Sharp Estimates and Extrapolation for Multilinear Weight Classes

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SHARP ESTIMATES AND EXTRAPOLATION FOR MULTILINEAR WEIGHT CLASSES

SHARP ESTIMATES AND EXTRAPOLATION FOR MULTILINEAR WEIGHT CLASSES

Proefschrift

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CONTENTS

1

Introduction

1

INTRODUCTION

The focus of this thesis is the development of theory of the multilinear and limited range analogues of the Muckenhoupt A_p weight classes and to develop methods of obtaining sharp weighted bounds for operators satisfying sparse bounds. This is facilitated by a quantitative study of multisublinear maximal operators which allows us to develop a sharp multilinear extrapolation theorem.

Through the domination of multilinear operators by sparse forms, which are intimately related to the multisublinear maximal operators, our techniques allow us to obtain sharp weighted bounds for these operators in both the scalar-valued and vector-valued settings. To this end we develop a multilinear analogue of the Hardy-Littlewood and UMD properties of Banach function spaces.

In this chapter we give a general introduction into each of these topics as well as provide a detailed outline of the thesis.

1.1. GENERAL INTRODUCTION

1.1.1. Weighted bounds for weights in the A_p classes

A positive function w in $\mathbf{R}^n$ is said to be an A_p weight for $p \in [1,\infty)$ when the Hardy-Littlewood maximal operator

$$Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f| \,\mathrm{d}x,$$

satisfies the weak-type bound $L^p(\mathbf{R}^n; w) \to L^{p,\infty}(\mathbf{R}^n; w)$. Here $L^p(\mathbf{R}^n; w)$ and $L^{p,\infty}(\mathbf{R}^n; w)$ are the respective strong- and weak-type L^p spaces over $\mathbf{R}^n$ with respect to the measure $w\,\mathrm{d}x$. In this case we have the equivalence

$$[w]_{A_p} := \sup_Q \left(\frac{1}{|Q|}\int_Q w \,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}} \,\mathrm{d}x\right)^{p-1} \eqsim \|M\|^p_{L^p(\mathbf{R}^n;w)\to L^{p,\infty}(\mathbf{R}^n;w)},$$

where for $p=1$ we use the interpretation $[w]_{A_1} = \sup_Q \left(\frac{1}{|Q|}\int_Q w\,\mathrm{d}x\right)\left(\operatorname{ess\,inf}_{y\in Q} w(y)\right)^{-1}$. This equivalence was shown by Muckenhoupt in [Muc72]. He proceeded to show that when $p \in (1,\infty)$, the condition $[w]_{A_p} < \infty$ is self-improving in the sense that in this case M also satisfies the strong bound $L^p(\mathbf{R}^n; w) \to L^p(\mathbf{R}^n; w)$.

Not long after this it was shown that the A_p condition is not only characterized by the strong boundedness of M, but also of certain singular integral operators. Indeed, in the works of Hunt, Muckenhoupt and Wheeden [HMW73] and in a simplified approach by Coifman and Fefferman [CF74] this was shown to be the case for the Hilbert transform. The latter also proved that the A_p condition is a sufficient condition to bound any Calderón-Zygmund operator.

A quantitative study of these bounds was initiated by Buckley in [Buc93]. He showed that

$$\|M\|^p_{L^p(\mathbf{R}^n;w)\to L^p(\mathbf{R}^n;w)} \lesssim_p [w]^{p'}_{A_p},$$

where the power of the weight constant is optimal. Moreover, he showed that if T is a Calderón-Zygmund operator, then

$$\|T\|^p_{L^p(\mathbf{R}^n;w)\to L^p(\mathbf{R}^n;w)} \lesssim_p [w]^{p'+p}_{A_p}.$$

While this bound is not optimal, he did show that the optimal power of the weight constant must lie between $\max\{p', p\}$ and $p' + p$.

This optimality became very relevant when Petermichl and Volberg [PV02] solved a long standing open problem on the regularity of solutions to Beltrami equations by showing that the Beurling-Ahlfors transform—a Calderón-Zygmund operator—satisfies this weighted bound with the sharp exponent $\max\{p', p\}$. Using a sharp version of Rubio de Francia extrapolation [Rub82, GR85] they reduced the problem to showing this bound in the case $p = 2$.

The problem of proving that the sharp bound with exponent $\max\{p', p\}$ holds for *all* Calderón-Zygmund operators became known as the A_2-conjecture. After a series of partial results, such as [Pet08, LPR10, NTV08, CMP10, Vag10, Ler11, PTV10, HLM+12], this conjecture was eventually settled by Hytönen in [Hyt12].

An alternative approach was developed by Lerner [Ler13], whose proof relied on dominating Calderón-Zygmund operators by the much simpler sparse operators. Subsequently, the idea of sparse domination was developed further and was broken down to its essentials by Lerner in [Ler16]. The literature on this topic is vast, see e.g., [BFP16, CR16, LN18, Lac17, HRT17, LO20, Lor19], and this technique can be applied to an increasingly general class of operators. By now, proving sharp weighted bounds has become more or less synonymous with proving sparse domination.

1.1.2. Weighted endpoint estimates

Sparse operators seem to very precisely capture the weighted behaviour of Calderón-Zygmund operator in the sense that any operator satisfying sparse domination also satisfies the result of the A_2-conjecture. At this point the question arises whether weighted endpoint bounds known for Calderón-Zygmund operators are also true when we consider the more general class of operators satisfying sparse domination.

In the work [BFP16] it was shown by Bernicot, Frey, and Petermichl that a large class of operators beyond the framework of Calderón-Zygmund operators satisfy sparse domination in form. The operators they considered are not necessarily bounded $L^p(\mathbf{R}^n) \to$

$L^p(\mathbf{R}^n)$ for all $p \in (1,\infty)$, but for a limited range $p \in (r,s)$, where $1 \le r < s \le \infty$. They showed that these operators satisfy the property that for every $f, g \in L^\infty_c(\mathbf{R}^n)$ there is a sparse collection of cubes $\mathscr{S}$ such that

$$\left|\int_{\mathbf{R}^n} (Tf) g \,\mathrm{d}x\right| \lesssim \sum_{Q\in\mathscr{S}} \langle f\rangle_{r,Q}\langle g\rangle_{s',Q}|Q|. \tag{1.1.1}$$

Moreover, they showed that this implies that T is bounded $L^p(\mathbf{R}^n;w) \to L^p(\mathbf{R}^n;w)$ for weights w that are in the intersection of the Muckenhoupt class $A_{\frac{p}{r}}$ and the Reverse Hölder class $\mathrm{RH}_{(s/p)'}$. This condition on the weight can be equivalently formulated as

$$[w]_{A_{p,(r,s)}} := \sup_Q \langle w^{\frac{1}{p}}\rangle^{\frac{1}{\frac{1}{p}-\frac{1}{s}}}_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} \langle w^{-\frac{1}{p}}\rangle^{\frac{1}{\frac{1}{p}-\frac{1}{s}}}_{\frac{1}{\frac{1}{r}-\frac{1}{p}},Q} < \infty, \tag{1.1.2}$$

and they showed that

$$\|T\|^{\frac{1}{\frac{1}{p}-\frac{1}{s}}}_{L^p(\mathbf{R}^n;w)\to L^p(\mathbf{R}^n;w)} \lesssim_{p,r,s} [w]^{\max\left\{\left(\frac{p}{r}\right)',\left(\frac{p'}{s'}\right)'\right\}}_{A_{p,(r,s)}} \tag{1.1.3}$$

for any T satisfying (1.1.1). Note that $A_{p,(1,\infty)} = A_p$, and in the case $r = 1$, $s = \infty$ the bound (1.1.3) recovers the bound from the A_2-Conjecture.

In Chapter 6 we extend weighted endpoint bounds known for Calderón-Zygmund operators to the setting of operators satisfying the sparse domination (1.1.1)

For Calderón-Zygmund operators, weighted weak type $(1,1)$ estimates were established by Lerner, Ombrosi, and Pérez [LOP09a] and later improved upon by Hytönen and Pérez [HP13], using mixed A_1–A_∞ type estimates. They showed that for all Calderón-Zygmund operators T and all $p \in (1,\infty)$ one has

$$\|T\|_{L^p(w)\to L^p(w)} \lesssim pp'[w]^{\frac{1}{p'}}_{A_\infty}[w]^{\frac{1}{p}}_{A_1}, \tag{1.1.4}$$

where

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)}\int_Q M(w\chi_Q)\,\mathrm{d}x,$$

is the Fujii-Wilson A_∞ constant, which characterizes the class $A_\infty = \cup_{p\in[1,\infty)} A_p$. Moreover, in the work of Lerner, Ombrosi, and Pérez [LOP08] it was shown that for all Calderón-Zygmund operators T and all weights $w \in A_1$ one has

$$\|T\|_{L^1(\mathbf{R}^n;w)\to L^{1,\infty}(\mathbf{R}^n;w)} \lesssim [w]_{A_1}\log(e+[w]_{A_1}). \tag{1.1.5}$$

This result is related to the weak Muckenhoupt-Wheeden conjecture, stating that one has linear dependence on $[w]_{A_1}$ on the right-hand side of (1.1.5), and the logarithm can be removed. This conjecture is now known to be false [NRVV10] and in fact, the estimate (1.1.5) is sharp for the Hilbert transform [LNO17]. The result (1.1.5) was improved by Hytönen and Pérez [HP13] to the mixed A_1–A_∞ type estimate

$$\|T\|_{L^1(\mathbf{R}^n;w)\to L^{1,\infty}(\mathbf{R}^n;w)} \lesssim [w]_{A_1}\log(e+[w]_{A_\infty}). \tag{1.1.6}$$

Both the proofs of (1.1.5) and (1.1.6) rely on introducing weights into the classical argument involving a Calderón-Zygmund decomposition $f = g + b$ and the vanishing mean value property of the 'bad' part b in combination with the Hörmander condition of the kernel of the operator. This is done through an argument that can already be found in [Pér94] (namely, they use [GR85, Lemma 3.3, p. 413]).

In general, the operators satisfying (1.1.1) need not be integral operators at all and for operators such as the Riesz transform associated to an elliptic operator, an argument by Blunck and Kunstmann [BK03] (see also [HM03]) proved a weak-type (r, r) boundedness using an adapted L^r Calderón-Zygmund decomposition, where a certain cancellation of the operator with respect to the semigroup generated by the elliptic operator replaces the regularity estimates of the kernel. Weights in the class $A_{p,(r,s)}$ were then introduced into this argument by Auscher and Martell [AM07], but their techniques do not seem to yield optimal bounds in terms of the constants of the weights. Therefore, in Chapter 6, which is based on the paper [FN19] by Frey and the author, we give a different argument to establish the corresponding bounds that are sharp in the sense that they recover the bounds found in [HP13].

Since we are making no assumptions on our operators other than the sparse domination (1.1.1), we need to carefully adapt the arguments to these sparse forms. To this end, we introduce weights into a weak boundedness argument for sparse operators where there exists a Calderón-Zygmund decomposition with the property that the 'bad' part b cancels completely. We then combine this with generalizations of the main lemmata used in [LOP09a]. Moreover, we leave the Euclidean setting and extend the results to more general spaces of homogeneous type in Chapter 7. This includes certain bounded domains and Riemannian manifolds that were also studied in [BK03] and [AM07, AM08]. In Subsection 5.4.3 we provide examples falling outside of the class of Calderón-Zygmund operators that our results are applicable to.

1.1.3. Rubio de Francia extrapolation

The reduction to the A_2 case for the A_2-conjecture was done through a quantitative version of Rubio de Francia's extrapolation theorem [Rub82, GR85]. In one of its forms, this theorem says that if an operator T is bounded $L^q(\mathbf{R}^n; w) \to L^q(\mathbf{R}^n; w)$ for a fixed $q \in [1,\infty)$ and for all $w \in A_q$, then T is in fact bounded $L^p(\mathbf{R}^n; w) \to L^p(\mathbf{R}^n; w)$ for all $p \in (1,\infty)$ and all $w \in A_p$. If the control of the initial bound in terms of the constant $[w]_{A_q}$ is known, then a sharp control of the bound for $p \in (1,\infty)$ in terms of $[w]_{A_p}$ can be obtained [DGPP05].

We point out that the range $p \in (1,\infty)$ in the conclusion of this result is sharp. Indeed, it need not be the case that an operator satisfying weighted bounds is bounded $L^\infty(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$, as is the case, for example, for the Hilbert transform. In particular, it is impossible to extrapolate estimates to this endpoint. However, this opens up the question if it is also possible to extrapolate an estimate starting from $q = \infty$, as is the case for $q = 1$. This question becomes particularly interesting in the multilinear setting.

For example, in the bilinear setting it may very well occur that singular integral operators are bounded $L^2(\mathbf{R}^n)\times L^\infty(\mathbf{R}^n)\to L^2(\mathbf{R}^n)$.

An example of such an operator is the bilinear Hilbert transform BHT given by

$$\mathrm{BHT}(f_1,f_2)(x):=\mathrm{p.v.}\int_{\mathbf{R}} f_1(x-y)f_2(x+y)\,\frac{\mathrm{d}y}{y},$$

which plays a central role in the theory of time-frequency analysis. This operator was introduced by A. Calderón [Cal77] and he conjectured that it has a bounded extension $L^2(\mathbf{R})\times L^\infty(\mathbf{R})\to L^2(\mathbf{R})$. This conjecture was finally settled by Lacey and Thiele [LT97, LT99], where they showed that BHT is bounded $L^{p_1}(\mathbf{R})\times L^{p_2}(\mathbf{R})\to L^p(\mathbf{R})$ for all $p_1,p_2\in(1,\infty]$ with $\frac{2}{3}<p<\infty$, where $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. It is an open problem whether we can obtain bounds for the remaining range $\frac{1}{2}<p\leq\frac{2}{3}$ or not. Weighted bounds for this operator were established through sparse domination in [CDO18], which caused an interest in proving a multilinear analogue of Rubio de Francia's extrapolation theorem. An added difficulty in this situation is the fact that only bounds in a limited range of p_1,p_2 are known for a restricted class of weights, so the multilinear extrapolation result of Grafakos and Martell [GM04] does not apply.

With an application to BHT in mind, a multilinear analogue of the limited range extrapolation result of Auscher and Martell [AM07] was obtained by Cruz-Uribe and Martell in [CM18]. They showed that if there are $r_j\in(0,\infty)$, $s_j\in(r_j,\infty]$, and $q_j\in[r_j,s_j]$, $q_j\neq\infty$, such that an m-linear operator T satisfies

$$\|T(\vec{f}\,)\|_{L^q(\mathbf{R}^n;w^q)}\lesssim\prod_{j=1}^m\|f_j\|_{L^{q_j}(\mathbf{R}^n;w_j^{q_j})}\tag{1.1.7}$$

for all weights $w_j^{q_j}\in A_{q_j,(r_j,s_j)}$ as in (1.1.2), where $w=\prod_{j=1}^m w_j$, $\frac{1}{q}=\sum_{j=1}^m\frac{1}{q_j}$, then T satisfies the same boundedness for all $p_j\in(r_j,s_j)$ and all $w_j^{p_j}\in A_{p_j,(r_j,s_j)}$, as well as certain vector-valued bounds.

Through the helicoidal method of Benea and Muscalu [BM16], vector-valued bounds of the form $L^{p_1}(\mathbf{R}^n;\ell^{q_1})\times L^{p_2}(\mathbf{R}^n;\ell^{q_2})\to L^p(\mathbf{R}^n;\ell^q)$ were established in this range of p_1, p_2, p for various choices of $1<q_1,q_2\leq\infty$, $\frac{2}{3}<q<\infty$ with $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$. However, the extrapolation result of Cruz-Uribe and Martell does not allow one to cover the full range of exponents. More precisely, their result cannot retrieve any of the vector-valued bounds involving ℓ^∞ spaces. The problem seems to be that the multilinear nature of the problem is not completely utilized when one imposes individual conditions on each of the weights rather than involving an interaction between the various weights.

In the work [LMO18] by Li, Martell, and Ombrosi an extrapolation result was presented for a limited range version of the multilinear weight condition introduced by Lerner, Ombrosi, Pérez, Torres, and Trujillo-González [LOP+09b]. These weight classes are characterized by boundedness of the multisublinear Hardy-Littlewood maximal operator as well as by boundedness of sparse forms, meaning the theory can be applied to a wealth of operators including multilinear Calderón-Zygmund operators and bilinear Hilbert transform. They introduced the weight class $A_{\vec{p},\vec{r}}$ where $\vec{p}=(p_1,\dots,p_m)$, $\vec{r}=(r_1,\dots,r_{m+1})$ with $1\le r_j\le p_j<\infty$, $r_{m+1}'>p$ with $\frac{1}{p}=\sum_{j=1}^m\frac{1}{p_j}$. Then $\vec{w}=(w_1,\dots,w_m)\in A_{\vec{p},\vec{r}}$ if

$$\sup_Q\left(\frac{1}{|Q|}\int_Q\Big(\prod_{j=1}^m w_j^{\frac{p}{p_j}}\Big)^{\frac{r_{m+1}'}{r_{m+1}'-p}}\,\mathrm{d}x\right)^{\frac{1}{p}-\frac{1}{r_{m+1}'}}\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q w_j^{\frac{r_j}{r_j-p_j}}\,\mathrm{d}x\right)^{\frac{1}{r_j}-\frac{1}{p_j}}<\infty, \tag{1.1.8}$$

which in the case $m=1$ coincides with the condition $[w]_{A_{p,(r,s)}}^{\frac{1}{p}-\frac{1}{s}}<\infty$. They showed that if (1.1.7) holds for a $\vec{q}=(q_1,\dots,q_m)$ with $1\le r_j\le q_j<\infty$, $r_{m+1}'>q$ and all $(w_1^{q_1},\dots,w_m^{q_m})\in A_{\vec{q},\vec{r}}$, then T satisfies the same boundedness for all $\vec{p}=(p_1,\dots,p_m)$ and $(w_1^{p_1},\dots,w_m^{p_m})\in A_{\vec{p},\vec{r}}$ with $r_j<p_j<\infty$ and $r_{m+1}'>p$. Furthermore, their result extends and reproves some of the vector-valued bounds found by Benea and Muscalu [BM18] for BHT. This class of weights does seem to be adapted to the situation even when $p_j=\infty$, but one needs to be careful in how the constant is interpreted in this case. Similar to the proof of the extrapolation result of Cruz-Uribe and Martell, their proof of this extrapolation result is based upon an off-diagonal extrapolation result, but in their work they left open exactly what happens in the case that some of the exponents are infinite. They eventually covered these cases in [LMM⁺19]. Here they show that, as a feature of off-diagonal extrapolation, it is also possible to obtain estimates that include the cases of infinite exponents.

In this work we again prove an extrapolation result using the multilinear weight classes, and our result includes these endpoint cases which, in particular, include the possibility of extrapolating from the cases where in the initial assumption the exponents can be infinite. This result was originally proven by the author in [Nie19] in the time before the paper [LMM⁺19] appeared. This proof is new and does not rely on any off-diagonal extrapolation result. Rather, we generalize the Rubio de Francia algorithm to a multilinear setting adapted to the multisublinear Hardy-Littlewood maximal operator. As a corollary, we are able to obtain vector-valued extensions of operators to spaces including ℓ^∞ spaces. Thus, applying this to BHT allows us to recover these endpoint bounds that were obtained earlier through the helicoidal method [BM18].

Our construction is quantitative in the sense that it allows us to track the dependence of the bounds on the weight constants. Such quantitative versions of extrapolation results were first formalized by Dragičević, Grafakos, Pereyra, and Petermichl in the linear setting in [DGPP05], but are completely new in the multilinear setting. In the linear setting this result is based on Buckley's sharp weighted bound for the Hardy-Littlewood maximal operator. This bound has been generalized to the multisublinear Hardy-Littlewood maximal operator by Damián, Lerner, and Pérez [DLP15] to a sharp estimate in the setting of a mixed type $A_{\vec{p}}-A_\infty$ estimates and a sharp $A_{\vec{p}}$ bound is found in [LMS14]. In Section 3.2 we give a different proof of this result for the more general limited range version of this maximal operator by generalizing a proof of Lerner [Ler08].

Unlike in the linear case $m = 1$, our quantitative extrapolation result in the case $m > 1$ is actually essential when it comes to obtaining the full range of sharp weighted bounds for multi(sub)linear operators dominated by sparse forms. The reason for this is that sparse domination initially yields sharp bounds for an operator for exponents $p_1, \dots, p_m$ only if $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_m} \le 1$, where one can appeal to duality. While in the linear setting $m = 1$ this covers the full range $p \in (1,\infty)$, in the multilinear setting $m > 1$ the exponent p can also satisfy $p \in (\frac{1}{m}, 1)$. Our extrapolation result allows us to show that the sharp bound for $p \ge 1$ obtained from sparse domination also holds when $p < 1$. We elaborate on this further in Chapter 5.

1.1.4. Symmetry in the A_p classes

For $p \in (1,\infty)$, a standard method of obtaining weighted L^p estimates with a weight w is by using the duality $(L^p(w))^* = L^{p'}(w^{1-p'})$ given through the integral pairing

$$\langle f, g\rangle = \int_{\mathbf{R}^n} f g \, \mathrm{d}x.$$

This duality in reflected in the definition of the Muckenhoupt A_p class, which is defined in terms of the weights w and $w^{1-p'} = w^{-\frac{1}{1-p}}$. One way to understand this definition better is by noting that we can relate the weights w and $w^{1-p'}$ through $w^{\frac{1}{p}}(w^{1-p'})^{\frac{1}{p'}} = 1$.

When we replace the weight w by the weight w^p we find, using the averaging notation $\langle h\rangle_{q,Q} := \left(\frac{1}{|Q|}\int_Q |h|^q \, \mathrm{d}x\right)^{\frac{1}{q}}$, that

$$[w^p]_{A_p}^{\frac{1}{p}} = \sup_Q \langle w\rangle_{p,Q} \langle w^{-1}\rangle_{p',Q}$$

for $p \in (1,\infty)$. The symmetry in this condition is much more prevalent and this condition seems to be more naturally adapted to the weighted L^p theory. Indeed, defining

$$[w]_p := [w^p]_{A_p}^{\frac{1}{p}},$$

we note that $[w]_p = [w^{-1}]_{p'}$. We define the bisublinear Hardy-Littlewood maximal operator $M_{(1,1)}$ by

$$M_{(1,1)}(f_1, f_2)(x) := \sup_{Q \ni x} \langle f_1\rangle_{1,Q} \langle f_2\rangle_{1,Q}.$$

Then, writing $\|f\|_{L^p_w(\mathbf{R}^n)} := \|f w\|_{L^p(\mathbf{R}^n)}$, we have the remarkable equivalences

$$\begin{aligned} [w]_p &\eqsim \|M\|_{L^p_w(\mathbf{R}^n) \to L^{p,\infty}(\mathbf{R}^n; w^p)} \eqsim \|M\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n) \to L^{p',\infty}(\mathbf{R}^n; w^{-p'})} \\ &\eqsim \|M_{(1,1)}\|_{L^p_w(\mathbf{R}^n) \times L^{p'}_{w^{-1}}(\mathbf{R}^n)) \to L^{1,\infty}(\mathbf{R}^n)}, \end{aligned} \tag{1.1.9}$$

see Chapter 3. Another way of thinking of these equivalences is by setting $w_1 := w$, $w_2 := w^{-1}$ and $p_1 := p$, $p_2 := p'$ so that we have the relations

$$w_1 w_2 = 1, \qquad \frac{1}{p_1} + \frac{1}{p_2} = 1. \tag{1.1.10}$$

Then one can impose a symmetric weight condition

$$[(w_1, w_2)]_{(p_1,p_2)} := \sup_Q \langle w_1\rangle_{p_1,Q}\langle w_2\rangle_{p_2,Q} < \infty$$

and note that

$$[(w_1, w_2)]_{(p_1,p_2)} = [w_1]_{p_1} = [w_2]_{p_2}.$$

The equivalences (1.1.9) can now be thought of as

$$\begin{aligned}\|M_{(1,1)}\|_{L^{p_1}_{w_1}(\mathbf{R}^n)\times L^{p_2}_{w_2}(\mathbf{R}^n)\to L^{1,\infty}(\mathbf{R}^n)} &\eqsim [(w_1, w_2)]_{(p_1,p_2)},\\ \|M\|_{L^{p_1}_{w_1}(\mathbf{R}^n)\to L^{p_1,\infty}(\mathbf{R}^n;w_1^{p_1})} &\eqsim [w_1]_{p_1},\\ \|M\|_{L^{p_2}_{w_2}(\mathbf{R}^n)\to L^{p_2,\infty}(\mathbf{R}^n;w_2^{p_2})} &\eqsim [w_2]_{p_2}.\end{aligned}$$

We can even make sense of these expressions when $p_1 = 1$ and $p_2 = \infty$ or $p_1 = \infty$ and $p_2 = 1$, since $f \in L^p_w(\mathbf{R}^n)$ (or $f \in L^{p,\infty}(\mathbf{R}^n; w^p)$) in the case $p = \infty$ just means that the function fw is essentially bounded. Writing $\langle h\rangle_{\infty,Q} = \operatorname{ess\,sup}_{x\in Q}|h(x)|$, we see that the condition $[w_1]_1 < \infty$ is equivalent to the usual A_1 condition imposed on the weight $w_1 = w$, while the condition $[w_1]_\infty < \infty$ is equivalent to the condition $w_2 = w^{-1} \in A_1$. We emphasize here that our condition $[w]_\infty < \infty$ is not equivalent to the condition $w \in A_\infty = \bigcup_{p\in[1,\infty)} A_p$ and these notions should not be confused. The condition $w^{-1} \in A_1$ seems to be a natural upper endpoint condition and we will show that this is equivalent to M being bounded $L^\infty_w(\mathbf{R}^n) \to L^\infty_w(\mathbf{R}^n)$. As a matter of fact, since M is an isometry in $L^\infty(\mathbf{R}^n)$, it behaves most naturally when $p = \infty$. Thus, even though this case was originally missed, this equivalence is the simplest case of Muckenhoupt's characterization of the A_p classes. We fill in this gap here.

Proposition 1.1.1. *Let w be a weight and $c \ge 0$. Then $[w]_\infty \le c$ if and only if*

$$\|Mf\|_{L^\infty_w(\mathbf{R}^n)} \le c\|f\|_{L^\infty_w(\mathbf{R}^n)}. \tag{1.1.11}$$

for all $f \in L^\infty_w(\mathbf{R}^n)$. In particular, $\|M\|_{L^\infty_w(\mathbf{R}^n)\to L^\infty_w(\mathbf{R}^n)} = [w]_\infty$.

Proof. We note that $[w]_\infty = [w^{-1}]_{A_1} \le c$ if and only if $M(w^{-1}) \le cw^{-1}$. Thus, if $[w]_\infty \le c$, then

$$Mf = M(fww^{-1}) \le M(w^{-1})\|f\|_{L^\infty_w(\mathbf{R}^n)} \le cw^{-1}\|f\|_{L^\infty_w(\mathbf{R}^n)}$$

so that $(Mf)w \le c\|f\|_{L^\infty_w(\mathbf{R}^n)}$. This proves (1.1.11).

For the converse, set $f := w^{-1}$. Then $\|f\|_{L^\infty_w(\mathbf{R}^n)} = 1$, so (1.1.11) implies that $M(w^{-1})w \le c$, or $M(w^{-1}) \le cw^{-1}$. The assertion follows. □

It also turns out that this condition allows us to extrapolate away from weighted L^∞ estimates. We point out that a version of this idea was used in the endpoint extrapolation result of Harboure, Macías and Segovia [HMS88, Theorem 3] involving weighted versions of the space $\mathrm{BMO}(\mathbf{R}^n)$. We now fill in the missing case with $q = \infty$ in the Rubio de Francia extrapolation theorem. We point out that this is the case that has the shortest argument.

Theorem 1.1.2. *Let T be a linear operator that is bounded $L^\infty_w(\mathbf{R}^n) \to L^\infty_w(\mathbf{R}^n)$ for all weights w satisfying $[w]_\infty < \infty$ with*

$$\|T\|_{L^\infty_w(\mathbf{R}^n)\to L^\infty_w(\mathbf{R}^n)} \lesssim [w]_\infty^{2020}. \tag{1.1.12}$$

Then for all $p \in (1,\infty]$ and all weights w with $[w]_p < \infty$, T is bounded $L^p_w(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)$ with

$$\|T\|_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} \lesssim [w]_p^{2020p'}. \tag{1.1.13}$$

Moreover, the operator $\widetilde{T}$ given by $\widetilde{T}(f_k)_{k\in\mathbf{N}} := (Tf_k)_{k\in\mathbf{N}}$ is bounded $L^p_w(\mathbf{R}^n;\ell^\infty) \to L^p_w(\mathbf{R}^n;\ell^\infty)$ with

$$\|\widetilde{T}\|_{L^p_w(\mathbf{R}^n;\ell^\infty)\to L^p_w(\mathbf{R}^n;\ell^\infty)} \lesssim [w]_p^{2020p'}.$$

Proof. Let $p \in (1,\infty]$, w a weight with $[w]_p < \infty$, and $f \in L^p_w(\mathbf{R}^n)$ non-zero. We define a weight W through

$$W^{-1} = \sum_{k=0}^\infty \frac{M^k f}{2^k \|M\|^k_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)}},$$

where $M^0 f := |f|$ and $M^k f := M(M^{k-1}f)$ for $k \in \mathbf{N}$. Then $|f| \le W^{-1}$, i.e., $\|f\|_{L^\infty_W(\mathbf{R}^n)} \le 1$. Moreover, we have $M(W^{-1}) \le 2\|M\|_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} W^{-1}$ so that $[W]_\infty \le 2\|M\|_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)}$, and, finally, we have $\|W^{-1}\|_{L^p_w(\mathbf{R}^n)} \le 2\|f\|_{L^p_w(\mathbf{R}^n)}$. Thus, combining these three properties of W with (1.1.12), we have

$$\begin{aligned}\|Tf\|_{L^p_w(\mathbf{R}^n)} &\le \|Tf\|_{L^\infty_W(\mathbf{R}^n)}\|W^{-1}\|_{L^p_w(\mathbf{R}^n)} \lesssim [W]_\infty^{2020}\|f\|_{L^p_w(\mathbf{R}^n)}\\ &\lesssim \|M\|^{2020}_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)}\|f\|_{L^p_w(\mathbf{R}^n)}.\end{aligned}$$

The result now follows from Buckley's bound $\|M\|_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} \lesssim [w]_p^{p'}$.

As for the bound of $\widetilde{T}$, we note that by interchanging the suprema, for all $f \in L^\infty_w(\mathbf{R}^n;\ell^\infty)$ we have

$$\|\widetilde{T}f\|_{L^\infty_w(\mathbf{R}^n;\ell^\infty)} = \sup_{k\in\mathbf{N}}\|Tf_k\|_{L^\infty_w(\mathbf{R}^n)} \lesssim [w]_\infty^{2020}\sup_{k\in\mathbf{N}}\|f_k\|_{L^\infty_w(\mathbf{R}^n)} = \|f\|_{L^\infty_w(\mathbf{R}^n;\ell^\infty)}.$$

Thus, the assertion follows by repeating the first part of the proof with Tf replaced by $\|\widetilde{T}f\|_{\ell^\infty}$ and f replaced by $\|f\|_{\ell^\infty}$. □

The construction of W in this proof uses the classical Rubio de Francia algorithm.

We wish to view our symmetric weight condition in the context of extrapolation for general $q \in [1,\infty]$. In proving Rubio de Francia's extrapolation theorem, one usually starts with an estimate of the form

$$\|Tf\|_{L^q_w(\mathbf{R}^n)} \lesssim \|f\|_{L^q_w(\mathbf{R}^n)} \tag{1.1.14}$$

for some $q \in [1,\infty]$ and all weights w satisfying $[w]_q < \infty$. The idea is then that given a $p \in (1,\infty)$ and a weight w satisfying $[w]_p < \infty$, one can construct a weight W, possibly depending on f, h, and w, so that W satisfies $[W]_q < \infty$ as well as some additional properties to ensure that we can use (1.1.14) with W to conclude that

$$\|Tf\|_{L^p_w(\mathbf{R}^n)} \lesssim \|f\|_{L^p_w(\mathbf{R}^n)}, \tag{1.1.15}$$

proving the desired boundedness for an operator T. For the proof one usually treats the two cases $p < q$ and $p > q$ separately. In the former case one can apply Hölder's inequality to move from L^p to L^q as we did in the proof of Theorem 1.1.2, and in the latter case one can use duality and a similar technique to move from $L^{p'}$ to $L^{q'}$. These cases are essentially the same, but due to the notation we use we have to deal with the cases separately. Here, we wish to come up with a formalization to avoid this redundancy.

The extrapolation theorem is essentially a consequence of to the following proposition:

Proposition 1.1.3. *Suppose we are given* $p_1, p_2 \in (1,\infty)$ *satisfying* $\frac{1}{p_1} + \frac{1}{p_2} = 1$ *and weights* w_1, w_2 *satisfying* $w_1 w_2 = 1$ *and* $[(w_1, w_2)]_{(p_1,p_2)} < \infty$. *Moreover, assume we have two functions* $f_1 \in L^{p_1}_{w_1}(\mathbf{R}^n)$ *and* $f_2 \in L^{p_2}_{w_2}(\mathbf{R}^n)$ *and* $q_1, q_2 \in [1,\infty]$ *with* $\frac{1}{q_1} + \frac{1}{q_2} = 1$. *Then there are weights* W_1, W_2 *satisfying* $W_1 W_2 = 1$,

$$\|f_1\|_{L^{q_1}_{W_1}(\mathbf{R}^n)} \|f_2\|_{L^{q_2}_{W_2}(\mathbf{R}^n)} \le 2\|f_1\|_{L^{p_1}_{w_1}(\mathbf{R}^n)} \|f_2\|_{L^{p_2}_{w_2}(\mathbf{R}^n)}$$

and

$$[(W_1, W_2)]_{(q_1,q_2)} \le C[(w_1, w_2)]_{(p_1,p_2)}^{\max\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}\right)}.$$

Indeed, the result of the extrapolation theorem follows by applying the proposition with $f_1 := f$, $q_1 := q$, $q_2 := q'$, $p_1 := p$, $p_2 := p'$, $w_1 := w$, $w_2 = w^{-1}$ and $W_1 := W$, $W_2 := W^{-1}$ so that, by (1.1.14), we have

$$\begin{aligned}\left|\int_{\mathbf{R}^n} (Tf) f_2 \, \mathrm{d}x\right| &\le \|Tf\|_{L^q_W(\mathbf{R}^n)} \|f_2\|_{L^{q'}_{W^{-1}}(\mathbf{R}^n)} \lesssim \|f\|_{L^q_W(\mathbf{R}^n)} \|f_2\|_{L^{q'}_{W^{-1}}(\mathbf{R}^n)} \\ &\lesssim \|f\|_{L^p_w(\mathbf{R}^n)} \|f_2\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)}.\end{aligned}$$

Thus, by duality, we obtain (1.1.15), as desired. Moreover, since by Fubini's Theorem we have the Bochner space equality $L^q(\mathbf{R}^n; \ell^q) = \ell^q(L^q(\mathbf{R}^n))$, as in the proof of Theorem 1.1.2 one can deduce bounds $\widetilde{T} : L^p_w(\mathbf{R}^n; \ell^q) \to L^p_w(\mathbf{R}^n; \ell^q)$.

The proof of Proposition 1.1.3 uses the classical construction using the Rubio de Francia algorithm and the novelty here is our symmetric formulation. We prove the full multilinear limited range generalization of this result in Chapter 4. The case $p < q$ in the proposition takes the form $p_1 < q_1$ and $p_2 > q_2$ while the case $p > q$ takes the form $p_1 > q_1$ and $p_2 < q_2$. The fact that the proposition is formulated completely symmetrically in terms of the parameters indexed over $\{1,2\}$, where we note that $[(w_1, w_2)]_{(p_1,p_2)} = [(w_2, w_1)]_{(p_2,p_1)}$, means that these respective cases can be proven using precisely the same argument, up to a permutation of the indices. Thus, without loss of generality, one only needs to prove one of the two cases.

These symmetries become especially important in the m-linear setting where we are dealing with parameters indexed over $\{1,\dots,m+1\}$ and the amount of cases we have to consider increases. Thanks to our formulation, we will be able to reduce these multiple cases back to a single case in our arguments again by permuting the indices.

1.1.5. Vector-valued extensions of operators

Vector-valued extensions of operators prevalent in the theory of harmonic analysis have been actively studied in the past decades. A centerpoint of the theory is the result of Burkholder [Bur83] and Bourgain [Bou83] which states that the Hilbert transform has a bounded tensor extension $\widetilde{H} : L^p(\mathbf{R}; X) \to L^p(\mathbf{R}; X)$ for some, or equivalently all, $p \in (1,\infty)$, if and only if the Banach space X has the so-called UMD property. From this connection one can derive the boundedness of the vector-valued extension of many operators in harmonic analysis, like Fourier multipliers and Littlewood-Paley operators.

In the specific case where X is a Banach function space, i.e. a lattice of functions over some measure space, very general extension theorems are known. These follow from the deep result of Bourgain [Bou84] and Rubio de Francia [Rub86] on the connection between the boundedness of the lattice Hardy-Littlewood maximal operator $\widetilde{M} : L^p(\mathbf{R}^n; X) \to L^p(\mathbf{R}^n; X)$ and the UMD property of X. The boundedness of the lattice Hardy-Littlewood maximal operator often allows one to use the scalar-valued arguments to show the boundedness of the vector-valued extension of an operator, using very elaborate Fubini-type techniques. Moreover it connects the extension problem to the theory of Muckenhoupt weights.

As we have seen, vector-valued extensions in sequence spaces ℓ^q of operators can be obtained from Rubio de Francia's extrapolation theorem through an argument using Fubini's Theorem. Rubio de Francia showed in [Rub85, Theorem 5] that one can take this even further. Indeed, again assuming that T is a linear operator satisfying the initial weighted estimate (1.1.14), then for each Banach function space X with the UMD property, T extends to an operator $\widetilde{T}$ satisfying the Bochner space bound $L^p(\mathbf{R}^n; X) \to L^p(\mathbf{R}^n; X)$ for all $p \in (1,\infty)$. This establishes a deep connection between the theory of Muckenhoupt weights, the theory of UMD Banach function spaces, and the theory of singular integral operators.

This vector-valued extrapolation result of Rubio de Francia was extended by Amenta, Lorist, and Veraar in [ALV19] to a rescaled setting and by Lorist and the author in [LN19] to a limited range multilinear setting.

In this latter result we proved that if there are $r_j \in (0,\infty)$, $s_j \in (r_j,\infty]$, and a multilinear operator T is bounded $L^{p_1}_{w_1}(\mathbf{R}^n) \times \cdots \times L^{p_m}_{w_m}(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)$ for all $p_j \in (r_j, s_j)$, and weights $w_j^{p_j} \in A_{p_j,(r_j,s_j)}$ for all $j \in \{1,\dots,m\}$, then it has a vector-valued extension $\widetilde{T}$ that is bounded $L^{p_1}_{w_1}(\mathbf{R}^n; X_1) \times \cdots \times L^{p_m}_{w_m}(\mathbf{R}^n; X_m) \to L^p_w(\mathbf{R}^n; X)$ for all $p_j \in (r_j, s_j)$, all weights $w_j^{p_j} \in A_{p_j,(r_j,s_j)}$, and all (quasi-)Banach function spaces X_j satisfying a UMD_{r_j,s_j} condition, which is a certain rescaled UMD condition. A version of this result with a slightly more general condition on the spaces is proven in Section 9.1.

In the linear case $m = 1$, our result extends the main result of [ALV19] in the sense that it allows for finite s_j, which can then be applied to any of the operators satisfying the sparse form domination (1.1.1) introduced in [BFP16].

As for the multilinear case $m > 1$, to place this result into context we point out that it appeared after the limited range multilinear extrapolation theorem of Cruz-Uribe and Martell [CM18], but before the realization of Li, Martell, and Ombrosi in [LMO18] that rather than assuming a condition on each individual weight, it is more appropriate to consider the multilinear weight classes defined through (1.1.8). Since the space ℓ^∞ is not a UMD space, bounds in this space can not be obtained through our vector-valued extrapolation theorem, even though these spaces can be obtained through the extrapolation techniques using multilinear weight classes [Nie19, LMM+19].

To unify the theory, a multilinear UMD condition for tuples of Banach spaces was introduced in the work [LN20] of Lorist and the author. We introduce these spaces in Chapter 8 and prove a multilinear extension theorem in which we use the multilinear structure to its fullest in Section 9.2. We impose a condition on the tuple of Banach function spaces $(X_1,\ldots,X_m)$ rather than a condition on each X_j individually. In parallel to the weighted theory, we will introduce this condition using the boundedness of a certain rescaled multisublinear Hardy-Littlewood maximal operator. In the linear case $m = 1$ this condition reads as follows:

$$\big\|\widetilde{M}_{(1,1)}(f,g)\big\|_{L^1(\mathbf{R}^n;L^1(\Omega))} \lesssim \|f\|_{L^p(\mathbf{R}^n;X)}\|g\|_{L^{p'}(\mathbf{R}^n;X^*)}$$

for all $f \in L^p(\mathbf{R}^n;X)$, $g \in L^{p'}(\mathbf{R}^n;X^*)$ and some $p \in (1,\infty)$, where $\widetilde{M}_{(1,1)}$ is the bisublinear lattice maximal operator that we introduce in Section 8.3. In Section 8.4 we will show that this condition is equivalent to the UMD condition for Banach function spaces and motivated by this result, we will call our multilinear analog a multilinear UMD condition, even though our definition only makes sense for tuples of Banach *function* spaces.

Both the Banach function space extension principle from [Rub86, ALV19, LN19] and the iterated L^q-space extension principle using the extrapolation results in [Nie19, LMM+19] use the weighted boundedness of a multilinear operator

$$T\colon L^{p_1}_{w_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)$$

to deduce the weighted boundedness of its extension

$$\widetilde{T}\colon L^{p_1}_{w_1}(\mathbf{R}^n;X_1)\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n;X_m)\to L^p_w(\mathbf{R}^n;X).$$

Usually these weighted bounds for T are deduced through sparse domination. Thus, to deduce the weighted boundedness of the vector-valued extension $\widetilde{T}$ of an operator T one typically goes through implications (1) and (3) in the following diagram.

$$
\begin{array}{ccc}
\text{Sparse domination for } T & \overset{(1)}{\Longrightarrow} & \text{Weighted bounds for } T \\
(2)\Big\Downarrow & & \Big\Downarrow(3) \\
\text{Sparse domination for } \widetilde{T} & \overset{(4)}{\Longrightarrow} & \text{Weighted bounds for } \widetilde{T}
\end{array}
$$

The implications (1) and (4) are respectively treated in Section 5.3 and Section 8.2. The vector-valued extrapolation theorem for implication (3) is proven in Section 9.1.

In Section 9.2 we will deduce the weighted boundedness of the vector-valued extension $\widetilde{T}$ of T through implications (2) and (4). To this end we will show that scalar-valued sparse domination implies vector-valued sparse domination (implication (2)) with respect to tuples of spaces satisfying our multilinear UMD-condition. Such a result was established by Culiuc, Di Plinio, and Ou in [CDO17] for sequence spaces ℓ^q with $q \geq 1$, which in particular satisfy our multilinear UMD condition. We point out that even in the linear case $m = 1$, the result of obtaining vector-valued extensions of operators in UMD Banach function spaces from sparse domination without appealing to a Rubio de Francia type extrapolation theorem is new.

The advantage of the route through implications (2) and (4) over the route through implications (1) and (3) is that for general tuples of quasi-Banach function spaces the Fubini-type techniques needed for implication (2) are a lot less technical than the ones needed for implication (3). Moreover implication (4) yields quantitative and in many cases sharp weighted estimates for $\widetilde{T}$, while the weight dependence in the arguments used for implication (3) is not easily tracked and certainly not sharp. A downside of our approach through implications (2) and (4) is the fact that we need sparse domination for T as a starting point, while one only needs weighted bounds in order to apply (3). We point out that it is an open question whether it is possible to prove implication (3) for tuples of spaces in our multilinear UMD classes, rather than for tuples of spaces that each satisfy a UMD condition as is the case in Section 9.1.

Our proof of implication (2) relies on two key ingredients. The first is the equivalence between sparse form and the L^1-norm of the multisublinear maximal function, which we treat in Section 3.2. This equivalence seems to have been used for the first time in [CDO17] by Culiuc, Di Plinio, and Ou. The second ingredient is a sparse domination result for the multisublinear lattice maximal operator under the multilinear UMD condition assumption, which we present in Section 8.3. This result is an extension of the idea of Hänninen and Lorist in [HL19], where a linear version of this result was obtained.

1.2. OUTLINE OF THE THESIS

Part $\frac{1}{1}$: Introduction

In Chapter 1 we give a general introduction for this thesis. In Chapter 2 we describe the setting as well as the notational conventions that are in force throughout the thesis.

Part $\frac{1}{2}$: Multilinear weight classes and Rubio de Francia extrapolation

In Chapter 3 we define the multilinear weight classes and the corresponding weight constants. The main results in this chapter are Theorem 3.2.3 and Theorem 3.2.11 in which the weight classes are characterized by the boundedness of the multisublinear maximal operator and sparse forms. Moreover, we obtain the sharp dependence of the their bounds in terms of the weight constant. Finally, we introduce multilinear analogues of the Fujii-Wilson constant and we prove a self-improvement property of the multilinear weight classes.

In Chapter 4 we prove the abstract version of the sharp multilinear limited range extrapolation theorem given in Theorem 4.1.1. This is done through the construction of a multilinear analogue of the Rubio de Francia algorithm in Lemma 4.1.3 and heavily utilizes the symmetry in the weight classes. A careful study of the dependence of the parameters and weight constants is done throughout the arguments.

Part $\frac{1}{3}$: Quantitative estimates for multilinear operators dominated by sparse forms

This part is dedicated to applying the theory from Part $\frac{1}{2}$ in order to obtain sharp weighted bound for multi(sub)linear operators.

In Chapter 5 we apply the extrapolation theorem from Chapter 4 to multi(sub)linear operators satisfying weighted bounds with respect to the multilinear weight classes. This is done in main result in the first section in Theorem 5.1.2. In the following section we apply Theorem 5.1.2 to prove Theorem 5.2.3, where we obtain a sharpness result for operators through the asymptotic behaviour of their unweighted operator norms. In the subsequent section we apply Theorem 5.1.2 to obtain the full range of sharp bounds for operators satisfying ℓ^q-type sparse domination in form in Theorem 5.3.6. In the last section of this chapter we introduce multilinear Calderón-Zygmund operators and the bilinear Hilbert transform, and apply Theorem 5.3.6 to obtain sharp bounds for these operators. Moreover, examples of operators satisfying sparse bounds are given in the linear case $m = 1$.

In Chapter 6 we first prove that multi(sub)linear operators satisfying sparse form domination are weakly bounded at the lower endpoint. The main results of the subsequent section are given in Theorem 6.2.1, Theorem 6.2.2, and Theorem 6.2.9, where weighted mixed type A_p–A_∞ endpoint bounds are proven for operators satisfying sparse form bounds in the linear setting $m = 1$. A main ingredient for these results is the sharp reverse Hölder inequality for Muckenhoupt weights.

In Chapter 7 we show that our results also hold in the setting of spaces of homogeneous type. To this end, we prove Calderón-Zygmund decompositions adapted to dyadic grids in these spaces in the separate cases where the space is either bounded or unbounded.

Part $\frac{1}{4}$: A multilinear UMD *condition and vector-valued extensions of multilinear operators*

In this part we introduce a multilinear analogue of the UMD condition for tuples of quasi-Banach function spaces, and prove vector-valued bounds for extensions of operators with respect to these spaces.

In Chapter 8 we introduce product quasi-Banach function spaces and use the extrapolation result to prove sharp weighted vector-valued bounds for operators satisfying a vector-valued sparse domination in Theorem 8.2.2. Moreover, we introduce the multisublinear lattice maximal operator and define a rescaled multilinear analogue of the Hardy-Littlewood property for tuples of quasi-Banach function spaces. The main result in this section is Theorem 8.3.3 in which sparse domination of the multisublinear lattice maximal operator is proven for such tuples of quasi-Banach function spaces. In the final section we introduce a limited range multilinear analogue of the UMD condition for tuples of quasi-Banach function spaces. Moreover, we provide basic properties and examples of these spaces.

In Chapter 9 we describe two methods of obtaining vector-valued bounds for extensions of multi(sub)linear operators. In the first section we prove a multilinear limited range analogue of Rubio de Francia's vector-valued extrapolation theorem in Theorem 9.1.1. A main ingredient here is a self-improvement property of our limited range UMD condition in the linear setting $m = 1$ proven in Proposition 9.1.7. In the next section we use the sparse domination result Theorem 8.3.3 for the multisublinear lattice maximal operator to prove Theorem 9.2.1 in which we show that if an operator satisfies sparse form domination, then it has a vector-valued extension satisfying vector-valued sparse domination for tuples of quasi-Banach function spaces satisfying our multilinear UMD condition. The results of Chapter 8 are then used to deduce sharp weighted vector-valued bounds of these operators. In the last section we describe how our methods can be used to prove optimal weighted vector-valued bounds in concrete situations and, in particular, we apply our results to multilinear Calderón-Zygmund operators and the bilinear Hilbert transform.

2

The setting and notational conventions

Since we are working in a multilinear setting, it is helpful to set some notational conventions in order to reduce the size and increase the readability of our expressions.

Throughout this work, m will denote an integer greater than or equal to 1. When $m = 1$, we will refer to this setting as the linear setting, while for general m we refer to the setting as an m-linear or multilinear setting. Moreover, we respectively refer to operators in these settings as linear and multilinear operators. We point out that this is somewhat inaccurate, since we are not only considering multilinear operators, but also multisublinear operators.

For most of this work we will be working with functions defined on the metric measure space $(\mathbf{R}^n, |\cdot|, \mathrm{d}x)$, where n is a positive integer, $|\cdot|$ is the Euclidean norm, and $\mathrm{d}x$ is the Lebesgue measure.

For $p \in (0,\infty]$, we denote by $L^p(\mathbf{R}^n)$ the complex Lebesgue space of measurable functions whose p-th power is integrable. We let $L^0(\mathbf{R}^n)$ denote the complex space of measurable functions. When we are working with a measure μ different from the Lebesgue measure on $\mathbf{R}^n$, we will denote these spaces by $L^p(\mathbf{R}^n;\mu)$. We use a similar convention for the weak-type spaces $L^{p,\infty}(\mathbf{R}^n)$.

For an m-tuple of parameters $p_1,\dots,p_m$, usually appearing in some subset of $(0,\infty]$, we will use the notation $\vec{p} = (p_1,\dots,p_m)$ for the vector that has the p_j as its components. We will often introduce such an m-tuple by simply writing $\vec{p} \in (0,\infty]^m$. Sometimes we will also write $\vec{1} = (1,\dots,1)$ and $\vec{\infty} = (\infty,\dots,\infty)$. Moreover, for $\vec{p} \in (0,\infty]^m$ we will, per convention, define the parameter $p \in (0,\infty]$ with the index j dropped through the Hölder relation

$$\frac{1}{p} := \sum_{j=1}^{m} \frac{1}{p_j}.$$

For $\vec{q} \in (0,\infty)^m$ we write $\vec{p} \geq \vec{q}$ if $p_j \geq q_j$ and write $\vec{p} > \vec{q}$ if $p_j > q_j$ for all $j \in \{1,\dots,m\}$. Note that $\vec{p} \geq \vec{q}$, $\vec{p} > \vec{q}$ respectively imply that $p \geq q$ and $p > q$. We define arithmetic operations on $\vec{p}$ and $\vec{q}$ coordinate wise, e.g., we may write $\frac{\vec{p}}{\vec{q}} := (\frac{p_1}{q_1},\dots,\frac{p_m}{q_m})$, $\vec{p}^{\alpha} := (p_1^{\alpha},\dots,p_m^{\alpha})$ for $\alpha > 0$, or $\frac{1}{\vec{p}} := (\frac{1}{p_1},\dots,\frac{1}{p_m})$. Moreover, we write $\max\{\vec{p}\} := \max\{p_1,\cdots,p_m\}$.

For $\vec{p} \in (0,\infty]^m$ we will use the shorthand notation

$$L^{\vec{p}}(\mathbf{R}^n) := L^{p_1}(\mathbf{R}^n) \times \cdots \times L^{p_m}(\mathbf{R}^n).$$

This way, we may write $\vec{f} \in L^{\vec{p}}(\mathbf{R}^n)$ to mean that $\vec{f} = (f_1,\dots,f_m)$ is an m-tuple of functions with $f_j \in L^{p_j}(\mathbf{R}^n)$ for all $j \in \{1,\dots,m\}$. When $p_j = \infty$ for all $j \in \{1,\dots,m\}$ we will sometimes also write $L^{\infty}(\mathbf{R}^n)^m$ rather than $L^{\vec{\infty}}(\mathbf{R}^n)$. Moreover, we use similar conventions when adding a subscript to the spaces, or when considering weak-type spaces e.g., for $L^{\vec{p}}_{\mathrm{loc}}(\mathbf{R}^n)$ and $L^{\vec{p}}_c(\mathbf{R}^n)$. We will later adopt similar conventions for weighted Lebesgue spaces $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$ and weighted mixed-norm Lebesgue spaces $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n;\vec{X})$ as soon as the related notions are introduced.

As for the dependance on parameters of constants appearing in inequalities, we will write $c_{a,b,\ldots}$ or $C_{a,b,\ldots}$ to denote a constant which only depends on the parameters $a, b, \ldots$ and possibly on m and the dimension n. By $\lesssim_{a,b,\ldots}$ we mean that there is a constant $c_{a,b,\ldots}$ such that inequality holds and by $\eqsim_{a,b,\ldots}$ we mean that both $\lesssim_{a,b,\ldots}$ and $\gtrsim_{a,b,\ldots}$ hold. Whenever possible, in the proofs of our results we will keep explicit track of the precise control of the constants other than m and the dimension n.

Finally, we set a convention on our notation for Lebesgue exponents. Since many of our estimates rely on Hölder's inequality and related convexity results, it is more convenient to think in terms of the parameter $\frac{1}{p}$ rather than p. To facilitate this, we aim to avoid using expressions such as, e.g., $q\big(\frac{p}{q}\big)'$, but rather write this as $\frac{1}{\frac{1}{q}-\frac{1}{p}}$. In this case, when $p = q$, then it is implied that $\frac{1}{\frac{1}{q}-\frac{1}{p}} = \infty$. Similarly we may write an expression such as $\frac{p_1 p_2}{p_1+p_2}$ as $\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}}$ to make it clearer that we have

$$\|\cdot\|_{L^{\frac{1}{\frac{1}{p_1}+\frac{1}{p_2}}}(\mathbf{R}^n)} \le \|\cdot\|_{L^{p_1}(\mathbf{R}^n)}\|\cdot\|_{L^{p_2}(\mathbf{R}^n)}$$

by Hölder's inequality. Since it often occurs that, e.g., $p_1 = \infty$, using our notational convention this way we do not need to treat this case separately, since we may simply take $\frac{1}{p_1} = 0$ in the expressions.

$\frac{1}{2}$

Multilinear weight classes and Rubio de Francia extrapolation

3

MULTILINEAR WEIGHT CLASSES

In this chapter we introduce the multilinear Muckenhoupt weight classes. The first two sections of this chapter are partly based on the first part of the paper

> B. Nieraeth. Quantitative estimates and extrapolation for multilinear weight classes. *Mathematische Annalen*, 375(1-2):453–507, 2019.

These sections are enhanced through the inclusion of various small results from unpublished drafts.

The third section contains the partial results from an original unpublished manuscript on multilinear reverse Hölder weight classes.

3.1. THE $A_{\vec{p},(\vec{r},s)}$ WEIGHT CLASSES

A *weight* w is a measurable function $w:\mathbf{R}^n\to(0,\infty)$. For a weight w and $p\in(0,\infty]$ we define the weighted Lebesgue space $L^p_w(\mathbf{R}^n)$ as the space of those measurable functions f satisfying $\|fw\|_{L^p(\mathbf{R}^n)}<\infty$. Note that if $p\in(0,\infty)$, then $L^p_w(\mathbf{R}^n)$ coincides with the space $L^p(\mathbf{R}^n;w^p)$, i.e., the L^p space over $\mathbf{R}^n$ with respect to the measure $w^p\,\mathrm{d}x$. It should be noted that our definition of $L^p_w(\mathbf{R}^n)$ is often denoted by $L^p(w^p)$ in the literature when $p<\infty$, but the advantage of our definition is that we also obtain a sensible definition when $p=\infty$.

When $p<\infty$ we use the notation $L^{p,\infty}_w(\mathbf{R}^n):=L^{p,\infty}(\mathbf{R}^n;w^p)$ for the weak-type L^p space over $\mathbf{R}^n$ with respect to the measure $w^p\,\mathrm{d}x$. Moreover, in the case that $p=\infty$ we set $L^{\infty,\infty}_w(\mathbf{R}^n):=L^\infty_w(\mathbf{R}^n)$.

For a vector of m weights $\vec{w}=(w_1,\ldots,w_m)$, per convention we will use the dropped index notation $w:=\prod_{j=1}^m w_j$ for the product of the weights. Moreover, for exponents $\vec{p}\in(0,\infty]^m$ we will also use the shorthand notation

$$L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n):=L^{p_1}_{w_1}(\mathbf{R}^n)\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n).$$

By a cube $Q\subseteq\mathbf{R}^n$ we mean a half-open cube whose sides are parallel to the coordinate axes. For a measurable function $f\in L^0(\mathbf{R}^n)$, a measurable set E of positive finite measure, and $q\in(0,\infty)$ we will write $\langle f\rangle_{q,E}:=\left(\frac{1}{|E|}\int_E|f|^q\,\mathrm{d}x\right)^{\frac{1}{q}}$ and $\langle f\rangle_{\infty,E}:=\operatorname{ess\,sup}_{x\in E}|f(x)|$.

Definition 3.1.1. Let $\vec{r}\in(0,\infty)^m$, $s\in(0,\infty]$ and let $\vec{p}\in(0,\infty]^m$ with $\vec{p}\geq\vec{r}$ and $p\leq s$. Let $\vec{w}$ be a vector of m weights. We call $\vec{w}$ a multilinear Muckenhoupt weight and write

$\vec{w} \in A_{\vec{p},(\vec{r},s)}$ if

$$[\vec{w}]_{\vec{p},(\vec{r},s)} := \sup_Q \Big(\prod_{j=1}^m \langle w_j^{-1} \rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q} \Big) \langle w \rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbf{R}^n$.

If we have an additional weight v we can replace the product weight w by v in the above definition. In this case we say that $(\vec{w}, v) \in A_{\vec{p},(\vec{r},s)}$ and denote the corresponding constant by $[\vec{w}, v]_{\vec{p},(\vec{r},s)}$.

We point out that the definition of the weight class is sensible as long as $\frac{1}{r_j} - \frac{1}{p_j} \geq 0$ and $\frac{1}{p} - \frac{1}{s} \geq 0$. Thus, we can also make sense of it when, e.g., $\frac{1}{s}$ is negative. Note that $\vec{p} \in (0,\infty]^m$ with $\vec{p} \geq \vec{r}$ and $p \leq s$ exist only when $r \leq s$.

By comparability of cubes and balls, we can equivalently define the weight classes in terms of balls with comparable constants.

Remark 3.1.2. The condition $[\vec{w}]_{\vec{p},(\vec{r},s)} < \infty$ coincides with (1.1.8) in the introduction when replacing w_j in that expression by $w_j^{p_j}$. Moreover, in the case $m = 1$ we have $[w]_{p,(r,s)} = [w^p]_{A_{p,(r,s)}}^{\frac{1}{p}-\frac{1}{s}}$, where the latter constant is defined in the introduction in (1.1.2). In particular, we have $[w]_{p,(1,\infty)} = [w^p]_{A_p}^{\frac{1}{p}}$.

We list some useful properties of the weight classes and weight constants.

Proposition 3.1.3. *Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$ and let $\vec{p} \in (0,\infty]^m$ with $\vec{p} \geq \vec{r}$ and $p \leq s$. Let $\vec{w} \in A_{\vec{p},(\vec{r},s)}$. Then we have the following result.*

(i) $[\vec{w}]_{\vec{p},(\vec{r},s)} \geq 1$.

(ii) Let $t > 0$. Then $\vec{w}^t \in A_{\frac{\vec{p}}{t},(\frac{\vec{r}}{t},\frac{s}{t})}$ with

$$[\vec{w}^t]_{\frac{\vec{p}}{t},(\frac{\vec{r}}{t},\frac{s}{t})} = [\vec{w}]^t_{\vec{p},(\vec{r},s)}.$$

(iii) Let $\vec{q} \in (0,\infty]^m$ with $\vec{q} \geq \vec{r}$ and $q \leq s$, $\theta \in [0,1]$, and $\vec{v} \in A_{\vec{q},(\vec{r},s)}$. Then $\vec{v}^\theta \vec{w}^{1-\theta} \in A_{\frac{1}{\theta\frac{1}{\vec{q}}+(1-\theta)\frac{1}{\vec{p}}},(\vec{r},s)}$ with

$$[\vec{v}^\theta \vec{w}^{1-\theta}]_{\frac{1}{\theta\frac{1}{\vec{q}}+(1-\theta)\frac{1}{\vec{p}}},(\vec{r},s)} \leq [\vec{v}]^\theta_{\vec{q},(\vec{r},s)} [\vec{w}]^{1-\theta}_{\vec{p},(\vec{r},s)}.$$

Proof. For (i), note that for any cube $Q \subseteq \mathbf{R}^n$ it follows from Hölder's inequality that

$$1 = \langle 1 \rangle_{\frac{1}{\frac{1}{r}-\frac{1}{s}},Q} \leq \Big(\prod_{j=1}^m \langle w_j^{-1} \rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q} \Big) \langle w \rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} \leq [\vec{w}]_{\vec{p},(\vec{r},s)}.$$

Hence, $[\vec{w}]_{\vec{p},(\vec{r},s)} \geq 1$, as asserted.

We note that (ii) is a consequence of the fact that for all cubes Q we have

$$\langle w_j^{-t}\rangle_{\frac{1}{\frac{t}{r_j}-\frac{t}{p_j}},Q}=\langle w_j^{-1}\rangle^t_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q},\quad \langle w^t\rangle_{\frac{1}{\frac{t}{p}-\frac{t}{s}},Q}=\langle w\rangle^t_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q}.$$

The result then follows from the definition of the weight constant.

For (ii) we note that

$$\frac{1}{r_j}-\left(\theta\frac{1}{q_j}+(1-\theta)\frac{1}{p_j}\right)=\theta\left(\frac{1}{r_j}-\frac{1}{q_j}\right)+(1-\theta)\left(\frac{1}{r_j}-\frac{1}{p_j}\right)$$
$$\theta\frac{1}{q}+(1-\theta)\frac{1}{p}-\frac{1}{s}=\theta\left(\frac{1}{q}-\frac{1}{s}\right)+(1-\theta)\left(\frac{1}{p}-\frac{1}{s}\right)$$

so that by Hölder's inequality we have

$$\langle v_j^{-\theta}w_j^{-(1-\theta)}\rangle_{\frac{1}{\frac{1}{r_j}-\left(\theta\frac{1}{q_j}+(1-\theta)\frac{1}{p_j}\right)},Q}\le\langle v_j^{-1}\rangle^{\theta}_{\frac{1}{\frac{1}{r_j}-\frac{1}{q_j}},Q}\langle w_j^{-1}\rangle^{1-\theta}_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}$$
$$\langle v^{\theta}w^{1-\theta}\rangle_{\frac{1}{\theta\frac{1}{q}+(1-\theta)\frac{1}{p}-\frac{1}{s}},Q}\le\langle v\rangle^{\theta}_{\frac{1}{\frac{1}{q}-\frac{1}{s}},Q}\langle w\rangle^{1-\theta}_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q}.$$

The result then follows from the definition of the weight constants. □

In the following result we show which power weights belong to the class $A_{\vec{p},(\vec{r},s)}$.

Proposition 3.1.4. *Let $\vec{r}\in(0,\infty)^m$, $s\in(0,\infty]$ and let $\vec{p}\in(0,\infty]^m$ with $\vec{p}\ge\vec{r}$ and $p\le s$. Let $\vec{\alpha}\in\mathbf{R}^m$ and let $w_j(x):=|x|^{\alpha_j n}$.*

We have $\vec{w}\in A_{\vec{p},(\vec{r},s)}$ if and only if for all $j\in\{1,\dots,m\}$

$$\alpha_j<\frac{1}{r_j}-\frac{1}{p_j},\quad \sum_{j=1}^m\alpha_j>-\left(\frac{1}{p}-\frac{1}{s}\right)$$

or respectively $\alpha_j\le0$ or $\sum_{j=1}^m\alpha_j\ge0$ when $p_j=r_j$ or $p=s$. In this case we have

$$[\vec{w}]_{\vec{p},(\vec{r},s)}\eqsim_{r,s}\left(\frac{1}{1+\frac{1}{\frac{1}{p}-\frac{1}{s}}\sum_{j=1}^m\alpha_j}\right)^{\frac{1}{p}-\frac{1}{s}}\prod_{j=1}^m\left(\frac{1}{1-\frac{\alpha_j}{\frac{1}{r_j}-\frac{1}{p_j}}}\right)^{\frac{1}{r_j}-\frac{1}{p_j}}.$$

where if $p_j=r_j$ or $p=s$, the corresponding term on the right should be replaced by 1.

Proof. Note that when $\frac{1}{r_j}-\frac{1}{p_j}>0$, $\frac{1}{p}-\frac{1}{s}>0$, the weights $w_j(x)^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}=|x|^{\frac{-\alpha_j n}{\frac{1}{r_j}-\frac{1}{p_j}}}$ and $w(x)^{\frac{1}{\frac{1}{p}-\frac{1}{s}}}=|x|^{\frac{n}{\frac{1}{p}-\frac{1}{s}}\sum_{j=1}^m\alpha_j}$ are locally integrable if and only if $\alpha_j<\frac{1}{r_j}-\frac{1}{p_j}$ and $\sum_{j=1}^m\alpha_j>-\left(\frac{1}{p}-\frac{1}{s}\right)$. The local integrability of these weights is necessary for the condition $[\vec{w}]_{\vec{p},(\vec{r},s)}<\infty$. In case $p_j=r_j$ or $p=s$ we note that respectively w_j or w is locally bounded if and only if respectively $\alpha_j\le0$ or $\sum_{j=1}^m\alpha_j\ge0$. We will prove that these conditions are also sufficient. To this end we consider the weight constants in terms of balls rather than cubes.

Set $c_n := |S^{n-1}|_{n-1}$ so that $|B(z;R)| = \frac{c_n}{n} R^n$. For the lower bound, note that

$$
\begin{aligned}
[\vec{w}]_{\vec{p},(\vec{r},s)} &\geq \left(\frac{n}{c_n} \int_{B(0;1)} |y|^{\frac{1}{\frac{1}{p}-\frac{1}{s}} \sum_{j=1}^m \alpha_j n} \,\mathrm{d}y \right)^{\frac{1}{p}-\frac{1}{s}} \prod_{j=1}^m \left(\frac{n}{c_n} \int_{B(0;1)} |y|^{\frac{-\alpha_j n}{\frac{1}{r_j}-\frac{1}{p_j}}} \,\mathrm{d}y \right)^{\frac{1}{r_j}-\frac{1}{p_j}} \\
&= \left(\frac{1}{1+\frac{1}{\frac{1}{p}-\frac{1}{s}} \sum_{j=1}^m \alpha_j} \right)^{\frac{1}{p}-\frac{1}{s}} \prod_{j=1}^m \left(\frac{1}{1-\frac{\alpha_j}{\frac{1}{r_j}-\frac{1}{p_j}}} \right)^{\frac{1}{r_j}-\frac{1}{p_j}},
\end{aligned}
\tag{3.1.1}
$$

where if $p_j = r_j$ or $p = s$, the corresponding integral should be replaced by $\sup_{y\in B(0;1)} |y|^{-\alpha_j n} = 1$ or $\sup_{y\in B(0;1)} |y|^{\sum_{j=1}^m \alpha_j n} = 1$.

For the upper bound, let $B(x_0;r_0)$ be a ball. We consider the two cases $|x_0| \geq 3r_0$ and $|x_0| < 3r_0$.

First assume that $|x_0| \geq 3r_0$. Then for any $y \in B(x_0;r_0)$ we have $|y| \eqsim |x_0|$. Indeed,

$$
|y| \leq |y-x_0| + |x_0| < r_0 + |x_0| \leq \frac{4}{3}|x_0|, \quad |y| \geq |x_0| - |y-x_0| > |x_0| - r_0 \geq \frac{2}{3}|x_0|.
$$

Then we have

$$
\begin{aligned}
&\left(\frac{1}{|B(x_0;r_0)|} \int_{B(x_0;r_0)} |y|^{\frac{1}{\frac{1}{p}-\frac{1}{s}} \sum_{j=1}^m \alpha_j n} \,\mathrm{d}y \right)^{\frac{1}{p}-\frac{1}{s}} \prod_{j=1}^m \left(\frac{1}{|B(x_0;r_0)|} \int_{B(x_0;r_0)} |y|^{\frac{-\alpha_j n}{\frac{1}{r_j}-\frac{1}{p_j}}} \,\mathrm{d}y \right)^{\frac{1}{r_j}-\frac{1}{p_j}} \\
&\eqsim |x_0|^{\sum_{j=1}^m \alpha_j n} \prod_{j=1}^m |x_0|^{-\alpha_j n} = 1,
\end{aligned}
$$

where a similar computation holds when $p_j = r_j$ or $p = s$. Since it follows from Hölder's inequality that any of the terms in the supremum taken to compute $[\vec{w}]_{\vec{p},(\vec{r},s)}$ are at least 1, this holds in particular for the term computed in (3.1.1). Thus,

$$
1 \leq \left(\frac{1}{1+\frac{1}{\frac{1}{p}-\frac{1}{s}} \sum_{j=1}^m \alpha_j} \right)^{\frac{1}{p}-\frac{1}{s}} \prod_{j=1}^m \left(\frac{1}{1-\frac{\alpha_j}{\frac{1}{r_j}-\frac{1}{p_j}}} \right)^{\frac{1}{r_j}-\frac{1}{p_j}},
$$

proving the desired upper bound in this case.

Finally, assume that $|x_0| < 3r_0$. Note that now $B(x_0;r_0) \subseteq B(0;4r_0)$, since whenever

$|y-x_0|<r_0$, we have $|y|\le|y-x_0|+|x_0|<4r_0$. Then

$$\begin{aligned}
&\left(\frac{1}{|B(x_0;r_0)|}\int_{B(x_0;r_0)}|y|^{\frac{1}{\frac{1}{p}-\frac{1}{s}}\sum_{j=1}^m\alpha_j n}\,\mathrm{d}y\right)^{\frac{1}{p}-\frac{1}{s}}\prod_{j=1}^m\left(\frac{1}{|B(x_0;r_0)|}\int_{B(x_0;r_0)}|y|^{\frac{-\alpha_j n}{\frac{1}{r_j}-\frac{1}{p_j}}}\,\mathrm{d}y\right)^{\frac{1}{r_j}-\frac{1}{p_j}}\\
&\le\left(\frac{n}{c_n r_0^n}\int_{B(0;4r_0)}|y|^{\frac{1}{\frac{1}{p}-\frac{1}{s}}\sum_{j=1}^m\alpha_j n}\,\mathrm{d}y\right)^{\frac{1}{p}-\frac{1}{s}}\prod_{j=1}^m\left(\frac{n}{c_n r_0^n}\int_{B(0;4r_0)}|y|^{\frac{-\alpha_j n}{\frac{1}{r_j}-\frac{1}{p_j}}}\,\mathrm{d}y\right)^{\frac{1}{r_j}-\frac{1}{p_j}}\\
&=\left(\frac{1}{r_0^n}\frac{(4r_0)^{n+\frac{1}{\frac{1}{p}-\frac{1}{s}}\sum_{j=1}^m\alpha_j n}}{1+\frac{1}{\frac{1}{p}-\frac{1}{s}}\sum_{j=1}^m\alpha_j}\right)^{\frac{1}{p}-\frac{1}{s}}\prod_{j=1}^m\left(\frac{1}{r_0^n}\frac{(4r_0)^{n-\frac{\alpha_j n}{\frac{1}{r_j}-\frac{1}{p_j}}}}{1-\frac{\alpha_j}{\frac{1}{r_j}-\frac{1}{p_j}}}\right)^{\frac{1}{r_j}-\frac{1}{p_j}}\\
&=4^{n\left(\frac{1}{r}-\frac{1}{s}\right)}\left(\frac{1}{1+\frac{1}{\frac{1}{p}-\frac{1}{s}}\sum_{j=1}^m\alpha_j}\right)^{\frac{1}{p}-\frac{1}{s}}\prod_{j=1}^m\left(\frac{1}{1-\frac{\alpha_j}{\frac{1}{r_j}-\frac{1}{p_j}}}\right)^{\frac{1}{r_j}-\frac{1}{p_j}},
\end{aligned}$$

where if $p_j=r_j$ or $p=s$, the corresponding integral estimate should be replaced by $\sup_{y\in B(x_0;r_0)}|y|^{-\alpha_j n}\le(4r_0)^{-\alpha_j}$ or $\sup_{y\in B(x_0;r_0)}|y|^{\sum_{j=1}^m\alpha_j n}\le(4r_0)^{\sum_{j=1}^m\alpha_j}$ in the computation. This proves the result. □

Proposition 3.1.5. *Let $\vec{r}\in(0,\infty)^m$ and $\vec{p}\in(0,\infty]^m$ with $\vec{p}\ge\vec{r}$. Let $\mathscr{I}$ be a partition of $\{1,\dots,m\}$ and let $\frac{1}{s_I}\in\mathbf{R}$ with $\frac{1}{p_I}:=\sum_{j\in I}\frac{1}{p_j}\ge\frac{1}{s_I}$ and $\frac{1}{r_I}:=\sum_{j\in I}\frac{1}{r_j}>\frac{1}{s_I}$ for each $I\in\mathscr{I}$. Then, if $\frac{1}{s}:=\sum_{I\in\mathscr{I}}\frac{1}{s_I}$, we have the inclusion*

$$\prod_{I\in\mathscr{I}}A_{(p_j)_{j\in I},((r_j)_{j\in I},s_I)}\subseteq A_{\vec{p},(\vec{r},s)}\tag{3.1.2}$$

with

$$[\vec{w}]_{\vec{p},(\vec{r},s)}\le\prod_{I\in\mathscr{I}}^m[(w_j)_{j\in I}]_{(p_j)_{j\in I},((r_j)_{j\in I},s_I)}.\tag{3.1.3}$$

Moreover, if $\mathscr{I}$ is a proper partition, then this inclusion is strict.

