{z} I_v(z)$$

One can evaluate the following limit

$$\lim_{z\to\infty} \frac{I_1(z)}{I_0(z)} \overset{\text{L'Hôpital}}{=} \lim_{z\to\infty} \frac{I_0(z) - \frac{1}{z}I_1(z)}{I_1(z)} = \lim_{z\to\infty} \frac{I_0(z)}{I_1(z)} - \frac{1}{z} = \lim_{z\to\infty} \frac{I_0(z)}{I_1(z)} \geq \lim_{z\to\infty} \frac{I_1(z)}{I_1(z)} = 1$$

$$\lim_{z\to\infty} \frac{I_1(z)}{I_0(z)} \leq \lim_{z\to\infty} \frac{I_0(z)}{I_0(z)} = 1$$

Conclusion:

$$\lim_{z\to\infty} \frac{I_1(z)}{I_0(z)} = 1$$

Therefore we also have:

$$\lim_{r_q\to\infty} \frac{I_1\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\eta)}\right)}{I_0\left(2\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\eta)}\right)} = 1$$

Now consider the limit

$$\begin{aligned}\lim_{r_q\to\infty} \frac{\theta r_q + h\cos(\eta)}{\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\eta)}} &\leq \lim_{r_q\to\infty} \frac{\theta r_q + h}{\sqrt{\theta^2 r_q^2 + h^2 - 2\theta r_q h}} \\ &= \lim_{r_q\to\infty} \frac{\theta r_q + h}{|\theta r_q - h|} = 1\end{aligned}$$

$$\begin{aligned}\lim_{r_q\to\infty} \frac{\theta r_q + h\cos(\eta)}{\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\eta)}} &= \lim_{r_q\to\infty} \frac{\theta r_q + h\cos(\eta)}{\sqrt{(\theta r_q + h\cos(\eta))^2 + (h\sin(\eta))^2}} \\ &\geq \lim_{r_q\to\infty} \frac{\theta r_q - h}{|\theta r_q + h\cos(\eta)| + |h\sin(\eta)|} \\ &\geq \lim_{r_q\to\infty} \frac{\theta r_q - h}{(\theta r_q + h) + h} = 1\end{aligned}$$

Therefore:

$$\lim_{r_q\to\infty} \frac{\theta r_q + h\cos(\eta)}{\sqrt{\theta^2 r_q^2 + h^2 + 2\theta r_q h\cos(\eta)}} = 1$$

To conclude:

$$\lim_{r_q\to\infty} F(r_q) = \int_0^{2\pi} 1\mu(\mathrm{d}\eta) = 1$$

The first and second derivative of F

Finding the first derivative is very straightforward.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r_q}F(r_q) &= \int_0^{2\pi}\int_0^{2\pi}\cos(x)\left[\frac{\mathrm{d}}{\mathrm{d}r_q}q(x,\eta)\right]\mathrm{d}x\mu(\mathrm{d}\eta) \\
&\overset{(13)}{=} \int_0^{2\pi}\int_0^{2\pi}\cos(x)\Bigg[\frac{\mathrm{d}}{\mathrm{d}r_q}\left(2\pi I_0\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta)}\right)\right)^{-1}\cdot \\
&\quad\cdot\exp(2\theta r_q\cos(x)+2h\cos(x+\eta))\Bigg]\mathrm{d}x\mu(\mathrm{d}\eta) \\
&= \int_0^{2\pi}\int_0^{2\pi}\cos(x)\Bigg[2\theta\cos(x)\exp(2\theta r_q\cos(x)+2h\cos(x+\eta))\cdot \\
&\quad\cdot\left(2\pi I_0\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta)}\right)\right)^{-1}+ \\
&\quad+\exp(2\theta r_q\cos(x)+2h\cos(x+\eta))\cdot-\left(2\pi I_0\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta)}\right)\right)^{-2}\cdot \\
&\quad\cdot(\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta))^{-\frac{1}{2}}(2r_q\theta^2+2h\theta\cos(\eta))\Bigg]\mathrm{d}x\mu(\mathrm{d}\eta) \\
&= \int_0^{2\pi}\int_0^{2\pi}\cos(x)\Bigg[2\theta\cos(x)q(x,\eta)+ \\
&\quad-2\theta q(x,\eta)\frac{\theta r_q+h\cos(\eta)}{\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta)}}\frac{I_1\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta)}\right)}{I_0\left(2\sqrt{\theta^2r_q^2+h^2+2\theta r_q h\cos(\eta)}\right)}\Bigg]\mathrm{d}x\mu(\mathrm{d}\eta) \\
&\overset{(40)}{=} \int_0^{2\pi}\int_0^{2\pi}\cos(x)\left[2\theta\cos(x)q(x,\eta)-2\theta q(x,\eta)\int_0^{2\pi}\cos(x)q(x,\eta)\mathrm{d}x\right]\mathrm{d}x\mu(\mathrm{d}\eta) \\
&= 2\theta\int_0^{2\pi}\left[\int_0^{2\pi}\cos^2(x)q(x,\eta)\mathrm{d}x-\left(\int_0^{2\pi}\cos(x)q(x,\eta)\mathrm{d}x\right)^2\right]\mu(\mathrm{d}\eta) \\
&= 2\theta\mathbb{E}_\mu\left[\mathbb{V}\mathrm{ar}_{q(x,\eta)}(\cos(X))\right]>0
\end{aligned}$$

