Axiomatic Projective Geometry

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April 2019

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Preface

Projective geometry is, in essence, a geometry in which parallel lines do not exist. In this way, it contrasts itself from Euclidean and hyperbolic geometry, and this difference causes many interesting results. In this paper, projective geometry will first be extensively introduced, before several definitions are introduced, often unique to this geometry, and many different theorems will be shown and proven.
To do so, it is important that the reader has some experience with common strategies for creating mathematical and geometrical proofs. This experience is assumed in this paper; strategies are not explained any further and instead simply applied.
It is also expected that the reader has some experience in axiomatic mathematics, preferably geometry. In a way, we will be going into this paper with no prior knowledge on the projective plane other than the axioms, so having an understanding as to how such a system works is crucial.
Lasty, experience in linear algebra is assumed. While it may not be obvious at first glance, the projective plane has deep links to the Euclidean three-dimensional space, so understanding the math and theorems that are applied is important.

This paper it is split up into four chapters. In the first, projective geometry will be introduced and defined, the second will describe and prove many theorems, the third will introduce projective maps and show many of their properties, and finally the fourth chapter will apply all we have learned to a new notion called harmonic additions.

This research was done as a final project in the bachelor Applied Mathematics at the Delft University of Technology, in collaboration with Dr. J. Vermeer. My thanks goes out to him.

Contents

1 Introduction to projective geometry

This chapter will be focused on introducing the basic concepts behind the projective plane. This is done through first defining it in the first section, introducing a core principle in the second, and finally showing what finite projective planes look like and how they function in the third.

1.1 Definitions and axioms

The very first thing we must do is to define a projective geometry. After all, as is always the case in math, we cannot talk about something we have not clearly defined. However, before we can do that, we'll have to define an axiomatic theory first:

Definition 1.1. An *axiomatic theory* is described by:

1. a system of fundamental notions $(P_1, P_2, ...)$,
2. a set of axioms about the fundamental notions.

Now, this may look very confusing, so we will illustrate it with a well-known example: our very own real number system. This is, after all, nothing more than an axiomatic theory. In this case, our system of fundamental notions are $(\mathbb{R}, +, *)$, in other words, they consist of the set of real numbers, the operation of addition, and the operation of multiplication. Meanwhile. our set of axioms consists of the well-known addition axioms, multiplication axioms, the order axiom, and the completeness axiom.
So now that we have a sense for what an axiomatic theory is, we can define our most important term:

Definition 1.2. A *projective geometry* is an axiomatic theory with the triple (Π, Λ, I) as its fundamental notions and axioms 1.1, 1.2, and 1.3 as the axioms. Here, Π and Λ are disjoint sets and I is a symmetric relation between Π and Λ (in other words, $a \ I \ b \Leftrightarrow b \ I \ a$ where $a \in \Pi$ and $b \in \Lambda$). The elements of Π are called 'points', the elements of Λ are called 'lines', and $a \ I \ b$ is read as 'a is incident with b'.

Now, again, this looks confusing, but effectively all it says is that in other to define a projective geometry, we need a set of points, a set of lines, and an incidence relation, as well as the following three axioms:

Axiom 1.1. *Given two distinct points, there is exactly one line incident with both points.*

Axiom 1.2. *Given two distinct lines, there is exactly one point incident with both lines.*

Axiom 1.3. Π *contains at least four points such that no three of them are incident with one and the same line.*

Before we move on, it should be noted that this axiomatic system is not unique; there is a wide variety of equivalent sets of axioms for projective geometry, but these 3 are the ones we will be working with.
Now, so far, we have talked about abstract sets for points and lines as well as an abstract relation for incidence.. The reason for this is that, often, we do not need to define these exactly to prove theorems. However, if we do, and the found sets and relation satisfy the axioms, we call it a *projective plane*. From here on out, we will often denote a projective plane as $\mathscr{P}$. In the case where we are talking about multiple different projective planes, we will use a subscript such as $\mathscr{P}_1$ or $\mathscr{P}_2$ to differentiate between them.
Next, we will provide a different way of defining lines. Note that every line l is determined uniquely by the set of points incident with it. If we call this set $\mathbb{P}$, then we can see there is no problem in saying $l = \mathbb{P}$. As such, from here on out, we will see lines as nothing more than sets of points. With this, we can also write '$P \in l$' for '$P\ I\ l$' and '$P \notin l$' for 'not $P\ I\ l$', where P is a point and l is a line. In addition to this, we can now denote the point of intersection for two different lines l and m as $l \cap m$; by axiom 1.2, this is exactly one point.
To continue, we will introduce a few more definitions:

Definition 1.3. Three (or more) points P_1, P_2, and P_3 are called *collinear* if and only if there is a line l such that $P_1, P_2, P_3 \in l$. In other words, there is a line such that all three points are incident with this line.

Definition 1.4. Three (or more) lines l_1, l_2, l_3 are called *concurrent* if and only if there is a point P such that $P \in l_1$, $P \in l_2$, and $P \in l_3$. In other words, there is a point such that all three lines are incident with this point.

Definition 1.5. Two projective planes $\mathscr{P}_0 = (\Pi_0, \Lambda_0, I_0)$ and $\mathscr{P}_1 = (\Pi_1, \Lambda_1, I_1)$ are called *isomorphic* if and only if there are one-to-one mappings $\pi : \Pi_0 \to \Pi_1$ and $\lambda : \Lambda_0 \to \Lambda_1$ such that $P \in l \Leftrightarrow \pi(P) \in \lambda(l)$. In other words, π and λ preserve incidence relations.

Finally, we have established almost every important definition. The final one will be discussed now.

1.2 Principle of duality

In the context of projective planes, duality refers to switching the words 'point' and 'line' in theorems, or interchanging the sets Π and Λ (recall that these are the sets of points and lines respectively). Before discussing what exactly this means, we will first show a proof: namely the proof of the dual theorem of axiom 1.3, or the theorem found by interchanging the words 'point' and 'line' in that axiom:

Theorem 1.1 (Dual theorem of axiom 1.3)**.** *Λ contains at least four lines such that no three of them are concurrent.*

Proof. By axiom 1.3, we can find four points such that no three of them are collinear. Clearly, these four points are different, so we will call them P_1, P_2, P_3, and P_4. By axiom 1.1, the lines P_1P_2, P_1P_3, P_2P_4, and P_3P_4 are unique. We will show no three of these are concurrent.
By way of contradiction, assume that P_1P_2, P_1P_3, and P_3P_4 are all incident with a point Q. Then, P_1P_2 and P_1P_3 are both incident with Q and with P_1. Since these lines are not the same (recall that P_1, P_2, and P_3 are not collinear), we must find $Q = P_1$ by axiom 1.2. But then P_3P_4 is incident with P_1, which means P_1, P_3, and P_4 are collinear, which is a contradiction. Thus, P_1P_2, P_1P_3, and P_3P_4 are not concurrent. This can be proven analogously for the other triples. □

So now, we have proven the dual theorem of axiom 1.3, and clearly, axioms 1.1 and 1.2 are dual to each other. This is very important, as what we have now shown is that for all of the axioms, their dual theorem is true. Thus now, for any theorem we proof using the axioms, we can also prove its dual theorem. This is known as the principle of duality:

Theorem 1.2 (Principle of duality)**.** *If, in a theorem that can be proven using the axioms of projective geometry, we interchange the words 'point' and 'line', we obtain another theorem that is true in projective geometry.*

Proof. Let P be a theorem that is true in a system with axioms 1.1, 1.2, and 1.3. Note that, in that system, theorem 1.1 is true. Therefore, we can use axioms 1.2 and 1.1 as well as theorem 1.1 to prove dP, the theorem dual to P. □

This is a very useful result, and we will illustrate using a simple example:

Theorem 1.3. *For each line l there are at least three points incident with l.*

Proof. We begin with a line l. By axiom 1.3, there exist 4 points P_1, P_2, P_3, and P_4 such that no three of them are collinear. We will consider three cases:

1. l is incident with two of these points. Without loss of generality, say $P_1, P_2 \in l$. Then, by axiom 1.2, $P_3P_4 \cap l$ exists. This point cannot be P_1 or P_2, because that would imply collinearity between that point and P_3 and P_4, which is a contradiction. Thus, we have found three points on l.

2. l is incident with one of these points. Without loss of generality, assume $P_1 \in l$. Then, l intersects P_2P_3, P_3P_4, and P_2P_4 by axiom 1.2. None of these points of intersection can be P_1, as otherwise we have a contradictory collinearity. In addition, no two of these lines can have the same point of intersection, as that would imply they are either the same line or have multiple points of intersection, both of which are contradictions. Thus, we have found four points on l.

3. l is incident with none of these points. In that case, by the fact that these points cannot be collinear and $P_1 \notin l$, the points $P_1P_2 \cap l$, $P_1P_3 \cap l$, $P_1P_4 \cap l$ are all different (after all, the lines that are intersecting l are all different, and they all intersect in P_1, which must be the only point of intersection by axiom 1.2). Thus, we have found three points on l.

□

This is a useful theorem to have, of course, but more importantly is how easily we can apply the principle of duality to it:

Theorem 1.4 (Dual theorem of theorem 1.3). *For each point P there are at least three lines incident with P.*

Thanks to the principle of duality, *we do not have to provide a proof to this.* Simply having the dual theorem is enough.
To close out this paragraph, I would like to make note of another result of the principle of duality:

Theorem 1.5. *For any projective plane $\mathscr{P} = (\Pi, \Lambda, I)$, its dual plane $d\mathscr{P} = (\Lambda, \Pi, I)$ is also a projective plane.*

1.3 Finite Projective Plane

In this paragraph, we will showcase a special case of projective planes, namely finite ones. A well-known one is Fano's Plane, which contains 7 points and 7 lines:

Definition 1.6. *Fano's Plane* is a projective plane with:

- $\Pi = \{A, B, C, D, E, F, G\}$
- $\Lambda = \{ADB, AGE, AFC, BEC, BGF, CGD, FDE\}$

The incidence relation is as expected.

See figure 1 for an image of what this plane looks like. Now, all of the axioms are easily verified: pick any two points and there will be a line through both, any two lines intersect in some point, and of the points A, C, and F, and G, no three are collinear.
Now, let us define the order of a finite projective plane:

Definition 1.7. A finite projective plane has *order* n if and only if there is at least one point such that there are exactly $n+1$ lines incident with that point.

