Delft University of Technology
Faculty of Electrical Engineering, Mathematics & Computer Science
Delft Institute of Applied Mathematics

Tanner's law in the case of partial wetting

(Dutch title: De wet van Tanner in het geval van onvolledige bevochtiging)

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“Tanner’s law in the case of partial wetting”

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Abstract

This thesis considers the thin-film equation in partial wetting. The mobility in this equation is given by $h^3 + \lambda^{3-n}h^n$, where h is the film height, λ is the slip length and n is the mobility exponent. The partial wetting regime implies the boundary condition $\frac{dh}{dz} > 0$ at the triple junction. The asymptotics as $h \downarrow 0$ are investigated. This is done by using a dynamical system for the error between the solution and the microscopic contact angle. Using the linearized version of the dynamical system, values for n when resonances occur are found. These resonances lead to a different behaviour for the solution as $h \downarrow 0$, so the asymptotics are found to be different for different values of n.

Together with the asymptotics for $h \to \infty$ as found in [Giacomelli et al., 2016], the solution to the thin-film equation in partial wetting can be characterized. Also, via this solution, the relation between the microscopic and macroscopic contact angles can be analyzed. From the main result of this thesis, it can be seen that the macroscopic Tanner law for the contact angle depends smoothly on the microscopic contact angle.

Contents

Chapter 1

Introduction

1.1 The problem

In this thesis the following thin-film equation with boundary conditions is studied:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial z}\left((h^3 + \lambda^{3-n}h^n)\frac{\partial^3 h}{\partial z^3}\right) = 0 \qquad \text{for } t > 0 \text{ and } z > Z(t) \tag{1.1a}$$

$$h = 0 \qquad \text{for } t > 0 \text{ and } z = Z(t) \tag{1.1b}$$

$$\frac{\partial h}{\partial z} = k > 0 \qquad \text{for } t > 0 \text{ and } z = Z(t) \tag{1.1c}$$

$$\lim_{z \to Z(t)^+} (h^2 + \lambda^{3-n}h^{n-1})\frac{\partial^3 h}{\partial z^3} = \frac{\partial Z}{\partial t}(t) \qquad \text{for } t > 0. \tag{1.1d}$$

Here, $h(t, z)$ gives the height of a liquid thin-film on a flat surface at time $t > 0$ and position $z \in \mathbb{R}$. This is visualised in figure 1.1. The point where the liquid, gas and solid meet is called the triple junction. Equation (1.1a) is a lubrication model, which means that it describes the flow of the fluid of a thin and viscous film. It has the form of a continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial z}(\rho u) = 0.$$

Here, ρ is the fluid density and u is the velocity of the flow of the fluid. In the case of equation (1.1a), the fluid density is equal to the film height h and the velocity of the flow is given by $(h^2 + \lambda^{3-n}h^{n-1})\frac{\partial^3 h}{\partial z^3}$. The equation can be derived from the Navier-Stokes equations, which has been done in detail in [Oron et al., 1997, chapter 2 section B].

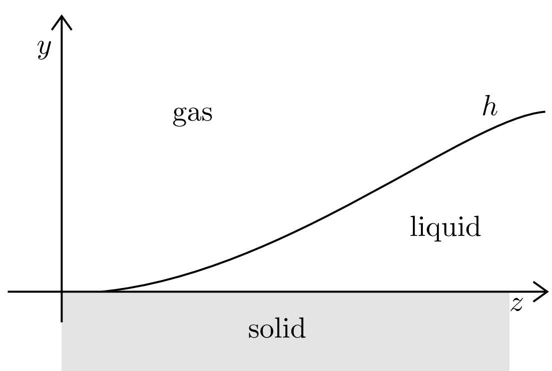


Figure 1.1: Example of a thin-film as described by (1.1)

The exponent n is called the mobility exponent. We choose $n \in [1, 3)$. This is because on the one hand, if $n \leq 0$ the speed of propagation of the film is infinite, and if $n \in (0, 1)$ the film height h can be negative, as demonstrated for instance in [Bowen and Witelski, 2019]. On the other hand, if $n \geq 3$, the boundary of the film cannot move. This is also the case if the slip length λ equals zero.

Translation invariance in the third space dimension is assumed, which is perpendicular to y and z. Also, it is assumed that the film covers the interval $(Z(t), \infty)$ and that it has a free boundary at $z = Z(t)$. The function $Z(t)$ depends on time and denotes the place of the triple junction. Condition (1.1b) gives that the height of the thin-film at the triple junction is zero. The next condition, (1.1c), says that the contact angle between the solid and the film at the triple junction is equal to $k > 0$. This tells us that we are dealing with a liquid thin-film in a partial wetting state, which will be explained in the following section. Condition (1.1d) implies that the transport velocity $(h^2 + \lambda^{3-n}h^n)\frac{\partial^3 h}{\partial z^3}$ of the flow has to match the velocity $\frac{dZ}{dt}$ of the free boundary, when approaching the contact line.

1.1.1 Partial and complete wetting

For a droplet on a surface, there are three surface tensions that will be looked into. The tensions are shown in figure 1.2. In [Bonn et al., 2009], Young's equation is stated, which gives the relation between the contact angle θ and these surface tensions. This relation is the following:

$$\sigma_{gs} = \sigma_{ls} + \cos(\theta)\sigma_{gl}.$$

If $\sigma_{gs} < \sigma_{ls} + \sigma_{gl}$, then the liquid thin-film is said to partially wet the solid. In this case, the contact angle is a finite number greater than zero. If on the other hand $\sigma_{gs} \geq \sigma_{ls} + \sigma_{gl}$, then the contact angle must be zero. The thin-film covers the entire solid, so this is called complete wetting.

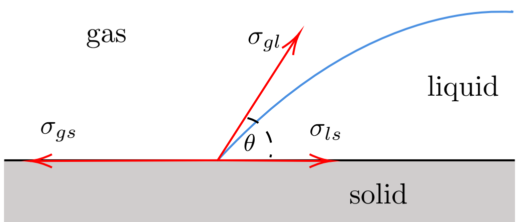
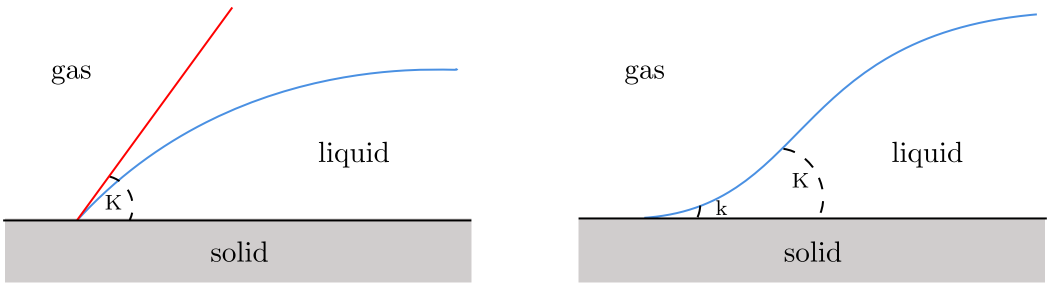


Figure 1.2: Example of a thin-film as described by (1.1)

1.1.2 Contact angles

Two different kinds of contact angles are distinguised, namely the microscopic contact angle and the macroscopic contact angle. The macroscopic contact angle is the angle at the triple junction that is fairly easy to measure optically, see figure 1.3a. This angle is dynamic, so it depends on the velocity of the film. When we look near the triple junction it can be seen that the fluid behaves differently there, see figure 1.3b. The microscopic contact angle is determined by Young's equation, as mentioned in the previous section. Because of the microscopic scale, it is difficult to measure this angle. The main goal of this thesis is to prove a theorem which gives a relation between the microscopic and the macroscopic contact angle.



(a) The macroscopic contact angle K

(b) The previous picture zoomed in near the triple junction. The macroscopic contact angle K and the microscopic contact angle k are shown.

Figure 1.3

1.2 Structure of this thesis

This thesis is mostly based on the paper "*Rigorous asymptotics of traveling-wave solutions to the thin-film equation and Tanner's law*" by [Giacomelli et al., 2016]. In this paper, the authors look at problem (1.1) in the case of complete wetting. This means that the boundary condition (1.1c) changes into $\frac{\partial h}{\partial z} = 0$ for $t > 0$ and $z = Z(t)$. Also, the mobility exponent is taken in the interval $n \in \left(\frac{3}{2}, \frac{7}{3}\right)$. We will use some of the methods used in this paper to look at problem (1.1), but the analysis at the triple junction is different. Because the microscopic contact angle is nonzero, the dependence of the macroscopic angle on the microscopic angle can be analyzed. This could not be done in the paper by [Giacomelli et al., 2016], because the contact angle in that case is equal to zero.

In chapter 2 of this thesis, the prerequisite knowledge from linear algebra and analysis is given for reference. Also, some theory of dynamical systems is introduced, which is needed in chapter 4. After introducing the prerequisite knowledge, we proceed in a similar manner as in section 1.1 of [Giacomelli et al., 2016] where the problem (1.1) is rewritten as a third order ordinary differential equation. This is explained in chapter 3. Also, a coordinate transformation is done and Tanner's law is explained. At the end of this chapter, the main theorem of this thesis is stated. This theorem will give a link between the macroscopic and microscopic contact angle. It will take the rest of this thesis to prove this theorem, which is done in chapter 7. This theorem is similar to [Giacomelli et al., 2016, theorem 2.1], but the asymptotic behavior when $H \downarrow 0$ changes.