Note that in particular this implies that for $\frac{1}{\vec{s}}\in\mathbf{R}^m$ with $\frac{1}{s_j}<\frac{1}{r_j}$ for all $j\in\{1,\dots,m\}$ we have $A_{p_1,(r_1,s_1)}\times\cdots\times A_{p_m,(r_m,s_m)}\subseteq A_{\vec{p},(\vec{r},s)}$ with a strict inclusion.

Proof. To prove the inclusion, note that it follows from Hölder's inequality that for any cube $Q\subseteq\mathbf{R}^n$ we have

$$\langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q}\le\prod_{I\in\mathscr{I}}\langle\prod_{j\in I}w_j\rangle_{\frac{1}{\frac{1}{p_I}-\frac{1}{s_I}},Q},$$

where $\frac{1}{p_I}:=\sum_{j\in I}\frac{1}{p_j}$. Thus, by the definition of the weight constants, this proves (3.1.3). We conclude from this that (3.1.2) holds.

To see that the inclusion is strict when $\mathscr{I}$ is a proper partition, we fix $I \in \mathscr{I}$ and define $\delta := \frac{1}{2}\left(\frac{1}{r_I} - \frac{1}{s_I}\right) + \frac{1}{2}\left(\frac{1}{r} - \frac{1}{s}\right) > \frac{1}{r_I} - \frac{1}{s_I} > 0$ and $\varepsilon := \frac{1}{2}\left(\frac{1}{r} - \frac{1}{s} - \delta\right) > 0$. Setting

$$\alpha_j := \begin{cases} \frac{1}{r_j} - \frac{1}{p_j} - \frac{\delta}{\#I} & \text{if } j \in I; \\ \frac{1}{r_j} - \frac{1}{p_j} - \frac{\varepsilon}{m-\#I} & \text{if } j \notin I, \end{cases}$$

and $w_j(x) := |x|^{\alpha_j n}$, we note that $\alpha_j < \frac{1}{r_j} - \frac{1}{p_j}$,

$$\sum_{j\in I} \alpha_j = \frac{1}{r_I} - \frac{1}{p_I} - \delta < \frac{1}{s_I} - \frac{1}{p_I}, \quad \sum_{j=1}^{m} \alpha_j = \frac{1}{r} - \frac{1}{p} - (\delta + \varepsilon) > \frac{1}{s} - \frac{1}{p}.$$

Hence, by Proposition 3.1.4 we have $(w_j)_{j\in I} \notin A_{(p_j)_{j\in I},((r_j)_{j\in I},s_I)}$ while $\vec{w} \in A_{\vec{p},(\vec{r},s)}$. This proves the assertion. □

The class $A_{\vec{p},(\vec{r},s)}$ has an alternative description in terms of individual conditions on the m weights $\vec{w}$ combined with a condition on the product weight w.

Proposition 3.1.6. *Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$ and $\vec{p} \in (0,\infty]^m$ with $\vec{p} \geq \vec{r}$ and $p \leq s$. Let $\vec{w}$ be an m-tuple of weights. The following are equivalent:*

(i) $\vec{w} \in A_{\vec{p},(\vec{r},s)}$;

(ii) $w_j \in A_{p_j,(r_j,\sigma_j)}$ *with* $\frac{1}{\sigma_j} = \frac{1}{r_j} - \left(\frac{1}{r} - \frac{1}{s}\right)$ *for all* $j \in \{1,\dots,m\}$ *and* $w \in A_{p,(r,s)}$.

Moreover, we have

$$\begin{aligned} \max\{[w_1]_{p_1,(r_1,\sigma_1)},\dots,[w_m]_{p_m,(r_m,\sigma_m)},[w]_{p,(r,s)}\} &\leq [\vec{w}]_{\vec{p},(\vec{r},s)}, \\ [\vec{w}]_{\vec{p},(\vec{r},s)} &\leq \left(\prod_{j=1}^{m} [w_j]_{p_j,(r_j,\sigma_j)}\right)[w]_{p,(r,s)}. \end{aligned} \tag{3.1.4}$$

Proof. Fix $j_0 \in \{1,\dots,m\}$. We first note that $\frac{1}{p_{j_0}} \geq \frac{1}{\sigma_{j_0}}$ so that the weight classes are well-defined. Indeed, this inequality is equivalent to $\frac{1}{r_{j_0}} - \frac{1}{p_{j_0}} \leq \frac{1}{r} - \frac{1}{s}$. For the latter, note that

$$\frac{1}{r_{j_0}} - \frac{1}{p_{j_0}} \leq \frac{1}{r} - \frac{1}{p} \leq \frac{1}{r} - \frac{1}{s},$$

since $\frac{1}{s} \leq \frac{1}{p}$, as desired.

For (i)⇒(ii), note that $\frac{1}{p_{j_0}} - \frac{1}{\sigma_{j_0}} = \frac{1}{p} - \frac{1}{s} + \sum_{\substack{j=1\\ j\neq j_0}}^{m} \frac{1}{r_j} - \frac{1}{p_j}$. Thus, by Hölder's inequality, we have

$$\begin{aligned} \langle w_{j_0}^{-1}\rangle_{\frac{1}{\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}},Q}\langle w_{j_0}\rangle_{\frac{1}{\frac{1}{p_{j_0}}-\frac{1}{\sigma_{j_0}}},Q} &= \langle w_{j_0}^{-1}\rangle_{\frac{1}{\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}},Q}\langle w \prod_{\substack{j=1\\ j\neq j_0}}^{m} w_j^{-1}\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}+\sum_{\substack{j=1\\ j\neq j_0}}^{m}\frac{1}{r_j}-\frac{1}{p_j}},Q} \\ &\leq \left(\prod_{j=1}^{m} \langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}\right)\langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} \end{aligned}$$

for all cubes Q. Taking a supremum over all cubes Q proves that $[w_{j_0}]_{p_{j_0},(r_{j_0},\sigma_{j_0})} \leq [\vec{w}]_{\vec{p},(\vec{r},s)}$. For the assertion about w, note that $\frac{1}{r}-\frac{1}{p}=\sum_{j=1}^m \frac{1}{r_j}-\frac{1}{p_j}$ so that

$$\langle w^{-1}\rangle_{\frac{1}{\frac{1}{r}-\frac{1}{p}},Q}\langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} \leq \Big(\prod_{j=1}^m \langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}\Big)\langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q}$$

for all cubes Q. By taking a supremum over all cubes Q we conclude that $[w]_{p,(r,s)} \leq [\vec{w}]_{\vec{p},(\vec{r},s)}$. Thus, we have proven (ii) and the first inequality in (3.1.4).

For (ii)⇒(i), note that if follows from Hölder's inequality that

$$\Big(\prod_{j=1}^m \langle w_j\rangle_{\frac{1}{\frac{1}{p_j}-\frac{1}{\sigma_j}},Q}\Big)\langle w^{-1}\rangle_{\frac{1}{\frac{1}{r}-\frac{1}{p}},Q} \geq \langle 1\rangle_{\frac{1}{m(\frac{1}{r}-\frac{1}{s})},Q}=1$$

and hence

$$\begin{aligned}\Big(\prod_{j=1}^m \langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}\Big)\langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} &\leq \Big(\prod_{j=1}^m \langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}\langle w_j\rangle_{\frac{1}{\frac{1}{p_j}-\frac{1}{\sigma_j}},Q}\Big)\langle w^{-1}\rangle_{\frac{1}{\frac{1}{r}-\frac{1}{p}},Q}\langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q}\\ &\leq \Big(\prod_{j=1}^m [w_j]_{p_j,(r_j,\sigma_j)}\Big)[w]_{p,(r,s)}\end{aligned}$$

for all cubes Q. Taking a supremum over Q proves (i) and the second inequality in (3.1.4). The result follows. □

In the case $m=1$, the class $A_{p,(r,s)}$ can also be described through a reverse Hölder condition.

Definition 3.1.7. Let $\beta \in (1,\infty]$ and let w be a weight. We write $w \in \mathrm{RH}_\beta$ when

$$[w]_{\mathrm{RH}_\beta} := \sup_Q \langle w\rangle_{\beta,Q}\langle w\rangle_{1,Q}^{-1} < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbf{R}^n$.

Proposition 3.1.8. *Let $r \in (0,\infty)$, $s \in (0,\infty]$, $p \in [r,s)$, and let w be a weight. The following are equivalent:*

(i) $w \in A_{p,(r,s)}$*;*

(ii) $w \in A_{p,(r,\infty)}$ *and* $w^p \in \mathrm{RH}_{\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}}$.

Moreover, in this case we have

$$\begin{aligned}\max\{[w^p]_{\mathrm{RH}_{\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}}}^{\frac{1}{p}}, [w]_{p,(r,\infty)}\} &\leq [w]_{p,(r,s)}\\ [w]_{p,(r,s)} &\leq [w^p]_{\mathrm{RH}_{\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}}}^{\frac{1}{p}}[w]_{p,(r,\infty)}\end{aligned} \tag{3.1.5}$$

Proof. For (i)⇒(ii), note that by Hölder's inequality we have $\langle w\rangle_{p,Q} \le \langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q}$ for all cubes Q so that $[w]_{p,(r,\infty)} \le [w]_{p,(r,s)}$. For the other assertion, note that by Hölder's inequality we have $1 = \langle ww^{-1}\rangle_{r,Q} \le \langle w\rangle_{p,Q}\langle w^{-1}\rangle_{\frac{1}{\frac{1}{r}-\frac{1}{p}},Q}$ so that

$$\langle w^p\rangle^{\frac{1}{p}}_{\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}},Q} = \langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q}\langle w^{-1}\rangle_{\frac{1}{\frac{1}{r}-\frac{1}{p}},Q}\langle w^{-1}\rangle^{-1}_{\frac{1}{\frac{1}{r}-\frac{1}{p}},Q} \le [w]_{p,(r,s)}\langle w^p\rangle^{\frac{1}{p}}_{1,Q}$$

for all cubes Q. Taking a supremum over all cubes Q proves the result and the first inequality in (3.1.5).

For (ii)⇒(i), note that since

$$\langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} = \langle w^p\rangle^{\frac{1}{p}}_{\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}},Q} \le [w^p]^{\frac{1}{p}}_{\mathrm{RH}_{\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}}}\langle w^p\rangle^{\frac{1}{p}}_{1,Q} = [w^p]^{\frac{1}{p}}_{\mathrm{RH}_{\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}}}\langle w\rangle_{p,Q},$$

we have

$$\langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q}\langle w^{-1}\rangle_{\frac{1}{\frac{1}{r}-\frac{1}{p}},Q} \le [w^p]^{\frac{1}{p}}_{\mathrm{RH}_{\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}}}\langle w\rangle_{p,Q}\langle w^{-1}\rangle_{\frac{1}{\frac{1}{r}-\frac{1}{p}},Q} \le [w^p]^{\frac{1}{p}}_{\mathrm{RH}_{\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}}}[w]_{p,(r,\infty)}$$

for all cubes Q. Taking a supremum over Q proves the result and the second inequality in (3.1.5), as desired. □

3.2. OPERATORS GOVERNING THE MULTILINEAR WEIGHT CLASSES

3.2.1. The multisublinear maximal operator

It is sometimes convenient to emphasize the separation of the parameter s from the r_j, as it often plays a different role from the other parameters in the proofs. The following lemma provides a way to deal with this parameter.

Lemma 3.2.1 (Translation lemma). *Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$ and $\vec{p} \in (0,\infty]^m$ with $\vec{p} \ge \vec{r}$ and $p \le s$ and let $\vec{w}$ be a vector of m weights. Then $\vec{w} \in A_{\vec{p},(\vec{r},s)}$ if and only if there are $\frac{1}{s_1},\dots \frac{1}{s_m} \in \mathbf{R}$ satisfying $\frac{1}{s_j} \le \frac{1}{p_j}$, $\sum_{j=1}^m \frac{1}{s_j} = \frac{1}{s}$, and $\vec{w} \in A_{\vec{p}(s),(\vec{r}(s),\infty)}$, where*

$$\vec{p}(s) = \left(\frac{1}{\frac{1}{p_1}-\frac{1}{s_1}},\dots,\frac{1}{\frac{1}{p_m}-\frac{1}{s_m}}\right), \qquad \vec{r}(s) = \left(\frac{1}{\frac{1}{r_1}-\frac{1}{s_1}},\dots,\frac{1}{\frac{1}{r_m}-\frac{1}{s_m}}\right).$$

Moreover, in this case we have

$$[\vec{w}]_{\vec{p},(\vec{r},s)} = [\vec{w}]_{\vec{p}(s),(\vec{r}(s),\infty)}. \tag{3.2.1}$$

Proof. We have

$$\frac{1}{p(s)} := \sum_{j=1}^m \left(\frac{1}{p_j}-\frac{1}{s_j}\right) = \frac{1}{p}-\frac{1}{s}.$$

it remains to note that

$$\left(\prod_{j=1}^m \langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}\right)\langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} = \left(\prod_{j=1}^m \langle w_j^{-1}\rangle_{\frac{1}{\left(\frac{1}{r_j}-\frac{1}{s_j}\right)-\left(\frac{1}{p_j}-\frac{1}{s_j}\right)},Q}\right)\langle w\rangle_{p(s),Q}.$$

Taking a supremum over all cubes Q yields (3.2.1), proving the assertion. □

We point out that the choice of the $\frac{1}{s_j}$ in the lemma is not necessarily unique if $m \neq 1$. One could, for example, take $\frac{1}{s_j} = \frac{p}{p_j}\frac{1}{s}$, but different choices are also possible. We note that this lemma can be used even if $\frac{1}{s} = 0$. In particular, in this case it can occur that some of the $\frac{1}{s_j}$ are negative.

When $s = \infty$, the weight classes are characterized by the boundedness of certain multisublinear maximal operators.

Definition 3.2.2. For $\vec{r} \in (0,\infty)^m$ and $\vec{f} \in L^{\vec{r}}_{\text{loc}}(\mathbf{R}^n)$ we define the m-sublinear Hardy-Littlewood maximal operator

$$M_{\vec{r}}(\vec{f})(x) := \sup_Q \prod_{j=1}^m \langle f_j\rangle_{r_j,Q}\chi_Q(x), \qquad x \in \mathbf{R}^n$$

where the supremum is taken over all cubes $Q \subseteq \mathbf{R}^n$. Similarly, for a collection of cubes $\mathscr{P}$ we define

$$M^{\mathscr{P}}_{\vec{r}}(\vec{f})(x) := \sup_{Q\in\mathscr{P}} \prod_{j=1}^m \langle f_j\rangle_{r_j,Q}\chi_Q(x), \qquad x \in \mathbf{R}^n.$$

Note that if $\mathscr{P}$ is countable, then $M^{\mathscr{P}}_{\vec{r}}(\vec{f})$ is a measurable function as it is a countable supremum of measurable functions. If we let $\mathscr{Q}$ denote the collection of cubes with rational center points and rational side length, then it follows from the regularity of the Lebesgue measure that $M_{\vec{r}}(\vec{f}) = M^{\mathscr{Q}}_{\vec{r}}(\vec{f})$. Hence, since $\mathscr{Q}$ is countable, we conclude that $M_{\vec{r}}(\vec{f})$ is also a measurable function.

The following proposition is the main result of this section.

Theorem 3.2.3. *Let $\vec{r} \in (0,\infty)^m$, $\vec{p} \in (0,\infty]^m$ with $\vec{p} \geq \vec{r}$ and let $\vec{w}$, v be $m+1$ weights. The following are equivalent:*

(i) $(\vec{w}, v) \in A_{\vec{p},(\vec{r},\infty)}$;

(ii) $M_{\vec{r}}$ *is bounded* $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^{p,\infty}_v(\mathbf{R}^n)$.

In this case we have

$$\|M_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^{p,\infty}_v(\mathbf{R}^n)} \eqsim_r [\vec{w}, v]_{\vec{p},(\vec{r},\infty)}. \tag{3.2.2}$$

Moreover, if $\vec{r} < \vec{p}$ and $v = w$, then (i) and (ii) are equivalent to

(iii) $M_{\vec{r}}$ *is bounded* $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)$

and we have

$$\|M_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} \lesssim_r c_{\vec{p},\vec{r}}[\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}}, \tag{3.2.3}$$

where

$$c_{\vec{p},\vec{r}} = \prod_{j=1}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}.$$

Moreover, the estimate (3.2.3) *is optimal in the sense that the power of the weight constant is the smallest possible one and in the unweighted case we have* $\|M_{\vec{r}}\|_{L^{\vec{p}}(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} \eqsim_r c_{\vec{p},\vec{r}}$.

To facilitate the proof of this result it is convenient to reduce to the case of dyadic grids. For $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$ we will consider the translated dyadic grids

$$\mathscr{D}^\alpha := \bigcup_{k\in\mathbf{Z}} \{2^{-k}\big([0,1)^n + m + (-1)^k\alpha\big) : m \in \mathbf{Z}^n\}.$$

An important property of these grids is the fact that for each cube $Q \subseteq \mathbf{R}^n$ there exists an $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$ and a cube $Q' \in \mathscr{D}^\alpha$ such that $Q \subseteq Q'$ and $|Q'| \le 6^n|Q|$. This so-called three lattice lemma will allow us to reduce our arguments to only having to consider dyadic grids. This property as well as further properties of dyadic grids can be found in [LN18]. An immediate consequence is the following:

Lemma 3.2.4. *Let* $\vec{r} \in (0,\infty)^m$. *Then for all* $\vec{f} \in L^{\vec{r}}_{\mathrm{loc}}(\mathbf{R}^n)$ *we have the pointwise equivalences*

$$M_{\vec{r}}(\vec{f}) \eqsim_{\vec{r}} \max_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} M^{\mathscr{D}^\alpha}_{\vec{r}}(\vec{f}) \eqsim \sum_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} M^{\mathscr{D}^\alpha}_{\vec{r}}(\vec{f}).$$

Proof. Note that the second equivalence follows from the equivalence of the ℓ^∞ and ℓ^1 norms in finite dimensions. It remains to prove the first equivalence.

The equality "≥" is clear, as the supremum on the right is taken over a smaller set of cubes. For the converse inequality, let $Q \subseteq \mathbf{R}^n$ be a cube. By the three lattice lemma there exists an $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$ and a cube $Q' \in \mathscr{D}^\alpha$ containing Q that satisfies $|Q'| \le 6^n|Q|$. Then

$$\prod_{j=1}^m \langle f_j\rangle_{r_j,Q}\chi_Q \le 6^{\frac{n}{r}} \prod_{j=1}^m \langle f_j\rangle_{r_j,Q'}\chi_{Q'} \le 6^{\frac{n}{r}} \max_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} M^{\mathscr{D}^\alpha}_{\vec{r}}(\vec{f}).$$

The result follows by taking a supremum over all cubes $Q \subseteq \mathbf{R}^n$. □

The fact that dyadic cubes can cover certain sets without overlapping each other allows us to essentially replace the Vitali covering lemma in the proof of the maximal theorem for dyadic grids. As a matter of fact, this allows us to prove bounds for weighted maximal operators independent of their reference weight. More precisely, for $r \in (0,\infty)$, $f \in L^r_{\mathrm{loc}}(\mathbf{R}^n)$, a weight w, and a cube Q, we define $\langle f\rangle^w_{r,Q} := \left(\frac{1}{w(Q)}\int_Q |f|^r w\,\mathrm{d}x\right)^{\frac{1}{r}}$. For a fixed dyadic grid $\mathscr{D} = \mathscr{D}^\alpha$ we may then define the weighted dyadic maximal operator $M_r^{\mathscr{D},w}(f)(x) := \sup_{Q\in\mathscr{D}} \langle f\rangle_{r,Q}^w \chi_Q(x)$. Letting $L^p(\mathbf{R}^n;w)$ denote the Lebesgue space over $\mathbf{R}^n$ with measure $w\,\mathrm{d}x$, we then have the following result:

Lemma 3.2.5. *Let $r \in (0,\infty)$, let w be a weight, and let $\mathscr{D} = \mathscr{D}^\alpha$ be a fixed dyadic grid. Then for all $p \in (r,\infty]$ the operator $M_r^{\mathscr{D},w}$ is bounded $L^p(\mathbf{R}^n;w) \to L^p(\mathbf{R}^n;w)$ with*

$$\|M_r^{\mathscr{D},w}\|_{L^p(\mathbf{R}^n;w)\to L^p(\mathbf{R}^n;w)} \le \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}.$$

Moreover, $M_r^{\mathscr{D},w}$ is bounded $L^r(\mathbf{R}^n;w) \to L^{r,\infty}(\mathbf{R}^n;w)$ with $\|M_r^{\mathscr{D},w}\|_{L^r(\mathbf{R}^n;w)\to L^{r,\infty}(\mathbf{R}^n;w)} \le 1$.

Proof. Note that for $p=\infty$ the bound is clear. If we can prove the weak-type, then the general result follows from the Marcinkiewicz Interpolation Theorem.

Let $\mathscr{F} \subseteq \mathscr{D}$ be a finite collection of cubes and set $M_r^{\mathscr{F},w}(h)(x) := \sup_{Q\in\mathscr{F}} \langle f\rangle_{r,Q}^w \chi_Q(x)$. Fix $f \in L^r(\mathbf{R}^n;w)$ and $\lambda > 0$. Per definition, for each $x \in \mathbf{R}^n$ such that $M_r^{\mathscr{F},w}(f)(x) > \lambda$, there is a cube $Q \in \mathscr{F}$ containing x such that $\langle f\rangle_{r,Q}^w > \lambda$. We pick the largest cube in $\mathscr{F}$ with this property and add it to the collection $\mathscr{P}$. We claim that the hereby obtained collection $\mathscr{P}$ is pairwise disjoint and satisfies

$$\{x \in \mathbf{R}^n : M_r^{\mathscr{F},w}(f)(x) > \lambda\} = \bigcup_{P\in\mathscr{P}} P. \tag{3.2.4}$$

For the first part of the claim, note that if $P_1, P_2 \in \mathscr{P}$, then there are $x_1, x_2 \in \mathbf{R}^n$ such that they were chosen as the maximal cube in $\mathscr{F}$ respectively containing $x_1, x_2 \in \mathbf{R}^n$. If $P_1 \cap P_2$, then by the properties of the dyadic system, we have $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$. Without loss of generality we assume the first. In that case P_2 contains x_1, and by maximality of P_1, this implies that $P_1 = P_2$. We conclude that $\mathscr{P}$ is indeed pairwise disjoint. For (3.2.4), the inclusion "$\subseteq$" holds per construction. For the other inclusion, suppose $P \in \mathscr{P}$ and $x \in P$. Then $M_r^{\mathscr{F},v}(h)(x) \ge \langle f\rangle_{r,P}^w > \lambda$ so that $x \in \{x \in \mathbf{R}^n : M_r^{\mathscr{F},w}(f)(x) > \lambda\}$, proving (3.2.4). This proves the claim.

Now, we have

$$\begin{aligned}\lambda^r w(\{x \in \mathbf{R}^n : M_r^{\mathscr{F},w}(f)(x) > \lambda\}) &= \sum_{P\in\mathscr{P}} \lambda^r w(P) \le \sum_{P\in\mathscr{P}} \left(\langle f\rangle_{r,P}^w\right)^r w(P)\\ &= \sum_{P\in\mathscr{P}} \int_P |f|^r w\,\mathrm{d}x = \int_{\{x\in\mathbf{R}^n : M_r^{\mathscr{F},w}(f)(x)>\lambda\}} |f|^r w\,\mathrm{d}x\\ &\le \|f\|_{L^r(\mathbf{R}^n;w)}^r.\end{aligned}$$

Thus, taking a supremum over $\lambda > 0$ yields $\|M_r^{\mathscr{F},w}(f)\|_{L^{r,\infty}(\mathbf{R}^n;w)} \le \|f\|_{L^r(\mathbf{R}^n;w)}$. Finally, for each $j \in \mathbf{N}$ we let $\mathscr{D}_j \subseteq \mathscr{D}$ denote a finite collection of cubes with the properties that $\mathscr{D}_j \subseteq \mathscr{D}_{j+1}$ and $\cup_{j\in\mathbf{N}} \mathscr{D}_j = \mathscr{D}$. Then $M_r^{\mathscr{D}_j,w}(f) \uparrow M_r^{\mathscr{D},w}(f)$ so that by monotonicity of the measure we have

$$\|M_r^{\mathscr{D},w}(f)\|_{L^{r,\infty}(\mathbf{R}^n;w)} = \lim_{j\to\infty} \|M_r^{\mathscr{D}_j,w}(f)\|_{L^{r,\infty}(\mathbf{R}^n;w)} \le \|f\|_{L^r(\mathbf{R}^n;w)}.$$

The assertion follows. □

Lemma 3.2.6. *Let $\vec{r}\in(0,\infty)^m$, $\vec{p}\in(0,\infty]^m$ with $\vec{r}<\vec{p}$ and let $\vec{w}\in A_{\vec{p},(\vec{r},\infty)}$. Then there exist sublinear operators $N_{p_j,r_j,\vec{w}}:L^{p_j}_{w_j}(\mathbf{R}^n)\to L^{p_j}_{w_j}(\mathbf{R}^n)$ so that for all $\vec{f}\in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$ we have*

$$M_{\vec{r}}(\vec{f})\le[\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}}\prod_{j=1}^m N_{p_j,r_j,\vec{w}}(f_j). \tag{3.2.5}$$

Moreover, $N_{p_j,r_j,\vec{w}}$ satisfies

$$\|N_{p_j,r_j,\vec{w}}\|_{L^{p_j}_{w_j}(\mathbf{R}^n)\to L^{p_j}_{w_j}(\mathbf{R}^n)}\lesssim_{r_j}\left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}.$$

Proof. We first prove this result for $M^{\mathscr{D}}_{\vec{r}}$, where $\mathscr{D}=\mathscr{D}^\alpha$ is a fixed dyadic grid, to obtain the appropriate operators $N^{\mathscr{D}}_{p_j,r_j,\vec{w}}$. Then it follows from Lemma 3.2.4 that

$$M_{\vec{r}}(\vec{f})\le 6^{\frac{n}{\vec{r}}}\max_{\alpha\in\left\{0,\frac13,\frac23\right\}^n}\prod_{j=1}^m N^{\mathscr{D}_\alpha}_{p_j,r_j,\vec{w}}(f_j)\le 6^{\frac{n}{\vec{r}}}\prod_{j=1}^m\max_{\alpha\in\left\{0,\frac13,\frac23\right\}^n}N^{\mathscr{D}_\alpha}_{p_j,r_j,\vec{w}}(f_j).$$

The result then follows by setting

$$N_{p_j,r_j,\vec{w}}:=6^{\frac{n}{r_j}}\max_{\alpha\in\left\{0,\frac13,\frac23\right\}^n}N^{\mathscr{D}_\alpha}_{p_j,r_j,\vec{w}}.$$

Now, let $\gamma:=\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}$, let $Q\in\mathscr{D}$, and set $v_j:=w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$. Since $\prod_{j=1}^m w_j^{-1}w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}}=\left(\prod_{j=1}^m w_j^{-1}\right)w=1$, it follows from Hölder's inequality that

$$\begin{aligned}
1=\langle 1\rangle^{\gamma-1}_{\frac{1}{\sum_{j=1}^m\frac{1}{r_j}},Q}&\le\prod_{j=1}^m\langle w_j^{-1}w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}}\rangle^{\gamma-1}_{r_j,Q}=\prod_{j=1}^m\langle w_j^{-1}w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}}\rangle_{r_j,Q}^{\gamma-\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}\langle w_j^{-1}w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}}\rangle_{r_j,Q}^{\frac{\frac{1}{p_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}\\
&\le\prod_{j=1}^m\left(\langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}\langle w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}}\rangle_{p_j,Q}\right)^{\gamma-\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}\langle w_j^{-1}w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}}\rangle_{r_j,Q}^{\frac{\frac{1}{p_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}\\
&=\left(\prod_{j=1}^m\left(\langle v_j\rangle_{1,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}\langle w^p\rangle_{1,Q}^{\frac{1}{p_j}}\right)^{\gamma-\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}\right)\prod_{j=1}^m\langle w_j^{-r_j}w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}r_j}\rangle_{1,Q}^{\frac{\frac{1}{p_j}\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}.
\end{aligned}$$

This implies that

$$\begin{aligned}
\prod_{j=1}^m \langle v_j\rangle_{1,Q}^{\frac{1}{r_j}} &\le \frac{[\vec{w}]^{\gamma}_{\vec{p},(\vec{r},\infty)}}{\Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q}^{\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\gamma-\frac{1}{r_j}}\Big)\langle w\rangle_{p,Q}^{\gamma}}\\
&= \frac{[\vec{w}]^{\gamma}_{\vec{p},(\vec{r},\infty)}}{\prod_{j=1}^m \left(\langle v_j\rangle_{1,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}\langle w^p\rangle_{1,Q}^{\frac{1}{p_j}}\right)^{\gamma-\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}}\prod_{j=1}^m\left(\frac{1}{\langle w^p\rangle_{1,Q}}\right)^{\frac{\frac{1}{p_j}\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}\\
&\le [\vec{w}]^{\gamma}_{\vec{p},(\vec{r},\infty)}\prod_{j=1}^m\left(\frac{\langle w_j^{-r_j} w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}r_j}\rangle_{1,Q}}{\langle w^p\rangle_{1,Q}}\right)^{\frac{\frac{1}{p_j}\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}.
\end{aligned}$$

Thus, for $f_j\in L^{p_j}_{w_j}(\mathbf{R}^n)$ and $x\in Q$, we have

$$\begin{aligned}
\prod_{j=1}^m\langle f_j\rangle_{r_j,Q} &= \prod_{j=1}^m \langle f_j v_j^{-\frac{1}{r_j}}\rangle^{v_j}_{r_j,Q}\langle v_j\rangle_{1,Q}^{\frac{1}{r_j}}\\
&\le [\vec{w}]^{\gamma}_{\vec{p},(\vec{r},\infty)}\prod_{j=1}^m\left(\frac{\inf_{y\in Q} M^{v_j,\mathscr{D}}_{r_j}(f_j v_j^{-\frac{1}{r_j}})(y)^{\frac{\frac{1}{r_j}-\frac{1}{p_j}}{\frac{1}{p_j}\frac{1}{r_j}}}\langle w_j^{-r_j} w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}r_j}\rangle_{1,Q}}{\langle w^p\rangle_{1,Q}}\right)^{\frac{\frac{1}{p_j}\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}\\
&\le [\vec{w}]^{\gamma}_{\vec{p},(\vec{r},\infty)}\prod_{j=1}^m M^{w^p,\mathscr{D}}_{\frac{\frac{1}{r_j}-\frac{1}{p_j}}{\frac{1}{p_j}\frac{1}{r_j}}}(M^{v_j,\mathscr{D}}_{r_j}(f_j v_j^{-\frac{1}{r_j}})v_j^{\frac{1}{p_j}} w^{-\frac{\frac{1}{p_j}}{\frac{1}{p}}})(x).
\end{aligned} \tag{3.2.6}$$

Setting

$$N^{\mathscr{D}}_{p_j,r_j,\vec{w}}(f_j) := M^{w^p,\mathscr{D}}_{\frac{\frac{1}{r_j}-\frac{1}{p_j}}{\frac{1}{p_j}\frac{1}{r_j}}}(M^{v_j,\mathscr{D}}_{r_j}(f_j v_j^{-\frac{1}{r_j}})v_j^{\frac{1}{p_j}} w^{-\frac{\frac{1}{p_j}}{\frac{1}{p}}})w^{\frac{\frac{1}{p_j}}{\frac{1}{p}}} w_j^{-1}$$

and by taking a supremum over all Q containing x in (3.2.6) we have proven (3.2.5) in the dyadic case. We remark here that in the case that $\frac{1}{p_j}=0$, we use the interpretation

$$N^{\mathscr{D}}_{\infty,r_j,\vec{w}}(f_j) = \|M^{v_j,\mathscr{D}}_{r_j}(f_j v_j^{-\frac{1}{r_j}})\|_{L^\infty} w_j^{-1}.$$

Noting that by Lemma 3.2.5 we have

$$\|M^{u,\mathscr{D}}_{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{q}\frac{1}{r}}}(h)\|_{L^q(\mathbf{R}^n;u)} \le \left(\frac{q}{r}\right)^{\frac{\frac{1}{q}\frac{1}{r}}{\frac{1}{r}-\frac{1}{q}}}\|h\|_{L^q(\mathbf{R}^n;u)} = e^{\frac{\log q-\log r}{q-r}}\|h\|_{L^q(\mathbf{R}^n;u)} \le e^{\frac{1}{r}}\|h\|_{L^q(\mathbf{R}^n;u)},$$

for the case $\frac{1}{p_j} > 0$, we compute

$$\begin{aligned}
\|N^{\mathscr{D}}_{p_j,r_j,\vec{w}}(f_j)\|_{L^{p_j}_{w_j}(\mathbf{R}^n)} &= \|M^{w^p,\mathscr{D}}_{\frac{\frac{1}{r_j}-\frac{1}{p_j}}{\frac{1}{p_j}\frac{1}{r_j}}}(M^{v_j,\mathscr{D}}_{r_j}(f_j v_j^{-\frac{1}{r_j}})v_j^{\frac{1}{p_j}}w^{-\frac{p}{p_j}})\|_{L^{p_j}(\mathbf{R}^n;w^p)}\\
&\lesssim_{r_j} \|M^{v_j,\mathscr{D}}_{r_j}(f_j v_j^{-\frac{1}{r_j}})v_j^{\frac{1}{p_j}}w^{-\frac{p}{p_j}}\|_{L^{p_j}(\mathbf{R}^n;w^p)}\\
&= \|M^{v_j,\mathscr{D}}_{r_j}(f_j v_j^{-\frac{1}{r_j}})\|_{L^{p_j}(\mathbf{R}^n;v_j)}\\
&\leq \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}} \|f_j v_j^{-\frac{1}{r_j}}\|_{L^{p_j}(\mathbf{R}^n;v_j)}\\
&= \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}} \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)},
\end{aligned}$$

and for the case $\frac{1}{p_j} = 0$, we compute

$$\|N^{\mathscr{D}}_{\infty,r_j,\vec{w}}(f_j)\|_{L^{\infty}_{w_j}(\mathbf{R}^n)} = \|M^{v_j,\mathscr{D}}_{r_j}(f_j v_j^{-\frac{1}{r_j}})\|_{L^\infty(\mathbf{R}^n)} \leq \|f_j v_j^{-\frac{1}{r_j}}\|_{L^\infty(\mathbf{R}^n)} = \|f_j\|_{L^\infty_{w_j}(\mathbf{R}^n)}.$$

The assertion follows. □

Proof of Theorem 3.2.3. We will prove the equivalence of (i) and (ii) by proving (3.2.2).

For "$\lesssim$", we note that it follows from Lemma 3.2.4 that it suffices to prove the estimate for $M^{\mathscr{D}}_{\vec{r}}$ for a fixed dyadic grid $\mathscr{D} = \mathscr{D}^\alpha$. Note that by Hölder's inequality we have $\langle f_j\rangle_{r_j,Q} \leq \langle f_j w_j\rangle_{p_j,Q}\langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}$ for a cube Q, so that

$$\prod_{j=1}^m \langle f_j\rangle_{r_j,Q} \leq [\vec{w},v]_{\vec{p},(\vec{r},\infty)}\langle v\rangle^{-1}_{p,Q}\prod_{j=1}^m\langle f_j w_j\rangle_{p_j,Q} = [\vec{w},v]_{\vec{p},(\vec{r},\infty)}\prod_{j=1}^m\langle f_j w_j v^{-\frac{p}{p_j}}\rangle^{v^p}_{p_j,Q}.$$

Thus, by Hölder's inequality for weak Lebesgue spaces and Lemma 3.2.5, we have

$$\begin{aligned}
\|M^{\mathscr{D}}_{\vec{r}}(\vec{f})\|_{L^{p,\infty}_v(\mathbf{R}^n)} &\leq [\vec{w},v]_{\vec{p},(\vec{r},\infty)}\|\prod_{j=1}^m M^{v^p,\mathscr{D}}_{p_j}(f_j w_j v^{-\frac{p}{p_j}})\|_{L^{p,\infty}(\mathbf{R}^n;v^p)}\\
&\lesssim [\vec{w},v]_{\vec{p},(\vec{r},\infty)}\prod_{j=1}^m\|M^{v^p,\mathscr{D}}_{p_j}(f_j w_j v^{-\frac{p}{p_j}})\|_{L^{p_j,\infty}(\mathbf{R}^n;v^p)}\\
&\leq [\vec{w},v]_{\vec{p},(\vec{r},\infty)}\prod_{j=1}^m\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)},
\end{aligned}$$

Thus, we have shown that

$$\|M_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^{p,\infty}_v(\mathbf{R}^n)} \lesssim [\vec{w},v]_{\vec{p},(\vec{r},\infty)}.$$

For the converse inequality, fix a cube Q and let $\vec{f}\in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$. Letting $0<\lambda<\prod_{j=1}^m\langle f_j\rangle_{r_j,Q}$, we have

$$M_{\vec{r}}(\vec{f})(x)\geq\prod_{j=1}^m\langle f_j\rangle_{r_j,Q}>\lambda$$

for all $x\in Q$ so that $Q\subseteq\{x\in\mathbf{R}^n:M_{\vec{r}}(\vec{f})(x)>\lambda\}$. Hence,

$$\begin{aligned}\lambda\langle v\rangle_{p,Q}&\leq|Q|^{-\frac{1}{p}}\lambda v^p(\{M_{\vec{r}}(\vec{f})>\lambda\})^{\frac{1}{p}}\\&\leq\|M_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^{p,\infty}_v(\mathbf{R}^n)}\prod_{j=1}^m|Q|^{-\frac{1}{p_j}}\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}.\end{aligned}$$

Taking a supremum over such λ and by replacing f_j with $\chi_Q f_j$, we conclude that

$$\left(\prod_{j=1}^m\langle f_j\rangle_{r_j,Q}\right)\langle v\rangle_{p,Q}\leq\|M_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^{p,\infty}_v(\mathbf{R}^n)}\prod_{j=1}^m\langle f_jw_j\rangle_{p_j,Q}. \tag{3.2.7}$$

Now set $f_j=w_j^{-\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}$ and assume for the moment that for those $j\in\{1,\dots,m\}$ with $p_j<\infty$ the function $f_j^{r_j}=f_j^{p_j}w_j^{p_j}=w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$ is locally integrable. Then the product on the right-hand side of (3.2.7) is positive and finite so that we may take it to the left-hand side. This yields

$$\left(\prod_{j=1}^m\langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}\right)\langle v\rangle_{p,Q}\leq\|M_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^{p,\infty}_v(\mathbf{R}^n)} \tag{3.2.8}$$

and taking a supremum over all cubes Q yields (3.2.2). To prove that $w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$ is indeed locally integrable, we choose f_j such that $f_j^{p_j}w_j^{p_j}=(w_j^{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}+\varepsilon)^{-1}$ for $\varepsilon>0$, the latter expression being bounded and thus locally integrable. Again taking the product on the right-hand side of (3.2.7) to the left, an appeal to the Monotone Convergence Theorem as $\varepsilon\downarrow 0$ yields (3.2.8). The assertion follows.

Since the implication (iii)⇒(ii) when $v=w$ is clear, we may finish the proof of the equivalences by showing (i)⇒(iii) through (3.2.3).

By Lemma 3.2.6, it follows from Hölder's inequality that

$$\begin{aligned}\|M_{\vec{r}}(\vec{f})\|_{L^p_w(\mathbf{R}^n)}&\leq[\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}}\prod_{j=1}^m\|N_{p_j,r_j,\vec{w}}f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}\\&\lesssim_r c_{\vec{p},\vec{r}}[\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}}\prod_{j=1}^m\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)},\end{aligned}$$

as desired.

Finally, we prove optimality of (3.2.3). Let $\alpha \geq 0$ denote the smallest possible constant in the estimate

$$\|M_{\vec{r}}(\vec{f})\|_{L^p_w(\mathbf{R}^n)} \lesssim [\vec{w}]^{\alpha}_{\vec{p},(\vec{r},\infty)} \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}.$$

We have shown that $\alpha \leq \max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}$ and it remains to prove the lower bound. Fix $j_0 \in \{1,\ldots,m\}$. For $\varepsilon \in (0,1)$ we define

$$w_{j_0}(x) := |x|^{n(1-\varepsilon)\left(\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}\right)}, \qquad w_j(x) := 1 \quad \text{for } j \in \{1,\ldots,m\}\backslash\{j_0\},$$
$$f_{j_0}(x) := |x|^{-\frac{n(1-\varepsilon)}{r_{j_0}}}\chi_{B(0;1)}(x), \qquad f_j(x) := |x|^{-\frac{n(1-\varepsilon)}{p_j}}\chi_{B(0;1)}(x) \quad \text{for } j \in \{1,\ldots,m\}\backslash\{j_0\}.$$

Then, by Proposition 3.1.4,

$$[\vec{w}]_{\vec{p},(\vec{r},\infty)} \eqsim \left(\frac{\frac{1}{p_{j_0}}}{\frac{1}{p_{j_0}}+(1-\varepsilon)\left(\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}\right)}\right)^{\frac{1}{p_{j_0}}} \varepsilon^{-\left(\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}\right)} \leq \varepsilon^{-\left(\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}\right)}.$$

Moreover, one computes

$$\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)} \eqsim \varepsilon^{-\frac{1}{p}}$$

so that

$$\|M_{\vec{r}}(\vec{f})\|_{L^p_w(\mathbf{R}^n)} \lesssim [\vec{w}]^{\alpha}_{\vec{p},(\vec{r},\infty)} \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)} \lesssim \varepsilon^{-\alpha\left(\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}\right)-\frac{1}{p}}. \tag{3.2.9}$$

Computing

$$\prod_{j=1}^m \langle f_j\rangle_{r_j,B(0;|x|)} \gtrsim \varepsilon^{-\frac{1}{r_{j_0}}} f_{j_0}(x) \prod_{\substack{j=1\\ j\neq j_0}}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-(1-\varepsilon)\frac{1}{p_j}}\right)^{\frac{1}{r_j}} f_j(x),$$

setting $g(x) := \prod_{j=1}^m f_j(x)w_j(x) = |x|^{-\frac{n(1-\varepsilon)}{p}}\chi_{B(0;1)}(x)$ yields

$$\|M_{\vec{r}}(\vec{f})\|_{L^p_w(\mathbf{R}^n)} \gtrsim \varepsilon^{-\frac{1}{r_{j_0}}}\|g\|_{L^p(\mathbf{R}^n)} \eqsim \varepsilon^{-\frac{1}{r_1}-\frac{1}{p}}.$$

By combining this with (3.2.9) we find that

$$1 \lesssim \varepsilon^{-\alpha\left(\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}\right)+\frac{1}{r_{j_0}}}.$$

Letting $\varepsilon \downarrow 0$ shows that we must have $-\alpha\left(\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}\right)+\frac{1}{r_{j_0}} \leq 0$, i.e.,

$$\alpha \geq \frac{\frac{1}{r_{j_0}}}{\frac{1}{r_{j_0}}-\frac{1}{p_{j_0}}}.$$

Taking a maximum over $j_0 \in \{1, \dots, m\}$ proves that $\alpha \geq \max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}} - \frac{1}{\vec{p}}}\right\}$, as desired.

For the last result it remains to prove that $\|M_{\vec{r}}\|_{L^{\vec{p}}(\mathbf{R}^n) \to L^p(\mathbf{R}^n)} \gtrsim c_{\vec{p},\vec{r}}$. For $\varepsilon \in (0,1)$ we set $f_j(x) := |x|^{-\frac{n(1-\varepsilon)}{p_j}} \chi_{B(0;1)}(x)$ so that

$$\prod_{j=1}^m \langle f_j \rangle_{r_j, B(0;|x|)} \gtrsim_{r_j} \prod_{j=1}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j} - (1-\varepsilon)\frac{1}{p_j}} \right)^{\frac{1}{r_j}} f_j(x)$$

and hence

$$\|M_{\vec{r}}(\vec{f})\|_{L^p(\mathbf{R}^n)} \gtrsim_r \|f\|_{L^p(\mathbf{R}^n)} \prod_{j=1}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j} - (1-\varepsilon)\frac{1}{p_j}} \right)^{\frac{1}{r_j}} \eqsim \varepsilon^{-\frac{1}{p}} \prod_{j=1}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j} - (1-\varepsilon)\frac{1}{p_j}} \right)^{\frac{1}{r_j}}. \tag{3.2.10}$$

Moreover, we have

$$\|M_{\vec{r}}(\vec{f})\|_{L^p(\mathbf{R}^n)} \leq \|M_{\vec{r}}\|_{L^{\vec{p}}(\mathbf{R}^n) \to L^p(\mathbf{R}^n)} \prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbf{R}^n)} \eqsim \|M_{\vec{r}}\|_{L^{\vec{p}}(\mathbf{R}^n) \to L^p(\mathbf{R}^n)} \varepsilon^{-\frac{1}{p}}.$$

Combining this with (3.2.10) yields

$$\|M_{\vec{r}}\|_{L^{\vec{p}}(\mathbf{R}^n) \to L^p(\mathbf{R}^n)} \gtrsim_r \prod_{j=1}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j} - (1-\varepsilon)\frac{1}{p_j}} \right)^{\frac{1}{r_j}}.$$

The assertion follows by letting $\varepsilon \downarrow 0$. □

3.2.2. Sparse forms and symmetry in the weight classes

In terms of symmetries, the definition of the weight constant $[\vec{w}]_{\vec{p},(\vec{r},s)}$ seems to be best suited to the case where $\frac{1}{p} \leq 1$. Indeed, if we set $\frac{1}{p_{m+1}} := 1 - \frac{1}{p} \geq 0$, $\frac{1}{r_{m+1}} := 1 - \frac{1}{s}$ and $w_{m+1} := w^{-1}$, then we have

$$\sum_{j=1}^{m+1} \frac{1}{p_j} = 1, \qquad \prod_{j=1}^{m+1} w_j = 1.$$

The conditions $\vec{p} \geq \vec{r}$, $p \leq s$ are then equivalent to $r_j \leq p_j$ for all $j \in \{1, \dots, m+1\}$ and the constant for the weight class now takes the form

$$[\vec{w}]_{\vec{p},(\vec{r},s)} = \sup_Q \prod_{j=1}^{m+1} \langle w_j^{-1} \rangle_{\frac{1}{\frac{1}{r_j} - \frac{1}{p_j}}, Q} = [(w_1, \dots, w_{m+1})]_{(p_1,\dots,p_{m+1}),((r_1,\dots,r_{m+1}),\infty)}, \tag{3.2.11}$$

where the last equality follows from the fact that the term involving the product weight in the $m+1$-linear weight class is equal to 1. The symmetry of this last expression also emphasizes a certain permutational invariance. Indeed, if $\pi \in S_{m+1}$ is a permutation, then, since

$$\sum_{j=1}^{m+1} \frac{1}{p_{\pi(j)}} = \sum_{j=1}^{m+1} \frac{1}{p_j} = 1, \qquad \prod_{j=1}^{m+1} w_{\pi(j)} = \prod_{j=1}^{m+1} w_j = 1,$$

we have

$$[\vec{w}]_{\vec{p},(\vec{r},s)} = [(w_{\pi(1)},\dots,w_{\pi(m)})]_{(p_{\pi(1)},\dots,p_{\pi(m)}),((r_{\pi(1)},\dots,r_{\pi(m)}),r'_{\pi(m+1)})}.$$

It will sometimes also be useful to redefine $v_j := w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$ for $j \in \{1,\dots,m+1\}$ so that

$$[(w_1,\dots,w_{m+1})]_{(p_1,\dots,p_{m+1}),((r_1,\dots,r_{m+1}),\infty)} = \sup_Q \prod_{j=1}^{m+1} \langle v_j\rangle_{1,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}.$$

Remark 3.2.7. While we have to restrict ourselves to the Banach range $\frac{1}{p} \le 1$ here, we point out that even when $\frac{1}{p} > 1$, we can reduce back to this case by the rescaling property Proposition 3.1.3(ii) applied, for example, with $t = r$. This way we are replacing $\frac{1}{p}$ by $\frac{r}{p} \le 1$, allowing us to use the results in this section.

In a way, we are now viewing the m-linear Muckenhoupt weight classes as the subclass of the $m+1$-linear Muckenhoupt weight classes where the $m+1$ weights satisfy the relation that their product weight is 1. To avoid confusion between the two viewpoints, we introduce a separate notation for $m+1$-tuples of parameters. We will use the following convention: for $m+1$ parameters $\alpha_1,\dots,\alpha_{m+1}$ we shall use the boldface notation $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_{m+1})$ for $m+1$-tuples while we will use the arrow notation $\vec{\alpha} = (\alpha_1,\dots,\alpha_m)$ for m-tuples. This means for example that the equality of the constants (3.2.11) will now be written as

$$[\vec{w}]_{\vec{p},(\vec{r},s)} = [\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)},$$

where as before $\frac{1}{p_{m+1}} := 1 - \frac{1}{p}$, $\frac{1}{r_{m+1}} := 1 - \frac{1}{s}$ and $w_{m+1} := w^{-1}$.

As it turns out, our weight classes are governed by sparse forms.

Definition 3.2.8. Let $R > 1$. A collection of cubes $\mathscr{S}$ is called R-*sparse* if there is pairwise disjoint collection $(E_Q)_{Q\in\mathscr{S}}$ of measurable sets satisfying $E_Q \subseteq Q$ and $|Q| \le R|E_Q|$. When $R = 2$, we simply call such a collection of cubes *sparse*.

Given $\boldsymbol{r} \in (0,\infty)^{m+1}$, for a sparse collection of cubes $\mathscr{S}$ we define the sparse form

$$\Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}) := \sum_{Q\in\mathscr{S}} \Big(\prod_{j=1}^{m+1} \langle f_j\rangle_{r_j,Q}\Big)|Q|$$

for $\boldsymbol{f} \in L^{\boldsymbol{r}}_{\mathrm{loc}}(\mathbf{R}^n)$.

We point out here that the sparsity constant R is not too important and it can usually be replaced by 2. More precisely, as a consequence of Proposition 3.2.10 below we have that for any $R, \widetilde{R} > 1$ and all $\boldsymbol{f} \in L^{\boldsymbol{r}}_{\mathrm{loc}}(\mathbf{R}^n)$ we have

$$\sup_{\substack{\mathscr{S} \\ \mathscr{S} \text{ is } R\text{-sparse}}} \Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}) \eqsim_{R,\widetilde{R}} \sup_{\substack{\mathscr{S} \\ \mathscr{S} \text{ is } \widetilde{R}\text{-sparse}}} \Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}).$$

Next, we note that we can use the three-lattice lemma to pass from general sparse collections to sparse collections in dyadic grids.

Lemma 3.2.9. *Let $R > 1$ and let $\mathscr{S}$ be an R-sparse collection of cubes. Then for each $\alpha \in \left\{0, \frac{1}{3}, \frac{2}{3}\right\}^n$ there exists a $6^n R$-sparse collection of cubes $\mathscr{S}^\alpha$ such that*

$$\Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}) \lesssim_r \sum_{\alpha \in \left\{0,\frac{1}{3},\frac{2}{3}\right\}^n} \Lambda_{\boldsymbol{r},\mathscr{S}^\alpha}(\boldsymbol{f})$$

for all $\boldsymbol{f} \in L^{\boldsymbol{r}}_{\mathrm{loc}}(\mathbf{R}^n)$.

Proof. For each $Q \in \mathscr{S}$ we use the three lattice lemma to choose an $\alpha(Q) \in \left\{0, \frac{1}{3}, \frac{2}{3}\right\}^n$ and $P(Q) \in \mathscr{D}^{\alpha(Q)}$ such that $Q \subseteq P(Q)$ and $|P(Q)| \leq 6^n |Q|$. We set $\mathscr{S}^\alpha := \{P(Q) : Q \in \mathscr{S}, \alpha(Q) = \alpha\}$ and $E_{P(Q)} := E_Q$. Then the $E_{P(Q)}$ are pairwise disjoint, and

$$|P(Q)| \leq 6^n |Q| \leq 6^n R |E_Q| = 6^n R |E_{P(Q)}|$$

so that $\mathscr{S}^\alpha$ is $6^n R$-sparse.

Finally, note that

$$\begin{aligned}
\Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}) &= \sum_{\alpha \in \left\{0,\frac{1}{3},\frac{2}{3}\right\}^n} \sum_{Q \in \mathscr{S} : \alpha(Q) = \alpha} \left(\prod_{j=1}^{m+1} \langle f_j \rangle_{r_j, Q} \right) |Q| \\
&\leq 6^{\frac{n}{r}} \sum_{\alpha \in \left\{0,\frac{1}{3},\frac{2}{3}\right\}^n} \sum_{Q \in \mathscr{S} : \alpha(Q) = \alpha} \left(\prod_{j=1}^{m+1} \langle f_j \rangle_{r_j, P(Q)} \right) |P(Q)| \\
&= 6^{\frac{n}{r}} \sum_{\alpha \in \left\{0,\frac{1}{3},\frac{2}{3}\right\}^n} \Lambda_{\boldsymbol{r},\mathscr{S}^\alpha}(\boldsymbol{f}).
\end{aligned}$$

The assertion follows. □

Sparse forms are deeply connected to multisublinear maximal operators. Note that for all $\boldsymbol{f} \in L^{\boldsymbol{r}}_{\mathrm{loc}}(\mathbf{R}^n)$ and any R-sparse collection of cubes $\mathscr{S}$ we have the estimate

$$\begin{aligned}
\Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}) &\leq R \sum_{Q \in \mathscr{S}} \inf_{y \in Q} M_{\boldsymbol{r}}(\boldsymbol{f})(y) |E_Q| \leq R \sum_{Q \in \mathscr{S}} \int_{E_Q} M_{\boldsymbol{r}}(\boldsymbol{f}) \, \mathrm{d}x \\
&\leq R \| M_{\boldsymbol{r}}(\boldsymbol{f}) \|_{L^1(\mathbf{R}^n)}.
\end{aligned}$$

Hence, we have

$$\sup_{\mathscr{S}} \Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}) \leq R \| M_{\boldsymbol{r}}(\boldsymbol{f}) \|_{L^1(\mathbf{R}^n)}, \tag{3.2.12}$$

where the supremum is taken over all R-sparse collections of cubes $\mathscr{S}$. From Theorem 3.2.3 it then follows that

$$\sup_{\mathscr{S}} \Lambda_{\boldsymbol{r},\mathscr{S}}(f) \lesssim_r c_{\boldsymbol{p},\boldsymbol{r}} [\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\max\left\{ \frac{\frac{1}{r}}{\frac{1}{r} - \frac{1}{p}} \right\}} \prod_{j=1}^{m+1} \| f_j \|_{L^{p_j}_{w_j}(\mathbf{R}^n)}$$

for all $\boldsymbol{p} > \boldsymbol{r}$ with $\sum_{j=1}^{m+1} \frac{1}{p_j} = 1$, all $\boldsymbol{w} \in A_{\boldsymbol{p},(\boldsymbol{r},\infty)}$ with $\prod_{j=1}^{m} w_j = 1$, and all $\boldsymbol{f} \in L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)$. In particular, we have shown that the sparse form is bounded with respect to the embedded weight classes. To show that the boundedness of the sparse form characterizes the embedded weight class, we will show that the converse estimate to (3.2.12) holds. This follows from a sparse domination result for the multisublinear maximal operator.

Proposition 3.2.10 (Sparse domination of the multisublinear maximal operator)**.** *Let $\vec{r} \in (0,\infty)^m$ and let $\vec{f} \in L^{\vec{r}}_{\mathrm{loc}}(\mathbf{R}^n)$. Let $\mathscr{D} = \mathscr{D}^\alpha$ be a dyadic grid and let $\mathscr{F} \subseteq \mathscr{D}$ be a finite collection of cubes. Then for all $R > 1$ there exists an R-sparse collection of cubes $\mathscr{S} \subseteq \mathscr{F}$ such that*

$$M^{\mathscr{F}}_{\vec{r}}(\vec{f}) \le (R')^{\frac{1}{r}} \sup_{Q \in \mathscr{S}} \prod_{j=1}^{m} \langle f_j \rangle_{r_j,Q} \chi_Q. \tag{3.2.13}$$

In particular, for all $\boldsymbol{r} \in (0,\infty)^{m+1}$, $\boldsymbol{f} \in L^{\boldsymbol{r}}_{\mathrm{loc}}(\mathbf{R}^n)$, and $R > 1$ we have

$$\|M_{\boldsymbol{r}}(\boldsymbol{f})\|_{L^1(\mathbf{R}^n)} \eqsim_{R,r} \sup_{\mathscr{S}} \Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}) \tag{3.2.14}$$

where the supremum is taken over all R-sparse collections of cubes $\mathscr{S}$.

Proof. We will define $\mathscr{S}$ recursively. For each $Q \in \mathscr{F}$ we define its stopping children $\mathrm{ch}_{\mathscr{F}}(Q)$ as follows. For each dyadic child Q' of Q we check if $Q' \in \mathscr{F}$ and

$$\prod_{j=1}^{m} \langle f_j \rangle_{r_j,Q'} > (R')^{\frac{1}{r}} \prod_{j=1}^{m} \langle f_j \rangle_{r_j,Q}. \tag{3.2.15}$$

If this is the case, then we add Q' to $\mathrm{ch}_{\mathscr{F}}(Q)$. If this is not the case, then we repeat this process to the dyadic children of Q'. The pairwise disjoint collection of cubes $\mathrm{ch}_{\mathscr{F}}(Q)$ thus obtained, are the maximal (with respect to inclusion) cubes in $\mathscr{F}$ strictly contained in Q satisfying (3.2.15). Now, let $\mathscr{S}_0$ denote the maximal cubes in $\mathscr{F}$. Then we recursively define $\mathscr{S}_{k+1} := \cup_{Q \in \mathscr{S}_k} \mathrm{ch}_{\mathscr{F}}(Q)$ and set $\mathscr{S} := \cup_{k=0}^{\infty} \mathscr{S}_k$.

To see that $\mathscr{S}$ is R-sparse, fix $Q \in \mathscr{S}$ and set $E_Q := Q \backslash \cup_{Q' \in \mathrm{ch}_{\mathscr{F}}(Q)} Q'$. By (3.2.15), Hölder's inequality, and disjointness of the cubes in $\mathrm{ch}_{\mathscr{F}}(Q)$, we have

$$\begin{aligned}\sum_{Q' \in \mathrm{ch}_{\mathscr{F}}(Q)} |Q'| &= \sum_{Q' \in \mathrm{ch}_{\mathscr{F}}(Q)} |Q'| \frac{\prod_{j=1}^{m} \langle f_j \rangle^r_{r_j,Q'}}{\prod_{j=1}^{m} \langle f_j \rangle^r_{r_j,Q'}} \le \frac{1}{R' \prod_{j=1}^{m} \langle f_j \rangle^r_{r_j,Q}} \sum_{Q' \in \mathrm{ch}_{\mathscr{F}}(Q)} \prod_{j=1}^{m} \Big(\int_{Q'} |f_j|^{r_j} \, \mathrm{d}x \Big)^{\frac{r}{r_j}} \\ &\le \frac{1}{R' \prod_{j=1}^{m} \langle f_j \rangle^r_{r_j,Q}} \prod_{j=1}^{m} \Big(\sum_{Q' \in \mathrm{ch}_{\mathscr{F}}(Q)} \int_{Q'} |f_j|^{r_j} \, \mathrm{d}x \Big)^{\frac{r}{r_j}} \le \frac{|Q|}{R'}.\end{aligned}$$

Hence, $|Q| \le |E_Q| + \sum_{Q' \in \mathrm{ch}_{\mathscr{F}}(Q)} |Q'| \le |E_Q| + |Q|/R'$ so that $|Q| \le R|E_Q|$. Since the E_Q are also pairwise disjoint, we conclude that $\mathscr{S}$ is R-sparse.

Next, we check that (3.2.13) holds for this collection $\mathscr{S}$. For each $Q' \in \mathscr{F}$ we let $\pi_{\mathscr{S}}(Q')$ denote the minimal cube in $\mathscr{S}$ containing Q'. Note that such a minimal cube exists, since any $Q' \in \mathscr{F}$ lies in one of the cubes in $\mathscr{S}_0$. Now, let $Q \in \mathscr{S}$ and $Q' \in \mathscr{F}$ with $\pi_{\mathscr{S}}(Q') = Q$. If $Q' \subsetneq Q$, then $Q' \notin \mathscr{S}$ and hence $Q' \notin \mathrm{ch}_{\mathscr{F}}(Q)$. Since $\pi_{\mathscr{S}}(Q') = Q$, Q'

cannot be contained in any of the cubes in $\operatorname{ch}_{\mathscr{F}}(Q)$. Thus, by maximality of the cubes in $\operatorname{ch}_{\mathscr{F}}(Q)$, Q' fails the estimate (3.2.15). Note that this is also the case when $Q' = Q$. Hence, we conclude that if $Q \in \mathscr{S}$ and $Q' \in \mathscr{F}$ satisfies $\pi_{\mathscr{S}}(Q') = Q$, then

$$\prod_{j=1}^{m} \langle f_j \rangle_{r_j,Q'} \le (R')^{\frac{1}{r}} \prod_{j=1}^{m} \langle f_j \rangle_{r_j,Q}. \tag{3.2.16}$$

Finally, note that

$$\mathscr{F} = \bigcup_{Q\in\mathscr{S}} \{P \in \mathscr{F} : \pi_{\mathscr{S}}(P) = Q\}$$

so that by (3.2.16) for all $x \in \mathbf{R}^n$ we have

$$M_{\vec{r}}^{\mathscr{F}}(\vec{f})(x) = \sup_{Q\in\mathscr{S}} \sup_{\substack{P\in\mathscr{F}\\ \pi_{\mathscr{S}}(P)=Q}} \prod_{j=1}^{m} \langle f_j \rangle_{r_j,P} \chi_P(x) \le (R')^{\frac{1}{r}} \sup_{Q\in\mathscr{S}} \prod_{j=1}^{m} \langle f_j \rangle_{r_j,Q} \chi_Q(x).$$

This proves (3.2.13).

For the second assertion, we have proven one of the inequalities in (3.2.12). For the converse inequality, note that by the inequality $\|\cdot\|_{\ell^\infty} \le \|\cdot\|_{\ell^1}$ it follows from (3.2.13) that for any finite collection of cubes $\mathscr{F}$ in a dyadic grid $\mathscr{D} = \mathscr{D}^\alpha$ there is an R-sparse collection $\mathscr{S}$ of cubes $\mathscr{S} \subseteq \mathscr{F}$ so that

$$\|M_{\boldsymbol{r}}^{\mathscr{F}}(\boldsymbol{f})\|_{L^1(\mathbf{R}^n)} \le (R')^{\frac{1}{r}} \Big\| \sum_{Q\in\mathscr{S}} \prod_{j=1}^{m} \langle f_j \rangle_{r_j,Q} \chi_Q \Big\|_{L^1(\mathbf{R}^n)} \le (R')^{\frac{1}{r}} \Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}).$$

Now note that it follows from the monotone convergence theorem that

$$\|M_{\boldsymbol{r}}^{\mathscr{D}}(\boldsymbol{f})\|_{L^1(\mathbf{R}^n)} \le \sup_{\mathscr{F}\subseteq\mathscr{D}:\mathscr{F}\text{ finite}} \|M_{\boldsymbol{r}}^{\mathscr{F}}(\boldsymbol{f})\|_{L^1(\mathbf{R}^n)} \le (R')^{\frac{1}{r}} \sup_{\mathscr{S}} \Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f}),$$

proving the result for the dyadic maximal operator. The assertion now follows from Lemma 3.2.4. □

The following result is a consequence of Theorem 3.2.3 and Proposition 3.2.10. For clarity, we formulate this result here in terms of m-tuples of weights. For this we will use the notation $M_{(\vec{r},s')}(\vec{f},g) = M_{(r_1,\dots,r_m,s')}(f_1,\dots,f_m,g)$ and similarly for $\Lambda_{(\vec{r},s')}$.

Theorem 3.2.11. *Let $\vec{r} \in (0,\infty)^m$, $s \in (1,\infty]$, and $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$ and $1 \le p < s$. Then the following are equivalent:*

(i) $\vec{w} \in A_{\vec{p},(\vec{r},s)}$*;*

(ii) $M_{(\vec{r},s')}$ *is bounded* $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \times L^{p'}_{w^{-1}}(\mathbf{R}^n) \to L^{1,\infty}(\mathbf{R}^n)$*;*

(iii) $M_{(\vec{r},s')}$ *is bounded* $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \times L^{p'}_{w^{-1}}(\mathbf{R}^n) \to L^{1}(\mathbf{R}^n)$*;*

(iv) $\sup_{\mathscr{S}} \Lambda_{(\vec{r},s'),\mathscr{S}}$ *is bounded* $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \times L^{p'}_{w^{-1}}(\mathbf{R}^n) \to \mathbf{R}$*, where the supremum is taken over all sparse collections* $\mathscr{S}$*;*

(v) $\Lambda_{\boldsymbol{r},\mathscr{S}}$ *is bounded* $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\times L^{p'}_{w^{-1}}(\mathbf{R}^n)\to\mathbf{R}$ *uniformly in all sparse collections* $\mathscr{S}$*.*

In this case we have

$$\|M_{(\vec{r},s')}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\times L^{p'}_{w^{-1}}(\mathbf{R}^n)\to L^{1,\infty}(\mathbf{R}^n)}\eqsim_{r,s}[\vec{w}]_{\vec{p},(\vec{r},s)},\tag{3.2.17}$$

$$\begin{aligned}\|M_{(\vec{r},s')}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\times L^{p'}_{w^{-1}}(\mathbf{R}^n)\to L^{1}(\mathbf{R}^n)}&\eqsim_{r,s}\|\sup_{\mathscr{S}}\Lambda_{(\vec{r},s'),\mathscr{S}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\times L^{p'}_{w^{-1}}(\mathbf{R}^n)\to\mathbf{R}}\\&\eqsim_{r,s}\sup_{\mathscr{S}}\|\Lambda_{(\vec{r},s'),\mathscr{S}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\times L^{p'}_{w^{-1}}(\mathbf{R}^n)\to\mathbf{R}}.\end{aligned}\tag{3.2.18}$$

Moreover, we have

$$\sup_{\mathscr{S}}\|\Lambda_{(\vec{r},s'),\mathscr{S}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\times L^{p'}_{w^{-1}}(\mathbf{R}^n)\to\mathbf{R}}\lesssim_{r,s}c_{\vec{p},\vec{r},s}[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}},\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}\tag{3.2.19}$$

with

$$c_{\vec{p},\vec{r},s}=\left(\prod_{j=1}^m\left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}\right)\left(\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}}.$$

This estimate is optimal in the sense that the power of the weight is the smallest possible one, and when $\vec{w}\equiv 1$*, the three quantities in* (3.2.18) *are equivalent to* $c_{\vec{p},\vec{r},s}$*.*

We point out that by Theorem 3.2.3, (3.2.17) holds more generally when $\vec{p}\geq\vec{r}$ and $1\leq p\leq s$. Note that the only part of this theorem that does not follow from Theorem 3.2.3 is (3.2.18), which we will see is a consequence of Proposition 3.2.10. We point out however, that rather than using Lemma 3.2.6 to prove the bound (3.2.19) as was done in the proof of Theorem 3.2.3, it is interesting in its own right to give an alternative proof of this bound via sparse forms. It is worth noting that this alternative approach is quantitatively worse than our previous proof in that we obtain an exponentially worse dependence in $\vec{p}$. Indeed, this constant appears in Lemma 3.2.13 below.

To facilitate our proof we will require two preparatory lemmata. The following is a reformulation of the definition of the weight class.

Lemma 3.2.12. *Let* $\boldsymbol{r}\in(0,\infty)^{m+1}$ *and let* $\boldsymbol{p}\in(0,\infty]^m$ *satisfy* $\boldsymbol{p}>\boldsymbol{r}$ *and* $\sum_{j=1}^{m+1}\frac{1}{p_j}=1$*. Moreover, let* $\boldsymbol{w}$ *be an* $m+1$ *tuple of weights satisfying* $\prod_{j=1}^{m+1}w_j=1$ *and define* $v_j:=w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$*. Then* $\boldsymbol{w}\in A_{\boldsymbol{p},(\boldsymbol{r},\infty)}$ *if and only if for those* $j\in\{1,\dots,m+1\}$ *for which* $p_j<\infty$ *the weight* v_j *is locally integrable, and there is a constant* $c>0$ *such that for all cubes* Q *we have*

$$\left(\prod_{j=1}^{m+1}\langle v_j\rangle_{1,Q}^{\frac{1}{r_j}}\right)|Q|\leq c\prod_{j=1}^{m+1}v_j(Q)^{\frac{1}{p_j}}.$$

In this case, the optimal constant c *in this inequality is given by* $[\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}$*.*

The next lemma allows us to deal with the sparseness condition.

Lemma 3.2.13. *Let $\boldsymbol{r} \in (0,\infty)^{m+1}$ and let $\boldsymbol{p} \in (0,\infty]^m$ satisfy $\boldsymbol{p} > \boldsymbol{r}$ and $\sum_{j=1}^{m+1} \frac{1}{p_j} = 1$. Moreover, let $\boldsymbol{w}$ be an $m+1$ tuple of weights satisfying $\prod_{j=1}^{m+1} w_j = 1$ and define $v_j := w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$. Let Q be a cube and let $E \subseteq Q$ such that $|Q| \le 2|E|$. If $\boldsymbol{w} \in A_{\boldsymbol{p},(\boldsymbol{r},\infty)}$, then*

$$\left(\prod_{j=1}^{m+1} \langle v_j \rangle_{1,Q}^{\frac{1}{r_j}}\right)|Q| \le 2^{\left(\frac{1}{r}-1\right)\max\left\{\frac{\frac{1}{\boldsymbol{r}}}{\frac{1}{\boldsymbol{r}}-\frac{1}{\boldsymbol{p}}}\right\}-\frac{1}{r}} [\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\max\left\{\frac{\frac{1}{\boldsymbol{r}}}{\frac{1}{\boldsymbol{r}}-\frac{1}{\boldsymbol{p}}}\right\}} \prod_{j=1}^{m+1} v_j(E)^{\frac{1}{p_j}}. \tag{3.2.20}$$

Remark 3.2.14. Having Lemma 3.2.12 in mind, it seems that the larger power of the weight constant in (3.2.20) comes from the fact that we are passing from the weighted measure of the set Q to the measure of the smaller set E. In fact, it seems like we are only using the full weight condition $\boldsymbol{w} \in A_{\boldsymbol{p},(\boldsymbol{r},\infty)}$ once and we are left with an estimate of the form

$$\prod_{j=1}^{m+1} v_j(Q)^{\frac{1}{p_j}} \lesssim \prod_{j=1}^{m+1} v_j(E)^{\frac{1}{p_j}},$$

where the implicit constant depends on the weights. This estimate seems to only require the weaker Fujii-Wilson A_∞ condition satisfied by the weight v_j, but we do not pursue this further here. We refer the reader to Section 3.3 for a discussion on improving estimates using multilinear analogues of Fujii-Wilson A_∞ conditions.