Clearly, Fano's Plane is of order 2. After all, every point has 3 lines incident with it. Interestingly, this is not a coincidence, as we will now prove in two steps:

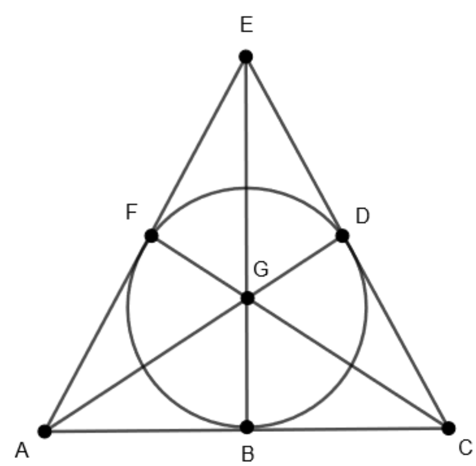

Figure 1: Fano's Plane

Theorem 1.6. *In a projective plane $\mathscr{P}$ of order n, every point has exactly $n+1$ lines incident with it.*

Proof. Let P be an arbitrary point in $\mathscr{P}$ and let l be a line with $n+1$ points on it. Label these points $P_1, ..., P_{n+1}$. We recognize two cases:

1. P is not on l. Then for every P_i, there is a line through P and P_i and since P is not on l, each of these are both distinct from l and from each other. After all, if there was a line that goes through P, P_i, and P_j, then that line would be distinct from l and still go through P_i and P_j, which violates uniqueness in axiom 1.1. Now, we have $n+1$ distinct lines going through P.

2. P is on l. Now, by axiom 1.3, there exist points Q and R not on l. Bt axiom 1.1, lines RP_1, RP_2, and RP_3 exist. Since R is not on l, at least two of these lines do not contain P, and similarly, at least two of these lines do not contain Q. Then, there is at least one line m that does not contain P or Q.
   Since Q is not on l, the first case gives us that there are exactly $n+1$ lines $m_1, ..., m_{n+1}$ through Q. And since Q is not on m, m intersects each of these lines in exactly one point by axiom 1.2, which means there are $n+1$ points $S_1, ..., S_{n+1}$. There cannot be another point on m either; after all, if there was another point T on m, then the line TQ would be distinct from the m_i. But then there are $n+2$ lines through Q, which is a contradiction. Thus, we now have a line m with exactly $n+1$ points on it and that is not incident with P. We can now apply case 1 to see there are exactly $n+1$ lines through P.

□

Now, we can trivially say there is a point with $n+1$ lines through it, which is the dual statement to the definition of a projective plane of order n. Therefore, the principle of duality still holds. Thanks to that, we can easily say

Theorem 1.7. *In a projective plane $\mathscr{P}$ of order n, every line has $n+1$ points incident with it.*

Proof. This theorem follows directly from theorem 1.6 and the principle of duality. □

Finally, we will showcase a proof of how many points and lines there are exactly:

Theorem 1.8. *In a projective plane of order n, there are exactly n^2+n+1 points and n^2+n+1 lines.*

Proof. By axiom 1.3, there exists at least one point P, as well as some number of points distinct from P. For every point, there must be exactly one line through P for each point distinct from P. By theorem 1.6, there are exactly $n+1$ lines through P. Note that every point in the plane must be on one of these lines. By theorem 1.7, each of these lines has exactly n points on it other than P. Thus, the total amount of points is $n(n+1)+1 = n^2+n+1$. By the principle of duality, the total amount of lines is the same. □

2 Theorems in the projective plane

This chapter will be focused on introducing many famous theorems of the projective plane, several of which will see use in later chapters too. To do so, we must first introduce certain notions in the first section before moving onto the theorems in later sections.

2.1 Definitions

In this section, we will repeat a few definitions from algebra and soon relate them to the projective planes that we have established before. To begin, recall the definitions of fields and division rings:

Definition 2.1. A *field* is an axiomative theory with $(L, +, *)$ as the fundamental notions, where L is a set, $+$ is an addition operation, and $*$ is a multiplication operation, and the following axioms:

- $a + (b + c) = (a + b) + c$ and $a * (b * c) = (a * b) * c$
- $a + b = b + a$ and $a * b = b * a$
- There exist elements 0_L and 1_L of L such that $a + 0_L = a$ and $b * 1_L = b$
- For any a, there exists an element $-a$ of L such that $a + (-a) = 0_L$
- For any $a \neq 0$, there exists an element a^{-1} such that $a * a^{-1} = 1_L$
- $a * (b + c) = (a * b) + (a * c)$

Here, a, b, and c are arbitrary elements of L. Such a field is often written as simply L if the addition and multiplication operations are clear.

Definition 2.2. A *division ring* is an axiomative theory with the same fundamental notions and axioms as a field, with one exception: $a * b = b * a$ does not have to be true. In other words, multiplication is not commutative.

A common example given for a division ring that is not a field are the quaternions. As it is just an example, we will not dive too deep into it, but in short, the set of quaternions is a number system where each number is written as $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$ and i, j, and k are the fundamental quaternion units, defined by

$$i^2 = j^2 = k^2 = -1,\ ij = -ji = k,\ jk = -kj = i, \text{ and } ki = -ik = j$$

It is not hard to show the quaternions form a division ring, but from how multiplication is defined, it is clearly not commutative. Thus, the set of quaternions is not a field.
However, of course, there are quite a few division rings that are, in fact, fields. It should be clear that the set of fields is a subset of the set of division rings, but we can say slightly more about it too:

Theorem 2.1. *Every finite division ring is a field.*

We present this theorem without a proof, as the proof, as it was originally given by MacLagan Wedderburn in 1905, calls upon concepts that we do not wish to introduce within this paper. However, it is certainly useful, as it means we know more about the context of section 1.3.
Now that we have this theorem and an example of when division rings aren't fields, we know that division rings is a relevant definition. Next, we recall the notions of vector spaces and modules:

Definition 2.3. An *vector space* over a field L is an axiomative theory with $(V, +, *)$ as the fundamental notions, where this time, V is a set of so-called vectors, $+$ is vector addition and $*$ is scalar multiplication. The axioms are as follows:

- $X + Y = Y + X$
- $(X + Y) + Z = X + (Y + Z)$
- There is an element 0_V of V such that $X + 0_{L^n} = X$
- For every X, there is an element $-X$ of V such that $X + -X = 0_V$
- $r * (s * X) = (rs) * X$
- $(r + s) * X = r * X + s * X$
- $r * (X + Y) = (r * Y) + (r * X)$
- $1_L * X = X$

Here, X, Y, and Z are arbitrary elements of V, r and s are arbitrary elements of L, and 1_L is as described in the field axioms.

Definition 2.4. An *module* over a division ring M is the generalization of the notion of a vector field to division rings. Their fundamental notions and axioms are equivalent.

Quickly, we will present two simple theorems without proof, as they are both easily shown:

Theorem 2.2. *For any field L, $(L^n, +, *)$ is a vector field.*

Theorem 2.3. *For any division ring M, $(M^n, +, *)$ is a module.*

These will both be useful, and in general, when we are talking about a vector field over L or a module over M, these will be the ones we are considering.
It is assumed common definitions from linear algebra, such as *subspaces*, *dependence*, *linear combinations*, *spans*, and *bases*, are known and understood. Note that each of these definitions is equivalent when used in the context of modules. Now, before we move on, we will show and prove an incredibly important theorem in vector spaces as well as some of its corollaries, though we first begin with a lemma:

Lemma 2.1. *If $u_1, u_2, ...u_k \in V$ are linearly independent, then so are $u_1, u_2 - c_2 * u_1, ...u_k - c_k * u_1$, where $c_2, ...c_k$ are scalars.*

The proof of this lemma is very simple and will not be presented, as it is more important to move on towards the following theorem:

Theorem 2.4. *If $u_1, ..., u_{k+1}$ are $k + 1$ vectors such that they are all included in the span of k vectors $v_1, ..., v_k$, then $u_1, ..., u_{k+1}$ are linearly dependent.*

Proof. Since u_i is contained in the span of $v_1, ..., v_k$, we can write each of them as $u_i = a_{i1}v_1 + ... + a_{ik}v_k$. Now, by way of contradiction, assume that $u_1, ..., u_{k+1}$ are linearly independent. Then, clearly, none of the u_i are multiples of u_1. Because of that, we can subtract a multiple of u_1 from each of them. If we do this appropriately, we can eliminate the term with v_1 in the expression of all u_i with $i \geq 2$. So, to recap, we have now written the u_i in the form $u_i = b_{i2}v_2 + ... + b_{ik}v_k$ for $i \geq 2$. However, we can continue on like this, constantly eliminate a term from the sums. Then, eventually, we will reach a sequence with just two sums: $u_k = \lambda_{kk}v_k$ and $u_{k+1} = \lambda_{k+1k}v_k$. But then, clearly, u_k and u_{k+1} are linearly dependent. This is a contradiction. Therefore, the u_i must be linearly dependent. □

Corollary 2.4.1. *If a basis of a module V contains k vectors, then every basis of V contains k vectors. We call k the dimension of V.*

Corollary 2.4.2. *Let V be a module with dimension k. Then, a set of k vectors $\{u_1, ..., u_k\}$ is linearly independent if and only if $Span\{u_1, ..., u_k\} = V$*

The proofs of both corollaries are trivial.
Now that we have these theorems, we can finally link this theory to projective planes:

Definition 2.5. Let L be a division ring and V be a module over L with dimension 3. A *projective plane over* V, written as $\mathscr{P}(V)$, is the projective plane where:

- The set of points Π is the set of one-dimensional subspaces of V;
- The set of lines Λ is the set of two-dimensional subspaces of V;
- For $\pi \in \Pi$ and $\lambda \in \Lambda$, $\pi \in \lambda$ if and only if π is a subspace of λ.

It can easily be shown that $\mathscr{P}(V)$ satisfies the three axioms of projective geometry, meaning it is indeed a projective plane.

Having this is nice, because it lets us think about projective planes as something other than abstract bodies, but we can actually make models. After all, it's important to realize that all fields are division rings and thus all vector spaces are modules. Therefore, this definition works just fine for vector spaces such as, for instance $\mathbb{R}^3$, where our points take the form of lines through the origin and our lines are planes through the origin.

2.2 Desargue's Theorem

It is great that we have managed to define all of this, but now, we have to move on and try to find theorems that are true in projective planes over modules. Two of the most important ones are Desargue's Theorem and Pappus's Theorem:

Desargue's Theorem. *Let A_1, A_2, A_3, B_1, B_2, and B_3 be points with the following properties:*

- *The lines A_1B_1, A_2B_2, and A_3B_3 are concurrent. Name their point of intersection C.*
- *No three of the points C, A_1, A_2, and A_3 and no three of the points C, B_1, B_2, and B_3 are collinear*

Let $P_{12} = A_1A_2 \cap B_1B_2$, $P_{23} = A_2A_3 \cap B_2B_3$, and $P_{31} = A_3A_1 \cap B_3B_1$. Then, P_{12}, P_{23}, and P_{31} are collinear.