In the next chapters, the necessary steps are taken to prove a proposition which is crucial for proving the main theorem. In chapter 4, the ordinary differential equation obtained in chapter 3 will be rewritten as a dynamical system. This system will be linearized and in chapter 5, the parameterization of the unstable manifold will be analyzed. In the next part of this chapter, we will look into the resonances of the system, that is when one eigenvalue of the linearized system is a multiple of another eigenvalue. These resonances occur because of the choice $n \in [1, 3)$. This will be done in the same way as in section 4.3 of [Belgacem et al., 2016]. We will find that for certain values of n resonances occur. This is why a 'resonant' and 'non-resonant' case will be distinguished for the propositions hereafter.

In chapter 6, two differential equations for the resonant and non-resonant case will be analyzed, which will both be rewritten as a fixed point problem. The fixed point theorem of Banach can be used to show the existence and uniqueness of solutions to these differential equations. This will be followed by proposition 7.1 and 7.2, which will be crucial to proving the main theorem. Using the solutions found in chapter 6, a proposition similar to [Giacomelli et al., 2016, prop 3.2] will be proven in chapter 7. The above result combined with lemma 7.3, lemma 7.4 and corollary 7.5 will lead to the proof of the main theorem.

Chapter 2

Prerequisite knowledge

In this chapter the necessary knowledge on linear algebra, analysis and dynamical systems is given. This is done by stating definitions and theorems, without giving the proofs. A complete introduction and proofs to the theorems can be found in [Fraleigh and Beauregard, 2014], [Sadun, 2008], [Carothers, 2000], [Teschl, 2012], and [de Pagter and Groenevelt, 2017].

2.1 Linear algebra

We start by looking into the necessary knowledge of linear algebra.

Definition 2.1. *We say that λ is an eigenvalue of an $n \times n$ matrix A if*

$$Av = \lambda v$$

for a nonzero column vector v. We call v an eigenvalue of A.
The set E_λ which consists of the zero vector and all eigenvectors of λ is called the eigenspace of λ. This is a subspace of $\mathbb{R}^n$.
The algebraic multiplicity of an eigenvalue λ is the multiplicity of λ as a root of the corresponding characteristic polynomial

$$\det(\lambda I - A).$$

The geometric multiplicity of an eigenvalue λ is the dimension of the eigenspace E_λ.

Theorem 2.2. *An $n \times n$ matrix A is diagonalizable if and only if the algebraic and geometric multiplicity of every eigenvalue are equal. That is, $A = PDP^{-1}$ for an invertible matrix P and a diagonal matrix D, where the i^{th} column of P gives the eigenvector corresponding to the eigenvalue in the i^{th} column of D.*

Definition 2.3. *The nonzero vector ξ is a generalized eigenvector corresponding to the eigenvalue λ of the $n \times n$ matrix A if*

$$(A - \lambda I)^p \xi = 0$$

for some positive integer p. The generalized eigenspace $\tilde{E}_\lambda$ is the subspace spanned by the generalized eigenvectors of λ.

Theorem 2.4. *For every $n \times n$ matrix A an invertible matrix T exists such that*

$$T^{-1}AT = J,$$

where J, the Jordan form, has a blockdiagonal structure

$$J = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & J_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & J_k \end{pmatrix}.$$

Each block J_i has the form

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_i \end{pmatrix},$$

where λ_i is an eigenvalue of A. The geometric multiplicity of eigenvalue λ_i gives the number of blocks with λ_i on the diagonal. The algebraic multiplicity of λ_i gives the total dimension of all blocks with this eigenvalue on the diagonal.

Theorem 2.5. (Neumann series) *For an $n \times n$ matrix A it holds that*

$$\sum_{n=0}^{\infty} A^n = (I - A)^{-1}$$

if and only if $||A|| < 1$, where $|| \cdot ||$ is the operator norm. This is the geometric series for matrices.

2.2 Analysis

The following two definitions are useful for analysing the behaviour of a function if its argument tends to some number in $\mathbb{R}$ or to infinity.

Definition 2.6. (Big-O notation). *We write*

$$f(x) = \mathcal{O}\big(g(x)\big) \quad \text{as } x \to \infty$$

if there exists an $M > 0$ and $x_1 \in \mathbb{R}$ such that $|f(x)| \leq M|g(x)|$ for all $x \geq x_1$.
Also,

$$f(x) = \mathcal{O}\big(g(x)\big) \quad \text{as } x \to 0$$

if there exist $r > 0$ and $M < \infty$ such that $|f(x)| \leq M|g(x)|$ for all $x \in [-r, r]$.

Definition 2.7. (Little-o notation). *We say that*

$$f(x) = o(g(x)) \quad \text{as } x \to a$$

if and only if $\lim_{x\to a} \frac{f(x)}{g(x)} = 0$, when $g(x) \neq 0$.

The following theorem is needed for proving the main theorem.

Theorem 2.8. (Implicit function theorem) *Let $U \subseteq \mathbb{R}^{n+k}$ be open and $\mathbf{f} : U \to \mathbb{R}^n$ continuously differentiable. Let $\mathbf{a} \in \mathbb{R}^k$ and $\mathbf{b} \in \mathbb{R}^n$ be such that $(\mathbf{a}, \mathbf{b}) \in U$ and*

$$\mathbf{f}(\mathbf{a}, \mathbf{b}) = 0, \quad \det \frac{\partial \mathbf{f}}{\mathbf{y}}(\mathbf{a}, \mathbf{b}) = \det \begin{pmatrix} \frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial y_1} & \cdots & \frac{\partial f_n}{\partial y_n} \end{pmatrix} \neq 0.$$

Then there exist open neighborhoods V of $\mathbf{a}$ and W of $\mathbf{b}$ and a continuously differentiable map $\mathbf{g} : V \to W$ such that

- $\mathbf{f}(\mathbf{x}, \mathbf{g}(\mathbf{x})) = \mathbf{0}$ *for all* $\mathbf{x} \in V$ *and* $\mathbf{g}(\mathbf{a}) = \mathbf{b}$*,*
- $\mathbf{f}(\mathbf{x}, \mathbf{y}) \neq \mathbf{0}$ *for all* $(\mathbf{x}, \mathbf{y}) \in V \times W$ *with* $\mathbf{y} \neq \mathbf{g}(\mathbf{x})$*.*

The next theorem will be used to show the existence and uniqueness of a solution to an ordinary differential equation.

Definition 2.9. *Let (M, d) be a metric space. The map $f : M \to M$ is called a contraction if there exists some constant $0 \leq \alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$.*

Theorem 2.10. (Banach's fixed point theorem) *Let (M, d) be a complete metric space. If $f : M \to M$ is a contraction, then f has a unique fixed point.*

2.3 Dynamical systems

In this section, some definitions from the theory of dynamical systems are given. These will be referred to when necessary in the next chapters of this thesis.

Definition 2.11. *A dynamical system is a semigroup G with an identity element e which acts on a set M. That is, there exists a map*

$$\begin{aligned} T : G \times M &\to M \\ (g, x) &\mapsto T_g(x) \end{aligned}$$

such that

$$T_g \circ T_h = T_{g \circ h}.$$

Also, $T_e = I$.

A dynamical system is continuous if $G = \mathbb{R}^+$ or $G = \mathbb{R}$.

Definition 2.12. *For the linear system $\dot{x} = Ax$ we distinguish the following manifolds:*

- *The linear manifold $E^+(e^A)$ is called the stable manifold, and is spanned by the (generalized) eigenvectors that correspond to eigenvalues with negative real part.*
- *The linear manifold $E^-(e^A)$ is called the unstable manifold, and is spanned by the (generalized) eigenvectors that correspond to eigenvalues with positive real part.*
- *The linear manifold $E^0(e^A)$ is called the center manifold, and is spanned by the (generalized) eigenvectors that correspond to eigenvalues with zero real part.*

Systems where all eigenvalues have nonzero real part are called hyperbolic systems.

Definition 2.13. *The point x_0 is a fixed point of the function f(x) if it holds that*

$$f(x_0) = x_0.$$

A fixed point of an autonomous system

$$\dot{x} = f(x)$$

is called hyperbolic if the linearized system is hyperbolic. That is, no eigenvalue of A in the linearized system $\dot{x} = Ax$ is equal to zero.

Definition 2.14. *For an autonomous system*

$$\dot{x} = f(x), \tag{2.1}$$

we say that a solution $\phi(t, x)$ of (2.1) converges exponentially to the fixed point x_0 if there exist constants $\alpha > 0, \delta$ and C such that

$$|\phi(t, x) - x_0| \leq Ce^{-\alpha t}|x - x_0|, \quad |x - x_0| \leq \delta.$$

Also, manifolds of systems that are not necessarily linear around a fixed point can be defined.

Definition 2.15. *For an autonomous system*

$$\dot{x} = f(x)$$

with fixed point x_0, we define

- *the local stable manifold $M^+(x_0)$ as the set of all points that converge exponentially to x_0 as $t \to \infty$.*
- *the local unstable manifold $M^-(x_0)$ as the set of all points that converge exponentially to x_0 as $t \to -\infty$.*

If the system is linear, then we have that $M^+(x_0) = E^+$ and $M^-(x_0) = E^-$.