Proof. We set $\gamma := \max\left\{\frac{\frac{1}{\boldsymbol{r}}}{\frac{1}{\boldsymbol{r}}-\frac{1}{\boldsymbol{p}}}\right\}$ and

$$\beta_j := \frac{1}{r_j} - \left(\frac{1}{r_j} - \frac{1}{p_j}\right)\gamma,$$

so that $\beta_j \le 0$ for all $j \in \{1,\dots,m+1\}$. Thus, since $\langle v_j\rangle_{1,E} \le 2\langle v_j\rangle_{1,Q}$ by the assumptions on E, we have $\langle v_j\rangle_{1,Q}^{\beta_j} \le 2^{-\beta_j}\langle v_j\rangle_{1,E}^{\beta_j}$. Then

$$\begin{aligned}\left(\prod_{j=1}^{m+1} \langle v_j \rangle_{1,Q}^{\frac{1}{r_j}}\right)|Q| &= \left(\prod_{j=1}^{m+1} \langle v_j \rangle_{1,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\gamma}\left(\prod_{j=1}^{m+1} \langle v_j \rangle_{1,Q}^{\beta_j}\right)|Q| \le [\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\gamma}\left(\prod_{j=1}^{m+1} \langle v_j \rangle_{1,Q}^{\beta_j}\right)|Q| \\ &\le 2^{\left(\frac{1}{r}-1\right)\gamma-\frac{1}{r}}[\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\gamma}\left(\prod_{j=1}^{m+1} \langle v_j \rangle_{1,E}^{\beta_j}\right)|E| \\ &= 2^{\left(\frac{1}{r}-1\right)\gamma-\frac{1}{r}}[\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\gamma}\left(\prod_{j=1}^{m+1} v_j(E)^{\beta_j}\right)|E|^{1-\sum_{j=1}^{m+1}\beta_j}.\end{aligned} \tag{3.2.21}$$

Next, set $\alpha := \sum_{j=1}^{m+1}\left(\frac{1}{r_j}-\frac{1}{p_j}\right) > 0$ and $k_j := \alpha\left(\frac{1}{r_j}-\frac{1}{p_j}\right)^{-1}$. Then

$$\sum_{j=1}^{m+1}\frac{1}{k_j} = \frac{1}{\alpha}\sum_{j=1}^{m+1}\left(\frac{1}{r_j}-\frac{1}{p_j}\right) = 1$$

and
$$1-\sum_{j=1}^{m+1}\beta_j=\sum_{j=1}^{m+1}\frac{1}{p_j}-\sum_{j=1}^{m+1}\frac{1}{r_j}+\gamma\sum_{j=1}^{m+1}\left(\frac{1}{r_j}-\frac{1}{p_j}\right)=(\gamma-1)\alpha$$
so that
$$\frac{1-\sum_{j=1}^{m+1}\beta_j}{k_j}=\left(\frac{1}{r_j}-\frac{1}{p_j}\right)(\gamma-1)=\frac{1}{p_j}-\beta_j.$$
Thus, since $\prod_{j=1}^{m+1}v_j^{\frac{1}{r_j}-\frac{1}{p_j}}=\prod_{j=1}^{m+1}w_j=1$, it follows from Hölder's inequality that
$$|E|^{1-\sum_{j=1}^{m+1}\beta_j}=\left(\int_E\prod_{j=1}^{m+1}v_j^{\frac{1}{\alpha}\left(\frac{1}{r_j}-\frac{1}{p_j}\right)}\,\mathrm{d}x\right)^{1-\sum_{j=1}^{m+1}\beta_j}\leq\prod_{j=1}^{m+1}v_j(E)^{\frac{1-\sum_{j=1}^{m+1}\beta_j}{k_j}}=\prod_{j=1}^{m+1}v_j(E)^{\frac{1}{p_j}-\beta_j}.$$
By combining this estimate with (3.2.21), we obtain (3.2.20). The assertion follows. □

Proof of Theorem 3.2.11. The equivalence of (i), (ii), and (iii) follows from Theorem 3.2.3. We will prove that (iii), (iv), (v) are equivalent by proving (3.2.18).

Set $\frac{1}{p_{m+1}}:=1-\frac{1}{p}\geq 0$, $\frac{1}{r_{m+1}}:=1-\frac{1}{s}$ and $w_{m+1}:=w^{-1}$ so that we are back in the $m+1$-tuple notation.

To prove (3.2.18), note that first equivalence follows from (3.2.14) with $R=2$. For the second equivalence we note that
$$\sup_{\mathscr{S}}\|\Lambda_{\boldsymbol{r},\mathscr{S}}\|_{L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)\to\mathbf{R}}\leq\|\sup_{\mathscr{S}}\Lambda_{\boldsymbol{r},\mathscr{S}}\|_{L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)\to\mathbf{R}},$$
so that to conclude the result it suffices to show that
$$\|M_{\boldsymbol{r}}\|_{L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)\to L^1(\mathbf{R}^n)}\lesssim\sup_{\mathscr{S}}\|\Lambda_{\boldsymbol{r},\mathscr{S}}\|_{L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)\to\mathbf{R}}.$$
Fix a dyadic grid $\mathscr{D}=\mathscr{D}^\alpha$, a finite collection of cubes $\mathscr{F}\subseteq\mathscr{D}$, and $\boldsymbol{f}\in L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)$ of norm 1. Then just as in the proof of Proposition 3.2.10 we can find a sparse collection $\mathscr{S}\subseteq\mathscr{F}$ such that
$$\begin{aligned}\|M_{\boldsymbol{r}}^{\mathscr{F}}(\boldsymbol{f})\|_{L^1(\mathbf{R}^n)}&\lesssim_{\boldsymbol{r}}\Lambda_{\boldsymbol{r},\mathscr{S}}(\boldsymbol{f})\leq\|\Lambda_{\boldsymbol{r},\mathscr{S}}\|_{L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)\to\mathbf{R}}\\&\leq\sup_{\mathscr{S}}\|\Lambda_{\boldsymbol{r},\mathscr{S}}\|_{L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)\to\mathbf{R}}.\end{aligned}$$
Hence, by the monotone convergence theorem, we have
$$\|M_{\boldsymbol{r}}^{\mathscr{D}}(\boldsymbol{f})\|_{L^1(\mathbf{R}^n)}\leq\sup_{\mathscr{F}\subseteq\mathscr{D}:\mathscr{F}\text{ finite}}\|M_{\boldsymbol{r}}^{\mathscr{F}}(\boldsymbol{f})\|_{L^1(\mathbf{R}^n)}\lesssim_{\boldsymbol{r}}\sup_{\mathscr{S}}\|\Lambda_{\boldsymbol{r},\mathscr{S}}\|_{L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)\to\mathbf{R}}.$$
The result now follows from taking a supremum over all $\boldsymbol{f}\in L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)$ of norm 1 and Lemma 3.2.4.

Finally, we prove (3.2.19). Let $\mathscr{S}$ be a sparse collection of cubes which, by Lemma 3.2.9, at the loss of a dimensional constant we may assume to be contained in a dyadic grid

$\mathscr{D} = \mathscr{D}^\alpha$. We set $v_j := w_j^{-\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}}$ for $j \in \{1, \ldots, m+1\}$ and $\gamma := \max\left\{\frac{\frac{1}{\boldsymbol{r}}}{\frac{1}{\boldsymbol{r}}-\frac{1}{\boldsymbol{p}}}\right\}$. Then it follows from Lemma 3.2.13 and Lemma 3.2.5 that

$$\begin{aligned}
\Lambda_{\boldsymbol{r},\mathscr{S}}(f_1,\ldots,f_{m+1}) &= \sum_{Q\in\mathscr{S}} \left(\prod_{j=1}^{m+1} \langle f_j\rangle_{r_j,Q}\right)|Q| \\
&= \sum_{Q\in\mathscr{S}} \left(\prod_{j=1}^{m+1} \langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j} \langle v_j\rangle_{1,Q}^{\frac{1}{r_j}}\right)|Q| \\
&\lesssim_{r,\gamma} [\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\gamma} \sum_{Q\in\mathscr{S}} \prod_{j=1}^{m+1} \langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j} v_j(E_Q)^{\frac{1}{p_j}} \\
&\le [\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\gamma} \sum_{Q\in\mathscr{P}} \prod_{j=1}^{m+1} \left(\int_{E_Q} M_{r_j}^{v_j,\mathscr{D}}(f_j v_j^{-\frac{1}{r_j}})^{p_j} v_j \,\mathrm{d}x\right)^{\frac{1}{p_j}} \\
&\le [\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\gamma} \prod_{j=1}^{m+1} \|M_{r_j}^{v_j,\mathscr{D}}(f_j v_j^{-\frac{1}{r_j}})\|_{L^{p_j}(\mathbf{R}^n;v_j)} \\
&\le c_{\boldsymbol{p},\boldsymbol{r}} [\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\gamma} \prod_{j=1}^{m+1} \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)},
\end{aligned}$$

As this estimate is uniform in the sparse collection $\mathscr{S}$, this proves (3.2.19). The assertion follows. □

3.3. MULTILINEAR FUJII-WILSON AND REVERSE HÖLDER CONSTANTS

A classical result for the A_p classes is the fact that every $w \in A_p$ satisfies a reverse Hölder estimate from which one can deduce that for some $\varepsilon > 0$ we have $w \in A_{p-\varepsilon}$. In our notation this can be equivalently formulated by saying that if $w \in A_{p,(r,s)}$, then there is an $\alpha > 1$ such that $w \in A_{p,(\alpha r,s)}$. In this section we establish a multilinear analogue of this result. Moreover, we include an open problem regarding the sharp control of the parameter α in terms of a multilinear analogue of the Fujii-Wilson A_∞ condition, as well as an open problem regarding an improved bound in the two weight setting for the multisublinear maximal operator with respect to this constant.

In the previous section we have established bounds for the multisublinear maximal operators that are optimal in terms of control by powers of the weight constant $[\cdot]_{\vec{p},(\vec{r},s)}$. In this section we define a smaller constant to try to obtain an even more precise control in terms of the weight.

For a collection of cubes $\mathscr{P}$ and a cube Q we will write $\mathscr{P}(Q) := \{P \in \mathscr{P} : P \subseteq Q\}$.

Definition 3.3.1. Let $\vec{r} \in (0,\infty)^m$, $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$, and let $\vec{w}$ be an m-tuple of weights. Setting $v_j := w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$, for each dyadic grid $\mathscr{D}^\alpha$, $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, we define

$$[\vec{w}]^{\mathrm{FW},\mathscr{D}^\alpha}_{\vec{p},\vec{r}} := \sup_{Q\in\mathscr{D}^\alpha} \frac{\langle M^{\mathscr{D}^\alpha(Q)}_{\vec{p}}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})\rangle_{p,Q}}{\prod_{j=1}^m \langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}}$$

and

$$[\vec{w}]^{\widetilde{\mathrm{FW}},\mathscr{D}^\alpha}_{\vec{p},\vec{r}} := \sup_{Q\in\mathscr{D}^\alpha} \frac{\langle M^{\mathscr{D}^\alpha}_{\vec{p},Q}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})\rangle_{p,Q}}{\langle \prod_{j=1}^m v_j^{\frac{1}{p_j}}\rangle_{p,Q}}$$

Moreover, we define $[\vec{w}]^{\mathrm{FW}}_{\vec{p},\vec{r}} := \max_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} [\vec{w}]^{\mathrm{FW},\mathscr{D}^\alpha}_{\vec{p},\vec{r}}$, $[\vec{w}]^{\widetilde{\mathrm{FW}}}_{\vec{p},\vec{r}} := \max_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} [\vec{w}]^{\widetilde{\mathrm{FW}},\mathscr{D}^\alpha}_{\vec{p},\vec{r}}$.

Note that by Hölder's inequality we have

$$[\vec{w}]^{\mathrm{FW}}_{\vec{p},\vec{r}} \le \max_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} \sup_{Q\in\mathscr{D}^\alpha} \frac{\prod_{j=1}^m \langle M^{\mathscr{D}^\alpha(Q)}_{p_j}(v_j^{\frac{1}{p_j}})\rangle_{p_j,Q}}{\prod_{j=1}^m \langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}} \le \prod_{j=1}^m [w_j]^{\mathrm{FW}}_{p_j,r_j} = \prod_{j=1}^m [v_j]_{A_\infty}^{\frac{1}{p_j}},$$

where $[\cdot]_{A_\infty}$ is the Fujii-Wilson A_∞ constant

$$[w]_{A_\infty} = \max_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} \sup_{Q\in\mathscr{D}^\alpha} \frac{1}{w(Q)} \int_Q M^{\mathscr{D}^\alpha(Q)}(w)\,\mathrm{d}x,$$

which first appeared (in an equivalent form) in the works [Fuj78, Wil87, Wil89, Wil08] and was later studied in [HPR12, HP13] in relation to a sharp reverse Hölder inequality. In this section we wish to generalize some of their results to a multilinear setting. Moreover, we give some alternative proofs or certain properties of the constant $[\cdot]^{\widetilde{\mathrm{FW}}}_{\vec{p},\vec{r}}$ which was first studied (in an alternative form) in [ZK19]. As we will see, the generally smaller constant $[\cdot]^{\mathrm{FW}}_{\vec{p},\vec{r}}$ seems to be better suited to study multilinear reverse Hölder inequalities, however the precise nature of this relationship remains open. We prove some partial results.

Remark 3.3.2. In a dyadic grid $\mathscr{D} = \mathscr{D}^\alpha$, it follows from the equivalence

$$\sup_{\substack{\mathscr{S}\subseteq\mathscr{D}\\ \mathscr{S}\text{ sparse}}} \Big(\sum_{Q'\in\mathscr{S}(Q)} \Big(\prod_{j=1}^m v_j(Q')^{\frac{1}{p_j}}\Big)^p\Big)^{\frac{1}{p}} \eqsim \Big(\int_Q M^{\mathscr{D}(Q)}_{\vec{p}}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})^p\,\mathrm{d}x\Big)^{\frac{1}{p}},$$

see Proposition 3.2.10, that the condition $[\vec{w}]^{\mathrm{FW},\mathscr{D}}_{\vec{p},\vec{r}} < \infty$ can be equivalently formulated through the inequality

$$\sum_{Q'\in\mathscr{S}(Q)} \prod_{j=1}^m v_j(Q')^{\frac{p}{p_j}} \lesssim \prod_{j=1}^m v_j(Q)^{\frac{p}{p_j}},$$

which should be valid uniformly for all sparse collections $\mathscr{S} \subseteq \mathscr{D}$ and cubes $Q \in \mathscr{D}$.

Note that when $m=1$ we have $[w]^{\mathrm{FW}}_{p,r}=[w]^{\widetilde{\mathrm{FW}}}_{p,r}$. In general, we have the following properties:

Proposition 3.3.3. *Let $\vec{r}\in(0,\infty)^m$, $\vec{p}\in(0,\infty]^m$ with $\vec{p}>\vec{r}$, and let $\vec{w}$ be an m-tuple of weights. The following properties hold:*

(i) $1\le[\vec{w}]^{\mathrm{FW}}_{\vec{p},\vec{r}}\le[\vec{w}]^{\widetilde{\mathrm{FW}}}_{\vec{p},\vec{r}}$;

(ii) If $\vec{w}\in A_{\vec{p},(\vec{r},\infty)}$, then $[\vec{w}]^{\mathrm{FW}}_{\vec{p},\vec{r}}, [\vec{w}]^{\widetilde{\mathrm{FW}}}_{\vec{p},\vec{r}}<\infty$ with

$$[\vec{w}]^{\widetilde{\mathrm{FW}}}_{\vec{p},\vec{r}}\le e^{\frac{1}{r}}[\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{\vec{p}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}}.$$

For the purpose of the proof of this result we extend the L^r-averaging notation to include the limiting case $r=0$ by setting $\langle f\rangle_{0,Q}:=e^{\frac{1}{|Q|}\int_Q\log|f|\,\mathrm{d}x}$ and set $M_0^{\mathscr{D}}(f)(x):=\sup_{Q\in\mathscr{D}}\langle f\rangle_{0,Q}\chi_Q(x)$. We will need the following lemma:

Lemma 3.3.4. *For all $p\in(0,\infty]$ we have $\|M_0^{\mathscr{D}}\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)}\le e^{\frac{1}{p}}$.*

Proof. Let $r\in(0,p)$. Since $M_0^{\mathscr{D}}(f)\le M_r^{\mathscr{D}}(f)$, it follows from Lemma 3.2.5 that

$$\|M_0^{\mathscr{D}}\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)}\le\|M_r^{\mathscr{D}}\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)}\le\left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}.$$

Letting $r\to0$, the right-hand side converges to $e^{\frac{1}{p}}$, proving the result. □

Proof of Proposition 3.3.3. Set $v_j:=w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$ and let $\mathscr{D}=\mathscr{D}^\alpha$ be a dyadic grid. To prove the first inequality in (i), note that for all $Q\in\mathscr{D}$ we have

$$1=\frac{\prod_{j=1}^m\langle v_j^{\frac{1}{p_j}}\rangle_{p_j,Q}}{\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}}\le\frac{\langle M_{\vec{p}}^{\mathscr{D}(Q)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})\rangle_{p,Q}}{\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}}\le[\vec{w}]^{\mathrm{FW}}_{\vec{p},\vec{r}}.$$

The second inequality follows from the inequality $\langle\prod_{j=1}^m v_j^{\frac{1}{p_j}}\rangle_{p,Q}\le\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}$ which follows from Hölder's inequality.

For (ii), define

$$[\vec{v}]:=\sup_{Q\in\mathscr{D}}\left(\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}\right)\langle\prod_{j=1}^m v_j^{-\frac{1}{p_j}}\rangle_{0,Q}=\sup_{Q\in\mathscr{D}}\left(\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}\right)\langle\prod_{j=1}^m v_j^{\frac{1}{p_j}}\rangle_{0,Q}^{-1}.$$

We note that the definition of $[\vec{v}]$ implies that for a fixed cube $Q\in\mathscr{D}$ we have

$$M_{\vec{p}}^{\mathscr{D}(Q)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})\le[\vec{v}]M_0^{\mathscr{D}}(\prod_{j=1}^m v_j^{\frac{1}{p_j}}\chi_Q).$$

Hence, by Lemma 3.3.4,

$$\langle M_{\vec{p}}^{\mathscr{D}(Q)}(v_1^{\frac{1}{p_1}},\ldots,v_m^{\frac{1}{p_m}})\rangle_{p,Q} \le [\vec{v}]\langle M_0^{\mathscr{D}}(\prod_{j=1}^m v_j^{\frac{1}{p_j}}\chi_Q)\rangle_{p,Q} \le [\vec{v}]|Q|^{-\frac{1}{p}}\|M_0^{\mathscr{D}}(\prod_{j=1}^m v_j^{\frac{1}{p_j}}\chi_Q)\|_{L^p(\mathbf{R}^n)}$$
$$\le e^{\frac{1}{p}}[\vec{v}]|Q|^{-\frac{1}{p}}\|\prod_{j=1}^m v_j^{\frac{1}{p_j}}\chi_Q\|_{L^p(\mathbf{R}^n)} = e^{\frac{1}{p}}[\vec{w}]_{\vec{p}}^{\gamma}\langle\prod_{j=1}^m v_j^{\frac{1}{p_j}}\rangle_{p,Q},$$

proving that

$$[\vec{w}]_{\vec{p}}^{\mathrm{FW}} \le e^{\frac{1}{r}}[\vec{v}]. \tag{3.3.1}$$

Setting $\gamma := \max\left\{\frac{\frac{1}{\vec{p}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}$, we claim that

$$[\vec{v}] \le [\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\gamma}. \tag{3.3.2}$$

Combining this with (3.3.1) then proves the desired result.

To prove the claim, fix a cube $Q\in\mathscr{D}$. Since $w\prod_{j=1}^m v_j^{\frac{1}{r_j}-\frac{1}{p_j}} = w\prod_{j=1}^m w_j^{-1} = 1$, we have

$$\prod_{j=1}^m v_j^{-\frac{1}{p_j}} = \Big(w\prod_{j=1}^m v_j^{\frac{1}{r_j}-\frac{1}{p_j}}\Big)^{\gamma}\prod_{j=1}^m v_j^{-\frac{1}{p_j}} = w^{\gamma}\prod_{j=1}^m v_j^{\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\gamma-\frac{1}{p_j}}.$$

Thus, setting $\frac{1}{q} := \frac{1}{p}\gamma + \sum_{j=1}^m\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\gamma - \frac{1}{p_j}$ and using the fact that $\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\gamma-\frac{1}{p_j}\ge 0$ for all $j\in\{1,\ldots,m\}$, it follows from Hölder's inequality that

$$\langle\prod_{j=1}^m v_j^{-\frac{1}{p_j}}\rangle_{0,Q} \le \langle\prod_{j=1}^m v_j^{-\frac{1}{p_j}}\rangle_{q,Q} \le \langle w^{\gamma}\rangle_{\frac{1}{\frac{1}{p}\gamma},Q}\prod_{j=1}^m\langle v_j^{\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\gamma-\frac{1}{p_j}}\rangle_{\frac{1}{\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\gamma-\frac{1}{p_j}},Q}$$
$$= \langle w\rangle_{p,Q}^{\gamma}\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\gamma-\frac{1}{p_j}}.$$

Then we find that

$$\Big(\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}\Big)\langle\prod_{j=1}^m v_j^{-\frac{1}{p_j}}\rangle_{0,Q} \le \langle w\rangle_{p,Q}^{\gamma}\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\gamma}$$
$$= \Big(\langle w\rangle_{p,Q}\prod_{j=1}^m\langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}\Big)^{\gamma} \le [\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\gamma}.$$

Taking a supremum over all cubes $Q\in\mathscr{D}$ yields (3.3.2), as desired. The assertion follows. □

We have the following result:

Theorem 3.3.5. *Let $\vec{r} \in (0,\infty)^m$, $\vec{p} \in (0,\infty]^m$ with $\vec{r} < \vec{p}$, and let $\vec{w}, v$ be $m+1$ weights. Then*

$$\|M_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_v(\mathbf{R}^n)} \lesssim_r c_{\vec{p},\vec{r}}[\vec{w},v]_{\vec{p},(\vec{r},\infty)}[\vec{w}]^{\widetilde{\mathrm{FW}}}_{\vec{p},\vec{r}}$$

with

$$c_{\vec{p},\vec{r}} = \prod_{j=1}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}.$$

This result was first proven in [ZK19] and we give an alternative version of this proof here, going through a Sawyer-type testing condition.

Note that when $v = w$ is the product weight, then Proposition 3.3.3(ii) implies that

$$[\vec{w}]_{\vec{p},(\vec{r},\infty)}[\vec{w}]^{\widetilde{\mathrm{FW}}}_{\vec{p},\vec{r}} \le [\vec{w}]_{\vec{p},(\vec{r},\infty)}^{1+\max\left\{\frac{\frac{1}{\vec{p}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}} = [\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}},$$

so that Theorem 3.3.5 improves the bound for $M_{\vec{r}}$ from Theorem 3.2.3.

In view of Proposition 3.3.3(i), we propose the following conjecture:

Conjecture 3.3.6. *Let $\vec{r} \in (0,\infty)^m$, $\vec{p} \in (0,\infty]^m$ with $\vec{r} < \vec{p}$, and let $\vec{w}, v$ be $m+1$ weights. Then*

$$\|M_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_v(\mathbf{R}^n)} \lesssim_{\vec{p},\vec{r}} [\vec{w},v]_{\vec{p},(\vec{r},\infty)}[\vec{w}]^{\mathrm{FW}}_{\vec{p},\vec{r}}.$$

The conjecture is true in the case $m = 1$, since then $[\cdot]^{\mathrm{FW}}_{p,r} = [\cdot]^{\widetilde{\mathrm{FW}}}_{p,r}$. We also provide an alternative way of proving the case $m = 1$ through a sharp reverse Hölder estimate.

To prove Theorem 3.3.5, we first prove the following result:

Theorem 3.3.7. *Let $\vec{r} \in (0,\infty)^m$, $\vec{p} \in (0,\infty]^m$ with $\vec{r} < \vec{p}$, let $\vec{w}, v$ be $m+1$ weights, and let $\mathscr{D} = \mathscr{D}^\alpha$ be a dyadic grid. Setting $v_j := w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$, we have*

$$\sup_{Q\in\mathscr{D}} \frac{\langle M^{\mathscr{D}(Q)}_{\vec{r}}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}})v\rangle_{p,Q}}{\prod_{j=1}^m \langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}} \le \|M^{\mathscr{D}}_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_v(\mathbf{R}^n)} \lesssim_r c_{\vec{p},\vec{r}} \sup_{Q\in\mathscr{D}} \frac{\langle M^{\mathscr{D}(Q)}_{\vec{r}}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}})v\rangle_{p,Q}}{\langle \prod_{j=1}^m v_j^{\frac{1}{p_j}}\rangle_{p,Q}},$$

with

$$c_{\vec{p},\vec{r}} = \prod_{j=1}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}.$$

Note that when $m = 1$ this result gives the equivalence

$$\|M^{\mathscr{D}}_r\|_{L^p_w(\mathbf{R}^n)\to L^p_v(\mathbf{R}^n)} \eqsim_{p,r} \sup_{Q\in\mathscr{D}} \frac{\langle M^{\mathscr{D}(Q)}_r(w^{-\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}})v\rangle_{p,Q}}{\langle w^{-\frac{1}{\frac{1}{r}-\frac{1}{p}}}\rangle_{1,Q}^{\frac{1}{p}}}.$$

It is however unlikely that when $m>1$ we have the equivalence

$$\|M_{\vec{r}}^{\mathscr{D}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_v(\mathbf{R}^n)} \eqsim_{\vec{p},\vec{r}} \sup_{Q\in\mathscr{D}} \frac{\langle M_{\vec{r}}^{\mathscr{D}(Q)}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}})v\rangle_{p,Q}}{\prod_{j=1}^m \langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}},$$

which would imply Conjecture 3.3.6 (as can be shown in a way analogous to the proof of Theorem 3.3.5 below). Indeed, a counterexample by Tuomas Hytönen[1] shows that this equivalence fails when the weights are replaced by more general measures and it seems that this counterexample can be adapted to this setting of weights. Thus, if Conjecture 3.3.6 is true, it needs to be proven using a different method.

Proof of Theorem 3.3.7. For the first inequality we have

$$\begin{aligned}|Q|^{\frac{1}{p}}\langle M_{\vec{r}}^{\mathscr{D}(Q)}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}})v\rangle_{p,Q} &\leq \|M_{\vec{r}}^{\mathscr{D}}(v_1^{\frac{1}{r_1}}\chi_Q,\dots,v_m^{\frac{1}{r_m}}\chi_Q)\|_{L^p_v(\mathbf{R}^n)}\\ &\leq \|M_{\vec{r}}^{\mathscr{D}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_v(\mathbf{R}^n)}\prod_{j=1}^m \|\chi_Q v_j^{\frac{1}{r_j}}\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}\\ &= |Q|^{\frac{1}{p}}\|M_{\vec{r}}^{\mathscr{D}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_v(\mathbf{R}^n)}\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}},\end{aligned}$$

as desired.

For the second, let $\mathscr{F}\subseteq\mathscr{D}$ be a finite collection of cubes. Then there is a sparse collection of cubes $\mathscr{S}\subseteq\mathscr{F}$ such that

$$\|M_{\vec{r}}^{\mathscr{F}}(\vec{f})\|_{L^p_v(\mathbf{R}^n)} \lesssim_r \Big\|\sum_{Q\in\mathscr{S}}\prod_{j=1}^m\langle f_j\rangle_{r_j,Q}\chi_{E_Q}\Big\|_{L^p_v(\mathbf{R}^n)},$$

see Proposition 3.2.10 and [Nie19, Lemma 2.9]. We proceed with a construction very similar to the one in the proof of Proposition 3.2.10. We will recursively define a collection of cubes $\mathscr{T}\subseteq\mathscr{S}$. For each $Q\in\mathscr{S}$ we define its stopping children $\operatorname{ch}_{\mathscr{S}}(Q)$ through the following procedure. For each dyadic child Q' of Q we check if $Q'\in\mathscr{S}$ and if

$$\prod_{j=1}^m\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q'}^{v_j} > 2^{\frac{1}{r}}\prod_{j=1}^m\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j}. \tag{3.3.3}$$

If this is the case, then we add Q' to $\operatorname{ch}_{\mathscr{S}}(Q)$. Otherwise, we repeat this process to the dyadic children of Q'. The pairwise disjoint collection of cubes$\operatorname{ch}_{\mathscr{S}}(Q)$ thus obtained are the cubes in $\mathscr{S}$ strictly contained in Q satisfying (3.3.3) that are maximal (with respect to inclusion). Now, let $\mathscr{T}_0$ denote the maximal cubes in $\mathscr{S}$. Then we recursively define $\mathscr{T}_{k+1}:=\cup_{Q\in\mathscr{T}_k}\operatorname{ch}_{\mathscr{S}}(Q)$ and set $\mathscr{T}:=\cup_{k=0}^\infty\mathscr{T}_k$.

[1]Personal communication, 2019

For a cube $Q \in \mathscr{F}$ we let $\pi_{\mathscr{T}}(Q)$ denote the minimal cube in $\mathscr{T}$ containing Q. As in the proof of Proposition 3.2.10 we then have $\prod_{j=1}^m \langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q'}^{v_j} \leq 2^{\frac{1}{r}} \prod_{j=1}^m \langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j}$ whenever $Q \in \mathscr{T}$ and $\pi_{\mathscr{T}}(Q') = Q$, and $\mathscr{S} = \cup_{Q\in\mathscr{T}}\{P \in \mathscr{S} : \pi_{\mathscr{T}}(P) = Q\}$. Thus, setting

$$\mathscr{M} := \sup_{Q\in\mathscr{D}} \frac{\langle M_{\vec{r}}^{\mathscr{F}(Q)}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}})v\rangle_{p,Q}}{\langle \prod_{j=1}^m v_j^{\frac{1}{p_j}}\rangle_{p,Q}}$$

and $v_{m+1} := v^p$, we have

$$\begin{aligned}
&\Big\| \sup_{Q\in\mathscr{S}} \prod_{j=1}^m \langle f_j\rangle_{r_j,Q}\chi_{E_Q}\Big\|_{L^p_v(\mathbf{R}^n)} = \Big\| \sum_{Q\in\mathscr{S}} \prod_{j=1}^m \langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j} \langle v_j^{\frac{1}{r_j}}\rangle_{r_j,Q}\chi_{E_Q}\Big\|_{L^p_v(\mathbf{R}^n)} \\
&= \left(\sum_{Q\in\mathscr{T}} \sum_{\substack{Q'\in\mathscr{S} \\ \pi_{\mathscr{T}}(Q')=Q}} \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q'}^{v_j} \langle v_j^{\frac{1}{r_j}}\rangle_{r_j,Q'}\big)^p v_{m+1}(E_{Q'})\right)^{\frac{1}{p}} \\
&\lesssim_r \left(\sum_{Q\in\mathscr{T}} \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j}\big)^p \sum_{\substack{Q'\in\mathscr{S} \\ \pi_{\mathscr{T}}(Q')=Q}} \Big(\prod_{j=1}^m \langle v_j^{\frac{1}{r_j}}\rangle_{r_j,Q'}\Big)^p v_{m+1}(E_{Q'})\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{Q\in\mathscr{T}} \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j}\big)^p \sum_{Q'\in\mathscr{S}(Q)} \int_{E_{Q'}} M_{\vec{r}}^{\mathscr{F}(Q)}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}})^p v_{m+1}\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{Q\in\mathscr{T}} \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j}\big)^p \int_Q M_{\vec{r}}^{\mathscr{F}(Q)}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}})^p v^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\leq \mathscr{M} \left(\sum_{Q\in\mathscr{T}} \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j}\big)^p \int_Q \prod_{k=1}^m v_k^{\frac{p}{p_k}}\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&= \mathscr{M} \left(\int_{\mathbf{R}^n} \Big(\sum_{Q\in\mathscr{T}} \chi_Q \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j}\big)^p\Big) \prod_{k=1}^m v_k^{\frac{p}{p_k}}\,\mathrm{d}x\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.3.4}$$

Now, fix a point $x \in \mathbf{R}^n$ and consider the (possibly finite) sequence of cubes $Q_0 \supseteq Q_1 \supseteq Q_2 \supseteq \cdots$ with $Q_k \in \mathscr{T}_k$ containing x. In case the sequence is finite, we denote the highest index by N_x. Otherwise we set $N_x = \infty$. By construction of the cubes, we have

$\prod_{j=1}^m \langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q_{k+1}}^{v_j} > 2^{\frac{1}{r}} \prod_{j=1}^m \langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q_k}^{v_j}$ for $k+1 \le N_x$ so that

$$\begin{aligned}\sum_{Q\in\mathcal{T}} \chi_Q(x) \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j}\big)^p &= \lim_{N\to\infty} \sum_{k=0}^{\min(N,N_x)} \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q_k}^{v_j}\big)^p \\ &\le \limsup_{N\to\infty} \sum_{k=0}^{\min(N,N_x)} 2^{-\frac{p}{r}(\min(N,N_x)-k)} \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q_N}^{v_j}\big)^p \\ &\le \frac{1}{1-2^{-\frac{p}{r}}} \prod_{j=1}^m \Big(M_{r_j}^{v_j,\mathcal{D}}(f_j v_j^{\frac{1}{r_j}})(x)\Big)^p \le 2 \prod_{j=1}^m \Big(M_{r_j}^{v_j,\mathcal{D}}(f_j v_j^{\frac{1}{r_j}})(x)\Big)^p.\end{aligned}$$

Hence, by Lemma 3.2.5 and Hölder's inequality,

$$\begin{aligned}\left(\int_{\mathbf{R}^n}\Big(\sum_{Q\in\mathcal{T}} \chi_Q \prod_{j=1}^m \big(\langle f_j v_j^{-\frac{1}{r_j}}\rangle_{r_j,Q}^{v_j}\big)^p\Big) \prod_{k=1}^m v_k^{\frac{p}{p_k}}\,\mathrm{d}x\right)^{\frac{1}{p}} &\lesssim \Big\| \prod_{j=1}^m M_{r_j}^{v_j,\mathcal{D}}(f_j v_j^{\frac{1}{r_j}}) v_j^{\frac{1}{p_j}} \Big\|_{L^p(\mathbf{R}^n)} \\ &\le \prod_{j=1}^m \|M_{r_j}^{v_j,\mathcal{D}}(f_j v_j^{-\frac{1}{r_j}})\|_{L^{p_j}(\mathbf{R}^n;v_j)} \\ &\le \prod_{j=1}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}} \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}.\end{aligned}$$

Combining this with (3.3.4) proves the assertion for $M_{\vec{r}}^{\mathcal{F}}$. The general result then follows from the fact that $\|M_{\vec{r}}^{\mathcal{D}}(\vec{f})\|_{L^p_v(\mathbf{R}^n)} \le \sup_{\substack{\mathcal{F}\subseteq\mathcal{D}\\ \mathcal{F}\text{ finite}}} \|M_{\vec{r}}^{\mathcal{F}}(\vec{f})\|_{L^p_v(\mathbf{R}^n)}$, which follows from the Monotone Convergence Theorem.

□

Proof of Theorem 3.3.5. By the three lattice lemma it suffices to prove this result in a dyadic grid $\mathcal{D} = \mathcal{D}^\alpha$ and by the Monotone Convergence Theorem it suffices to consider the result for finite collections $\mathcal{F} \subseteq \mathcal{D}$. Fix $Q \in \mathcal{F}$ and let $\mathcal{S} \subseteq \mathcal{F}(Q)$ be a sparse collection of cubes such that $M_{\vec{r}}^{\mathcal{F}(Q)}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}}) \lesssim_r \sum_{Q'\in\mathcal{S}} \prod_{j=1}^m \langle v_j\rangle_{1,Q'}^{\frac{1}{r_j}} \chi_{E_{Q'}}$. Then

$$\begin{aligned}|Q|\langle M_{\vec{r}}^{\mathcal{F}(Q)}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}})v\rangle_{p,Q}^p &\lesssim_r |Q| \sum_{Q'\in\mathcal{S}} \Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q'}^{\frac{1}{r_j}}\Big)^p \langle v\chi_{E_{Q'}}\rangle_{p,Q'}^p |Q'| \\ &\le 2[\vec{w},v]_{\vec{p},(\vec{r},\infty)}^p |Q| \sum_{Q'\in\mathcal{S}} \Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q'}^{\frac{1}{p_j}}\Big)^{\frac{1}{p}} |E_{Q'}| \\ &\le 2[\vec{w},v]_{\vec{p},(\vec{r},\infty)}^p |Q| \sum_{Q'\in\mathcal{S}} \int_{E_{Q'}} M_{\vec{p}}^{\mathcal{F}(Q)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})^p\,\mathrm{d}x \\ &\le 2[\vec{w},v]_{\vec{p},(\vec{r},\infty)}^p |Q| \langle M_{\vec{p}}^{\mathcal{F}(Q)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})\rangle_{p,Q}^p.\end{aligned}$$

Hence,

$$\frac{\langle M_{\vec{r}}^{\mathscr{F}(Q)}(v_1^{\frac{1}{r_1}},\dots,v_m^{\frac{1}{r_m}})v\rangle_{p,Q}}{\langle\prod_{j=1}^m v_j^{\frac{1}{p_j}}\rangle_{p,Q}} \lesssim_r [\vec{w},v]_{\vec{p},(\vec{r},\infty)}\frac{\langle M_{\vec{p}}^{\mathscr{F}(Q)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})\rangle_{p,Q}}{\langle\prod_{j=1}^m v_j^{\frac{1}{p_j}}\rangle_{p,Q}}$$
$$\le [\vec{w},v]_{\vec{p},(\vec{r},\infty)}[\vec{w}]_{\vec{p},\vec{r}}^{\widetilde{\mathrm{FW}},\mathscr{D}}.$$

Taking a supremum over $Q\in\mathscr{F}$, the assertion then follows from Theorem 3.3.7. □

An alternate approach to try to prove Conjecture 3.3.6 will be through a multilinear reverse Hölder condition.

Definition 3.3.8. Let $\vec{r}\in(0,\infty)^m$, $\vec{p}\in(0,\infty]^m$ with $\vec{p}>\vec{r}$, let $\vec{w}$ be an m-tuple of weights, and let $\beta\in[1,\infty]$. Setting $v_j:=w_j^{-\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}$, we write $\vec{w}\in\mathrm{RH}_{\vec{p},\vec{r},\beta}$ if there is a $c>0$ such that for all cubes Q we have

$$\prod_{j=1}^m\langle v_j\rangle_{\beta,Q}^{\frac{1}{p_j}}\le c\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}.$$

We denote the smallest possible constant c by $[\vec{w}]_{\mathrm{RH}_{\vec{p},\vec{r},\beta}}$.

In the following result we show that if $\vec{w}\in\mathrm{RH}_{\vec{p},\beta}$, then $[\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW}}<\infty$.

Proposition 3.3.9. *Let $\vec{r}\in(0,\infty)^m$, $\vec{p}\in(0,\infty]^m$ with $\vec{p}>\vec{r}$, let $\vec{w}$ be an m-tuple of weights, and let $\beta\in[1,\infty]$. If $\vec{w}\in\mathrm{RH}_{\vec{p},\vec{r},\beta}$, then*

$$[\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW}}\lesssim(\beta')^{\frac{1}{p}}[\vec{w}]_{\mathrm{RH}_{\vec{p},\vec{r},\beta}}.$$

Proof. By Theorem 3.2.3 we have $\|M_{\vec{p}}\|_{L^{\beta\vec{p}}(\mathbf{R}^n)\to L^{\beta p}(\mathbf{R}^n)}\eqsim\prod_{j=1}^m\left[\frac{\frac{1}{p_j}}{\frac{1}{p_j}-\frac{1}{\beta p_j}}\right]^{\frac{1}{p_j}}=(\beta')^{\frac{1}{p}}$. Thus, for a dyadic grid $\mathscr{D}=\mathscr{D}^\alpha$ and a cube $Q\in\mathscr{D}$ we have

$$\langle M_{\vec{p}}^{\mathscr{D}(Q)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})\rangle_{p,Q}\le\langle M_{\vec{p}}^{\mathscr{D}(Q)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})\rangle_{\beta p,Q}\le|Q|^{-\frac{1}{\beta p}}\|M_{\vec{p}}^{\mathscr{D}}(v_1^{\frac{1}{p_1}}\chi_Q,\dots,v_m^{\frac{1}{p_m}}\chi_Q)\|_{L^{\beta p}(\mathbf{R}^n)}$$
$$\lesssim(\beta')^{\frac{1}{p}}|Q|^{-\frac{1}{\beta p}}\prod_{j=1}^m\|v_j^{\frac{1}{p_j}}\chi_Q\|_{L^{\beta p_j}(\mathbf{R}^n)}=(\beta')^{\frac{1}{p}}\prod_{j=1}^m\langle v_j\rangle_{\beta,Q}^{\frac{1}{p_j}}$$
$$\le(\beta')^{\frac{1}{p}}[\vec{w}]_{\mathrm{RH}_{\vec{p},\vec{r},\beta}}\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}.$$

Hence, $[\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW}}\lesssim(\beta')^{\frac{1}{p}}[\vec{w}]_{\mathrm{RH}_{\vec{p},\vec{r},\beta}}$, as desired. □

Proposition 3.3.10. *Let $\vec{r}\in(0,\infty)^m$, $\vec{p}\in(0,\infty]^m$ with $\vec{p}>\vec{r}$, let $\vec{w},v$ be $m+1$ weights, and let $\beta\in[1,\infty]$. If $\vec{w}\in\mathrm{RH}_{\vec{p},\vec{r},\beta}$, then*

$$\|M_{\vec{r}}\|_{L_{\vec{w}}^{\vec{p}}(\mathbf{R}^n)\to L_v^p(\mathbf{R}^n)}\lesssim_r c_{\vec{p},\vec{r}}(\beta')^{\frac{1}{p}}[\vec{w}]_{\mathrm{RH}_{\vec{p},\vec{r},\beta}}^{\max_{j\in J}\left\{\frac{\frac{1}{r_j}-\frac{1}{p_j}}{\frac{1}{p_j}}\right\}}[\vec{w},v]_{\vec{p},(\vec{r},\infty)}.$$

where $J := \{j \in \{1,\dots,m\} : p_j < \infty\}$ *and*

$$c_{\vec{p},\vec{r}} = \prod_{j=1}^{m} \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j} - \frac{1}{p_j}} \right)^{\frac{1}{r_j}}.$$

Proof. By the three lattice lemma, it suffices to prove the result in a dyadic grid $\mathscr{D} = \mathscr{D}^\alpha$ and by the Monotone Convergence Theorem we only need to consider finite collections $\mathscr{F} \subseteq \mathscr{D}$. Let $\vec{f} \in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$ and let $\mathscr{S} \subseteq \mathscr{F}$ be a sparse collection of cubes such that $M^{\mathscr{F}}_{\vec{r}}(\vec{f}\,) \lesssim_r \sum_{Q\in\mathscr{S}} \prod_{j=1}^m \langle f_j\rangle_{r_j,Q}\chi_{E_Q}$. Setting $v_{m+1} := v^p$, we have

$$\begin{aligned}
\|M^{\mathscr{F}}_{\vec{r}}(\vec{f}\,)\|_{L^p_v(\mathbf{R}^n)} &\lesssim_r \Big\| \sum_{Q\in\mathscr{S}} \prod_{j=1}^m \langle f_j\rangle_{r_j,Q}\chi_{E_Q} \Big\|_{L^p_v(\mathbf{R}^n)} = \left(\sum_{Q\in\mathscr{S}} \prod_{j=1}^m \langle f_j\rangle^p_{r_j,Q} v_{m+1}(E_Q) \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{Q\in\mathscr{S}} \prod_{j=1}^m \big(\langle f_j w_j\rangle_{\frac{1}{\frac{1}{\beta'}\frac{1}{r_j}+\frac{1}{\beta}\frac{1}{p_j}},Q} \langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{\beta}\left(\frac{1}{r_j}-\frac{1}{p_j}\right)},Q} \big)^p v_{m+1}(E_Q) \right)^{\frac{1}{p}} \qquad (3.3.5) \\
&\leq \left(\sum_{Q\in\mathscr{S}} \prod_{j=1}^m \langle f_j w_j\rangle^p_{\frac{1}{\frac{1}{\beta'}\frac{1}{r_j}+\frac{1}{\beta}\frac{1}{p_j}},Q} |Q| \langle v_{m+1}\rangle_{1,Q} \Big(\prod_{j=1}^m \langle v_j\rangle_{\beta,Q}^{\frac{1}{r_j}-\frac{1}{p_j}} \Big)^p \right)^{\frac{1}{p}},
\end{aligned}$$

where whenever $p_j = \infty$ we replace the estimate of $\langle f_j\rangle_{r_j,Q}$ by $\langle f_j\rangle_{r_j,Q} \leq \langle f_j w_j\rangle_{\infty,Q}\langle w_j^{-1}\rangle_{r_j,Q}$ so that in the final product we have $\langle v_j\rangle_{1,Q}^{\frac{1}{r_j}} = \langle v_j\rangle_{1,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}$ instead of $\langle v_j\rangle_{\beta,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}$.

Setting $\gamma := \max_{j\in J}\left\{ \frac{\frac{1}{r_j}-\frac{1}{p_j}}{\frac{1}{p_j}} \right\}$, we have $\gamma\frac{1}{p_j} - \left(\frac{1}{r_j}-\frac{1}{p_j}\right) \geq 0$ and thus $\langle v_j\rangle_{\beta,Q}^{\gamma\frac{1}{p_j}-\left(\frac{1}{r_j}-\frac{1}{p_j}\right)} \geq \langle v_j\rangle_{1,Q}^{\gamma\frac{1}{p_j}-\left(\frac{1}{r_j}-\frac{1}{p_j}\right)}$ for all $j \in J$. Then we have

$$\prod_{j\in J} \langle v_j\rangle_{\beta,Q}^{\frac{1}{r_j}-\frac{1}{p_j}} = \frac{\left(\prod_{j\in J}\langle v_j\rangle_{\beta,Q}^{\frac{1}{p_j}}\right)^\gamma}{\prod_{j\in J}\langle v_j\rangle_{\beta,Q}^{\gamma\frac{1}{p_j}-\left(\frac{1}{r_j}-\frac{1}{p_j}\right)}} \leq \frac{[\vec{w}]^\gamma_{\mathrm{RH}_{\vec{p},\vec{r},\beta}}\left(\prod_{j\in J}\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}\right)^\gamma}{\prod_{j\in J}\langle v_j\rangle_{1,Q}^{\gamma\frac{1}{p_j}-\left(\frac{1}{r_j}-\frac{1}{p_j}\right)}} = [\vec{w}]^\gamma_{\mathrm{RH}_{\vec{p},\vec{r},\beta}} \prod_{j\in J} \langle v_j\rangle_{1,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}.$$

Thus,

$$\begin{aligned}
\langle v_{m+1}\rangle_{1,Q}\Big(\prod_{j\in J}\langle v_j\rangle_{\beta,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}\Big)^p\Big(\prod_{j\notin J}\langle v_j\rangle_{1,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}\Big)^p &\leq [\vec{w}]^{\gamma p}_{\mathrm{RH}_{\vec{p},\vec{r},\beta}} \langle v_{m+1}\rangle_{1,Q}\Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q}^{\frac{1}{r_j}-\frac{1}{p_j}}\Big)^p \\
&\leq [\vec{w}]^{\gamma p}_{\mathrm{RH}_{\vec{p},\vec{r},\beta}} [\vec{w}, v]^p_{\vec{p},\vec{r}}.
\end{aligned}$$

Combining this with (3.3.5) yields

$$\begin{aligned}\|M^{\mathscr{F}}_{\vec r}(\vec f)\|_{L^p_v(\mathbf{R}^n)} &\lesssim_r [\vec w]^{\gamma}_{\mathrm{RH}_{\vec p,\vec r,\beta}}[\vec w,v]_{\vec p,(\vec r,\infty)}\Big(\sum_{Q\in\mathscr{S}}\Big(\prod_{j\in J}\langle f_j w_j\rangle_{\frac{1}{\frac{1}{\beta'}\frac{1}{r_j}+\frac{1}{\beta}\frac{1}{p_j}},Q}\prod_{j\notin J}\langle f_j w_j\rangle_{\infty,Q}\Big)^p|Q|\Big)^{\frac1p}\\
&\le 2[\vec w]^{\gamma}_{\mathrm{RH}_{\vec p,\vec r,\beta}}[\vec w,v]_{\vec p,(\vec r,\infty)}\|M^{\mathscr{D}}_{((\frac{1}{\frac{1}{\beta'}\frac{1}{r_j}+\frac{1}{\beta}\frac{1}{p_j}})_{j\in J},(\infty)_{j\notin J})}((f_jw_j)_{j\in J},(f_jw_j)_{j\notin J})\|_{L^p(\mathbf{R}^n)}\\
&\lesssim_r (\beta')^{\frac1p}[\vec w]^{\gamma}_{\mathrm{RH}_{\vec p,\vec r,\beta}}[\vec w,v]_{\vec p,(\vec r,\infty)}\prod_{j=1}^m\left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}\|f_j\|_{L^{p_j}_{w_j}},\end{aligned}$$

where in the last step we used Lemma 3.2.5 so that

$$\begin{aligned}\|M^{\mathscr{D}}_{\frac{1}{\frac{1}{\beta'}\frac{1}{r_j}+\frac{1}{\beta}\frac{1}{p_j}}}\|_{L^{p_j}(\mathbf{R}^n)\to L^{p_j}(\mathbf{R}^n)} &\le \left(\frac{\frac{1}{\beta'}\frac{1}{r_j}+\frac{1}{\beta}\frac{1}{p_j}}{\frac{1}{\beta'}\left(\frac{1}{r_j}-\frac{1}{p_j}\right)}\right)^{\frac{1}{\beta'}\frac{1}{r_j}+\frac{1}{\beta}\frac{1}{p_j}} \le (\beta')^{\frac{1}{\beta'}\frac{1}{r_j}+\frac{1}{\beta}\frac{1}{p_j}}\left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}\\
&\le e^{\frac1e\frac{1}{r_j}}(\beta')^{\frac{1}{p_j}}\left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}.\end{aligned}$$

for $j\in J$. The assertion follows. □

In view of this result, Conjecture 3.3.6 is a consequence of the following conjecture:

Conjecture 3.3.11. *Let $\vec r\in(0,\infty)^m$, $\vec p\in(0,\infty]^m$ with $\vec p>\vec r$, let $\vec w$ be an m-tuple of weights, and let $\beta\in[1,\infty]$. If $[\vec w]^{\mathrm{FW}}_{\vec p,\vec r}<\infty$ and $\beta'\ge 2^{n+1}([\vec w]^{\mathrm{FW}}_{\vec p,\vec r})^p$, then $\vec w\in\mathrm{RH}_{\vec p,\vec r,\beta}$ with*

$$[\vec w]_{RH_{\vec p,\beta}}\lesssim 2^{\frac{1}{\beta p}}.$$

This conjecture is motivated by the following weaker result.

Theorem 3.3.12. *Let $\vec r\in(0,\infty)^m$, $\vec p\in(0,\infty]^m$ with $\vec p>\vec r$, let $\vec w$ be an m-tuple of weights, let $\beta\in[1,\infty]$, and let $\mathscr{D}=\mathscr{D}^\alpha$ be a dyadic grid. If $[\vec w]^{\mathrm{FW},\mathscr{D}}_{\vec p,\vec r}<\infty$ and $\beta'\ge 2^{n+1}([\vec w]^{\mathrm{FW},\mathscr{D}}_{\vec p,\vec r})^p$, then*

$$\langle\prod_{j=1}^m v_j^{\frac{1}{p_j}}\rangle_{\beta p,Q}\le 2^{\frac{1}{\beta p}}\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}$$

for all $Q\in\mathscr{D}$.

For the proof we require a lemma.

Lemma 3.3.13. *Let $\vec r\in(0,\infty)^m$, $\vec p\in(0,\infty]^m$ with $\vec p>\vec r$, let $\vec w$ be an m-tuple of weights, let $\beta\in[1,\infty]$, and let $\mathscr{D}=\mathscr{D}^\alpha$ be a dyadic grid. For $\beta'\ge 2^{n+1}([\vec w]^{\mathrm{FW},\mathscr{D}}_{\vec p,\vec r})^p$ we have*

$$\sup_{Q\in\mathscr{D}}\frac{\langle M^{\mathscr{D}(Q)}_{\vec p}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})\rangle_{\beta p,Q}}{\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}}\le 2^{\frac{1}{\beta p}}([\vec w]^{\mathrm{FW},\mathscr{D}}_{\vec p,\vec r})^{\frac1\beta}.$$

Proof. Let $Q_0 \in \mathscr{D}$ and set $\Omega_\lambda := \{x \in Q_0 : M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})(x) > \lambda\}$ and

$$\mathrm{d}\mu = M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})^p \,\mathrm{d}x.$$

Then

$$\int_{Q_0} M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})^{\beta p} \,\mathrm{d}x = \int_0^\infty (\beta-1) p \lambda^{(\beta-1)p-1} \mu(\Omega_\lambda) \,\mathrm{d}\lambda.$$

If $\prod_{j=1}^m \langle v_j \rangle_{1,Q_0}^{\frac{1}{p_j}} > \lambda$ we have $\Omega_\lambda = Q_0$. Hence,

$$\begin{aligned}\int_0^{\prod_{j=1}^m \langle v_j \rangle_{1,Q_0}^{\frac{1}{p_j}}} (\beta-1) p \lambda^{(\beta-1)p-1} \mu(\Omega_\lambda) \,\mathrm{d}\lambda &= \left(\int_{Q_0} M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})^p \,\mathrm{d}x\right) \left(\prod_{j=1}^m \langle v_j \rangle_{1,Q_0}^{\frac{1}{p_j}}\right)^{(\beta-1)p} \\ &\le |Q_0| ([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p \left(\prod_{j=1}^m \langle v_j \rangle_{1,Q_0}^{\frac{1}{p_j}}\right)^{\beta p}.\end{aligned}$$

Moreover, when $\prod_{j=1}^m \langle v_j \rangle_{1,Q_0}^{\frac{1}{p_j}} \le \lambda$ the collection $\mathscr{P}_\lambda \subseteq \mathscr{D}(Q_0)$ of cubes Q that are maximal with respect to the inequality $\prod_{j=1}^m \langle v_j \rangle_{1,Q}^{\frac{1}{p_j}} > \lambda$ is pairwise disjoint, has each of its members strictly contained in Q_0, and satisfies $\Omega_\lambda = \bigcup_{Q \in \mathscr{P}_\lambda}$. Hence,

$$\begin{aligned}&\int_{\prod_{j=1}^m \langle v_j \rangle_{1,Q_0}^{\frac{1}{p_j}}}^\infty (\beta-1) p \lambda^{(\beta-1)p-1} \mu(\Omega_\lambda) \,\mathrm{d}\lambda \\ &\qquad = \int_{\prod_{j=1}^m \langle v_j \rangle_{1,Q_0}^{\frac{1}{p_j}}}^\infty (\beta-1) p \lambda^{(\beta-1)p-1} \sum_{Q \in \mathscr{P}_\lambda} \int_Q M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})^p \,\mathrm{d}x \,\mathrm{d}\lambda.\end{aligned} \tag{3.3.6}$$

By maximality of the $Q \in \mathscr{P}_\lambda$ we find that $M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})(x) = M_{\vec{p}}^{\mathscr{D}(Q)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})(x)$ for all $x \in Q$. Thus, denoting the dyadic parent of $Q \in \mathscr{P}_\lambda$ by $\widehat{Q}$, by maximality we have

$$\begin{aligned}\int_Q M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})^p \,\mathrm{d}x &\le ([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p \prod_{j=1}^m v_j(Q)^{\frac{p}{p_j}} \le ([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p |\widehat{Q}| \prod_{j=1}^m \langle v_j \rangle_{1,\widehat{Q}}^{\frac{p}{p_j}} \\ &\le 2^n \lambda^p ([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p |Q|.\end{aligned}$$

Hence, the right-hand side of (3.3.6) can be estimated by

$$\begin{aligned}([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p \int_{\prod_{j=1}^m \langle v_j \rangle_{1,Q_0}^{\frac{1}{p_j}}}^\infty (\beta-1) p \lambda^{\beta p-1} |\Omega_\lambda| \,\mathrm{d}\lambda &\le \frac{2^n ([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p}{\beta'} \int_{Q_0} M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})^{\beta p} \,\mathrm{d}x \\ &\le \frac{1}{2} \int_{Q_0} M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}})^{\beta p} \,\mathrm{d}x\end{aligned}$$

whenever $\beta' \ge 2^{n+1}([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p$. Collecting the results yields

$$\frac{1}{2} \langle M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}}, \dots, v_m^{\frac{1}{p_m}}) \rangle_{\beta p, Q_0}^{\beta p} \le ([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p \left(\prod_{j=1}^m \langle v_j \rangle_{1,Q_0}^{\frac{1}{p_j}}\right)^{\beta p}.$$

Thus,

$$\sup_{Q\in\mathscr{D}} \frac{\langle M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})\rangle_{\beta p,Q_0}}{\prod_{j=1}^m \langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}} \le 2^{\frac{1}{\beta p}}([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^{\frac{1}{\beta}},$$

whenever $\beta' \ge 2^{n+1}([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p$, as asserted. □

Proof of Theorem 3.3.12. Set $v_{\vec{p}} := \prod_{j=1}^m v_j^{\frac{p}{p_j}}$ and let $Q_0 \in \mathscr{D}$. Since $v_{\vec{p}} \le M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})^p$, we have

$$\int_{Q_0} v_{\vec{p}}^{\beta}\,\mathrm{d}x \le \int_{Q_0} M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})^{(\beta-1)p} v_{\vec{p}}\,\mathrm{d}x = \int_0^\infty (\beta-1)p\lambda^{(\beta-1)p-1} v_{\vec{p}}(\Omega_\lambda)\,\mathrm{d}\lambda \tag{3.3.7}$$

where $\Omega_\lambda = \{x\in Q_0 : M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})(x) > \lambda\}$. Then

$$\begin{aligned}\int_0^{\prod_{j=1}^m\langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}} (\beta-1)p\lambda^{(\beta-1)p-1} v_{\vec{p}}(\Omega_\lambda)\,\mathrm{d}\lambda &= |Q_0|\langle v_{\vec{p}}\rangle_{1,Q_0}\Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}\Big)^{(\beta-1)p} \\ &\le |Q_0|\Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}\Big)^{\beta p}\end{aligned} \tag{3.3.8}$$

and, with $\mathscr{P}_\lambda$ as in the proof of Lemma 3.3.13,

$$\int_{\prod_{j=1}^m\langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}}^\infty (\beta-1)p\lambda^{(\beta-1)p-1} v_{\vec{p}}(\Omega_\lambda)\,\mathrm{d}\lambda = \int_{\prod_{j=1}^m\langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}}^\infty (\beta-1)p\lambda^{(\beta-1)p-1} \sum_{Q\in\mathscr{P}_\lambda} v_{\vec{p}}(Q)\,\mathrm{d}\lambda, \tag{3.3.9}$$

where

$$\begin{aligned}\sum_{Q\in\mathscr{P}_\lambda} v_{\vec{p}}(Q) \le \sum_{Q\in\mathscr{P}_\lambda} |\widehat{Q}|\langle v_{\vec{p}}\rangle_{1,\widehat{Q}} &\le 2^n \sum_{Q\in\mathscr{P}_\lambda} |Q|\Big(\prod_{j=1}^m \langle v_j\rangle_{1,\widehat{Q}}^{\frac{1}{p_j}}\Big)^p \\ &\le 2^n\lambda^p \sum_{Q\in\mathscr{P}_\lambda} |Q| = 2^n\lambda^p|\Omega_\lambda|.\end{aligned}$$

Thus, (3.3.9) can be estimated by

$$2^n \int_{\prod_{j=1}^m\langle v_j\rangle_{1,Q}^{\frac{1}{p_j}}}^\infty (\beta-1)p\lambda^{\beta p-1}|\Omega_\lambda|\,\mathrm{d}\lambda \le \frac{2^n}{\beta'}\int_{Q_0} M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}},\dots,v_m^{\frac{1}{p_m}})^{\beta p}\,\mathrm{d}x.$$

Combining this with (3.3.8) and (3.3.7), it follows from Lemma 3.3.13 that

$$\begin{aligned}\int_{Q_0} v_{\vec{p}}^{\beta}\,\mathrm{d}x &\leq |Q_0|\Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}\Big)^{\beta p} + \frac{2^n}{\beta'}\int_{Q_0} M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}},\ldots,v_m^{\frac{1}{p_m}})^{\beta p}\,\mathrm{d}x\\ &= |Q_0|\Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}\Big)^{\beta p} + \frac{2^n}{\beta'}|Q_0|\langle M_{\vec{p}}^{\mathscr{D}(Q_0)}(v_1^{\frac{1}{p_1}},\ldots,v_m^{\frac{1}{p_m}})\rangle_{\beta p,Q_0}^{\beta p}\\ &\leq |Q_0|\Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}\Big)^{\beta p} + \frac{2^{n+1}([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p}{\beta'}|Q_0|\Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}\Big)^{\beta p}\\ &\leq 2|Q_0|\Big(\prod_{j=1}^m \langle v_j\rangle_{1,Q_0}^{\frac{1}{p_j}}\Big)^{\beta p}.\end{aligned}$$

whenever $\beta' \geq 2^{n+1}([\vec{w}]_{\vec{p},\vec{r}}^{\mathrm{FW},\mathscr{D}})^p$. This proves the result. □

Note that in the case $m = 1$, Theorem 3.3.12 implies Conjecture 3.3.11. Indeed, in this case we have

Corollary 3.3.14. *Let $r \in (0,\infty)$, $p \in (0,\infty]$ with $p > r$, let w be a weight, and let $\beta \in [1,\infty]$. If $[w]_{p,r}^{\mathrm{FW}} < \infty$ and $\beta' \geq 2^{n+1}([w]_{p,r}^{\mathrm{FW}})^p = 2^{n+1}[w^{-\frac{1}{\frac{1}{r}-\frac{1}{p}}}]_{A_\infty}$, then $w \in \mathrm{RH}_{p,r,\beta}$ with $[w]_{\mathrm{RH}_{p,r,\beta}} \lesssim 2^{\frac{1}{\beta p}}$ and hence,*

$$\langle w^{-\frac{1}{\frac{1}{r}-\frac{1}{p}}}\rangle_{\beta,Q} \lesssim 2\langle w^{-\frac{1}{\frac{1}{r}-\frac{1}{p}}}\rangle_{1,Q}.$$

for all cubes Q.

This result was proven in [HPR12] and was used to obtain a sharp self-improvement result for the Muckenhoupt classes. We show here that it can also be used to prove the following multilinear self-improvement result:

Proposition 3.3.15 (Self-improvement of multilinear weight classes)**.** *Let $\vec{w} \in A_{\vec{p},(\vec{r},s)}$. Then there is an $1 < \tilde{\alpha} < \min\{\frac{\vec{p}}{\vec{r}}\}$ such that for all $0 < \alpha \leq \tilde{\alpha}$ we have $\vec{w} \in A_{\vec{p},(\alpha\vec{r},s)}$ with*

$$[\vec{w}]_{\vec{p},(\alpha\vec{r},s)} \lesssim_{r,s} [\vec{w}]_{\vec{p},(\vec{r},s)}. \tag{3.3.10}$$

Proof. By Proposition 3.1.6 and Lemma 3.2.1 we have $w_j \in A_{\frac{1}{\frac{1}{p_j}-\frac{1}{\sigma_j}},(\frac{1}{\frac{1}{r_j}-\frac{1}{\sigma_j}},\infty)}$ with $\frac{1}{\sigma_j} = \frac{1}{r_j} - (\frac{1}{r} - \frac{1}{s})$. Hence, by Proposition 3.3.3 we have $[w_j]^{\mathrm{FW}}_{\frac{1}{\frac{1}{p_j}-\frac{1}{\sigma_j}},\frac{1}{\frac{1}{r_j}-\frac{1}{\sigma_j}}} < \infty$. Thus, since $\frac{1}{r_j} - \frac{1}{\sigma_j} - (\frac{1}{p_j} - \frac{1}{\sigma_j}) = \frac{1}{r_j} - \frac{1}{p_j}$, by Corollary 3.3.14 there exists a $\beta_j > 1$ such that

$$\langle w_j^{-1}\rangle_{\frac{\beta_j}{\frac{1}{r_j}-\frac{1}{p_j}},Q} \lesssim 2^{\frac{1}{r_j}-\frac{1}{p_j}}\langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q}.$$

We define $\tilde{\alpha} := \min_j \frac{\frac{1}{r_j}}{\frac{1}{\beta_j'}\frac{1}{p_j}+\frac{1}{\beta_j}\frac{1}{r_j}}$ so that $1 < \tilde{\alpha} < \min\{\frac{\vec{p}}{\vec{r}}\}$ and $\frac{\frac{1}{r_j}-\frac{1}{p_j}}{\frac{1}{\alpha r_j}-\frac{1}{p_j}} \le \beta_j$ for any $0 < \alpha \le \tilde{\alpha}$. Then

$$[\vec{w}]_{\vec{p},(\alpha\vec{r},s)} = \sup_Q \langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} \prod_{j=1}^m \langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{\alpha r_j}-\frac{1}{p_j}},Q} \le \sup_Q \langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} \prod_{j=1}^m \langle w_j^{-1}\rangle_{\frac{\beta_j}{\frac{1}{r_j}-\frac{1}{p_j}},Q}$$

$$\lesssim 2^{\frac{1}{r}-\frac{1}{s}} \sup_Q \langle w\rangle_{\frac{1}{\frac{1}{p}-\frac{1}{s}},Q} \prod_{j=1}^m \langle w_j^{-1}\rangle_{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}},Q} = 2^{\frac{1}{r}-\frac{1}{s}} [\vec{w}]_{\vec{p},(\vec{r},s)}.$$

The assertion follows. □

4

THE MULTILINEAR RUBIO DE FRANCIA ALGORITHM AND EXTRAPOLATION

In this chapter we prove the abstract version of the sharp multilinear limited range extrapolation theorem. This is based on the main result from the paper

> B. Nieraeth. Quantitative estimates and extrapolation for multilinear weight classes. *Mathematische Annalen*, 375(1-2):453–507, 2019.

4.1. MULTILINEAR EXTRAPOLATION

The main theorem of this chapter is as follows:

Theorem 4.1.1 (Quantitative multilinear limited range extrapolation)**.** *Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$, $t \in (0,s)$, and let $\vec{q} \in (0,\infty]^m$ satisfying $\vec{q} \geq \vec{r}$, $t \leq q \leq s$.*

Suppose we are given $\vec{p} \in (0,\infty]^m$ satisfying $\vec{p} > \vec{r}$, $t \leq p < s$, (or $p_j = q_j$ for some $j \in \{1,\dots,m\}$ or $p = q$), $\vec{w} \in A_{\vec{p},(\vec{r},s)}$, and $\vec{f} \in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$, $g \in L^{\frac{1}{\frac{1}{t}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)$. Then there is a $\vec{W} \in A_{\vec{q},(\vec{r},s)}$ such that

$$\left(\prod_{j=1}^m \|f_j\|_{L^{q_j}_{W_j}(\mathbf{R}^n)}\right)\|g\|_{L^{\frac{1}{\frac{1}{t}-\frac{1}{q}}}_{W^{-1}}(\mathbf{R}^n)} \leq 2^{\frac{m^2}{t}}\left(\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}\right)\|g\|_{L^{\frac{1}{\frac{1}{t}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)} \tag{4.1.1}$$

and

$$[\vec{W}]_{\vec{q},(\vec{r},s)} \leq C_{\vec{p},\vec{q},\vec{r},s}[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{r}-\frac{1}{p}},\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}. \tag{4.1.2}$$

We note that the addendum $p_j = q_j$ or $p = q$ is only relevant when we have equality in one of the components in $\vec{q} \geq \vec{r}$ or $q \leq s$, i.e., if $q = s$ or $q_j = r_j$ for some $j \in \{1,\dots,m\}$, and in this case we may indeed include the respective cases with $p = s$ or $p_j = r_j$ to the conclusions of the result. In this case one should respectively use the interpretation $\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}} = 1$ or $\frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}} = 1$.

Qualitatively, one can think of this result as the inclusion

$$\bigcup_{\vec{p}>\vec{r},\, p<s}\ \bigcup_{\vec{w}\in A_{\vec{p},(\vec{r},s)}} L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \subseteq \bigcap_{\vec{p}\geq\vec{r},\, p\leq s}\ \bigcup_{\vec{w}\in A_{\vec{p},(\vec{r},s)}} L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n). \tag{4.1.3}$$

After a rescaling argument, the extrapolation theorem follows from the symmetric version of the theorem below. In this theorem we deal with $m+1$-tuples as well as m-tuples of the same parameters, which can be notationally confusing. To circumvent this problem, we shall use the convention from Subsection 3.2.2 that for $m+1$ parameters $\alpha_1,\dots,\alpha_{m+1}$ we shall use the boldface notation $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_{m+1})$ for $m+1$-tuples while we will use the arrow notation $\vec{\alpha}=(\alpha_1,\dots,\alpha_m)$ for m-tuples.

Theorem 4.1.2. *Let* $\boldsymbol{r}\in(0,\infty)^{m+1}$ *and let* $\boldsymbol{q}\in(0,\infty]^{m+1}$ *satisfying* $\boldsymbol{q}\ge\boldsymbol{r}$ *and* $\sum_{j=1}^{m+1}\frac{1}{q_j}=1$.

Suppose we are given $\boldsymbol{p}\in(0,\infty]^{m+1}$ *satisfying* $\boldsymbol{p}>\boldsymbol{r}$ *(or* $p_j=q_j$*) and* $\sum_{j=1}^{m+1}\frac{1}{p_j}=1$*, an* $m+1$ *tuple of weights* $\boldsymbol{w}$ *satisfying* $\prod_{j=1}^{m+1}w_j=1$ *and* $\boldsymbol{w}\in A_{\boldsymbol{p},(\boldsymbol{r},\infty)}$*, and* $\boldsymbol{f}\in L^{\boldsymbol{p}}_{\boldsymbol{w}}(\mathbf{R}^n)$*. Then there is an* $m+1$*-tuple of weights* $\boldsymbol{W}$ *satisfying* $\prod_{j=1}^{m+1}W_j=1$ *and* $\boldsymbol{W}\in A_{\boldsymbol{q},(\boldsymbol{r},\infty)}$ *such that*

$$\prod_{j=1}^{m+1}\|f_j\|_{L^{q_j}_{W_j}(\mathbf{R}^n)}\le 2^{m^2}\prod_{j=1}^{m+1}\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)} \tag{4.1.4}$$

and

$$[\boldsymbol{W}]_{\boldsymbol{q},(\boldsymbol{r},\infty)}\le C_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}[\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\max\left\{\frac{\frac{1}{\boldsymbol{r}}-\frac{1}{\boldsymbol{q}}}{\frac{1}{\boldsymbol{r}}-\frac{1}{\boldsymbol{p}}}\right\}}. \tag{4.1.5}$$

The proof of this theorem relies on a multilinear generalization of the Rubio de Francia algorithm.

Lemma 4.1.3 (Multilinear Rubio de Francia algorithm)**.** *Let* $\vec{r}\in(0,\infty)^m$, $\vec{p}\in(0,\infty]^m$ *with* $\vec{r}<\vec{p}$*. Then for each* $\vec{w}\in A_{\vec{p},(\vec{r},\infty)}$ *there exist operators* $R_{p_j,r_j,\vec{w}}:L^{p_j}_{w_j}(\mathbf{R}^n)\to L^{p_j}_{w_j}(\mathbf{R}^n)$ *satisfying*

(i) $|f_j|\le R_{p_j,r_j,\vec{w}}f_j$;

(ii) $\|R_{p_j,r_j,\vec{w}}f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}\le 2\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}$;

(iii) $\prod_{j=1}^{m}\langle R_{p_j,r_j,\vec{w}}f_j\rangle_{r_j,Q}\lesssim_r c_{\vec{p},\vec{r}}[\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}}\inf_{y\in Q}\prod_{j=1}^{m}R_{p_j,r_j,\vec{w}}f_j(y)$ *for all cubes* Q*, where*

$$c_{\vec{p},\vec{r}}=\prod_{j=1}^{m}\left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}.$$

Remark 4.1.4. When $f_j\neq0$, then we note that $R_{p_j,r_j,\vec{w}}f_j$ is strictly positive. Setting $R_j:=(R_{p_j,r_j,\vec{w}}f_j)^{-1}$, we point out that property (iii) is then equivalent to the condition $\vec{R}\in A_{\vec{\infty},(\vec{r},\infty)}$ with

$$[\vec{R}]_{\vec{\infty},(\vec{r},\infty)}\lesssim_r c_{\vec{p},\vec{r}}[\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}}.$$

Proof. Letting $N_{p_j,r_j,\vec{w}}$ be as in Lemma 3.2.6, we define

$$R_{p_j,r_j,\vec{w}} f_j := \sum_{k=0}^{\infty} \frac{N^k_{p_j,r_j,\vec{w}}(f_j)}{2^k \|N_{p_j,r_j,\vec{w}}\|^k_{L^{p_j}_{w_j}(\mathbf{R}^n)\to L^{p_j}_{w_j}(\mathbf{R}^n)}},$$

where $N^0_{p_j,r_j,\vec{w}}(f_j) := |f_j|$ and $N^k_{p_j,r_j,\vec{w}}(f_j) := N_{p_j,r_j,\vec{w}}(N^{k-1}_{p_j,r_j,\vec{w}}(f_j))$.

To prove property (i), it suffices to note that the $k = 0$ term in the sum is equal to $|f_j|$.