Calculating the second derivative can be handled similarly.

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}r_q^2}F(r_q) &= \int_0^{2\pi}\int_0^{2\pi} 2\theta\cos^2(x)\left[2\theta\cos(x)q(x,\eta) - 2\theta q(x,\eta)\int_0^{2\pi}\cos(x)q(x,\eta)\mathrm{d}x\right]\mathrm{d}x + \\
&\quad - \left[\frac{\mathrm{d}}{\mathrm{d}r_q}2\theta\left(\int_0^{2\pi}\cos(x)q(x,\eta)\mathrm{d}x\right)^2\right]\mu(\mathrm{d}\eta) \\
&= \int_0^{2\pi} 4\theta^2 \int_0^{2\pi}\cos^3(x)q(x,\eta)\mathrm{d}x + \\
&\quad - 4\theta^2\left(\int_0^{2\pi}\cos^2(x)q(x,\eta)\mathrm{d}x\right)\left(\int_0^{2\pi}\cos(x)q(x,\eta)\mathrm{d}x\right) \\
&\quad - 4\theta\left(\int_0^{2\pi}\cos(x)q(x,\eta)\mathrm{d}x\right)\left[\frac{\mathrm{d}}{\mathrm{d}r_q}\int_0^{2\pi}\cos(x)q(x,\eta)\mathrm{d}x\right]\mu(\mathrm{d}\eta) \\
&= \int_0^{2\pi}\left[4\theta^2\int_0^{2\pi}\cos^3(x)q(x,\eta)\mathrm{d}x + \right. \\
&\quad - 12\theta^2\left(\int_0^{2\pi}\cos^2(x)q(x,\eta)\mathrm{d}x\right)\left(\int_0^{2\pi}\cos(x)q(x,\eta)\mathrm{d}x\right) \\
&\quad \left. + 8\theta^2\left(\int_0^{2\pi}\cos(x)q(x,\eta)\mathrm{d}x\right)^3\right]\mu(\mathrm{d}\eta) \\
&= 4\theta^2\mathbb{E}_\mu\left[\mathbb{E}_{q(x,\eta)}(\cos^3(X)) - 3\mathbb{E}_{q(x,\eta)}(\cos^2(X))\mathbb{E}_{q(x,\eta)}(\cos(X)) + 2\mathbb{E}_{q(x,\eta)}(\cos(X))^3\right] \\
&= 4\theta^2\mathbb{E}_\mu\left[\mathbb{E}_{q(x,\eta)}\left(\left[\cos(X) - \mathbb{E}_{q(x,\eta)}(\cos(X))\right]^3\right)\right]
\end{aligned}
$$