Chapter 3

Coordinate transformation and main theorem

In this chapter, the thin-film equation is rewritten into an ordinary differential equation. Also, the obtained differential equation will be rescaled, so that it will become easier to work with. After that, a coordinate transformation will be done and the main theorem of this thesis will be stated.

3.1 Reformulation of the thin-film equation

The thin-film equation and its boundary conditions as explained in chapter 1 are given by

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial z}\left((h^3 + \lambda^{3-n}h^n)\frac{\partial^3 h}{\partial z^3}\right) = 0 \qquad \text{for } t > 0 \text{ and } z > Z(t) \tag{3.1a}$$

$$h = 0 \qquad \text{for } t > 0 \text{ and } z = Z(t) \tag{3.1b}$$

$$\frac{\partial h}{\partial z} = k > 0 \qquad \text{for } t > 0 \text{ and } z = Z(t) \tag{3.1c}$$

$$\lim_{z \to Z(t)^+} (h^2 + \lambda^{3-n}h^{n-1})\frac{\partial^3 h}{\partial z^3} = \frac{\partial Z}{\partial t}(t) \qquad \text{for } t > 0. \tag{3.1d}$$

Letting $h(z,t) = H(x)$ with $x = z + Vt$, where V is the velocity of the film, the following problem follows. Note that the problem is now an ordinary differential equation instead of a partial differential equation.

Lemma 3.1. *Problem* (3.1) *can be simplified to*

$$(H^2(x) + \lambda^{3-n}H^{n-1}(x))\frac{d^3H(x)}{dx^3} = -V \qquad \textit{for } x > 0, \tag{3.2a}$$

with boundary conditions

$$H(x) = 0 \qquad \textit{if } x = 0 \tag{3.2b}$$

$$\frac{dH(x)}{dx} = k > 0 \qquad \textit{if } x = 0 \tag{3.2c}$$

$$\frac{d^2H(x)}{dx^2} \to 0 \qquad \textit{as } x \to \infty. \tag{3.2d}$$

Proof. We have that

$$\frac{\partial H}{\partial t} = \frac{dH}{dx}\frac{\partial x}{\partial t},$$

and from $x = z + Vt$ it follows that $\frac{\partial x}{\partial t} = V$. Also, by using the chain rule it is obtained that

$$\frac{\partial^3 H}{\partial z^3} = \frac{d^3 H}{dx^3}$$

and

$$\frac{\partial}{\partial z}(H^3 + \lambda^{3-n}H^n) = \frac{d}{dx}(H^3 + \lambda^{3-n}H^n).$$

The differential equation now becomes

$$\frac{dH}{dx}\cdot V + \frac{d}{dx}\left((H^3 + \lambda^{3-n}H^n)\frac{d^3H}{dx^3}\right) = 0.$$

When $\frac{d}{dx}$ is taken out and the equation is integrated once,

$$H(x)\cdot V + (H^3(x) + \lambda^{3-n}H^n)\frac{d^3H(x)}{dx^3} = c \quad \text{for } x > 0$$

follows, where c is constant. Because of the translation invariance in x of the above equation it follows that $c = 0$.

Now, since it is known that $H \neq 0$, equation (3.2a) follows. The first two boundary conditions follow trivially from the boundary conditions (3.1b) and (3.1c). The third boundary condition follows from theory in [Chiricotto and Giacomelli, 2011]. Here, the authors show that for the differential equation (3.2a) a unique classical solution exists that satisfies this condition. □

3.2 Rescaling

The goal of this section is to get rid of the parameters V and λ of problem (3.2) by rescaling x and H. The rescaling is motivated in the following lemma.

Lemma 3.2. *Equation* (3.2a) *can be scaled so that* $\lambda = 1$ *and* $V = \frac{1}{3}$, *hence the differential equation and boundary conditions become*

$$(H^2(x) + H^{n-1}(x))\frac{d^3H(x)}{dx^3} = -\frac{1}{3} \qquad \textit{for } x > 0 \qquad (3.3a)$$

$$H(x) = 0 \qquad \textit{for } x = 0 \qquad (3.3b)$$

$$\frac{dH(x)}{dx} = k\,(3V)^{-\frac{1}{3}} =: \tilde{k} > 0 \quad \textit{(since } V,\ k > 0\textit{)} \qquad \textit{for } x = 0 \qquad (3.3c)$$

$$\frac{d^2H(x)}{dx^2} \to 0 \qquad \textit{as } x \to \infty. \qquad (3.3d)$$

Proof. Let

$$H = \lambda\tilde{H} \;\text{ and }\; x = (3V)^{\frac{1}{3}}\,\tilde{x}\lambda. \qquad (3.4)$$

From this it follows that

$$\frac{dH}{dx} = \frac{d\tilde{H}}{d\tilde{x}}\frac{dH}{d\tilde{H}}\frac{d\tilde{x}}{dx} = \frac{d\tilde{H}}{d\tilde{x}}\,(3V)^{\frac{1}{3}}$$

and
$$\frac{d^3H}{dx^3} = \frac{d^3\tilde{H}}{d\tilde{x}^3}\frac{dH}{d\tilde{H}}\left(\frac{d\tilde{x}}{dx}\right)^3 = \frac{d^3\tilde{H}}{d\tilde{x}^3}\frac{3V}{\lambda^2}.$$
Plugging these into equation (3.2a) we obtain
$$\left(\lambda^2\tilde{H}^2 + \lambda^{3-n}\lambda^{n-1}\tilde{H}^{n-1}\right)\frac{3V}{\lambda^2}\frac{d^3\tilde{H}}{d\tilde{x}^3} = -V$$
and so equation (3.2a) becomes
$$\left(\tilde{H}^2 + \tilde{H}^{n-1}\right)\frac{d^3\tilde{H}}{d\tilde{x}^3} = -\frac{1}{3}.$$
The boundary conditions follow directly from the definitions of $\tilde{H}$ and $\tilde{x}$ in (3.4). □

3.2.1 Tanner's law

Recall that we chose $n \in [1, 3)$. So, when x tends to infinity, the term H^2 dominates H^{n-1} in equation (3.3a). This is why the expected behaviour of the differential equation (3.3a) is determined by
$$H^2\frac{d^3H}{dx^3} = -\frac{1}{3} \tag{3.5}$$
for large values of x. In section 1.2 from [Giacomelli et al., 2016] it is shown that
$$H = x(\ln x)^{\frac{1}{3}}(1 + o(1)) \quad \text{as } x \to \infty$$
is a solution to the differential equation (3.5). Differentiating this equation with respect to x, raising it to the power of three and undoing the normalization of the speed V gives
$$\left(\frac{dH}{dx}\right)^3 = 3V\ln(x)(1 + o(1)). \tag{3.6}$$
Equation (3.6) says that the cube of the macroscopic contact angle is, up to a logarithmic correction, proportional to the speed of the contact line. This is referred to as Tanner's law.

3.3 Coordinate transformation

It can easily be seen that equation (3.3a) is invariant in x. For (3.3) we know that

- $H > 0$ for all $x > 0$
- $\frac{d^3H}{dx^3} < 0$ for all $x > 0$ by (3.3a)
- $\frac{d^2H}{dx^2} > 0$ for all $x > 0$ by (3.3d)
- $\frac{dH}{dx} > 0$ for all $x > 0$ by (3.3b). This is because
$$\left.\frac{dH}{dx}\right|_x = \underbrace{\left.\frac{dH}{dx}\right|_0}_{=\tilde{k}} + \int_0^x \underbrace{\left.\frac{d^2H}{dx^2}\right|_{\tilde{x}}}_{>0 \text{ for } \tilde{x}>0} d\tilde{x} > \tilde{k} > 0$$
for all $x > 0$.

The above shows that H is a strictly monotone function. Because of this property, equation (3.3a) can be rewritten in terms of x as a function of H, getting rid of the translation invariance in x. This will be changed slightly, as

$$\psi := \left(\frac{dH}{dx}\right)^2 > 0 \text{ as a function of } H \tag{3.7}$$

will be used. This coordinate transformation is chosen because, as we will see in the next lemma, the ordinary differential equation will change to be of order two instead of three.