For (ii) we have

$$\begin{aligned}
\|R_{p_j,r_j,\vec{w}} f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)} &\le \sum_{k=0}^{\infty} \frac{\|N^k_{p_j,r_j,\vec{w}}(f_j)\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}}{2^k \|N_{p_j,r_j,\vec{w}}\|^k_{L^{p_j}_{w_j}(\mathbf{R}^n)\to L^{p_j}_{w_j}(\mathbf{R}^n)}} \\
&\le \sum_{k=0}^{\infty} \frac{\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}}{2^k} = 2\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}.
\end{aligned}$$

To prove (iii), we first note that

$$\begin{aligned}
N_{p_j,r_j,\vec{w}}(R_{p_j,r_j,\vec{w}} f_j) &\le \sum_{k=0}^{\infty} \frac{N^{k+1}_{p_j,r_j,\vec{w}}(f_j)}{2^k \|N_{p_j,r_j,\vec{w}}\|^k_{L^{p_j}_{w_j}(\mathbf{R}^n)\to L^{p_j}_{w_j}(\mathbf{R}^n)}} \\
&\le 2\|N_{p_j,r_j,\vec{w}}\|_{L^{p_j}_{w_j}(\mathbf{R}^n)\to L^{p_j}_{w_j}(\mathbf{R}^n)} R_{p_j,r_j,\vec{w}} f_j.
\end{aligned}$$

Thus, it follows from Lemma 3.2.6 that

$$\begin{aligned}
M_{\vec{r}}(R_{p_1,r_1,\vec{w}} f_1, \dots, R_{p_m,r_m,\vec{w}} f_m) &\le [\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right\}} \prod_{j=1}^{m} N_{p_j,r_j,\vec{w}}(R_{p_j,r_j,\vec{w}} f_j) \\
&\lesssim_r 2^m c_{\vec{p},\vec{r}} [\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right\}} \prod_{j=1}^{m} R_{p_j,r_j,\vec{w}} f_j,
\end{aligned}$$

as desired. The assertion follows. □

Remark 4.1.5. We can obtain a more precise control in terms of the weight constant in (iii) in the case $m = 1$. Indeed, in this case we do not need to pass to the operators $N_{p,r,w}$ to define $R_{p,r,w}$, but we can instead define

$$R_{p,r,w} f := \sum_{k=0}^{\infty} \frac{M^k_r(f)}{2^k \|M_r\|^k_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)}}.$$

We now obtain (iii) with $c_{p,r}[w]_{p,(r,\infty)}^{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}}$ replaced by $\|M_r\|_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)}$. We could then use Theorem 3.3.5 instead of Theorem 3.2.3 to obtain a more precise control in terms of the weight.

Proof of Theorem 4.1.2. The proof will consist of two steps. In the first step we prove the result for very specific $\boldsymbol{q}$. In the second step we iterate the first step to obtain the desired result.

Step 1. In this step we assume that there is some $j_0 \in \{1,\dots,m+1\}$ such that

$$\frac{1}{p_{j_0}} < \frac{1}{q_{j_0}}, \qquad \frac{1}{p_j} \geq \frac{1}{q_j} \quad \text{for } j \neq j_0.$$

Since none of the statements in the formulation of the proposition depend on the order of the indices, we may assume without loss of generality that $j_0 = m+1$.

We define $\frac{1}{s} := 1 - \frac{1}{r_{m+1}} \geq 0$, $\frac{1}{p} := 1 - \frac{1}{p_{m+1}} > 0$, $\frac{1}{q} := 1 - \frac{1}{q_{m+1}} \geq 0$, and $w := w_{m+1}^{-1}$ so that $w = \prod_{j=1}^m w_j$. For an $m+1$-tuple $(\alpha_1,\dots,\alpha_{m+1})$ we will use the notation $\vec{\alpha} = (\alpha_1,\dots,\alpha_m)$ so that the arrow notation will always refer to an m-tuple. Thus, we have now reduced the problem to proving that there exist m weights $\vec{W} \in A_{\vec{q},(\vec{r},s)}$ such that $f_j \in L^{q_j}_{w_j}(\mathbf{R}^n)$, $f_{m+1} \in L^{q'}_{W^{-1}}(\mathbf{R}^n)$, where $W := \prod_{j=1}^m W_j$, with

$$\left(\prod_{j=1}^m \|f_j\|_{L^{q_j}_{W_j}(\mathbf{R}^n)}\right) \|f_{m+1}\|_{L^{q'}_{W^{-1}}(\mathbf{R}^n)} \leq 2^m \left(\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}\right) \|f_{m+1}\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)}, \tag{4.1.6}$$

and

$$[\vec{W}]_{\vec{q},(\vec{r},s)} \lesssim_{r,s} \left(\prod_{j=1}^m \left(\frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{q_j}}\right)^{\frac{1}{r_j}-\frac{1}{q_j}}\right)[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}}. \tag{4.1.7}$$

Indeed, the result then follows by setting $W_{m+1} := W^{-1}$ and by noting that

$$[\boldsymbol{W}]_{\boldsymbol{q},(\boldsymbol{r},\infty)} = [\vec{W}]_{\vec{q},(\vec{r},s)}, \qquad [\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)} = [\vec{w}]_{\vec{p},(\vec{r},s)}.$$

The construction of the m weights $W_1,\dots,W_m$ relies on the multilinear Rubio de Francia algorithm as well as a clever usage of the translation lemma to deal with the parameter s. Setting

$$\frac{1}{s_j} := \frac{\left(\frac{1}{p}-\frac{1}{s}\right)\frac{1}{q_j} - \left(\frac{1}{q}-\frac{1}{s}\right)\frac{1}{p_j}}{\frac{1}{p}-\frac{1}{q}},$$

we have

$$\frac{1}{s_j} \leq \frac{\left(\frac{1}{p}-\frac{1}{s}\right)\frac{1}{q_j} - \left(\frac{1}{q}-\frac{1}{s}\right)\frac{1}{q_j}}{\frac{1}{p}-\frac{1}{q}} = \frac{1}{q_j}$$

with equality if and only if $\frac{1}{q_j} = \frac{1}{p_j}$ and so that $\frac{1}{s_j} \leq \frac{1}{q_j} \leq \frac{1}{p_j}$, and

$$\sum_{j=1}^m \frac{1}{s_j} = \frac{\left(\frac{1}{p}-\frac{1}{s}\right)\frac{1}{q} - \left(\frac{1}{q}-\frac{1}{s}\right)\frac{1}{p}}{\frac{1}{p}-\frac{1}{q}} = \frac{1}{s}.$$

We set

$$\frac{1}{p_j(s)} := \frac{1}{p_j} - \frac{1}{s_j}, \qquad \frac{1}{q_j(s)} := \frac{1}{q_j} - \frac{1}{s_j}, \qquad \frac{1}{r_j(s)} := \frac{1}{r_j} - \frac{1}{s_j}$$

and $\frac{1}{p(s)} := \sum_{j=1}^{m} \frac{1}{p_j(s)} = \frac{1}{p} - \frac{1}{s}$, $\vec{p}(s) := (p_1(s), \ldots, p_m(s))$, and similarly for $\frac{1}{q(s)}$, $\vec{q}(s)$, and $\vec{r}(s)$.

We emphasize here that $\frac{1}{p_j(s)} = 0$ if and only if $\frac{1}{p_j} = \frac{1}{q_j}$ and we encourage the reader to verify that the remaining steps in this proof remain valid in this particular case.

We may compute

$$\frac{\frac{1}{p(s)} - \frac{1}{q(s)}}{\frac{1}{p(s)}} = \frac{\frac{1}{p_j} - \frac{1}{q_j}}{\frac{1}{p_j(s)}}, \qquad \frac{\frac{1}{q(s)}}{\frac{1}{p(s)}} = \frac{\frac{1}{q_j(s)}}{\frac{1}{p_j(s)}}. \tag{4.1.8}$$

We set $g_j := |f_j|^{\frac{\frac{1}{p_j(s)}}{\frac{1}{p_j}}} w_j^{-\frac{\frac{1}{s_j}}{\frac{1}{p_j}}}$ so that

$$\|g_j\|_{L^{p_j(s)}_{w_j}(\mathbf{R}^n)} = \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}^{\frac{\frac{1}{p_j(s)}}{\frac{1}{p_j}}}$$

and, using the notation from Lemma 4.1.3, we set $R_j := R_{p_j(s), r_j(s), \vec{w}}(g_j)^{-1}$ and

$$W_j := R_j^{\frac{\frac{1}{p(s)} - \frac{1}{q(s)}}{\frac{1}{p(s)}}} w_j^{\frac{\frac{1}{q(s)}}{\frac{1}{p(s)}}}.$$

By Lemma 4.1.3(i) we have $R_j \le |g_j|^{-1}$ so that by (4.1.8) we have

$$|f_j| W_j \le |g_j|^{\frac{\frac{1}{p_j}}{\frac{1}{p_j(s)}} - \frac{\frac{1}{p(s)} - \frac{1}{q(s)}}{\frac{1}{p(s)}}} w_j^{\frac{\frac{1}{s_j}}{\frac{1}{p_j(s)}} + \frac{\frac{1}{q(s)}}{\frac{1}{p(s)}}} = (g_j w_j)^{\frac{\frac{1}{q_j}}{\frac{1}{p_j(s)}}}.$$

Hence,

$$\|f_j\|_{L^{q_j}_{W_j}(\mathbf{R}^n)} \le \|g_j\|_{L^{p_j(s)}_{w_j}(\mathbf{R}^n)}^{\frac{\frac{1}{q_j}}{\frac{1}{p_j(s)}}} = \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}^{\frac{\frac{1}{q_j}}{\frac{1}{p_j}}}. \tag{4.1.9}$$

Next, it follows from (4.1.8), Hölder's inequality, and Lemma 4.1.3(ii) that

$$
\begin{aligned}
\|f_{m+1}\|_{L^{q'}_{W^{-1}}(\mathbf{R}^n)} &\le \|f_{m+1}w^{-1}\|_{L^{p'}}\|W^{-1}w\|_{L^{\frac{1}{\frac{1}{p}-\frac{1}{q}}}}\\
&= \|f_{m+1}\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)}\Big\|\Big(\prod_{j=1}^m R_{p_j(s),r_j(s),\vec{w}}(g_j)\Big)^{\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}} w^{\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}}\Big\|_{L^{\frac{1}{\frac{1}{p(s)}-\frac{1}{q(s)}}}}\\
&= \|f_{m+1}\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)}\Big\|\prod_{j=1}^m R_{p_j(s),r_j(s),\vec{w}}(g_j)\Big\|_{L^{p(s)}_w(\mathbf{R}^n)}^{\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}}\\
&\le \|f_{m+1}\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)}\prod_{j=1}^m\|R_{p_j(s),r_j(s),\vec{w}}(g_j)\|_{L^{p_j(s)}_{w_j}(\mathbf{R}^n)}^{\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}}\\
&\le 2^m\|f_{m+1}\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)}\prod_{j=1}^m\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}^{\frac{\frac{1}{p_j}-\frac{1}{q_j}}{\frac{1}{p_j}}}.
\end{aligned}
$$

By combining this estimate with (4.1.9), we have proven (4.1.6).

Finally, we prove (4.1.7). By Remark 4.1.4, Proposition 3.1.3(iii), and Lemma 3.2.1, we have

$$
\begin{aligned}
[\vec{W}]_{\vec{q},(\vec{r},s)} = [\vec{W}]_{\vec{q}(s),(\vec{r}(s),\infty)} &\le [\vec{R}]_{\vec{\infty},(\vec{r}(s),\infty)}^{\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}}[\vec{w}]_{\vec{p}(s),(\vec{r}(s),\infty)}^{\frac{\frac{1}{q(s)}}{\frac{1}{p(s)}}}\\
&\lesssim_{r,s} c_{\vec{p}(s),\vec{r}(s)}^{\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}}[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}(s)}}{\frac{1}{\vec{r}(s)}-\frac{1}{\vec{p}(s)}}\right\}\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}+\frac{\frac{1}{q(s)}}{\frac{1}{p(s)}}}.
\end{aligned}
\tag{4.1.10}
$$

Using (4.1.8), we compute

$$
\begin{aligned}
\frac{\frac{1}{r_j(s)}}{\frac{1}{r_j(s)}-\frac{1}{p_j(s)}}\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}+\frac{\frac{1}{q(s)}}{\frac{1}{p(s)}} &= \frac{\left(\frac{1}{p_j(s)}-\frac{1}{q_j(s)}\right)\frac{1}{r_j(s)}+\left(\frac{1}{r_j(s)}-\frac{1}{p_j(s)}\right)\frac{1}{q_j(s)}}{\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\frac{1}{p_j(s)}}\\
&= \frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}},
\end{aligned}
$$

which we interpret as being equal to 1 when $\frac{1}{q_j}=\frac{1}{p_j}=\frac{1}{r_j}$, so that

$$
\max\left\{\frac{\frac{1}{\vec{r}(s)}}{\frac{1}{\vec{r}(s)}-\frac{1}{\vec{p}(s)}}\right\}\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}+\frac{\frac{1}{q(s)}}{\frac{1}{p(s)}} = \max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}\right\}. \tag{4.1.11}
$$

Next, we compute

$$
\frac{1}{r_j}-\frac{1}{s_j} = \frac{\left(\frac{1}{r_j}-\frac{1}{q_j}\right)\frac{1}{p(s)}-\left(\frac{1}{r_j}-\frac{1}{p_j}\right)\frac{1}{q(s)}}{\frac{1}{p(s)}-\frac{1}{q(s)}} \le \left(\frac{1}{r_j}-\frac{1}{q_j}\right)\frac{\frac{1}{p(s)}}{\frac{1}{p(s)}-\frac{1}{q(s)}}.
$$

Hence,

$$
\begin{aligned}
c_{\vec{p}(s),\vec{r}(s)}^{\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}} &\le \prod_{j=1}^{m}\left(\frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}-\frac{1}{q_j}}\left(\frac{\frac{1}{p(s)}}{\frac{1}{p(s)}-\frac{1}{q(s)}}\right)^{\left(\frac{1}{r_j}-\frac{1}{s_j}\right)\frac{\frac{1}{p(s)}-\frac{1}{q(s)}}{\frac{1}{p(s)}}} \\
&\le e^{\frac{1}{e}\left(\frac{1}{r}-\frac{1}{s}\right)}\prod_{j=1}^{m}\left(\frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}-\frac{1}{q_j}}.
\end{aligned} \tag{4.1.12}
$$

Thus, combining (4.1.11) and (4.1.12) with (4.1.10) proves (4.1.7). This concludes Step 1.

Step 2. Now suppose $\boldsymbol{q}$ is arbitrary. For each j we either have $\frac{1}{p_j} < \frac{1}{q_j}$ or $\frac{1}{p_j} \ge \frac{1}{q_j}$. Assume without loss of generality that there is a $j_1 \in \{1,\dots,m\}$ such that

$$\frac{1}{p_j} \ge \frac{1}{q_j} \quad \text{if } j \in \{1,\dots,j_1\}, \qquad \frac{1}{p_j} < \frac{1}{q_j} \quad \text{if } j \in \{j_1+1,\dots,m+1\}. \tag{4.1.13}$$

Indeed, if this is not the case then, just as in Step 1, we may permute the indices to reduce back to this case.

The strategy will be to construct the $m+1$ weights $\boldsymbol{W}$ in $m-j_1+1$ steps through repeated application of Step 1.

We define

$$\theta_k := \begin{cases} \dfrac{\sum_{j=m-k+2}^{m+1} \frac{1}{q_j}-\frac{1}{p_j}}{\sum_{j=j_1+1}^{m+1} \frac{1}{q_j}-\frac{1}{p_j}} & \text{if } k \in \{1,\dots,m-j_1+1\}; \\ 0 & \text{if } k = 0, \end{cases}$$

so that $0 = \theta_0 \le \theta_1 \le \cdots \le \theta_{m-j_1+1} = 1$. Thus, defining,

$$\frac{1}{q_j^k} := \frac{1}{q_j} + \theta_k\left(\frac{1}{p_j}-\frac{1}{q_j}\right),$$

we have

$$\frac{1}{q_j} = \frac{1}{q_j^0} \le \frac{1}{q_j^1} \le \cdots \le \frac{1}{q_j^{m-j_1}} \le \frac{1}{q_j^{m-j_1+1}} = \frac{1}{p_j}.$$

Now, we define

$$
\begin{aligned}
\boldsymbol{q}^1 &:= (q_1^1,\dots,q_{j_1}^1, q_{j_1+1},\dots,q_m, p_{m+1}) \\
\boldsymbol{q}^2 &:= (q_1^2,\dots,q_{j_1}^2, q_{j_1+1},\dots,q_{m-1}, p_m, p_{m+1}) \\
&\vdots \\
\boldsymbol{q}^{m-j_1} &:= (q_1^{m-j_1},\dots,q_{j_1}^{m-j_1}, q_{j_1+1}, p_{j_1+2},\dots,p_{m+1}).
\end{aligned}
$$

First we will check that the reciprocals of the coordinates of these $m+1$-tuples sum to 1. Indeed, using $\sum_{j=1}^{m+1}\frac{1}{p_j}=\sum_{j=1}^{m+1}\frac{1}{q_j}=1$, we have

$$\begin{aligned}\sum_{j=1}^{j_1}\frac{1}{q_j^k}&=\sum_{j=1}^{j_1}\frac{1}{q_j}+\theta_k\sum_{j=1}^{j_1}\frac{1}{p_j}-\frac{1}{q_j}=\sum_{j=1}^{j_1}\frac{1}{q_j}+\theta_k\left(1-\sum_{j=j_1+1}^{m+1}\frac{1}{p_j}\right)-\theta_k\left(1-\sum_{j=j_1+1}^{m+1}\frac{1}{q_j}\right)\\&=\sum_{j=1}^{j_1}\frac{1}{q_j}+\sum_{j=m-k+2}^{m+1}\frac{1}{q_j}-\frac{1}{p_j}=1-\sum_{j=j_1+1}^{m-k+1}\frac{1}{q_j}-\sum_{j=m-k+2}^{m+1}\frac{1}{p_j}\end{aligned}$$

so that

$$\sum_{j=1}^{j_1}\frac{1}{q_j^k}+\sum_{j=j_1+1}^{m-k+1}\frac{1}{q_j}+\sum_{j=m-k+2}^{m+1}\frac{1}{p_j}=1,$$

as desired.

Now, for $k\in\{1,\dots,m-j_1+1\}$ we define

$$\gamma_k:=\max_{j=1,\dots,j_1}\left\{\frac{\frac{1}{r_j}-\frac{1}{q_j^{k-1}}}{\frac{1}{r_j}-\frac{1}{q_j^k}}\right\},$$

where the terms should be interpreted as being equal to 1 when $\frac{1}{q_j^k}=\frac{1}{r_j}$, and we write $\vec{q}^k=(q_1^k,\dots,q_m^k)$ for the m-tuple given by the first m coordinates of $\boldsymbol{q}^k$, with $\frac{1}{q^k}:=\sum_{j=1}^m\frac{1}{q_j^k}$.

We may apply Step 1 with $j_0=j_1+1$ to obtain weights $\boldsymbol{W}^{m-j_1}=(W_1^{m-j_1},\dots,W_{m+1}^{m-j_1})$ such that

$$\prod_{j=1}^{m+1}\|f_j\|_{L^{q_j^{m-j_1}}_{W_j^{m-j_1}}(\mathbf{R}^n)}\le 2^m\prod_{j=1}^{m+1}\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)} \tag{4.1.14}$$

and

$$[\boldsymbol{W}^{m-j_1}]_{\boldsymbol{q}^{m-j_1},(\boldsymbol{r},\infty)}\le C_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}[\boldsymbol{w}]^{\gamma_{m-j_1+1}}_{\boldsymbol{p},(\boldsymbol{r},\infty)}. \tag{4.1.15}$$

Next, we apply Step 1 with $j_0=j_1+2$ to obtain weights $\boldsymbol{W}^{m-j_1-1}$ with

$$\prod_{j=1}^{m+1}\|f_j\|_{L^{q_j^{m-j_1-1}}_{W_j^{m-j_1-1}}(\mathbf{R}^n)}\le 2^m\prod_{j=1}^{m+1}\|f_j\|_{L^{q_j^{m-j_1}}_{W_j^{m-j_1}}(\mathbf{R}^n)}$$

and

$$[\boldsymbol{W}^{m-j_1-1}]_{\boldsymbol{q}^{m-j_1-1},(\boldsymbol{r},\infty)}\le C_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}[\boldsymbol{W}^{m-j_1}]^{\gamma_{m-j_1}}_{\boldsymbol{q}^{m-j_1},(\boldsymbol{r},\infty)}.$$

Combining these estimates with (4.1.14) and (4.1.15) we obtain

$$\prod_{j=1}^{m+1}\|f_j\|_{L^{q_j^{m-j_1-1}}_{W_j^{m-j_1-1}}(\mathbf{R}^n}\le(2^m)^2\prod_{j=1}^{m+1}\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}$$

and

$$[\boldsymbol{W}^{m-j_1-1}]_{\boldsymbol{q}^{m-j_1-1},(\boldsymbol{r},\infty)} \leq C_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}[\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\gamma_{m-j_1}\gamma_{m-j_1+1}}.$$

Continuing this process, applying Step 1 with $j_0 = j_1 + k$ for $k = 3, \dots, m - j_1 + 1$, we conclude, setting $\boldsymbol{W} := \boldsymbol{W}^0$, that

$$\prod_{j=1}^{m+1} \|f_j\|_{L^{q_j}_{W_j}(\mathbf{R}^n)} = \prod_{j=1}^{m+1} \|f_j\|_{L^{q_j^0}_{W_j^0}(\mathbf{R}^n)} \leq (2^m)^{m-j_1+1} \prod_{j=1}^{m+1} \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)} \tag{4.1.16}$$

and

$$[\boldsymbol{W}]_{\boldsymbol{q},(\boldsymbol{r},\infty)} = [\boldsymbol{W}^0]_{\boldsymbol{q}^0,(\boldsymbol{r},\infty)} \leq C_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}}[\boldsymbol{w}]_{\boldsymbol{p},(\boldsymbol{r},\infty)}^{\prod_{k=1}^{m-j_1+1}\gamma_k}. \tag{4.1.17}$$

Since $(2^m)^{m-j_1+1} \leq 2^{m^2}$, we note that (4.1.4) now follows from (4.1.16). Finally, we note that (4.1.5) follows from (4.1.17), provided we can show that

$$\prod_{k=1}^{m-j_1+1} \gamma_k = \max_{j=1,\dots,m+1} \left\{ \frac{\frac{1}{r_j} - \frac{1}{q_j}}{\frac{1}{r_j} - \frac{1}{p_j}} \right\}. \tag{4.1.18}$$

Note that by our initial assumption (4.1.13), this maximum is attained at some $j_2 \in \{1, \dots, j_1\}$.

We claim that

$$\gamma_k = \frac{\frac{1}{r_{j_2}} - \frac{1}{q_{j_2}^{k-1}}}{\frac{1}{r_{j_2}} - \frac{1}{q_{j_2}^{k}}}$$

for all $k \in \{1, \dots, m - j_1 + 1\}$. Assuming for the moment that the claim is true, we find that

$$\prod_{k=1}^{m-j_1+1} \gamma_k = \prod_{k=1}^{m-j_1+1} \frac{\frac{1}{r_{j_2}} - \frac{1}{q_{j_2}^{k-1}}}{\frac{1}{r_{j_2}} - \frac{1}{q_{j_2}^{k}}} = \frac{\frac{1}{r_{j_2}} - \frac{1}{q_{j_2}^{0}}}{\frac{1}{r_{j_2}} - \frac{1}{q_{j_2}^{m-j_1+1}}} = \frac{\frac{1}{r_{j_2}} - \frac{1}{q_{j_2}}}{\frac{1}{r_{j_2}} - \frac{1}{p_{j_2}}},$$

proving (4.1.18).

To prove the claim, we compute

$$\begin{aligned}\frac{1}{r_j} - \frac{1}{q_j^k} &= \frac{1}{r_j} - \frac{1}{q_j} - \theta_k\left(\frac{1}{r_j} - \frac{1}{q_j}\right) + \theta_k\left(\frac{1}{r_j} - \frac{1}{p_j}\right)\\ &= \left(\frac{1}{r_j} - \frac{1}{p_j}\right)\left((1-\theta_k)\frac{\frac{1}{r_j} - \frac{1}{q_j}}{\frac{1}{r_j} - \frac{1}{p_j}} + \theta_k\right)\end{aligned}$$

so that

$$\frac{\frac{1}{r_j} - \frac{1}{q_j^{k-1}}}{\frac{1}{r_j} - \frac{1}{q_j^{k}}} = \frac{(1-\theta_{k-1})\frac{\frac{1}{r_j} - \frac{1}{q_j}}{\frac{1}{r_j} - \frac{1}{p_j}} + \theta_{k-1}}{(1-\theta_k)\frac{\frac{1}{r_j} - \frac{1}{q_j}}{\frac{1}{r_j} - \frac{1}{p_j}} + \theta_k} = \psi_k\left(\frac{\frac{1}{r_j} - \frac{1}{q_j}}{\frac{1}{r_j} - \frac{1}{p_j}}\right),$$

where

$$\psi_k(x) = \frac{(1-\theta_{k-1})x+\theta_{k-1}}{(1-\theta_k)x+\theta_k}.$$

We note that proving the claim is equivalent to proving the equality

$$\max_{j=1,\dots,m+1} \psi_k\left(\frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right) = \psi_k\left(\max_{j=1,\dots,m+1} \frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right).$$

The inequality

$$\psi_k\left(\max_{j=1,\dots,m+1} \frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right) = \psi_k\left(\frac{\frac{1}{r_{j_2}}-\frac{1}{q_{j_2}}}{\frac{1}{r_{j_2}}-\frac{1}{p_{j_2}}}\right) \le \max_{j=1,\dots,m+1} \psi_k\left(\frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)$$

is clear. To prove the converse inequality, it suffices to show that ψ_k is an increasing function for all $k \in \{1,\dots,m-j_1+1\}$. Computing

$$\begin{aligned}\psi_k'(x) &= \frac{(1-\theta_{k-1})((1-\theta_k)x+\theta_k)-(1-\theta_k)((1-\theta_{k-1})x+\theta_{k-1})}{((1-\theta_k)x+\theta_k)^2}\\ &= \frac{\theta_k-\theta_{k-1}}{((1-\theta_k)x+\theta_k)^2} \ge 0,\end{aligned}$$

we have proven the desired result. This concludes Step 2. The assertion follows. □

Proof of Theorem 4.1.1. By Proposition 3.1.3(ii), the result follows from applying Theorem 4.1.2 with $\boldsymbol{r} = (\frac{\vec{r}}{t}, (\frac{s}{t})')$, $\boldsymbol{q} = (\frac{\vec{q}}{t}, (\frac{q}{t})')$, $\boldsymbol{p} = (\frac{\vec{p}}{t}, (\frac{p}{t})')$, $\boldsymbol{w} = (\vec{w}^t, w^{-t})$, and $\boldsymbol{f} = (|\vec{f}|^t, |g|^t)$. □

3
QUANTITATIVE ESTIMATES FOR MULTILINEAR OPERATORS DOMINATED BY SPARSE FORMS

5

WEIGHTED BOUNDS FOR MULTILINEAR OPERATORS

This third part is dedicated to applying the theory we have developed so far to operators satisfying sparse domination.

This chapter is based on parts of the papers

B. Nieraeth. Quantitative estimates and extrapolation for multilinear weight classes. *Mathematische Annalen*, 375(1-2):453–507, 2019;

E. Lorist and B. Nieraeth. Sparse domination implies vector-valued sparse domination. arXiv:2003.02233, 2020,

with the exception of the third section, which is based on the optimality result in the paper

D. Frey and B. Nieraeth. Weak and Strong Type A_1–A_∞ Estimates for Sparsely Dominated Operators. *Journal of Geometric Analysis*, 29(1):247–282, 2019.

We point out that the result we present here is actually a new multilinear version of that result which can be obtained through a careful tracking of the constants in the proof of the extrapolation theorem in Chapter 4.

5.1. EXTRAPOLATION FOR MULTILINEAR OPERATORS

In this section we will be considering operators T defined on m-tuples of functions in the weighted Lebesgue spaces $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$. For fixed $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$ it is a consequence of the extrapolation theorem (in particular, of (4.1.3)) that if there is a $\vec{p} \in (0,\infty]^m$ with $\vec{p} \geq \vec{r}$, $p \leq s$, and T is defined on $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$ for all $\vec{w} \in A_{\vec{p},(\vec{r},s)}$, then T is actually defined on $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$ for all $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$, $p < s$, and $\vec{w} \in A_{\vec{p},(\vec{r},s)}$. Thus, for the results in this section we will not need to assume any additional structure on T such as (sub)linearity in its components. These notions will come in to play in the next section where we will be considering operators satisfying sparse domination.

Definition 5.1.1. Let $\vec{U}$, V be $m+1$ quasi-normed linear subspaces of $L^0(\mathbf{R}^n)$ and $T : \vec{U} \to V$. We say that T is *bounded* when there is a constant $c \geq 0$ such that for all $\vec{f} \in \vec{U}$ we have $\|T(\vec{f})\|_V \leq c\prod_{j=1}^m \|f_j\|_{U_j}$. The smallest possible c is denoted by $\|T\|_{\vec{U}\to V}$.

Using the extrapolation result Theorem 3.2.11 we can give a detailed quantitative bound for the operators under consideration. Using Fubini's Theorem we can also extend the extrapolation theorem to a vector-valued setting. In the following result we are considering the space $L^p_w(\mathbf{R}^n; L^t(\Omega))$ for $p, t \in (0,\infty]$, a weight w, and Ω a σ-finite measure space. This space consists of the measurable functions $f : \mathbf{R}^n \times \Omega \to \mathbf{C}$ such that the function $x \mapsto \|f(x,\cdot)\|_{L^t(\Omega)}$ lies in $L^p_w(\mathbf{R}^n)$, with $\|f\|_{L^p_w(\mathbf{R}^n;L^t(\Omega))} := \big\| x \mapsto \|f(x,\cdot)\|_{L^t(\Omega)} \big\|_{L^p_w(\mathbf{R}^n)}$. In the case when $p = t$, we can use Fubini's Theorem to find that

$$\|f\|_{L^t_w(\mathbf{R}^n;L^t(\Omega))} = \big\| \|f\|_{L^t_w(\mathbf{R}^n)} \big\|_{L^t(\Omega)},$$

valid for any $t \in (0,\infty]$, allowing us to carry over scalar-valued estimates to this vector-valued setting.

Theorem 5.1.2 (Multilinear Rubio de Francia extrapolation). *Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$, $\vec{q} \in (0,\infty]^m$ with $\vec{q} \geq \vec{r}$, $q \leq s$, and let T be an operator that is bounded $L^{\vec{q}}_{\vec{w}}(\mathbf{R}^n) \to L^q_w(\mathbf{R}^n)$ for all $\vec{w} \in A_{\vec{q},(\vec{r},s)}$. Moreover, suppose that there exists an increasing function $\phi_{\vec{q}}$ such that*

$$\|T\|_{L^{\vec{q}}_{\vec{w}}(\mathbf{R}^n) \to L^q_w(\mathbf{R}^n)} \leq \phi_{\vec{q}}([\vec{w}]_{\vec{q},(\vec{r},s)}) \tag{5.1.1}$$

for all $\vec{w} \in A_{\vec{q},(\vec{r},s)}$.

Then for all $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$, $p < s$, (or $p_j = q_j$ for some $j \in \{1,\dots,m\}$ or $p = q$), all weights $\vec{w} \in A_{\vec{p},(\vec{r},s)}$, T is bounded $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)$ with

$$\|T\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)} \leq 2^{\frac{m^2}{r}} \phi_{\vec{q}}\Big(C_{\vec{p},\vec{q},\vec{r},s} [\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{ \frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}} \right\}} \Big). \tag{5.1.2}$$

Moreover, suppose (Ω,μ) is a σ-finite measure space, $\vec{t} \in (0,\infty]^m$ with $\vec{t} > \vec{r}$ and $t < s$ (or $t_j = q_j$, $t = q$). Then

$$\widetilde{T}(\vec{f}\,)(x,\omega) := T(\vec{f}(\cdot,\omega))(x)$$

is well-defined for all $f_j \in L^{p_j}_{w_j}(\mathbf{R}^n; L^{t_j}(\Omega))$. Moreover, for all $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$, $p < s$, (or $p_j = q_j$, $p = q$), all weights $\vec{w} \in A_{\vec{p},(\vec{r},s)}$, and all $f_j \in L^{p_j}_{w_j}(\mathbf{R}^n; L^{t_j}(\Omega))$ for which $\widetilde{T}(\vec{f}\,)$ is measurable,

$$\|\widetilde{T}(\vec{f}\,)\|_{L^p_w(\mathbf{R}^n;L^t(\Omega))} \leq 2^{2\frac{m^2}{r}} \phi_{\vec{q}}\Big(C_{\vec{p},\vec{q},\vec{r},s,\vec{t}} [\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{ \frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{t}-\frac{1}{s}} \right\} \cdot \max\left\{ \frac{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}, \frac{\frac{1}{t}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}} \right\}} \Big) \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;L^{t_j}(\Omega))}. \tag{5.1.3}$$

We again note that the addendum $p_j = q_j$, $p = q$ is only relevant when we have equality in any of the components in $\vec{q} \geq \vec{r}$ or $q \leq s$, i.e., if $q = s$ or $q_j = r_j$ for some $j \in \{1,\dots,m\}$, in which case we may indeed include the respective cases with $p = s$ or $p_j = r_j$ to the conclusions of the result. In this case one should respectively use the interpretation $\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}} = 1$ or $\frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}} = 1$.

Remark 5.1.3. In certain specific cases we have a precise control of the constant $C_{\vec{p},\vec{q},\vec{r},s}$ in (5.1.2). Indeed, the proof is based on the extrapolation theorem, Theorem 4.1.2, and in Step 1 of the proof of this result we computed a precise control of this constant in (4.1.7). More precisely, in Step 1 of this proof we have the following situations:

(i) If there is a $j_0 \in \{1,\dots,m\}$ such that $\frac{1}{p_{j_0}} < \frac{1}{q_{j_0}}$, $\frac{1}{p_j} \geq \frac{1}{q_j}$ for all $j \neq j_0$, and $\frac{1}{p} \leq \frac{1}{q}$, then

$$C_{\vec{p},\vec{q},\vec{r},s} \lesssim_{r,s} \Big(\prod_{\substack{j=1 \\ j\neq j_0}}^m \Big(\frac{\frac{1}{r_j} - \frac{1}{q_j}}{\frac{1}{r_j} - \frac{1}{p_j}} \Big)^{\frac{1}{r_j} - \frac{1}{q_j}} \Big) \Big(\frac{\frac{1}{q} - \frac{1}{s}}{\frac{1}{p} - \frac{1}{s}} \Big)^{\frac{1}{q} - \frac{1}{s}}.$$

(ii) If $\frac{1}{p} > \frac{1}{q}$ and $\frac{1}{p_j} \geq \frac{1}{q_j}$ for all $j \in \{1,\dots,m\}$, then

$$C_{\vec{p},\vec{q},\vec{r},s} \lesssim_{r,s} \prod_{j=1}^m \Big(\frac{\frac{1}{r_j} - \frac{1}{q_j}}{\frac{1}{r_j} - \frac{1}{p_j}} \Big)^{\frac{1}{r_j} - \frac{1}{q_j}}.$$

Remark 5.1.4. We point out that the measurable assumption on $\widetilde{T}$ is redundant when T is m-linear. Indeed, any element of $L^{p_j}_{w_j}(\mathbf{R}^n) \otimes L^{t_j}(\Omega)$ is spanned by functions of the form $(f_j \otimes \xi_j)(x,\omega) = f_j(x)\xi_j(\omega)$ for $f_j \in L^{p_j}_{w_j}(\mathbf{R}^n)$, $\xi_j \in L^{t_j}(\Omega)$, and on these functions $\widetilde{T}$ coincides with the tensor extension of T, i.e.,

$$\widetilde{T}(\vec{f} \otimes \vec{\xi}\,) = T(\vec{f}\,) \otimes \prod_{j=1}^m \xi_j$$

is measurable in $\mathbf{R}^n \times \Omega$. Since $L^{p_j}_{w_j}(\mathbf{R}^n) \otimes L^{t_j}(\Omega)$ is dense in $L^{p_j}_{w_j}(\mathbf{R}^n; L^{t_j}(\Omega))$, measurability of $\widetilde{T}(\vec{f}\,)$ for general $f_j \in L^{p_j}_{w_j}(\mathbf{R}^n; L^{t_j}(\Omega))$ follows from an approximation argument, see also Lemma 5.3.2 below.

Remark 5.1.5. Let $(\Omega_1,\mu_1),\dots,(\Omega_K,\mu_K)$ be σ-finite measure spaces and for $j \in \{1,\cdots,m\}$ we set $X_j := L^{t^K_j}(\Omega_K;\cdots;L^{t^1_j}(\Omega_1)))$ for $\vec{t}^k > \vec{r}$ and $t^k < s$ (or $t^k_j = q_j$, $t^k = q$) as in the theorem. By induction it is also possible to obtain vector-valued estimates for T in the theorem for functions in the spaces $L^{p_j}(\mathbf{R}^n; X_j)$. We however do not pursue this further here, since this method does not give us optimal quantitative weighted bounds. In Chapter 9.2 we provide a different method that gives sharp vector-valued weighted bounds for operators satisfying sparse domination and, in particular, we can replace the exponent of the weight constant in (5.1.3) in certain instances by the smaller exponent

$$\max\left\{ \frac{\frac{1}{\vec{r}} - \frac{1}{\vec{q}}}{\frac{1}{\vec{r}} - \frac{1}{\vec{p}}}, \frac{\frac{1}{q} - \frac{1}{s}}{\frac{1}{p} - \frac{1}{s}} \right\}.$$

Proof of Theorem 5.1.2. Fix $\vec{w} \in A_{\vec{p},(\vec{r},s)}$ and $\vec{f} \in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$, $g \in L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)$. By applying Theorem 4.1.1 with $t = r$ we can pick a $\vec{W} \in A_{\vec{q},(\vec{r},s)}$ such that

$$\Big(\prod_{j=1}^m \|f_j\|_{L^{q_j}_{W_j}(\mathbf{R}^n)} \Big) \|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{q}}}_{W^{-1}}(\mathbf{R}^n)} \leq 2^{\frac{m^2}{r}} \Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)} \Big) \|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)}$$

and

$$[\vec{W}]_{\vec{q},(\vec{r},s)} \le C_{\vec{p},\vec{q},\vec{r},s}[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}},\frac{\frac{1}{\vec{q}}-\frac{1}{s}}{\frac{1}{\vec{p}}-\frac{1}{s}}\right\}}.$$

Hence, we obtain

$$\begin{aligned}\|T(\vec{f})\cdot g\|_{L^r(\mathbf{R}^n)} &\le \phi_{\vec{q}}([\vec{W}]_{\vec{q},(\vec{r},s)})\Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}_{W_j}(\mathbf{R}^n)}\Big)\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{W^{-1}}(\mathbf{R}^n)}\\ &\le 2^{\frac{m^2}{r}}\phi_{\vec{q}}(C_{\vec{p},\vec{q},\vec{r},s}[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}},\frac{\frac{1}{\vec{q}}-\frac{1}{s}}{\frac{1}{\vec{p}}-\frac{1}{s}}\right\}})\Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}\Big)\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)}.\end{aligned}$$

The assertion now follows from the duality result

$$\|T(\vec{f})\|_{L^p_w(\mathbf{R}^n)} = \||T(\vec{f})|^r\|^{\frac{1}{r}}_{L^{\frac{p}{r}}_{w^r}(\mathbf{R}^n)} = \sup_{\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)}=1}\|T(\vec{f})\cdot g\|_{L^r(\mathbf{R}^n)}.$$

For the second result, fix $\vec{w}\in A_{\vec{p},(\vec{r},s)}$, $f_j\in L^{p_j}_{w_j}(\mathbf{R}^n;L^{t_j}(\Omega))$ for all $j\in\{1,\dots,m\}$, and $g\in L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)$. Then by applying Theorem 4.1.1 with $q_j=t_j$, $t=r$, and the f_j replaced by $\|f_j\|_{L^{t_j}(\Omega)}\in L^{p_j}_{w_j}(\mathbf{R}^n)$, we can pick a $\vec{W}\in A_{\vec{t},(\vec{r},s)}$ such that, by Fubini's Theorem,

$$\begin{aligned}\Big(\prod_{j=1}^m \|f_j\|_{L^{t_j}_{W_j}(\mathbf{R}^n;L^{t_j}(\Omega))}\Big)\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{t}}}_{W^{-1}}(\mathbf{R}^n)} &= \Big(\prod_{j=1}^m \|x\mapsto\|f_j(x,\cdot)\|_{L^{t_j}(\Omega)}\|_{L^{t_j}_{W_j}(\mathbf{R}^n)}\Big)\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{t}}}_{W^{-1}}(\mathbf{R}^n)}\\ &\le 2^{\frac{m^2}{r}}\Big(\prod_{j=1}^m \|x\mapsto\|f_j(x,\cdot)\|_{L^{t_j}(\Omega)}\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}\Big)\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)}\\ &= 2^{\frac{m^2}{r}}\Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;L^{t_j}(\Omega))}\Big)\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)}\end{aligned} \tag{5.1.4}$$

and

$$[\vec{W}]_{\vec{t},(\vec{r},s)} \le C_{\vec{p},\vec{r},s,\vec{t}}[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}},\frac{\frac{1}{\vec{t}}-\frac{1}{s}}{\frac{1}{\vec{p}}-\frac{1}{s}}\right\}}.$$

In particular we note that by Fubini's Theorem, we have that $\omega\mapsto\|f_j(\cdot,\omega)\|_{L^{t_j}_{W_j}(\mathbf{R}^n)}$ lies in $L^{t_j}(\Omega)$ and is therefore finite a.e. This implies that $f_j(\cdot,\omega)\in L^{t_j}_{W_j}(\mathbf{R}^n)$ for a.e. $\omega\in\Omega$ and thus $\widetilde{T}(\vec{f})$ is well-defined. Now, assuming that $\widetilde{T}(\vec{f})$ is measurable in $\mathbf{R}^n\times\Omega$, it follows from (5.1.2) with $\vec{p} = \vec{t}$, (5.1.4), Fubini's Theorem, and Hölder's inequality, that

$$
\begin{aligned}
\|\|\widetilde{T}(\vec{f}\,)\|_{L^t(\Omega)} \cdot g\|_{L^r(\mathbf{R}^n)} &\le \|\omega \mapsto \|T(\vec{f}(\cdot,\omega)\,)\|_{L^t_W(\mathbf{R}^n)}\|_{L^t(\Omega)} \|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{t}}}_{W^{-1}}(\mathbf{R}^n)} \\
&\le 2^{\frac{m^2}{r}} \phi_{\vec{q}}(C_{\vec{q},\vec{r},s,\vec{t}}[\vec{W}]_{\vec{t},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{t}-\frac{1}{s}}\right\}}) \Big\|\omega \mapsto \Big(\prod_{j=1}^m \|f_j(\cdot,\omega)\|_{L^{t_j}_{W_j}(\mathbf{R}^n)}\Big)\Big\|_{L^t(\Omega)} \|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{t}}}_{W^{-1}}(\mathbf{R}^n)} \\
&\le 2^{\frac{m^2}{r}} \phi_{\vec{q}}(C_{\vec{p},\vec{q},\vec{r},s,\vec{t}}[\vec{w}]_{\vec{t},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{t}-\frac{1}{s}}\right\}\cdot\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}, \frac{\frac{1}{t}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}) \Big(\prod_{j=1}^m \|f_j\|_{L^{t_j}_{W_j}(\mathbf{R}^n;L^{t_j}(\Omega))}\Big) \|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{t}}}_{W^{-1}}(\mathbf{R}^n)} \\
&\le 2^{2\frac{m^2}{r}} \phi_{\vec{q}}(C_{\vec{p},\vec{q},\vec{r},s,\vec{t}}[\vec{w}]_{\vec{t},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{t}-\frac{1}{s}}\right\}\cdot\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}, \frac{\frac{1}{t}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}) \Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;L^{t_j}(\Omega))}\Big) \|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)} .
\end{aligned}
$$

The assertion now follows from the duality result

$$
\|\widetilde{T}(\vec{f}\,)\|_{L^p_w(\mathbf{R}^n;L^t(\Omega))} = \|\|\widetilde{T}(\vec{f}\,)\|^r_{L^t(\Omega)}\|^{\frac{1}{r}}_{L^{\frac{p}{r}}_{w^r}(\mathbf{R}^n)} = \sup_{\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)}=1} \|\|T(\vec{f}\,)\|_{L^t(\Omega)} \cdot g\|_{L^r(\mathbf{R}^n)}.
$$

□

In applying the extrapolation theorem, one can obtain further results by making appropriate choices for the initial operator T. The following is an extrapolation result involving weak-type estimates. The trick used to obtain this result is well-known and can be found already in [GM04].

Theorem 5.1.6 (Weak type extrapolation)**.** *Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$, $\vec{q} \in (0,\infty]^m$ with $\vec{q} \ge \vec{r}$, $q \le s$, and let T be an operator that is bounded $L^{\vec{q}}_{\vec{w}}(\mathbf{R}^n) \to L^{q,\infty}_w(\mathbf{R}^n)$ for all $\vec{w} \in A_{\vec{q},(\vec{r},s)}$. Moreover, suppose that there exists an increasing function $\phi_{\vec{q}}$ such that*

$$
\|T\|_{L^{\vec{q}}_{\vec{w}}(\mathbf{R}^n)\to L^{q,\infty}_w(\mathbf{R}^n)} \le \phi_{\vec{q}}([\vec{w}]_{\vec{q},(\vec{r},s)}) \tag{5.1.5}
$$

for all $\vec{w} \in A_{\vec{q},(\vec{r},s)}$.

Then for all $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$, $p < s$, (or $p_j = q_j$, $p = q$), all weights $\vec{w} \in A_{\vec{p},(\vec{r},s)}$, T is bounded $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^{p,\infty}_w(\mathbf{R}^n)$ with

$$
\|T\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^{p,\infty}_w(\mathbf{R}^n)} \le 2^{\frac{m^2}{r}} \phi_{\vec{q}}\Big(C_{\vec{p},\vec{q},\vec{r},s}[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{q}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}\Big). \tag{5.1.6}
$$

Proof. Let $\lambda > 0$ and for $\vec{f} \in L^{\vec{q}}_{\vec{w}}(\mathbf{R}^n)$ we set $E_\lambda := \{x \in \mathbf{R}^n : |T(\vec{f}\,)(x)| > \lambda\}$. Define

$$
T_\lambda(\vec{f}\,) := \lambda \chi_{E_\lambda}
$$

and note that by (5.1.5) we have

$$\|T_\lambda(\vec{f})\|_{L^q_w(\mathbf{R}^n)} = \lambda\big(w^q(E_\lambda)\big)^{\frac{1}{q}} \le \|T(\vec{f})\|_{L^{q,\infty}(w^q)} \le \phi_{\vec{q}}([\vec{w}]_{\vec{q},(\vec{r},s)}) \prod_{j=1}^m \|f_j\|_{L^{q_j}_{w_j}(\mathbf{R}^n)}.$$

Thus, by applying Theorem 4.1.1 with T replaced by T_λ we find that

$$\|T_\lambda(\vec{f})\|_{L^p_w(\mathbf{R}^n)} \le 2^{\frac{m^2}{r}} \phi_{\vec{q}}\Big(C_{\vec{p},\vec{q},\vec{r},s}[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{r}-\frac{1}{p}},\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}\Big) \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}$$

for all $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$, $p < s$, (or $p_j = q_j$, $p = q$), all weights $\vec{w} \in A_{\vec{p},(\vec{r},s)}, \vec{w} \in A_{\vec{p},(\vec{r},s)}$, and all $\vec{f} \in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$. As $\lambda > 0$ was arbitrary, noting that $\sup_{\lambda>0}\|T_\lambda(\vec{f})\|_{L^p_w(\mathbf{R}^n)} = \|T(\vec{f})\|_{L^{p,\infty}_w(\mathbf{R}^n)}$ proves (5.1.6). The assertion follows. □

As a consequence we can extrapolate from weak lower endpoint estimates in cases where strong bounds are not available. Writing $\vec{1}$ for the vector consisting of m components all equal to 1, passing to the full-range case where $\vec{r} = \vec{1}$ and $s = \infty$, we obtain the following corollary:

Corollary 5.1.7. *Let T be an operator that is bounded $L^{\vec{1}}_{\vec{w}}(\mathbf{R}^n) \to L^{\frac{1}{m},\infty}_w(\mathbf{R}^n)$ for all $\vec{w} \in A_{\vec{1},(\vec{1},\infty)}$. Moreover, suppose there is an increasing function ϕ such that*

$$\|T\|_{L^{\vec{1}}_{\vec{w}}(\mathbf{R}^n)\to L^{\frac{1}{m},\infty}_w(\mathbf{R}^n)} \le \phi([\vec{w}]_{\vec{1},(\vec{1},\infty)})$$

for all $\vec{w} \in A_{\vec{1},(\vec{1},\infty)}$.

Then for all $\vec{p} \in [1,\infty]^m$ with $p < \infty$ and $\vec{w} \in A_{\vec{p},(\vec{1},\infty)}$, T is bounded $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^{p,\infty}_w(\mathbf{R}^n)$ with

$$\|T\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^{p,\infty}_w(\mathbf{R}^n)} \le 2^{m^3}\phi\Big(C_{\vec{p}}[\vec{w}]^{pm}_{\vec{p},(\vec{1},\infty)}\Big).$$

We can also extrapolate from the upper endpoints. An application of Theorem 4.1.1 in the case $s = \infty$ with $\vec{q} = \vec{\infty}$, where $\vec{\infty}$ is the vector consisting of m components all equal to ∞, together with Remark 5.1.3 yields the following:

Theorem 5.1.8 (Upper endpoint extrapolation)**.** *Let $\vec{r} \in (0,\infty)^m$ and let T be an operator that is bounded $L^{\vec{\infty}}_{\vec{w}}(\mathbf{R}^n) \to L^{\infty}_w(\mathbf{R}^n)$ for all $\vec{w} \in A_{\vec{\infty},(\vec{r},\infty)}$. Moreover, suppose there is an increasing function ϕ such that*

$$\|T\|_{L^{\vec{\infty}}_{\vec{w}}(\mathbf{R}^n)\to L^{\infty}_w(\mathbf{R}^n)} \le \phi([\vec{w}]_{\vec{\infty},(\vec{r},\infty)})$$

for all $\vec{w} \in A_{\vec{\infty},(\vec{r},\infty)}$.

Then for all $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$,

$$\|T\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^{p}_w(\mathbf{R}^n)} \le 2^{\frac{m}{r}}\phi\Big(C_{\vec{p},\vec{r}}[\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right\}}\Big).$$

where

$$C_{\vec{p},\vec{r}} \lesssim_r \prod_{j=1}^{m} \Big(\frac{\frac{1}{r_j}}{\frac{1}{r_j} - \frac{1}{p_j}} \Big)^{\frac{1}{r_j}}.$$

We point out that this result is the simplest case in the extrapolation result. Indeed, in this case we have $\vec{W} = \vec{R}$, where $\vec{R}$ are the weights from Remark 4.1.4 obtained directly from the multilinear Rubio de Francia algorithm.

An interesting application of this result is related to the space $\mathrm{BMO}(\mathbf{R}^n)$ of functions of bounded mean oscillation. We define the sharp maximal operator $M^{\#}$ by

$$M^{\#} f = \sup_{Q} \langle |f - \langle f \rangle_{1,Q}| \rangle_{1,Q} \chi_Q$$

for $f \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$, where the supremum is taken over all cubes $Q \subseteq \mathbf{R}^n$. The definition of $\mathrm{BMO}(\mathbf{R}^n)$ in the unweighted setting can be given in terms of $M^{\#}$ by saying a measurable function f is in $\mathrm{BMO}(\mathbf{R}^n)$ if $M^{\#} f \in L^\infty$, with $\|f\|_{\mathrm{BMO}(\mathbf{R}^n)} := \|M^{\#} f\|_{L^\infty(\mathbf{R}^n)}$. This suggests the following definition of a weighted version of the $\mathrm{BMO}(\mathbf{R}^n)$ space:

Definition 5.1.9. Given a weight w, we define the space $\mathrm{BMO}_w(\mathbf{R}^n)$ as those locally integrable functions f such that

$$\|f\|_{\mathrm{BMO}_w(\mathbf{R}^n)} := \|(M^{\#} f)\|_{L^\infty_w(\mathbf{R}^n)} < \infty.$$

Weighted BMO spaces also appeared in the work of Muckenhoupt and Wheeden in [MW76], and they showed that the estimate

$$\|Tf\|_{\mathrm{BMO}_w(\mathbf{R}^n)} \lesssim \|f\|_{L^\infty_w(\mathbf{R}^n)}, \tag{5.1.7}$$

with an explicit constant depending on w, is satisfied when T is the Hilbert transform, if and only if $w^{-1} \in A_1$. We recall here that the condition $w^{-1} \in A_1$ is equivalent to the condition $w \in A_{\infty,(1,\infty)}$ with $[w]_{\infty,(1,\infty)} = [w^{-1}]_{A_1}$. Later it was shown by Harboure, Macías and Segovia in [HMS88] that one can extrapolate from the estimate (5.1.7) for an operator T to obtain that T is bounded on $L^p_w(\mathbf{R}^n)$ for all $w^p \in A_p$. As a consequence of Theorem 5.1.8 we obtain a multilinear version of this result.

Corollary 5.1.10 (Extrapolation from BMO estimates)**.** *Let $\vec{r} \in (0,\infty)^m$ and let T be an operator that is bounded $L^{\vec{\infty}}_{\vec{w}}(\mathbf{R}^n) \to \mathrm{BMO}_w(\mathbf{R}^n)$ for all $\vec{w} \in A_{\vec{\infty},(\vec{r},\infty)}$. Moreover, suppose there is an increasing function ϕ such that*

$$\|T\|_{L^{\vec{\infty}}_{\vec{w}}(\mathbf{R}^n) \to \mathrm{BMO}_w(\mathbf{R}^n)} \leq \phi([\vec{w}]_{\vec{\infty},(\vec{r},\infty)})$$

for all $\vec{w} \in A_{\vec{\infty},(\vec{r},\infty)}$.

Then for all $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$, there is an increasing function $\phi_{\vec{p},\vec{r}}$ such that

$$\|T(\vec{f}\,)\|_{L^p_w(\mathbf{R}^n)} \leq \phi_{\vec{p},\vec{r}}([\vec{w}]_{\vec{p},(\vec{r},\infty)}) \prod_{j=1}^{m} \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}$$

for all $\vec{w} \in A_{\vec{p},(\vec{r},\infty)}$ and all $\vec{f} \in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$ for which the left-hand side is finite.

Proof. We apply Theorem 5.1.8 with T replaced by $M^{\#}T$ to find that for all $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$,

$$\|M^{\#}T\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} \le 2^{\frac{m}{r}}\phi\Big(C_{\vec{p},\vec{r}}[\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right\}}\Big)$$

where

$$C_{\vec{p},\vec{r}} \lesssim_r \prod_{j=1}^m \left(\frac{\frac{1}{r_j}}{\frac{1}{r_j}-\frac{1}{p_j}}\right)^{\frac{1}{r_j}}.$$

By the Fefferman-Stein inequality for the sharp maximal operator, see [FS72], we find that

$$\|T(\vec{f})\|_{L^p_w(\mathbf{R}^n)} \lesssim_{w,p} \|M^{\#}(T(\vec{f}))\|_{L^p_w(\mathbf{R}^n)},$$

for all $\vec{f} \in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$ for which the left-hand side is finite, where the implicit constant depends on w only through an increasing function in the constant $[w^p]_{A_\infty}$. It remains to note that by Proposition 3.1.6,

$$[w^p]_{A_\infty} \lesssim [w^p]_{A_{\frac{p}{r}}} = [w]^p_{p,(r,\infty)} \le [\vec{w}]^p_{\vec{p},(\vec{r},\infty)},$$

see also [Gra14a, Chapter 7]. The assertion follows. □

Examples of multilinear operators satisfying weak-type and BMO endpoint estimates are multilinear Calderón-Zygmund operators, see also [Gra14b, Section 7.4.1]. Weighted estimates in these situations can be found in [LOP+09b].

5.2. OPTIMALITY OF WEIGHTED BOUNDS

In this section we describe a way to use the extrapolation theorem to deduce when weighted bounds of an operator T are optimal, given a certain asymptotic behaviour of the unweighted operator norms $\|T\|_{L^{\vec{p}}(\mathbf{R}^n)\to L^p(\mathbf{R}^n)}$.

First we define the critical exponents we need that determine a certain asymptotic behaviour of $\|T\|_{L^{\vec{p}}(\mathbf{R}^n)\to L^p(\mathbf{R}^n)}$.

Definition 5.2.1. Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$, $\vec{q} \in (0,\infty]^m$ with $\vec{q} \ge \vec{r}$, $q \le s$, and let T be an operator that is bounded $L^{\vec{p}}(\mathbf{R}^n) \to L^p(\mathbf{R}^n)$ for all $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$, $p < s$ (or $p_j = q_j$ for some $j \in \{1,\dots,m\}$, or $p = q$). Setting $\|T\|_{\vec{p}} := \|T\|_{L^{\vec{p}}(\mathbf{R}^n)\to L^p(\mathbf{R}^n)}$, for $j \in \{1,\dots,m\}$ we define

$$\alpha_{j,\vec{q}}(T) := \sup\Big\{\alpha \in [0,\infty) : \forall \varepsilon > 0, \limsup_{\frac{1}{p_j}\to\frac{1}{r_j}} \Big(\frac{1}{r_j}-\frac{1}{p_j}\Big)^{\alpha-\varepsilon}\|T\|_{(q_1,\dots,q_{j_0-1},p_{j_0},q_{j_0+1},\dots,q_m)} = \infty\Big\}$$

with $\vec{\alpha}_{\vec{q}}(T) := (\alpha_{1,\vec{q}}(T),\dots,\alpha_{m,\vec{q}}(T))$ and

$$\omega_{j,\vec{q}}(T) := \sup\Big\{\omega \in [0,\infty) : \forall \varepsilon > 0, \limsup_{\frac{1}{p}\to\frac{1}{s}} \Big(\frac{1}{p}-\frac{1}{s}\Big)^{\omega-\varepsilon}\|T\|_{(q_1,\dots,q_{j-1},\frac{1}{\frac{1}{q_j}-(\frac{1}{q}-\frac{1}{p})},q_{j+1},\dots,q_m)} = \infty\Big\}.$$

with $\omega_{\vec{q}}(T) := \max_{j\in\{1,\dots,m\}} \omega_{j,\vec{q}}(T)$.

Note that when $m = 1$, the quantities $\alpha_{j,\vec{q}}(T)$, $\omega_{\vec{q}}(T)$ do not depend on $\vec{q}$.

Remark 5.2.2. We note that we are considering m different ways of letting $\frac{1}{p} \to \frac{1}{s}$ in the definition of $\omega_{\vec{q}}(T)$, while we are only considering one way of letting $\frac{1}{p_j} \to \frac{1}{r_j}$ in the definition of $\alpha_{j,\vec{q}}(T)$. This is for notational simplicity only and our results can be refined by also considering these other directions.

Theorem 5.2.3 (Optimality of weighted bounds for multilinear operators)**.** *Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$, $\vec{q} \in (0,\infty]^m$ with $\vec{q} \geq \vec{r}$, $q \leq s$, and let T be an operator that is bounded $L^{\vec{q}}_{\vec{w}}(\mathbf{R}^n) \to L^q_w(\mathbf{R}^n)$ for all $\vec{w} \in A_{\vec{q},(\vec{r},s)}$. Moreover, suppose that there is a $\beta \in [0,\infty)$ such that*

$$\|T\|_{L^{\vec{q}}_{\vec{w}}(\mathbf{R}^n) \to L^q_w(\mathbf{R}^n)} \lesssim [\vec{w}]^{\beta}_{\vec{q},(\vec{r},s)}$$

for all $\vec{w} \in A_{\vec{q},(\vec{r},s)}$.

Then

$$\beta \geq \max\Big\{ \frac{\vec{\alpha}_{\vec{q}}(T)}{\frac{1}{\vec{r}} - \frac{1}{\vec{q}}}, \frac{\omega_{\vec{q}}(T)}{\frac{1}{q} - \frac{1}{s}} \Big\},$$

where we interpret $\frac{\alpha_{j,\vec{q}}(T)}{\frac{1}{r_j} - \frac{1}{q_j}}$ as 0 when $q_j = r_j$ and $\frac{\omega_{\vec{q}}(T)}{\frac{1}{q} - \frac{1}{s}}$ as 0 when $q = s$.

Proof. Fix $j \in \{1, \dots, m\}$. By Remark 5.1.3, applying Theorem 5.1.2 with $\frac{1}{p_k} = \frac{1}{q_k}$ for $k \neq j$ and $\frac{1}{p_j} \leq \frac{1}{q_j}$, $\vec{w} \equiv 1$ yields

$$\|T\|_{(q_1,\dots,q_{j-1},p_j,q_{j+1},\dots,p_m)} \lesssim_{r,s} \left(\frac{\frac{1}{r_j} - \frac{1}{q_j}}{\frac{1}{r_j} - \frac{1}{p_j}} \right)^{\beta(\frac{1}{r_j} - \frac{1}{q_j})}. \tag{5.2.1}$$

Now let $\varepsilon > 0$ and let $\alpha \in [0,\infty)$ satisfy $\limsup_{\frac{1}{p_j} \to \frac{1}{r_j}} \left(\frac{1}{r_j} - \frac{1}{p_j} \right)^{\alpha - \varepsilon} \|T\|_{(q_1,\dots,q_{j-1},p_j,q_{j+1},\dots,q_m)} = \infty$. By (5.2.1) this implies that

$$\limsup_{\frac{1}{p_j} \to \frac{1}{r_j}} \Big(\frac{1}{r_j} - \frac{1}{p_j} \Big)^{\alpha - \varepsilon - \beta(\frac{1}{r_j} - \frac{1}{q_j})} \gtrsim_{r,s} \limsup_{\frac{1}{p_j} \to \frac{1}{r_j}} \Big(\frac{1}{r_j} - \frac{1}{p_j} \Big)^{\alpha - \varepsilon} \|T\|_{(q_1,\dots,q_{j-1},p_j,q_{j+1},\dots,q_m)} = \infty$$

which implies that $\alpha - \varepsilon - \beta(\frac{1}{r_j} - \frac{1}{q_j}) < 0$. Since $\varepsilon > 0$ is arbitrary, this implies that

$$\beta \geq \frac{\alpha}{\frac{1}{r_j} - \frac{1}{q_j}}$$

when $\frac{1}{q_j} \neq \frac{1}{r_j}$ and $\alpha = 0$ when $\frac{1}{q_j} = \frac{1}{r_j}$. Taking a supremum over such α and a maximum over $j \in \{1, \dots, m\}$ yields

$$\beta \geq \max\Big\{ \frac{\vec{\alpha}_{\vec{q}}(T)}{\frac{1}{\vec{r}} - \frac{1}{\vec{q}}} \Big\}. \tag{5.2.2}$$

By Remark 5.1.3, another application of Theorem 5.1.2 with $\frac{1}{p_k} = \frac{1}{q_k}$ for $k \neq j$ and $\frac{1}{p_j} := \frac{1}{\frac{1}{q_j} - (\frac{1}{q} - \frac{1}{p})}$ with $\frac{1}{q} \leq \frac{1}{p}$, $\vec{w} \equiv 1$ yields

$$\|T\|_{(q_1,\dots,q_{j-1},\frac{1}{\frac{1}{q_j}-(\frac{1}{q}-\frac{1}{p})},q_{j+1},\dots,q_m)} \lesssim_{r,s} \left(\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{\beta(\frac{1}{q}-\frac{1}{s})}.$$

With an argument analogous to our previous one, this implies that

$$\beta \geq \frac{\omega_{j,\vec{q}}(T)}{\frac{1}{q}-\frac{1}{s}}.$$

Taking a maximum over $j \in \{1,\dots,m\}$ and combining this result with (5.2.2), the assertion follows. □

Remark 5.2.4. By using Theorem 5.1.6 instead of Theorem 5.1.2 we can obtain the same result for weak-type bounds with an analogous argument.

In the case $m = 1$ this result reduces to the following:

Corollary 5.2.5. *Let $r \in (0,\infty)$, $s \in (0,\infty]$, $q \in [r,s]$, and let T be an operator that is bounded $L^q_w(\mathbf{R}^n) \to L^q_w(\mathbf{R}^n)$ for all $w \in A_{q,(r,s)}$. Moreover, suppose that there is a $\beta \in [0,\infty)$ such that*

$$\|T\|_{L^q_w(\mathbf{R}^n)\to L^q_w(\mathbf{R}^n)} \lesssim [w]^{\beta}_{q,(r,s)}$$

for all $w \in A_{q,(r,s)}$.

Then

$$\beta \geq \max\left\{\frac{\alpha(T)}{\frac{1}{r}-\frac{1}{q}}, \frac{\omega(T)}{\frac{1}{q}-\frac{1}{s}}\right\},$$

where

$$\alpha(T) = \sup\left\{\alpha \in [0,\infty) : \forall \varepsilon > 0, \limsup_{\frac{1}{p}\to\frac{1}{r}} \left(\frac{1}{r}-\frac{1}{p}\right)^{\alpha-\varepsilon} \|T\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} = \infty\right\},$$

$$\omega(T) = \sup\left\{\omega \in [0,\infty) : \forall \varepsilon > 0, \limsup_{\frac{1}{p}\to\frac{1}{s}} \left(\frac{1}{p}-\frac{1}{s}\right)^{\omega-\varepsilon} \|T\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} = \infty\right\},$$

and where we interpret $\frac{\alpha(T)}{\frac{1}{r}-\frac{1}{q}}$ as 0 when $q = r$ and $\frac{\omega(T)}{\frac{1}{q}-\frac{1}{s}}$ as 0 when $q = s$.

Finally, we present the following variant of this result:

Theorem 5.2.6. *Let $r \in (0,\infty)$, $s \in (0,\infty]$, $q \in [r,s)$, and let T be an operator that is bounded $L^q_w(\mathbf{R}^n) \to L^q_w(\mathbf{R}^n)$ for all $w \in A_{q,(q,s)}$. Moreover, suppose that there is a $\beta \in [0,\infty)$ such that*

$$\|T\|_{L^q_w(\mathbf{R}^n)\to L^q_w(\mathbf{R}^n)} \lesssim [w]^{\beta}_{q,(q,s)}$$

for all $w \in A_{q,(q,s)}$.

Then

$$\beta \geq \frac{\omega(T)}{\frac{1}{q}-\frac{1}{s}},$$

where

$$\omega(T) = \sup\Big\{\omega \in [0,\infty) : \forall \varepsilon > 0, \limsup_{\frac{1}{p}\to\frac{1}{s}} \Big(\frac{1}{p}-\frac{1}{s}\Big)^{\omega-\varepsilon} \|T\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} = \infty\Big\}.$$

We note that $[w]_{q,(r,s)} \leq [w]_{q,(q,s)}$ by Hölder's inequality, and

$$[w]_{q,(q,s)} = [w^{\frac{1}{\frac{1}{q}-\frac{1}{s}}}]_{A_1}^{\frac{1}{q}-\frac{1}{s}}.$$

The key observation for the proof of this result is that the Rubio de Francia algorithm in the case $m = 1$ produces an A_1 weight.

Proof. Fix $p \in [q,s)$ and $g \in L^{\frac{1}{\frac{1}{q}-\frac{1}{p}}}(\mathbf{R}^n)$ of norm 1. By applying the Rubio de Francia algorithm, Lemma 4.1.3, to g, we obtain a weight $w := R_{\frac{1}{\frac{1}{q}-\frac{1}{p}},\frac{1}{\frac{1}{q}-\frac{1}{s}},1}g$ satisfying $|g| \leq w$, and

$$[w]_{q,(q,s)} = \sup_Q \langle R_{\frac{1}{\frac{1}{q}-\frac{1}{p}},\frac{1}{\frac{1}{q}-\frac{1}{s}},1}g\rangle_{\frac{1}{\frac{1}{q}-\frac{1}{s}},Q} \langle (R_{\frac{1}{\frac{1}{q}-\frac{1}{p}},\frac{1}{\frac{1}{q}-\frac{1}{s}},1}g)^{-1}\rangle_{\infty,Q} \lesssim_r \left(\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{\frac{1}{q}-\frac{1}{s}}.$$

Hence, for all $f \in L^q_w(\mathbf{R}^n)$ of norm 1 we have

$$\|(Tf)g\|_{L^q(\mathbf{R}^n)} \leq \|Tf\|_{L^q_w(\mathbf{R}^n)} \lesssim [w]^\beta_{q,(q,s)} \lesssim_r \left(\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{\beta(\frac{1}{q}-\frac{1}{s})}.$$

By the duality result

$$\|Tf\|_{L^p(\mathbf{R}^n)} = \||Tf|^q\|^{\frac{1}{q}}_{L^{\frac{p}{q}}(\mathbf{R}^n)} = \sup_{\|g\|_{L^{\frac{1}{\frac{1}{q}-\frac{1}{p}}}(\mathbf{R}^n)}=1} \|(Tf)g\|_{L^q(\mathbf{R}^n)},$$

we now obtain

$$\|T\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} \lesssim_r \left(\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{\beta(\frac{1}{q}-\frac{1}{s})}.$$

With an argument analogous to the one in the proof of Theorem 5.2.3 this implies that $\beta \geq \frac{\omega(T)}{\frac{1}{q}-\frac{1}{s}}$. The assertion follows. □

The initial weighted boundedness that we need to apply these results can be obtained through sparse domination.

5.3. SPARSE DOMINATION OF ℓ^q-TYPE

There are many different variants of sparse forms and operators and there are various ways that an operator can be bounded by them. In this section we will consider bounds obtained for multilinear operators satisfying a general ℓ^q-type sparse form domination, covering a variety of examples presented in the next section. Since sparse domination is usually proven for functions of bounded support, we will first need to discuss extensions of operators.

Typically, unlike in our assumptions on the operators in the previous sections, operators will not initially be defined on $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$. Rather, they will be defined on an m-tuple of spaces $\vec{U}$ where each U_j is an appropriately large subspace of the of measurable functions $L^0(\mathbf{R}^n)$ such as the space of bounded functions with bounded support $L^\infty_c(\mathbf{R}^n)$, the space of simple functions with characteristic functions over finite sets, the space of compactly supported smooth functions $C^\infty_c(\mathbf{R}^n)$, the space of Schwartz functions $\mathscr{S}(\mathbf{R}^n)$, etc. If we then prove that T is bounded for these functions between weighted Lebesgue spaces, we can use density to extend T to a bounded operator on these spaces, as long as we assume some additional structure on the operator. What is noteworthy is that in the multilinear setting this argument is slightly more technical than in the linear setting.

Definition 5.3.1. Let $\vec{U}$, V be $m+1$ quasi-normed linear subspaces of $L^0(\mathbf{R}^n)$ and $T : \vec{U} \to V$. We say that T is *m-linear* if it is linear in each of its components, i.e., if for all $\vec{f} \in \vec{U}$ and $j \in \{1,\dots,m\}$ the map $U_j \to V$, $g \mapsto T(f_1,\dots,f_{j-1},g,f_{j+1},\dots,f_m)$ is linear. We say that T is *m-sublinear* if it is positive-valued and subadditive in each of its components, i.e., if for all $\vec{f} \in \vec{U}$ the function $T(\vec{f}\,)$ takes values in the positive reals, and for all $j \in \{1,\dots,m\}$ and $g \in U_j$,

$$T(f_1,\dots,f_{j-1},f_j+g,f_{j+1},\dots,f_m) \le T(\vec{f}) + T(f_1,\dots,f_{j-1},g,f_{j+1},\dots,f_m).$$

We will generally consider operators that are either m-linear or m-sublinear, which we shorten by saying that the operator is m-(sub)linear. In the case $m=1$, a bounded (sub)linear operator satisfies a reverse triangle inequality type estimate and thus, in particular, is uniformly continuous. Therefore, if it takes values in a complete space, it extends to an operator on the closure of its domain. For $m>2$ this uniform continuity needs to be replaced by a local uniform continuity. This again suffices to extend the operator to the closure of its domain. While this result may be straightforward, we include it here. For the definition of a quasi-Banach function space we refer the reader to Chapter 8.

Lemma 5.3.2. *Let $\vec{Y}$ be an m-tuple of quasi-normed vector spaces, let V be a quasi-Banach function space, and let $U_j \subseteq Y_j$ be a dense subspace for each $j \in \{1,\dots,m\}$. If $T : \vec{U} \to V$ is bounded and satisfies the pointwise a.e. estimate*

$$\begin{aligned} |T(\vec{f}\,) - T(\vec{g}\,)| \le \sum_{j=1}^m &|T(f_1,\dots,f_{j-1},f_j-g_j,g_{j+1},\dots,g_m)| \\ &+ |T(g_1,\dots,g_{j-1},g_j-f_j,f_{j+1},\dots,f_m)| \end{aligned} \tag{5.3.1}$$

for all $\vec{f}, \vec{g} \in \vec{U}$, then T uniquely extends to a bounded operator $\vec{Y} \to V$ with a comparable bound. Moreover, if T is m-(sub)linear, then it satisfies (5.3.1)*, and if V has the property that every convergent sequence has a pointwise a.e. convergent subsequence, then the extension of T is again an m-(sub)linear operator.*

The extension of T will again be denoted by T.