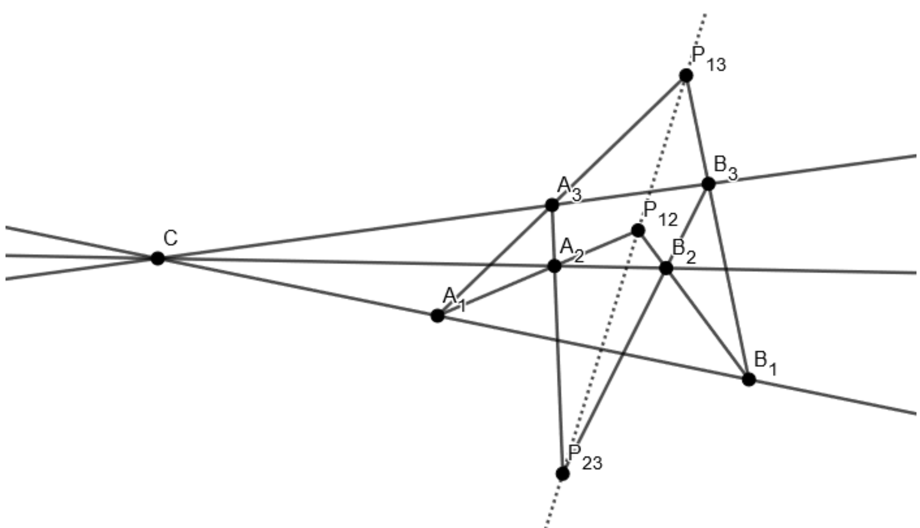

Figure 2: Desargue's Theorem

Now, as can be seen, we have not yet provided a proof for this theorem. The reason for that is that, unfortunately, Desargue's theorem does not hold in every projective plane. However, it does hold in the most common ones:

Theorem 2.5. *Let V be a module of dimension 3 over a division ring M. Then, Desargue's Theorem holds in $\mathscr{P}(V)$.*

Proof. Let V and M be as described and assume points A_1, A_2, A_3, B_1, B_2, and B_3 are as in the hypothesis for Desargue's Theorem. Now, remember that these points are one-dimensional vector spaces. This means there are vectors v_1, v_2, and v_3 such that $A_1 = \langle v_1 \rangle$, $A_2 = \langle v_2 \rangle$, and $A_3 = \langle v_3 \rangle$. In addition, since

A_1, A_2, and A_3 are not collinear, v_1, v_2, and v_3 are linearly independent. Thus, these three vectors span V and thus, C lies in the span of v_1, v_2, and v_3. Then, by definition, there are $a_1, a_2, a_3 \in M$ such that $C = a_1v_1 + a_2v_2 + a_3v_3$. Since we have that no three of A_1, A_2, A_3, and C are collinear, we have $a_1, a_2, a_3 \neq 0$, as otherwise, C would be a linear combination of two of the vectors, thus making it collinear with two of the points. Now, remember that $A_i = \langle v_i \rangle$. Because this is a span we're talking about, we can easily define $w_i = a_iv_i$ and say $A_i = \langle w_i \rangle$, meaning we get $C = \langle w_1 + w_2 + w_3 \rangle$.
We know that for $i \in \{1, 2, 3\}$, C, A_i, and B_i are collinear. Thus, there are $b_1, b_2, b_3 \in M$ such that

$$B_1 = \langle w_1 + w_2 + w_3 + b_1w_1 \rangle = \langle (b_1 + 1)w_1 + w_2 + w_3 \rangle$$

$$B_2 = \langle w_1 + (b_2 + 1)w_2 + w_3 \rangle$$

$$B_3 = \langle w_1 + w_2 + (b_3 + 1)w_3 \rangle$$

Now, we want to find the points P_{ij}. We begin with P_{12}:

$$P_{12} = A_1A_2 \cap B_1B_2 = \langle w_1, w_2 \rangle \cap \langle (b_1 + 1)w_1 + w_2 + w_3, w_1 + (b_2 + 1)w_2 + w_3 \rangle$$

Clearly, $\langle b_1w_1 - b_2w_2 \rangle$ is on both A_1A_2 and B_1B_2. Then, by axiom 1.2, this is the only point on both lines (as the lines are distinct). Therefore

$$P_{12} = \langle b_1w_1 - b_2w_2 \rangle$$

and similarly

$$P_{23} = \langle b_2w_2 - b_3w_3 \rangle$$

$$P_{31} = \langle b_3w_3 - b_1w_1 \rangle$$

Clearly, $P_{31} = -P_{12} - P_{23}$, meaning these three points are collinear. □

Interestingly, this theorem has another side to it, which is quite fascinating:

Theorem 2.6. *If Desargue's Theorem holds in a projective plane $\mathscr{P}$, then there is a module V such that $\mathscr{P} = \mathscr{P}(V)$.*

We will not provide a proof for this statement, as it is quite technical and is not fit for the scope of this paper.
Next, we will discuss the dual of Desargue's theorem:

Dual of Desargue's Theorem. *Let l_1, l_2, l_3, m_1, m_2, and m_3 be lines with the following properties:*

- *The points $P_1 = l_1 \cap m_1$, $P_2 = l_2 \cap m_2$, and $P_3 = l_3 \cap m_3$ are collinear. Call their common line n.*
- *No three of the lines n, l_1, l_2, and l_3 and no three of the lines n, m_1, m_2, m_3 are concurrent.*

Let $c_{12} = (l_1 \cap l_2)(m_1 \cap m_2)$*,* $c_{23} = (l_2 \cap l_3)(m_2 \cap m_3)$*, and* $c_{31} = (l_3 \cap l_1)(m_3 \cap m_1)$*. Then* c_{12}*,* c_{23}*, and* c_{31} *are concurrent.*

Now, this may sound exceptionally confusing, and that's understandable. However, in fact, it is no more than the converse of Desargue's Theorem! You can check this for yourself too, but in the end, the point of intersection between the line c_{12}, c_{23}, and c_{31} is equivalent to the point C from Desargue's Theorem. Now, interestingly, we will show the following:

Theorem 2.7. *If Desargue's Theorem holds in a projective plane, then so does the dual of Desargue's Theorem.*

Proof. Let l_1, l_2, l_3, m_1, m_2, m_3, P_1, P_2, P_3, n, c_{12}, c_{23}, and c_{31} be as described. Let C be the point of intersection between c_{23} and c_{31}, which exists by axiom 1.2. Call $A_{ij} = l_i \cap l_j$ and $B_{ij} = m_i \cap m_j$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Note that, by definition, $A_{ij}, B_{ij} \in c_{ij}$. See also figure 3. Now, we will apply Desargue's theorem, but we have to determine which points we use for this. First, P_3 will be the point of intersection between the concurrent lines (called C in Desargue's Theorem). Next, P_2, A_{23}, and B_{23} will play the roles of the A_i from Desargue's Theorem, while P_1, A_{31}, and B_{31} are the B_i. It is easily checked that these points fulfill the conditions from Desargue's theorem.
Now, clearly, $A_{23}B_{23} \cap A_{31}B_{31} = c_{23} \cap c_{31} = C$. Furthermore, it is not hard to see that $P_2A_{23} \cap P_1A_{31} = A_{12}$ and $P_2B_{23} \cap P_1B_{31} = B_{12}$. By Desargue's Theorem, these are collinear. But then $c_{12} = A_{12}B_{12}$ goes through C, which was defined as $c_{23} \cap c_{31}$! Therefore, c_{12}, c_{23}, and c_{31} all go through C and are thus concurrent. □

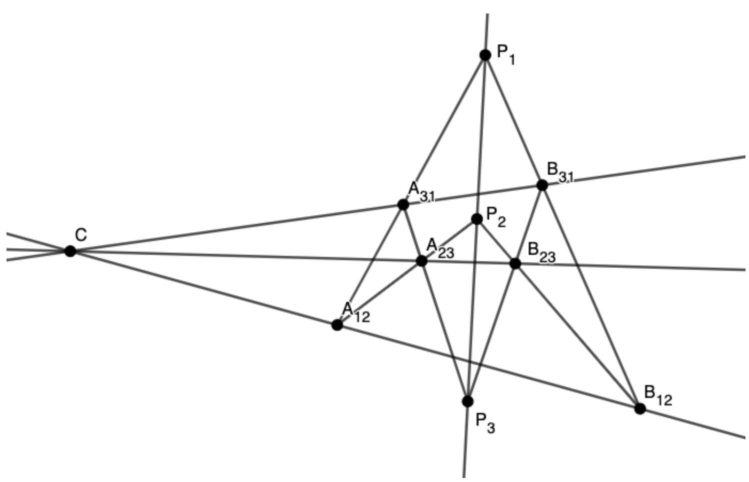

Figure 3: Dual of Desargue's Theorem

It is great to see that the Dual of Desargue's Theorem holds if Desargue's Theorem does, because that means that the Principle of Duality also holds

in that case. This is important, as otherwise, we'd already have ruined that important principle. Luckily, we have not, and so we can continue working with it.
Now, a question may arise now as to in what sort of plane Desargue's Theorem does not hold. After all, we know it holds in the most regular ones we know, vector spaces, so what would a plane look like where it does not? For that, we turn towards the so-called Moulton Plane:

Definition 2.6. The *Moulton Plane* is a projective plane with:

- $\Pi = \mathbb{R}^2$
- $\Lambda = (\mathbb{R} \cup \{\infty\} \times \mathbb{R}$
- Let $\pi = (x, y) \in \Pi$ and $\lambda = (m, b) \in \Lambda$. Then

$$\pi I \lambda \Leftrightarrow \begin{cases} x = b & \text{if } m = \infty \\ y = \frac{1}{2}mx + b & \text{if } m \leq 0, x \leq 0 \\ y = mx + b & \text{if } m > 0 \text{ or } x > 0 \end{cases}$$

Of course, at first glance, this means nothing. The incidence relation is very complicated and thus very unintuitive. Thus, in figure 4, an image is shown. Here, you can see that as lines cross the y-axis, the bend somewhat. It is this bend that causes Desargue's Theorem not to hold in the Moulton Plane: the line through two of the P_{ij} would bend away before reaching the third, thus causing a lack of collinearity.

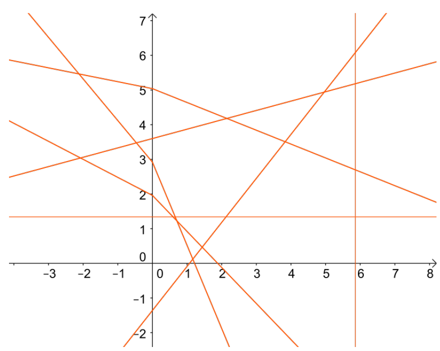

Figure 4: Moulton's Plane
Source: By Kmhkmh - Own work, CC BY 4.0,
https://commons.wikimedia.org/w/index.php?curid=53588728

Clearly, Moulton's Plane is an infinite plane. But what about finite planes, like the ones in section 1.3? Well, as it turns out, it is possible to construct finite projective planes where Desargue's Theorem doesn't hold. The catch is that this is only possible for planes of order 9 or higher. However, we do not provide a proof of this, as it goes beyond the scope of this paper.