Lemma 3.3. *Using the coordinate transformation* (3.7) *the differential equation with boundary conditions* (3.3) *turns into*

$$\frac{d^2\psi}{dH^2} + \psi^{-\frac{1}{2}}\phi(H) = 0 \quad \textit{for } H > 0 \tag{3.8a}$$
$$\textit{where } \phi(H) = \frac{2}{3}(H^2 + H^{n-1})^{-1}.$$

With boundary conditions

$$\psi = \tilde{k}^2 \ \textit{at } H = 0 \tag{3.8b}$$
$$\frac{d\psi}{dH} \to 0 \ \textit{as } H \to \infty. \tag{3.8c}$$

Proof. Because of the definition of $\psi(H)$ in equation (3.7), it can be seen that $\frac{d\psi}{dx} = \frac{d\psi}{dH}\frac{dH}{dx}$. Thus $\frac{d\psi}{dH} = \frac{d\psi}{dx}\left(\frac{dH}{dx}\right)^{-1}$ follows. It is easy to see that $\frac{d\psi}{dx} = 2 \cdot \frac{dH}{dx}\left(\frac{d^2H}{dx^2}\right)$. Combining this gives

$$\frac{d\psi}{dH} = 2 \cdot \frac{d^2H}{dx^2}.$$

In a similar way we find that

$$\frac{d^2\psi}{dH^2} = 2 \cdot \left(\frac{dH}{dx}\right)^{-1}\frac{d^3H}{dx^3}. \tag{3.9}$$

Equation (3.3a) can be rewritten as follows:

$$\begin{aligned}
&\frac{d^3H}{dx^3} + \frac{1}{3}(H^2 + H^{n-1})^{-1} = 0 \\
\Longrightarrow \quad &\left(\frac{dH}{dx}\right)^{-1}\left(2\frac{d^3H}{dx^3} + \frac{2}{3}\left(H^2 + H^{n-1}\right)^{-1}\right) = 0 \\
\Longrightarrow \quad &2\left(\frac{dH}{dx}\right)^{-1}\frac{d^3H}{dx^3} + \left(\frac{dH}{dx}\right)^{-1}\left(\frac{2}{3}(H^2 + H^{n-1})^{-1}\right) = 0
\end{aligned}$$

and this equation, using equations (3.9) and (3.7), can be written as equation (3.8a). The boundary conditions (3.8b) and (3.8c) follow directly from the definition of $\psi(H)$ in (3.7) and the boundary conditions of (3.3). □

3.4 The main result

The rest of this thesis is devoted to proving the following theorem, which is often referred to as the main theorem.

Theorem 3.4. *Let $n \in [1,3)$. The unique classical solution $\psi(H)$ of (3.8) obeys the following asymptotic behavior:*
there exists a parameter B and a function $R(H)$ such that

$$\psi(H) = \psi_T(BH)(1 + R(H)) \quad \textit{for } BH \geq C,\ C > 0 \tag{3.10}$$

where

$$R(H) = O\big((\log(H))^{-1} H^{-(3-n)}\big) \quad \textit{as } H \to \infty.$$

Here, B and R are C^1-functions of $\tilde{k}$. Also, in the non-resonant case,

$$\psi(H) = \tilde{k}^2(1 + O(H^{\alpha})) \quad \textit{as } H \downarrow 0, \tag{3.11}$$

where $\alpha = \min\{1, 3-n\}$. In the resonant case, it follows that

$$\psi(H) = \tilde{k}^2\big(1 + O(H^{3-n} - H\log(H))\big) \quad \textit{as } H \downarrow 0. \tag{3.12}$$

In this theorem, equation (3.10) tells us what the behaviour of the solution $\psi(H)$ to equation (3.8a) is, with its boundary conditions as $H \to \infty$. Since $\psi(H)$ is defined as $\psi(H) = \big(\frac{dH}{dx}\big)^2$, this behaviour gives information about the macroscopic contact angle. Because the parameter B depends on $\tilde{k}^2$, it follows that the macroscopic contact angle is dependent on the microscopic contact angle.

On the other hand, equation (3.11) and (3.12) give us information about the behaviour of the solution for values of H close to zero. We can see from the equations that the value of $\psi(H)$ as $H \downarrow 0$ is equal to $\tilde{k}^2$ with a correction. This equation gives information about the microscopic contact angle.

Chapter 4

Reformulation as a dynamical system

In this chapter a dynamical system (see definition 2.11) will be formulated to characterize the error between ψ and $\tilde{k}^2$ as $H \downarrow 0$. In order to do this, first the contact line that is now at $H = 0$ is shifted to $s = -\infty$. This is done using the coordinate transformation $s = \ln(H)$. A new unknown

$$1 + \mu = \frac{\psi}{\tilde{k}^2} \tag{4.1}$$

is introduced. Here, μ is the error between ψ and $\tilde{k}^2$. The coordinate transform and the new unknown $1 + \mu$ are used to rewrite the problem found in lemma 3.3

$$\frac{d^2\psi}{dH^2} + \psi^{-\frac{1}{2}}\phi(H) = 0 \quad \text{for } H > 0 \tag{4.2a}$$

$$\text{where } \phi(H) = \frac{2}{3}(H^2 + H^{n-1})^{-1}.$$

$$\psi = \tilde{k}^2 \text{ at } H = 0 \tag{4.2b}$$

$$\frac{d\psi}{dH} \to 0 \text{ as } H \to \infty \tag{4.2c}$$

as a differential equation for μ. The new ordinary differential equation with boundary condition is given by:

$$\left(\frac{d^2\mu}{ds^2} + \frac{d\mu}{ds}\right)(1+\mu)^{\frac{1}{2}} + \frac{2}{3\tilde{k}^3(1 - e^{(n-3)s})} = 0 \quad \text{for } s \in \mathbb{R} \tag{4.3a}$$

$$\lim_{s \to -\infty} \mu = 0. \tag{4.3b}$$

This differential equation is obtained by using the definition of μ in equation (4.1), differentiating this with respect to s (or to H using the coordinate transform), and plugging it into equation (4.2a).

4.1 The dynamical system

Equation (4.3a) will now be reformulated as an autonomous three-dimensional continuous dynamical system, using the additional functions

$$r(s) = e^{(3-n)s} \quad \text{and} \quad p(s) = \frac{d\mu}{ds}.$$

The dynamical system becomes

$$\frac{d}{ds}\begin{pmatrix} r \\ \mu \\ p \end{pmatrix} = F(r,\mu,p), \tag{4.4}$$

where

$$F(r,\mu,p) = \begin{pmatrix} (3-n)r \\ p \\ p - \frac{2r}{3\tilde{k}^3(1+\mu)^{\frac{1}{2}}(r-1)} \end{pmatrix}.$$

This system is obtained by using the functions r and p, plugging them into equation (4.3a) and rewriting this as a system of first order ordinary differential equations. It can be easily checked that $(0,0,0)$ is a fixed point of the system (4.4). In the next lemma we will see that the unique solution of equation (4.2a) converges to the fixed point when $s \to -\infty$.

Lemma 4.1. *$(r,\mu,p) \to (0,0,0)$ as $s \to -\infty$ for the unique solution of* (4.2a).

Proof. For r, the following holds: $r \to 0$ as $s \to -\infty$, since $r = e^{(3-n)s}$, and $3-n > 0$ for $n \in [1,3)$.
Then for μ we have that: $\mu = \frac{\psi}{\tilde{k}^2} - 1$, and $\frac{\psi}{\tilde{k}^2} \to 1$ as $s \to -\infty$. So $\mu \to 0$ as $s \to -\infty$.
To find out what happens to p when $s \to -\infty$, equation (4.1) is differentiated. This gives:

$$p = \frac{d\mu}{ds} = \frac{1}{\tilde{k}^2}\frac{d\psi}{ds} = \frac{1}{\tilde{k}^2}H\frac{d\psi}{dH}.$$

It remains to show that $H\frac{d\psi}{dH}$ converges to 0. To do this, note that from equation (4.3b) it follows that $\psi = \tilde{k}^2(1+o(1))$ as $H \downarrow 0$ and equation (4.2a) gives

$$\frac{d^2\psi}{dH^2} = -\psi^{-\frac{1}{2}}\phi(H) = -\tilde{k}^{-1}(1+o(1))H^{1-n} = \tilde{k}^{-1}H^{1-n}(1+o(1))$$

as $H \downarrow 0$. To get an expression for $\frac{d\psi(H)}{dH}$, we do the following. Let $\varepsilon > 0$ be small. Then

$$\frac{d\psi(H)}{dH} = \frac{d\psi(\varepsilon)}{dH} - \int_H^\varepsilon \frac{d^2\psi(\tilde{H})}{dH^2}d\tilde{H} = \frac{d\psi(\varepsilon)}{dH} + \tilde{k}^{-1}(1+o(1))\int_H^\varepsilon \tilde{H}^{1-n}d\tilde{H}$$

$$= \begin{cases} C(\varepsilon) - \frac{1}{\tilde{k}(2-n)}H^{2-n}(1+o(1)) & \text{for } n \neq 2 \\ C(\varepsilon) - \tilde{k}^{-1}\log(H)(1+o(1)) & \text{for } n = 2 \end{cases}$$

as $H \downarrow 0$, where $C(\varepsilon)$ is a constant depending on ε. This can be rewritten as

$$\begin{cases} C(\varepsilon)(1+o(1)) & \text{for } 0 < n < 2 \\ -\tilde{k}^{-1}\log(H)(1+o(1)) & \text{for } n = 2 \\ \frac{1}{\tilde{k}(n-2)}H^{2-n}(1+o(1)) & \text{for } 2 < n < 3 \end{cases} \quad \text{as } H \downarrow 0.$$

We can see that for $n \neq 2$, $\frac{d\psi}{dH}$ does not diverge as $H \downarrow 0$. If $n = 2$, $H\log(H) \to 0$ as $H \downarrow 0$. Now it is clear that $H\frac{d\psi}{dH} \to 0$ as $H \downarrow 0$. So $p \to 0$ as $s \to -\infty$. □

4.2 Linearization

Equation (4.4) can be linearized around the fixed point (0, 0, 0), resulting in

$$DF(0,0,0) = \begin{pmatrix} 3-n & 0 & 0 \\ 0 & 0 & 1 \\ \frac{2}{3\tilde{k}^3} & 0 & 1 \end{pmatrix}, \tag{4.5}$$

where DF denotes the Jacobian matrix of F, evaluated in $(0,0,0)$. The eigenvalues are equal to $0,1$ and $3-n$. Because of these eigenvalues, it follows that the dimension of the center manifold is one and the dimension of the unstable manifold is two (see definition 2.12) . When $n \neq 2$, all eigenvalues are distinct, so the linearized system can be diagonalized as in theorem 2.2. Then it follows that

$$DF(0,0,0) = PDP^{-1},$$

where $P = \begin{pmatrix} 0 & 0 & -\frac{3(n-2)\tilde{k}^3}{2} \\ 1 & 1 & -\frac{1}{n-3} \\ 0 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3-n \end{pmatrix}$.