Proof. By the compatibility of the norm on V with pointwise estimates, (5.3.1) and boundedness of T yields for $\vec{f}, \vec{g} \in \vec{U}$

$$\begin{aligned}\|T(\vec{f}) - T(\vec{g})\|_V &\lesssim \sum_{j=1}^m \|T(f_1,\dots,f_{j-1},f_j-g_j,g_{j+1},\dots,g_m)\|_V \\ &\quad + \|T(g_1,\dots,g_{j-1},g_j-f_j,f_{j+1},\dots,f_m)\|_V \\ &\lesssim \sum_{j=1}^m \Big(\prod_{\substack{l=1\\ l\neq j}}^m (\|f_l\|_{Y_l} + \|g_l\|_{Y_l})\Big)\|f_j-g_j\|_{Y_j}.\end{aligned} \tag{5.3.2}$$

Now, if $\vec{f} \in \vec{Y}$ and $(f_j^k)_{k\in\mathbf{N}}$ is a sequence in U_j converging to f_j in Y_j for all $j \in \{1,\dots,m\}$, then (5.3.2) implies that $(T(\vec{f}^k))_{k\in\mathbf{N}}$ is a Cauchy sequence in V. The first assertion then follows by defining $T(\vec{f})$ to be the limit of this sequence in V. Note that this is well-defined since it follows from another application of (5.3.2) that this limit does not depend on the approximating sequences of the f_j. For the bound we have

$$\|T(\vec{f})\|_V \le \beta \liminf_{k\to\infty} \|T(\vec{f}^k)\|_V \le \beta c \prod_{j=1}^m \limsup_{k\to\infty} \|f_j^k\|_{Y_j} \le \beta\Big(\prod_{j=1}^m \alpha_j\Big) c \prod_{j=1}^m \|f_j^k\|_{Y_j}.$$

where c, α_j, and β are respectively the bound for T, the quasi-triangle inequality constant for Y_j, and the quasi-triangle inequality constant for V.

For the second assertion, if T is m-sublinear, then it follows from iterating the inequality

$$T(\vec{f}) \le T(g_1, f_2, \dots, f_m) + T(f_1 - g_1, f_2, \dots, f_m)$$

for all f_j in the first term on the right for $j = 2$ to $j = m$, that

$$T(\vec{f}) \le T(\vec{g}) + \sum_{j=1}^m T(f_1,\dots,f_{j-1},f_j-g_j,g_{j+1},\dots,g_m).$$

By symmetry, we obtain

$$T(\vec{g}) \le T(\vec{f}) + \sum_{j=1}^m T(g_1,\dots,g_{j-1},g_j-f_j,f_{j+1},\dots,f_m)$$

and by combining these two estimates we obtain (5.3.1). If T is m-linear, these first two inequalities are actually equalities, so we can proceed analogously.

The final assertion is a consequence of the fact that m-(sub)linearity is a pointwise property. $\square$

Note that if $V = L^p_w(\mathbf{R}^n)$ for a weight w and a $p \in (0,\infty]$, then V satisfies the property that every convergent sequence has a pointwise a.e. convergent subsequence. Moreover, by continuity of the (quasi-)norm $\|\cdot\|_{L^p_w(\mathbf{R}^n)}$, the bound of the extension T will not only be comparable, but it will be equal to the bound of the original T.

Consider an operator $T : \vec{U} \to L^0(\mathbf{R}^n)$, where $\vec{U}$ is an m-tuple of quasi-normed linear subspaces of $L^0(\mathbf{R}^n)$. Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$ and $q \in (0,s)$. We then impose the condition on T that for all $\vec{f} \in \vec{U}$ there exists a sparse collection $\mathscr{S}$ such that

$$\|T(\vec{f}\,)g\|_{L^q(\mathbf{R}^n)} \lesssim \Big(\sum_{Q\in\mathscr{S}} \Big(\prod_{j=1}^m \langle f_j\rangle_{r_j,Q}\Big)^q \langle g\rangle^q_{\frac{1}{\frac{1}{q}-\frac{1}{s}},Q}|Q|\Big)^{\frac{1}{q}}. \tag{5.3.3}$$

It turns out that it is convenient to reformulate this sparse domination in terms of a domination by a multisublinear maximal operator.

Proposition 5.3.3. *Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$ and $q \in (0,s)$. Then*

$$\|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\vec{f},g)\|_{L^q(\mathbf{R}^n)} \eqsim \sup_{\mathscr{S}}\Big(\sum_{Q\in\mathscr{S}} \Big(\prod_{j=1}^m \langle f_j\rangle_{r_j,Q}\Big)^q \langle g\rangle^q_{\frac{1}{\frac{1}{q}-\frac{1}{s}},Q}|Q|\Big)^{\frac{1}{q}},$$

for all $\vec{f} \in L^{\vec{r}}_{\mathrm{loc}}(\mathbf{R}^n)$, $g \in L^{\frac{1}{\frac{1}{q}-\frac{1}{s}}}_{\mathrm{loc}}(\mathbf{R}^n)$, where the supremum is over all sparse collections $\mathscr{S}$.

Proof. Note that the left-hand side can be written as

$$\|M_{(\frac{\vec{r}}{q},\frac{\frac{1}{q}}{\frac{1}{q}-\frac{1}{s}})}(|f_1|^q,\ldots,|f_m|^q,|g|^q)\|^{\frac{1}{q}}_{L^1(\mathbf{R}^n)}$$

while the right-hand side can be written as

$$\sup_{\mathscr{S}}\Big(\sum_{Q\in\mathscr{S}} \Big(\prod_{j=1}^m \langle |f_j|^q\rangle_{\frac{r_j}{q},Q}\Big)\langle |g|^q\rangle_{\frac{\frac{1}{q}}{\frac{1}{q}-\frac{1}{s}},Q}|Q|\Big)^{\frac{1}{q}} = \Big(\sup_{\mathscr{S}}\Lambda_{\mathscr{S},(\frac{\vec{r}}{q},\frac{\frac{1}{q}}{\frac{1}{q}-\frac{1}{s}})}(|\vec{f}|^q,|g|^q)\Big)^{\frac{1}{q}}$$

Thus, the result follows from Proposition 3.2.10. □

By this result we can write (5.3.3) as

$$\|T(\vec{f}\,)g\|_{L^q(\mathbf{R}^n)} \lesssim \|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\vec{f},g)\|_{L^q(\mathbf{R}^n)},$$

which is not only notationally convenient, but as we will see in Chapter 9.2, gives us the right point of view to extend sparse domination to a vector-valued setting.

One of the reasons we are considering sparse domination in this general form is because pointwise sparse domination by an ℓ^q sparse operator implies the sparse form domination we are considering for $s=\infty$.

Proposition 5.3.4. *Let $\vec{r} \in (0,\infty)^m$, $q \in (0,\infty)$ and let T be an operator defined on $L^\infty_c(\mathbf{R}^n)^m$. Suppose that for each bounded set B and all $\vec{f} \in L^\infty_c(\mathbf{R}^n)^m$ supported in B, for each $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$ there exists a sparse collection $\mathscr{S}^\alpha \subseteq \mathscr{D}^\alpha$ such that*

$$|T(\vec{f})| \le C_T \sum_{\alpha \in \{0,\frac{1}{3},\frac{2}{3}\}^n} \Big(\sum_{Q \in \mathscr{S}^\alpha} \Big(\prod_{j=1}^m \langle f_j \rangle_{r_j,Q} \Big)^q \chi_Q \Big)^{\frac{1}{q}} \tag{5.3.4}$$

pointwise a.e. in B. Then

$$\|T(\vec{f}) \cdot g\|_{L^q(\mathbf{R}^n)} \lesssim_q C_T \|M_{(\vec{r},q)}(\vec{f}, g)\|_{L^q(\mathbf{R}^n)}$$

for all $\vec{f} \in L^\infty_c(\mathbf{R}^n)^m$, $g \in L^\infty_c(\mathbf{R}^n)$.

Proof. Let $\vec{f} \in L^\infty_c(\mathbf{R}^n)^m$, $g \in L^\infty_c(\mathbf{R}^n)$. Since the set $B := \Big(\cup_{j=1}^m \operatorname{supp} f_j\Big) \cup \operatorname{supp} g$ is a bounded set, letting $\mathscr{S}^\alpha$ be sparse collections such that (5.3.4) holds pointwise a.e. in B, it follows from Proposition 5.3.3 that

$$\begin{aligned}\|T(\vec{f}) \cdot g\|_{L^q(\mathbf{R}^n)} &\lesssim_q C_T \sum_{\alpha \in \{0,\frac{1}{3},\frac{2}{3}\}^n} \Big\| \Big(\sum_{Q \in \mathscr{S}^\alpha} \Big(\prod_{j=1}^m \langle f_j \rangle_{r_j,Q} \Big)^q \chi_Q \Big)^{\frac{1}{q}} g \Big\|_{L^q(\mathbf{R}^n)} \\ &= C_T \sum_{\alpha \in \{0,\frac{1}{3},\frac{2}{3}\}^n} \Big(\sum_{Q \in \mathscr{S}^\alpha} \Big(\prod_{j=1}^m \langle f_j \rangle_{r_j,Q} \Big)^q \langle g \rangle_{q,Q}^q |Q| \Big)^{\frac{1}{q}} \\ &\lesssim_q C_T \|M_{(\vec{r},q)}(\vec{f}, g)\|_{L^q(\mathbf{R}^n)}.\end{aligned}$$

The assertion follows. □

Remark 5.3.5. We point out that if $\widetilde{q} \in (0,q]$, then the inequality $\|\cdot\|_{\ell^q} \le \|\cdot\|_{\ell^{\widetilde{q}}}$ implies that if (5.3.4) holds, then it also holds with q replaced by $\widetilde{q}$. Thus, we actually find that (5.3.4) implies that

$$\|T(\vec{f}) \cdot g\|_{L^{\widetilde{q}}(\mathbf{R}^n)} \le 2^{\frac{1}{\widetilde{q}}} C_T \|M_{(\vec{r},\widetilde{q})}(\vec{f}, g)\|_{L^{\widetilde{q}}(\mathbf{R}^n)}$$

for all $\widetilde{q} \in (0,q]$ and $g \in L^{\widetilde{q}}_{\text{loc}}(\mathbf{R}^n)$.

In the following result we will deduce weighted bounds from domination by the multisublinear Hardy–Littlewood operator. We recall here that $\alpha_{j,\vec{p}}(T)$, $\omega_{\vec{p}}(T)$ are defined in Definition 5.2.1.

Theorem 5.3.6. *Let T be an m-(sub)linear operator initially defined on $L^\infty_c(\mathbf{R}^n)^m$. Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$ and $q \in (0,s)$ and suppose that*

$$\|T(\vec{f}) \cdot g\|_{L^q(\mathbf{R}^n)} \le C_T \|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\vec{f}, g)\|_{L^q(\mathbf{R}^n)} \tag{5.3.5}$$

for all $\vec{f} \in L^\infty_c(\mathbf{R}^n)^m$, $g \in L^\infty_c(\mathbf{R}^n)$. Then for all $\vec{p} \in (0,\infty]^m$ with $\vec{r} < \vec{p}$ and $p < s$, all $\vec{w} \in A_{\vec{p},(\vec{r},s)}$, T has a bounded extension $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)$ with

$$\|T\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)} \lesssim_{\vec{p},q,\vec{r},s} C_T [\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}. \tag{5.3.6}$$

If for all $j \in \{1, \dots, m\}$

$$\alpha_{j,\vec{p}}(T) \geq \frac{1}{r_j}, \quad \omega_{\vec{p}}(T) \geq \frac{1}{q} - \frac{1}{s},$$

then the exponent of the weight constant in (5.3.6) *is the smallest possible one.*

Moreover, suppose (Ω, μ) *is a* σ*-finite measure space,* $\vec{t} \in (0,\infty]^m$ *with* $\vec{t} > \vec{r}$ *and* $t < s$. *Then*

$$\widetilde{T}(\vec{f}\,)(x,\omega) := T(\vec{f}(\cdot,\omega))(x)$$

is well-defined for all $\vec{w} \in A_{\vec{p},(\vec{r},s)}$ *and* $f_j \in L^{p_j}_{w_j}(\mathbf{R}^n; L^{t_j}(\Omega))$. *Furthermore, for all* $f_j \in L^{p_j}_{w_j}(\mathbf{R}^n; L^{t_j}(\Omega))$ *for which* $\widetilde{T}(\vec{f}\,)$ *is measurable,*

$$\|\widetilde{T}(\vec{f}\,)\|_{L^p_w(\mathbf{R}^n;L^t(\Omega))} \lesssim_{\vec{p},q,\vec{r},s,\vec{t}} [\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{t}-\frac{1}{s}}\right\}\cdot\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}, \frac{\frac{1}{t}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}} \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;L^{t_j}(\Omega))}. \tag{5.3.7}$$

Remark 5.3.7. As was noted in Remark 5.1.4, when T is m-linear the measurability assumption on $\widetilde{T}(\vec{f}\,)$ is redundant and in this case we have the boundedness result

$$\|\widetilde{T}\|_{L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n;L^{t_m}(\Omega))\to L^p_w(\mathbf{R}^n;L^t(\Omega))} \lesssim_{\vec{p},q,\vec{r},s,\vec{t}} [\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{t}-\frac{1}{s}}\right\}\cdot\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}, \frac{\frac{1}{t}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}.$$

We moreover point out that the weight constant here is in general not optimal. We will show in Section 9.2 that, at least when $t \geq 1$, it is possible to replace the exponent by the smaller exponent

$$\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\},$$

which does not depend on $\vec{t}$.

Theorem 5.3.6 is essentially a consequence of Theorem 3.2.11 and, in certain cases, the quantitative multilinear extrapolation theorem. The reason we might have to use extrapolation is because sparse domination by forms yields, a priori, weighted bounds for the range of exponents where one can dualize the operator. Typically, in the multilinear case, this does not yield the full range of exponents where the operator satisfies weighted bounds. To recover this full range of exponents, we will use Theorem 5.1.2. Before we can do this however, we need to use Lemma 5.3.2 to extend T to weighted Lebesgue spaces. Since we are working with weights that are not necessarily locally integrable, it is not a-priori clear that the bounded functions of bounded suppose are dense in these spaces. We prove that this density result does indeed hold.

Lemma 5.3.8. *Let* w *be a weight and* $p \in (0,\infty)$. *Then* $L^p_w(\mathbf{R}^n) \cap L^\infty_c(\mathbf{R}^n)$ *is dense in* $L^p_w(\mathbf{R}^n)$.

Proof. First consider the case $p > 1$ and suppose $g \in L^{p'}_{w^{-1}}(\mathbf{R}^n)$ satisfies the property that $\int_{\mathbf{R}^n} fg\,\mathrm{d}x = 0$ for all $f \in L^p_w(\mathbf{R}^n) \cap L^\infty_c(\mathbf{R}^n)$. If $f \in L^\infty_c(\mathbf{R}^n)$, then $\frac{f}{1+w} \in L^p_w(\mathbf{R}^n) \cap L^\infty_c(\mathbf{R}^n)$, so

$$\int_{\mathbf{R}^n} f \frac{g}{1+w}\,\mathrm{d}x = \int_{\mathbf{R}^n} \frac{f}{1+w} g\,\mathrm{d}x = 0,$$

i.e., $\frac{g}{1+w} \in L^{p'}(\mathbf{R}^n)$ is annihilated by all $f \in L^\infty_c(\mathbf{R}^n)$. Since $L^\infty_c(\mathbf{R}^n)$ is dense in $L^p(\mathbf{R}^n)$, we have $\frac{g}{1+w} = 0$ and thus $g = 0$. We conclude that $L^p_w(\mathbf{R}^n) \cap L^\infty_c(\mathbf{R}^n)$ is dense in $L^p_w(\mathbf{R}^n)$.

Now consider the case $p \le 1$. Fix $k \in \mathbf{N}$ so that $2^k p > 1$. If $f \in L^p_w(\mathbf{R}^n)$, then we can pick a positive $g \in L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n)$ with $g^{2^k} = |f|$. By our previous result we can find a sequence $(g_j)_{j\in\mathbf{N}}$ in $L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n) \cap L^\infty_c(\mathbf{R}^n)$ converging to g. Setting $f_j := |g_j|^{2^k}\operatorname{sgn}(f) \in L^p_w(\mathbf{R}^n) \cap L^\infty_c(\mathbf{R}^n)$ we compute

$$|f_j - f| = ||g_j|^{2^k} - g^{2^k}| = ||g_j| - g| \prod_{l=0}^{k-1} ||g_j|^{2^l} + g^{2^l}|$$

so that by Hölder's inequality

$$\|f_j - f\|_{L^p_w(\mathbf{R}^n)} \le \|g_j - g\|_{L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n)} \prod_{l=0}^{k-1} \||g_j|^{2^l} + g^{2^l}\|_{L^{2^{k-l}p}_{w^{2^{-(k-l)}}}(\mathbf{R}^n)}.$$

Since $\||g_j|^{2^l} + g^{2^l}\|_{L^{2^{k-l}p}_{w^{2^{-(k-l)}}}(\mathbf{R}^n)} \lesssim \|g_j\|^{2^l}_{L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n)} + \|g\|^{2^l}_{L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n)}$ is bounded in j, we conclude that $f_j \to f$ in $L^p_w(\mathbf{R}^n)$. Hence, $L^p_w(\mathbf{R}^n) \cap L^\infty_c(\mathbf{R}^n)$ is dense in $L^p_w(\mathbf{R}^n)$, as desired. □

We are now ready to prove Theorem 5.3.6.

Proof of Theorem 5.3.6. Set $\frac{1}{t_j} := \frac{\tau}{r_j}$ with $\frac{1}{\tau} = \frac{\frac{1}{r} - \frac{1}{s} + \frac{1}{q}}{\frac{1}{q}} > 1$. Noting that $\frac{1}{t} = \frac{\tau}{r} = \frac{\frac{1}{r}}{\frac{1}{r} + \frac{1}{q} - \frac{1}{s}} \frac{1}{q} \in (\frac{1}{s}, \frac{1}{q})$,

$$\frac{\frac{1}{r_1}}{\frac{1}{r_1} - \frac{1}{t_1}} = \ldots = \frac{\frac{1}{r_m}}{\frac{1}{r_m} - \frac{1}{t_m}} = \frac{\frac{1}{q} - \frac{1}{s}}{\frac{1}{t} - \frac{1}{s}} = \frac{1}{1-\tau},$$

it follows from Theorem 3.2.11 and Proposition 3.1.3(ii) that for all $\vec{w} \in A_{\vec{t},(\vec{r},s)}$ we have

$$\begin{aligned}
\|T(\vec{f}\,) \cdot g\|_{L^q(\mathbf{R}^n)} &\le C_T \|M_{(\vec{r}, \frac{1}{\frac{1}{q} - \frac{1}{s}})}(\vec{f}, g)\|_{L^q(\mathbf{R}^n)} = C_T \|M_{\left(\frac{\vec{r}}{q}, (\frac{s}{q})'\right)}(|\vec{f}|^q, |g|^q)\|^{\frac{1}{q}}_{L^1(\mathbf{R}^n)} \\
&\lesssim_{q,\vec{r},s} C_T [\vec{w}^q]^{\frac{1}{q}\cdot\max\left\{\frac{\frac{q}{r}}{\frac{q}{r} - \frac{q}{t}}, \frac{1-\frac{q}{s}}{\frac{q}{t} - \frac{q}{s}}\right\}}_{\frac{\vec{t}}{q}, (\frac{\vec{r}}{q}, \frac{s}{q})} \left(\prod_{j=1}^m \||f_j|^q\|^{\frac{1}{q}}_{L^{\frac{t_j}{q}}_{w_j^q}(\mathbf{R}^n)}\right) \||g|^q\|^{\frac{1}{q}}_{L^{\left(\frac{t}{q}\right)'}_{w^{-q}}(\mathbf{R}^n)} \\
&= C_T [\vec{w}]^{\frac{1}{1-\tau}}_{\vec{t},(\vec{r},s)} \left(\prod_{j=1}^m \|f_j\|_{L^{t_j}_{w_j}(\mathbf{R}^n)}\right) \|g\|_{L^{\frac{1}{\frac{1}{q} - \frac{1}{t}}}_{w^{-1}}(\mathbf{R}^n)}
\end{aligned} \tag{5.3.8}$$

for all $f_j \in L^{t_j}_{w_j}(\mathbf{R}^n) \cap L^\infty_c(\mathbf{R}^n)$, $g \in L^{\frac{1}{\frac{1}{q}-\frac{1}{t}}}_{w^{-1}}(\mathbf{R}^n) \cap L^\infty_c(\mathbf{R}^n)$. By Lemma 5.3.8 and Lemma 5.3.2 we see that (5.3.8) extends to an inequality valid for all $\vec{f} \in L^{\vec{t}}_{\vec{w}}(\mathbf{R}^n)$, $g \in L^{\frac{1}{\frac{1}{q}-\frac{1}{t}}}_{w^{-1}}(\mathbf{R}^n)$.

Hence, by duality, we have

$$\|T(\vec{f}\,)\|_{L^t_w(\mathbf{R}^n)} = \big\||T(\vec{f}\,)|^q\big\|^{\frac{1}{q}}_{L^{\frac{t}{q}}_{w^q}(\mathbf{R}^n)} = \sup_{\|g\|_{L^{\frac{1}{\frac{1}{q}-\frac{1}{t}}}_{w^{-1}}(\mathbf{R}^n)}=1} \|T(\vec{f}\,)\cdot g\|_{L^q(\mathbf{R}^n)},$$

we have proven (5.1.1) with $\vec{q} = \vec{t}$ and $\phi_{\vec{t}}([\vec{w}]_{\vec{t},(\vec{r},s)}) \eqsim_{q,\vec{r},s} C_T[\vec{w}]^{\frac{1}{1-\tau}}_{\vec{t},(\vec{r},s)}$. Noting that

$$\max\left\{\frac{\frac{1}{\overline{r}}-\frac{1}{\overline{t}}}{\frac{1}{\overline{r}}-\frac{1}{\overline{p}}}, \frac{\frac{1}{t}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\} = (1-\tau)\cdot\max\left\{\frac{\frac{1}{\overline{r}}}{\frac{1}{\overline{r}}-\frac{1}{\overline{p}}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\},$$

the asserted bounds follow from Theorem 5.1.2. Finally, the optimality assertion follows from Theorem 5.2.3. □

5.4. EXAMPLES OF OPERATORS SATISFYING SPARSE DOMINATION AND APPLICATIONS

In this section we apply Theorem 5.3.6 to multilinear Calderón-Zygmund operators and the bilinear Hilbert transform, both of which having an intriguing history in terms of obtaining weighted bounds. Moreover, we give some examples of operators satisfying limited range sparse domination in the linear case $m = 1$.

5.4.1. Multilinear Calderón-Zygmund operators

Let T be an m-linear operator, initially defined for m-tuples $\vec{f} \in C^\infty_c(\mathbf{R}^n)^m$, that satisfies

$$T(f_1,\ldots,f_m)(x) = \int_{(\mathbf{R}^n)^m} K(x,y_1,\ldots,y_m)\prod_{j=1}^m f_j(y_j)\,\mathrm{d}y,$$

whenever $x \notin \cap_{j=1}^m \operatorname{supp} f_j$, where K is a kernel defined in $(\mathbf{R}^n)^{m+1}\backslash\Delta$, with $\Delta := \{(y_0,\ldots,y_m) \in (\mathbf{R}^n)^{m+1} : y_0 = y_1 = \cdots = y_m\}$. Suppose K satisfies the estimate

$$|K(y_0,\ldots,y_m)| \lesssim \frac{1}{\left(\sum_{j,k=0}^m |y_j-y_k|\right)^{mn}}$$

for all $(y_0,\ldots,y_m) \in (\mathbf{R}^n)^{m+1}\backslash\Delta$, and suppose that for all $l \in \{0,\ldots,m\}$ we have

$$|K(y_0,\ldots,y_l,\ldots,y_m) - K(y_0,\ldots,y_l',\ldots,y_m)| \lesssim \omega\left(\frac{|y_l-y_l'|}{\sum_{j,k=0}^m |y_j-y_k|}\right)\frac{1}{\left(\sum_{j,k=0}^m |y_j-y_k|\right)^{mn}}$$

for all $(y_0,\dots,y_l,\dots,y_m) \in (\mathbf{R}^n)^{m+1}\setminus\Delta$, and y'_l with $|y_l - y'_l| \le \frac{1}{2}\max_{k\in\{0,\dots,m\}}|y_l - y_k|$, where $\omega : [0,\infty) \to [0,\infty)$ is a nondecreasing continuous doubling function. If there exist $\vec{q} \in (1,\infty]^m$ so that T extends to a bounded operator $L^{\vec{q}}(\mathbf{R}^n) \to L^q(\mathbf{R}^n)$, then T is called an *m-linear Calderón-Zygmund operator* with modulus of continuity ω.

Multilinear Calderón-Zygmund operators with modulus of continuity $\omega(t) = t^\varepsilon$ for some $\varepsilon > 0$ first appeared in the work [CM75] by Coifman and Meyer. Weighted estimates for these operators have been considered for example by Grafakos and Torres in [GT02] and subsequently by Grafakos and Martell in [GM04]. In the work [LOP+09b] by Lerner, Ombrosi, Pérez, Torres, and Trujillo-González, it was realized that the appropriate weight classes to study these operators are the multilinear weight classes $A_{\vec{p},(\vec{1},\infty)}$ associated to the multisublinear maximal operator. Sharp weighted bounds for the specific exponents $p_1 = \ldots = p_m = m+1$ were found by Damián, Lerner, and Pérez in [DLP15]. These bounds were extended to all $\vec{p} \in (1,\infty)^m$ in the Banach range $1 \le p < \infty$ by Li, Moen, and Sun in [LMS14]. They proved that the same bounds hold also in the case $\frac{1}{p} > 1$ for multilinear sparse operators, leading them to conjecture that the bounds for multilinear Calderón-Zygmund operators should also extend to the case $\frac{1}{p} > 1$. Through a pointwise sparse domination result, this conjecture was independently proven to be true by Conde-Alonso and Rey [CR16] and Lerner and Nazarov [LN18] who considered moduli of continuity ω satisfying a log–Dini condition. More precisely, they proved that if

$$\int_0^1 \omega(t)\log\frac{1}{t}\frac{\mathrm{d}t}{t} < \infty, \tag{5.4.1}$$

then for all bounded sets B and all $\vec{f} \in L^\infty_c(\mathbf{R}^n)^m$ supported in B, for each $\alpha \in \{0,\frac{1}{3},\frac{2}{3}\}^n$ there exists a sparse collection $\mathscr{S}^\alpha \subseteq \mathscr{D}^\alpha$ such that

$$|T(\vec{f})| \le C_T \sum_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n}\sum_{Q\in\mathscr{S}}\Big(\prod_{j=1}^m \langle f_j\rangle_{r_j,Q}\Big)\chi_Q$$

pointwise a.e. in B. We also refer the reader to [Lac17, HRT17] for the linear case $m = 1$, where the weaker Dini condition was assumed on ω. The Dini condition was used in the bilinear setting $m = 2$ by Damián, Hormozi, and Li [DHL18] where, in addition, quantitative mixed multilinear $A_{\vec{p}}$–A_∞ bounds were considered.

To see how the pointwise sparse domination can be used to obtain sharp weighted bounds, note that by Proposition 5.3.4 we have

$$\|T(\vec{f})\cdot g\|_{L^1(\mathbf{R}^n)} \lesssim C_T \|M_{(\vec{r},1)}(\vec{f},g)\|_{L^1(\mathbf{R}^n)}$$

for all $\vec{f} \in L^\infty_c(\mathbf{R}^n)^m$, $g \in L^\infty_c(\mathbf{R}^n)$. Hence, by Theorem 5.3.6 and Remark 5.1.4, we obtain the following result:

Theorem 5.4.1. *Let T be an m-linear Calderón-Zygmund operator with modulus of continuity ω satisfying the* log–*Dini condition* (5.4.1). *Then for all $\vec{p} \in (1,\infty]^m$ with $p < \infty$*

and all $\vec{w} \in A_{\vec{p},(\vec{1},\infty)}$, *T has a bounded extension* $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)$ *with*

$$\|T\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} \lesssim_{\vec{p}} C_T\, [\vec{w}]_{\vec{p},(\vec{1},\infty)}^{\max\{p_1',\dots,p_m',p\}}.$$

Moreover, let (Ω,μ) *be a* σ*-finite measure space, and let* $\vec{t} \in (1,\infty]^m$ *with* $t<\infty$. *Then for all* $\vec{w} \in A_{\vec{p},(\vec{r},s)}$ *the tensor extension* $\widetilde{T}$ *of* T *is bounded* $L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n;L^{t_m}(\Omega)) \to L^p_w(\mathbf{R}^n;L^t(\Omega))$ *with*

$$\|\widetilde{T}\|_{L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n;L^{t_m}(\Omega))\to L^p_w(\mathbf{R}^n;L^t(\Omega))} \lesssim_{\vec{p},\vec{t}} [\vec{w}]_{\vec{p},(\vec{1},\infty)}^{\max\{t_1',\dots,t_m',t\}\cdot\max\left\{\frac{p_1'}{t_1'},\dots,\frac{p_m'}{t_m'},\frac{p}{t}\right\}}.$$

We point out that the extrapolation result, Theorem 5.1.2, yields another proof of this result using only the bound at $p_1=\ldots=p_m=m+1$ obtained in [DLP15] and hence, gives an alternative method of proving the conjecture from [LMS14] to obtain bounds in the cases $\frac{1}{p}>1$. This follows from the observation that in the proof of Theorem 5.4.1 we only require the bound at $p_1=\ldots=p_m=m+1$.

The exponent $\max\{t_1',\dots,t_m',t\}\cdot\max\left\{\frac{p_1'}{t_1'},\dots,\frac{p_m'}{t_m'},\frac{p}{t}\right\}$ in the vector-valued bound is not optimal and, in fact, we will see in Chapter 9 that it can be replaced by the sharp bound $\max\{p_1',\dots,p_m',p\}$ when $t\geq 1$, which coincides with the exponent in the scalar estimate.

Finally, we note that our bounds in the cases where $p_j=\infty$ are completely new.

5.4.2. The bilinear Hilbert transform

The bilinear Hilbert transform BHT, initially defined for $f_1,f_2\in\mathscr{S}(\mathbf{R})$, is given by

$$\mathrm{BHT}(f_1,f_2)(x) := \mathrm{p.v.}\int_{\mathbf{R}} f_1(x-y)f_2(x+y)\,\frac{\mathrm{d}y}{y},$$

and is an integral operator falling outside of the theory of bilinear Calderón-Zygmund operators. The reason for this is that its symbol $\mathrm{sgn}(\xi_1-\xi_2)$ in its representation as a Fourier multiplier

$$\mathrm{BHT}(f_1,f_2)(x) = -i\pi\int_{\mathbf{R}^2}\mathrm{sgn}(\xi_1-\xi_2)\hat{f}(\xi_1)\hat{g}(\xi_2)e^{2\pi i(\xi_1+\xi_2)x}\,\mathrm{d}\xi_1\mathrm{d}\xi_2$$

has a singularity along the line $\xi_1=\xi_2$ rather than in a single point, as is the case for Calderón-Zygmund operators.

This operator was introduced by A. Calderón and he wanted to know if it has a bounded extension $L^2(\mathbf{R})\times L^\infty(\mathbf{R})\to L^2(\mathbf{R})$. This question was answered by Lacey and Thiele [LT97, LT99] and they showed that BHT is bounded $L^{p_1}(\mathbf{R})\times L^{p_2}(\mathbf{R})\to L^p(\mathbf{R})$ for all $p_1,p_2\in(1,\infty]$ with $\frac{2}{3}<p<\infty$. It is an open problem whether we can obtain bounds for the remaining range $\frac{1}{2}<p\leq\frac{2}{3}$ or not. However, in the range of Lacey and Thiele several weighted bounds and vector-valued extensions have been obtained.

Let $r_1,r_2,s\in(1,\infty)$ satisfy one of the following equivalent properties:

(i) $\max\{\frac{1}{r_1},\frac{1}{2}\}+\max\{\frac{1}{r_2},\frac{1}{2}\}+\max\{\frac{1}{s'},\frac{1}{2}\}<2$;

(ii) There exist $\theta_1,\theta_2,\theta_3\in[0,1)$ with $\theta_1+\theta_2+\theta_3=1$ so that

$$\frac{1}{r_1}<\frac{1+\theta_1}{2},\qquad \frac{1}{r_2}<\frac{1+\theta_2}{2},\qquad \frac{1}{s}>\frac{1-\theta_3}{2}.$$

Using characterization (i), it was shown by Culiuc, Di Plinio and Ou in [CDO18] that

$$\|\operatorname{BHT}(f_1,f_2)\cdot g\|_{L^1(\mathbf{R})}\lesssim\|M_{(r_1,r_2,s')}(f_1,f_2,g)\|_{L^1(\mathbf{R})} \tag{5.4.2}$$

for all $f_1,f_2,g\in L^\infty_c(\mathbf{R})$.

Using characterization (ii), it was later shown by Benea and Muscalu in [BM17] that for $q\in(0,s)$ we also have the ℓ^q-type sparse domination

$$\|\operatorname{BHT}(f_1,f_2)\cdot g\|_{L^q(\mathbf{R})}\lesssim\|M_{(r_1,r_2,\frac{1}{\frac{1}{q}-\frac{1}{s}})}(f_1,f_2,g)\|_{L^q(\mathbf{R})} \tag{5.4.3}$$

for all $f_1,f_2,g\in L^\infty_c(\mathbf{R})$, as well as more general vector-valued sparse domination results. While we only require (5.4.2) to obtain bounds in the scalar-valued setting, we will see in Section 9.3 that allowing for this smaller q in (5.4.3) is important to obtain bounds in the vector-valued setting.

In [CDO18], it was deduced from (5.4.2) that for all $p_1,p_2\in(1,\infty)$ with $\vec{p}>\vec{r}$, in the Banach range $1<p<s$ and for all $\vec{w}\in A_{\vec{p},(\vec{r},s)}$ we have the weighted bounds $\operatorname{BHT}: L^{p_1}_{w_1}(\mathbf{R}^n)\times L^{p_2}_{w_2}(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)$. These weighted bounds were used in [CM18] to obtain weighted and vector-valued estimates in the range $p\le 1$ through extrapolation using certain product $A_{p_1,(r_1,s_1)}\times A_{p_2,(r_2,s_2)}$ weight classes. This result was extended in [LMO18] where the full multilinear weight classes $A_{(p_1,p_2),((r_1,r_2),s)}$ were used, but only the cases for finite p_j were treated. However, as shown in [LMM+19], their methods can be used to also obtain the cases with $p_j=\infty$. By applying Theorem 5.3.6 and Remark 5.1.4 to (5.4.2), we obtain the following result:

Theorem 5.4.2. *Let $r_1,r_2,s\in(1,\infty)$ satisfy one of the equivalent conditions (i), (ii). Then for all $p_1,p_2\in(1,\infty]$ with $\vec{p}>\vec{r}$, $p<s$ and all $\vec{w}\in A_{\vec{p},(\vec{r},s)}$,* BHT *extends to a bounded operator $L^{p_1}_{w_1}(\mathbf{R})\times L^{p_2}_{w_2}(\mathbf{R})\to L^p_w(\mathbf{R})$ with*

$$\|\operatorname{BHT}\|_{L^{p_1}_{w_1}(\mathbf{R})\times L^{p_2}_{w_2}(\mathbf{R})\to L^p_w(\mathbf{R})}\lesssim[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{r_1}}{\frac{1}{r_1}-\frac{1}{p_1}},\frac{\frac{1}{r_2}}{\frac{1}{r_2}-\frac{1}{p_2}},\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}.$$

Moreover, let (Ω,μ) be a σ-finite measure space, and let $t_1,t_2\in(1,\infty]$ with $\vec{t}>\vec{r}$, $t<s$. Then for all $\vec{w}\in A_{\vec{p},(\vec{r},s)}$ the tensor extension $\widetilde{\operatorname{BHT}}$ of BHT *is bounded $L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times L^{p_2}_{w_2}(\mathbf{R}^n;L^{t_2}(\Omega))\to L^p_w(\mathbf{R}^n;L^t(\Omega))$ with*

$$\begin{aligned}&\|\widetilde{\operatorname{BHT}}\|_{L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times L^{p_2}_{w_2}(\mathbf{R}^n;L^{t_2}(\Omega))\to L^p_w(\mathbf{R}^n;L^t(\Omega))}\\ &\qquad\lesssim_{\vec{p},\vec{t}}[\vec{w}]_{\vec{p},(\vec{1},\infty)}^{\max\left\{\frac{\frac{1}{r_1}}{\frac{1}{r_1}-\frac{1}{t_1}},\frac{\frac{1}{r_2}}{\frac{1}{r_2}-\frac{1}{t_2}},\frac{1-\frac{1}{s}}{\frac{1}{t}-\frac{1}{s}}\right\}\cdot\max\left\{\frac{\frac{1}{r_1}-\frac{1}{t_1}}{\frac{1}{r_1}-\frac{1}{p_1}},\frac{\frac{1}{r_2}-\frac{1}{t_2}}{\frac{1}{r_2}-\frac{1}{p_2}},\frac{\frac{1}{t}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}.\end{aligned} \tag{5.4.4}$$

As we noted in Remark 5.3.7, the quantitative bound (5.4.4) is not sharp and will be improved in Chapter 9.

5.4.3. Examples in the linear case $m = 1$

There is a wealth of examples of sparsely dominated operators in the case $m = 1$. Going beyond the class of Calderón-Zygmund operators, a general class of examples associated to semigroups was found in the work of Bernicot, Frey, and Petermichl [BFP16]. A very general sparse domination principle was established by Lerner in [Ler16] and was further generalized by Lerner and Ombrosi in [LO20] and by Lorist [Lor19], who also considered ℓ^q-type sparse domination in spaces of homogeneous type.

We point out several interesting examples here.

Example 5.4.3 (Rough homogeneous singular integral operators). Let (S^{d-1}, σ) denote the unit sphere in $\mathbf{R}^d$ with its Euclidean surface measure. For $\Omega \in L^\infty(S^{d-1})$ with $\int_{S^{d-1}} \Omega \, \mathrm{d}\sigma = 0$ we define the *rough homogeneous singular integral operator* T_Ω as

$$T_\Omega f(x) := \mathrm{p.v.} \int_{\mathbf{R}^d} f(x-y) \frac{\Omega(y/|y|)}{|y|^d} \, \mathrm{d}y.$$

One of the main results in the work [CCDO17] of Conde-Alonso, Culiuc, Di Plinio, and Ou is that for all $s \in (1,\infty)$ and all $f, g \in L^\infty_c(\mathbf{R}^d)$ there exists a sparse collection $\mathscr{S}$ such that

$$\|(T_\Omega f) g\|_{L^1(\mathbf{R}^d)} \lesssim s \|\Omega\|_{L^\infty(S^{d-1})} \sum_{Q \in \mathscr{S}} \langle f \rangle_{1,Q} \langle g \rangle_{s',Q} |Q|.$$

An alternative proof of this result was given by Lerner [Ler19].

Adapting the technique of Lerner from [Ler19], it was shown by Canto, Li, Roncal, and Tapiola in [CLRT19, Theorem 5.1], that for all $s \in (1,\infty)$, $q \in (0,1]$, and all $f, g \in L^\infty_c(\mathbf{R}^d)$ we have the ℓ^q-type sparse domination

$$\|(T_\Omega f) g\|_{L^q(\mathbf{R}^d)} \lesssim \frac{s}{q} \|\Omega\|_{L^\infty(S^{d-1})} \|M_{(r, \frac{1}{\frac{1}{q} - \frac{1}{s}})}(f,g)\|_{L^q(\mathbf{R}^n)}.$$

Example 5.4.4 (Riesz transform associated with elliptic second order divergence form operators). Let A be a complex, bounded, measurable matrix-valued function in $\mathbf{R}^n$ satisfying the ellipticity condition $\mathrm{Re}(A(x)\xi \cdot \overline{\xi}) \geq \lambda |\xi|^2$ for all $\xi \in \mathbf{C}^n$ and a.e. $x \in \mathbf{R}^n$. Then one can define a maximal accretive operator

$$Lf := -\operatorname{div}(A \nabla f)$$

which generates a semigroup $(e^{-tL})_{t>0}$. If $r \in [1,\infty)$, $s \in (1,\infty]$, then if both the semigroup and the family $(\sqrt{t} \nabla e^{-tL})_{t>0}$ satisfy L^r–L^s off-diagonal estimates, then it is shown in [BFP16] that the Riesz transform $R := \nabla L^{-1/2}$ satisfies

$$\|Rf \cdot g\|_{L^1(\mathbf{R}^n)} \lesssim \|M_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)}$$

for all $f, g \in L^\infty_c(\mathbf{R}^n)$. Moreover, if we are in dimension $n = 1$, then we can take $r = 1$ and $s = \infty$ so that R satisfies sparse domination in the full range. We refer the reader to [Aus07] for more values of r and s in other dimensions.

Example 5.4.5 (Riesz transform associated to the Neumann Laplacian). Suppose Δ is the Laplace operator associated with Neumann boundary conditions in a bounded convex doubling domain $U \subseteq \mathbf{R}^n$. As studied in [WY13], the Riesz transform $R := \nabla\Delta^{-1/2}$ will not in general have a kernel satisfying pointwise regularity estimates and is thus not in the class of Calderón-Zygmund operators. Nonetheless, this operator satisfies

$$\|Rf \cdot g\|_{L^1(U)} \lesssim \|M^{\mathscr{B}}_{(1,1)}(f,g)\|_{L^1(U)}$$

for all $f, g \in L^\infty_c(\mathbf{R}^n)$, where $\mathscr{B}$ is the collection of balls in U. Note that to apply our results to this example, we need to show that they remain valid when replacing $\mathbf{R}^n$ by the doubling metric measure space U. We refer the reader to Chapter 7, where we show how our results can be extended to general spaces of homogeneous type.

Example 5.4.6 (The Bochner-Riesz multiplier). For each $\delta \geq 0$, the Bochner-Riesz multiplier B_δ is defined as the Fourier multiplier $\mathscr{F}(B_\delta f) = (1-|\xi|^2)^\delta_+ \mathscr{F} f$, where $t_+ = \max(t, 0)$. For $\delta \geq (n-1)/2$, B_δ satisfies weighted bounds $\|B_\delta\|_{L^p_W(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)} < \infty$ for any $p \in (1, \infty)$ and any $w \in A_{p,(1,\infty)}$, see [Buc93, DR86, SS92].

The situation is more complicated when $0 < \delta < (n-1)/2$ and weighted bounds for such δ have, for example, been considered in [CDL12, Chr85, DMOS08]. The idea to quantify weighted bounds for B_δ for $0 < \delta < (n-1)/2$ through sparse domination was initiated by Benea, Bernicot, and Luque [BBL17]. It was shown by Lacey, Mena, and Reguera in [LMR19] that for this range of δ there are explicit subsets $R_{\delta,n}$ of the plane such that for any $(r, s) \in R_{\delta,n}$ we have

$$\|B_\delta f \cdot g\|_{L^1(\mathbf{R}^n)} \lesssim \|M_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)}.$$

We also refer the reader to the recent work by Kesler and Lacey [KL18] containing certain sparse endpoint bounds in dimension $n = 2$.

Example 5.4.7 (Spherical maximal operators). Let (S^{n-1}, σ) denote the unit sphere in $\mathbf{R}^n$ equipped with its normalized Euclidean surface measure σ. For a smooth function f on $\mathbf{R}^n$ we denote by $A_\rho f(x)$ the average of f over the sphere centered at x of radius $\rho > 0$, i.e.,

$$A_\rho f(x) := \int_{S^{d-1}} f(x - \rho\omega)\, \mathrm{d}\sigma(\omega).$$

We respectively define the lacunary spherical maximal operator and the full spherical maximal operator by

$$M_{\text{lac}} f := \sup_{k \in \mathbf{Z}} |A_{2^k} f|, \quad M_{\text{full}} f := \sup_{\rho > 0} |A_\rho f|,$$

the latter having been introduced by Stein [Ste76] and the former having been studied by Calderón [Cal79]. It was shown by Lacey [Lac19] that for explicit subsets L_n, F_n of the plane we have

$$\begin{aligned}\|M_{\text{lac}} f \cdot g\|_{L^1(\mathbf{R}^n)} &\lesssim \|M_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)}, \quad \text{for } (p_-,p_+) \in L_d,\\ \|M_{\text{full}} f \cdot g\|_{L^1(\mathbf{R}^n)} &\lesssim \|M_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)}, \quad \text{for } (p_-,p_+) \in F_d.\end{aligned}$$

In the recent work [RSS20] by Roncal, Shrivastava, and Shuin, the ideas of [Lac19] were adapted to prove sparse domination result for the bisublinear analogues of M_{lac} and M_{full}.

6

WEIGHTED ENDPOINT ESTIMATES

In this chapter we will be proving mixed A_1–A_∞ type endpoint estimates as a consequence of sparse domination in the case $m=1$.

This chapter as well as the next one are based on the paper

> D. Frey and B. Nieraeth. Weak and Strong Type A_1–A_∞ Estimates for Sparsely Dominated Operators. *Journal of Geometric Analysis*, 29(1):247–282, 2019.

6.1. WEAK-TYPE BOUNDS FOR MULTILINEAR OPERATORS FROM SPARSE DOMINATION

In this section we prove how one can obtain unweighted weak-type bounds from sparse domination. In the next section we adapt this proof in the linear case $m=1$ to a weighted setting. The proofs are based on the fact that for any σ-finite measure space (Ω,μ), any $r\in(0,\infty)$ we have the equivalence

$$\|f\|_{L^{r,\infty}(\Omega)}\eqsim\sup_{\substack{E\subseteq\Omega\\ 0<\mu(E)<\infty}}\inf_{\substack{E'\subseteq E\\ \mu(E)\leq 2\mu(E')}}\mu(E)^{\frac{1}{r}-1}\|f\chi_{E'}\|_{L^1(\Omega)} \tag{6.1.1}$$

for all $f\in L^{r,\infty}(\Omega)$, see [Gra14a, Exercise 1.4.14]. This description of the $L^{r,\infty}$ quasinorm in terms of L^1 norms allows us to utilize the sparse domination assumption to deduce weak-type bounds.

We have the following result:

Proposition 6.1.1. *Let T be an m-(sub)linear operator initially defined on $L^\infty_c(\mathbf{R}^n)^m$. Let $\vec{r}\in(0,\infty)^m$, $s\in(1,\infty]$, and suppose that*

$$\|T(\vec{f})g\|_{L^1(\mathbf{R}^n)}\leq C_T\|M_{(\vec{r},s')}(\vec{f},g)\|_{L^1(\mathbf{R}^n)}$$

for all $\vec{f}\in L^\infty_c(\mathbf{R}^n)^m$, $g\in L^\infty(\mathbf{R}^n)\cap L^1(\mathbf{R}^n)$. Then T has a bounded extension $L^{\vec{r}}(\mathbf{R}^n)\to L^{r,\infty}(\mathbf{R}^n)$ with

$$\|T\|_{L^{\vec{r}}(\mathbf{R}^n)\to L^{r,\infty}(\mathbf{R}^n)}\lesssim_{r,s}C_T.$$

Proof. Let $\vec{f}\in L^\infty_c(\mathbf{R}^n)^m$ with $\|f_j\|_{L^{r_j}(\mathbf{R}^n)}=1$. By the equivalence (6.1.1) and Lemma 3.2.4

we have

$$\begin{aligned}\|T(\vec{f})\|_{L^{r,\infty}(\mathbf{R}^n)} &\eqsim \sup_{\substack{E\subseteq\mathbf{R}^n\\0<|E|<\infty}} \inf_{\substack{E'\subseteq E\\|E|\le 2|E'|}} |E|^{\frac{1}{r}-1}\|T(\vec{f})\chi_{E'}\|_{L^1(\mathbf{R}^n)}\\ &\le C_T \sup_{\substack{E\subseteq\mathbf{R}^n\\0<|E|<\infty}} \inf_{\substack{E'\subseteq E\\|E|\le 2|E'|}} |E|^{\frac{1}{r}-1}\|M_{(\vec{r},s')}(\vec{f},\chi_{E'})\|_{L^1(\mathbf{R}^n)}\\ &\eqsim C_T \sup_{\substack{E\subseteq\mathbf{R}^n\\0<|E|<\infty}} \max_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} \inf_{\substack{E'\subseteq E\\|E|\le 2|E'|}} |E|^{\frac{1}{r}-1}\|M^{\mathscr{D}^\alpha}_{(\vec{r},s')}(\vec{f},\chi_{E'})\|_{L^1(\mathbf{R}^n)}.\end{aligned} \tag{6.1.2}$$

Fix a dyadic grid $\mathscr{D}=\mathscr{D}^\alpha$ and $E\subseteq\mathbf{R}^n$ with $0<|E|<\infty$. Define

$$\Omega_j := \Big\{x\in\mathbf{R}^n; M^{\mathscr{D}}_{r_j}(f_j)(x) > \Big(\frac{2m}{|E|}\Big)^{\frac{1}{r_j}}\Big\},\quad \Omega := \bigcup_{j=1}^m \Omega_j.$$

Then, since $\|M^{\mathscr{D}}_{r_j}\|_{L^{r_j}(\mathbf{R}^n)\to L^{r_j,\infty}(\mathbf{R}^n)}\le 1$ by Lemma 3.2.5, we have

$$|\Omega| \le \sum_{j=1}^m |\Omega_j| \le \sum_{j=1}^m \frac{|E|}{2m} = \frac{|E|}{2}.$$

Setting $E' := E\backslash\Omega$, we find $|E|\le|E'|+|\Omega|\le|E'|+|E|/2$ so that $|E|\le 2|E'|$.

Now, using an analogous argument as in the proof of Lemma 3.2.5 we can write $\Omega_j = \bigcup_{P_j\in\mathscr{P}_j} P_j$ where $\mathscr{P}_j\subseteq\mathscr{D}$ is the pairwise disjoint collection cubes P_j that are maximal with respect to the inequality $\langle f_j\rangle_{r_j,P_j} > \big(\frac{2m}{|E|}\big)^{\frac{1}{r_j}}$. Note that unlike in Lemma 3.2.5 where we had to consider finite collections of cubes, the fact that these maximal cubes exist follows from the fact that Ω_j has finite measure. Indeed, since the increasing sequence of cubes in $\mathscr{D}$ containing a point $x\in\mathbf{R}^n$ has a strictly increasing sequence of measures converging to ∞, the cubes P_j in this sequence satisfying $\langle f_j\rangle_{r_j,P_j} > \big(\frac{2m}{|E|}\big)^{\frac{1}{r_j}}$ must be bounded from above as they are contained in Ω_j, proving that there is a maximal one.

Now, we write

$$g_j := |f_j|\chi_{\Omega_j^c} + \sum_{P_j\in\mathscr{P}_j}\langle f_j\rangle_{r_j,P_j}\chi_{P_j},\quad b_j := \sum_{P_j\in\mathscr{P}_j} b_{j,P_j} := \sum_{P_j\in\mathscr{P}_j}\big(|f_j|^{r_j} - \langle f_j\rangle_{r_j,P_j}^{r_j}\big)\chi_{P_j}$$

so that $|f_j|^{r_j} = g_j^{r_j} + b_j$. Then for all $P_j\in\mathscr{P}_j$ we have

$$\operatorname{supp} b_{j,P_j}\subseteq P_j,\quad \int_{P_j} b_{j,P_j}\,\mathrm{d}x = 0, \tag{6.1.3}$$

$$\|g_j\|_\infty \le \Big(\frac{2^{n+1}m}{|E|}\Big)^{\frac{1}{r_j}},\quad \|g_j\|_{L^{r_j}(\mathbf{R}^n)} = 1. \tag{6.1.4}$$

Here, for the first bound on g_j we used the fact that on Ω_j^c we have $|f_j|\le M^{\mathscr{D}}_{r_j}(f_j)(x)\le \big(\frac{2m}{|E|}\big)^{\frac{1}{r_j}}$ by the Lebesgue Differentiation Theorem, while for the parent $\widehat{P_j}$ of a cube $P_j\in\mathscr{P}_j$ we have $\langle f_j\rangle_{r_j,P_j}\le 2^{\frac{n}{r_j}}\langle f_j\rangle_{r_j,\widehat{P_j}}\le 2^{\frac{n}{r_j}}\big(\frac{2m}{|E|}\big)^{\frac{1}{r_j}}$ by maximality.

Next, we claim that

$$M^{\mathscr{D}}_{(\vec{r},s')}(\vec{f},\chi_{E'}) = M^{\mathscr{D}}_{(\vec{r},s')}(\vec{g},\chi_{E'}). \tag{6.1.5}$$

Indeed, fix a cube $Q \in \mathscr{D}$. If $Q \cap E' = \varnothing$, then $\langle \chi_{E'}\rangle_{s',Q} = 0$. For the other case, suppose $Q \cap E' \neq \varnothing$. If $P_j \in \mathscr{P}_j$ satisfies $P_j \cap Q \neq \varnothing$, we must have $P_j \subseteq Q$ (otherwise $Q \subseteq P_j \subseteq \Omega_j \subseteq \Omega$, but $E' \cap \Omega = \varnothing$). But then, by (6.1.3), we have $\langle b_{j,P_j}\rangle_{1,Q} = 0$. If on the other hand $P_j \cap Q = \varnothing$, then we similarly have $\langle b_{j,P_j}\rangle_{1,Q} = 0$. Hence,

$$\langle f_j\rangle^{r_j}_{r_j,Q} = \langle g_j\rangle^{r_j}_{r_j,Q} + \sum_{P_j\in\mathscr{P}_j} \langle b_{j,P_j}\rangle_{1,Q} = \langle g_j\rangle^{r_j}_{r_j,Q}.$$

The claim (6.1.5) now follows from the definition of $M^{\mathscr{D}}_{(\vec{r},s')}$.

Finally, we note that for $\frac{1}{p_j} := \frac{\tau}{r_j}$ with $\frac{1}{\tau} := \frac{1}{s'} + \frac{1}{r}$ it follows from (6.1.4) that

$$\|g_j\|_{L^{p_j}(\mathbf{R}^n)} \lesssim_{r_j} |E|^{-\frac{1}{r_j}\left(1-\frac{r_j}{p_j}\right)} \|g_j\|^{\frac{r_j}{p_j}}_{L^{r_j}(\mathbf{R}^n)} = |E|^{\frac{1}{p_j}-\frac{1}{r_j}}$$

so that by Hölder's inequality and Lemma 3.2.5 we have

$$\begin{aligned}\|M^{\mathscr{D}}_{(\vec{r},s')}(\vec{g},\chi_{E'})\|_{L^1(\mathbf{R}^n)} &\leq \left(\frac{\frac{1}{r}+\frac{1}{s'}}{\frac{1}{r}-\frac{1}{s}}\right)^{\frac{1}{r}+\frac{1}{s'}} \Big(\prod_{j=1}^m \|g_j\|_{L^{p_j}(\mathbf{R}^n)}\Big)\|\chi_{E'}\|_{L^{p'}(\mathbf{R}^n)}\\ &\lesssim_{r,s} |E'|^{\frac{1}{p'}}|E|^{\frac{1}{p}-\frac{1}{r}} \leq |E|^{1-\frac{1}{r}}.\end{aligned}$$

The assertion now follows from (6.1.2) and (6.1.5). □

6.2. WEIGHTED ENDPOINT BOUNDS FOR LINEAR OPERATORS

We will be considering estimates for (sub)linear operators at the endpoint $p = r$ with weights in the class $A_{r,(r,s)}$. Our bounds will be in terms of the constant $[w]_{r,(r,s)} = [w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]^{\frac{1}{r}-\frac{1}{s}}_{A_1}$ and the Fujii-Wilson constant $[w^{-1}]^{\mathrm{FW}}_{r',s'} = [w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]^{\frac{1}{r'}}_{A_\infty}$ which was introduced in Section 3.3. It follows from Proposition 3.1.8 that $w \in A_{r,(r,s)}$ if and only if $[w^r]^{\frac{1}{r}}_{A_1} = [w]_{r,(r,\infty)} < \infty$ and $w^r \in \mathrm{RH}_{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{s}}}$. We will establish the following weak-type bounds:

Theorem 6.2.1. *Let T be a (sub)linear operator initially defined on $L^\infty_c(\mathbf{R}^n)$. Let $r \in [1,\infty)$, $s \in (1,\infty]$, and suppose that*

$$\|Tf\cdot g\|_{L^1(\mathbf{R}^n)} \leq C_T\|M_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)}$$

for all $f \in L^\infty_c(\mathbf{R}^n)$, $g \in L^\infty(\mathbf{R}^n)\cap L^1(\mathbf{R}^n)$. Then for all $w \in A_{r,(r,s)}$, T has a bounded extension $L^r_w(\mathbf{R}^n) \to L^{r,\infty}_w(\mathbf{R}^n)$ with

$$\|T\|_{L^r_w(\mathbf{R}^n)\to L^{r,\infty}_w(\mathbf{R}^n)} \lesssim_{r,s} C_T\psi(w),$$

where

$$\psi(w) = \begin{cases} [w]_{A_1} \log(e + [w]_{A_\infty}) & \text{if } r = 1,\ s = \infty; \\ [w^r]_{A_1}^{\frac{1}{r}} [w^r]_{A_\infty}^{\frac{1}{r'}} \log(e + [w^r]_{A_\infty})^{\frac{2}{r}} & \text{if } r > 1,\ s = \infty; \\ [w^{s'}]_{A_\infty} [w]_{A_1} [w]_{\mathrm{RH}_{s'}} & \text{if } r = 1,\ s < \infty; \\ [w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}^{1+\frac{1}{r}} [w^r]_{A_1}^{\frac{1}{r}} [w^r]_{\mathrm{RH}_{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{s}}}}^{\frac{1}{r}} & \text{if } r > 1,\ s < \infty. \end{cases}$$

We will also prove the following result:

Theorem 6.2.2. *Let T be a (sub)linear operator initially defined on $L^\infty_c(\mathbf{R}^n)$. Let $r \in [1,\infty)$, $s \in (1,\infty]$, and suppose that*

$$\|Tf \cdot g\|_{L^1(\mathbf{R}^n)} \le C_T \|M_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)}$$

for all $f, g \in L^\infty_c(\mathbf{R}^n)$. Then for all $p \in (r,s)$, $w \in A_{p,(p,s)}$, T has a bounded extension $L^p_w(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)$ with

$$\|Tf\|_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} \lesssim_{r,s} C_T c_{p,r,s} [w^{-1}]^{\mathrm{FW}}_{p',s'} [w]_{p,(p,s)}, \tag{6.2.1}$$

where

$$c_{p,r,s} = \left(\frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}} \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}} \left(\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}}.$$

In particular, for all $p \in (r,s)$, $w \in A_{p,(p,s)}$ we have

$$\|T\|_{L^p_w(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} \lesssim_{p,r,s} C_T [w]_{p,(p,s)}^{\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}} = C_T [w^{\frac{1}{\frac{1}{p}-\frac{1}{s}}}]_{A_1}^{\frac{1}{s'}}. \tag{6.2.2}$$

Moreover, if $\omega(T) \ge 1 - \frac{1}{s}$, where

$$\omega(T) = \sup\left\{\omega \in [0,\infty) : \forall \varepsilon > 0, \limsup_{\frac{1}{p}\to\frac{1}{s}} \left(\frac{1}{p} - \frac{1}{s}\right)^{\omega-\varepsilon} \|T\|_{L^p(\mathbf{R}^n)\to L^p(\mathbf{R}^n)} = \infty\right\},$$

then the exponent of the weight constant in (6.2.2) *is the smallest possible one.*

For the proofs of these results we will require several lemmata. Throughout these results we will work in a fixed dyadic grid $\mathscr{D} = \mathscr{D}^\alpha$.

As an analogue to [LOP08, Lemma 3.2] and [HP13, Lemma 6.1], our main lemma is the following:

Lemma 6.2.3. *Let $r \in (0,\infty)$, $s \in (1,\infty]$, $p \in (r,s)$ and $\frac{1}{q} \in (0, \frac{1}{p} - \frac{1}{s})$. Then*

$$\begin{aligned} &\|M^{\mathscr{D}}_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)} \\ &\lesssim \left(\frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}} \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}} \left(\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}} \left[q\left(\frac{1}{p}-\frac{1}{s}\right)\right]'^{\frac{1}{p'}} \|f\|_{L^p_{M^{\mathscr{D}}_q w}(\mathbf{R}^n)} \|g\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)} \end{aligned}$$

for all $f \in L^r_{\mathrm{loc}}(\mathbf{R}^n)$, $g \in L^{s'}_{\mathrm{loc}}(\mathbf{R}^n)$ and $w \in L^q_{\mathrm{loc}}(\mathbf{R}^n)$ non-zero on the support of g.

We point out that a similar type of result is established in [DHL17, Theorem B].

Remark 6.2.4. In the unweighted case it follows from Hölder's inequality and 3.2.5 that

$$\|M^{\mathscr{D}}_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)} \le \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}\left(\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}}\|f\|_{L^p(\mathbf{R}^n)}\|g\|_{L^{p'}(\mathbf{R}^n)}.$$

Thus, it appears that adding the weight accounts for the extra term $\left(\frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}$, which depends on p when $r \neq 1$. This extra term appears in Lemma 6.2.6 below and it causes the additional terms in the quantitative bounds for $r \neq 1$ in Theorem 6.2.1. At this moment it is not clear whether this term can be removed or not.

We break up the proof of the main lemma into another series of lemmata.

Lemma 6.2.5. *For all $f \in L^r_{\mathrm{loc}}(\mathbf{R}^n)$, $g \in L^{s'}_{\mathrm{loc}}(\mathbf{R}^n)$ and $\beta \in (0,1]$ we have the pointwise estimate*

$$M^{\mathscr{D}}_{(r,s')}(f,g) \le M^{\mathscr{D}}_r\left((M^{\mathscr{D}}_{s'}g)^{1-\beta}f\right)(M^{\mathscr{D}}_{s'}g)^{\beta}.$$

Proof. Fix $x \in \mathbf{R}^n$ and let $Q \in \mathscr{D}$ with $x \in Q$. Then

$$\langle g\rangle_{s',Q} = \langle g\rangle_{s',Q}^{\beta}\langle g\rangle_{s',Q}^{1-\beta} \le \langle g\rangle_{s',Q}^{\beta}\operatorname*{ess\,inf}_{y\in Q}(M^{\mathscr{D}}_{s'}g)(y)^{1-\beta}$$

so that

$$\langle f\rangle_{r,Q}\langle g\rangle_{s',Q} \le \langle (M^{\mathscr{D}}_{s'}g)^{1-\beta}f\rangle_{r,Q}\langle g\rangle_{s',Q}^{\beta} \le M^{\mathscr{D}}_r\left((M^{\mathscr{D}}_{s'}g)^{1-\beta}f\right)(x)(\mathscr{M}_{s'}g)(x)^{\beta}.$$

Taking a supremum over all $Q \in \mathscr{D}$ with $x \in Q$ proves the result. □

Lemma 6.2.6. *Let $r \in (0,\infty)$, $s \in (1,\infty]$, $p \in (r,s)$, and $q \in (p,\infty)$. Then*

$$\|M^{\mathscr{D}}_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)} \lesssim \left(\frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}\left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}\|f\|_{L^p_{M^{\mathscr{D}}_q w}(\mathbf{R}^n)}\|M^{\mathscr{D}}_{s'}g\|_{L^{p'}_{(M^{\mathscr{D}}_q w)^{-1}}(\mathbf{R}^n)},$$

for all $f \in L^r_{\mathrm{loc}}(\mathbf{R}^n)$, $g \in L^{s'}_{\mathrm{loc}}(\mathbf{R}^n)$, and $w \in L^q_{loc}(\mathbf{R}^n)$.

For the proof of this lemma we require two results on dyadic maximal operators. The first is a version of a classical result of Fefferman and Stein [FS71].

Lemma 6.2.7. *Let $r \in (0,\infty)$, and let w be a weight. Then for all $p \in (r,\infty]$ the operator $M^{\mathscr{D}}_r$ is bounded $L^p_{M^{\mathscr{D}}_p w}(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)$ with*

$$\|M^{\mathscr{D}}_r f\|_{L^p_{M^{\mathscr{D}}_p w}(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} \le \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{q}}.$$

Moreover, $M^{\mathscr{D}}_r$ is bounded $L^r_{M^{\mathscr{D}}_r w}(\mathbf{R}^n) \to L^{r,\infty}_w(\mathbf{R}^n)$ with $\|M^{\mathscr{D}}_r\|_{L^r_{M^{\mathscr{D}}_r w}(\mathbf{R}^n)\to L^{r,\infty}_w(\mathbf{R}^n)} \le 1$.

Proof. The proof of this result is very similar to the proof of Lemma 3.2.5. By the same reasoning as in that proof, it suffices to prove the weak-type bound and by a rescaling argument, it suffices to prove this in the case $r=1$. Let $\mathscr{F}\subseteq\mathscr{D}$ be a finite collection of cubes and fix $f\in L^1(\mathbf{R}^n;M^{\mathscr{D}}w)$, $\lambda>0$. Let $\mathscr{P}$ be the pairwise disjoint collection of cubes P that are maximal with respect to the inequality $\langle f\rangle_{1,P}>\lambda$ so that $\{x\in\mathbf{R}^n:M^{\mathscr{F}}(f)(x)>\lambda\}=\cup_{P\in\mathscr{P}}P$.

Now, since

$$\langle f\rangle_{1,P}w(P)\leq\big(\operatorname*{ess\,inf}_{y\in P}M^{\mathscr{D}}w(y)\big)\langle f\rangle_{1,P}|P|\leq\int_P fM^{\mathscr{D}}w\,\mathrm{d}x,$$

we have

$$\begin{aligned}\lambda w(\{x\in\mathbf{R}^n:M^{\mathscr{F}}(f)(x)>\lambda\})&=\sum_{P\in\mathscr{P}}\lambda w(P)\leq\sum_{P\in\mathscr{P}}\langle f\rangle_{1,P}w(P)\\&\leq\sum_{P\in\mathscr{P}}\int_P fM^{\mathscr{D}}w\,\mathrm{d}x=\int_{\{x\in\mathbf{R}^n:M^{\mathscr{F}}(f)(x)>\lambda\}}fM^{\mathscr{D}}w\,\mathrm{d}x\\&\leq\|f\|_{L^1(\mathbf{R}^n;M^{\mathscr{D}}w)}.\end{aligned}$$

Thus, taking a supremum over $\lambda>0$ yields $\|M^{\mathscr{F}}(f)\|_{L^{1,\infty}(\mathbf{R}^n;w)}\leq\|f\|_{L^1(\mathbf{R}^n;M^{\mathscr{D}}w)}$. By monotonicity of the measure we have

$$\|M^{\mathscr{D}}(f)\|_{L^{1,\infty}(\mathbf{R}^n;w)}\leq\sup_{\substack{\mathscr{F}\subseteq\mathscr{D}\\ \mathscr{F}\text{ finite}}}\|M^{\mathscr{F}}(f)\|_{L^{1,\infty}(\mathbf{R}^n;w)}\leq\|f\|_{L^1(\mathbf{R}^n;M^{\mathscr{D}}w)},$$

proving the assertion follows. □

The second result we need can be found in [CR80, Proposition 2], see also [Gra14a, Theorem 7.2.7], and states that

$$M^{\mathscr{D}}((M^{\mathscr{D}}f)^\delta)\lesssim\frac{(M^{\mathscr{D}}f)^\delta}{1-\delta}\tag{6.2.3}$$

for all $\delta\in(0,1)$ and $f\in L^1_{\mathrm{loc}}(\mathbf{R}^n)$.

Proof of Lemma 6.2.6. We will prove the stronger assertion

$$\|M^{\mathscr{D}}_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)}\lesssim\left(\frac{1-\frac1t}{\frac1r-\frac1t}\right)^{\frac1r}\left(\frac{\frac1r}{\frac1r-\max\left(\frac1t,\frac1p\right)}\right)^{\frac1r}\|f\|_{L^p_{(M^{\mathscr{D}}_pw)^{\frac{1-t'}{1-p'}}}(\mathbf{R}^n)}\|M^{\mathscr{D}}_{s'}g\|_{L^{p'}_{(M^{\mathscr{D}}_pw)^{-\frac{1-t'}{1-p'}}}(\mathbf{R}^n)},$$

valid for all $t>r$, generalizing a version of the result [Ler10, Theorem 1.7] in which the case $r=1$, $s=\infty$ is treated. The result of the lemma follows with $\frac1t:=\frac{\frac1q}{\frac1{p'}+\frac1q}<\frac1r$.

We set

$$\beta:=\min\left(p'\frac{\frac1t\frac1{r'}+\frac1{t'}\frac1r}{\frac1t+\frac1r},1\right)$$

so that $0 < \beta \le 1$.

By Lemma 6.2.5 and by Hölder's inequality we find that

$$\begin{aligned}\|M^{\mathscr{D}}_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)} &\le \|M^{\mathscr{D}}_r\left((M^{\mathscr{D}}_{s'}g)^{1-\beta}f\right)(M^{\mathscr{D}}_{s'}g)^{\beta}\|_{L^1(\mathbf{R}^n)}\\ &\le I\|M^{\mathscr{D}}_{s'}g\|^{\beta}_{L^{p'}_{(M^{\mathscr{D}}_p w)^{-\frac{1-t'}{1-p'}}}(\mathbf{R}^n)},\end{aligned} \tag{6.2.4}$$

where

$$I = \left\|M^{\mathscr{D}}_r\left((M^{\mathscr{D}}_{s'}g)^{1-\beta}f\right)\right\|_{L^{\frac{\frac{1}{\beta}}{\frac{1}{\beta}-\frac{1}{p'}}}_{(M^{\mathscr{D}}_p w)^{\beta\frac{1-t'}{1-p'}}}(\mathbf{R}^n)}.$$

We will consider two cases. First assume that

$$p'\frac{\frac{1}{t}\frac{1}{r'}+\frac{1}{t'}\frac{1}{r}}{\frac{1}{t}+\frac{1}{r}} \ge 1$$

and $\beta = 1$. Then

$$(p'-1)\left(\frac{1}{t}\frac{1}{r'}+\frac{1}{t'}\frac{1}{r}\right) \ge 2\frac{1}{t}\frac{1}{r}$$

so that

$$\frac{1-t'}{1-p'} \le \frac{1}{2}\left(1+\frac{t'-1}{r'-1}\right) < 1$$

by the assumption $r < t$. Then it follows from Lemma 6.2.7 and (6.2.3) that

$$\begin{aligned}I = \|M^{\mathscr{D}}_r f\|_{L^p_{(M^{\mathscr{D}}_p w)^{\frac{1-t'}{1-p'}}}(\mathbf{R}^n)} &\le \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}\|f\|_{L^p_{M^{\mathscr{D}}_p((M^{\mathscr{D}}_p w)^{\frac{1-t'}{1-p'}})}(\mathbf{R}^n)}\\ &\lesssim \left(\frac{1}{1-\frac{1-t'}{1-p'}}\right)^{\frac{1}{p}}\left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}\|f\|_{L^p_{(M^{\mathscr{D}}_p w)^{\frac{1-t'}{1-p'}}}(\mathbf{R}^n)}\\ &\le \left(\frac{2}{r}\frac{1-\frac{1}{t}}{\frac{1}{r}-\frac{1}{t}}\right)^{\frac{1}{p}}\left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}\|f\|_{L^p_{(M^{\mathscr{D}}_p w)^{\frac{1-t'}{1-p'}}}(\mathbf{R}^n)},\end{aligned}$$

as desired.

For the second case we assume that

$$p'\frac{\frac{1}{t}\frac{1}{r'}+\frac{1}{t'}\frac{1}{r}}{\frac{1}{t}+\frac{1}{r}} < 1 \quad\text{and}\quad \beta = p'\frac{\frac{1}{t}\frac{1}{r'}+\frac{1}{t'}\frac{1}{r}}{\frac{1}{t}+\frac{1}{r}}.$$

Then, using $r < t$, we note that

$$\frac{\frac{1}{\beta}-\frac{1}{p'}}{\frac{1}{\beta}} = \frac{2\frac{1}{t}\frac{1}{r}}{\frac{1}{r}+\frac{1}{t}} < \frac{1}{r} \quad\text{and}\quad \frac{(1-t')\frac{1}{p'}}{\frac{1}{p'}-\frac{1}{\beta}} = \frac{1}{2}\left(1+\frac{t'-1}{r'-1}\right) < 1.$$

Hence, we may apply Lemma 6.2.7 and (6.2.3) so that

$$
\begin{aligned}
I &\lesssim \left(\frac{1}{1-\frac{(1-t')\frac{1}{p'}}{\frac{1}{p'}-\frac{1}{\beta}}}\right)^{\frac{\frac{1}{\beta}-\frac{1}{p'}}{\frac{1}{\beta}}} \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{\frac{1}{\beta}-\frac{1}{p'}}{\frac{1}{\beta}}}\right)^{\frac{1}{r}} \|(M^{\mathscr{D}}_{s'}g)^{1-\beta}f\|_{L^{\frac{\frac{1}{\beta}}{\frac{1}{\beta}-\frac{1}{p'}}}_{(M^{\mathscr{D}}_{p}w)^{\beta\frac{1-t'}{1-p'}}}(\mathbf{R}^n)} \qquad (6.2.5)\\
&= \left(\frac{2}{r}\frac{1-\frac{1}{t}}{\frac{1}{r}-\frac{1}{t}}\right)^{\frac{\frac{1}{\beta}-\frac{1}{p'}}{\frac{1}{\beta}}} \left(\frac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r}-\frac{1}{t}}\right)^{\frac{1}{r}} \|(M^{\mathscr{D}}_{s'}g)^{1-\beta}f\|_{L^{\frac{\frac{1}{\beta}}{\frac{1}{\beta}-\frac{1}{p'}}}_{(M^{\mathscr{D}}_{p}w)^{\beta\frac{1-t'}{1-p'}}}(\mathbf{R}^n)}.
\end{aligned}
$$

By Hölder's inequality we find that

$$
\begin{aligned}
\|(M^{\mathscr{D}}_{s'}g)^{1-\beta}f\|_{L^{\frac{\frac{1}{\beta}}{\frac{1}{\beta}-\frac{1}{p'}}}_{(M^{\mathscr{D}}_{p}w)^{\beta\frac{1-t'}{1-p'}}}(\mathbf{R}^n)} &\le \|(M^{\mathscr{D}}_{s'}g)^{1-\beta}\|_{L^{\frac{p'}{1-\beta}}_{(M^{\mathscr{D}}_{p}w)^{(\beta-1)\frac{1-t'}{1-p'}}}(\mathbf{R}^n)} \|f\|_{L^p_{(M^{\mathscr{D}}_{p}w)^{\frac{1-t'}{1-p'}}}(\mathbf{R}^n)}\\
&= \|M^{\mathscr{D}}_{s'}g\|^{1-\beta}_{L^{p'}_{(M^{\mathscr{D}}_{p}w)^{-\frac{1-t'}{1-p'}}}(\mathbf{R}^n)} \|f\|_{L^p_{(M^{\mathscr{D}}_{p}w)^{\frac{1-t'}{1-p'}}}(\mathbf{R}^n)}.
\end{aligned}
$$

Hence, by (6.2.5) we have

$$
I \lesssim \left(\frac{1-\frac{1}{t}}{\frac{1}{r}-\frac{1}{t}}\right)^{\frac{1}{r}} \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{t}}\right)^{\frac{1}{r}} \|f\|_{L^p_{(M^{\mathscr{D}}_{p}w)^{\frac{1-t'}{1-p'}}}(\mathbf{R}^n)} \|M^{\mathscr{D}}_{s'}g\|^{1-\beta}_{L^{p'}_{(M^{\mathscr{D}}_{p}w)^{-\frac{1-t'}{1-p'}}}(\mathbf{R}^n)}.
$$

Thus, the result follows from (6.2.4).