2.3 Pappus's Theorem

In the last section, we mentioned one of two important and well-known theorems in projective planes. In this section, we will discuss the other:

Pappus's Theorem. *Let l and m be distinct lines. Let A_1, A_2, and A_3 be distinct points on l, while B_1, B_2, and B_3 are distinct points on m. These 6 points are also all different from $l \cap m$. Let $P_{12} = A_1B_2 \cap B_1A_2$, $P_{23} = A_2B_3 \cap B_2A_3$, and $P_{31} = A_3B_1 \cap B_3A_1$. Then, P_{12}, P_{23}, and P_{31} are collinear.*

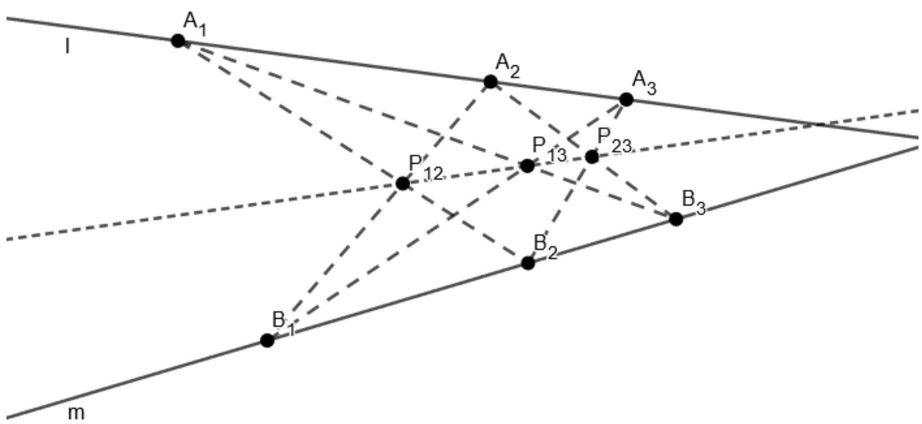

Figure 5: Pappus's Theorem

Theorem 2.8. *Let V be a module of dimension 3 over a division ring M. Then Pappus's Theorem holds on $\mathscr{P}(V)$ if and only if M is a field.*

Proof. Let V and M be as in the theorem and let the points A_1, A_2, A_3, B_1, B_2, and B_3 as well as lines l and m be as in the hypothesis for Pappus's Theorem. We begin by only looking at C, A_1, A_2, B_1, and B_2, where $C = l \cap m$.
Let $u, v, w \in V$ such that $C = \langle u \rangle$, $A_1 = \langle v \rangle$, and $B_1 = \langle w \rangle$. By definition, $A_2 = \langle u + av \rangle$ and $B_2 = \langle u + bw \rangle$ with $a, b \in M \setminus \{0\}$. Through appropriate rescaling, we can set $A_1 = \langle v' \rangle$, $A_2 = \langle u + v' \rangle$, $B_1 = \langle w' \rangle$, and $B_2 = \langle u + w' \rangle$.
Let $p, q \in M$. Since we are aiming to show that M is a field, we must show commutativity of multiplication. Since 0 and 1 commute with any element of M, we can safely assume $p, q \neq 0, 1$. Define $A_3 = u + pv'$ and $B_3 = u + qw'$. Since $p, q \neq 0, 1$, $A_3 \neq A_1, A_2, C$ and $B_3 \neq B_1, B_2, C$.
Claim: The points P_{12}, P_{23}, and P_{31} are collinear if and only if $pq = qp$.
To do so, we must first compute expressions for the points P_{ij}.

$$P_{12} = A_1B_2 \cap B_1A_2 = \langle v', u + w' \rangle \cap \langle w', u + v' \rangle = \langle u + v' + w' \rangle$$

$$P_{31} = A_3B_1 \cap B_3A_1 = \langle u + pv', w' \rangle \cap \langle u + qw', w' \rangle = \langle u + pv' + qw' \rangle$$

$$P_{23} = A_2B_3 \cap B_2A_3 = \langle u+v', u+qw'\rangle \cap \langle u+w', u+pv'\rangle$$
$$= \langle (p+(p-1)(q-1)^{-1})u + pv' + (p-1)(q-1)^{-1}qw'$$

This last one seems to come out of nowhere, so we will do a quick calculation to show it is correct:

$$(p+(p-1)(q-1)^{-1})u + pv' + (p-1)(q-1)^{-1}qw'$$
$$= p(u+v) + (p-1)(q-1)^{-1}(u+qw') \in \langle u+v', u+qw'\rangle$$

and

$$(p+(p-1)(q-1)^{-1})u + pv' + (p-1)(q-1)^{-1}qw'$$
$$= (p+(p-1)(q-1)^{-1} + (p-1)(q-1)^{-1}q - (p-1)(q-1)^{-1}q)u$$
$$+pv' + (p-1)(q-1)^{-1}qw'$$
$$= (p+(p-1)(q-1)^{-1}(1-q) + (p-1)(q-1)^{-1}q)u + pv' + (p-1)(q-1)^{-1}qw'$$
$$= (p-(p-1) + (p-1)(q-1)^{-1}q)u + pv' + (p-1)(q-1)^{-1}qw'$$
$$= (1+(p-1)(q-1)^{-1}q)u + pv' + (p-1)(q-1)^{-1}qw'$$
$$= (u+pv') + ((p-1)(q-1)^{-1}q)(u+w') \in \langle u+w', u+pv'\rangle$$

By axiom 1.2, this is the only point on both lines and therefore equal to P_{23}. Now, we first show what collinearity of these three points would mean:

$$P_{23} \in P_{12}P_{31}$$
$$\Leftrightarrow$$
$$(p+(p-1)(q-1)^{-1})u + pv' + (p-1)(q-1)^{-1}qw' \subseteq \langle u+v'+w', u+pv'+qw'\rangle$$
$$\Leftrightarrow$$

There exist $x, y \in M$ such that

$$(p+(p-1)(q-1)^{-1})u+pv'+(p-1)(q-1)^{-1}qw' = xu+xv'+xw'+yu+ypv'+yqw'$$
$$\Leftrightarrow$$

The following equations hold:

$$p+(p-1)(q-1)^{-1} = x+y$$
$$p = x+yp$$
$$(p-1)(q-1)^{-1}q = x+yq$$

Now, we must solve these equations. From the first one we get $x = p+(p-1)(q-1)^{-1} - y$, which allows the second one to give us

$$y = (p-1)(q-1)^{-1}(1-p)^{-1}$$

and thus

$$x = p+(p-1)(q-1)^{-1}(1-(1-p)^{-1})$$

Using these results and the third equation, we can finally find our equivalence after a long manipulation of formulae:

$$P_{23} \in P_{12}P_{31}$$

$$\Leftrightarrow$$

$$(p-1)(q-1)^{-1}q =$$
$$= p + (p-1)(q-1)^{-1}(1-(1-p))^{-1} + (p-1)(q-1)^{-1}(1-p)^{-1}q$$

$$\Leftrightarrow$$

$$q = (q-1)(p-1)^{-1}p + (1-(1-p)^{-1}) + (1-p)^{-1}q$$

$$\Leftrightarrow$$

$$(1-p)q = (1-p)(q-1)(p-1)^{-1}p + (1-p) - 1 + q$$

$$\Leftrightarrow$$

$$(1-p)q =$$
$$= (1-p)(q-1)(p-1)^{-1}p + (1-p)(q-1)(p-1)^{-1} - (1-p)(q-1)(p-1)^{-1} - p + q$$

$$\Leftrightarrow$$

$$(1-p)q = (1-p)(q-1)(p-1)^{-1}(p-1) + (1-p)(q-1)(p-1)^{-1} - p + q$$

$$\Leftrightarrow$$

$$(1-p)q = (1-p)(q-1) - p + q + (1-p)(q-1)(p-1)^{-1}$$

$$\Leftrightarrow$$

$$0 = (p-1) - p + q + (1-p)(q-1)(p-1)^{-1}$$

$$\Leftrightarrow$$

$$1 - q = (1-p)(q-1)(p-1)^{-1}$$

$$\Leftrightarrow$$

$$(1-q)(p-1) = (1-p)(q-1) \Leftrightarrow p - 1 - qp + q = q - 1 - pq + p$$

$$\Leftrightarrow$$

$$qp = pq$$

Since p and q are arbitrary, this is true for all $p, q \in M$. Therefore, Pappus's theorem holds if and only if M is a field. □

Now, it is important to point out that Pappus's Theorem and Desargue's Theorem are not independent. In fact:

Theorem 2.9. *If Pappus's Theorem holds in a projective plane, then so does Desargue's Theorem.*

As of right now, we do not have everything we need to prove this theorem. As such, the proof will come in a future section. For now, based on what we already know, we can actually say something else:

Theorem 2.10. *If $\mathscr{P}$ is a finite projective plane and Desargue's Theorem holds in $\mathscr{P}$, then so does Pappus's Theorem.*

Proof. From theorem 2.6, we know $\mathscr{P} = \mathscr{P}(V)$ for some module V and division ring M. But then M is a finite division ring, and thus a field by theorem 2.1. Finally, by theorem 2.8, Pappus's Theorem holds. □

Interestingly, this proof relies entirely on the algebra we've used earlier, which may feel weird for a paper on geometry. The reason for this, however, is that there is currently no geometrical proof known for theorem 2.10.
To conclude this section, let us take a look at the theorem dual to Pappus and show their relation, in a similar manner to what we did for Desargue's Theorem.