If $n = 2$ the eigenspace for eigenvalue 1, which turns up twice now, has dimension 1. A generalized eigenvector is needed to get a generalized eigenspace of dimension 2. The linearized system can now be written in Jordan form, as in theorem 2.4.

$$DF(0,0,0) = TJT^{-1},$$

where $T = \begin{pmatrix} 0 & 0 & \frac{3\tilde{k}^3}{2} \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ and $J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

From the linearization, we can see that the fixed point $(0,0,0)$ is not hyperbolic (see definition 2.13).

This linearization will be used in the following chapter to see which eigenvalues and (generalized) eigenvectors of this matrix correspond to the unstable manifold.

Chapter 5

Resonances

In this chapter, the resonances of the unstable manifold of the linearized system as found in the previous chapter will be calculated. For this, the next lemma is needed, where the unstable manifold (see definitions 2.12 and 2.15) will be characterized.

Lemma 5.1. *For $n \in [1,3)$ the unstable manifold $M^-(0,0,0)$ of the dynamical system (4.4) can be parameterized by (r,μ). In particular we have*

$$p = g(r,\mu), \tag{5.1}$$

where g satisfies

$$g(0,0) = 0, \quad \frac{\partial g}{\partial r}(0,0) = \frac{-n_1}{n_3} \quad \text{and} \quad \frac{\partial g}{\partial \mu}(0,0) = 1. \tag{5.2}$$

Here n_1, n_2 and n_3 are given by the cross product of the vectors spanning the tangent space to the unstable manifold. $g(r,\mu)$ is analytic on $[0,\varepsilon]\times[0,\varepsilon]$ for ε small enough. ε depends on both $\tilde{k}^2$ and n.

Proof. There are two cases which need to be distinguished, namely $n=2$ and $n \neq 2$. First, look at $n \neq 2$.
The tangent space to the unstable manifold, $M^-(0,0,0)$, is given by

$$\operatorname{span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}-\frac{3(n-2)\tilde{k}^2}{2}\\ -\frac{1}{n-3}\\ 1\end{pmatrix}\right\} = \operatorname{span}\{v_1, v_2\}.$$

If we say that

$$\mathbf{n} = v_1 \times v_2 = \begin{pmatrix}\frac{1}{n-3}+1\\ -\frac{\tilde{k}^3(3n-6)}{2}\\ \frac{\tilde{k}^3(3n-6)}{2}\end{pmatrix} =: \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix},$$

the equation

$$0 = \mathbf{n}\cdot\begin{pmatrix}r\\ \mu\\ p\end{pmatrix} = n_1\cdot r + n_2\cdot\mu + n_3\cdot p$$

can be rewritten so that p is expressed in terms of r and μ. When this is done, it follows that $p = \frac{-n_1 r - n_2\mu}{n_3}$. Because of this, the unstable manifold around the stationary point $(0,0,0)$ can be parameterized by r and μ. We can say that $p = g(r,\mu)$.

Now for $n = 2$. the tangent space to the unstable manifold is given by

$$\text{span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}\frac{3\tilde{k}^3}{2}\\1\\2\end{pmatrix}\right\} = \text{span}\{v_1, v_3\}.$$

Similarly as before, we can use

$$\mathbf{n} = v_1 \times v_3 = \begin{pmatrix}1\\ \frac{3\tilde{k}^3}{2}\\ -\frac{3\tilde{k}^3}{2}\end{pmatrix} =: \begin{pmatrix}n_1\\n_2\\n_3\end{pmatrix},$$

to find that $p = \frac{-n_1 r - n_2\mu}{n_3}$. Also, $M^-(0,0,0)$ can be parameterized by r and μ, and we can say that $p = g(r,\mu)$. It is now clear that the unstable manifold can be parameterized by r and μ in both cases. Note that the function $g(r,\mu)$ can be found for both $n = 2$ and $n \neq 2$, but that it is different in both cases. This is because the tangent spaces are different. The partial derivatives of g in $(0,0)$ can easily be computed using the expressions found for p. □

The flow on the unstable manifold is now given by

$$\frac{d}{ds}\begin{pmatrix}r\\ \mu\end{pmatrix} = \mathcal{A}\begin{pmatrix}r\\ \mu\end{pmatrix} + \mathcal{N}(r,\mu) \tag{5.3}$$

because of lemma 5.1. Here, $\mathcal{A}$ denotes the linear part and $\mathcal{N}$ the nonlinear part of the flow. In the next definition, the notion of a resonance is explained.

Definition 5.2. *Let $\Lambda = (\lambda_1, \lambda_2)^\top$ contain the eigenvalues of $\mathcal{A}$. Let $q \in (\mathbb{N}_0)^2$ and define $|q| = q_1 + q_2$. The system in equation (5.3) has a resonance if there exist a vector $q = (q_1, q_2)^\top$ and index k such that $q^\top\Lambda = \lambda_k$ and $|q| \geq 2$.*

For the system (5.3) the resonances are characterized in the following lemma. The structure of this lemma is similar to [Belgacem et al., 2016, lemma 4.8], and the same strategy of the proof is used.

Lemma 5.3. *If $n \in [1,3)$ then resonances of (5.3) appear if*

- *$n = 3 - \frac{1}{m}$ with $m = 2, 3, \ldots$ for $q = (0, q_2)$ and $q_2 \geq 2$,*
- *$n = 1$ for $q = (2,0)^\top$.*

Proof. Since Λ contains two eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 3 - n$, there are two possibilities for resonances to occur:

1. $q_1\lambda_1 + q_2\lambda_2 = \lambda_1$
2. $q_1\lambda_1 + q_2\lambda_2 = \lambda_2$

We will now look at the first case. If $q_1 \geq 1$, it follows that

$$q_1\lambda_1 + q_2\lambda_2 \overset{\text{def 5.2}}{\geq} \min\{\lambda_1 + \lambda_2, 2\lambda_1\} > \lambda_1.$$

This means that no resonances can occur if $q_1 \geq 1$, so q_1 has to be equal to 0. Now the equation becomes

$$(3-n)q_2 = 1, \quad q_2 \geq 2,$$

and this implies that

$$n = 3 - \frac{1}{q_2}, \quad q_2 \geq 2$$

and this proves the first point of lemma 5.3.
Now, for the second possibility. If $q_2 \geq 1$, then

$$q_1\lambda_1 + q_2\lambda_2 \overset{\text{def 5.2}}{\geq} \min\{\lambda_1 + \lambda_2, 2\lambda_2\} > \lambda_2.$$

So no resonances can occur if $q_2 \geq 1$, hence $q_2 = 0$. Now, it follows that

$$\begin{aligned}
& q_1 = 3 - n, \quad q_1 \geq 2 \\
\implies \quad & n = 3 - q_1, \quad q_1 \geq 2 \text{ and } n \in [1, 3) \\
\implies \quad & n = 1 \text{ and } q_1 = 2.
\end{aligned}$$

This proves the second point in lemma 5.3. □

Note that a resonance also occurs if $n = 2$, this follows from the linearization in the previous chapter. When $n = 2$, not all eigenvalues are distinct and a generalized eigenvector is needed to get a generalized eigenspace of dimension 2. The found resonances will be used to distinguish a resonant and non-resonant case in the next chapter. This will be done by considering values for n where resonances of the system (5.3) do or do not occur.

Chapter 6

Fixed point problem

In this chapter, the existence of μ is shown. This is split in the non-resonant case in section 6.1 and the resonant case in section 6.2.

6.1 Non-resonant case

For the non-resonant case, consider $n \in [1, 3)$ with $n \neq 3 - \frac{1}{m}$ for $m \in \mathbb{N}$. Note that the case $n = 1$ is included in this section. The reason for this is stated in the proof of proposition 6.1. From lemma 5.1 it follows that $p = H\frac{d\mu}{dH} = g(H^{3-n}, \mu)$. This can be rewritten as

$$\left(H\frac{d}{dH} - 1\right)\mu = G(H^{3-n}, \mu), \tag{6.1}$$

where $G(H^{3-n}, \mu) = g(H^{3-n}, \mu) - \mu$.