By combining the two cases, the assertion follows. □

Proof of Lemma 6.2.3. Since $\frac{1}{q} < \frac{1}{p} - \frac{1}{s} \le \frac{1}{s'}$, we note that by Hölder's inequality we have $M^{\mathscr{D}}_{s'}g \le (M^{\mathscr{D}}_{q}w)(M^{\mathscr{D}}_{\frac{1}{\frac{1}{s'}-\frac{1}{q}}}(gw^{-1}))$ where gw^{-1} is well-defined since w is non-zero on the support of g. Then, setting $\frac{1}{\widetilde{q}} := \frac{\frac{1}{q}}{\frac{1}{p}-\frac{1}{s}}$ we have

$$
\begin{aligned}
\|M^{\mathscr{D}}_{s'}g\|_{L^{p'}_{(M^{\mathscr{D}}_{q}w)^{-1}}(\mathbf{R}^n)} &\le \|M^{\mathscr{D}}_{\frac{1}{\frac{1}{s'}-\frac{1}{q}}}(gw^{-1})\|_{L^{p'}(\mathbf{R}^n)}\\
&\le \left(\frac{\frac{1}{s'}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{s}-\frac{1}{q}}\right)^{\frac{1}{s'}-\frac{1}{q}} \|g\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)} \qquad (6.2.6)\\
&\le \left(\frac{\frac{1}{s'}}{\frac{1}{p}-\frac{1}{s}}\right)^{\frac{1}{s'}} (\widetilde{q}')^{\frac{1}{s'}-\frac{1}{q}} \|g\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)}.
\end{aligned}
$$

Now, note that since $\frac{1}{p}-\frac{1}{s}\leq\frac{1}{p}\leq 1$, we have

$$\frac{1}{s'}-\frac{1}{q}=\frac{1}{p'}+\frac{1}{p}-\frac{1}{s}-\frac{1}{q}\leq\frac{1}{p'}+\frac{\frac{1}{p}-\frac{1}{s}-\frac{1}{q}}{\frac{1}{p}-\frac{1}{s}}=\frac{1}{p'}+\frac{1}{\widetilde{q}'}$$

so that $(\widetilde{q}')^{\frac{1}{s'}-\frac{1}{q}}\leq(\widetilde{q}')^{\frac{1}{p'}+\frac{1}{\widetilde{q}'}}$. By maximizing the function $t\mapsto t^{1/t}$ for $t\geq 1$, we note that $(\widetilde{q}')^{\frac{1}{\widetilde{q}'}}\leq e^{\frac{1}{e}}$. By combining this with (6.2.6) and Lemma 6.2.6, the result follows. □

Finally, we need the following result:

Lemma 6.2.8. *Let $r\in[1,\infty)$, $s\in(1,\infty]$, $w\in A_{r,(r,s)}$, let $q\in(1,\infty)$ with $q'=2^{n+1}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}$. Then*

$$M_{\frac{q}{\frac{1}{r}-\frac{1}{s}}}w\lesssim_{r,s}[w]_{r,(r,s)}w.$$

Proof. Let Q be a cube. By applying Corollary 3.3.14 with w replaced by w^{-1}, $p=s$, and $\beta=q$, we obtain

$$\langle w\rangle_{\frac{q}{\frac{1}{r}-\frac{1}{s}},Q}\lesssim_{r,s}2^{\frac{1}{r}-\frac{1}{s}}\langle w\rangle_{\frac{1}{\frac{1}{r}-\frac{1}{s}},Q}\leq[w]_{r,(r,s)}\operatorname*{ess\,inf}_{y\in Q}w(y).$$

Picking $x\in\mathbf{R}^n$ and taking a supremum over all cubes Q containing x proves the assertion. □

Proof of Theorem 6.2.2. Let $\frac{1}{q}\in(0,\frac{1}{p}-\frac{1}{s})$ be such that $\left[q\left(\frac{1}{p}-\frac{1}{s}\right)\right]'=2^{n+1}[w^{\frac{1}{\frac{1}{p}-\frac{1}{s}}}]_{A_\infty}$. Then it follows from Lemma 6.2.3 and Lemma 6.2.8 that

$$\|Tf\cdot g\|_{L^1(\mathbf{R}^n)}\lesssim_{r,s}C_T\left(\frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}\left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}}\left(\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}}[w^{\frac{1}{\frac{1}{p}-\frac{1}{s}}}]_{A_\infty}^{\frac{1}{p'}}[w]_{p,(p,s)}\|f\|_{L^p_w(\mathbf{R}^n)}\|g\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n)},$$

for all $f,g\in L^\infty_c(\mathbf{R}^n)$. Then, since $[w^{\frac{1}{\frac{1}{p}-\frac{1}{s}}}]_{A_\infty}^{\frac{1}{p'}}=[w^{-1}]^{\mathrm{FW}}_{p',s'}$, (6.2.1) follows from duality and density. For the next assertion, note that by Proposition 3.3.3(ii) we have

$$[w^{-1}]^{\mathrm{FW}}_{p',s'}\lesssim_s[w^{-1}]_{p',(s',\infty)}^{\frac{1-\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}}=[w]_{p,(1,s)}^{\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}-1}.$$

Since $[w]_{p,(1,s)}\leq[w]_{p,(p,s)}$ by Hölder's inequality, combining this result with (6.2.1) proves (6.2.2). Finally, the optimality result is a consequence of Theorem 5.2.6. □

Proof of Theorem 6.2.1. The proof uses arguments similar to the ones presented in the proof of Proposition 6.1.1. Let $f \in L^r_w(\mathbf{R}^n)$ with $\|f\|_{L^r_w(\mathbf{R}^n)} = 1$. We have

$$\begin{aligned}
\|Tf\|_{L^{r,\infty}_w(\mathbf{R}^n)} &\eqsim \sup_{\substack{E\subseteq \mathbf{R}^n\\ 0<w^r(E)<\infty}} \inf_{\substack{E'\subseteq E\\ w^r(E)\le 2w^r(E')}} w^r(E)^{\frac{1}{r}-1}\|Tf\cdot w^r\chi_{E'}\|_{L^1(\mathbf{R}^n)}\\
&\le C_T \sup_{\substack{E\subseteq \mathbf{R}^n\\ 0<w^r(E)<\infty}} \inf_{\substack{E'\subseteq E\\ w^r(E)\le 2w^r(E')}} w^r(E)^{\frac{1}{r}-1}\|M_{(r,s')}(f,w^r\chi_{E'})\|_{L^1(\mathbf{R}^n)}\\
&\eqsim C_T \max_{\alpha\in\left\{0,\frac13,\frac23\right\}^n} \sup_{\substack{E\subseteq \mathbf{R}^n\\ 0<w^r(E)<\infty}} \inf_{\substack{E'\subseteq E\\ w^r(E)\le 2w^r(E')}} w^r(E)^{\frac{1}{r}-1}\|M^{\mathscr{D}^\alpha}_{(r,s')}(f,w^r\chi_{E'})\|_{L^1(\mathbf{R}^n)}.
\end{aligned} \tag{6.2.7}$$

Fix a dyadic grid $\mathscr{D} = \mathscr{D}^\alpha$ and $E\subseteq \mathbf{R}^n$ with $0 < w^r(E) < \infty$. We define

$$\Omega := \{x\in\mathbf{R}^n : M^{\mathscr{D}}_r f(x) > \big(\frac{2[w^r]_{A_1}}{w^r(E)}\big)^{\frac1r}\}$$

so that, since $\|M^{\mathscr{D}}_r\|_{L^r_w(\mathbf{R}^n)\to L^{r,\infty}_w(\mathbf{R}^n)} \le [w]_{r,(r,\infty)} = [w^r]^{\frac1r}_{A_1}$ by Theorem 3.2.3, we have

$$w^r(\Omega) \le \frac{[w^r]_{A_1} w^r(E)}{2[w^r]_{A_1}} = \frac{w^r(E)}{2}.$$

Setting $E' := E\backslash\Omega$ this implies that $w^r(E') \ge w^r(E) - w^r(\Omega) \ge w^r(E)/2$.

By applying Lemma 7.2.3 with $|f|^r \in L^1(\mathbf{R}^n)$, we obtain a pairwise disjoint collection $\mathscr{P}\subseteq\mathscr{D}$ of cubes so that $\Omega = \cup_{P\in\mathscr{P}} P$ and functions g, b so that $|f|^r = g + b$, where

$$g = |f|^r\chi_{\Omega^c} + \sum_{P\in\mathscr{P}} \langle |f|^r\rangle_{1,P}\chi_P$$

and

$$\|g\|_\infty \lesssim \frac{[w^r]_{A_1}}{w^r(E)}.$$

Using Lemma 6.2.3 with the weight $\chi_{E'} w^{\frac{r}{p}}$, for all $p\in(r,s)$ and $\frac1q \in (0,\frac1p - \frac1s)$ with $\frac{1}{\widetilde{q}} := \frac{\frac1q}{\frac1p-\frac1s}$ we have

$$\begin{aligned}
&\|M^{\mathscr{D}}_{(r,s')}(f,w^r\chi_{E'})\|_{L^1(\mathbf{R}^n)} = \|M^{\mathscr{D}}_{(r,s')}(|g|^{\frac1r},w^r\chi_{E'})\|_{L^1(\mathbf{R}^n)}\\
&\lesssim \left(\frac{1-\frac1p}{\frac1r-\frac1p}\right)^{\frac1r}\left(\frac{\frac1r}{\frac1r-\frac1p}\right)^{\frac1r}\left(\frac{1-\frac1s}{\frac1p-\frac1s}\right)^{1-\frac1s}(\widetilde{q}')^{\frac{1}{p'}}\||g|^{\frac1r}\|_{L^p_{M^{\mathscr{D}}_q(w^{\frac rp}\chi_{E'})}(\mathbf{R}^n)}\|w^r\chi_{E'}\|_{L^{p'}_{w^{-\frac rp}}(\mathbf{R}^n)}\\
&\lesssim \left(\frac{1-\frac1p}{\frac1r-\frac1p}\right)^{\frac1r}\left(\frac{\frac1r}{\frac1r-\frac1p}\right)^{\frac1r}\left(\frac{1-\frac1s}{\frac1p-\frac1s}\right)^{1-\frac1s}(\widetilde{q}')^{\frac{1}{p'}}[w^r]^{\frac1r-\frac1p}_{A_1} w^r(E)^{\frac1p-\frac1r}\|g\|^{\frac1p}_{L^1_{M^{\mathscr{D}}_{\frac qp}(w^r\chi_{E'})}(\mathbf{R}^n)} w^r(E')^{\frac{1}{p'}}.
\end{aligned} \tag{6.2.8}$$

Note here that we have used the fact that the terms involving b cancel in the exact same way as they do in the proof of Proposition 6.1.1.

Similar to what is done in [Pér94, LOP08, HP13], we deal with the term involving g as follows: We remark that for a cube $P \in \mathscr{D}$ and a function $\phi \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$ we have

$$M^{\mathscr{D}}(\phi\chi_{P^c})(x) = \inf_{y\in P} M^{\mathscr{D}}(\phi\chi_{P^c})(y) \tag{6.2.9}$$

for all $x \in P$. Indeed, let $x, y \in P$ and let $R \in \mathscr{D}$ so that $x \in R$. Then either $R \subseteq P$ or $P \subseteq R$. In the first case we have $\langle \phi\chi_{P^c}\rangle_{1,R} = 0$ while in the second case we have $y \in R$ and thus $\langle \phi\chi_{P^c}\rangle_{1,R} \le M^{\mathscr{D}}(\phi\chi_{P^c})(y)$. Thus, we may conclude that $M^{\mathscr{D}}(\phi\chi_{P^c})(x) \le M^{\mathscr{D}}(\phi\chi_{P^c})(y)$, proving (6.2.9). Using this result, we find, since $E' \subseteq P^c$ for all $P \in \mathscr{P}$, that

$$\begin{aligned}\|g\chi_\Omega\|_{L^1_{M^{\mathscr{D}}_{\frac{q}{p}}(w^r\chi_{E'})}(\mathbf{R}^n)} &\le \sum_{P\in\mathscr{P}} \inf_{y\in P} M^{\mathscr{D}}_{\frac{q}{p}}(\chi_{P^c} w^r)(y) \int_P |f|^r \,\mathrm{d}x \\ &\le \|f\chi_\Omega\|^r_{L^r_{M^{\mathscr{D}}_{\frac{qr}{p}} w}(\mathbf{R}^n)}.\end{aligned}$$

Since $g = |f|^r$ on Ω^c, we conclude that

$$\|g\|_{L^1_{M^{\mathscr{D}}_{\frac{q}{p}}(\chi_{E'} w^r)}(\mathbf{R}^n)} \le \|f\|^r_{L^r_{M^{\mathscr{D}}_{\frac{qr}{p}} w}(\mathbf{R}^n)}. \tag{6.2.10}$$

We choose $\frac{1}{t} \in (0,1)$ such that $t' = 2^{n+1}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}$ and set

$$\frac{1}{q} := \frac{\frac{1}{r}-\frac{1}{s}}{\frac{1}{r}} \frac{1}{p} \frac{1}{t}$$

so that $\frac{1}{q} \in (0, \frac{1}{p} - \frac{1}{s})$ whenever

$$\frac{1}{p} > \frac{\frac{1}{s}\frac{1}{r}}{\left(\frac{1}{r}-\frac{1}{s}\right)\left(2^{n+1}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}\right)^{-1} + \frac{1}{s}}. \tag{6.2.11}$$

Then it follows from Lemma 6.2.8 that

$$M^{\mathscr{D}}_{\frac{qr}{p}} w = M^{\mathscr{D}}_{\frac{t}{\frac{1}{r}-\frac{1}{s}}} w \lesssim_{r,s} [w]_{r,(r,s)} w. \tag{6.2.12}$$

Moreover, we compute

$$\frac{1}{\widetilde{q}'} = \frac{1}{2^{n+1}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}} \frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}} \frac{\frac{1}{r}-\frac{1}{s}}{\frac{1}{r}} + \frac{\frac{1}{s}}{\frac{1}{r}} \ge \frac{1}{2^{n+1}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}}.$$

Thus, it follows from (6.2.8), (6.2.10), and (6.2.12) that

$$\begin{aligned}&w^r(E)^{\frac{1}{r}-1}\|M^{\mathscr{D}^\alpha}_{(r,s')}(f, w^r\chi_{E'})\|_{L^1(\mathbf{R}^n)} \\ &\lesssim_{r,s} \left(\frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}} \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}} \left(\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}} [w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]^{\frac{1}{p'}}_{A_\infty} [w^r]^{\frac{1}{r}-\frac{1}{p}}_{A_1} [w]^{\frac{r}{p}}_{r,(r,s)}.\end{aligned} \tag{6.2.13}$$

We treat the cases $\frac{1}{s}=0$ and $\frac{1}{s}>0$ separately.

First assume $\frac{1}{s}>0$. We define

$$\frac{1}{p}:=\frac{\frac{1}{s}\frac{1}{r}}{\left(\frac{1}{r}-\frac{1}{s}\right)\left(2^{n+2}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}\right)^{-1}+\frac{1}{s}}$$

which satisfies (6.2.11). Then

$$\frac{1}{p'}=\frac{\left(\frac{1}{r}-\frac{1}{s}\right)\left(2^{n+2}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}\right)^{-1}+\frac{1}{s}\frac{1}{r'}}{\left(\frac{1}{r}-\frac{1}{s}\right)\left(2^{n+2}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}\right)^{-1}+\frac{1}{s}}\leq\frac{1}{r'}+\frac{\frac{1}{r}-\frac{1}{s}}{\frac{1}{s}}\frac{1}{2^{n+2}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}}$$

so that

$$[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}^{\frac{1}{p'}}\lesssim_{r,s}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}^{\frac{1}{r'}}.$$

Moreover, we compute

$$\begin{aligned}
\frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}}&=r+\frac{\frac{1}{s}\left(1-\frac{1}{r}\right)}{\frac{1}{r}\left(\frac{1}{r}-\frac{1}{s}\right)}2^{n+2}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty},\\
\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}&=1+\frac{\frac{1}{s}}{\frac{1}{r}-\frac{1}{s}}2^{n+2}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty},\\
\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}&=\frac{1-\frac{1}{s}}{\frac{1}{s}}\left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{s}}\frac{1}{1-\left(2^{n+2}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}\right)^{-1}}-1\right)\leq\frac{1-\frac{1}{s}}{\frac{1}{s}}\frac{\frac{1}{r}+\frac{1}{s}}{\frac{1}{r}-\frac{1}{s}}.
\end{aligned}$$

Hence, by combining these estimates with (6.2.13), it follows from (6.2.7) that

$$\begin{aligned}
\|Tf\|_{L^{r,\infty}_w(\mathbf{R}^n)}&\lesssim_{r,s}[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}\left(1+(1-\frac{1}{r})[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}\right)^{\frac{1}{r}}[w^r]_{A_1}^{\frac{1}{r}-\frac{1}{p}}[w]_{r,(r,s)}^{\frac{r}{p}}\\
&\leq[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}\left(1+(1-\frac{1}{r})[w^{\frac{1}{\frac{1}{r}-\frac{1}{s}}}]_{A_\infty}\right)^{\frac{1}{r}}[w^r]_{A_1}^{\frac{1}{r}}[w^r]_{\mathrm{RH}_{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{s}}}}^{\frac{1}{r}}.
\end{aligned}$$

The result follows by considering the cases $\frac{1}{r}=1$ and $\frac{1}{r}<1$ separately.

Now we assume that $\frac{1}{s}=0$. Note that (6.2.11) no longer imposes any restrictions on $\frac{1}{p}$. We set

$$\frac{1}{p}:=\frac{1}{r}\left(1-\frac{1}{\log(e+[w^r]_{A_\infty})}\right)$$

and compute

$$\frac{1}{p'}=\frac{1}{r'}+\frac{1}{r}\frac{1}{\log(e+[w^r]_{A_\infty})}$$

so that

$$[w^r]_{A_\infty}^{\frac{1}{p'}}\lesssim[w^r]_{A_\infty}^{\frac{1}{r'}}.$$

Moreover, we compute

$$\frac{1-\frac{1}{p}}{\frac{1}{r}-\frac{1}{p}} = 1+\frac{r}{r'}\log(e+[w^r]_{A_\infty}),$$

$$\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}} = \log(e+[w^r]_{A_\infty}),$$

$$\frac{1}{\frac{1}{p}} = r\left(1+\frac{1}{\log(e+[w^r]_{A_\infty})-1}\right) \lesssim r.$$

combining this with (6.2.13), we conclude from (6.2.7) that

$$\|Tf\|_{L^{r,\infty}_w(\mathbf{R}^n)} \lesssim_r [w^r]_{A_\infty}^{\frac{1}{r'}} \log(e+[w^r]_{A_\infty})^{\frac{1}{r}} \big(1+(1-\frac{1}{r})\log(e+[w^r]_{A_\infty})\big)^{\frac{1}{r}} [w^r]_{A_1}^{\frac{1}{r}}.$$

By considering the cases $\frac{1}{r}=1$ and $\frac{1}{r}<1$ separately, the assertion follows. □

Finally, we establish a dual result of the type first studied in [LOP09a], generalizing the result [HP13, Theorem 1.23].

Theorem 6.2.9. *Let T be a (sub)linear operator initially defined on $L^\infty_c(\mathbf{R}^n)$. Let $r\in[1,\infty)$, $s\in(1,\infty]$, $p\in(r,s)$, and suppose that*

$$\|Tf\cdot g\|_{L^1(\mathbf{R}^n)} \le C_T \|M_{(r,s')}(f,g)\|_{L^1(\mathbf{R}^n)}$$

for all $f\in L^\infty_c(\mathbf{R}^n)$, $g\in L^\infty(\mathbf{R}^n)\cap L^1(\mathbf{R}^n)$.

Then for all $w^r\in A_1$, T has an extension to $L^r(\mathbf{R}^n)$ satisfying

$$\left\|\frac{Tf}{w}\right\|_{L^{r,\infty}_w(\mathbf{R}^n)} \lesssim_{r,s} C_T[w^r]_{A_\infty}^{\frac{1}{r}} \log(e+[w^r]_{A_1})^{\frac{1}{r}} \|f\|_{L^r(\mathbf{R}^n)}$$

for all $f\in L^r(\mathbf{R}^n)$,

Proof of Theorem 6.2.9. Let $f\in L^\infty_c(\mathbf{R}^n)$ with $\|f\|_{L^r(\mathbf{R}^n)}=1$. Then

$$\begin{aligned}\left\|\frac{Tf}{w}\right\|_{L^{r,\infty}(\mathbf{R}^n;w^r)} &\eqsim \sup_{\substack{E\subseteq\mathbf{R}^n\\ 0<w^r(E)<\infty}} \inf_{\substack{E'\subseteq E\\ w^r(E)\le 2w^r(E')}} w^r(E)^{\frac{1}{r}-1}\|Tf\,w^{r-1}\chi_{E'}\|_{L^1(\mathbf{R}^n)}\\ &\le C_T \max_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n} \sup_{\substack{E\subseteq\mathbf{R}^n\\ 0<w^r(E)<\infty}} \inf_{\substack{E'\subseteq E\\ w^r(E)\le 2w^r(E')}} w^r(E)^{\frac{1}{r}-1}\|M^{\mathscr{D}^\alpha}_{(r,s')}(f,w^{r-1}\chi_{E'})\|_{L^1(\mathbf{R}^n)}.\end{aligned} \tag{6.2.14}$$

Fix a dyadic grid $\mathscr{D}=\mathscr{D}^\alpha$ and $E\subseteq\mathbf{R}^n$ with $0<w^r(E)<\infty$. We define

$$\Omega := \left\{x\in\mathbf{R}^n : M_r^{w^r}\Big(\frac{f}{w}\Big)(x) > \|M_r^{w^r}\|_{L^r(\mathbf{R}^n;w^r)\to L^{r,\infty}(\mathbf{R}^n;w^r)}\Big(\frac{2}{w^r(E)}\Big)^{\frac{1}{r}}\right\}$$

so that

$$w^r(\Omega) \leq \frac{w^r(E)}{2} \left\| \frac{f}{w} \right\|^r_{L^r_w(\mathbf{R}^n)} = \frac{w^r(E)}{2}$$

which, setting $E' := E \backslash \Omega$, implies that $w^r(E') \geq w^r(E) - w^r(\Omega) \geq w^r(E)/2$. We also note here that $\|M_r^{w^r}\|_{L^r(\mathbf{R}^n;w^r)\to L^{r,\infty}(\mathbf{R}^n;w^r)} \lesssim 1$ by the three lattice lemma and Lemma 3.2.5.

By applying the Whitney Decomposition Theorem to Ω we obtain a disjoint collection $\mathscr{P} \subseteq \mathscr{D}^\alpha$ of cubes so that $\Omega = \cup_{P\in\mathscr{P}} P$ with the property that for each $P \in \mathscr{P}$ there exists a cube $Q(P)$ containing P so that $Q(P) \cap \Omega^c \neq \varnothing$ and $|Q(P)| \lesssim |P|$. Then we can write $|f|^r = g + b$, where

$$g = |f|^r \chi_{\Omega^c} + \sum_{P\in\mathscr{P}} \langle |f|^r \rangle_{1,P} \chi_P.$$

Fix $p \in (r,s)$ to be chosen later. By applying Lemma 6.2.3 with the weight $w^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}}$, we find that for all $\frac{1}{q} \in (0, \frac{1}{r} - \frac{1}{p})$ with $\frac{1}{\widetilde{q}} := \frac{\frac{1}{q}}{\frac{1}{r}-\frac{1}{p}}$ we have

$$\begin{aligned}
\|M^{\mathscr{D}}_{(r,s')}(f, w^{\frac{1-\frac{1}{r}}{\frac{1}{r}}}\chi_{E'})\|_{L^1(\mathbf{R}^n)} &= \|M^{\mathscr{D}}_{(s',r)}(w^{\frac{1-\frac{1}{r}}{\frac{1}{r}}}\chi_{E'}, |g|^{\frac{1}{r}})\|_{L^1(\mathbf{R}^n)} \\
&\lesssim \left(\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}} \left(\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}} \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}} (\widetilde{q}')^{\frac{1}{p}} \|w^{\frac{1-\frac{1}{r}}{\frac{1}{r}}}\chi_{E'}\|_{L^{p'}_{M^{\mathscr{D}}_q(w^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}})}(\mathbf{R}^n)} \||g|^{\frac{1}{r}}\|_{L^p_{w^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}}}(\mathbf{R}^n)}
\end{aligned} \tag{6.2.15}$$

where the terms involving b cancel in the same way as in the previous proofs.

Let $\frac{1}{q} \in (0, \frac{1}{r} - \frac{1}{p})$ be such that $\widetilde{q}' = 2^{n+1}[w^r]_{A_\infty}$. Then it follows from Lemma 6.2.8 that

$$M^{\mathscr{D}}_q(w^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}}) = M^{\mathscr{D}}_{r\widetilde{q}}(w)^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}} \lesssim_{r,s} [w^r]_{A_1}^{\frac{1}{r}-\frac{1}{p}} w^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}}$$

so that

$$\begin{aligned}
(\widetilde{q}')^{\frac{1}{p}} \|w^{\frac{1-\frac{1}{r}}{\frac{1}{r}}}\chi_{E'}\|_{L^{p'}_{M^{\mathscr{D}}_q(w^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}})}(\mathbf{R}^n)} &\lesssim_{r,s} [w^r]_{A_\infty}^{\frac{1}{p}} [w^r]_{A_1}^{\frac{1}{r}-\frac{1}{p}} \|w^{\frac{1-\frac{1}{r}}{\frac{1}{r}}} w^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}} \chi_{E'}\|_{L^{p'}(\mathbf{R}^n)} \\
&= [w^r]_{A_\infty}^{\frac{1}{p}} [w^r]_{A_1}^{\frac{1}{r}-\frac{1}{p}} w^r(E')^{\frac{1}{p'}}.
\end{aligned} \tag{6.2.16}$$

Next, since in Ω^c we have $|f| \leq M_r^{w^r}\left(\frac{f}{w}\right) w \lesssim_r w^r(E)^{-1/r} w$, we have

$$\|f\chi_{\Omega^c}\|_{L^p_{w^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}}}(\mathbf{R}^n)} \lesssim_r w^r(E)^{\frac{1}{p}-\frac{1}{r}} \||f|^{\frac{r}{p}} w^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}} w^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}} \chi_{\Omega^c}\|_{L^p(\mathbf{R}^n)} \leq w^r(E)^{\frac{1}{p}-\frac{1}{r}}. \tag{6.2.17}$$

Furthermore, fixing a $P \in \mathscr{P}$ and $x \in Q(P) \cap \Omega^c$, we have

$$\begin{aligned}\langle f\rangle_{r,P}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}\frac{1}{p}}} \lesssim \langle f\rangle_{r,Q(P)}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}\frac{1}{p}}} \le M_r^{w^r}\Big(\frac{f}{w}\Big)(x)^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}\frac{1}{p}}} \langle w\rangle_{r,B(P)}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}\frac{1}{p}}} \\ \lesssim_r w^r(E)^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}} \langle w\rangle_{r,B(P)}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}\frac{1}{p}}}\end{aligned}$$

and

$$\langle w^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}}\rangle_{r,P}^r \lesssim_r [w^r]_{A_1}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}} \langle (M_r w)^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}}\rangle_{r,Q(P)}^r \le [w^r]_{A_1}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}} \langle w\rangle_{r,Q(P)}^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}\frac{1}{p}}}$$

so that

$$\begin{aligned}\sum_{P\in\mathscr{P}} \langle f\rangle_{r,P}^p \int_P w^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}\frac{1}{p}}}\,\mathrm{d}x &= \sum_{P\in\mathscr{P}} \langle f\rangle_{r,P}^r \langle f\rangle_{r,P}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}\frac{1}{p}}} \langle w^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}}\rangle_{r,P}^r |P| \\ &\lesssim_r [w^r]_{A_1}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}} w^r(E)^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}} \sum_{P\in\mathscr{P}} \langle f\rangle_{r,P}^r |P| \\ &\le [w^r]_{A_1}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}} w^r(E)^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}}.\end{aligned} \tag{6.2.18}$$

Hence, by (6.2.17) and (6.2.18) we have

$$\begin{aligned}\||g|^{\frac{1}{r}}\|^p_{L^p_{w^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}}}(\mathbf{R}^n)} &= \|f\chi_{\Omega^c}\|^p_{L^p_{w^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}}}}(\mathbf{R}^n)} + \sum_{P\in\mathscr{P}} \langle f\rangle_{r,P}^p \int_P w^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}\frac{1}{p}}}\,\mathrm{d}x \\ &\lesssim_r [w^r]_{A_1}^{\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}} w^r(E)^{-\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{p}}}.\end{aligned} \tag{6.2.19}$$

Thus, by combining (6.2.16) and (6.2.19) with (6.2.15), we conclude that

$$\begin{aligned}& w^r(E)^{\frac{1}{r}-1}\|M^{\mathscr{D}^\alpha}_{(r,s')}(f, w^{r-1}\chi_{E'})\|_{L^1(\mathbf{R}^n)} \\ &\lesssim_{r,s} \left(\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}} \left(\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right)^{1-\frac{1}{s}} \left(\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}\right)^{\frac{1}{r}} [w^r]_{A_\infty}^{\frac{1}{p}} [w^r]_{A_1}^{2\left(\frac{1}{r}-\frac{1}{p}\right)}\end{aligned} \tag{6.2.20}$$

By writing $L := \log(e + [w^r]_{A_1})$ and choosing

$$\frac{1}{p} = \frac{1}{L}\frac{1}{s} + \left(1-\frac{1}{L}\right)\frac{1}{r}$$

we have

$$[w^r]_{A_1}^{2\left(\frac{1}{r}-\frac{1}{p}\right)} = [w^r]_{A_1}^{2\left(\frac{1}{r}-\frac{1}{s}\right)\frac{1}{\log(e+[w^r]_{A_1})}} \le e^{2\left(\frac{1}{r}-\frac{1}{s}\right)\frac{1}{e}}$$

and

$$\frac{\frac{1}{p}}{\frac{1}{p}-\frac{1}{s}} = 1+\frac{\frac{1}{s}}{\frac{1}{r}-\frac{1}{s}}L' \lesssim \frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{s}},$$

$$\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}} = \frac{1-\frac{1}{s}}{\frac{1}{r}-\frac{1}{s}}L' \lesssim \frac{1-\frac{1}{s}}{\frac{1}{r}-\frac{1}{s}}$$

$$\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}} = \frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{s}}L.$$

Thus, by (6.2.14) and (6.2.20) we have

$$\left\| \frac{Tf}{w} \right\|_{L^{r,\infty}(\mathbf{R}^n;w^r)} \lesssim_{r,s} C_T [w^r]_{A_\infty}^{\frac{1}{r}} \log(e+[w^r]_{A_1})^{\frac{1}{r}},$$

as desired. The assertion follows. □

7

Extensions of the results to spaces of homogeneous type

7.1. Dyadic grids in spaces of homogeneous type

So far we have formulated and proven our results in $\mathbf{R}^n$ equipped with the Lebesgue measure. This section is dedicated to extending our results to more general quasimetric measure spaces (S,d,μ), commonly referred to as spaces of homogeneous type and introduced in [CW71]. Here S is a set equipped with a quasimetric d, i.e., a mapping satisfying the usual properties of a metric except for the triangle inequality, which is replaced by the estimate

$$d(x,y) \leq A(d(x,z)+d(z,y))$$

for a constant $A \geq 1$, and μ is a Borel measure on S satisfying the doubling property, i.e., there is a $C > 0$ such that

$$\mu(B(x;2r)) \leq C\mu(B(x;r)) \tag{7.1.1}$$

for all $x \in S$, $r > 0$. Note that for (7.1.1) to make sense, we need to assume that μ is defined on all balls. To facilitate this we assume that all balls in S are Borel sets. Note that this condition is restrictive, since in general quasimetric spaces balls may fail to be Borel sets as is shown in [Ste15, Example 1.1].

Taking the smallest admissible C in (7.1.1) we set $\nu := \log_2 C$, which we will refer to as the doubling dimension of S. Note that in $\mathbf{R}^n$ we have $|B(x;2r)| = 2^n|B(x;r)|$ and hence, its doubling dimension is n.

We will write $|E| := \mu(E)$ for all Borel sets $E \subseteq S$. The doubling property implies that for $x \in S$ and $R \geq r > 0$ we have

$$|B(x;R)| \leq C\left(\frac{R}{r}\right)^{\nu}|B(x;r)|. \tag{7.1.2}$$

In turn, this implies that if $y \in B(x;R)$ for $x \in S$, then for $0 < r \leq 2AR$ we have

$$|B(x;R)| \leq C\left(\frac{2AR}{r}\right)^{\nu}|B(y;r)|. \tag{7.1.3}$$

We make the additional assumption that $0 < |B| < \infty$ for all balls $B \subseteq S$. This property ensures that S is separable [BB11, Proposition 1.6]. Moreover, this implies that the averages of $\langle f\rangle_{r,B}$ are well-defined and, as shown in [AM15, Section 3.3], the Lebesgue Differentiation Theorem holds in (S,d,μ).

The essential structure of $\mathbf{R}^n$ that we used so far is its decomposition into dyadic grids. The three lattice lemma then allowed us to reduce our arguments to a single dyadic grid. The reason we are able to extend our results to (S,d,μ) is because such a space also admits a version of dyadic grids, as well as a version of the three lattice lemma.

We will use the following definition of a dyadic system in (S,d,μ):

Definition 7.1.1. Let $0 < c_0 \le C_0 < \infty$ and $0 < \delta < 1$. If for each $k \in \mathbf{Z}$ we have a pairwise disjoint collection $\mathscr{D}_k = (Q_j^k)_{j\in J_k}$ of Borel subsets of S and a collection of points $(z_j^k)_{j\in J_k}$, then we call $(\mathscr{D}_k)_{k\in\mathbf{Z}}$ a *dyadic system* in S with parameters c_0, C_0, δ, if it satisfies the following properties:

(i) for all $k \in \mathbf{Z}$ we have
$$S = \bigcup_{j\in J_k} Q_j^k;$$

(ii) for $l \ge k$, if $Q \in \mathscr{D}_l$ and $Q' \in \mathscr{D}_k$, we have that either $Q \cap Q' = \varnothing$ or $Q \subseteq Q'$;

(iii) for each $k \in \mathbf{Z}$ and $j \in J_k$ we have
$$B(z_j^k; c_0\delta^k) \subseteq Q_j^k \subseteq B(z_j^k; C_0\delta^k);$$

(iv) for $l \ge k$, if $Q_{j'}^l \subseteq Q_j^k$, then $B(z_{j'}^l; C_0\delta^k) \subseteq B(z_j^k; C_0\delta^k)$.

The elements of a dyadic system are called cubes. We call z_j^k the *center* of Q_j^k. If $Q \in \mathscr{D}_k$, then we call the unique cube $Q' \in \mathscr{D}_{k-1}$ so that $Q \subseteq Q'$ the *parent* of Q. Furthermore, we say that Q is a *child* of Q'. Note that it is possible that for a cube Q there exists more than one $k \in \mathbf{Z}$ so that $Q \in \mathscr{D}_k$. Hence, when speaking of a child or the parent of Q, this should be with respect to a specific $k \in \mathbf{Z}$ where $Q \in \mathscr{D}_k$ to avoid ambiguity.

For a detailed discussion on the construction of dyadic systems as well as the following version of the three lattice lemma we refer the reader to [HK12] and references therein.

Theorem 7.1.2. *There exist $0 < c_0 < C_0 < \infty$, $0 < \delta < 1$, $\rho > 0$ and a positive integer K, so that there are dyadic system $(\mathscr{D}^\alpha)_{\alpha=1}^K$ in S with parameters c_0, C_0, δ so that for each $x \in S$ and $r > 0$ there exists an $\alpha \in \{1,\dots,K\}$ and $Q \in \mathscr{D}^\alpha$ so that*
$$B(x;r) \subseteq Q \quad \textit{and} \quad \operatorname{diam}(Q) \le \rho r.$$

Using these systems to replace the systems $(\mathscr{D}^\alpha)_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n}$ in $\mathbf{R}^n$ almost all of our results up to this as well as the results in the next part will go through. The only exception to this are the weak-type result in Chapter 6, i.e., Proposition 6.1.1, Theorem 6.2.1, and Theorem 6.2.9. The reason for this can be found in the proof of Proposition 6.1.1, where in a maximal cube selection argument for a Calderón-Zygmund decomposition we used the fact that in $\mathbf{R}^n$, the increasing sequence of cubes in a dyadic grid containing a fixed point has the property that the corresponding sequence of measures converges to ∞. This need not be the case in (S,d,μ), since, for example, (S,d) could be bounded. In the following section we provide additional restrictions on (S,d,μ) in order to recover the results from Chapter 6 in this setting.

Rather than allowing for dimensional constants in the implicit constants in our estimates, we now allow for implicit constants depending on the parameters of the dyadic system, the doubling dimension ν, and the quasimetric constant A.

7.2. CALDERÓN-ZYGMUND DECOMPOSITIONS IN SPACES OF HOMOGENEOUS TYPE

We will consider the situations where the underlying quasimetric space (S,d) is unbounded and where (S,d) is bounded separately. More precisely, we assume that (S,d) has exactly one of the following properties:

(I) All balls in (S,d) are open and there is a constant $\gamma > 0$ so that

$$\operatorname{diam}(B(x;r)) \geq \gamma r \tag{7.2.1}$$

for all $x \in S$, $r > 0$;

(II) $\operatorname{diam} S < \infty$.

We note that property (I) and property (II) are mutually exclusive, since (I) implies that S is unbounded. When (S,d) is an unbounded connected metric space, then it satisfies (I):

Proposition 7.2.1. *Suppose (S,d) is an unbounded connected metric space. Then (I) holds with $\gamma = 1$.*

Proof. Since d is a metric, all balls in (S,d) are open. For the second assertion, let $x \in S$ and $r > \varepsilon > 0$. Since S is connected and the closed ball $\overline{B}(x;r-\varepsilon)$ and $B(x;r)^c$ are disjoint, S is not equal to the union of these sets. Thus, there is a $y \in B(x;r)\backslash\overline{B(x;r-\varepsilon)}$ from which it follows that $\operatorname{diam}(B(x;r)) \geq d(x,y) \geq r-\varepsilon$. The result follows by letting $\varepsilon \to 0$. □

A non-connected example where (I) holds with $\gamma = 1/2$ is the subset $(-\infty,0)\cup(1,2)$ of the real line. An example where (I) fails is any metric space that has an isolated point.

From now on we consider a fixed dyadic system $\mathscr{D} = \cup_{k\in\mathbf{Z}}\mathscr{D}_k$ in S with parameters c_0, C_0, δ.

We first consider the case (II).

Lemma 7.2.2 (Calderón-Zygmund Lemma in the case (II))**.** *Let $f \in L^1(S)$, $\lambda > 0$, and let $\Omega := \{x \in S : M^{\mathscr{D}} f(x) > \lambda\}$. If $\Omega \neq S$, then we can find a pairwise disjoint collection of cubes $\mathscr{P} \subseteq \mathscr{D}$ and a constant $c > 0$, depending only on the parameters of the dyadic system, the doubling dimension ν, and the quasimetric constant A, so that*

$$\Omega = \bigcup_{P\in\mathscr{P}} P,$$

and

$$\lambda < \langle f\rangle_{1,P} \lesssim \lambda.$$

for all $P \in \mathscr{P}$.

Proof. Fix $k_0 \in \mathbf{Z}$ small enough so that $c_0\delta^{k_0} > \operatorname{diam} S$. Then for any $x \in S$ we have $B(x; c_0\delta^{k_0}) = S$. Hence, it follows from property (iii) of dyadic systems that $\mathscr{D}_{k_0} = \{S\}$.

Note that $\Omega \neq S$ implies that $\langle f\rangle_{1,S} \leq \lambda$. Let $x \in \Omega$. Then the set

$$K_x := \{k > k_0 \mid \text{there is a } Q \in \mathscr{D}_k,\ x \in Q,\ \langle f\rangle_{1,Q} > \lambda\}$$

is non-empty. Thus, by well-orderedness there is a minimal $k_x \in K_x$, and thus a cube $P_x \in \mathscr{D}_{k_x}$ that contains x so that $\langle f\rangle_{1,P_x} > \lambda$. By minimality of k_x, it follows that $\langle f\rangle_{1,p(P_x)} \leq \lambda$, where $p(P_x) \in \mathscr{D}_{k_x-1}$ denotes the parent of P_x. By (7.1.3) and property (iii) of dyadic systems this implies that

$$\lambda < \langle f\rangle_{1,P_x} \leq c\langle f\rangle_{1,p(P_x)} \leq c\lambda,$$

with $c = C(2AC_0/(c_0\delta))^\nu$.

It remains to show that the collection $\mathscr{P} = (P_x)_{x\in S}$ is pairwise disjoint. Indeed, assume that $P_1, P_2 \in \mathscr{P}$ so that $P_1 \cap P_2 \neq \varnothing$. We have either $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ by property (ii) of dyadic systems. Without loss of generality we assume the first. Pick $x \in S$ so that $P_1 = P_x$. Since $x \in P_2$ and $\langle f\rangle_{1,P_2} > \lambda$, minimality of k_x implies that $P_2 \in \mathscr{D}_l$ for some $l \geq k_x$. Again by property (ii) of dyadic systems, this implies that $P_2 \subseteq P_1$, proving that $P_1 = P_2$. The assertion follows. □

Next, we consider the case (I). We define the uncentered maximal operator with respect to the collection of balls $\mathscr{B}$ in S by $M^{\mathscr{B}} f(x) := \sup_B \langle f\rangle_{1,B}\chi_B(x)$.

Lemma 7.2.3 (Calderón-Zygmund Lemma in the case (I))**.** *Let* $f \in L^1(S)$, $\lambda > 0$, *and let* $\Omega := \{x \in S : M^{\mathscr{B}} f(x) > \lambda\}$. *If* $\Omega \neq S$, *then we can find a pairwise disjoint collection of cubes* $\mathscr{P} \subseteq \mathscr{D}$ *such that*

$$\Omega = \bigcup_{P\in\mathscr{P}} P,$$

and

$$\langle f\rangle_{1,P} \lesssim \lambda.$$

for all $P \in \mathscr{P}$.

For the proof we use a version of the Whitney Decomposition Theorem. Note that the diameter assumption (7.2.1) together with property (iii) of dyadic systems implies that for any $Q \in \mathscr{D}_k$ we have

$$\gamma c_0\delta^k \leq \operatorname{diam} Q \leq 2AC_0\delta^k. \tag{7.2.2}$$

Theorem 7.2.4 (Whitney Decomposition Theorem for Dyadic Cubes)**.** *Let $\Omega \subsetneq S$ be open. Then there exists a pairwise disjoint collection of cubes $\mathscr{P} \subseteq \mathscr{D}$ such that*

$$\Omega = \bigcup_{P \in \mathscr{P}} P$$

and for each $P \in \mathscr{P}$,

$$\operatorname{diam} P \le d(P, \Omega^c) \le \frac{4A^2 C_0}{\gamma c_0 \delta} \operatorname{diam} P.$$

In particular, for each $P \in \mathscr{P}$ there is a ball $B(P)$ containing P satisfying $|B(P)| \lesssim |P|$ and $B(P) \cap \Omega^c \neq \varnothing$.

Proof. We define

$$\mathscr{E} := \{Q \in \mathscr{D} \mid Q \subseteq \Omega, \operatorname{diam} Q \le d(Q, \Omega^c)\}.$$

Moreover we set

$$\mathscr{P} := \{Q \in \mathscr{E} \mid \text{there is a } k \in \mathbf{Z} \text{ so that } Q \in \mathscr{D}_k,\ p(Q) \notin \mathscr{E}\},$$

where $p(Q) \in \mathscr{D}_{k-1}$ denotes the parent of $Q \in \mathscr{D}_k$. We will show that

$$\bigcup_{P \in \mathscr{P}} P = \Omega.$$

Indeed, any $P \in \mathscr{P}$ is contained in Ω. Conversely, if $x \in \Omega$, Let $(Q_x^k)_{k \in \mathbf{Z}}$ be the sequence of cubes in $\mathscr{D}$ with $x \in Q_x^k$ and $Q_x^k \in \mathscr{D}_k$ for all $k \in \mathbf{Z}$. Since Ω is open, there is a ball $B = B(x; r)$ contained in Ω. Picking k_0 large enough so that $2AC_0\delta^{k_0} < r$, we find that

$$Q_x^k \subseteq B(x; r) \subseteq \Omega$$

for all $k \ge k_0$ by (7.2.2). Moreover, since $d(Q_x^k, \Omega^c) \ge A^{-1}(d(x, \Omega^c) - 2A^2 C_0 \delta^k) \uparrow A^{-1} d(x, \Omega^c)$ as $k \to \infty$, while $\operatorname{diam}(Q_x^k) \le 2AC_0\delta^k \downarrow 0$ as $k \to \infty$, we can find a $k_1 \in \mathbf{Z}$ so that $\operatorname{diam}(Q_x^k) \le d(Q_x^k, \Omega^c)$ whenever $k \ge k_1$. Hence, for all $k \ge \max(k_0, k_1)$ we have $Q_x^k \in \mathscr{E}$. Thus, the set

$$K_x := \{k \in \mathbf{Z} \mid Q_x^k \in \mathscr{E}\}$$

is non-empty. We also claim that K_x is bounded from below. Indeed, if we choose $k_2 \in \mathbf{Z}$ small enough so that $\gamma c_0 \delta^{k_2} > d(x, \Omega^c)$, then

$$d(Q_x^k, \Omega^c) \le d(x, \Omega^c) < \operatorname{diam}(Q_x^k)$$

for all $k \le k_2$ by (7.2.2), and hence $Q_x^k \notin \mathscr{E}$ for $k \le k_2$, proving the claim.

We set $k_x := \min K_x \in \mathbf{Z}$. Then $Q_x^{k_x} \in \mathscr{E}$ while $p(Q_x^{k_x}) = Q_x^{k_x - 1} \notin \mathscr{E}$. Hence, $Q_x^{k_x} \in \mathscr{P}$, proving that $x \in \cup_{P \in \mathscr{P}} P$, as desired.

Next we will show that $\mathscr{P}$ is pairwise disjoint. Suppose for a contradiction that we have $P_1, P_2 \in \mathscr{P}$ so that $P_1 \cap P_2 \neq \varnothing$ and $P_1 \neq P_2$. Let $l_1, l_2 \in \mathbf{Z}$ so that $P_1 \in \mathscr{D}_{l_1}$, $P_2 \in \mathscr{D}_{l_2}$ and $p(P_1), p(P_2) \notin \mathscr{E}$. Without loss of generality we assume that $l_1 > l_2$ and thus $P_1 \subseteq P_2$

by property (ii) of the dyadic systems. Then also $p(P_1) \subseteq P_2$. Since $p(P_1) \notin \mathscr{E}$, we must have that either $p(P_1) \nsubseteq \Omega$ or $d(p(P_1), \Omega^c) < d(p(P_1))$. The first case implies that $P_2 \nsubseteq \Omega$, contradicting the fact that $P_2 \in \mathscr{E}$. The second case implies that

$$\operatorname{diam}(P_2) \geq \operatorname{diam}(p(P_1)) > d(p(P_1), \Omega^c) \geq d(P_2, \Omega^c),$$

again contradicting $P_2 \in \mathscr{E}$. We conclude that $\mathscr{P}$ is pairwise disjoint, as desired.

It remains to show that $d(P, \Omega^c) < 4A^2 C_0/(\gamma c_0 \delta) \operatorname{diam} P$ for all $P \in \mathscr{P}$. Let $P \in \mathscr{P}$, $P \in \mathscr{D}_k$ so that $p(P) \notin \mathscr{E}$. Then either $p(P) \nsubseteq \Omega$ or $d(p(P), \Omega^c) < \operatorname{diam}(p(P))$. In the first case we have $d(p(Q), \Omega^c) = 0$, so in both cases we have

$$d(p(P), \Omega^c) < \operatorname{diam}(p(P)) \leq 2AC_0 \delta^{k-1} = \frac{2AC_0}{\gamma c_0 \delta} \gamma c_0 \delta^k \leq \frac{2AC_0}{\gamma c_0 \delta} \operatorname{diam} P.$$

by (7.2.2). Hence,

$$d(P, \Omega^c) \leq A(d(p(P), \Omega^c) + \operatorname{diam}(p(P))) < \frac{4A^2 C_0}{\gamma c_0 \delta} \operatorname{diam} P,$$

as desired.

For the final assertion, note that if $P \in \mathscr{P}$, $P \in \mathscr{D}_k$ with center z_P, we have

$$\begin{aligned} 2d(z_P, \Omega^c) &\leq 2A \operatorname{diam} P + 2Ad(P, \Omega^c) \leq \left(2A + \frac{8A^3 C_0}{\gamma c_0 \delta}\right) \operatorname{diam} P \\ &\leq \left(4A + \frac{16A^3 C_0}{\gamma c_0 \delta}\right) C_0 \delta^k =: \tau C_0 \delta^k \end{aligned}$$

so that

$$\varnothing \neq B(z_P; 2d(z_p, \Omega^c)) \cap \Omega^c \subseteq B(z_P; \tau C_0 \delta^k) \cap \Omega^c.$$

Since

$$\left| B(z_P; \tau C_0 \delta^k) \right| \leq C \left(\frac{\tau C_0}{c_0} \right)^{\nu} |B(z_P; c_0 \delta^k)| \lesssim |P|$$

by (7.1.2), this proves the assertion with $B(P) := B(z_p; \tau C_0 \delta^k)$. □

Proof of Lemma 7.2.3. To see that Ω is open, note that for each $x \in \Omega$ there is a ball B containing x such that $\langle f \rangle_{1,B} > \lambda$. But then for every $y \in B$ we have $M^{\mathscr{B}} f(y) \geq \langle f \rangle_{1,B} > \lambda$ so that $y \in \Omega$. Hence, $B \subseteq \Omega$. Since we assumed that all balls in S are open, this proves that Ω is open. Thus, we may apply the Whitney Decomposition Theorem to write $\Omega = \cup_{P \in \mathscr{P}} P$.

If $P \in \mathscr{P}$, we may pick a point $x \in B(P) \cap \Omega^c$ to conclude that

$$\langle f \rangle_{1,P} \lesssim \langle f \rangle_{1,B(P)} \leq M^{\mathscr{B}} f(x) \leq \lambda.$$

The assertion follows. □

Proposition 7.2.5. *Suppose that* (S,d) *satisfies either property (I) or (II). Then the results of Proposition 6.1.1 and Theorem 6.2.1 remain true when replacing* $\mathbf{R}^n$ *by* (S,d,μ).

If (S,d) *satisfies property (I), then the results of Theorem 6.2.9 remains true when replacing* $\mathbf{R}^n$ *by* (S,d,μ).

Proof. In the case (I), by replacing $M_r^{\mathscr{D}}$ by a constant multiple of $M_r^{\mathscr{B}}$ in the definition of Ω_j and Ω in Proposition 6.1.1 and Theorem 6.2.1 respectively, using Lemma 7.2.3 the proofs of these results run mutatis mutandis. For Theorem 6.2.9, one has to replace $M_r^{w^r}$ by $M_r^{\mathscr{B},w^r}$ in the definition of Ω and then apply Theorem 7.2.4, noting that Ω is open in the same way as is done in the proof of Lemma 7.2.3.

For Theorem 6.2.1 in the case (II), we note that since S is bounded we have $w^r(S) < \infty$. Thus, since $w^r(\Omega) \leq w^r(E)/2 \leq w^r(S)/2$, the set Ω has strictly smaller w^r-measure than S and hence, $\Omega \neq S$. Thus, we may apply Lemma 7.2.2 to decompose Ω, and the proof runs analogously. An analogous reasoning works for the sets Ω_j in Proposition 6.1.1. The assertion follows. □

1/4 A multilinear UMD condition and vector-valued extensions of multilinear operators

8

A MULTILINEAR UMD CONDITION

This chapter is based on the preliminary sections of the papers

> E. Lorist and B. Nieraeth. Vector-valued extensions of operators through multilinear limited range extrapolation. *Journal of Fourier Analysis and Applications*, 25(5):2608–2634, 2019.
>
> E. Lorist and B. Nieraeth. Sparse domination implies vector-valued sparse domination. arXiv:2003.02233, 2020.

8.1. QUASI-BANACH FUNCTION SPACES

Let (Ω,μ) be a measure space. A subspace $X \subseteq L^0(\Omega)$ equipped with a quasi-norm $\|\cdot\|_X$ is called a *quasi-Banach function space* if it satisfies the following properties:

- *Ideal property:* If $\xi \in L^0(\Omega)$ and $\eta \in X$ with $|\xi| \le |\eta|$, then $\xi \in X$ and $\|\xi\|_X \le \|\eta\|_X$.
- *Weak order unit:* There is a $\xi \in X$ with $\xi > 0$ a.e.
- *Fatou property:* If $0 \le \xi_j \uparrow \xi$ pointwise a.e. for $(\xi_j)_{j\in\mathbf{N}}$ in X and $\sup_{j\in\mathbf{N}}\|\xi_j\|_X < \infty$, then $\xi \in X$ and $\|\xi\|_X = \sup_{j\in\mathbf{N}}\|\xi_j\|_X$.

If $\|\cdot\|_X$ is a norm then X is called a *Banach function space.*

A quasi-Banach function space X is called *order-continuous* if for any sequence $0 \le \xi_j \uparrow \xi \in X$ we have $\|\xi_j - \xi\|_X \to 0$. As an example we note that all reflexive Banach function spaces are order-continuous. If X is order-continuous, then the Bochner space $L^p(\mathbf{R}^n;X)$ for $p \in (0,\infty)$ coincides with the *mixed-norm space* of all measurable functions $f : \mathbf{R}^n \times \Omega \to \mathbf{C}$ such that

$$\big\| x \mapsto \|f(x,\cdot)\|_X \big\|_{L^p(\mathbf{R}^n)} < \infty.$$

Moreover if X is an order-continuous Banach function space, then its dual X^* is also a Banach function space. For an introduction to Banach function spaces we refer the reader to [LT79, Section 1.b] or [BS88].

A quasi-Banach function space X is said to be *p-convex* for $p \in (0,\infty)$ if for any $\xi_1,\cdots,\xi_K \in X$ we have

$$\Big\| \Big(\sum_{k=1}^K |\xi_k|^p \Big)^{1/p} \Big\|_X \le \Big(\sum_{k=1}^K \|\xi_k\|^p \Big)^{1/p}.$$

Moreover, X is said to be *p-concave* when the reverse inequality holds. Usually the defining inequalities for p-convexity and p-concavity include a constant depending on p and X, but as shown in [LT79, Theorem 1.d.8], X can be renormed equivalently such that these constants equal 1. The *p-concavification* of X for $p \in (0,\infty)$ is defined as

$$X^p := \{|\xi|^p \operatorname{sgn}\xi : \xi \in X\} = \{\xi \in L^0(\Omega) : |\xi|^{1/p} \in X\}$$

equipped with the quasinorm $\|\xi\|_{X^p} := \||\xi|^{1/p}\|_X^p$. Note that $\|\cdot\|_{X^p}$ is a norm if and only if X is p-convex. In particular for $f \in L^p_{\text{loc}}(\mathbf{R}^n; X)$ and a set $E \subseteq \mathbf{R}^n$ of positive finite measure the p-convexity of X ensures that $\langle f\rangle_{p,E}$ is well-defined as a Bochner integral. See [LT79, Section 1.d] and [Kal84] for a further introduction to p-convexity and related notions.

8.1.1. Product quasi-Banach function spaces

For m quasi-Banach function spaces $X_1,\ldots,X_m$ over the same measure space we, wish to define their product $\prod_{j=1}^m X_j$. This space is essentially defined as the pointwise product of functions in the factors. More precisely:

Definition 8.1.1. Let $X_1,\ldots,X_m$ be m quasi-Banach function spaces over a measure space (Ω,μ). We define

$$\prod_{j=1}^m X_j := \Big\{\xi \in L^0(\Omega) : \text{there exist } 0 \le \xi_j \in X_j\ 1 \le j \le m \text{ such that } |\xi| \le \prod_{j=1}^m \xi_j\Big\}.$$

Moreover, for $\xi \in \prod_{j=1}^m X_j$ we define

$$\|\xi\|_{\prod_{j=1}^m X_j} := \inf\Big\{\prod_{j=1}^m \|\xi_j\|_{X_j} : |\xi| \le \prod_{j=1}^m \xi_j, 0 \le \xi_j \in X_j,\ 1 \le j \le m\Big\}.$$

We call $\vec{X} = (X_1,\cdots,X_m)$ an *m-tuple of quasi-Banach function spaces* if $X_1,\cdots,X_m$ are quasi-Banach function spaces over the same measure space and the product $\prod_{j=1}^m X_j$ equipped with $\|\cdot\|_{\prod_{j=1}^m X_j}$ is also a quasi-Banach function space.

We use the convention that for an m-tuple of quasi-Banach function spaces we write $X := \prod_{j=1}^m X_j$. We extend our convention of the vector notation for the weighted mixed-norm spaces by writing

$$L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n;\vec{X}) := L^{p_1}_{w_1}(\mathbf{R}^n;X_1) \times \cdots \times L^{p_m}_{w_m}(\mathbf{R}^n;X_m).$$

Moreover, we say that $\vec{X}$ is $\vec{r}$-convex for $\vec{r} \in (0,\infty)^m$ if X_j is r_j-convex for all $j \in \{1,\ldots,m\}$.

For a pair of quasi-Banach spaces X_1 and X_2 we will sometimes also write $X_1 \cdot X_2$ for their product. We point out that taking products of quasi-Banach spaces is associative in the sense that $X_1 \cdot (X_2 \cdot X_3) = (X_1 \cdot X_2) \cdot X_3$ with equal (quasi)norms and therefore Definition 8.1.1 is consistently defined under changes in m. Moreover, we point out that $\left(\prod_{j=1}^m X_j\right)^p = \prod_{j=1}^m X_j^p$ for all $p \in (0,\infty)$.

We show that the space $\prod_{j=1}^m X_j$ is a vector-space.

Proposition 8.1.2. *Let $X_1,\ldots,X_m$ be m quasi-Banach function spaces over the same measure space. Then $\prod_{j=1}^m X_j$ is a vector-space. Moreover, we have*

$$\prod_{j=1}^m X_j = \Big\{\xi \in L^0(\Omega) : \text{there exist } \xi_j \in X_j\ 1 \le j \le m \text{ such that } \xi = \prod_{j=1}^m \xi_j\Big\} \tag{8.1.1}$$

and

$$\|\xi\|_{\prod_{j=1}^m X_j} = \inf\Big\{\prod_{j=1}^m \|\xi_j\|_{X_j} : |\xi| = \prod_{j=1}^m \xi_j, 0 \le \xi_j \in X_j, 1 \le j \le m\Big\}. \tag{8.1.2}$$

Proof. It is clear that $\prod_{j=1}^m X_j$ is closed under scalar multiplication. To see that it is closed under addition, note that if $\xi,\eta \in \prod_{j=1}^m X_j$, then $|\xi| \le \prod_{j=1}^m \xi_j$ for some $0 \le \xi_j \in X_j$, and $|\eta| \le \prod_{j=1}^m \eta_j$ for some $0 \le \eta_j \in X_j$. Then $0 \le \xi_j + \eta_j \in X_j$ and

$$|\xi+\eta| \le |\xi| + |\eta| \le \prod_{j=1}^m \xi_j + \prod_{j=1}^m \eta_j \le \prod_{j=1}^m (\xi_j + \eta_j),$$

proving that $\xi+\eta \in \prod_{j=1}^m X_j$, as desired.

For (8.1.1) and (8.1.2), the inclusion " $\supseteq$" for (8.1.1) together with the norm inequality " $\le$" for (8.1.2) are clear. For the converse, note that if $|\xi| \le \prod_{j=1}^m \xi_j$ for $0 \le \xi_j \in X_j$, then we can define $\widetilde{\xi}_1$ by 0 where $\xi = 0$ and by $\xi \prod_{j=2}^m \xi_j^{-1}$ where $\xi \neq 0$. Then we have $\xi = \widetilde{\xi}_1 \prod_{j=2}^m \xi_j$, and $|\widetilde{\xi}_1| \le \xi_1$, so that by the ideal property of X_1 we have $\widetilde{\xi}_1 \in X_1$ with $\|\widetilde{\xi}_1\|_{X_1} \le \|\xi_1\|_{X_1}$. This proves the inclusion " $\subseteq$" in (8.1.1), proving the equality. For the norm equality, note that since $|\xi| = |\widetilde{\xi}_1| \prod_{j=2}^m \xi_j$, we have

$$\inf\Big\{\prod_{j=1}^m \|\xi_j\|_{X_j} : |\xi| = \prod_{j=1}^m \xi_j, 0 \le \xi_j \in X_j, 1 \le j \le m\Big\} \le \|\widetilde{\xi}_1\| \prod_{j=2}^m \|\xi_j\|_{X_j} \le \prod_{j=1}^m \|\xi_j\|_{X_j}.$$

Taking an infimum over all $0 \le \xi_j \in X_j$ with $|\xi| \le \prod_{j=1}^m \xi_j$ then proves (8.1.2). □

We refer the reader to [Cal64, Loz69, Sch10] for a further elaboration on product Banach function spaces. Let us give a few examples:

Proposition 8.1.3. *Let (Ω,μ) be a σ-finite measure space.*

(i) For any quasi-Banach function space X we have $X \cdot L^\infty(\Omega) = X$.

(ii) Lebesgue spaces: $L^p(\Omega) = \prod_{j=1}^m L^{p_j}(\Omega)$ for $\vec{p} \in (0,\infty]^m$.

(iii) Lorentz spaces: $L^{p,q}(\Omega) = \prod_{j=1}^m L^{p_j,q_j}(\Omega)$ for $\vec{p} \in (0,\infty)^m$, $\vec{q} \in (0,\infty]^m$.

(iv) Orlicz spaces: $L^\Phi(\Omega) = \prod_{j=1}^m L^{\Phi_j}(\Omega)$ for Young functions Φ_j and $\Phi^{-1} = \prod_{j=1}^m \Phi_j^{-1}$.

In all these cases the (quasi)-norm of the product is equivalent to the usual (quasi)-norm.

Proof. For (i), if $\xi \in X \cdot L^\infty(\Omega)$, pick $0 \le \xi_1 \in X$, $0 \le \xi_2 \in L^\infty(\Omega)$ such that $|\xi| \le \xi_1 \xi_2$. Then $|\xi| \le \|\xi_2\|_{L^\infty(\Omega)} \xi_1$, so by the ideal property of X we have $\xi \in X$ with $\|\xi\|_X \le \|\xi_2\|_{L^\infty(\Omega)} \|\xi_1\|_X$. Taking an infimum over all such decompositions ξ_1, ξ_2 yields $\|\xi\|_X \le \|\xi\|_{X \cdot L^\infty(\Omega)}$. Conversely, if $\xi \in X$ then $|\xi| = |\xi| \cdot 1$ so that $\xi \in X \cdot L^\infty(\Omega)$ with $\|\xi\|_{X \cdot L^\infty(\Omega)} \le \|\xi\|_X \|1\|_{L^\infty(\Omega)} = \|\xi\|_X$. The assertion follows.

For (ii), (iii), and (iv), the inclusion $\prod_{j=1}^m X_j \subseteq X$ with X respectively equal to $L^p(\Omega)$, $L^{p,q}(\Omega)$, and $L^\Phi(\Omega)$ and X_j respectively equal to $L^{p_j}(\Omega)$, $L^{p_j,q_j}(\Omega)$, and $L^{\Phi_j}(\Omega)$, follows from the generalized Hölder's inequality $\|\prod_{j=1}^m \xi_j\|_X \lesssim \prod_{j=1}^m \|\xi_j\|_{X_j}$ valid for these spaces, see [O'N63, O'N65].

For the converse in (ii) and (iii) in the case that $q = q_1 = \cdots = q_m = \infty$, let $\xi \in L^p(\Omega)$ or $\xi \in L^{p,\infty}(\Omega)$ respectively. If $p = p_1 = \cdots = p_m = \infty$, the result follows from (i). Otherwise, we set $\xi_j := |\xi|^{\frac{p}{p_j}}$. Then $\xi_j \in L^{p_j}(\Omega)$ or $\xi_j \in L^{p_j,\infty}(\Omega)$ respectively, $|\xi| = \prod_{j=1}^m \xi_j$, and $\prod_{j=1}^m \|\xi_j\|_{L^{p_j}(\Omega)} = \|\xi\|_{L^p(\Omega)}$ or similarly in the weak case, proving the result. The converse for (iv) is proven analogously with $\xi_j := \Phi_j^{-1}(\Phi(|\xi|))$.

Finally, for (iii) in the case $q_k < \infty$ for some $1 \le k \le j$ we take $\alpha > 0$ such that $X_j := L^{p_j/\alpha, q_j/\alpha}(\Omega)$ are all reflexive Banach spaces. Then by [Tri78, Theorem 1.10.3 and 1.18.6] we can identify the product space $\prod_{j=1}^m L^{p_j/\alpha, q_j/\alpha}(\Omega)$ with an iterated complex interpolation space by [Cal64]. So $\prod_{j=1}^m L^{p_j/\alpha, q_j/\alpha}(\Omega) = L^{p/\alpha, q/\alpha}(\Omega)$. The assertion now follows by taking a $\frac{1}{\alpha}$-concavification of both sides and the fact that the concavification of a product is the product of concavifications. □

Next, we present several useful results for when our spaces are Banach function spaces and not merely quasi-Banach spaces. We will be working with so called *Calderón-Lozanovskii* products which are products of the form $X_0^{1-\theta} \cdot X_1^\theta$ for some $\theta \in (0,1)$, see [Cal64, Loz69].

We have the following properties of product Banach function spaces:

Proposition 8.1.4. *Let $X, X_1, \ldots, X_m$ be Banach function spaces over a σ-finite measure space (Ω, μ) and let $\theta, \theta_1, \ldots, \theta_m \in (0,1)$ with $\sum_{j=1}^m \theta_j = 1$.*

(i) If one of $X_1, \ldots, X_m$ is reflexive, then $\prod_{j=1}^m X_j^{\theta_j}$ is reflexive.

(ii) If X is reflexive, then so is X^θ.

(iii) $\left(\prod_{j=1}^m X_j^{\theta_j}\right)^* = \prod_{j=1}^m (X_j^{\theta_j})^*$.

(iv) $(X^\theta)^* = (X^*)^\theta \cdot L^{1/(1-\theta)}(\Omega)$.

Proof. Part (i) is proven in [Loz69, Theorem 3], and it also follows from [Cal64] through complex interpolation. Part (ii) follows from (i) by noting that by Proposition 8.1.3(i) we have $X^\theta = L^\infty(\Omega)^{1-\theta} \cdot X^\theta$. Part (iii) is proven in [Loz69, Theorem 2] and for (iv) see [Sch10, Theorem 2.9]. □

Next we prove a result for the products of weighted mixed-norm Lebesgue spaces.

Lemma 8.1.5. *Let $\vec{X}$ be an m-tuple of quasi-Banach function spaces, let $\vec{p}\in(0,\infty]^m$ with $p<\infty$, and let $\vec{w}$ be an m-tuple of weights. If there is a $q\in(0,\infty)$ such that X is q-convex and order-continuous, then*

$$L^p_w(\mathbf{R}^n;X)=\prod_{j=1}^m L^{p_j}_{w_j}(\mathbf{R}^n;X_j).$$

Proof. For the inclusion $\prod_{j=1}^m L^{p_j}_{w_j}(\mathbf{R}^n;X_j)\subseteq L^p_w(\mathbf{R}^n;X)$, note that if $f\in\prod_{j=1}^m L^{p_j}_{w_j}(\mathbf{R}^n;X_j)$ and $|f|\leq\prod_{j=1}^m f_j$ for $0\leq f_j\in L^{p_j}_{w_j}(\mathbf{R}^n;X_j)$, then $\|f\|_X\leq\prod_{j=1}^m\|f_j\|_{X_j}$ so that by Hölder's inequality we have $f\in L^p_w(\mathbf{R}^n;X)$ with

$$\|f\|_{L^p_w(\mathbf{R}^n;X)}\leq\prod_{j=1}^m\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}.$$

Moreover, taking an infimum over all $0\leq f_j\in L^{p_j}_{w_j}(\mathbf{R}^n;X_j)$ such that $|f|\leq\prod_{j=1}^m f_j$, we conclude that $\|f\|_{L^p_w(\mathbf{R}^n;X)}\leq\|f\|_{\prod_{j=1}^m L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}$.

For the converse, we first reduce to the case $q=1$, $p>1$. For all $\alpha\in(0,\infty)$ we have $L^p_w(\mathbf{R}^n;X)^\alpha=L^{\frac{p}{\alpha}}_{w^\alpha}(\mathbf{R}^n;X^\alpha)$ and $\left(\prod_{j=1}^m L^{p_j}_{w_j}(\mathbf{R}^n;X_j)\right)^\alpha=\prod_{j=1}^m L^{\frac{p_j}{\alpha}}_{w_j^\alpha}(\mathbf{R}^n;X_j^\alpha)$, so that for the result it suffices to prove that $L^{\frac{p}{\alpha}}_{w^\alpha}(\mathbf{R}^n;X^\alpha)=\prod_{j=1}^m L^{\frac{p_j}{\alpha}}_{w_j^\alpha}(\mathbf{R}^n;X_j^\alpha)$. By taking $\alpha<\min\{p,q\}$, replacing $\frac{p_j}{\alpha}$ by p_j, X_j^α by X_j, and w_j^α by w_j, we have reduced to the case $q=1$, $p>1$.

Now, let $f\in L^p_w(\mathbf{R}^n;X)$ be a function such that fw is a simple function, say $fw=\sum_{k=1}^K\chi_{A_k}\otimes\xi_k$ with non-zero $\xi_k\in X$, and $(A_k)_{k=1}^K$ a pairwise disjoint collection of measurable sets in $\mathbf{R}^n$ such that, since $p<\infty$, $|A_k|<\infty$ for all $k\in\{1,\ldots,K\}$. Since $\xi_k\in X$, we can find $0\leq\xi_{j,k}\in X_j$ such that $|\xi_k|\leq\prod_{j=1}^m\xi_{j,k}$ for all $k\in\{1,\ldots,K\}$. We define

$$\eta_{j,k}:=\|\xi_{j,k}\|_{X_j}^{-1}\Big(\prod_{l=1}^m\|\xi_{l,k}\|_{X_l}\Big)^{\frac{p}{p_j}}\xi_{j,k}$$

so that $\|\eta_{j,k}\|_{X_j}=\left(\prod_{l=1}^m\|\xi_{l,k}\|_{X_l}\right)^{\frac{p}{p_j}}$ for all $j\in\{1,\ldots,m\}$, $k\in\{1,\ldots,K\}$ and $\prod_{j=1}^m\eta_{j,k}=\prod_{j=1}^m\xi_{j,k}$ for all $k\in\{1,\ldots,K\}$.