Dual to Pappus's Theorem. *Let P and Q be distinct points. Let l_1, l_2, and l_3 be distinct lines through P, while m_1, m_2, and m_3 are distinct lines through Q. Let these six lines also be different from PQ. Let $n_{12} = (l_1 \cap m_2)(l_2 \cap m_1)$, $n_{23} = (l_2 \cap m_3)(l_3 \cap m_2)$, and $n_{31} = (l_3 \cap m_1)(l_1 \cap m_3)$. Then n_{12}, n_{23}, and n_{31} are concurrent.*

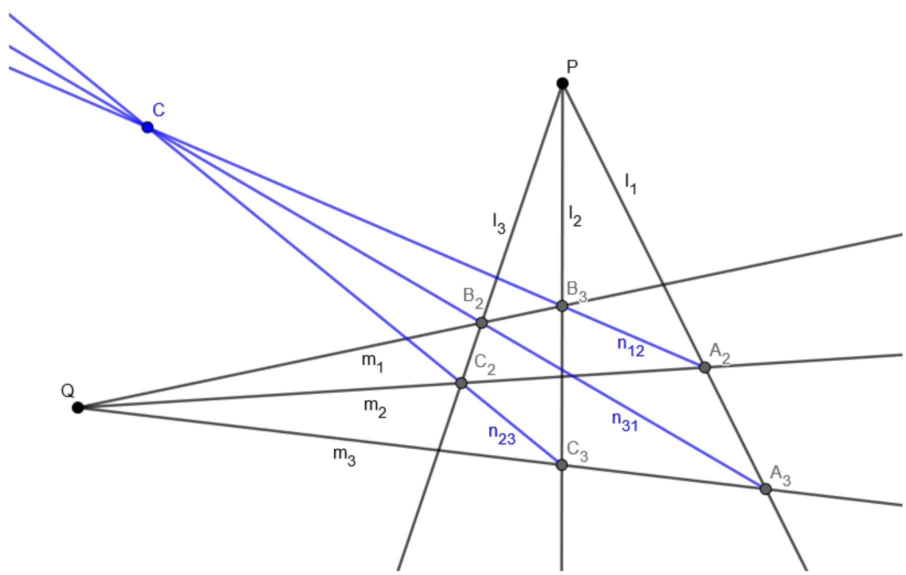

Figure 6: Dual to Pappus's Theorem

Theorem 2.11. *If Pappus's Theorem holds in a projective plane, then so does the Dual to Pappus's Theorem.*

Proof. Let $A_2 = l_1 \cap m_2$, $A_3 = l_1 \cap m_3$, $B_2 = l_3 \cap m_1$, and $B_3 = l_2 \cap m_1$, as in figure 6. In addition, let $C = n_{12} \cap n_{31} = A_2B_3 \cap A_3B_2$. Now, we apply Pappus's Theorem to the lines l_1 and m_1, with the points P, A_2, and A_3 on l_1 and Q, B_2, and B_3 on m_1. We find $C_2 = PB_2 \cap A_2Q = l_3 \cap m_2$ and $C_3 = PB_3 \cap A_3Q = l_2 \cap m_3$. By Pappus's Theorem, C, C_2, and C_3 are collinear. But $C_2C_3 = n_{23}$. Since this line is unique by axiom 1.1, $C \in n_{23}$. Since C was defined as $n_{12} \cap n_{31}$, we have that n_{12}, n_{31}, and n_{23} are concurrent. □

Finally, we will present a different way to state the Dual to Pappus's Theorem:

Pappus's Hexagon Theorem. *Let $ABCDEF$ be a hexagon. If the lines AB, CF, and DE are concurrent, and the lines BC, AD, and EF are concurrent, then the lines AF, BE, and CD are concurrent.*

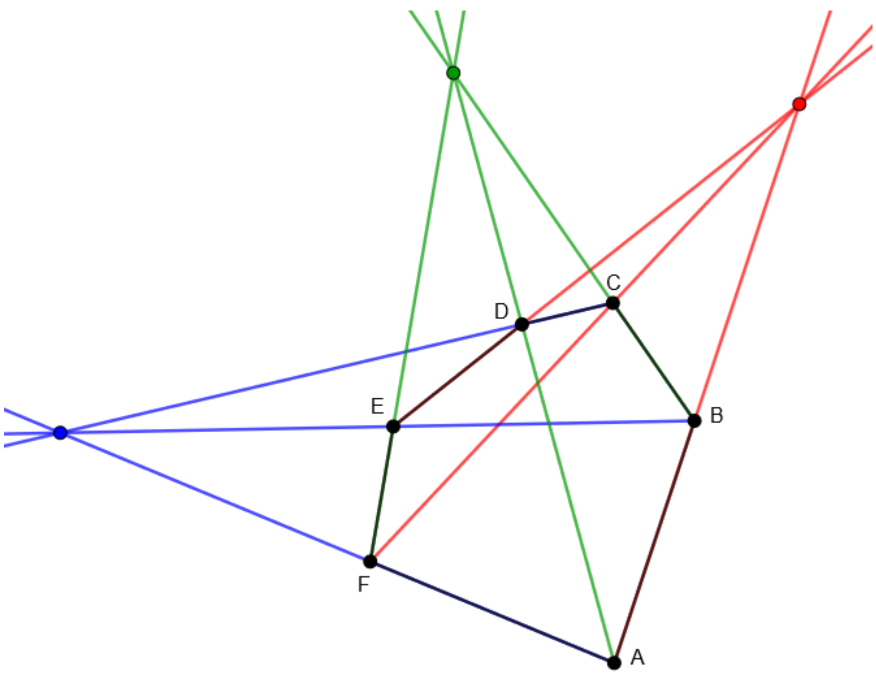

Figure 7: Pappus's Hexagon Theorem

It is very easily shown that this theorem is equivalent with the dual to Pappus's Theorem. See also figure 7 for an aid in visualizing what it means.

2.4 Fano's Axiom

Next, we will present another theorem, this time about quadrilaterals. While we all have an intuition regarding what a quadrilateral is, it is important to define it clearly:

Definition 2.7. A *quadrilateral $ABCD$* consists of four points A, B, C, and D, no three of which are collinear, and the four lines AB, BC, CD, and DA.

It is important to note that the order of writing matters: the quadrilateral $ABCD$ is different from the quadrilateral $BADC$.
With that out of the way, we will need another definition:

Definition 2.8. The *diagonal points* of a quadrilateral $ABCD$ are the three points $X_1 = AB \cap CD$, $X_2 = AC \cap BD$, and $X_3 = AD \cap BC$.

Intuitively, the diagonal points as described here are simply the points of intersection of the lines of the quadrilateral, excluding A, B, C, and D.
Now, with these few definitions written, we can move on to presenting Fano's Axiom:

Fano's Weak Axiom. *In a projective plane, there exists a quadrilateral $ABCD$ such that its diagonal points are non-collinear.*

Fano's Strong Axiom. *For any quadrilateral $ABCD$ in a projective plane, its diagonal points are non-collinear.*

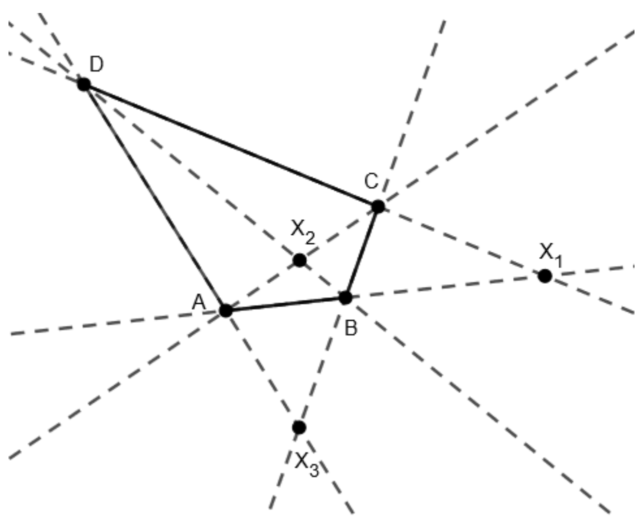

Figure 8: Fano's Axiom

We introduce the theorem in two steps, as that is how it was originally presented. It should be clear that Fano's Strong Axiom implies Fano's Weak Axiom, though we will return to the reverse implication later.
You may notice the discrepancy between the name of the theorem, which implies it is an axiom, and its status as a theorem. The reason for this is that in certain geometries, Fano's Axiom is used as an axiom. In this paper, however, Fano's

Axiom will be seen as a theorem, as it is not always true, instead requiring a certain condition:

Theorem 2.12. *Let V be a module over a division ring M. Then Fano's Strong Axiom holds in $\mathscr{P}(V)$ if and only if $1+1 \neq 0$ in M.*

Proof. Let V and M be as described. To start off, notice that the theorem is equivalent to
Fano's Axiom does not hold in $\mathscr{P}(V)$ if and only if $1+1=0$ in M.
This is easier to prove, and therefore will be what we go for instead.
First, let $ABCD$ be a quadrilateral $ABCD$ with diagonal points X_1, X_2, and X_3. Let v_1, v_2, v_3, and v_4 be vectors in V so that $A = \langle v_1 \rangle$, $B = \langle v_2 \rangle$, $C = \langle v_3 \rangle$, and $D = \langle v_4 \rangle$. Then, through appropriate rescaling, we can get

$$X_1 = \langle v_1 + v_2 \rangle = \langle v_3 + v_4 \rangle$$

$$X_2 = \langle v_1 - v_3 \rangle = \langle v_4 - v_2 \rangle$$

$$X_3 = \langle v_1 - v_4 \rangle = \langle v_3 - v_2 \rangle$$

Now that we have these coordinates, we will prove the theorem in both directions. Assume $1+1=0$. Then $1=-1$ and

$$X_1 = \langle v_1 + v_2 \rangle$$

$$X_2 = \langle v_1 + v_3 \rangle$$

And

$$X_2 - X_1 = \langle v_3 - v_2 \rangle = X_3$$

Thus, the three diagonal points are collinear.

Next, assume the three points are on a line. Then, for some $a_1, a_2 \in M$

$$X_3 = a_1 X_1 + a_2 X_2$$

$$v_3 - v_2 = a_1(v_1 + v_2) + a_2(v_1 - v_3)$$

$$(a_1 + a_2)v_1 + (a_1 + 1)v_2 + (-a_2 - 1)v_3 = 0$$

Since A, B, and C are, by definition of a quadrilateral, non-collinear, we have v_1, v_2, and v_3 linearly independent. Thus, by definition

$$a_1 + 1 = 0 \Rightarrow a_1 = -1$$

$$-a_2 - 1 = 0 \Rightarrow a_2 = -1$$

$$a_1 + a_2 = 0 \Rightarrow -1 - 1 = 0 \Rightarrow 1 + 1 = 0$$

□

Note that the first part of this proof implies that if $1 + 1 = 0$, then for any quadrilateral, its diagonal points must be on a line. However, the second part of this proof implies that if there exists a quadrilateral with the diagonal points on a line, then $1 + 1 = 0$. From that, we can conclude

Theorem 2.13. *Let V be a module over a division ring M. Then, in $\mathscr{P}(V)$, Fano's Weak Axiom implies Fano's Strong Axiom. In other words, if there is a quadrilateral $ABCD$ with non-collinear diagonal points, all quadrilaterals have non-collinear diagonal points.*

The proof for this requires no more work than we have already done.
Now, we have a nice equivalence relation between Fano's Weak and Strong Axioms in projective planes over modules, but we cannot say anything about other projective planes. In fact, it is currently still an open question whether or not Fano's Weak Axiom implies Fano's Strong Axiom in all projective planes.
Before moving on, let us, just as with the theorems of Desargue and Pappus, take a look at the dual theorem to Fano's Strong Axiom:

Dual to Fano's Strong Axiom. *Let $ABCD$ be a quadrilateral in a projective plane. Let us rename the lines of the quadrilateral as $l_1 = AB$, $l_2 = BC$, $l_3 = CD$, and $l_4 = AD$. Now, we define the diagonal lines as $x_1 = (l_1 \cap l_2)(l_3 \cap l_4) = BD$, $x_2 = (l_1 \cap l_3)(l_2 \cap l_4)$, and $x_3 = (l_1 \cap l_4)(l_2 \cap l_3) = AC$. Then, x_1, x_2, and x_3 are non-concurrent.*

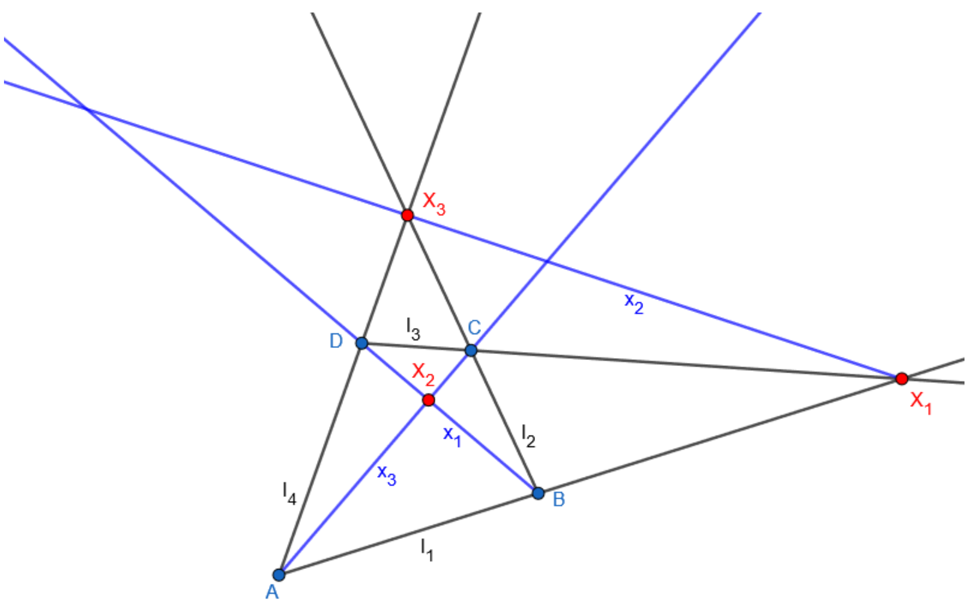

Figure 9: Dual to Fano's Strong Axiom

Theorem 2.14. *If Fano's Strong Axiom holds in a projective plane, then so does the dual theorem to Fano's Strong Axiom.*

Proof. Let $ABCD$ and its diagonal lines x_1, x_2, and x_3 be as defined in the dual theorem to Fano's Strong Axiom. Also, let X_1, X_2, and X_3 be the diagonal points of $ABCD$. By definition, $X_2 = AC \cap BD = x_1 \cap x_3$. Thus, if the diagonal lines are concurrent, then x_2 must go through X_2. On the other hand, if x_2 does not go through X_2, then the diagonal lines are non-concurrent.
Now, take note that $X_1 = AB \cap CD = l_1 \cap l_3$ and $X_3 = AD \cap BC = l_2 \cap l_4$. Thus, by definition, $x_2 = X_1X_3$. By Fano's Strong Axiom, the diagonal points of $ABCD$ are non-collinear. Thus, $X_2 \notin x_2$ and the diagonal lines are non-concurrent. □

2.5 Theorems of Menelaus and Ceva

In this section, we will showcase two more theorems, as well as proofs for both. Throughout this section, V is a module of dimension 3 over a division ring M, and we are discussing the theorems within the context of $\mathscr{P}(V)$.

Theorem 2.15 (Theorem of Menelaus)**.** *Let $P = \langle u \rangle$, $Q = \langle v \rangle$, and $R = \langle w \rangle$ be three non-collinear points. Let $P' = \langle v + aw \rangle$ be a point on QR, $Q' = \langle w + bu \rangle$ be a point on RP, and $R' = \langle u + cv \rangle$ be a point on PQ. Then, P', Q', and R' are collinear if and only if $abc = -1$.*

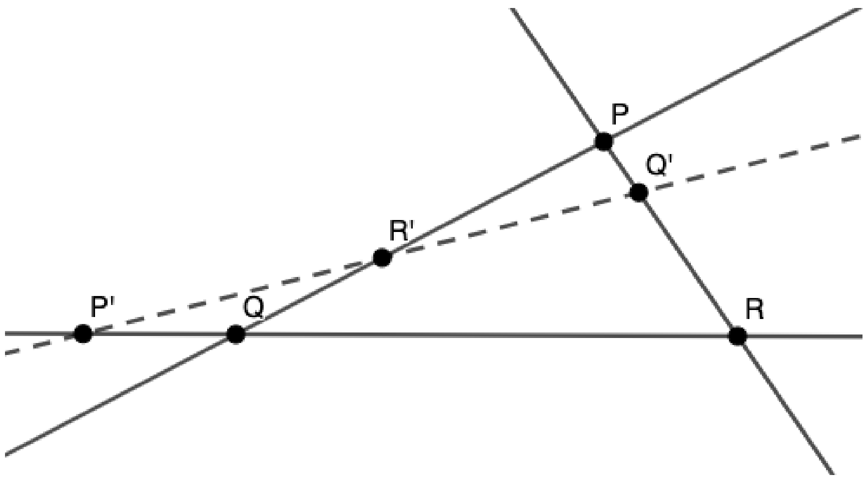

Figure 10: Theorem of Menelaus

Proof. Note that, if P', Q', and R' are collinear, we have $v + aw = p(w + bu) + q(u + cv)$. Because P, Q, and R are non-collinear, we have u, v, and w independent. Thus, we find

$$P', Q', \text{ and } R' \text{ are collinear}$$

$$\Leftrightarrow v = qcv,\ aw = pw,\ 0 = (pb + q)u$$

$$\Leftrightarrow q = c^{-1},\ p = a,\ pb + q = 0$$

$$\Leftrightarrow ab + c^{-1} = 0$$

$$\Leftrightarrow ab = -c^{-1}$$

$$\Leftrightarrow abc = -c^{-1}c = -1$$

□

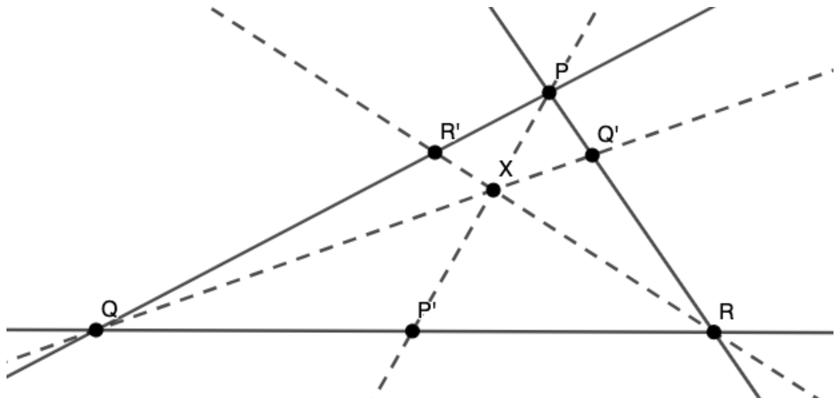

Figure 11: Theorem of Ceva

Theorem 2.16 (Theorem of Ceva)**.** *Let $P = \langle u \rangle$, $Q = \langle v \rangle$, and $R = \langle w \rangle$ be three non-collinear points. Let $P' = \langle v + aw \rangle$ be a point on QR, $Q' = \langle w + bu \rangle$ be a point on RP, and $R' = \langle u + cv \rangle$ be a point on PQ. Then, the lines PP', QQ', and RR' are concurrent if and only if $abc = 1$.*

Proof. Let $X = \langle x \rangle$ be the point of intersection between PP' and QQ', which exists and is unique by axiom 1.2. Then, for some scalars p_1, p_2, q_1, q_2, we have $p_1 u + p_2(v + aw) = x = q_1 v + q_2(w + bu)$. Now, note that RR' goes through X if and only if, for some scalars r_1, r_2, $x = r_1 w + r_2(u + cv)$. Thus, we get

$$PP', QQ', \text{ and } RR' \text{ are concurrent}$$
$$\Leftrightarrow p_1 u = q_2 bu = r_2 u,\ p_2 v = q_1 v = r_2 cv,\ p_2 aw = q_2 = r_1$$
$$\Leftrightarrow p_2 = p_1 c,\ q_2 = p_2 a,\ p_1 = q_2 b$$
$$\Leftrightarrow p_1 = p_1 cab$$
$$\Leftrightarrow cab = 1$$
$$\Leftrightarrow ab = c^{-1}$$
$$\Leftrightarrow abc = c^{-1} c = 1$$

□

What makes both of these theorems interesting is that they have a way of being described in Euclidean Geometry as well; this is not true for the theorems of Pappus and Desargue, as they rely on axiom 1.2, but Ceva and Menelaus do not, at least no heavily, meaning we get:

Remark 2.1 (Theorem of Menelaus in Euclidean Geometry)**.** *Let ΔABC be a triangle and let $D \in BC$, $E \in AC$, and $F \in AB$, with all three distinct from A, B, and C. Then, D, E, and F are collinear if and only if*

$$\frac{AF}{FB} * \frac{BD}{DC} * \frac{CE}{EA} = -1$$

Here, we use signed lengths of segments.

Remark 2.2. *Let ΔABC be a triangle and let $D \in BC$, $E \in AC$, and $F \in AB$, with all three distinct from A, B, and C. Then, AD, BE, and CF are concurrent if and only if*

$$\frac{AF}{FB} * \frac{BD}{DC} * \frac{CE}{EA} = 1$$

Here, we use signed lengths of segments.

To conclude, it should be noted that the theorems of Menelaus and Ceva are very close to being each other's dual theorems, with only the difference between 1 and -1 being in the way, as well as some of the wording. This wording is much more convenient, however, as writing the dual of the theorem of Menelaus results in unintuitive use of spans to indicate lines, while being equivalent with the Theorem of Ceva.

3 Perspectives and projective maps

This chapter will consider the notions of perspectives and projective maps, which will be defined in the first section. Afterwards, the next two sections will focus on stating and proving several incredibly important theorems.

3.1 Definitions

As stated, this section will define perspectives and projective maps. However, each of these take into account one very important fact that we will restate here again before moving on: lines can be seen as sets of points. This is crucial, as you will see very quickly. If this weren't the case, our definitions would not make sense.
Without further ado, let us introduce the two most important terms for this chapter:

Definition 3.1. Let l and m be lines and let A be a point that not incident with l or m. The *perspective from l to m through A* is the function $\sigma_A : l \longrightarrow m$ such that for every $B \in l$, $\sigma_A(B) = AB \cap m$.