To continue, let $x_1 = H$, $x_2 = H^{3-n}$ and $\mu(H) = v(x_1, x_2)$. For $n \notin \left\{3 - \frac{1}{m},\ m = 1, 2, 3, \dots\right\}$, we can look at equation (6.1) by treating x_1 and x_2 as independent variables. Then we can write

$$\left(x_1\frac{\partial}{\partial x_1} + (3-n)x_2\frac{\partial}{\partial x_2} - 1\right)v = G(x_2, v) \quad \text{around} \quad (x_1, x_2) = (0, 0). \tag{6.2}$$

Now, x_1 is in the kernel of the linear operator in (6.2). The boundary conditions are taken as

$$\left(v, \frac{\partial v}{\partial x_1}\right) = (0, b) \quad \text{at } (x_1, x_2) = (0, 0).$$

Note that the dependence of v on b is implicit, this is made explicit using

$$w(x_1, x_2, x_3) + x_3 = v(x_1, x_2) \quad \text{with } x_3 = bx_1.$$

Then it follows that

$$\left(x_1\frac{\partial}{\partial x_1} + (3-n)x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)w + x_3 = \left(x_1\frac{\partial}{\partial x_1} + (3-n)x_2\frac{\partial}{\partial x_2} - 1\right)v.$$

The new differential equation becomes

$$\left(x_1\frac{\partial}{\partial x_1} + (3-n)x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\right)w + x_3 = G(x_2, w + x_3) \quad \text{around} \quad (x_1, x_2, x_3) = (0, 0, 0) \tag{6.3a}$$

$$\left(w, \frac{\partial w}{\partial x_1}, \frac{\partial w}{\partial x_3}\right) = (0, 0, 0) \quad \text{at} \quad (x_1, x_2, x_3) = (0, 0, 0) \tag{6.3b}$$

Proposition 6.1. (non-resonant case). *Let* $n \in [1,3)$, $n \neq 3 - \frac{1}{m}$ *for* $m \in \mathbb{N}$. *Let* $0 < \varepsilon \leq C$, $C > 0$. *Then* (6.3) *has an analytic solution* $w = w(x_1, x_2, x_3)$ *for* $(x_1, x_2, x_3) \in [0, \varepsilon] \times [0, \varepsilon^2] \times [-\varepsilon, \varepsilon]$.

Proof. In this proof, equation (6.3) is rewritten as a fixed point problem in the following way:

$$w = \mathcal{T}G(x_1, x_2, w + x_3). \tag{6.4}$$

Here, the linear operator $\mathcal{T}$ is defined by

$$\mathcal{T}g(x_1, x_2, x_3) := \sum_{(k,l,p)\in\mathcal{I}} \frac{\frac{\partial^k}{\partial x_1^k}\frac{\partial^l}{\partial x_2^l}\frac{\partial^p}{\partial x_3^p} g(0,0,0)}{k + (3-n)l + p - 1} x_1^k x_2^l x_3^p$$

where $\mathcal{I} = (\mathbb{N}_0)^3 \backslash \{(0,0,0), (1,0,0), (0,0,1)\}$. Since the denominator does not vanish for $n = 1$, this proposition also holds for this value of n. To construct a solution for w using Banach's fixed point theorem 2.10, the following sub-multipicative norm is used:

$$||w|| := \sum_{(k,l,p)\in\mathcal{I}} \frac{\varepsilon^{k+2l+p}}{k!l!p!} \left| \frac{\partial^k}{\partial x_1}\frac{\partial^l}{\partial x_2}\frac{\partial^p}{\partial x_3} w(0,0,0) \right| \tag{6.5}$$

The rest of the proof can be found in [Belgacem et al., 2016, proposition 4.9]. □

6.2 Resonant case

For the resonant case, consider $n \in \left\{3 - \frac{1}{m},\ m = 1, 2, 3, \dots\right\}$. Let $y = H^{\frac{1}{m}}$ and $G(y, \mu) = m(g(y, \mu) - \mu)$, then equation (6.1) becomes

$$\left(y\frac{d}{dy} - m\right)\mu = G(y, \mu).$$

Similar to the non-resonant case, let $y_1 = y^m \log y$, $y_2 = y$ and $v(y_1, y_2) = \mu(y)$. Writing $y\frac{d}{dy}$ as $(my_1 + y_2^m)\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}$, equation (6.2) becomes

$$\left((my_1 + y_2^m)\frac{\partial}{\partial y_1} - y_2\frac{\partial}{\partial y_2} - m\right)v = G(y, v) \quad \text{around} \quad (y_1, y_2) = (0,0). \tag{6.6}$$

Now y_2^m is in the kernel of the linear operator in (6.6). The boundary conditions are taken as

$$(v, \frac{\partial^m}{\partial y_2^m} v) = (0, bm!) \quad \text{at} \quad (y_1, y_2) = (0,0).$$

The dependence on b is made explicit by using

$$w(y_1, y_2, y_3) + y_3 = v(y_1, y_2) \quad \text{if} \quad y_3 = by_2^m.$$

Then it follows that

$$\left((my_1 + y_2^m)\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + my_3\frac{\partial}{\partial y_3} - m\right)w + my_3 = \left((my_1 + y_2^m)\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2}\right)v.$$

We get a differential equation of the form

$$\left((my_1 + y_2^m)\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} + my_3\frac{\partial}{\partial y_3} - m\right)w = G(y_2, \mu) \quad \text{around } (y_1, y_2, y_3) = (0,0,0) \tag{6.7a}$$

$$\left(w, \frac{\partial^m w}{\partial y_2^m}, \frac{\partial w}{\partial y_3}\right) = (0,0,0) \quad \text{at} \quad (y_1, y_2, y_3) = (0,0,0). \tag{6.7b}$$

Proposition 6.2. (resonant case). *Suppose* $n = 3 - \frac{1}{m}$, $m \in \mathbb{N}$. *Let* $0 < \varepsilon \leq C$, $C > 0$. *Then* (6.7) *has an analytic solution* $w = w(y_1, y_2, y_3)$ *for* $(y_1, y_2, y_3) \in [0, \varepsilon^2] \times [0, \varepsilon^2] \times [-\varepsilon, \varepsilon]$.

Proof. Equation (6.7a) is rewritten as a fixed point problem as follows:

$$w = \mathcal{T}G(y_1, w + y_3).$$

The definition of the linear operator $\mathcal{T}$ is different from the definition of $\mathcal{T}$ in the proof of proposition 6.1. The $\mathcal{T}$ that is needed here is defined inductively in the proof of [Belgacem et al., 2016, proposition 4.10]. The norm that is needed for using Banach's fixed point theorem is

$$||w|| := \sum_{(k,l,p)\in\mathcal{I}} \frac{\varepsilon^{2mk+2l+p}}{k!l!p!} \left| \frac{\partial^k}{\partial y_1} \frac{\partial^l}{\partial y_2} \frac{\partial^p}{\partial y_3} w(0,0,0) \right|.$$

This norm is also sub-multiplicative. The rest of the proof can be found in [Belgacem et al., 2016, proposition 4.10]. $\square$

Chapter 7

Proof of the main result

To be able to prove the main theorem, a few more results are needed. This is split into two sections, a section where two important propositions are given, and a section about the linear independence of $\left(\frac{\partial \psi_b}{\partial b}, \frac{\partial}{\partial H}\frac{\partial \psi_b}{\partial b}\right)$ and $\left(\frac{\partial \psi_B}{\partial B}, \frac{\partial}{\partial H}\frac{\partial \psi_B}{\partial B}\right)$. This result will be needed for the use of the implicit function theorem in the proof of the main theorem. The propositions characterize two solution manifolds to

$$\frac{d^2\psi}{dH^2} + \psi^{-\frac{1}{2}}\phi(H) = 0 \quad \text{for } H > 0 \tag{7.1a}$$
$$\text{where } \phi(H) = \frac{2}{3}(H^2 + H^{n-1})^{-1}.$$

with boundary conditions

$$\psi = \tilde{k}^2 \text{ at } H = 0 \tag{7.1b}$$
$$\frac{d\psi}{dH} \to 0 \text{ as } H \to \infty. \tag{7.1c}$$

7.1 Two important propositions

In this section, we look into two propositions. The first proposition characterizes the solution curve for values of H close to zero. The second proposition does the same as H goes to infinity. For this proposition, the results are the same as in [Giacomelli et al., 2016], since in partial and complete wetting the film behaves the same near infinity.

Proposition 7.1. *For all $b \in \mathbb{R}$ there exists a function $\mu_b(H)$ for $H < c \cdot \max\{1, b^{-1}\}$, c constant, such that*

$$\psi_b(H) = \tilde{k}^2(1 + \mu_b(H)) \tag{7.2}$$

is a solution to (7.1a) *and its first boundary condition. Also*

$$\mu_b(H) = bH + w(H, H^{3-n}, bH), \tag{7.3}$$

where w is a solution of (6.3) *in the non-resonant case. In the resonant case we have*

$$\mu_b(H) = bH + w(H\log(H^{3-n}), H^{3-n}, bH), \tag{7.4}$$

where w is a solution of (6.7)*. Furthermore, μ_b depends smoothly on $\tilde{k}$. Also, the boundary condition*

$$\frac{\partial \psi_b}{\partial b}(H) = \tilde{k}^2(H + \frac{\partial w(H, H^{3-n}, bH)}{\partial bH}H) \quad \text{as } H \downarrow 0 \tag{7.5}$$

holds.

Proof. In the way μ was constructed in subsections 6.1 and 6.2 it follows that μ_b has the form as in (7.3) and (7.4). From propositions 6.1 and 6.2 it follows that in both the resonant and non-resonant case an analytic solution w exists for all $b \in \mathbb{R}$. From the definition of μ, $1 + \mu = \frac{\psi}{\tilde{k}^2}$, it follows that $\psi_b(H)$ has the form as in (7.2). Thus, for all $b \in \mathbb{R}$ there exists a $\mu_b(H)$ such that $\psi_b(H)$ is a solution of (7.1a) and its first boundary condition.