Defining $f_j:=w_j^{-1}\sum_{k=1}^K\chi_{A_k}\otimes\eta_{j,k}$, it follows from the fact that the A_k are pairwise disjoint that $f=\prod_{j=1}^m f_j$. Since $f_j\in L^{p_j}_{w_j}(\mathbf{R}^n;X_j)$ with

$$\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}=\Big(\sum_{k=1}^K|A_k|\|\eta_{j,k}\|_{X_j}^{p_j}\Big)^{\frac{1}{p_j}}=\Big(\sum_{k=1}^K|A_k|\Big(\prod_{l=1}^m\|\xi_{l,k}\|_{X_l}\Big)^p\Big)^{\frac{1}{p_j}},$$

we conclude that $f\in\prod_{j=1}^m L^{p_j}_{w_j}(\mathbf{R}^n;X_j)$ with

$$\|f\|_{\prod_{j=1}^m L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}\leq\prod_{j=1}^m\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}=\Big(\sum_{k=1}^K|A_k|\Big(\prod_{l=1}^m\|\xi_{l,k}\|_{X_l}\Big)^p\Big)^{\frac{1}{p}}.$$

Taking an infimum over all possible $0 \le \xi_{j,k} \in X_j$ such that $|\xi_k| \le \prod_{j=1}^m \xi_{j,k}$ for all $k \in \{1,\dots,K\}$, we conclude that

$$\|f\|_{\prod_{j=1}^m L^{p_j}_{w_j}(\mathbf{R}^n;X_j)} \le \Big(\sum_{k=1}^K |A_k| \|\xi_k\|_X^p\Big)^{\frac{1}{p}} = \|f\|_{L^p_w(\mathbf{R}^n;X)}. \tag{8.1.3}$$

Since X is an order-continuous Banach function space and $p > 1$, the mixed-norm space $L^p(\mathbf{R}^n;X)$ coincides with the corresponding Bochner space and hence, the simple functions are dense in this space. Thus, the functions f for which fw is a simple function are dense in $L^p_w(\mathbf{R}^n;X)$ so that we can extend the inequality (8.1.3) to all $f \in L^p_w(\mathbf{R}^n;X)$. We conclude that $\prod_{j=1}^m L^{p_j}_{w_j}(\mathbf{R}^n;X_j) = L^p_w(\mathbf{R}^n;X)$ with equal norm, as desired. □

8.2. VECTOR-VALUED SPARSE DOMINATION

This section serves as a vector-valued analogue of Section 5.3. We will be considering operators satisfying vector-valued sparse domination in one of the two equivalent senses presented in the following proposition. The first uses duality in X, which is useful as it allows one to apply Fubini's theorem. The second is domination with the norm of X on the inside, which allows one to deduce weighted bounds with a simpler argument.

Proposition 8.2.1. *Let $\vec{r} \in (0,\infty)^m$, $q \in (0,\infty)$, $s \in (q,\infty]$ and let $\vec{X}$ be an m-tuple of quasi-Banach function spaces over a measure space (Ω,μ) such that X is q-convex and order-continuous. Let $\widetilde{T}$ be an operator defined on an m-tuple $\vec{f} \in L^{\vec{r}}_{\mathrm{loc}}(\mathbf{R}^d;\vec{X})$ with $\widetilde{T}(\vec{f}) \in L^0(\mathbf{R}^d;X)$. Then the following are equivalent:*

(i) For all $g \in L^\infty_c(\mathbf{R}^d;((X^q)^*)^{\frac{1}{q}})$

$$\big\|\widetilde{T}(\vec{f})\cdot g\big\|_{L^q(\mathbf{R}^d;L^q(\Omega))} \le C\,\big\|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\|\vec{f}\|_{\vec{X}},\|g\|_{((X^q)^*)^{\frac{1}{q}}})\big\|_{L^q(\mathbf{R}^d)}.$$

(ii) For all $g \in L^\infty_c(\mathbf{R}^d)$

$$\big\|\|\widetilde{T}(\vec{f})\|_X\cdot g\big\|_{L^q(\mathbf{R}^d)} \le C\,\big\|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\|\vec{f}\|_{\vec{X}},g)\big\|_{L^q(\mathbf{R}^d)}.$$

Proof. For (ii)⇒(i), note that

$$\begin{aligned}\|\widetilde{T}(\vec{f})\cdot g\|_{L^q(\Omega)} &= \Big(\int_\Omega |\widetilde{T}(\vec{f})|^q|g|^q\,\mathrm{d}\mu\Big)^{\frac{1}{q}} \le \big\||\widetilde{T}(\vec{f})|^q\big\|_{X^q}^{\frac{1}{q}}\big\||g|^q\big\|_{(X^q)^*}^{\frac{1}{q}}\\ &= \big\|\widetilde{T}(\vec{f})\big\|_X\|g\|_{((X^q)^*)^{\frac{1}{q}}}\end{aligned}$$

so that $\big\|\widetilde{T}(\vec{f})\cdot g\big\|_{L^q(\mathbf{R}^d;L^q(\Omega))} \le \big\|\|\widetilde{T}(\vec{f})\|_X\cdot\|g\|_{((X^q)^*)^{\frac{1}{q}}}\big\|_{L^q(\mathbf{R}^d)}$. Since for a simple function $g \in L^\infty_c(\mathbf{R}^d;((X^q)^*)^{\frac{1}{q}})$ we have $\|g\|_{((X^q)^*)^{\frac{1}{q}}} \in L^\infty_c(\mathbf{R}^d)$, applying (ii) with g replaced by $\|g\|_{((X^q)^*)^{\frac{1}{q}}}$ proves (i).

For (i)⇒(ii) we note that by duality (see e.g. [HNVW16, Proposition 1.3.1]) we have

$$\begin{aligned}
\big\| \|\widetilde{T}(\vec{f}\,)\|_X \cdot g \big\|_{L^q(\mathbf{R}^d)} &= \big\| \| |\widetilde{T}(\vec{f}\,)|^q \|_{X^q} \cdot |g|^q \big\|_{L^1(\mathbf{R}^d)}^{\frac{1}{q}} \\
&= \big\| |\widetilde{T}(\vec{f}\,)|^q |g|^q \big\|_{L^1(\mathbf{R}^d;X^q)}^{\frac{1}{q}} \\
&= \sup_{\|h\|_{L^\infty(\mathbf{R}^d;((X^q)^*)^{1/q})}=1} \big\| |\widetilde{T}(\vec{f}\,)|^q \cdot |g|^q \cdot |h|^q \big\|_{L^1(\mathbf{R}^d;L^1(\Omega))}^{\frac{1}{q}} \\
&= \sup_{\|h\|_{L^\infty(\mathbf{R}^d;((X^q)^*)^{1/q})}=1} \big\| \widetilde{T}(\vec{f}\,) \cdot gh \big\|_{L^q(\mathbf{R}^d;L^q(\Omega))}.
\end{aligned} \tag{8.2.1}$$

Since $gh \in L^\infty_c(\mathbf{R}^d;((X^q)^*)^{\frac{1}{q}})$ for any $g \in L^\infty_c(\mathbf{R}^d)$ and $h \in L^\infty(\mathbf{R}^d;((X^q)^*)^{\frac{1}{q}})$ of norm 1 with $\|gh\|_{((X^q)^*)^{\frac{1}{q}}} \le |g| \|h\|_{L^\infty(\mathbf{R}^d;((X^q)^*)^{\frac{1}{q}})} = |g|$, it follows from (i) that

$$\begin{aligned}
\big\| \widetilde{T}(\vec{f}\,) \cdot gh \big\|_{L^q(\mathbf{R}^d;L^q(\Omega))} &\le C \big\| M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\|\vec{f}\|_{\vec{X}}, \|gh\|_{((X^q)^*)^{\frac{1}{q}}}) \big\|_{L^q(\mathbf{R}^d)} \\
&\le C \big\| M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\|\vec{f}\|_{\vec{X}}, g) \big\|_{L^q(\mathbf{R}^d)}.
\end{aligned}$$

By combining this result with (8.2.1) we have proven (ii). □

In the following result we will deduce weighted bounds from vector-valued sparse domination.

Theorem 8.2.2. *Let $\vec{r} \in (0,\infty)^m$, $q \in (0,\infty)$, $s \in (q,\infty]$ and let $\vec{X}$ be an m-tuple of quasi-Banach function spaces over a measure space (Ω,μ) such that X is q-convex and order-continuous. Let $\widetilde{T}$ be an m-(sub)linear operator initially defined for all simple functions $\vec{f} \in L^\infty_c(\mathbf{R}^d;\vec{X})$. Suppose that*

$$\big\| \|\widetilde{T}(\vec{f}\,)\|_X \cdot g \big\|_{L^q(\mathbf{R}^d)} \le C_T \big\| M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\|\vec{f}\|_{\vec{X}}, g) \big\|_{L^q(\mathbf{R}^d)}. \tag{8.2.2}$$

for all simple $\vec{f} \in L^\infty_c(\mathbf{R}^d;\vec{X})$, $g \in L^\infty_c(\mathbf{R}^d)$. Then for all $\vec{p} \in (0,\infty]^m$ with $\vec{r} < \vec{p}$ and $p < s$, all $\vec{w} \in A_{\vec{p},(\vec{r},s)}$, $\widetilde{T}$ has a unique extension satisfying

$$\|\widetilde{T}(\vec{f}\,)\|_{L^p_w(\mathbf{R}^d;X)} \lesssim_{\vec{p},q,\vec{r},s} C_T\, [\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}} \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^d;X_j)}$$

for all $\vec{f} \in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^d;\vec{X})$.

For the proof, we first require a density result. Note that we are only considering simple functions $g \in L^\infty_c(\mathbf{R}^d)$, while in Proposition 8.2.1 we are considering all $g \in L^\infty_c(\mathbf{R}^d)$. However, as another consequence of the following density result, this is equivalent.

Lemma 8.2.3. *Let w be a weight, $p, q \in (0,\infty)$ and X a q-convex quasi-Banach function space. Then the simple functions in $L^p_w(\mathbf{R}^n;X) \cap L^\infty_c(\mathbf{R}^n;X)$ are dense in $L^p_w(\mathbf{R}^n;X)$.*

Proof. First suppose that $p,q \geq 1$ and fix $f \in L^p_w(\mathbf{R}^n;X)$. By [HNVW16, Corollary 1.1.21] and the dominated convergence theorem there exists a sequence of simple functions $(f_j)_{j\in\mathbf{N}}$ such that $f_j \to f$ in $L^p_w(\mathbf{R}^n;X)$, and $f_j(x) \to f(x)$ and $\|f_j(x)\|_X \leq \|f\|_X$ for a.e. $x \in \mathbf{R}^n$. Setting $(g_j)_{j\in\mathbf{N}} = (f_j\chi_{B(0,j)})_{j\in\mathbf{N}}$ it follows that $g_j \in L^p_w(\mathbf{R}^n;X)\cap L^\infty_c(\mathbf{R}^n;X)$ for all $j \in \mathbf{N}$ and $g_j \to f$ in $L^p_w(\mathbf{R}^n;X)$ by the dominated convergence theorem, proving the lemma.

Now consider the case $p < 1$ and/or $q < 1$. Fix $k \in \mathbf{N}$ so that $2^k p, 2^k q > 1$. For $f \in L^p_w(\mathbf{R}^n;X)$ we can pick a positive $g \in L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n;X^{2^{-k}})$ with $g^{2^k} = |f|$. By our previous result we can find a positive sequence of simple functions $(g_j)_{j\in\mathbf{N}}$ in $L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n;X^{2^{-k}})\cap L^\infty_c(\mathbf{R}^n;X^{2^{-k}})$ converging to g. Setting $f_j := g_j^{2^k}\operatorname{sgn}(f) \in L^p_w(\mathbf{R}^n;X)\cap L^\infty_c(\mathbf{R}^n;X)$ we compute

$$|f_j - f| = |g_j^{2^k} - g^{2^k}| = |g_j - g|\prod_{l=0}^{k-1}|g_j^{2^l} + g^{2^l}|$$

so that by Hölder's inequality

$$\|f_j - f\|_{L^p_w(\mathbf{R}^n;X)} \leq \|g_j - g\|_{L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n;X^{2^{-k}})}\prod_{l=0}^{k-1}\|g_j^{2^l} + g^{2^l}\|_{L^{2^{k-l}p}_{w^{2^{-(k-l)}}}(\mathbf{R}^n;X^{2^{-k-l}})}.$$

Since $\|g_j^{2^l} + g^{2^l}\|_{L^{2^{k-l}p}_{w^{2^{-(k-l)}}}(\mathbf{R}^n;X^{2^{-k-l}})} \lesssim \|g_j\|^{2^l}_{L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n;X^{2^{-k}})} + \|g\|^{2^l}_{L^{2^k p}_{w^{2^{-k}}}(\mathbf{R}^n;X^{2^{-k}})}$ is bounded in j, we conclude that $f_j \to f$ in $L^p_w(\mathbf{R}^n;X)$, proving the result. □

Proof of Theorem 8.2.2. The proof is completely analogous to the proof of Theorem 5.3.6, replacing T by $\|\widetilde{T}(\vec{f})\|_X$ and f_j by $\|f_j\|_{X_j}$, and by using the density result Lemma 8.2.3 rather than Lemma 5.3.8. □

8.3. THE MULTISUBLINEAR LATTICE MAXIMAL OPERATOR

In this section we will introduce and study properties of the multisublinear lattice maximal operator. We begin with an overview of the case $m = 1$. Let X be a Banach function space and let $\mathscr{F}$ be a finite collection of cubes. Since X is a Banach function space over a measure space (Ω,μ), it is a Banach lattice with respect to the partial order $\leq$ given by $\xi \leq \eta$ if and only if $\xi(\omega) \leq \eta(\omega)$ for all $\omega \in \Omega$. Thus, for any $f \in L^1_{\text{loc}}(\mathbf{R}^n;X)$, $x \in \mathbf{R}^n$, we may define

$$\widetilde{M}^{\mathscr{F}} f(x) := \sup_{Q\in\mathscr{F}}\langle f\rangle_{1,Q}\chi_Q(x),$$

where the supremum is taken in the lattice sense in X. Since this supremum is taken over the finitely many values $(\langle f\rangle_{1,Q})_{Q\in\mathscr{F}}$ in X, this means that $\widetilde{M}^{\mathscr{F}} f$ can take at most $2^{\#\mathscr{F}}$ values and hence, is an X-valued simple function. Moreover, since $\langle f\rangle_{1,Q}(\omega) = \langle f(\cdot,\omega)\rangle_{1,Q}$ for $\omega \in \Omega$ and the supremum of functions in X is given by their pointwise supremum, we have

$$\widetilde{M}^{\mathscr{F}} f(x,\omega) = \max_{Q\in\mathscr{F}} \langle f(\cdot,\omega)\rangle_{1,Q}\chi_Q(x) = M^{\mathscr{F}}(f(\cdot,\omega))(x)$$

for all $(x,\omega) \in \mathbf{R}^n \times \Omega$.

We say that X has the *Hardy–Littlewood property* and write $X \in \mathrm{HL}$ if there is a $p \in (1,\infty)$ such that

$$\|\widetilde{M}\|_{p,X} := \sup_{\mathscr{F}} \|\widetilde{M}^{\mathscr{F}}\|_{L^p(\mathbf{R}^n;X)\to L^p(\mathbf{R}^n;X)} < \infty,$$

where the supremum is taken over all finite collections of cubes $\mathscr{F}$. This property is independent of the exponent p and the dimension n, see [GMT93], and even the quantity $\|\widetilde{M}\|_{p,X}$ can be bounded by a constant independent of n, see [DK19].

As an example we note that (iterated) L^p-spaces for $p \in (1,\infty]$ have the Hardy–Littlewood property. Moreover by a deep result of Bourgain [Bou84] and Rubio de Francia [Rub86, Theorem 3] we have that both X and X^* have the Hardy–Littlewood property if and only if X has the so-called UMD property. We will elaborate on the connection between the Hardy–Littlewood property and the UMD property in Section 8.4.

If X is an order-continuous Banach function space with the Hardy–Littlewood property and $p \in [1,\infty)$, we define the *lattice Hardy–Littlewood maximal operator* for $f \in L^p(\mathbf{R}^n;X)$, $x \in \mathbf{R}^n$, by

$$\widetilde{M} f(x) := \sup_{Q} \langle f\rangle_{1,Q}\chi_Q(x),$$

where the supremum is taken in the lattice sense over all cubes $Q \subseteq \mathbf{R}^n$. We will show that $\widetilde{M} f : \mathbf{R}^n \to X$ is strongly measurable. By regularity of the Lebesgue measure, to compute $\widetilde{M} f$ it is equivalent to take the supremum over the countable collection of cubes with rational center points and rational side lengths. Thus, we can find finite collections of cubes $\mathscr{F}_j$ for $j \in \mathbf{N}$ such that $\mathscr{F}_j \subseteq \mathscr{F}_{j+1}$ and for a.e. $x \in \mathbf{R}^n$

$$\sup_{j\in\mathbf{N}} \widetilde{M}^{\mathscr{F}_j} f(x) \uparrow \sup_{j\in\mathbf{N}} \widetilde{M}^{\mathscr{F}_j} f(x) = \widetilde{M} f(x)$$

pointwise a.e. Since X has the Hardy–Littlewood property (where we use the fact that $\sup_{j\in\mathbf{N}} \|\widetilde{M}^{\mathscr{F}_j} f\|_{L^{1,\infty}(\mathbf{R}^d;X)} < \infty$ by [GMT93, Theorem 1.7] for the case $p = 1$), it follows from the Fatou property of X that $\widetilde{M} f(x) \in X$ for a.e. $x \in \mathbf{R}^n$. By order-continuity of X, $\big(\widetilde{M}^{\mathscr{F}_j} f(x)\big)_{j\in\mathbf{N}}$ converges in X to $\widetilde{M} f(x)$ for a.e. $x \in \mathbf{R}^n$. As $\widetilde{M}^{\mathscr{F}_j} f$ is a simple function for each $j \in \mathbf{N}$, we conclude that $\widetilde{M} f$ is strongly measurable. We also point out that since

$$\widetilde{M}^{\mathscr{F}_j}(f(\cdot,\omega))(x) = \widetilde{M}^{\mathscr{F}_j} f(x,\omega) \to \widetilde{M} f(x,\omega)$$

for a.e. $(x,\omega) \in \mathbf{R}^n \times \Omega$, we also have

$$\widetilde{M} f(x,\omega) = M(f(\cdot,\omega))(x)$$

for a.e. $(x,\omega) \in \mathbf{R}^n \times \Omega$.

For the multisublinear analogue of the lattice Hardy–Littlewood maximal operator, let $\vec{r} \in (0,\infty)^m$ and let $\vec{X}$ be an $\vec{r}$-convex m-tuple of quasi-Banach function spaces. For a finite collection of cubes $\mathscr{F}$, $\vec{f} \in L^{\vec{r}}_{\mathrm{loc}}(\mathbf{R}^n;\vec{X})$, and $x \in \mathbf{R}^n$, we define

$$\widetilde{M}^{\mathscr{F}}_{\vec{r}}(\vec{f})(x) := \sup_{Q\in\mathscr{D}} \prod_{j=1}^{m} \langle f_j\rangle_{r_j,Q}\chi_Q(x),$$

where the supremum is taken in the lattice sense.

Definition 8.3.1. Let $\vec{r} \in (0,\infty)^m$ and let $\vec{X}$ be an m-tuple of quasi-Banach function spaces. We say that $\vec{X}$ has the *$\vec{r}$-Hardy–Littlewood property* and write $\vec{X} \in \mathrm{HL}_{\vec{r}}$ if $\vec{X}$ is $\vec{r}$-convex and there is a $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$ such that

$$\|\widetilde{M}_{\vec{r}}\|_{\vec{p},\vec{X}} := \sup_{\mathscr{F}} \|\widetilde{M}^{\mathscr{F}}_{\vec{r}}\|_{L^{\vec{p}}(\mathbf{R}^n;\vec{X})\to L^{p}(\mathbf{R}^n;X)} < \infty,$$

where the supremum is taken over all finite collection of cubes $\mathscr{F}$.

As in the linear case $m = 1$, the definition of $\mathrm{HL}_{\vec{r}}$ is independent of the exponents $\vec{p}$ and the dimension n. The independence of n can be shown using the method of rotations (see e.g. [GMT93, Remark 1.3]), and the independence of $\vec{p}$ follows from Corollary 8.3.5 below.

We also point out that we have the rescaling property that if $\vec{X} \in \mathrm{HL}_{\vec{r}}$, then $\vec{X}^{\alpha} \in \mathrm{HL}_{\frac{\vec{r}}{\alpha}}$ for all $\alpha \in (0,\infty)$ with $\|\widetilde{M}_{\frac{\vec{r}}{\alpha}}\|_{\frac{\vec{p}}{\alpha},\vec{X}^{\alpha}} = \|\widetilde{M}_{\vec{r}}\|^{\alpha}_{\vec{p},\vec{X}}$. For the case $m = 1$ this means that X^r has the Hardy-Littlewood property if and only if $X \in \mathrm{HL}_r$.

The multilinear Hardy–Littlewood satisfies the following partition result:

Proposition 8.3.2. *Let $\vec{r} \in (0,\infty)^m$, let $\vec{X}$ be an $\vec{r}$-convex m-tuple of quasi-Banach function spaces, and let $\mathscr{I}$ be a partition of $\{1,\dots,m\}$. If $(X_j)_{j\in I} \in \mathrm{HL}_{(r_j)_{j\in I}}$ for all $I \in \mathscr{I}$, then $\vec{X} \in \mathrm{HL}_{\vec{r}}$.*

Proof. Fix a finite collection of cubes $\mathscr{F}$. For each $I \in \mathscr{I}$, let $(p_j)_{j\in I} \in (0,\infty]^{\#I}$ be such that $(p_j)_{j\in I} \geq (r_j)_{j\in I}$ and $\|\widetilde{M}\|_{(p_j)_{j\in I},(X_j)_{j\in I}} < \infty$. Let $\vec{f} \in L^{\vec{p}}(\mathbf{R}^n,\vec{X})$ of norm 1. Writing $\frac{1}{p_I} := \sum_{j\in I}\frac{1}{p_j}$ and $X_I := \prod_{j\in I} X_j$, it follows from the associativity of taking products of quasi-Banach function spaces that $\|\cdot\|_X \leq \prod_{I\in\mathscr{I}}\|\cdot\|_{X_I}$. Hence, by Hölder's inequality we have

$$\big\|\widetilde{M}^{\mathscr{F}}_{\vec{r}}(\vec{f})\big\|_{L^p(\mathbf{R}^n;X)} \leq \prod_{I\in\mathscr{I}} \big\|\widetilde{M}^{\mathscr{F}}_{(r_j)_{j\in I}}((f_j)_{j\in I})\big\|_{L^{p_I}(\mathbf{R}^n;X_I)} \leq \prod_{I\in\mathscr{I}} \|\widetilde{M}\|_{(p_j)_{j\in I},(X_j)_{j\in I}}.$$

Thus, taking the supremum over all $\vec{f}$ of norm 1 and all finite collection of cubes $\mathscr{D}$ yields $\vec{X} \in \mathrm{HL}_{\vec{r}}$ with $\|\widetilde{M}_{\vec{r}}\|_{\vec{p},\vec{X}} \leq \prod_{I\in\mathscr{I}}\|\widetilde{M}\|_{(p_j)_{j\in I},(X_j)_{j\in I}}$, as desired. □

This result implies in particular that if $X_j^{r_j} \in \mathrm{HL}$ for all $j \in \{1,\dots,m\}$, then $\vec{X} \in \mathrm{HL}_{\vec{r}}$. We note that in general, this does not provide a necessary condition. Indeed, for $m = 3$ we can take $X_1 = \ell^2(\ell^\infty)$, $X_2 = \ell^\infty(\ell^2)$ and $X_3 = \ell^2(\ell^2)$. It is shown in [NVW15, Proposition

8.1] that X_2 does not satisfy the Hardy-Littlewood property. However, noting that $X_3 = (X_1 \cdot X_2)^*$, it follows from Corollary 8.4.8 below that $\vec{X} \in \mathrm{HL}_{(1,1,1)}$.

Let $\vec{r}, \vec{p} \in (0,\infty]^m$ with $\vec{p} \geq \vec{r}$, let $\vec{X} \in \mathrm{HL}_{\vec{r}}$, and assume that X is order-continuous. For $\vec{f} \in L^{\vec{p}}(\mathbf{R}^n; \vec{X})$ we define the *multisublinear lattice maximal operator*

$$\widetilde{M}_{\vec{r}}(\vec{f})(x) := \sup_Q \prod_{j=1}^m \langle f_j \rangle_{r_j,Q} \chi_Q(x),$$

where the supremum is taken in the lattice sense over all cubes $Q \subseteq \mathbf{R}^n$. By an analogous argument as in the case $m = 1$ (using Lemma 8.3.4 below for when $p_j = r_j$ for some $j \in \{1, \ldots, m\}$), the order-continuity of X and ensures that $\widetilde{M}_{\vec{r}}(\vec{f}) \in L^0(\mathbf{R}^n; X)$ and, if (Ω, μ) is the underlying measure space of X, we have

$$\widetilde{M}_{\vec{r}}(\vec{f})(x,\omega) = M_{\vec{r}}(\vec{f}(\cdot,\omega))(x)$$

for a.e. $(x,\omega) \in \mathbf{R}^n \times \Omega$.

Next we will prove vector-valued sparse domination of $\widetilde{M}_{\vec{r}}$ in a vector-valued analogue of Proposition 3.2.10. Since we are now dealing with the order structure of X, the selection procedure of the maximal cubes requires a more involved argument than what is presented for $X = \mathbf{C}$ in Proposition 3.2.10. This result in the case $m = 1$ was studied in [HL19] and the argument here is a multilinear analogue of their proof.

Theorem 8.3.3. *Let $\vec{r} \in (0,\infty)^m$, let $\vec{X}$ be an m-tuple of quasi-Banach function spaces, and let $q \in [r,\infty)$. Suppose that $\vec{X} \in \mathrm{HL}_{\vec{r}}$ and that X is an order-continuous q-convex quasi-Banach function space. Let $\mathscr{D} = \mathscr{D}^\alpha$ be a dyadic grid and let $\mathscr{F} \subseteq \mathscr{D}$ be a finite collection of cubes. Then for all $\vec{f} \in L^{\vec{r}}_{\mathrm{loc}}(\mathbf{R}^n; \vec{X})$ there is a sparse collection of cubes $\mathscr{S} \subseteq \mathscr{F}$ such that*

$$\big\| M^{\mathscr{F}}_{\vec{r}}(\vec{f}) \big\|_X \lesssim_r \sup_{Q \in \mathscr{S}} \prod_{j=1}^m \langle \|f_j\|_{X_j} \rangle_{r_j,Q} \chi_Q.$$

Moreover, for any $\vec{f} \in L^{\vec{r}}_{\mathrm{loc}}(\mathbf{R}^n; \vec{X})$ and $g \in L^q_{\mathrm{loc}}(\mathbf{R}^n)$ we have

$$\big\| \|\widetilde{M}_{\vec{r}}(\vec{f})\|_X \cdot g \big\|_{L^q(\mathbf{R}^n)} \lesssim_{\vec{X},\vec{r}} \big\| M_{(\vec{r},q)}\big(\|\vec{f}\|_{\vec{X}}, g\big) \big\|_{L^q(\mathbf{R}^n)},$$

In particular, we have

$$\big\| \widetilde{M}_{\vec{r}}(\vec{f}) \big\|_{L^q(\mathbf{R}^n;X)} \lesssim_{\vec{X},\vec{r}} \big\| M_{\vec{r}}\big(\|\vec{f}\|_{\vec{X}}\big) \big\|_{L^q(\mathbf{R}^n)}.$$

Note that X in Theorem 8.3.3 is automatically r-convex, which follows from the fact that X_j is r_j-convex for $1 \leq j \leq m$ and X^r is equal to the Calderón-Lozanovskii product $\prod_{j=1}^m (X_j^{r_j})^{\frac{r}{r_j}}$. If X is q-convex for $q > r$ we get a sparse domination result with a smaller sparse operator, which, as we will see in Corollary 8.3.5), yields better weighted bounds.

For the proof we will first show that $\widetilde{M}^{\mathscr{D}}_{\vec{r}}$ satisfies a weak endpoint estimate.

Lemma 8.3.4. *Let $\vec r \in (0,\infty)^m$, let $\vec X$ be an m-tuple of quasi-Banach function spaces. Suppose that $\vec X \in \mathrm{HL}_{\vec r}$ and that X and let $\mathscr{D} = \mathscr{D}^\alpha$ be a dyadic grid. Then for all $\vec p \in (0,\infty]^m$ with $\vec p > \vec r$ we have*

$$\sup_{\substack{\mathscr{F}\subseteq\mathscr{D}\\ \mathscr{F}\text{ finite}}} \|\widetilde{M}^{\mathscr{F}}_{\vec r}\|_{L^{\vec r}(\mathbf{R}^n;\vec X)\to L^{r,\infty}(\mathbf{R}^n;X)} \lesssim_{\vec p,\vec r} \|\widetilde{M}_{\vec r}\|_{\vec p,X}$$

Proof. Fix $\mathscr{F}\subseteq\mathscr{D}$ finite and let $\vec f \in L^{\vec r}(\mathbf{R}^n;\vec X)$ of norm 1. For $\lambda>0$ and $j\in\{1,\ldots,m\}$ we let $\mathscr{P}_j$ denote the collection of maximal cubes in $\mathscr{D}$ satisfying $\big\langle \|f_j\|_{X_j}\big\rangle_{r_j,Q} > \lambda^{\frac{r}{r_j}}$ so that $\Omega_j := \{x\in\mathbf{R}^n : M^{\mathscr{D}}_{r_j}(\|f_j\|_{X_j})(x) > \lambda^{\frac{r}{r_j}}\} = \bigcup_{Q\in\mathscr{P}_j} Q$. For a fixed $P\in\mathscr{F}$ and $j\in\{1,\ldots,m\}$ we find that if $P\setminus\Omega_j\neq\varnothing$, then, since the collection $\mathscr{P}_j$ is pairwise disjoint and since

$$\big\langle \langle f_j\rangle_{r_j,Q}\chi_Q\big\rangle_{r_j,P} = \langle f_j\chi_Q\rangle_{r_j,P}$$

for all $Q\subseteq P$, we have

$$\begin{aligned}\langle f_j\rangle_{r_j,P}\chi_P &= \Big\langle f_j\chi_{\Omega_j^c} + \sum_{\substack{Q\in\mathscr{P}_j:\\ Q\subseteq P}} f_j\chi_Q\Big\rangle_{r_j,P}\chi_P\\ &= \Big\langle f_j\chi_{\Omega_j^c} + \sum_{Q\in\mathscr{P}_j} \langle f_j\rangle_{r_j,Q}\chi_Q\Big\rangle_{r_j,P}\chi_P.\end{aligned}$$

Taking the product over $j\in\{1,\ldots,m\}$ and the supremum over $P\in\mathscr{F}$ this yields

$$\begin{aligned}\widetilde{M}^{\mathscr{F}}_{\vec r}(\vec f\,) &\le \sup_{P\in\mathscr{F}}\prod_{j=1}^m\Big(\Big\langle f_j\chi_{\Omega_j^c} + \sum_{Q\in\mathscr{P}_j}\langle f_j\rangle_{r_j,Q}\chi_Q\Big\rangle_{r_j,P}\chi_P + \langle f_j\rangle_{r_j,P}\chi_{\Omega_j}\Big)\\ &\le \widetilde{M}^{\mathscr{F}}_{\vec r}(\vec g\,) + b,\end{aligned}$$

where

$$g_j := g_j^1 + g_j^2 := f_j\chi_{\Omega_j^c} + \sum_{Q\in\mathscr{P}_j}\langle f_j\rangle_{r_j,Q}\chi_Q,$$

and $b:\mathbf{R}^n\to X$ is the sum of all terms of the product over $j\in\{1,\ldots,m\}$ other than $\widetilde{M}^{\mathscr{F}}_{\vec r}(\vec g)$. Since $\mathscr{P}_j$ is pairwise disjoint, we have $\|g_j\|_{L^{r_j}(\mathbf{R}^n;X_j)} = \|f_j\|_{L^{r_j}(\mathbf{R}^n;X_j)} = 1$. Moreover since

$$\operatorname{supp} b \subseteq \bigcup_{j=1}^m \Omega_j = \bigcup_{j=1}^m \big\{x\in\mathbf{R}^n : M^{\mathscr{D}}_{r_j}(\|f_j\|_{X_j})(x) > \lambda^{\frac{r}{r_j}}\big\}$$

and since $\|M^{\mathscr{D}}_{r_j}\|_{L^{r_j}(\mathbf{R}^n)\to L^{r_j,\infty}(\mathbf{R}^n)}\le 1$ by Lemma 3.2.5, we have

$$\big|\{x\in\mathbf{R}^n : \|b(x)\|_X > \lambda\}\big| \le \sum_{j=1}^m \big|\{x\in\mathbf{R}^n : M^{\mathscr{D}}_{r_j}\big(\|f_j\|_{X_j}\big)(x) > \lambda^{\frac{r}{r_j}}\}\big| \le \frac{m}{\lambda^r}.$$

To estimate $\vec g$, note that by the Lebesgue differentiation theorem we have

$$\|g_j^1\|_{X_j} = \|f_j\|_{X_j}\chi_{\Omega_j^c} \le M^{\mathscr{D}}_{r_j}\big(\|f_j\|_{X_j}\big)\chi_{\Omega_j^c} \le \lambda^{\frac{r}{r_j}}.$$

and, by pairwise disjointness of $\mathscr{P}_j$, r_j-convexity of X_j, and the maximality of the cubes in $\mathscr{P}_j$, we have

$$\|g_j^2\|_{X_j} = \Big\|\Big(\sum_{Q\in\mathscr{P}_j} \langle f_j\rangle_{r_j,Q}^{r_j}\chi_Q\Big)^{\frac{1}{r_j}}\Big\|_{X_j} \leq 2^{\frac{n}{r_j}}\Big(\sum_{Q\in\mathscr{P}_j} \langle \|f_j\|_{X_j}\rangle_{r_j,\widehat{Q}}^{r_j}\chi_Q\Big)^{\frac{1}{r_j}} \leq 2^{\frac{n}{r_j}}\lambda^{\frac{r}{r_j}},$$

where $\hat{Q}$ is the dyadic parent of $Q\in\mathscr{P}_j$. Thus we have $\|g_j\|_{L^\infty(\mathbf{R}^n;X_j)} \lesssim_{r_j} \lambda^{\frac{r}{r_j}}$.

Combining the estimates for $\vec{g}$ and b we obtain for $\vec{r} < \vec{p} < \infty$

$$\begin{aligned}
\Big|\{x\in\mathbf{R}^n : \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{f})(x)\|_X > 2\lambda\}\Big| &\leq \Big|\{x\in\mathbf{R}^n : \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{g})(x)\|_X > \lambda\}\Big| + \big|\{x\in\mathbf{R}^n : \|b(x)\|_X > \lambda\}\big| \\
&\leq \|\widetilde{M}_{\vec{r}}\|_{\vec{p},X} \frac{\prod_{j=1}^m \|g_j\|_{L^{p_j}(\mathbf{R}^n;X_j)}^p}{\lambda^p} + \frac{m}{\lambda^r} \\
&\lesssim_{\vec{p},\vec{r}} \|\widetilde{M}_{\vec{r}}\|_{\vec{p},X} \frac{\prod_{j=1}^m \|g_j\|_{L^{r_j}(\mathbf{R}^n;X_j)}^{\frac{p_j}{r_j}p} \lambda^{(\frac{1}{r_j}-\frac{1}{p_j})pr}}{\lambda^p} + \frac{1}{\lambda^r} \leq \|\widetilde{M}_{\vec{r}}\|_{\vec{p},X}\frac{2}{\lambda^r},
\end{aligned}$$

and the case where $p_j = \infty$ for some (or all) $1 \leq j \leq m$ is similar. Taking the supremum over $\vec{f} \in L^{\vec{r}}(\mathbf{R}^n;\vec{X})$ of norm 1 and all finite collections of cubes $\mathscr{F} \subseteq \mathscr{D}$ proves the result. □

Proof of Theorem 8.3.3. Let $\vec{f} \in L^{\vec{r}}_{\text{loc}}(\mathbf{R}^n;\vec{X})$ and set

$$A_0 := \sup_{\substack{\mathscr{F}\subseteq\mathscr{D}\\ \mathscr{F}\text{ finite}}} \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}\|_{L^{\vec{r}}(\mathbf{R}^n;\vec{X})\to L^{r,\infty}(\mathbf{R}^n;X)},$$

which is finite by Lemma 8.3.4. For a cube $Q\in\mathscr{F}$, we define its stopping children $\mathrm{ch}_{\mathscr{F}}(Q)$ to be the collection of maximal cubes $Q'\in\mathscr{F}$ such that $Q' \subsetneq Q$ and

$$\Big\|\sup_{\substack{P\in\mathscr{F}\\ Q'\subseteq P\subseteq Q}} \prod_{j=1}^m \langle f_j\rangle_{r_j,P}\Big\|_X > 2^{\frac{1}{r}} A_0 \prod_{j=1}^m \big\langle \|f_j\|_{X_j}\big\rangle_{r_j,Q}. \tag{8.3.1}$$

We let $\mathscr{S}_0$ be the maximal cubes in $\mathscr{D}$, recursively define $\mathscr{S}_{k+1} := \bigcup_{Q\in\mathscr{S}_k} \mathrm{ch}_{\mathscr{D}}(Q)$, and set $\mathscr{S} := \bigcup_{k=0}^\infty \mathscr{S}_k$.

Fix $Q\in\mathscr{S}$, set $E_Q := Q\setminus\bigcup_{Q'\in\mathrm{ch}_{\mathscr{D}}(Q)} Q'$, and define

$$Q^* := \Big\{x\in\mathbf{R}^n : \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{f}\chi_Q)(x)\|_X > 2^{\frac{1}{r}} A_0 \prod_{j=1}^m \big\langle \|f_j\|_{X_j}\big\rangle_{r_j,Q}\Big\}.$$

Then by the definition of A_0 we have

$$|Q^*|^{\frac{1}{r}} \leq \frac{1}{2^{\frac{1}{r}}} \frac{\prod_{j=1}^m \|f_j\chi_Q\|_{L^{r_j}(\mathbf{R}^n;X_j)}}{\prod_{j=1}^m \big\langle \|f_j\|_{X_j}\big\rangle_{r_j,Q}} = \frac{|Q|^{1/r}}{2^{\frac{1}{r}}}. \tag{8.3.2}$$

Moreover, for $Q' \in \operatorname{ch}_{\mathscr{S}}(Q)$ and $x \in Q'$, it follows from (8.3.1) that

$$\big\|\widetilde{M}^{\mathscr{F}}_{\vec{r}}(\vec{f}\chi_Q)(x)\big\|_X \geq \Big\| \sup_{\substack{Q\in\mathscr{F}\\ Q'\subseteq P\subseteq Q}} \prod_{j=1}^m \langle f_j\rangle_{r_j,P}\Big\|_X > 2^{1/r} A_0 \prod_{j=1}^m \big\langle \|f_j\|_{X_j}\big\rangle_{r_j,Q}$$

so $x \in Q^*$ and thus $Q' \subseteq Q^*$. Since $\operatorname{ch}_{\mathscr{F}}(Q)$ is pairwise disjoint, it follows from (8.3.2) that

$$\sum_{Q'\in \operatorname{ch}_{\mathscr{S}}(Q)} |Q'| \leq |Q^*| \leq \frac{1}{2}|Q|$$

so that $|E_Q| \geq \frac{1}{2}|Q|$. We conclude that $\mathscr{S}$ is a sparse collection of cubes.

Next, we check that $\widetilde{M}^{\mathscr{F}}_{\vec{r}}(\vec{f}\,)$ is pointwise dominated by the sparse operator associated to $\mathscr{S}$. For each $P \in \mathscr{F}$ we denote by $\pi_{\mathscr{S}}(P)$ the minimal $Q \in \mathscr{S}$ satisfying $P \subseteq Q$ so that we can partition $\mathscr{F}$ as

$$\mathscr{F} = \bigcup_{Q\in\mathscr{S}} \big\{P \in \mathscr{F} : \pi_{\mathscr{S}}(P) = Q\big\}.$$

Fix $Q \in \mathscr{S}$, $x \in Q$ and let $Q' \in \mathscr{F}$ be the minimal cube such that $x \in Q'$ and $\pi_{\mathscr{S}}(Q') = Q$. If $Q' \subsetneq Q$, we have

$$\begin{aligned}\Big\| \sup_{\substack{P\in\mathscr{F}:\\ \pi_{\mathscr{S}}(P)=Q}} \prod_{j=1}^m \langle f_j\rangle_{r_j,P}\chi_P(x)\Big\|_X &= \Big\| \sup_{\substack{P\in\mathscr{F}:\\ Q'\subseteq P\subseteq Q}} \prod_{j=1}^m \langle f_j\rangle_{r_j,P}\Big\|_X \\ &\leq 2^{\frac{1}{r}} A_0 \prod_{j=1}^m \big\langle \|f_j\|_{X_j}\big\rangle_{r_j,Q}\chi_Q(x).\end{aligned}$$

If $Q' = Q$ the same estimate follows from the r_j-convexity of the X_j. Using the fact that $\|\cdot\|_{\ell^\infty} \leq \|\cdot\|_{\ell^q}$ and the q-convexity of X we can conclude for any $x \in \mathbf{R}^n$

$$\begin{aligned}\big\|\widetilde{M}^{\mathscr{F}}_{\vec{r}}(\vec{f}\,)(x)\big\|_X &= \Big\|\sup_{Q\in\mathscr{S}} \sup_{\substack{P\in\mathscr{F}:\\ \pi_{\mathscr{S}}(P)=Q}} \prod_{j=1}^m \langle f_j\rangle_{r_j,P}\chi_P(x)\Big\|_X \\ &\leq \Big(\sum_{Q\in\mathscr{S}} \Big\| \sup_{\substack{P\in\mathscr{F}:\\ \pi_{\mathscr{S}}(P)=Q}} \prod_{j=1}^m \langle f_j\rangle_{r_j,P}\chi_P(x)\Big\|_X^q\Big)^{\frac{1}{q}} \\ &\leq 2^{\frac{1}{r}} A_0 \Big(\sum_{Q\in\mathscr{S}} \prod_{j=1}^m \big\langle \|f_j\|_{X_j}\big\rangle_{r_j,Q}^q \chi_Q(x)\Big)^{\frac{1}{q}},\end{aligned}$$

as desired.

For the second assertion, note that by the Fatou property of X we have

$$\big\|\widetilde{M}^{\mathscr{D}}_{\vec{r}}(\vec{f}\,)(x)\big\|_X \leq \sup_{\mathscr{F}\subseteq\mathscr{D}:\mathscr{F}\text{ finite}} \big\|\widetilde{M}^{\mathscr{F}}_{\vec{r}}(\vec{f}\,)(x)\big\|_X$$

for $x \in \mathbf{R}^n$. Hence, the form domination result for $\widetilde{M}^{\mathscr{D}}$ follows from an argument analogous to the one in the proof of Proposition 5.3.4, and the result for $\widetilde{M}_{\vec{r}}$ then follows from the three lattice lemma. The final statement follows by setting $g = \chi_{\mathbf{R}^n}$. □

By combining Theorem 8.3.3 with Theorem 8.2.2, we can now directly conclude weighted estimates for $\widetilde{M}_{\vec{r}}$. In particular this proves the $\vec{p}$-independence of the $\vec{r}$-Hardy–Littlewood property.

Corollary 8.3.5. *Let $\vec{X}$ be an m-tuple of quasi-Banach function spaces, take $\vec{r} \in (0,\infty)^m$ and $q \in [r,\infty)$. Suppose $\vec{X} \in \mathrm{HL}_{\vec{r}}$ and assume X is an order-continuous q-convex quasi-Banach function space. Then for $\vec{p} \in (0,\infty]^m$ with $\vec{r} < \vec{p}$ and $p < \infty$ and any $w \in A_{\vec{p},(\vec{r},\infty)}$*

$$\|\widetilde{M}_{\vec{r}}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n;\vec{X}) \to L^p_w(\mathbf{R}^n;X)} \lesssim_{\vec{X},\vec{p},q,\vec{r}} [\vec{w}]_{\vec{p},(\vec{r},\infty)}^{\max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}, \frac{p}{q}\right\}}.$$

We point out that the condition $p < \infty$ here is necessary. Indeed, it is shown in [GMT93, Remark 2.9] that $\widetilde{M}$ is not bounded on $L^\infty(\mathbf{R};\ell^2)$.

8.4. LIMITED RANGE MULTILINEAR UMD CLASSES OF QUASI-BANACH FUNCTION SPACES

A Banach space has the UMD property if the martingale difference sequence of any finite martingale in $L^p(\Omega;X)$ is unconditional for some (equivalently all) $p \in (1,\infty)$, i.e. if for $(f_k)_{k=0}^K$ any finite martingale in $L^p(\Omega;X)$ for some (equivalently all) $p \in (1,\infty)$ and a probability space $(\Omega,\mathbb{P})$ and all scalars $|\epsilon_1| = \cdots = |\epsilon_K| = 1$ we have

$$\Big\|\sum_{k=1}^K \epsilon_k df_k\Big\|_{L^p(\Omega;X)} \lesssim \Big\|\sum_{k=1}^K df_k\Big\|_{L^p(\Omega;X)}, \tag{8.4.1}$$

where $(df_k)_{k=1}^K$ is the difference sequence of $(f_k)_{k=0}^K$. The least admissible constant in (8.4.1) is denoted by $\beta_{p,X}$. The class of UMD Banach function spaces includes for example all reflexive Lebesgue, Lorentz and Musielak-Orlicz spaces. As the UMD property implies reflexivity, L^1 and L^∞ do not have the UMD property. For an introduction to the UMD property we refer the reader to [HNVW16, Pis16].

As already noted in the previous section, for Banach function spaces the UMD property is intimately connected to the Hardy–Littlewood property. As shown by Bourgain [Bou84] and Rubio de Francia [Rub86, Theorem 3], a Banach function space X has the UMD property if and only if both X and X^* have the Hardy–Littlewood property. This connection between the Hardy–Littlewood property and the UMD property is made quantitative in [KLW20], where it is shown that $\|\widetilde{M}\|_{p,X} \lesssim (\beta_{p,X})^2$.

Motivated by this connection between the Hardy–Littlewood property and the UMD property and using the extension of the Hardy–Littlewood property to the rescaled, multilinear setting from Section 8.3, we will now define a limited range, multilinear version of the UMD property for m-tuples of quasi-Banach function spaces.

Definition 8.4.1. Let $\vec{X}$ be an m-tuple of quasi-Banach function spaces, take $\vec{r} \in (0,\infty)^m$ and $s \in (r,\infty]$. We say that $\vec{X}$ has the $(\vec{r},s)$-UMD property and write $\vec{X} \in \mathrm{UMD}_{\vec{r},s}$ if $X = \prod_{j=1}^m X_j$ is an order-continuous Banach function space and $(\vec{X}, X^*) \in \mathrm{HL}_{(\vec{r},s')}$.

Note that while the UMD property is well-defined in terms of martingale difference sequences for any Banach space, our limited range multilinear version is only given for quasi-Banach *function* spaces and has no immediate connection to martingales. It would be interesting to have an equivalent characterization of either the limited range or the multilinear generalization (for example in terms of martingale difference sequences) that does not use the lattice structure of $\vec{X}$.

As a first result on the limited range multilinear UMD property we will show that our nomenclature makes sense, i.e. that the $\mathrm{UMD}_{\vec{r},s}$ property is actually related to the UMD property for Banach function spaces. If X is a Banach function space, then X has the UMD property if and only if $X \in \mathrm{UMD}_{1,\infty}$. This follows directly from the result of Bourgain and Rubio de Francia and the case $m = r = s' = 1$, of the following proposition.

Proposition 8.4.2. *Let $\vec{X}$ be an m-tuple of quasi-Banach function spaces and let $\vec{r} \in [1,\infty)^m$ and $s \in (1,\infty]$. The following are equivalent:*

(i) $\vec{X} \in \mathrm{UMD}_{\vec{r},s}$*;*

(ii) $\vec{X} \in \mathrm{HL}_{\vec{r}}$ *and* $(X_1,\ldots,X_{j-1},X_{j+1},\ldots,X_m,X^*) \in \mathrm{HL}_{(r_1,\ldots,r_{j-1},r_{j+1},\ldots r_m,s')}$ *for all* $j \in \{1,\ldots,m\}$.

Proof. For (i)⇒(ii) we only prove $\vec{X} \in \mathrm{HL}_{\vec{r}}$. The other results with $j \in \{1,\ldots,m\}$ follow from an analogous argument by interchanging the roles of X^* and X_j and the roles of s' and r_j.

Let (Ω,μ) denote the underlying measure space over which the $\vec{X}$ are defined and fix $\vec{p} \in (0,\infty]^m$ with $\vec{r} < \vec{p}$, $1 \le p < s$ and a finite collection of cubes $\mathscr{F}$. By the pointwise sparse domination result for $M^{\mathscr{F}}_{\vec{r}}$, it follows from Proposition 5.3.4 that $\|M^{\mathscr{F}}_{\vec{r}}(\vec{f})g\|_{L^1(\mathbf{R}^n)} \lesssim_{\vec{r}} \|M^{\mathscr{F}}_{(\vec{r},1)}(\vec{f},g)\|_{L^1(\mathbf{R}^n)}$ for $\vec{f} \in L^{\vec{r}}_{\mathrm{loc}}(\mathbf{R}^n)$, $g \in L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Since $\widetilde{M}^{\mathscr{F}}_{\vec{r}}(\vec{f})(x,\omega) = M^{\mathscr{F}}_{\vec{r}}(\vec{f}(\cdot,\omega))(x)$, combining this with Fubini's theorem we obtain for $\vec{f} \in L^{\vec{p}}_c(\mathbf{R}^n;\vec{X})$ and $g \in L^{p'}(\mathbf{R}^n;X^*)$

$$
\begin{aligned}
\left|\int_{\mathbf{R}^n}\int_\Omega \widetilde{M}^{\mathscr{F}}_{\vec{r}}(\vec{f})g\,\mathrm{d}\mu\,\mathrm{d}x\right| &\le \int_\Omega \|M^{\mathscr{F}}_{\vec{r}}(\vec{f}(\cdot,\omega))g(\cdot,\omega)\|_{L^1(\mathbf{R}^n)}\,\mathrm{d}\mu(\omega)\\
&\lesssim_{\vec{r}} \int_\Omega \left\|M^{\mathscr{F}}_{(\vec{r},1)}(\vec{f}(\cdot,\omega),g(\cdot,\omega))\right\|_{L^1(\mathbf{R}^n)}\,\mathrm{d}\mu(\omega)\\
&= \left\|\widetilde{M}^{\mathscr{F}}_{(\vec{r},1)}(\vec{f},g)\right\|_{L^1(\mathbf{R}^n;L^1(\Omega))} \le \left\|\widetilde{M}^{\mathscr{F}}_{(\vec{r},s')}(\vec{f},g)\right\|_{L^1(\mathbf{R}^n;L^1(\Omega))}\\
&\le \left\|\widetilde{M}_{(\vec{r},s')}\right\|_{(\vec{p},p'),(\vec{X},X^*)}\Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbf{R}^n;X_j)}\Big)\|g\|_{L^{p'}(\mathbf{R}^n;X^*)},
\end{aligned}
$$

where in the second to last step we used Hölder's inequality with $s' \ge 1$ and (i) and Corollary 8.3.5 in the last. Taking a supremum over all $g \in L^{p'}(\mathbf{R}^n;X^*)$ with $\|g\|_{L^{p'}(\mathbf{R}^n;X^*)} = 1$ proves that $\vec{X} \in \mathrm{HL}_{\vec{r}}$, as asserted.

The proof of (ii)⇒(i) relies on some combinatorics. To facilitate this, we set $r_{m+1} := s'$ and $X_{m+1} := X^*$. Fix $\vec{p} \in (0,\infty]^{m+1}$ with $\min\vec{p} > \max\vec{r}$, $\vec{f} \in L^{\vec{p}}(\mathbf{R}^n;\vec{X})$, and a finite collection of cubes $\mathscr{F}$. Note that

$$\prod_{j=1}^{m+1} \langle f_j \rangle_{r_j,Q} \chi_Q = \prod_{j=1}^{m+1} \Big(\prod_{\substack{k=1\\k\neq j}}^{m+1} \langle f_k \rangle_{r_k,Q} \chi_Q \Big)^{\frac{1}{m}}$$

for all $Q \in \mathscr{F}$ so that

$$\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{f}\,) \leq \prod_{j=1}^{m+1} \widetilde{M}_{\vec{q}}^{\mathscr{F}}(\vec{g})^{\frac{1}{m}}$$

with

$$\begin{aligned}\vec{q} &= (r_1, \dots, r_{j-1}, r_{j+1}, \dots r_{m+1})\\ \vec{g} &= (f_1, \dots, f_{j-1}, f_{j+1}, \dots, f_{m+1})\end{aligned}$$

Furthermore setting $\vec{Y}_j = (X_1, \cdots, X_{j-1}, X_{j+1}, \cdots, X_{m+1})$, we have

$$\prod_{j=1}^{m+1} Y_j^{\frac{1}{m}} = \prod_{j=1}^{m+1} \prod_{\substack{k=1\\k\neq j}}^{m+1} X_k^{\frac{1}{m}} = \prod_{j=1}^{m+1} X_j = L^1(\Omega).$$

Thus setting $A_j := \|\widetilde{M}_{\vec{q}}\|_{(p_j,\dots,p_j),\vec{Y}_j}$, which is finite by Corollary 8.3.5, we have

$$\begin{aligned}\big\|M_{\vec{r}}^{\mathscr{F}}(\vec{f}\,)\big\|_{L^p(\mathbf{R}^n;L^1(\Omega))} &\leq \prod_{j=1}^{m+1} \Big\|\widetilde{M}_{\vec{q}}^{\mathscr{F}}(\vec{g})^{\frac{1}{m}}\Big\|_{L^{p_j}(\mathbf{R}^n;Y_j^{\frac{1}{m}})} = \prod_{j=1}^{m+1} \big\|\widetilde{M}_{\vec{q}}^{\mathscr{F}}(\vec{g})\big\|^{\frac{1}{m}}_{L^{\frac{p_j}{m}}(\mathbf{R}^n;Y_j)}\\ &\leq \prod_{j=1}^{m+1} A_j^{\frac{1}{m}} \prod_{\substack{k=1\\k\neq j}}^{m+1} \|f_k\|^{\frac{1}{m}}_{L^{p_j}(\mathbf{R}^n;X_k)} = \prod_{j=1}^{m+1} A_j^{\frac{1}{m}} \|f_k\|_{L^{p_j}(\mathbf{R}^n;X_k)},\end{aligned}$$

proving (i). The assertion follows. □

In particular, in the case $m = 1$ this result says that $X \in \mathrm{UMD}_{r,s}$ if and only if $X^r \in \mathrm{HL}$ and $(X^*)^{s'} \in \mathrm{HL}$.

We have the following result on the product space X:

Proposition 8.4.3. *Let $\vec{X}$ be an m-tuple of quasi-Banach function spaces and let $\vec{r} \in [1,\infty)^m$ and $s \in (1,\infty]$. If $\vec{X} \in \mathrm{UMD}_{\vec{r},s}$ then $X \in \mathrm{UMD}_{r,s}$.*

Proof. Fix $\vec{p} \in (0,\infty)^m$ with $\vec{p} > \vec{r}$, $1 < p < s$ and $f \in L^p(\mathbf{R}^n;X)$. Since X is assumed to be order-continuous, By Lemma 8.1.5 we have $L^p(\mathbf{R}^n;X) = \prod_{j=1}^m L^{p_j}(\mathbf{R}^n;X_j)$. Hence, we can find positive $f_j \in L^{p_j}(\mathbf{R}^n;X_j)$ such that $|f| \leq \prod_{j=1}^m f_j$. Let (Ω,μ) denote the underlying σ-finite measure space of $\vec{X}$ and fix a fixed finite collection of cubes $\mathscr{F}$. By Hölder's inequality we have $\langle f \rangle_{r,Q} \leq \prod_{j=1}^m \langle f_j \rangle_{r_j,Q}$ for all $Q \in \mathscr{F}$. This implies that for $g \in L^{p'}(\mathbf{R}^n;X^*)$ we have

$$\begin{aligned}\|\widetilde{M}_{(r,s')}^{\mathscr{F}}(f,g)\|_{L^1(\mathbf{R}^n;L^1(\Omega))} &\leq \|\widetilde{M}_{(\vec{r},s')}^{\mathscr{F}}(\vec{f},g)\|_{L^1(\mathbf{R}^n;L^1(\Omega))}\\ &\leq \|\widetilde{M}\|_{(\vec{p},p'),(\vec{X},X^*)} \Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}(\mathbf{R}^n;X_j)}\Big)\|g\|_{L^{p'}(\mathbf{R}^n;X^*)}.\end{aligned}$$

Taking an infimum over all positive $f_j \in L^{p_j}(\mathbf{R}^n; X_j)$ such that $|f| \leq \prod_{j=1}^m f_j$ and a supremum over all finite collections $\mathscr{F}$ we conclude that $X \in \mathrm{UMD}_{r,s}$ with $\|\widetilde{M}_{(r,s')}\|_{(p,p'),(X,X^*)} \leq \|\widetilde{M}_{(\vec{r},s')}\|_{(\vec{p},p'),(\vec{X},X^*)}$. This proves the assertion. □

Example 8.4.4. Let (Ω,μ) be a σ-finite measure space. In the case $m=1$, it follows from Proposition 8.4.2 that $X \in \mathrm{UMD}_{r,s}$ for $1 \leq r < s \leq \infty$ if and only if $X^r \in \mathrm{HL}$ and $(X^*)^{s'} \in \mathrm{HL}$. This implies the following:

(i) If $X = L^p(\Omega)$ with $p \in (r,s)$, then $X \in \mathrm{UMD}_{r,s}$.

(ii) If $X = L^{p,q}(\Omega)$ with $p,q \in (r,s)$, then $X \in \mathrm{UMD}_{r,s}$.

(iii) If $X = L^{\Phi}(\Omega)$ is a Musielak-Orlicz space such that $(\omega,t) \mapsto \Phi(\omega, t^{\frac{1}{r}})$ and $(\omega,t) \mapsto \Phi^*(\omega, t^{\frac{1}{s'}})$ are Young functions satisfying the Δ_2 condition, then $X \in \mathrm{UMD}_{r,s}$. See [FG91, LVY19] for the UMD (and thus the HL) property of these spaces.

In [LN19] vector-valued extensions of multilinear operators in quasi-Banach function spaces were constructed through weighted techniques. In that work the condition that $((X_j^{r_j})^*)^{(s_j/r_j)'}$ has the UMD property for $1 \leq j \leq m$ was imposed. In the next proposition we wish to compare this assumption to our limited range multilinear UMD property.

Proposition 8.4.5. *Let $\vec{X}$ be an m-tuple of quasi-Banach function spaces, let $\vec{r} \in (0,\infty)^m$ and take $\vec{r} < \vec{s} \leq \infty$. Suppose that X_j is r_j-convex, s_j-concave and $\big((X_j^{r_j})^*\big)^{(s_j/r_j)'}$ has the* UMD *property for $1 \leq j \leq m$. Then for all $q \in (0,r]$ we have $\vec{X}^q \in \mathrm{UMD}_{\frac{\vec{r}}{q},\frac{s}{q}}$. In particular, $\vec{X} \in \mathrm{UMD}_{\vec{r},s}$ if $r \geq 1$.*

Proof. Note that $\vec{X}^q \in \mathrm{UMD}_{\frac{\vec{r}}{q},\frac{s}{q}}$ per definition means that

$$(X_1^q,\ldots,X_m^q,(X^q)^*) \in \mathrm{HL}_{\left(\frac{\vec{r}}{q},\left(\frac{s}{q}\right)'\right)}.$$

So by Proposition 8.3.2 it suffices to show $(X_j^q)^{\frac{r_j}{q}} = X_j^{r_j} \in \mathrm{HL}$ for $j = 1,\cdots,m$ and $((X^q)^*)^{(s/q)'} \in \mathrm{HL}$. Since $(s_j/r_j)' \geq 1$, we know that $(X_j^{r_j})^*$ has the UMD property (see [Rub86, Theorem III.4]) and thus $X_j^{r_j} \in \mathrm{HL}$ for $j = 1,\cdots,m$. To show $((X^q)^*)^{(s/q)'} \in \mathrm{HL}$ we note that by [LN19, Proposition 3.4] we have $((X^r)^*)^{(s/r)'} \in \mathrm{UMD}$. Then, by [LN19, Proposition 3.3(iii)] this implies that also $((X^q)^*)^{(s/q)'} \in \mathrm{UMD}$ for all $q \in (0,r]$. In particular, we have $((X^q)^*)^{(s/q)'} \in \mathrm{HL}$, as desired. The assertion follows. □

To end this section we will give some examples of tuples in the $\mathrm{UMD}_{\vec{r},s}$-class and provide some methods to generate new tuples from old ones. We start with a family of examples in the form of Lebesgue spaces.

Proposition 8.4.6. *Let $\vec{r} \in (0,\infty)^m$, $s \in (1,\infty]$ and $\vec{t} \in (0,\infty]^m$ with $\vec{t} > \vec{r}$ and $1 \leq t < s$. Let (Ω,μ) be a σ-finite measure space. Then $L^{\vec{t}}(\Omega) \in \mathrm{UMD}_{\vec{r},s}$.*

Proof. Write $X_j = L^{t_j}(\Omega)$ so that $X = \prod_{j=1}^m X_j = L^t(\Omega)$ by Proposition 8.1.3(ii). Note that since $\frac{t_1}{r_1}, \dots, \frac{t_m}{r_m}, \frac{t'}{s'} \in (1,\infty]$, we have $X_j^{r_j} = L^{\frac{t_j}{r_j}}(\Omega) \in \mathrm{HL}$ for all $j \in \{1,\dots,m\}$ and $(X^*)^{s'} = L^{\frac{t'}{s'}}(\Omega) \in \mathrm{HL}$. Thus, it follows from Proposition 8.3.2 that $\vec{X} \in \mathrm{UMD}_{\vec{r},s}$. The assertion follows. □

To extend this example to iterated L^p-spaces, we will show that the $\mathrm{UMD}_{\vec{r},s}$ class is stable under iteration. For quasi-Banach function spaces X and Y respectively over measure spaces (Ω_1,μ_1) and (Ω_2,μ_2) the mixed-norm space $X(Y)$ is given by all measurable functions $f\colon \Omega_1 \times \Omega_2 \to \mathbf{C}$ such that

$$\big\| \omega_1 \mapsto \|f(\omega_1,\cdot)\|_Y \big\|_X < \infty.$$

Proposition 8.4.7. *Let $\vec{r} \in (0,\infty)^m$ and $s \in (1,\infty]$ and let $\vec{X}$ and $\vec{Y}$ be m-tuples of quasi-Banach function spaces. If $\vec{X}, \vec{Y} \in \mathrm{UMD}_{\vec{r},s}$, then $\vec{X}(\vec{Y}) \in \mathrm{UMD}_{\vec{r},s}$.*

Proof. Denote by (Ω_1,μ_1), (Ω_2,μ_2) the σ-finite measure spaces that $\vec{X}$, $\vec{Y}$ are respectively defined over and write

$$A_1 := \sup_{\mathscr{F} \text{ finite}} \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}\|_{L^{\vec{r}}(\mathbf{R}^n;\vec{X}) \to L^{r,\infty}(\mathbf{R}^n;X)}, \quad A_2 := \sup_{\mathscr{F} \text{ finite}} \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}\|_{L^{\vec{r}}(\mathbf{R}^n;\vec{Y}) \to L^{r,\infty}(\mathbf{R}^n;Y)}.$$

Let $\mathscr{F}$ denote a finite collection of cubes and let $\vec{f} \in L^{\vec{r}}(\mathbf{R}^n; \vec{X}(\vec{Y}))$. By Fubini's Theorem and by applying Theorem 8.3.3 twice we obtain

$$\begin{aligned}
\|\widetilde{M}_{\vec{r},s}^{\mathscr{F}}(\vec{f},g)\|_{L^1(\mathbf{R}^n;L^1(\Omega_1\times\Omega_2))} &= \int_{\Omega_1} \|\widetilde{M}_{\vec{r},s}^{\mathscr{F}}(\vec{f}(\cdot,\omega_1,\cdot),g(\cdot,\omega_1,\cdot))\|_{L^1(\mathbf{R}^n;L^1(\Omega_1))} \,\mathrm{d}\mu_1(\omega_1) \\
&\lesssim A_2 \int_{\Omega_1} \big\|\widetilde{M}_{\vec{r},s}^{\mathscr{F}}(\|\vec{f}(\cdot,\omega_1,\cdot)\|_{\vec{Y}}, \|g(\cdot,\omega_1,\cdot)\|_{Y^*})\big\|_{L^1(\mathbf{R}^n)} \,\mathrm{d}\mu_1(\omega_1) \\
&= A_1 \big\|\widetilde{M}_{\vec{r},s}^{\mathscr{F}}(\|\vec{f}\|_{\vec{Y}}, \|g\|_{Y^*})\big\|_{L^1(\mathbf{R}^n;L^1(\Omega_1))} \\
&\lesssim A_1 A_2 \big\|M_{\vec{r},s}^{\mathscr{F}}(\|\vec{f}\|_{\vec{X}(\vec{Y})}, \|g\|_{X^*(Y^*)})\big\|_{L^1(\mathbf{R}^n)}.
\end{aligned}$$

Thus, by Theorem 3.2.3 we conclude that $\vec{X}(\vec{Y}) \in \mathrm{UMD}_{\vec{r},s}$, as desired. □

Applying Proposition 8.4.7 to the result in Proposition 8.4.6 we obtain the announced result for iterated L^p-spaces.

Corollary 8.4.8. *Let $\vec{r} \in (0,\infty)^m$ and $s \in (1,\infty]$. Let $K \in \mathbf{N}$ and let $\vec{t}^{\,1}, \dots, \vec{t}^{\,K} \in (0,\infty]^m$ with $\vec{t}^{\,k} > \vec{r}$ and $1 \le t^k < s$ for all $k \in \{1,\cdots,K\}$. Let (Ω_k,μ_k) for $k \in \{1,\cdots,K\}$ be σ-finite measure spaces and for $j \in \{1,\cdots,m\}$ we set*

$$X_j := L^{t_j^1}(\Omega_1;\cdots;L^{t_j^K}(\Omega_K)).$$

Then $\vec{X} \in \mathrm{UMD}_{\vec{r},s}$.

A point of interest in the above result is that it shows that we can go beyond assuming that each individual X_j has the UMD property. We can even consider spaces such as $\ell^\infty(\ell^2)$, which by [NVW15, Proposition 8.l] does not even satisfy the Hardy-Littlewood property.

Remark 8.4.9. By mimicking the proof of Proposition 8.4.6 we can also obtain a version of Corollary 8.4.8 for Lorentz and Orlicz spaces. We point out however that it is not clear if we can consider the appropriate endpoint cases outside of the range of UMD spaces. More precisely, in the case of Lorentz spaces it is unknown whether $L^{p,\infty}(\Omega)$ for $p \in (1,\infty)$ satisfies the Hardy-Littlewood property. Similarly it is unknown whether there are Orlicz spaces that are not UMD, but satisfy the Hardy-Littlewood property. If there are such spaces, we obtain more examples beyond the setting of individual UMD conditions that fall within our range.

In the next result we show that we can add L^∞ spaces to existing UMD tuples to create new ones.

Proposition 8.4.10. *Let $\vec{r} \in (0,\infty)^m$ and $s \in (1,\infty]$. Let $\vec{X}$ be an $m-1$-tuple of quasi-Banach function spaces over a measure space Ω. If $\vec{X} \in \mathrm{UMD}_{(r_1,\dots,r_{m-1}),s}$, then*

$$(X_1,\dots,X_{m-1},L^\infty(\Omega)) \in \mathrm{UMD}_{\vec{r},s}.$$

Proof. We first note that by Proposition 8.1.3(i) we have $\big(\prod_{j=1}^{m-1} X_j\big)\cdot L^\infty(\Omega) = X\cdot L^\infty(\Omega) = X$. Next, let $\mathscr{F}$ denote a finite collection of cubes and fix $\vec{p} \in (1,\infty]^m$ with $p_m = \infty$ and $\vec{p} > \vec{r}$, $p < s$. For $\vec{f} \in L^{\vec{p}}(\mathbf{R}^n;\vec{X})$, $g \in L^{p'}(\mathbf{R}^n;X^*)$ we have

$$\widetilde{M}^{\mathscr{F}}_{(\vec{r},s')}(\vec{f},g) \le \widetilde{M}^{\mathscr{F}}_{(r_1,\dots,r_{m-1},s')}(f_1,\dots,f_{m-1},g)\widetilde{M}^{\mathscr{F}}_{r_m}(f_m).$$

Hence,

$$\begin{aligned}
&\|\widetilde{M}^{\mathscr{F}}_{(\vec{r},s')}(\vec{f},g)\|_{L^1(\mathbf{R}^n;L^1(\Omega))}\\
&\le \|\widetilde{M}^{\mathscr{F}}_{(r_1,\dots,r_{m-1},s')}(f_1,\dots,f_{m-1},g)\|_{L^1(\mathbf{R}^n;L^1(\Omega))}\|\widetilde{M}^{\mathscr{F}}_{r_m}(f_m)\|_{L^\infty(\mathbf{R}^n;L^\infty(\Omega))}\\
&\le \|\widetilde{M}_{(r_1,\dots,r_{m-1},s')}\|_{(p_1,\dots,p_{m-1},p'),\vec{X}}\Big(\prod_{j=1}^{m-1}\|f_j\|_{L^{p_j}(\mathbf{R}^n;X_j)}\Big)\|f_m\|_{L^\infty(\mathbf{R}^n;L^\infty(\Omega))}\|g\|_{L^{p'}(\mathbf{R}^n;X^*)},
\end{aligned}$$

proving that $(X_1,\dots,X_{m-1},L^\infty(\Omega)) \in \mathrm{UMD}_{\vec{r},s}$. The assertion follows. □

Note in particular that in the case $m = 2$, this result implies that if X has the UMD property, then $(X,L^\infty(\Omega)) \in \mathrm{UMD}_{(1,1),\infty}$.

Finally, we prove that the $\mathrm{HL}_{\vec{r}}$ and $\mathrm{UMD}_{\vec{r},s}$ properties are stable under taking Calderón-Lozanovskii products.

Proposition 8.4.11. *Let $\vec{X},\vec{Y}$ be m-tuples of quasi-Banach function defined over the same σ-finite measure space and let $\vec{r} \in (0,\infty)^m$, $s \in (1,\infty]$, and $\theta \in (0,1)$.*

(i) If $\vec{X},\vec{Y} \in HL_{\vec{r}}$ and for all $j \in \{1,\dots,m\}$ either $X_j^{1-\theta}\cdot Y_j^\theta$ is order-continuous, or X_j or Y_j is equal to $L^\infty(\Omega)$, then $\vec{X}^{1-\theta}\cdot\vec{Y}^\theta \in \mathrm{HL}_{\vec{r}}$.

(ii) If $\vec{X}, \vec{Y} \in \mathrm{UMD}_{\vec{r},s}$ *with* X_j, Y_j *as in (i), then* $\vec{X}^{1-\theta} \cdot \vec{Y}^{\theta} \in \mathrm{UMD}_{\vec{r},s}$.

Proof. We first prove (i). Let $\mathscr{F}$ be a finite collection of cubes and let $\vec{p} \in (0,\infty]^m$ with $\vec{p} > \vec{r}$, where $p_j = \infty$ whenever X_j or Y_j is equal to $L^\infty(\Omega)$, and $p_j < \infty$ otherwise. By Lemma 8.1.5 we have

$$L^{p_j}(\mathbf{R}^n; X_j^{1-\theta} \cdot Y_j^{\theta}) = L^{\frac{p_j}{1-\theta}}(\mathbf{R}^n; X_j^{1-\theta}) \cdot L^{\frac{p_j}{\theta}}(\mathbf{R}^n; Y_j^{\theta}) = L^{p_j}(\mathbf{R}^n; X_j)^{1-\theta} \cdot L^{p_j}(\mathbf{R}^n; Y_j)^{\theta}$$

when $X_j^{1-\theta} \cdot Y_j^{\theta}$ is order-continuous. We prove that this also holds when X_j or Y_j is equal to $L^\infty(\Omega)$. Assume that $X_j = L^\infty(\Omega)$, the case $Y_j = L^\infty(\Omega)$ being analogous. Since $p_j = \infty$, it follows from Proposition 8.1.3(i) that

$$\begin{aligned} L^{p_j}(\mathbf{R}^n; X_j^{1-\theta} \cdot Y_j^{\theta}) &= L^\infty(\mathbf{R}^n; Y_j^{\theta}) = L^\infty(\mathbf{R}^n; Y_j)^{\theta} = L^\infty(\mathbf{R}^n \times \Omega)^{1-\theta} \cdot L^\infty(\mathbf{R}^n; Y_j)^{\theta} \\ &= L^{p_j}(\mathbf{R}^n; X_j)^{1-\theta} \cdot L^{p_j}(\mathbf{R}^n; Y_j)^{\theta}, \end{aligned}$$

as desired.