In figure 12, an example of a perspective is given. Clearly, $\sigma_A(B_1) = C_1$ and $\sigma_A(B_3) = C_3$. However, note what is happening at $B_2 = l \cap m$. Clearly, $\sigma_A(B_2) = AB_2 \cap m = B_2$, meaning it is a fixed point. This is not a coincidence:

Theorem 3.1. *For any perspective $\sigma_A : l \longrightarrow m$, $l \cap m$ is a fixed point.*

The proof of this is trivial.
This theorem will prove important later on, but for now, let us simply move on to the next important definition for the time being:

Definition 3.2. A *projective map of order n* is a composition of n perspectives.

This seems simply and not particularly useful, but as we will see, it is quite nice to have a name for this.

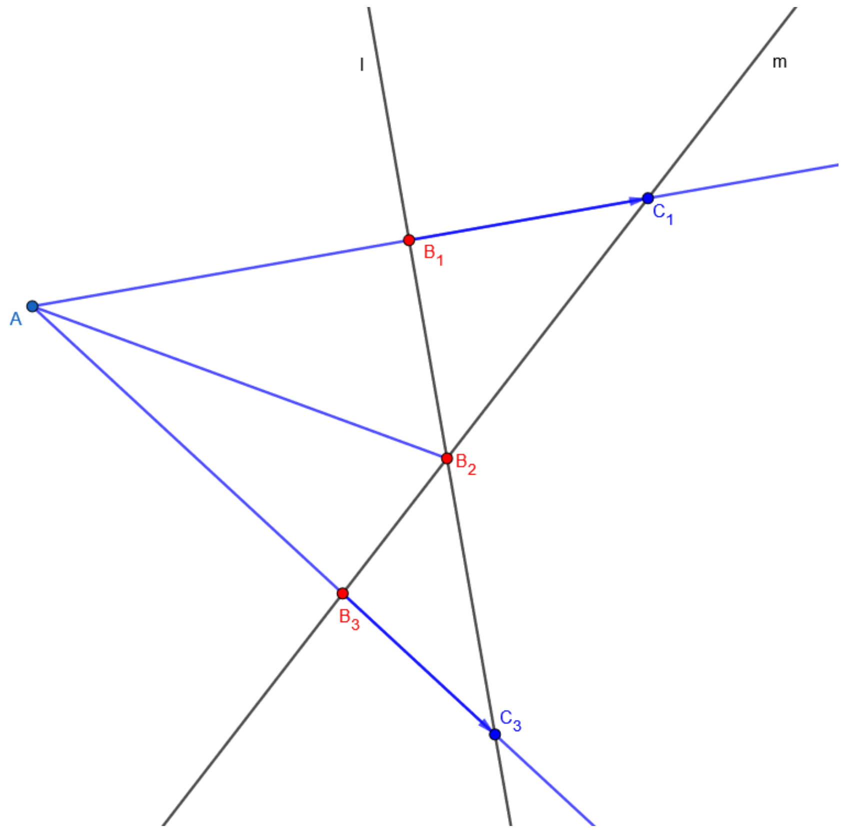

Figure 12: The perspective $\sigma_A : l \longrightarrow m$

3.2 Double Perspective Theorem

In this section, we will introduce a certain theorem that will immediately show why we defined projective maps, but more importantly, we will show its relation to the theorems of Desargue and Pappus and, finally, prove theorem 2.9, which states that Pappus's Theorem implies Desargue's Theorem.
Of course, we will need several steps to get there, so first, let's look at the Double Perspective Theorem:

Double Perspective Theorem. *We will split this theorem into two cases:*

1. *Let l, m, and n be three non-concurrent lines and let $\sigma : l \longrightarrow n$ be a projective map of order 2 with $\sigma = \sigma_A \circ \sigma_B$ for some perspectives $\sigma_A : l \longrightarrow m$ and $\sigma_B : m \longrightarrow n$, such that $l \cap n$ is a fixed point for σ. Then there is a point C such that $\sigma = \sigma_C$, where $\sigma_C : l \longrightarrow n$ is the perspective from l to n through C.*

2. *Let l, m, and n be three concurrent lines and let $\sigma : l \longrightarrow n$ be a projective*

map of order 2 with $\sigma = \sigma_A \circ \sigma_B$ for some perspectives $\sigma_A : l \longrightarrow m$ and $\sigma_B : m \longrightarrow n$, such that $l \cap n$ is a fixed point for σ. Then there is a point C such that $\sigma = \sigma_C$, where $\sigma_C : l \longrightarrow n$ is the perspective from l to n through C.

In other words, any projective map $\sigma : l \longrightarrow n$ of order 2 with $l \cap n$ a fixed point is a perspective.

Do not let the separate cases confuse you: in the end, this theorem says little more than the last line does. The reason for splitting it up is for use in the proof that Pappus's Theorem implies Desargue's Theorem. For notational purposes, from here in out, we will refer to the Double Perspective Theorem as 'DPT'. In addition, if we require one of the two cases, we will write it as 'DPT.1' or 'DPT.2' respectively.
So, why is this important at all? At first glance, this theorem seems completely separate from the ones we've shown before, focusing entirely on projective maps and perspectives instead of lines, points, and collinearity. However, as we will see, this is not the case at all:

Theorem 3.2. *In a projective plane, DPT.1 holds if and only if Pappus's Theorem does.*

Proof. As this is an 'if and only if'-statement, we will need to prove it both ways. As such, we'll split it in two:

1. First, assume DPT.1 holds. We will prove Pappus's Theorem does as well. To start off, choose l, m, A_1, A_2, A_3, B_1, B_2, B_3, P_{12}, P_{23}, and P_{31} such that the hypothesis of Pappus's Theorem is satisfied. We will consider the perspectives $\sigma_{A_1} : B_1A_2 \longrightarrow m$ and $\sigma_{A_3} : m \longrightarrow A_2B_3$, as well as the projective map $\sigma = \sigma_{A_1} \circ \sigma_{A_3}$ of order 2. It should be clear that the 3 lines B_1A_2, m, and A_2B_3 are non-concurrent, since $A_2 \notin m$. In addition, $\sigma_{A_1}(A_2) = l \cap m$ and $\sigma_{A_3}(l \cap m) = A_2$, meaning $A_2 = B_1A_2 \cap A_2B_3$ is a fixed point for σ. Thus, by DPT.1, there is a point S such that $\sigma = \sigma_S : B_1A_2 \longrightarrow A_2B_3$.
   Next, our goal will be to find where S is. To accomplish that, let's take a look at what happens to P_{12} when we apply σ to it. Clearly, $\sigma_{A_1}(P_{12}) = B_2$ and $\sigma_{A_3}(B_2) = P_{23}$. Thus, by definition of a perspective, $S \in P_{12}P_{23}$.
   Now, to continue on, we will define two more points: $Q_1 = A_1B_3 \cap B_1A_2$ and $Q_2 = B_1A_3 \cap A_2B_3$. It is easily checked that we now have $\sigma_S(Q_1) = B_3$ and $\sigma_S(B_1) = Q_2$. But then S is on $Q_1B_3 = A_1B_3$ and on $B_1Q_2 = B_1A_3$. By axiom 1.2 and the definition of P_{31}, the only point that satisfies both of these requirements is P_{31}. And now, as we have found earlier, $P_{31} = S \in P_{12}P_{23}$. Thus, P_{12}, P_{23}, and P_{31} are collinear.

2. Now, we assume Pappus's Theorem holds and will use this to prove DPT.1. Let there be two perspectives $\sigma_{A_1} : l_1 \longrightarrow l_2$ and $\sigma_{A_3} : l_2 \longrightarrow l_3$ such that for $A_2 = l_1 \cap l_3$ and $\sigma = \sigma_{A_3} \circ \sigma_{A_1}$, we have $\sigma(A_2) = A_2$. In addition, let l_1, l_2, and l_3 be non-concurrent. Then, clearly, both A_1 and A_3 are on the

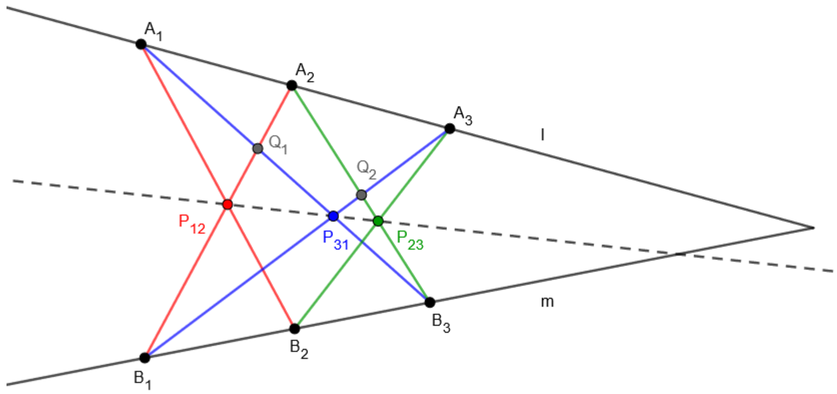

Figure 13: DPT.1 implies Pappus's Theorem

line between A_2 and $\sigma_A(A_2)$, so A_1, A_2 and A_3 are collinear.
Next, let us define some points: $B_1 = l_1 \cap l_2$, $B_3 = l_2 \cap l_3$, and $P_{31} = A_1B_3 \cap A_3B_1$. Then choose an arbitrary point P_{12} on l_1 and let $B_2 = A_1P_{12} \cap l_2$, such that $\sigma_{A_1}(P_{12}) = B_2$, and let $P_{23} = A_2B_3 \cap A_3B_2 = l_3 \cap A_3B_2$, such that $\sigma_{A_3}(B_2) = P_{23}$. Now we have $\sigma(P_{12}) = P_{23}$.
Now, we can use Pappus's Theorem to say P_{12}, P_{23}, and P_{31} are on one line, so if we define $\sigma_{P_{23}} : l_1 \longrightarrow l_3$, we get $\sigma_{P_{31}}(P_{12}) = P_{23}$. But since P_{12} was chosen arbitrarily, we can conclude that for every point $C \in l_1$, we get $\sigma_{P_{31}}(C) = \sigma(C)$. Thus, $\sigma = \sigma_{P_{31}}$.