To look at the smooth dependence of μ_b on $\tilde{k}$, the fixed point problem from propositions 6.1 and 6.2 is used. For readability, write $G(x_2, w + x_2) = G(w)$ in the case of no resonances, and write $G(y_2, w + y_3) = G(w)$ in the resonant case. The proof works the same in the resonant and non-resonant case, so the same notation can be used. The fixed point problem can then be written as

$$w_{\tilde{k}} = \mathcal{T} G_{\tilde{k}}(w_{\tilde{k}}),$$

where $\mathcal{T}$ is a linear operator as defined in the proof of proposition 6.1 or 6.2. The subscript $\tilde{k}$ is used to denote the dependence of w and G on that parameter. Let $\tilde{k}_1$ and $\tilde{k}_2$ be different values of $\tilde{k} > 0$. Then

$$\begin{aligned}
w_{\tilde{k}_1} - w_{\tilde{k}_2} &= \mathcal{T} G_{\tilde{k}_1}(w_{\tilde{k}_1}) - \mathcal{T} G_{\tilde{k}_2}(w_{\tilde{k}_2}) \\
&= \mathcal{T}\left[G_{\tilde{k}_1}(w_{\tilde{k}_1}) - G_{\tilde{k}_2}(w_{\tilde{k}_2})\right] \quad \text{(because } \mathcal{T} \text{ is a linear operator)} \\
&= \mathcal{T}\left[G_{\tilde{k}_1}(w_{\tilde{k}_1}) - G_{\tilde{k}_2}(w_{\tilde{k}_1})\right] + \mathcal{T}\left[G_{\tilde{k}_2}(w_{\tilde{k}_1}) - G_{\tilde{k}_2}(w_{\tilde{k}_2})\right] \\
\iff \quad \frac{w_{\tilde{k}_1} - w_{kt}}{\tilde{k}_1 - \tilde{k}_2} &= \mathcal{T}\left[\frac{G_{\tilde{k}_1}(w_{\tilde{k}_1}) - G_{\tilde{k}_2}(w_{\tilde{k}_1})}{\tilde{k}_1 - \tilde{k}_2}\right] + \mathcal{T}\left[\frac{G_{\tilde{k}_2}(w_{\tilde{k}_1}) - G_{\tilde{k}_2}(w_{\tilde{k}_2}))}{\tilde{k}_1 - \tilde{k}_2}\right].
\end{aligned} \tag{7.6}$$

When in equation (7.6) the limit of $\tilde{k}_2$ to $\tilde{k}_1$ is taken, it follows that

$$\begin{aligned}
\frac{\partial w_{\tilde{k}_1}}{\partial \tilde{k}_1} &= \mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial w_{\tilde{k}_1}} \frac{\partial w_{\tilde{k}_1}}{\partial \tilde{k}_1}\right] + \mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial \tilde{k}_1}\right] \\
&= \mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial w_{\tilde{k}_1}} \frac{\partial w_{\tilde{k}_1}}{\partial \tilde{k}_1} + \frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial \tilde{k}_1}\right].
\end{aligned}$$

This is rewritten as a fixed point equation for $\phi := \frac{\partial w_{\tilde{k}_1}}{\partial \tilde{k}_1}$, which gives

$$\begin{aligned}
\phi &= \mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial w_{\tilde{k}_1}} \phi + \frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial \tilde{k}_1}\right] \\
\iff \quad \left(I - \mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\partial \tilde{k}_1})}{\tilde{k}_1}\right]\right)\phi &= \mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial \tilde{k}_1}\right]
\end{aligned}$$

Using the Neumann series, see theorem 2.5, we find that

$$\phi = \left(I - \mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\partial \tilde{k}_1})}{\tilde{k}_1}\right]\right)^{-1} \mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial \tilde{k}_1}\right]$$

if $\left|\left|\mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial \tilde{k}_1}\right]\phi\right|\right| \leq C||\phi||$ and $C < 1$, where the sub-multiplicative norm is taken as in the proof of proposition 6.1 or 6.2 in respectively the non-resonant and resonant case. Because of the sub-multiplicativity, and because for the linear operator $\mathcal{T}$ we know that $||\mathcal{T}g|| \leq D||g||$ for some constant $D > 0$, we can write

$$\left|\left|\mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial \tilde{k}_1}\right]\phi\right|\right| \leq D\left|\left|\frac{\partial G}{\partial w}\right|\right| \cdot ||\phi||. \tag{7.7}$$

From the definition of G and lemma 5.1 it follows that $\frac{\partial G}{\partial \mu} = 0$ for all values of $\tilde{k}$. So also $\frac{\partial G}{\partial w} = 0$, which leads to the estimate $||\frac{\partial G}{\partial w}|| < \varepsilon$ for all $\varepsilon > 0$. If we choose $\varepsilon < \frac{1}{D}$, then it follows that $C = D\varepsilon < 0$, so

$$\left|\left|\mathcal{T}\left[\frac{\partial G_{\tilde{k}_1}(w_{\tilde{k}_1})}{\partial \tilde{k}_1}\right]\phi\right|\right| \leq C||\phi||.$$

Hence, the dependence of w on $\tilde{k}$ is smooth, so also the dependence of $\mu_b = w_b + bH$ on $\tilde{k}$ is smooth.

The boundary condition of equation (7.5) can be obtained by differentiating (7.2) with respect to b. □

Proposition 7.2. *For all $B > 0$, there exists a function $R_B(H)$ for $H \geq C(1 + B^{-1})$, $C > 0$, such that*

$$\psi_B(H) = \psi_T(BH)(1 + R_B(H)) \quad \text{for } H \geq C(1 + B^{-1}),\ C > 0$$

gives a solution of (7.1a) *and its second boundary condition. $\psi_T(H)$ is the unique classical solution of*

$$\frac{d^2\psi}{dH^2} + \frac{2}{3}\psi^{-\frac{1}{2}}H^{-2} = 0 \quad \text{for } H > 0$$
$$\frac{d\psi}{dH} \to 0 \quad \text{as } H \to \infty,$$

where

$$(\psi_T(H))^{\frac{3}{2}} = \ln H - \frac{1}{3}\ln\ln H + o(1) \quad \text{as } H \to \infty.$$

Also

$$|R_B(H)| \leq D\big(B^{3-n}(\ln(H))^{-1}H^{-(3-n)}\big),\ D > 0,\ H \geq C(1 + B^{-1}),\ C > 0.$$

The correction $R_B(H)$ has, locally in H, a C^1-dependence on B, $\tilde{k}$ and the boundary condition

$$\frac{\partial}{\partial H}\frac{\partial \psi_B}{\partial B} = -\frac{2}{9B}(\ln(H))^{-\frac{4}{3}}H^{-1}(1 + o(1)) \quad \text{as } H \to \infty$$

is satisfied. Furthermore, there exists a $B > 0$ such that the unique solution ψ of (4.2a) *is the same as ψ_B.*

Proof. The proof can be found in [Giacomelli et al., 2016, prop 3.1]. □

7.2 Linear independence

The solutions that are constructed in propositions 7.1 and 7.2 fulfill the following boundary conditions, see (7.1b) and (7.1c):

$$\psi_b = \tilde{k}^2 \quad \text{at } H = 0 \tag{7.9a}$$

$$\frac{\partial \psi_B}{\partial H} \to 0 \quad \text{as } H \to \infty. \tag{7.9b}$$

When ψ_b is differentiated with respect to b, we find that $\frac{\partial \psi_b}{\partial b} = 0$ at $H = 0$. Let $(\psi, \eta) \in \Big\{(\psi_b, \frac{\partial \psi_b}{\partial b}), (\psi_B, \frac{\partial \psi_B}{\partial B})\Big\}$. If b and B are chosen such that $\psi_b = \psi_B =: \psi$, then both $\frac{\partial \psi_b}{\partial b}$ and $\frac{\partial \psi_B}{\partial B}$ exist globally and satisfy

$$\frac{d^2\eta}{dH^2} - \frac{1}{2}\psi^{\frac{3}{2}}\phi(H)\eta = 0. \tag{7.10}$$

Equation (7.10) is obtained by differentiating (7.1a) to b or B, depending on the choice of η. That the derivative of ψ to b and B exist globally is because of the C^1-dependence of ψ_b and ψ_B on b and B respectively. For equation (7.10) we have the following result:

Lemma 7.3. *Suppose that ψ is the unique classical solution of* (7.1a) *and its boundary conditions, and $\eta \in C^0([0,\infty)) \cap C^2((0,\infty))$ is a solution of* (7.10) *fulfilling*

$$\eta = 0 \quad \text{at } H = 0 \tag{7.11}$$

and

$$\frac{d\eta}{dH} \to 0 \quad \text{at } H \to \infty. \tag{7.12}$$

Then $\eta \equiv 0$.