Now, let $\vec{f} \in L^{\vec{p}}(\mathbf{R}^n; \vec{X})$. Then we can pick positive $g_j \in L^{p_j}(\mathbf{R}^n; X_j)$, $h_j \in L^{p_j}(\mathbf{R}^n; Y_j)$ so that $|f| \le g_j^{1-\theta} h_j^{\theta}$ for all $j \in \{1, \dots, m\}$. Then $\langle f_j \rangle_{r_j,Q} \le \langle g_j \rangle_{r_j,Q}^{1-\theta} \langle h_j \rangle_{r_j,Q}^{\theta}$ for all $Q \in \mathscr{F}$ by Hölder's inequality so that

$$\|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{f})\|_{X^{1-\theta} \cdot Y^{\theta}} \le \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{g})^{1-\theta} \cdot \widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{h})^{\theta}\|_{X^{1-\theta} \cdot Y^{\theta}} \le \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{g})\|_X^{1-\theta} \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{h})\|_Y^{\theta}$$

a.e. in $\mathbf{R}^n$. Hence, by Hölder's inequality,

$$\begin{aligned} \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{f})\|_{L^p(\mathbf{R}^n; X^{1-\theta} \cdot Y^{\theta})} &\le \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{g})\|_{L^p(\mathbf{R}^n; X)}^{1-\theta} \|\widetilde{M}_{\vec{r}}^{\mathscr{F}}(\vec{h})\|_{L^p(\mathbf{R}^n; Y)}^{\theta} \\ &\le \|\widetilde{M}\|_{\vec{p},\vec{X}}^{1-\theta} \|\widetilde{M}\|_{\vec{p},\vec{Y}}^{\theta} \prod_{j=1}^m \|g_j\|_{L^{p_j}(\mathbf{R}^n; X_j)}^{1-\theta} \|h_j\|_{L^{p_j}(\mathbf{R}^n; Y_j)}^{\theta}. \end{aligned}$$

Taking an infimum over all positive $g_j \in L^{p_j}(\mathbf{R}^n; X_j)$, $h_j \in L^{p_j}(\mathbf{R}^n; Y_j)$ with $|f_j| \le g_j^{1-\theta} h_j^{\theta}$ and a supremum over all finite collections of cubes $\mathscr{F}$ proves that $\vec{X}^{1-\theta} \cdot \vec{Y}^{\theta} \in \mathrm{HL}_{\vec{r}}$ with $\|\widetilde{M}\|_{\vec{p},\vec{X}^{1-\theta} \cdot \vec{Y}^{\theta}} \le \|\widetilde{M}\|_{\vec{p},\vec{X}}^{1-\theta} \|\widetilde{M}\|_{\vec{p},\vec{Y}}^{\theta}$. This proves (i).

For (ii), note that by Proposition 8.4.3 we have that X, Y are r-convex and X^*, Y^* are s'-convex. This implies that X, Y are reflexive and hence, X^* and Y^* are reflexive. By Proposition 8.1.4(iii),(i) we have $(X^{1-\theta} \cdot Y^{\theta})^* = (X^*)^{1-\theta} \cdot (Y^*)^{\theta}$ and this space is reflexive and hence order-continuous. Thus, since $(\vec{X}, X^*), (\vec{Y}, Y^*) \in \mathrm{HL}_{(\vec{r},s')}$, it follows from part (i) that $(\vec{X}^{1-\theta} \vec{Y}^{\theta}, (X^{1-\theta} Y^{\theta})^*) \in \mathrm{HL}_{(\vec{r},s')}$. Hence, $\vec{X}^{1-\theta} \cdot \vec{Y}^{\theta} \in \mathrm{UMD}_{\vec{r},s}$, as asserted. □

9

VECTOR-VALUED EXTENSIONS OF MULTILINEAR OPERATORS

In the first section of this chapter we prove a slightly more general version of the main result in

> E. Lorist and B. Nieraeth. Vector-valued extensions of operators through multilinear limited range extrapolation. *Journal of Fourier Analysis and Applications*, 25(5):2608–2634, 2019.

Our result here is more general in the sense that we are considering a more general condition on the spaces.

The remaining sections of this chapter are based on the main result and the applications from the paper

> E. Lorist and B. Nieraeth. Sparse domination implies vector-valued sparse domination. arXiv:2003.02233, 2020.

9.1. VECTOR-VALUED EXTRAPOLATION

The goal of this section is to prove the following theorem:

Theorem 9.1.1 (Multilinear limited range vector-valued extrapolation). *Let $\vec{r} \in (0,\infty)^m$, $\vec{s} \in (0,\infty]^m$ with $\vec{r} < \vec{s}$. Suppose T is an m-(sub)linear operator such that for all $\vec{p} \in (0,\infty)^m$ satisfying $\vec{r} < \vec{p} < \vec{s}$ there exists a function $\phi_{\vec{p}} : [1,\infty)^m \to [0,\infty)$, increasing in each variable, such that T is bounded $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n) \to L^p_w(\mathbf{R}^n)$ with*

$$\|T\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)\to L^p_w(\mathbf{R}^n)} \le \phi_{\vec{p}}([w_1]_{p_1,(r_1,s_1)},\dots,[w_m]_{p_m,(r_m,s_m)}) \tag{9.1.1}$$

for all weights $\vec{w}$ satisfying $w_j \in A_{p_j,(r_j,s_j)}$ for all $j \in \{1,\dots,m\}$.

Let $\vec{X}$ be an m-tuple of quasi-Banach function spaces over a σ-finite measure space (Ω,μ) and assume that for all simple functions $\vec{f} \in L^\infty_c(\mathbf{R}^n;\vec{X})$ the function

$$\widetilde{T}(\vec{f}\,)(x,\omega) := T(\vec{f}(\cdot,\omega))(x) \tag{9.1.2}$$

is strongly measurable.

If $X_j^{r_j} \in \mathrm{UMD}_{1,\frac{s_j}{r_j}}$ *for all* $j \in \{1,\dots,m\}$, *then for all* $\vec{r} < \vec{p} < \vec{s}$ *there is a function* $\phi_{\vec{X},\vec{p},\vec{r},\vec{s}}: [1,\infty)^m \to [0,\infty)$, *increasing in each variable, such that* $\widetilde{T}$ *is bounded* $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n;\vec{X}) \to L^p_w(\mathbf{R}^n;X)$ *with*

$$\|\widetilde{T}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n;\vec{X})\to L^p_w(\mathbf{R}^n;X)} \le \phi_{\vec{X},\vec{p},\vec{r},\vec{s}}([w_1]_{p_1,(r_1,s_1)},\dots,[w_m]_{p_m,(r_m,s_m)})$$

for all weights $\vec{w}$ *satisfying* $w_j \in A_{p_j,(r_j,s_j)}$ *for all* $j \in \{1,\dots,m\}$.

Remark 9.1.2. In the same way as is explained in Remark 5.1.4, the strong measurability assumption on $\widetilde{T}$ is redundant for m-linear T, since $\widetilde{T}$ then coincides with the tensor extension.

Remark 9.1.3. This theorem can be equivalently formulated if we only assume that (9.1.1) holds for some $\vec{q} \in (0,\infty)^m$ satisfying $\vec{r} < \vec{q} < \vec{s}$ rather than all $\vec{r} < \vec{p} < \vec{s}$. This is a consequence of the limited range multilinear extrapolation theorem of Cruz-Uribe and Martell [CM18]. However, this can also be seen by using the extrapolation theorem, Theorem 4.1.1, in the case $m = 1$ to each of the m component functions. This actually yields an improved version of the result by Cruz-Uribe and Martell in the sense that we obtain a sharp dependence on the weight constants. We sketch the proof here.

Given $\vec{r} < \vec{p} < \vec{s}$ and weights $\vec{w}$ satisfying $w_j \in A_{p_j,(r_j,s_j)}$ for all $j \in \{1,\dots,m\}$ as in the theorem and $\vec{f} \in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n)$, $g \in L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)$, we set $g_j := w_j|gw^{-1}|^{\frac{\frac{1}{r_j}-\frac{1}{p_j}}{\frac{1}{r}-\frac{1}{p}}} \in L^{\frac{1}{\frac{1}{r_j}-\frac{1}{p_j}}}_{w_j^{-1}}(\mathbf{R}^n)$. By applying Theorem 4.1.1 with $m = 1$ to the pairs f_j, g_j, we find weights $\vec{W}$ with

$$[W_j]_{q_j,(r_j,s_j)} \lesssim_{p_j,q_j,r_j,s_j} [w_j]_{p_j,(r_j,s_j)}^{\max\left\{\frac{\frac{1}{r_j}-\frac{1}{q_j}}{\frac{1}{r_j}-\frac{1}{p_j}},\frac{\frac{1}{q_j}-\frac{1}{s_j}}{\frac{1}{p_j}-\frac{1}{s_j}}\right\}}$$

and

$$\begin{aligned}\Big(\prod_{j=1}^m \|f_j\|_{L^{q_j}_{W_j}(\mathbf{R}^n)}\Big)\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{q}}}_{W^{-1}}(\mathbf{R}^n)} &\le \prod_{j=1}^m \|f_j\|_{L^{q_j}_{W_j}(\mathbf{R}^n)}\|g_j\|_{L^{\frac{1}{\frac{1}{r_j}-\frac{1}{q_j}}}_{W_j^{-1}}(\mathbf{R}^n)} \\ &\le 2^{\frac{1}{r}}\Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n)}\Big)\|g\|_{L^{\frac{1}{\frac{1}{r}-\frac{1}{p}}}_{w^{-1}}(\mathbf{R}^n)}.\end{aligned}$$

Analogously to what we did in the proof of Theorem 5.1.2 we can then show that if (9.1.1) holds for some $\vec{r} < \vec{q} < \vec{s}$, then it holds for all $\vec{r} < \vec{p} < \vec{s}$, with a quantitative control of the weighted bound.

Remark 9.1.4. The condition $X_j^{r_j} \in \mathrm{UMD}_{1,\frac{s_j}{r_j}}$ rather than $X_j \in \mathrm{UMD}_{r_j,s_j}$ might seem unnatural, but is actually merely a consequence of our choice of definition for these classes, which is not scaling invariant. In the latter definition, we require X_j to be a Banach function space, while for the former we require X_j to be r_j-convex, which is more natural in the situation. It would have also been reasonable to define $\vec{X} \in \mathrm{UMD}_{\vec{r},s}$ by $(\vec{X}^r, (X^r)^*) \in \mathrm{HL}_{\frac{\vec{r}}{r},\left(\frac{s}{r}\right)'}$ rather than our current definition $(\vec{X}, (X)^*) \in \mathrm{HL}_{\vec{r},s'}$. Had we defined $\mathrm{UMD}_{\vec{r},s}$ in this former manner, then the conditions $X_j^{r_j} \in \mathrm{UMD}_{1,\frac{s_j}{r_j}}$ and $X_j \in \mathrm{UMD}_{r_j,s_j}$ would have been equivalent.

Remark 9.1.5. This result is not sharp in the sense that our method does not yield optimal weighted bounds for $\widetilde{T}$. Part of the reason is that our result relies on a certain self-improvement of the $\mathrm{UMD}_{1,s}$ class, see Proposition 9.1.7 below, and unlike for the self-improvement of the weight classes from Proposition 3.3.15, we do not know how this self-improvement is quantified precisely. Another issue is that in our vector-valued Rubio de Francia algorithm, see Lemma 9.1.8 below, the control we obtain of the weight constants are determined by the geometry of the spaces $\vec{X}$. As we shall see in the following chapter, it need not be the case at all that the weighted bounds of $\widetilde{T}$ depend on the spaces $\vec{X}$.

Remark 9.1.6. In this result we are considering weights in the class $A_{p_1,(r_1,s_1)} \times \cdots \times A_{p_m,(r_m,s_m)}$ which, by Proposition 3.1.5, is contained in the multilinear weight class $A_{\vec{p},(\vec{r},s)}$ with

$$[\vec{w}]_{\vec{p},(\vec{r},s)} \leq \prod_{j=1}^{m} [w_j]_{p_j,(r_j,s_j)},$$

and with a strict inclusion whenever $m > 1$. This means that a bound on T in terms of the multilinear weight class $A_{\vec{p},(\vec{r},s)}$ implies (9.1.1). A version of Theorem 9.1.1 for the classes $A_{\vec{p},(\vec{r},s)}$ rather than for $A_{p_1,(r_1,s_1)} \times \cdots \times A_{p_m,(r_m,s_m)}$ would be of great interest, but it seems a closer study of the condition $\vec{X}^r \in \mathrm{UMD}_{\frac{\vec{r}}{r},\frac{s}{r}}$ is required to attain this. Thus, this result finds most of its use in the linear case $m = 1$, while for the multilinear cases $m > 1$ it is mostly overshadowed by the vector-valued sparse domination result we present in Section 9.2. We elaborate on this further in Subsection 9.3.1.

For the proof of Theorem 9.1.1 we require several preparatory results. We first need a certain self-improvement result for the $\mathrm{UMD}_{1,s}$ class, which we can then combine with the self-improvement property of weights from Proposition 3.3.15.

Proposition 9.1.7. *Let $s \in (1,\infty]$ and $X \in \mathrm{UMD}_{1,s}$. Then there exists an $\widetilde{\alpha} \in (1,s)$ such that for all $\alpha \in (1,\widetilde{\alpha}]$ we have $X^\alpha \in \mathrm{UMD}_{1,\frac{s}{\alpha}}$.*

In the case $s = \infty$, this is a result by Rubio de Francia, see [Rub86, Theorem 4]. This result is a main ingredient for our proof for the cases $s < \infty$. We also require an analogous self-improvement result for the HL property which can be found in [GMT93].

Proof of Proposition 9.1.7. Since $X \in \mathrm{UMD}_{1,s}$, we have $X \in \mathrm{UMD}$ and $(X^*)^{s'} \in \mathrm{HL}$. Thus, by [Rub86, Theorem 4] and [GMT93] we can find a $\widetilde{\beta} > 1$ such that for all $\beta \in (1,\widetilde{\beta}]$ we respectively have $X^\beta \in \mathrm{UMD}$ and $(X^*)^{s'\beta} \in \mathrm{HL}$. In particular, we have $Y_1 := (X^\beta)^* \in \mathrm{HL}$

and $Y_2 := (X^*)^{s'\beta} \in \mathrm{HL}$. Set

$$\frac{1}{\widetilde{\alpha}} := \frac{\frac{1}{s} + \frac{1}{\widetilde{\beta}}\frac{1}{\widetilde{\beta}'}\frac{1}{s'}}{\frac{1}{s} + \frac{1}{\widetilde{\beta}'}\frac{1}{s'}} \in (\frac{1}{s}, 1)$$

and let $\alpha \in (1, \widetilde{\alpha}]$. Then there is a $\beta \in (1, \widetilde{\beta}]$ such that $\frac{1}{\alpha} = \frac{\frac{1}{s}+\frac{1}{\beta}\frac{1}{\beta'}\frac{1}{s'}}{\frac{1}{s}+\frac{1}{\beta'}\frac{1}{s'}} \in [\frac{1}{\beta}, 1)$. By Proposition 8.4.2, showing that $X^\alpha \in \mathrm{UMD}_{1,\frac{s}{\alpha}}$ is equivalent to showing that $X^\alpha \in \mathrm{HL}$ and $((X^\alpha)^*)^{(s/\alpha)'} \in \mathrm{HL}$. For the first assertion, note that since $\alpha \le \beta \le \widetilde{\beta}$ we have $X^\alpha \in \mathrm{UMD}$ and thus, $X^\alpha \in \mathrm{HL}$. It remains to prove the second assertion. Let (Ω, μ) be the measure space that X is defined over. Then, by Proposition 8.1.4(iv), we have

$$\begin{aligned}
((X^\alpha)^*)^{(s/\alpha)'} &= \big((X^\beta)^*\big)^{\frac{\frac{1}{\beta}}{\frac{1}{\alpha}-\frac{1}{s}}} \cdot L^{\frac{\frac{1}{\alpha}-\frac{1}{s}}{\frac{1}{\alpha}-\frac{1}{\beta}}}(\Omega) \\
&= \big((X^\beta)^*\big)^{\frac{\frac{1}{\beta}}{1-\frac{1}{\beta}}\frac{1-\frac{1}{\alpha}}{\frac{1}{\alpha}-\frac{1}{s}}} \cdot \big(((X^\beta)^*)^{\frac{1}{\beta}} \cdot L^{\beta'}(\Omega)\big)^{\frac{\frac{1}{\alpha}-\frac{1}{\beta}}{(\frac{1}{\alpha}-\frac{1}{s})(1-\frac{1}{\beta})}} \\
&= \big((X^\beta)^*\big)^{1-\theta} \cdot \big((X^*)^{s'\beta}\big)^{\theta}.
\end{aligned}$$

with

$$\theta = \frac{(1-\frac{1}{s})\frac{1}{\beta}(\frac{1}{\alpha}-\frac{1}{\beta})}{(\frac{1}{\alpha}-\frac{1}{s})(1-\frac{1}{\beta})} \in [0,1).$$

The result then follows from applying Proposition 8.4.11(i) in the case $m = 1$. □

Next, we need a vector-valued version of the Rubio de Francia iteration algorithm.

Lemma 9.1.8. *Let $s \in (1,\infty)$, $p \in (1,s)$, $X \in \mathrm{UMD}_{1,s}$ defined over a σ-finite measure space (Ω,μ), and $w \in A_{p,(1,s)}$. If $h \in L^{p'}_{w^{-1}}(\mathbf{R}^n;(X^*))$ is a positive function, then there is a positive function $H \in L^{p'}_{w^{-1}}(\mathbf{R}^n;(X^*))$ satisfying:*

(i) $h \le H$ *pointwise a.e.;*

(ii) $\|H\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n;(X^*))} \le 2\|h\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n;(X^*))}$*;*

(iii) $H(\cdot,\omega) \in A_{1,(1,s)}$ *for a.e.* $\omega \in \Omega$ *with*

$$[H(\cdot,\omega)]_{1,(1,s)} \lesssim_{X,p,q,s} [w]_{p,(1,s)}^{\max\left\{\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}, \frac{1-\frac{1}{q}}{1-\frac{1}{p}}\right\}}$$

for all $q \in (1,s]$ such that X is q-concave.

Proof. Since $X \in \mathrm{UMD}_{1,s}$, it follows from Proposition 8.4.2 that $X^* \in \mathrm{HL}_{s'}$. Hence, by Corollary 8.3.5 we have

$$\|\widetilde{M}_{s'}\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n;X^*)\to L^{p'}_{w^{-1}}(\mathbf{R}^n;X^*)} \lesssim_{X,p,q,s} [w^{-1}]_{p',(s',\infty)}^{\max\left\{\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}, \frac{1-\frac{1}{q}}{1-\frac{1}{p}}\right\}} = [w]_{p,(1,s)}^{\max\left\{\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}, \frac{1-\frac{1}{q}}{1-\frac{1}{p}}\right\}} \qquad (9.1.3)$$

for all $q \in (1,s]$ such that X^* is q'-convex.

We define

$$H := \sum_{k=0}^{\infty} \frac{\widetilde{M}_{s'}^k(h)}{2^k \|\widetilde{M}_{s'}\|^k_{L^{p'}_{w^{-1}}(\mathbf{R}^n;X^*)\to L^{p'}_{w^{-1}}(\mathbf{R}^n;X^*)}}.$$

Then (i) and (ii) follow in the same way as in the proof of Lemma 4.1.3. For (iii), note that by (9.1.3) we have

$$\begin{aligned} M_{s'}(H(\cdot,\omega)) = \widetilde{M}_{s'}(H)(\cdot,\omega) &\le 2\|\widetilde{M}_{s'}\|_{L^{p'}_{w^{-1}}(\mathbf{R}^n;X^*)\to L^{p'}_{w^{-1}}(\mathbf{R}^n;X^*)} H(\cdot,\omega) \\ &\lesssim_{X,p,q,s} [w]_{p,(1,s)}^{\max\left\{\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}},\frac{1-\frac{1}{q}}{1-\frac{1}{p}}\right\}} H(\cdot,\omega). \end{aligned}$$

so that $[H(\cdot,\omega)]_{1,(1,s)} \lesssim_{X,p,q,s} [w]_{p,(1,s)}^{\max\left\{\frac{1-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}},\frac{1-\frac{1}{q}}{1-\frac{1}{p}}\right\}}$ for a.e. $\omega \in \Omega$. The assertion follows. □

We are now ready to prove Theorem 9.1.1.

Proof of Theorem 9.1.1. Fix $\vec{r} < \vec{p} < \vec{s}$ and weights $\vec{w}$ satisfying $w_j \in A_{p_j,(r_j,s_j)}$ for all $j \in \{1,\ldots,m\}$. By Proposition 3.3.15 and Proposition 9.1.7 we can pick $1 < \alpha < \min\{\frac{\vec{p}}{\vec{r}}\}$ such that $w_j \in A_{p_j,(\alpha r_j,s_j)}$ with $[w_j]_{p_j,(\alpha r_j,s_j)} \lesssim [w_j]_{p_j,(r_j,s_j)}$ and $X_j^{\alpha r_j} \in \mathrm{UMD}_{1,\frac{s_j}{\alpha r_j}}$ for all $j \in \{1,\ldots,m\}$.

Since $\sum_{j=1}^m \frac{r}{r_j} = 1$, it follows from Proposition 8.1.4(iii) that

$$\big((X^{\alpha r})^*\big)^{\frac{1}{\alpha r}} = \Big(\Big(\prod_{j=1}^m (X_j^{\alpha r_j})^{\frac{r}{r_j}}\Big)^*\Big)^{\frac{1}{\alpha r}} = \Big(\prod_{j=1}^m \big((X_j^{\alpha r_j})^*\big)^{\frac{r}{r_j}}\Big)^{\frac{1}{\alpha r}} = \prod_{j=1}^m \big((X_j^{\alpha r_j})^*\big)^{\frac{1}{\alpha r_j}}.$$

Since $X_j^{\alpha r_j} \in \mathrm{UMD}$, the space $(X_j^{\alpha r_j})^*$ is reflexive for all $j \in \{1,\ldots,m\}$ so that $(X^{\alpha r})^*$ is also reflexive by Proposition 8.1.4(i) and hence order-continuous. Thus, it follows from Lemma 8.1.5 that

$$L_{w^{-1}}^{\frac{1}{\frac{1}{\alpha r}-\frac{1}{p}}}(\mathbf{R}^n;((X^{\alpha r})^*)^{\frac{1}{\alpha r}}) = \prod_{j=1}^m L_{w_j^{-1}}^{\frac{1}{\frac{1}{\alpha r_j}-\frac{1}{p_j}}}(\mathbf{R}^n;((X_j^{\alpha r_j})^*)^{\frac{1}{\alpha r_j}}). \tag{9.1.4}$$

Thus, fixing $g \in L_{w^{-1}}^{\frac{1}{\frac{1}{\alpha r}-\frac{1}{p}}}(\mathbf{R}^n;((X^{\alpha r})^*)^{\frac{1}{\alpha r}})$, we can pick positive $g_j \in L_{w_j^{-1}}^{\frac{1}{\frac{1}{\alpha r_j}-\frac{1}{p_j}}}(\mathbf{R}^n;((X_j^{\alpha r_j})^*)^{\frac{1}{\alpha r_j}})$ such that $|g| \le \prod_{j=1}^m g_j$.

For the functions H_j obtained from applying Lemma 9.1.8 with $s = \frac{s_j}{\alpha r_j}$, $p = \frac{p_j}{\alpha r_j}$, $X = X_j^{\alpha r_j}$, $w = w_j^{\alpha r_j}$, $h_j = g_j^{\alpha r_j}$, we set $W_j := H_j^{\frac{1}{\alpha r_j}}$ so that

$$|g| \le \prod_{j=1}^m g_j = \prod_{j=1}^m h_j^{\frac{1}{\alpha r_j}} \le \prod_{j=1}^m H_j^{\frac{1}{\alpha r_j}} = \prod_{j=1}^m W_j = W \tag{9.1.5}$$

and

$$\begin{aligned}\|W_j\|_{L^{\frac{1}{\frac{1}{\alpha r_j}-\frac{1}{p_j}}}_{w_j^{-1}}(\mathbf{R}^n;((X_j^{\alpha r_j})^*)^{\frac{1}{\alpha r_j}})} &= \|H_j\|^{\frac{1}{\alpha r_j}}_{L^{(p_j/\alpha r_j)'}_{w_j^{-\alpha r_j}}(\mathbf{R}^n;(X_j^{\alpha r_j})^*)} \leq 2^{\frac{1}{\alpha r_j}}\|h_j\|^{\frac{1}{\alpha r_j}}_{L^{(p_j/\alpha r_j)'}_{w_j^{-\alpha r_j}}(\mathbf{R}^n;(X_j^{\alpha r_j})^*)}\\ &= 2^{\frac{1}{\alpha r_j}}\|g_j\|_{L^{\frac{1}{\frac{1}{\alpha r_j}-\frac{1}{p_j}}}_{w_j^{-1}}(\mathbf{R}^n;((X_j^{\alpha r_j})^*)^{\frac{1}{\alpha r_j}})}.\end{aligned} \tag{9.1.6}$$

Moreover, by Proposition 3.1.3(ii), Hölder's inequality, and the definition of α, we have

$$\begin{aligned}[W_j(\cdot,\omega)]_{\alpha r_j,(r_j,s_j)} &= [H_j(\cdot,\omega)]^{\frac{1}{\alpha r_j}}_{1,(\frac{1}{\alpha},\frac{s_j}{\alpha r_j})} \leq [H_j(\cdot,\omega)]^{\frac{1}{\alpha r_j}}_{1,(1,\frac{s_j}{\alpha r_j})}\\ &\lesssim_{X,p_j,q,s_j} [w_j^{\alpha r_j}]^{\frac{1}{\alpha r_j}\cdot\max\left\{\frac{1-\frac{\alpha r_j}{s_j}}{\frac{\alpha r_j}{p_j}-\frac{\alpha r_j}{s_j}},\frac{1-\frac{\alpha r_j}{q_j}}{1-\frac{\alpha r_j}{p_j}}\right\}}_{\frac{p_j}{\alpha r_j},(1,\frac{s_j}{\alpha r_j})}\\ &= [w_j]^{\max\left\{\frac{\frac{1}{\alpha r_j}-\frac{1}{s_j}}{\frac{1}{p_j}-\frac{1}{s_j}},\frac{\frac{1}{\alpha r_j}-\frac{1}{q_j}}{\frac{1}{\alpha r_j}-\frac{1}{p_j}}\right\}}_{p_j,(\alpha r_j,s_j)} \lesssim [w_j]^{\max\left\{\frac{\frac{1}{\alpha r_j}-\frac{1}{s_j}}{\frac{1}{p_j}-\frac{1}{s_j}},\frac{\frac{1}{\alpha r_j}-\frac{1}{q_j}}{\frac{1}{\alpha r_j}-\frac{1}{p_j}}\right\}}_{p_j,(r_j,s_j)}\end{aligned} \tag{9.1.7}$$

for a.e. $\omega\in\Omega$ and all $q_j\in(\alpha r_j,s_j]$ such that X_j is q_j-concave.

Let $\vec{f}\in L^\infty_c(\mathbf{R}^n;\vec{X})$ be simple functions. Then $\vec{f}(\cdot,\omega)\in L^\infty_c(\mathbf{R}^n)$ for a.e. $\omega\in\Omega$ so that $T(\vec{f}\,)$ is well-defined. By (9.1.1) and (9.1.5) we have

$$\begin{aligned}\|T(\vec{f}(\cdot,\omega))g(\cdot,\omega)\|_{L^{\alpha r}(\mathbf{R}^n)} &\leq \|T(\vec{f}(\cdot,\omega))\|_{L^{\alpha r}_{W(\cdot,\omega)}(\mathbf{R}^n)}\\ &\leq \phi_{\alpha\vec{r}}([W_1(\cdot,\omega)]_{\alpha r_1,(r_1,s_1)},\ldots,[W_m(\cdot,\omega)]_{\alpha r_m,(r_m,s_m)})\prod_{j=1}^m\|f_j(\cdot,\omega)\|_{L^{\alpha r_j}_{W_j(\cdot,\omega)}(\mathbf{R}^n)}\\ &\leq \phi_{\vec{X},\vec{p},\vec{r},\vec{s}}([w_1]_{p_1,(r_1,s_1)},\ldots,[w_m]_{p_m,(r_m,s_m)})\prod_{j=1}^m\|f_j(\cdot,\omega)\|_{L^{\alpha r_j}_{W_j(\cdot,\omega)}(\mathbf{R}^n)},\end{aligned}$$

where $\phi_{\vec{X},\vec{p},\vec{r},\vec{s}}$ is a componentwise increasing function determined by (9.1.7). Then, abbreviating $\phi:=\phi_{\vec{X},\vec{p},\vec{r},\vec{s}}([w_1]_{p_1,(r_1,s_1)},\ldots,[w_m]_{p_m,(r_m,s_m)})$, by Fubini's Theorem, Hölder's inequality, and (9.1.6), we have

$$\begin{aligned}\|\widetilde{T}(\vec{f}\,)\cdot g\|_{L^{\alpha r}(\mathbf{R}^n;L^{\alpha r}(\Omega))} &= \|\|T(\vec{f}(\cdot,\omega))g(\cdot,\omega)\|_{L^{\alpha r}(\mathbf{R}^n)}\|_{L^{\alpha r}(\Omega)} \leq \phi\left\|\prod_{j=1}^m\|f_j(\cdot,\omega)\|_{L^{\alpha r_j}_{W_j(\cdot,\omega)}(\mathbf{R}^n)}\right\|_{L^{\alpha r}(\Omega)}\\ &\leq \phi\prod_{j=1}^m\|f_jW_j\|_{L^{\alpha r_j}(\mathbf{R}^n;L^{\alpha r_j}(\Omega))} \leq \phi\prod_{j=1}^m\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}\|W_j\|_{L^{\frac{1}{\frac{1}{\alpha r_j}-\frac{1}{p_j}}}_{w_j^{-1}}(\mathbf{R}^n;((X_j^{\alpha r_j})^*)^{\frac{1}{\alpha r_j}})}\\ &\leq 2^{\frac{1}{\alpha r}}\phi\prod_{j=1}^m\|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}\|g_j\|_{L^{\frac{1}{\frac{1}{\alpha r_j}-\frac{1}{p_j}}}_{w_j^{-1}}(\mathbf{R}^n;((X_j^{\alpha r_j})^*)^{\frac{1}{\alpha r_j}})}.\end{aligned}$$

Taking an infimum over all possible $g_j \in L_{w_j^{-1}}^{\frac{1}{\frac{1}{\alpha r_j}-\frac{1}{p_j}}}(\mathbf{R}^n;((X_j^{\alpha r_j})^*)^{\frac{1}{\alpha r_j}})$ with $|g| \leq \prod_{j=1}^m g_j$, we conclude that

$$\|\widetilde{T}(\vec{f}\,)\cdot g\|_{L^{\alpha r}(\mathbf{R}^n;L^{\alpha r}(\Omega))} \lesssim_r \phi\Big(\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}\Big)\|g\|_{L_{w^{-1}}^{\frac{1}{\frac{1}{\alpha r}-\frac{1}{p}}}(\mathbf{R}^n;((X^{\alpha r})^*)^{\frac{1}{\alpha r}})}.$$

Thus, by duality in Bochner spaces, we have

$$\begin{aligned}\|\widetilde{T}(\vec{f}\,)\|_{L^p_w(\mathbf{R}^n;X)} = \||\widetilde{T}(\vec{f}\,)|^{\alpha r}\|^{\frac{1}{\alpha r}}_{L^{\frac{p}{\alpha r}}_{w^{\alpha r}}(\mathbf{R}^n;X^{\alpha r})} &= \sup_{\|g\|_{L_{w^{-1}}^{\frac{1}{\frac{1}{\alpha r}-\frac{1}{p}}}(\mathbf{R}^n;((X^{\alpha r})^*)^{\frac{1}{\alpha r}})}=1} \|\widetilde{T}(\vec{f}\,)\cdot g\|_{L^{\alpha r}(\mathbf{R}^n;L^{\alpha r}(\Omega))} \\ &\lesssim_r \phi\prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}\end{aligned}$$

for all simple functions $\vec{f} \in L^\infty_c(\mathbf{R}^n;\vec{X})$. The assertion now follows from the density result Lemma 8.2.3 and the extension result Lemma 5.3.2. □

9.2. VECTOR-VALUED SPARSE DOMINATION FROM SCALAR-VALUED SPARSE DOMINATION

This section is dedicated to proving vector-valued sparse domination of operators satisfying scalar-valued sparse domination. Moreover, we use this to deduce sharp vector-valued weighted bounds for these operators.

Note that we introduce the parameter q into the theorem here, which is essential in obtaining the full range of vector-valued bounds, including the quasi-Banach range. We elaborate further on this in Section 9.3.

Theorem 9.2.1. *Let $\vec{r} \in (0,\infty)^m$, $q \in (0,\infty)$, $s \in (q,\infty]$ and let T be an operator defined on m-tuples of functions such that for any $\vec{f}, g \in L^\infty_c(\mathbf{R}^n)$*

$$\big\|T(\vec{f}\,)\cdot g\big\|_{L^q(\mathbf{R}^n)} \leq C_T\big\|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\vec{f},g)\big\|_{L^q(\mathbf{R}^n)}. \tag{9.2.1}$$

Let $\vec{X}$ be and m-tuple of quasi-Banach function spaces over a measure space (Ω,μ) such that $\vec{X}^q \in \mathrm{UMD}_{\frac{\vec{r}}{q},\frac{s}{q}}$. Furthermore suppose that for all simple functions $\vec{f} \in L^\infty_c(\mathbf{R}^n;\vec{X})$ the function $\widetilde{T}(\vec{f}\,):\mathbf{R}^n \to X$ given by

$$\widetilde{T}(\vec{f}\,)(x,\omega) := T(\vec{f}(\cdot,\omega))(x), \qquad (x,\omega) \in \mathbf{R}^n \times \Omega$$

is well-defined and strongly measurable. Then for all simple functions $\vec{f} \in L^\infty_c(\mathbf{R}^n;\vec{X})$ and $g \in L^\infty_c(\mathbf{R}^n)$

$$\big\|\|\widetilde{T}(\vec{f}\,)\|_X\cdot g\big\|_{L^q(\mathbf{R}^n)} \lesssim_{\vec{X},q,\vec{r},s} C_T\big\|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\|\vec{f}\|_{\vec{X}},g)\big\|_{L^q(\mathbf{R}^n)}. \tag{9.2.2}$$

As in Remark 9.1.2, if T is m-linear, then $\widetilde{T}(\vec{f})$ is always well-defined and strongly measurable for simple functions $\vec{f} \in L^\infty_c(\mathbf{R}^n;\vec{X})$ as it is given by the tensor extension of T.

Proof. The proof essentially consists of applying Fubini's Theorem twice and then using the vector-valued sparse domination result for the multisublinear maximal operator. Let $\vec{f} \in L^\infty_c(\mathbf{R}^n;\vec{X})$ and $g \in L^\infty_c(\mathbf{R}^n;((X^q)^*)^{\frac{1}{q}})$ be simple. Then for a.e. $\omega \in \Omega$ we have $f_j(\cdot,\omega), g(\cdot,\omega) \in L^\infty_c(\mathbf{R}^n)$. Thus, using Fubini's Theorem and (9.2.1), we have

$$\begin{aligned}\big\|\widetilde{T}(\vec{f})\cdot g\big\|_{L^q(\mathbf{R}^n;L^q(\Omega))} &= \big\|\omega \mapsto \|T(\vec{f}(\cdot,\omega),g(\cdot,\omega)\|_{L^q(\mathbf{R}^n)}\big\|_{L^q(\Omega)}\\ &\leq C_T\big\|\omega \mapsto \|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\vec{f}(\cdot,\omega),g(\cdot,\omega))\|_{L^q(\mathbf{R}^n)}\big\|_{L^q(\Omega)}\\ &= C_T\big\|\widetilde{M}_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\vec{f},g)\big\|_{L^q(\mathbf{R}^n;L^q(\Omega))}.\end{aligned} \tag{9.2.3}$$

We set $X_{m+1} := ((X^q)^*)^{\frac{1}{q}}$ so that

$$\prod_{j=1}^{m+1} X_j = (X^q\cdot(X^q)^*)^{\frac{1}{q}} = L^1(\Omega)^{\frac{1}{q}} = L^q(\Omega),$$

which is an order-continuous q-convex quasi-Banach function space. Then it follows from the sparse domination result in Theorem 8.3.3 that

$$\big\|\widetilde{M}_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\vec{f},g)\big\|_{L^q(\mathbf{R}^n;L^q(\Omega))} \lesssim_{\vec{X},q,\vec{r},s} \big\|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}\big(\|\vec{f}\|_{\vec{X}},\|g\|_{((X^q)^*)^{\frac{1}{q}}}\big)\big\|_{L^q(\mathbf{R}^n)}.$$

By combining this with (9.2.3) and Proposition 8.2.1, the assertion follows. □

We will now use Theorem 9.2.1 to deduce weighted boundedness for the vector-valued extension of an operator T from a scalar-valued sparse domination result for T, which is new even in the unweighted setting.

Theorem 9.2.2. *Let $\vec{r} \in (0,\infty)^m$, $q \in (0,\infty)$, $s \in (q,\infty]$ and let T an m-linear or positive-valued m-sublinear operator satisfying* (9.2.1) *and let $\vec{X}$ satisfy the assumptions in Theorem 9.2.1. Then for all $p \in (0,\infty]^m$ with $\vec{p} > \vec{r}$ and $p < s$, and all $\vec{w} \in A_{\vec{p},(\vec{r},s)}$ we have*

$$\|\widetilde{T}(\vec{f})\|_{L^p_w(\mathbf{R}^n;X)} \lesssim_{\vec{X},\vec{p},q,\vec{r},s} C_T[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}},\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}} \prod_{j=1}^m \|f_j\|_{L^{p_j}_{w_j}(\mathbf{R}^n;X_j)}$$

for all $\vec{f} \in L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n;\vec{X})$.

Proof. This follows from combining Theorem 9.2.1 with Theorem 8.2.2. □

9.3. APPLICATIONS

In this section we provide a discussion regarding utilizing ℓ^q-type sparse domination in order to obtain vector-valued sparse domination in spaces beyond the Banach range. Moreover, we wish to compare the utility of our vector-valued extrapolation and our vector-valued sparse domination methods. Furthermore, we compare the results of Section 5.4 for multilinear Calderón-Zygmund operator and the bilinear Hilbert transform to the results obtained in this chapter. We point out that our results are of course applicable far beyond these examples as they include all operators satisfying sparse domination.

9.3.1. Vector-valued estimates in the quasi-Banach range

In the multilinear setting it is a natural occurrence that an operator maps into a Lebesgue space with exponents smaller than 1 and hence, no longer in the Banach range. For this reason one also expects the vector-valued extensions of the operator to map into spaces in the quasi-Banach range. However, in our multilinear UMD condition we assume that the product of the spaces is a Banach space. This is partly because we are obtaining our estimates after a dualization argument which is usually not possible in the quasi-Banach setting. As we have seen in Theorem 8.2.2, it is thanks to the quantitative extrapolation theorem from Chapter 4 that this dualization does not hinder us in obtaining sharp bounds in the mixed-norm spaces $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n;\vec{X})$ in the full range of exponents $\vec{p}$. We are however still hindered in how much convexity we are allowed to assume on the tuple $\vec{X}$.

In this subsection we explain how the parameter q in the results in Section 9.2 can be used to recover the expected results in the quasi-Banach range, at the cost of a worse exponent in the weighted estimate. We illustrate this in the following proposition:

Proposition 9.3.1. *Let $\vec{r}\in(0,\infty)^m$, $q_0\in(0,\infty)$, and let T be an m-linear operator initially defined on $L^\infty_c(\mathbf{R}^n)^m$. Suppose that for each bounded set B and all $\vec{f}\in L^\infty_c(\mathbf{R}^n)^m$ supported in B, for each $\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n$ there exists a sparse collection $\mathscr{S}^\alpha\subseteq\mathscr{D}^\alpha$ such that*

$$|T(\vec{f}\,)|\leq C_T\sum_{\alpha\in\{0,\frac{1}{3},\frac{2}{3}\}^n}\Big(\sum_{Q\in\mathscr{S}^\alpha}\Big(\prod_{j=1}^m\langle f_j\rangle_{r_j,Q}\Big)^{q_0}\chi_Q\Big)^{\frac{1}{q_0}} \tag{9.3.1}$$

pointwise a.e. in B. If (Ω,μ) is a σ-finite measure space, then for all $\vec{p}\in(0,\infty]^m$, $\vec{t}\in(0,\infty]^m$ with $\vec{r}<\vec{p},\vec{t}$ and $p,t<\infty$ and all $\vec{w}\in A_{\vec{p},(\vec{r},\infty)}$ the tensor extension $\widetilde{T}$ of T has a bounded extension $L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n;L^{t_m}(\Omega))\to L^p_w(\mathbf{R}^n;L^t(\Omega))$ with

$$\|\widetilde{T}\|_{L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n;L^{t_m}(\Omega))\to L^p_w(\mathbf{R}^n;L^t(\Omega))}\lesssim_{\vec{p},q_0,\vec{r},\vec{t}} C_T[\vec{w}]^\gamma_{\vec{p},(\vec{r},\infty)}, \tag{9.3.2}$$

where

$$\gamma=\begin{cases}\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}},\frac{\frac{1}{q_0}}{\frac{1}{p}}\right\} & \text{if } t\in[q_0,\infty);\\[2ex] \max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}},\frac{\frac{1}{t}}{\frac{1}{p}}\right\} & \text{if } t\in(r,q_0].\end{cases}$$

The above result also holds for m-sublinear T as in Theorem 9.2.1. Of course, our methods go beyond the setting of L^t-spaces, but we restrict our attention to this particular case for now for the sake of clarity and for the sake of comparing our result to the results in Section 5.4.

Proof. Write $\vec{X} = (L^{t_1}(\Omega),\dots,L^{t_m}(\Omega))$. We consider the two cases separately.

For the case $t \in [q_0,\infty)$, we note that $\vec{X}^{q_0} = (L^{\frac{t_1}{q_0}}(\Omega),\dots,L^{\frac{t_m}{q_0}}(\Omega)) \in \mathrm{UMD}_{\frac{\vec{r}}{q_0},\infty}$ by Proposition 8.4.6. Thus, the result follows from an application of Proposition 5.3.4 and Theorem 9.2.2 with $q = q_0$ and $s = \infty$.

In the other case $t \in (r,q_0]$ we have $\vec{X}^t = (L^{\frac{t_1}{t}}(\Omega),\dots,L^{\frac{t_m}{t}}(\Omega)) \in \mathrm{UMD}_{\frac{\vec{r}}{t},\infty}$ by Proposition 8.4.6. Note that the inequality $\|\cdot\|_{\ell^{q_0}} \le \|\cdot\|_{\ell^t}$ implies that (9.3.1) holds with q_0 replaced by t, see also Remark 5.3.5. Hence, an application of Proposition 5.3.4 and Theorem 9.2.2 with $q = t$ and $s = \infty$ proves the result. □

Note that in case $t \in [q_0,\infty)$ we did not need to assume the pointwise sparse domination (9.3.1) in our proof, but it would have sufficed to assume domination in form. For example, if we instead assumed that for an $s \in (q_0,\infty]$ and all $\vec{f}, g \in L^\infty_c(\mathbf{R}^n)$ we have

$$\big\|T(\vec{f})\cdot g\big\|_{L^{q_0}(\mathbf{R}^n)} \le C_T \big\|M_{(\vec{r},\frac{1}{\frac{1}{q_0}-\frac{1}{s}})}(\vec{f},g)\big\|_{L^{q_0}(\mathbf{R}^n)}, \tag{9.3.3}$$

then exactly as in the proof we obtain (9.3.2) for $t \in [q_0,\infty)$ with

$$\gamma = \max\left\{\frac{\frac{1}{r}}{\frac{1}{r}-\frac{1}{p}}, \frac{\frac{1}{q_0}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}. \tag{9.3.4}$$

However, at this point it is not clear how to deal with the cases $t \in (r,q_0]$. In the case of (9.3.1) we can simply apply the estimate $\|\cdot\|_{\ell^{q_0}} \le \|\cdot\|_{\ell^t}$ to obtain the domination required to complete the argument. However, if we only assume the sparse domination in form (9.3.3), it is unknown whether we automatically also have (9.3.3) with q_0 replaced by a smaller exponent $0 < q \le q_0$, meaning that it is not clear whether we have the flexibility to cover the cases $t \in (r,q_0]$ or not without *assuming* that (9.3.3) also holds with q_0 replaced by t.

We point out that replacing q_0 by $0 < q \le q_0$ qualitatively yields the same weighted bounds, but the result is quantitatively worse in that it yields a worse exponent γ in the bound. Thus, on all accounts it seems that the following conjecture should hold:

Conjecture 9.3.2 (Sparse form domination implies worse sparse form domination)**.** *Let $\vec{r} \in (0,\infty)^m$, $q_0 \in (0,\infty)$, and $s \in (q_0,\infty]$. Let T be an m-(sub)linear operator initially defined on $L^\infty_c(\mathbf{R}^n)^m$ and suppose that for any $\vec{f} \in L^\infty_c(\mathbf{R}^n)^m$ we have*

$$\|T(\vec{f})\cdot g\|_{L^{q_0}(\mathbf{R}^n)} \lesssim \big\|M_{(\vec{r},\frac{1}{\frac{1}{q_0}-\frac{1}{s}})}(\vec{f},g)\big\|_{L^{q_0}(\mathbf{R}^n)}.$$

Then the same estimate also holds when we replace q_0 by any $q \in (0,q_0]$.

We point out that even the simplest case $m = 1$, $r = 1$, $q_0 = 1$, $s = \infty$ is unknown. For specific cases of T one can usually verify the conjecture by going back to the proof of (9.3.3) and insert the estimate $\|\cdot\|_{\ell^q} \le \|\cdot\|_{\ell^{q_0}}$ at the right place in the proof. Examples where this is the case include:

- In [Lor19, Theorem 3.5] a general theorem to obtain sparse domination for an operator T is shown. In this theorem a *localized ℓ^q-estimate* is imposed on T to deduce (9.3.3) with $q_0 = q$. The localized ℓ^q-estimate for T becomes weaker for smaller q, so any operator whose sparse domination can be proven through [Lor19, Theorem 3.5] also satisfies the result of Conjecture 9.3.2.
- As mentioned in Subsection 5.4.2, sparse domination with $q_0 = 1$ for the bilinear Hilbert transform BHT was proven by Culiuc, Di Plinio and Ou in [CDO18]. The result of Conjecture 9.3.2 for BHT was verified later by Benea and Muscalu in [BM17].
- One of the main results in [CCDO17] is (9.3.3) with $q_0 = 1$ for rough homogeneous singular operators T_Ω, see also [Ler19] for an alternative proof. As mentioned in Example 5.4.3, adapting the technique in [Ler19], Conjecture 9.3.2 was verified for these operators in [CLRT19, Theorem 5.1], which has implications for weighted norm inequalities for T_Ω with so-called C_p-weights.

Now, if Conjecture 9.3.2 is false and, e.g., there is an m-linear operator T such that for all $\vec{f} \in L^\infty_c(\mathbf{R}^n)^m$ we have

$$\|T(\vec{f}\,)\cdot g\|_{L^1(\mathbf{R}^n)} \lesssim \left\|M_{(\vec{1},1)}(\vec{f},g)\right\|_{L^1(\mathbf{R}^n)},$$

but T does not satisfy the corresponding ℓ^q-type sparse domination for any $q \in (0,1)$. Then T has vector-valued extensions that can be obtained from the vector-valued extrapolation result Theorem 9.1.1, but not from the vector-valued sparse domination result Theorem 9.2.2. For example, bounds for any m-tuple of quasi-Banach function spaces $\vec{X}$ with $X_j \in \mathrm{UMD}$ for all $j \in \{1,\ldots,m\}$ whose product X is not a Banach space can be obtained from the extrapolation theorem, but not from the vector-valued sparse domination theorem. This includes, for example, bounds with respect to $L^{t_j}(\Omega)$ spaces with $\vec{t} \in (1,\infty)^m$, but with $t \in (0,1)$. Similar examples for Lorentz and Orlicz spaces can be given using Example 8.4.4. Moreover, we point out that if an operator satisfies weighted bounds, but no sparse domination, then Theorem 9.1.1 can still be used, while Theorem 9.2.1 can not. It is however not clear if there are examples of operators satisfying weighted bounds, but no form of sparse domination.

To conclude this subsection, we wish to compare the results of Section 9.2 with Theorem 5.3.6, where vector-valued extensions for Lebesgue spaces $L^{t_j}(\Omega)$ for a σ-finite measure space (Ω,μ) were proven as a result of sparse domination and scalar-valued extrapolation. Let $\vec{r} \in (0,\infty)^m$, $s \in (0,\infty]$, $q \in (0,s)$, and let T be an m-linear operator initially defined on $L^\infty_c(\mathbf{R}^n)^m$ such that for all $\vec{f} \in L^\infty_c(\mathbf{R}^n)^m$ we have

$$\|T(\vec{f}\,)\cdot g\|_{L^q(\mathbf{R}^n)} \le C_T \left\|M_{(\vec{r},\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\vec{f},g)\right\|_{L^q(\mathbf{R}^n)}.$$

Then, by Theorem 5.3.6, we find that for all $\vec{p}\in(0,\infty]^m$, $\vec{t}\in(0,\infty]^m$ with $\vec{r}<\vec{p},\vec{t}$ and $p,t<s$ and all $\vec{w}\in A_{\vec{p},(\vec{r},s)}$ we have

$$\|\widetilde{T}\|_{L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n;L^{t_m}(\Omega))\to L^p_w(\mathbf{R}^n;L^t(\Omega))}\lesssim C_T[\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}},\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{t}-\frac{1}{s}}\right\}\cdot\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}},\frac{\frac{1}{t}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}}.$$

Since

$$\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}},\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\}\le\max\left\{\frac{\frac{1}{\vec{r}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}},\frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{t}-\frac{1}{s}}\right\}\cdot\max\left\{\frac{\frac{1}{\vec{r}}-\frac{1}{\vec{t}}}{\frac{1}{\vec{r}}-\frac{1}{\vec{p}}},\frac{\frac{1}{t}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\right\},$$

the exponent (9.3.4) obtained from vector-valued sparse domination improves this result in the Banach range $t\in[1,\infty)$, as was noted in Remark 5.3.7.

9.3.2. Multilinear Calderón-Zygmund operators

As discussed in Subsection 5.4.1, it was shown in [CR16, LN18] that multilinear Calderón-Zygmund operators with a modulus of continuity satisfying a log-Dini condition satisfy the sparse domination (9.3.1) for $\vec{r}=\vec{1}$, $q_0=1$, and $s=\infty$. Hence, by applying Proposition 9.3.1 we find that if (Ω,μ) is a σ-finite measure space, then for all $\vec{p},\vec{t}\in(1,\infty]^m$, $p,t<\infty$ and all $\vec{w}\in A_{\vec{p},(\vec{1},\infty)}$ the tensor extension $\widetilde{T}$ of T is bounded $L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n;L^{t_m}(\Omega))\to L^p_w(\mathbf{R}^n;L^t(\Omega))$ with

$$\|\widetilde{T}\|_{L^{p_1}_{w_1}(\mathbf{R}^n;L^{t_1}(\Omega))\times\cdots\times L^{p_m}_{w_m}(\mathbf{R}^n;L^{t_m}(\Omega))\to L^p_w(\mathbf{R}^n;L^t(\Omega))}\lesssim_{\vec{p},q_0,\vec{r},\vec{t}} C_T[\vec{w}]^{\gamma}_{\vec{p},(\vec{1},\infty)}$$

with

$$\gamma=\begin{cases}\max\{p_1',\dots,p_m',p\} & \text{if } t\in[1,\infty);\\ \max\{p_1',\dots,p_m',\frac{p}{t}\} & \text{if } t\in(\frac{1}{m},1].\end{cases}$$

Hence, in the case $t\in[1,\infty)$ our quantitative bound improves the one from Theorem 5.4.1.

By applying Proposition 5.3.4, Remark 5.3.5, Theorem 9.2.1, and Theorem 9.2.2, the full result for the tensor extension $\widetilde{T}$ of an m-linear Calderón-Zygmund operator T we obtain is as follows:

Theorem 9.3.3. *Let T be an m-linear Calderón-Zygmund operator with a modulus of continuity ω satisfying the* log*-Dini condition* (5.4.1)*. Let $\vec{X}$ be an m-tuple of quasi-Banach function spaces such that $\vec{X}^q\in\mathrm{UMD}_{\frac{\vec{1}}{q},\infty}$ for some $q\in(0,1]$. Then for all simple functions $\vec{f}\in L^\infty_c(\mathbf{R}^n;\vec{X})$, $g\in L^\infty_c(\mathbf{R}^n)$ we have*

$$\big\|\|\widetilde{T}(\vec{f})\|_X\cdot g\big\|_{L^q(\mathbf{R}^n)}\lesssim_{\vec{X},q} C_T\big\|M_{(\vec{r},q)}(\|\vec{f}\|_{\vec{X}},g)\big\|_{L^q(\mathbf{R}^n)}.$$

Moreover, for all $\vec{p}\in(1,\infty]^m$ with $p<\infty$ and all $\vec{w}\in A_{\vec{p},(\vec{1},\infty)}$, $\widetilde{T}$ has a bounded extension $L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n;\vec{X})\to L^p_w(\mathbf{R}^n;X)$ with

$$\|\widetilde{T}\|_{L^{\vec{p}}_{\vec{w}}(\mathbf{R}^n;\vec{X})\to L^p_w(\mathbf{R}^n;X)}\lesssim_{\vec{X},\vec{p},q} C_T[\vec{w}]_{\vec{p},(\vec{1},\infty)}^{\max\left\{p_1',\dots,p_m',\frac{p}{q}\right\}}.$$

To optimize the weighted bound, for each tuple of spaces $\vec{X}$ one should determine the largest $q \in (0,1]$ such that $\vec{X}^q \in \mathrm{UMD}_{\frac{1}{q},\infty}$. For $q = 1$ our bound coincides with the known sharp bound in the scalar case, so in this case our bound is optimal.

In conclusion, our result recovers the full known range of vector-valued extensions of multilinear Calderón-Zygmund operators and proves new ones with new sharp weighted bounds.

Remark 9.3.4. In the linear case $m = 1$, the sharpness of the $T(b)$ theorem in [NTV02] enabled Hytönen in [Hyt14, Theorem 3] to prove boundedness of the tensor extension of a Calderón–Zygmund operator T on $L^p(\mathbf{R}^n; X)$ for general UMD Banach spaces X from scalar-valued boundedness of T. It would be of great interest to develop techniques to extend more general multilinear operators beyond the function space setting.

9.3.3. The bilinear Hilbert transform

As mentioned in Subsection 5.4.2, it was shown in [BM17] that the bilinear Hilbert transform

$$\mathrm{BHT}(f_1, f_2)(x) := \mathrm{p.v.}\int_{\mathbf{R}} f_1(x-y) f_2(x+y)\,\frac{\mathrm{d}y}{y}$$

satisfies the ℓ^q-type sparse domination

$$\|\mathrm{BHT}(f_1, f_2)\cdot g\|_{L^q(\mathbf{R})} \lesssim \|M_{(r_1,r_2,\frac{1}{\frac{1}{q}-\frac{1}{s}})}(f_1, f_2, g)\|_{L^q(\mathbf{R})} \tag{9.3.5}$$

for all $f_1, f_2, g \in L^\infty_c(\mathbf{R})$, $q \in (0, s)$, whenever $r_1, r_2, s \in (1,\infty)$ satisfy the property that there exist $\theta_1,\theta_2,\theta_3 \in [0,1)$ with $\theta_1+\theta_2+\theta_3 = 1$ such that

$$\frac{1}{r_1} < \frac{1+\theta_1}{2}, \quad \frac{1}{r_2} < \frac{1+\theta_2}{2}, \quad \frac{1}{s} > \frac{1-\theta_3}{2} \tag{9.3.6}$$

or equivalently

$$\max\Big\{\frac{1}{r_1}, \frac{1}{2}\Big\} + \max\Big\{\frac{1}{r_2}, \frac{1}{2}\Big\} + \max\Big\{\frac{1}{s'}, \frac{1}{2}\Big\} < 2.$$

Hence, by Theorem 9.2.1 and Theorem 9.2.2, we obtain the following result for the tensor extension $\widetilde{\mathrm{BHT}}$ of BHT:

Theorem 9.3.5. *Let $r_1, r_2, s \in (1,\infty)$ satisfy* (9.3.6) *and let (X_1, X_2) be a pair of quasi-Banach function spaces such that $\vec{X}^q \in \mathrm{UMD}_{\frac{\vec{r}}{q},\frac{s}{q}}$ for some $q \in (0,1]$. Then for all simple functions $\vec{f} \in L^\infty_c(\mathbf{R}; \vec{X})$ and $g \in L^\infty_c(\mathbf{R})$ we have*

$$\big\| \|\widetilde{\mathrm{BHT}}(f_1, f_2)\|_X \cdot g\big\|_{L^q(\mathbf{R})} \lesssim_{\vec{X},q,\vec{r},s} C_T \big\| M_{(r_1,r_2,\frac{1}{\frac{1}{q}-\frac{1}{s}})}(\|f_1\|_{X_1}, \|f_2\|_{X_2}, g)\big\|_{L^q(\mathbf{R})}. \tag{9.3.7}$$

Moreover, for all $\vec{p} \in (0,\infty]^2$ with $\vec{r} < \vec{p}$ and $p < s$, all $\vec{w} \in A_{\vec{p},(\vec{r},s)}$, $\widetilde{\mathrm{BHT}}$ has a bounded extension $L^{p_1}_{w_1}(\mathbf{R}; X_1) \times L^{p_2}_{w_2}(\mathbf{R}; X_2) \to L^p_w(\mathbf{R}; X)$ with

$$\|\widetilde{\mathrm{BHT}}\|_{L^{p_1}_{w_1}(\mathbf{R};X_1)\times L^{p_2}_{w_2}(\mathbf{R};X_2)\to L^p_w(\mathbf{R};X)} \lesssim_{\vec{X},\vec{p},\vec{r},s} [\vec{w}]_{\vec{p},(\vec{r},s)}^{\max\Big\{\frac{\frac{1}{r_1}}{\frac{1}{r_1}-\frac{1}{p_1}}, \frac{\frac{1}{r_2}}{\frac{1}{r_2}-\frac{1}{p_2}}, \frac{\frac{1}{q}-\frac{1}{s}}{\frac{1}{p}-\frac{1}{s}}\Big\}}.$$

Note that in particular we find that for all $r_1, r_2, s \in (1,\infty)$ satisfying (9.3.6) and all $\vec{X} \in \mathrm{UMD}_{\vec{r},s}$ we have

$$\|\widetilde{\mathrm{BHT}}(f_1, f_2)\|_{L^p(\mathbf{R};X)} \lesssim_{\vec{X},\vec{p},\vec{r},s} \|f_1\|_{L^{p_1}(\mathbf{R};X_1)} \|f_2\|_{L^{p_2}(\mathbf{R};X_2)}$$

for all $f_j \in \mathscr{S}(\mathbf{R}; X_j)$.

We point out here that [BM17] actually proved the vector-valued sparse domination (9.3.7) in the cases where the X_j are iterated Lebesgue spaces with the same range of exponents we obtain (see Corollary 8.4.8), through the helicoidal method. It is worth to note that Theorem 9.3.5 extends the main result of [BM17] to our more general vector spaces by only using their scalar-valued sparse domination (9.3.5) as an input.

To end this section, we compare our results to the results obtained by Amenta and Uraltsev [AU19] and Di Plinio, Li, Martikainen, and Vuorinen [DLMV19]. In their works they prove vector-valued bounds for BHT for triples of complex Banach spaces (X_1, X_2, X_3) that are not necessarily Banach function spaces, but that are compatible in the sense that there is a bounded trilinear form $\Pi : X_1 \times X_2 \times X_3 \to \mathbf{C}$. Then the trilinear form $\mathrm{BHF}(f_1, f_2, f_3) := \langle \mathrm{BHT}(f_1, f_2) f_3 \rangle$ has the vector-valued analogue

$$\mathrm{BHF}_\Pi(f_1, f_2, f_3) := \int_{\mathbf{R}} \mathrm{p.v.} \int_{\mathbf{R}} \Pi(f_1(x-y), f_2(x+y), f_3(x)) \frac{\mathrm{d}y}{y} \,\mathrm{d}x,$$

whose boundedness properties can then be studied. We point out that the main result in [DLMV19] considers estimates for the same tuples of spaces as in [AU19], but for a larger range of exponents. Since our main interest is in the spaces, for simplicity we compare our result to the main result of [AU19]. To state the result we need to introduce the notion of intermediate UMD spaces. We say that a Banach space X is a u-intermediate UMD space for $u \in [2,\infty]$ if it is isomorphic to the complex interpolation space $[E, H]_{\frac{2}{u}}$, where E is a UMD space and H is a Hilbert space and the couple (E, H) is compatible. For $\vec{u} \in [2,\infty]^m$ We say that a tuple of Banach spaces $\vec{X}$ is $\vec{u}$-intermediate UMD if X_j is u_j-intermediate UMD for $1 \le j \le m$.

Theorem 9.3.6 ([AU19, Theorem 1.1])**.** *Let $\vec{u} \in [2,\infty]^3$, let $\vec{X}$ be a triple of $\vec{u}$-intermediate Banach spaces, and let $\Pi : X_1 \times X_2 \times X_3 \to \mathbf{C}$ be a bounded trilinear form. For all $p_1, p_2 \in (1,\infty)$ with $p \in (1,\infty)$ satisfying*

$$1 < \frac{1}{u_1} \min\Big\{\frac{u_1'}{p_1'}, 1\Big\} + \frac{1}{u_2} \min\Big\{\frac{u_2'}{p_2'}, 1\Big\} + \frac{1}{u_3} \min\Big\{\frac{u_3'}{p}, 1\Big\}, \tag{9.3.8}$$

we have

$$|\mathrm{BHF}_\Pi(f_1, f_2, g)| \lesssim \|f_1\|_{L^{p_1}(\mathbf{R};X_1)} \|f_2\|_{L^{p_2}(\mathbf{R};X_2)} \|g\|_{L^{p'}(\mathbf{R};X_3)} \tag{9.3.9}$$

for all $f_j \in \mathscr{S}(\mathbf{R}; X_j)$, $g \in \mathscr{S}(\mathbf{R}; X_3)$.

Even though we are not able to recover any of their results for spaces that are not Banach function spaces, in the setting of Banach function spaces our results go much beyond theirs. Indeed, consider a pair of complex quasi-Banach function spaces (X_1, X_2)

over (Ω,μ). Then we define

$$\Pi : X_1 \times X_2 \times X^* \to \mathbf{C}, \quad \Pi(f_1, f_2, g) := \int_\Omega f_1 f_2 g \,\mathrm{d}\mu.$$

By an application of Fubini's Theorem, we find that for all $f_j \in \mathscr{S}(\mathbf{R}; X_j)$, $g \in \mathscr{S}(\mathbf{R}; X^*)$ we have

$$\begin{aligned} |\mathrm{BHF}_\Pi(f_1, f_2, g)| &= \left| \int_\mathbf{R} \int_\Omega \mathrm{BHT}(f_1(\cdot,\omega), f_2(\cdot,\omega))(x) g(x,\omega) \,\mathrm{d}\mu(\omega) \,\mathrm{d}x \right| \\ &\leq \|\widetilde{\mathrm{BHT}}(f_1, f_2) g\|_{L^1(\mathbf{R}; L^1(\Omega))}. \end{aligned} \tag{9.3.10}$$

This means that the sparse domination result in Theorem 9.3.5 combined with Proposition 8.2.1 implies that whenever $r_1, r_2, s \in (1,\infty)$ satisfy (9.3.6) and $\vec{X} \in \mathrm{UMD}_{\vec{r},s}$, we obtain (9.3.9) for all $\vec{p} \in (0,\infty]^2$ with $\vec{r} < \vec{p}$ and $p < s$, as well as weighted bounds.

Since intermediate UMD spaces are themselves UMD spaces, any of our results where X_1 or X_2 is not UMD improve on Theorem 9.3.6 in the function space setting. This includes examples such as $X_1 = L^\infty(\Omega)$, $X_2 = L^2(\Omega)$, or $X_1 = \ell^2(\ell^\infty)$, $X_2 = \ell^\infty(\ell^2)$, see Corollary 8.4.8.

Next, let $\vec{t} \in (0,\infty]^2$ with $\vec{r} < \vec{t}$, $1 \leq t < s$ and consider the case

$$X_1 = L^{t_1}(\Omega), \quad X_2 = L^{t_2}(\Omega), \quad X^* = L^{t'}(\Omega).$$

Then by (9.3.10) and Theorem 9.3.5 with $q = 1$ we obtain

$$|\mathrm{BHF}_\Pi(f_1, f_2, g)| \lesssim \|f_1\|_{L^{p_1}(\mathbf{R}; L^{t_1}(\Omega))} \|f_2\|_{L^{p_2}(\mathbf{R}; L^{t_2}(\Omega))} \|g\|_{L^{p'}(\mathbf{R}; L^{t'}(\Omega))} \tag{9.3.11}$$

for all $f_j \in \mathscr{S}(\mathbf{R}; L^{t_j}(\Omega))$, $g \in \mathscr{S}(\mathbf{R}; L^{t'}(\Omega))$ and $\vec{p} \in (0,\infty]^2$ with $\vec{r} < \vec{p}$, $p < s$. This is beyond the reach of Theorem 9.3.6, as Theorem 9.3.6 does not include Lebesgue space over non-atomic measure spaces because of the restrictions in (9.3.8), see [AU19, Example 6.2.3].

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SUMMARY

The subject of this thesis is the study of the multilinear Muckenhoupt weight classes and the quantitative boundedness of operators with respect to these weights in both the scalar-valued and the vector-valued setting. This includes the study of multisublinear Hardy-Littlewood maximal operators, sparse forms, and multilinear Rubio de Francia extrapolation methods.

After giving a historical overview of the theory in the first part, in the second part we introduce the limited range multilinear Muckenhoupt weight classes and the corresponding weight constants. We show that these weight classes are characterized by the boundedness of the multisublinear Hardy-Littlewood maximal operator and sparse forms, and we obtain the sharp dependence of these bounds in terms of the weight constants. We also define multilinear reverse Hölder and Fujii-Wilson constants and prove a self-improvement property of the multilinear Muckenhoupt weight classes. Finally, we prove an abstract quantitative multilinear limited range extrapolation theorem which allows us to extrapolate from weighted estimates that include the cases where some of the exponents are infinite. To this end we develop a multilinear analogue of the Rubio de Francia algorithm adapted to the multisublinear maximal operator.

In the third part we prove weighted bounds for multi(sub)linear operators satisfying sparse domination by using the sharp extrapolation theorem to extend quantitative estimates obtained from sparse domination in the Banach space setting to the quasi-Banach space setting. We provide a criterium on the unweighted operator norm of the operators which ensures that the obtained bounds are sharp. Moreover, we obtain vector-valued estimates for Lebesgue spaces including L^∞. As a corollary, we obtain multilinear extrapolation results for some upper and lower endpoints estimates in weak-type and BMO spaces. We apply our results to multilinear Calderón-Zygmund operators and the bilinear Hilbert transform.

In the fourth part we introduce a multilinear and limited range analogues of the UMD condition for tuples of quasi-Banach function spaces and prove vector-valued bounds for operators in these spaces through two separate methods. The first is through a multilinear limited range version of Rubio de Francia's vector-valued extrapolation theorem. Through the second method we show that if an operator has scalar-valued sparse domination, then this operator has a vector-valued extension satisfying vector-valued sparse domination with respect to our tuples of spaces satisfying the multilinear UMD condition. For the proof of this result, we introduce the multisublinear Hardy-Littlewood lattice maximal operator and define a rescaled multilinear analogue of the Hardy-Littlewood property for tuples of quasi-Banach function spaces. We show that if a tuple of quasi-Banach function spaces has this property, then the multisublinear Hardy-

Littlewood maximal operator satisfies vector-valued sparse bounds in these spaces, which is the main ingredient in proving that scalar-valued sparse domination implies vector-valued sparse domination. Finally, we apply our results to multilinear Calderón-Zygmund operators and the bilinear Hilbert transform.

SAMENVATTING

Het onderwerp van dit proefschrift is de studie van multilineaire Muckenhoupt gewichtsklassen en kwantitatieve begrensdheid van operatoren in termen van deze gewichten in het scalair en vectorwaardige geval. Hiervoor komen onder andere multisublineaire Hardy-Littlewood maximaal operatoren, schaarse vormen, en multilineaire Rubio de Francia extrapolatie methoden aan bod.

Na het geven van een historisch overzicht in het eerste deel, geven we een introductie over de begrensde bereik multilineaire Muckenhoupt gewichtsklassen en de bijbehorende gewichts constanten in het tweede deel. We laten zien dat deze gewichtsklassen gekarakteriseerd worden door de begrensdheid van de multisublineaire Hardy-Littlewood maximaal operator en de schaarse vormen. We tonen een scherpe afhankelijkheid van deze grens aan in termen van de gewichts constanten. Ook definiëren we multilineaire omgekeerde Hölder en Fujii-Wilson constanten en bewijzen we een zelfverbeterings eigenshcap van de multilineaire gewichtsklassen. Tot slot bewijzen we een abstracte kwantitatieve multilineaire begrensde bereik extrapolatie stelling waardoor we gewogen afschattingen kunnen extrapoleren in onder andere de gevallen waar een aantal van de exponenten oneindig zijn. Om dit te doen ontwikkelen we een multilineaire analogon van het Rubio de Francia algoritme die aangepast is op de multisublineaire maximaal operator.

In het derde deel bewijzen we gewogen afschattingen voor multi(sub)lineaire operatoren die aan schaarse dominatie voldoen door gebruik te maken van de scherpe extrapolatie stelling om kwantitatieve afschattingen in het Banach bereik naar het quasi-Banach bereik uit te breiden. We bepalen een criterium op de ongewogen operator normen van de operatoren die ervoor zorgt dat deze verkregen afschattingen scherp zijn. Verder verkrijgen we vectorwaardige afschattingen voor Lebesgue ruimtes waaronder L^∞. Als gevolg krijgen we eindpunt extrapolatie stellingen voor zwakke Lebesgue en BMO ruimtes. We passen ons resultaat toe op multilineaire Calderón-Zygmund operatoren en de bilineaire Hilbert transform.

In het vierde deel introduceren we een multilineaire begrensde bereik analogon van de UMD eis voor tupels van quasi-Banach ruimtes en bewijzen we vectorwaardige afschattingen voor operatoren in deze ruimtes door middel van twee verschillende methoden. De eerste methode maakt gebruik van een multilineaire begrensde bereik versie van Rubio de Francia's vectorwaardige extrapolatie stelling. Via de tweede methode laten we zien dat als een operator scalairwaardige schaarse dominatie heeft, dan heeft deze operator een vectorwaardige uitbreiding met vectorwaardige schaarse dominatie voor tupels van ruimtes die voldoen aan de multilineaire UMD eis. Voor het bewijs van dit resultaat introduceren we de multisublineaire Hardy-Littlewood rooster maximaal operator en definiëren we een hergeschaalde multilineaire analogon van de Hardy-Littlewood eigenschap voor tupels van quasi-Banach functie ruimtes. We laten zien dat als een tupel van quasi-Banach functie ruimtes deze eigenschap heeft, dan heeft de multisublineaire Hardy-Littlewood maximaal operator schaarse begrensdheid in deze ruimtes, wat het hoofdingrediënt is om aan te tonen dat scalairwaardige schaarse dominatie vectorwaardige schaarse dominatie impliceert. Tot slot passen we onze resultaten toen op multilineaire Calderón-Zygmund operatoren en de bilineaire Hilbert transform.

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Writing the acknowledgements for this dissertation is something I've put off for months now. As a matter of fact, at the time of writing this I am already working as a postdoc in Bilbao. I've been stuck in my apartment here a lot thanks to the pandemic that is going on. Perhaps this is a good opportunity to reflect on the past four years. As it is impossible to write about everyone, I will just mention some of the people and related stories that have left an impression. My apologies to anyone I have left out.

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Bilbao, November 2020

Zoe Nieraeth

Curriculum Vitæ

Zoe Nieraeth

Zoe Nieraeth was born in Maarssen, the Netherlands on February 11, 1992. She completed her secondary education in 2011. From 2011 to 2014 she studied for her B.Sc. degree in Mathematics at Utrecht University and wrote a thesis on 'The spectrum of the Dirichlet Laplacian' under the supervision of prof. dr. E.P. van den Ban. She obtained her B.Sc. degree 'cum laude' in 2014. From 2014 to 2016 she studied for her M.Sc. degree in Mathematical Sciences with specialization in Pure Analysis at Utrecht University and wrote a thesis 'Iwaniec's conjecture on the Beurling-Ahlfors transform' under the supervision of prof. dr. ir. M.C. Veraar from the Delft University of Technology. She obtained her M.Sc. degree 'cum laude' in 2016. In the same year she started her PhD research under the daily supervision of prof. dr. D. Frey at the Delft University of Technology. In 2019 she relocated to the Karlsruhe Institute of Technology to continue her PhD research under the supervision of prof. dr. D. Frey.

List of Publications

(1) D. Frey and B. Nieraeth. Weak and Strong Type A_1–A_∞ Estimates for Sparsely Dominated Operators. *Journal of Geometric Analysis*, 29(1):247–282, 2019.

(2) E. Lorist and B. Nieraeth. Vector-valued extensions of operators through multilinear limited range extrapolation. *Journal of Fourier Analysis and Applications*, 25(5):2608–2634, 2019.

(3) B. Nieraeth. Quantitative estimates and extrapolation for multilinear weight classes. *Mathematische Annalen*, 375(1-2):453–507, 2019.

(4) E. Lorist and Z. Nieraeth. Sparse domination implies vector-valued sparse domination. arXiv:2003.02233, 2020.