Proof. Consider the function $\eta^2 \geq 0$. Note that

$$\frac{1}{2}\frac{d^2\eta^2}{dH^2} = \frac{1}{2}\frac{d}{dH}\left[2\eta\frac{d\eta}{dH}\right] = \eta\frac{d^2\eta}{dH^2} + \left(\frac{d\eta}{dH}\right)^2$$

and thus

$$\frac{1}{2}\frac{d^2\eta^2}{dH^2} = \eta\frac{d^2\eta}{dH^2} + \left(\frac{d\eta}{dH}\right)^2 \overset{(7.10)}{=} \frac{1}{2}\psi^{-\frac{3}{2}}\phi(H)\eta^2 + \left(\frac{d\eta}{dH}\right)^2 \geq 0.$$

From this it follows that $\frac{d\eta^2}{dH}$ is increasing. Also, $\eta^2 = 0$ at $H = 0$ by (7.11) and $\eta^2 \geq 0$ for $H > 0$, which implies that $\frac{d\eta^2}{dH} \geq 0$ at $H = 0$. Hence, $\frac{d\eta^2}{dH} \geq 0$ for all $H \geq 0$. Note that

$$\frac{d\eta^2}{dH} = 2\eta\frac{d\eta}{dH} \quad \Longrightarrow \quad \eta\frac{d\eta}{dH} = \frac{1}{2}\frac{d\eta^2}{dH},$$

which gives that

$$\frac{d}{dH}\left(\frac{d\eta}{dH}\right)^2 = 2\frac{d\eta}{dH}\frac{d^2\eta}{dH^2} \overset{(7.10)}{=} \psi^{-\frac{3}{2}}\phi(H)\eta\frac{d\eta}{dH} = \frac{1}{2}\psi^{-\frac{3}{2}}\phi(H)\frac{d\eta^2}{dH} \geq 0.$$

So $\left(\frac{d\eta}{dH}\right)^2 \geq 0$ is increasing. Condition (7.12) implies that $\left(\frac{d\eta}{dH}\right)^2 = 0$ when H approaches infinity. Hence, $\left(\frac{d\eta}{dH}\right)^2 \equiv 0$ and also $\frac{d\eta}{dH} \equiv 0$. This means that η must be constant and from (7.11) it follows that $\eta \equiv 0$. $\square$

For the next corollary, the following lemma is needed:

Lemma 7.4. *For the first order linear system*

$$y'(t) = A(t)y(t), \tag{7.13}$$

where A is continuous in time, the following holds: two solutions of the system are linearly dependent for one value t_0 if and only if the two solutions are linearly dependent for all t.

Proof. For the first order linear system (7.13) it is known that, with initial condition $y_0 = y(t_0)$, the solution is unique. Suppose y_1 and y_2 are solutions of (7.13), and that they are linearly independent for time t_0, i.e., there exist $a_1, a_2 \in \mathbb{R}$ such that

$$a_1y_1(t_0) + a_2y_2(t_0) = 0.$$

Here, a_1 and a_2 cannot both be equal to zero. Define $y(t) = a_1y_1(t)+a_2y_2(t)$. By linearity, $y(t)$ is a solution of (7.13) with $y(t_0) = 0$. Because also the constant solution 0 solves (7.13) with initial condition 0, it is necessary that $y(t) = 0$ for all t. This means that

$$a_1y_1(t) + a_2y_2(t) = 0$$

for all t, so y_1 and y_2 are linearly dependent. The converse is trivial. □

From lemma 7.4 it also follows that two solutions of the system (7.13) are linearly independent for one value t_0 if and only if the two solutions are linearly independent for all t. This result is needed in the following corollary.

Corollary 7.5. *Suppose that for every $n \in [1, 3)$ the parameters $b, B \in \mathbb{R}$ are chosen so that $\psi_b = \psi_B =: \psi$, where ψ is the unique classical solution of* (7.1a)*. Then, the vectors*

$$\left(\frac{\partial\psi_b}{\partial b}, \frac{\partial}{\partial H}\frac{\partial\psi_b}{\partial b}\right) \quad \text{and} \quad \left(\frac{\partial\psi_B}{\partial B}, \frac{\partial}{\partial H}\frac{\partial\psi_B}{\partial B}\right) \tag{7.14}$$

are linearly independent for all $H > 0$.

Proof. Because of propositions 7.1 and 7.2 these values of b and B can be chosen. From lemma 7.4 we know that for the vectors (7.14) to be linearly independent for all $H > 0$, we only have to show that they are linearly independent for one $H > 0$. Looking into the structure of these vectors, it is clear that they can only be independent if the functions $\frac{\partial\psi_b}{\partial b}$ and $\frac{\partial\psi_B}{\partial B}$ are linearly independent. This will now be proven in the following way: suppose that

$$\alpha_b\frac{\partial\psi_b}{\partial b} + \alpha_B\frac{\partial\psi_B}{\partial B} \equiv 0 \tag{7.15}$$

for some $\alpha_b, \alpha_B \in \mathbb{R}$. By proposition 7.1 we know that $\frac{\partial\psi_b}{\partial b} \not\equiv 0$, and because $\frac{\partial\psi_b}{\partial b}$ fulfills (7.11) it cannot fulfill (7.12). Because $\frac{\partial\psi_B}{\partial B}$ fulfills (7.12), it must hold that $\alpha_b = 0$. Since it is also known that $\frac{\partial\psi_B}{\partial B} \not\equiv 0$, it is also necessary that $\alpha_B = 0$. □

This lemma tells us that geometrically the solution manifolds

$$\left\{\left(H, \psi_b, \frac{\partial\psi_b}{\partial b}\right)\right\} \quad \text{and} \tag{7.16a}$$

$$\left\{\left(H, \psi_B, \frac{\partial\psi_B}{\partial B}\right)\right\} \tag{7.16b}$$

are transversal, i.e. the manifolds are not tangent along their intersection line.

7.3 Proof of the main theorem

The main theorem is

Theorem 7.6. *Let $n \in [1, 3)$. The unique classical solution $\psi(H)$ of* (7.1) *obeys the following asymptotic behavior:*
there exists a parameter B and a function $R(H)$ such that

$$\psi(H) = \psi_T(BH)(1 + R(H)) \quad \text{for } BH \geq C,\ C > 0 \tag{7.17}$$

where

$$R(H) = O\big((\log(H))^{-1}H^{-(3-n)}\big) \quad \text{as } H \to \infty.$$

Here, B and R are C^1-functions of $\tilde{k}$. Also, in the non-resonant case,

$$\psi(H) = \tilde{k}^2(1 + O(H^\alpha)) \quad \text{as } H \downarrow 0, \tag{7.18}$$

where $\alpha = \min\{1, 3-n\}$. In the resonant case, it follows that

$$\psi(H) = \tilde{k}^2\big(1 + O(H^{3-n} - H\log(H))\big) \quad \text{as } H \downarrow 0. \tag{7.19}$$

Proof. Because of the uniqueness result in [Chiricotto and Giacomelli, 2011], we know that the solution manifolds (7.16a) and (7.16b) intersect in exactly one curve that defines the unique solution of the dynamical system $\left(H, \frac{d\psi}{dH}, \frac{d^2\psi}{dH^2}\right)$ corresponding to (7.1a). So, this means that there exist a unique b and B such that $\psi_b(H) = \psi_B(H) =: \psi(H)$, where $\psi_b(H)$ and $\psi_B(H)$ are as in propositions 7.1 and 7.2. The order of the errors in the non-resonant and resonant case follow from (7.3) and (7.4) respectively.

It remains to show that $b(\tilde{k})$ and $B(\tilde{k})$ are C^1-functions of $\tilde{k}$. Define $f_1 = \psi_B(H) - \psi_b(H)$ and $f_2 = \frac{\partial\psi_B(H)}{\partial H} - \frac{\partial\psi_b(H)}{\partial H}$. Note that $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ equals zero on the solution curve. Because of corollary 7.5, we have that

$$\det\begin{pmatrix} \frac{\partial\psi_B(H)}{\partial B} & \frac{\partial}{\partial B}\frac{\partial\psi_B(H)}{\partial H} \\ \frac{\partial\psi_b(H)}{\partial b} & \frac{\partial}{\partial b}\frac{\partial\psi_b(H)}{\partial H} \end{pmatrix} \neq 0.$$

Now, the implicit function theorem (theorem 2.8) gives a locally continuously differentiable map $g(\tilde{k}) = \begin{pmatrix} B(\tilde{k}) \\ b(\tilde{k}) \end{pmatrix}$, so B and b are locally C^1-functions of $\tilde{k}$. $\square$

7.4 Conclusion

In this thesis the goal was to investigate the asymptotic behaviour of the solution to the thin-film equation in partial wetting. To do this, the thin-film equation was first rewritten as a third order ordinary differential equation. After a coordinate transformation, this differential equation was reduced to second order. The obtained problem is given by

$$\frac{d^2\psi}{dH^2} + \psi^{-\frac{1}{2}}\phi(H) = 0 \quad \text{for } H > 0$$
$$\text{where } \phi(H) = \frac{2}{3}(H^2 + H^{n-1})^{-1}$$

with boundary conditions

$$\psi = \tilde{k}^2 \text{ at } H = 0$$
$$\frac{d\psi}{dH} \to 0 \text{ as } H \to \infty.$$

Here the boundary condition $\psi = \tilde{k}^2 > 0$ at $H = 0$ follows from the partial wetting state.

A class of solutions $\psi_B(H)$ as $H \to \infty$ was constructed in [Giacomelli et al., 2016]. In this thesis a class of solutions $\psi_b(H)$ was constructed that obeys the boundary condition at $H = 0$. This class of solutions has the form $\psi_b(H) = \tilde{k}^2(1 + \mu_b(H))$. Here, $\mu_b(H)$ is different depending on the (resonant or non-resonant) value of n. The intersection of $\psi_B(H)$ and $\psi_b(H)$ defines the unique solution $\psi(H)$ to the problem defined above. This solution has a local C^1-dependence on $\tilde{k}^2$.

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