□

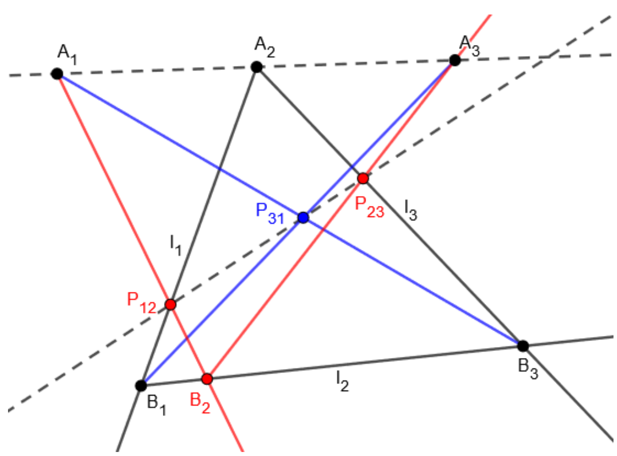

Figure 14: Pappus's Theorem implies DPT.1

This alone is already special, but of course, we aren't done. Because in order to work with Pappus's and Desargue's Theorem, we need to have a link to the latter:

Theorem 3.3. *In a projective plane, DPT.2 holds if and only if Desargue's Theorem does.*

Proof. As before, we have an 'if and only if'-statement, so we will split this in two.

1. First, assume DPT.2 holds. We will show Desargue's Theorem. Let l_1, l_2, l_3, A_1, A_2, A_3, B_1, B_2, B_3, C, P_{12}, P_{23}, and P_{31} be such that the conditions for Desargue's Theorem are fulfilled. Consider $\sigma_{P_{12}} : l_1 \longrightarrow l_2$ and $\sigma_{P_{23}} : l_2 \longrightarrow l_3$. By DPT.2, there is a point S such that $\sigma_S = \sigma_{P_{23}} \circ \sigma_{P_{12}}$, since clearly $\sigma_{P_{23}} \circ \sigma_{P_{12}}(C) = C$ and l_1, l_2, and l_3 are concurrent. Note that $\sigma_S(A_1) = A_3$ and $\sigma_S(B_1) = B_3$. Thus $S = A_1A_3 \cap B_1B_3 = P_{31}$. Now, let $Q_1 = P_{12}P_{23} \cap l_1$ and $Q_3 = P_{12}P_{23} \cap l_3$. Then, clearly, $\sigma_S(Q_1) = Q_3$, so $P_{31} = S \in Q_1Q_3 = P_{12}P_{23}$. Thus, P_{12}, P_{23}, and P_{31} are collinear.

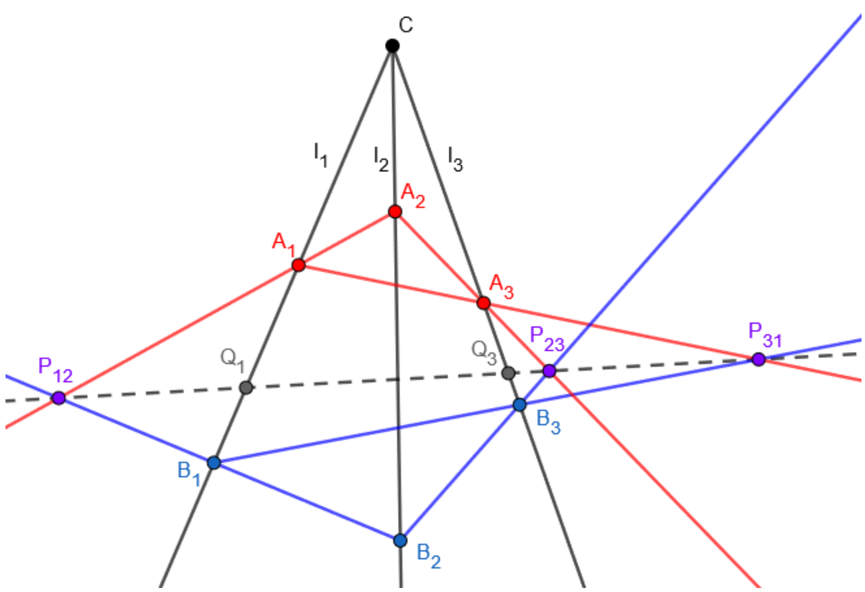

Figure 15: DPT.2 implies Desargue's Theorem

2. Next, assume Desargue's Theorem holds. We will show DPT.2. To start, we look at two perspectives, $\sigma_F : l_1 \longrightarrow l_2$ and $\sigma_E : l_2 \longrightarrow l_3$, where l_1, l_2, and l_3 are concurrent. If DPT.2 holds, then there is some S such that $\sigma_S = \sigma_E \circ \sigma_F$.
   To show this, we fix $A_1 \in l_1$ and call $A_2 = \sigma_F(A_1)$ and $A_3 = \sigma_E(A_2)$. We also set $S = EF \cap A_1A_3$. Now, we pick an arbitrary $B_1 \in l_1$, with

$B_2 = \sigma_F(B_1)$ and $B_3 = \sigma_E(B_2)$. By applying Desargue's Theorem to the triangles $\Delta A_1A_2A_3$ and $\Delta B_1B_2B_3$, we see that regardless of our choice of B_1, EF goes through $A_1A_3 \cap B_1B_3$. This also means that B_1B_3 goes through $EF \cap A_1A_3 = S$. Thus, $\sigma_S(B_1) = B_3 = (\sigma_E \circ \sigma_F)(B_1)$ for any $B_1 \in l_1$. Thus, $\sigma_S = \sigma_E \circ \sigma_F$.

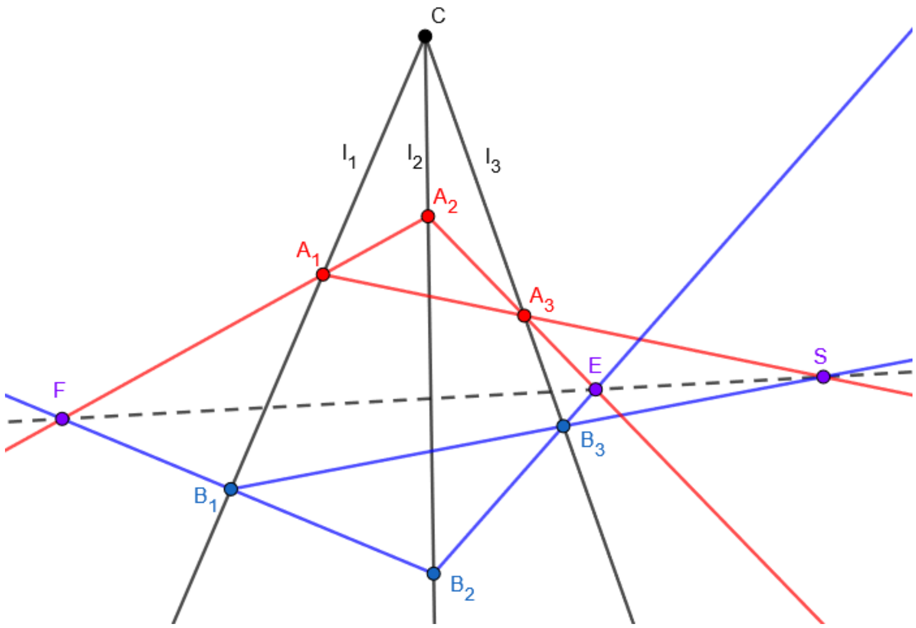

Figure 16: Desargue's Theorem implies DPT.2

□

Now, we finally have links to Pappus's Theorem and to Desargue's Theorem. With that information, we can finally complete the proof we've been working towards:

Theorem 3.4. *If Pappus's Theorem holds in a projective plane, then so does Desargue's Theorem.*

Proof. Let A_1, A_2, A_3, B_1, B_2, B_3, C, P_{12}, P_{23}, and P_{31} be so that the conditions for Desargue's Theorem are met. Name $l_1 = A_1B_1$, $l_2 = A_2B_2$, and $l_3 = A_3B_3$. We will split this up into a few cases:

1. Assume $C \notin P_{12}P_{23}$ and $P_{23} \notin l_1$. We will look at $P_{12}P_{23}$ and $G = l_1 \cap P_{12}P_{23}$. Consider $\sigma_{P_{12}} : l_1 \longrightarrow l_2$ and $\sigma_{P_{23}} : l_2 \longrightarrow l_3$. Now, by theorem 3.2, we know DPT.1 holds. Unfortunately, l_1, l_2, and l_3 are concurrent, so we cannot apply it to $\sigma_{P_{23}} \circ \sigma_{P_{12}}$ yet.

Now, let m be an arbitrary line through G with $m \neq l_1$ and $m \neq P_{12}P_{23}$. Consider the perspectives $\mu_{P_{23}} : l_2 \longrightarrow m$ and $\nu_{P_{23}} : m \longrightarrow l_3$. Note that here, we use that $P_{23} \notin l_1$. Clearly, $\sigma_{P_{23}} = \nu_{P_{23}} \circ \mu_{P_{23}}$, so $\sigma_{P_{23}} \circ \sigma_{P_{12}} = \nu_{P_{23}} \circ \mu_{P_{23}} \circ \sigma_{P_{12}}$.
Let's first take a closer look at $\mu_{P_{23}} \circ \sigma_{P_{12}}$. The three lines that are being considered here are l_1, l_2, and m. It is easily checked that these are non-concurrent, and in addition, $G = l_1 \cap m$ is a fixed point. This means that by DPT.1, there is a point S_1 such that $\sigma_{S_1} = \mu_{P_{23}} \circ \sigma_{P_{12}}$.
So now we have $\sigma_{P_{23}} \circ \sigma_{P_{12}} = \nu_{P_{23}} \circ \sigma_{S_1}$. Considering the right portion, we are looking at the lines l_1, m, and l_3, which are clearly non-concurrent. In addition, we know $(\nu_{P_{23}} \circ \sigma_{S_1})(C) = (\sigma_{P_{23}} \circ \sigma_{P_{12}})(C) = C$. Since $C = l_1 \cap l_3$, we can apply DPT.1 to find that there is a point S_2 such that $\sigma_{S_2} = \sigma_{P_{23}} \circ \sigma_{P_{12}}$.
So all we now have to do is find where S_2 is. First off, it's very easily seen that $\sigma_{S_2}(A_1) = (\sigma_{P_{23}} \circ \sigma_{P_{12}})(A_1) = A_3$ and $\sigma_{S_2}(B_1) = (\sigma_{P_{23}} \circ \sigma_{P_{12}})(B_1) = B_3$. Thus, $S_2 = A_1A_3 \cap B_1B_3 = P_{31}$ by axiom 1.2 and the definition of P_{31}.
Let $G_1 = P_{12}P_{23} \cap l_2$ and $G_2 = P_{12}P_{23} \cap l_3$. Then $\sigma_{P_{23}}(G) = G_1$ and $\sigma_{P_{23}}(G_1) = G_2$. Thus, $\sigma_{S_2}(G) = G_2$, meaning $P_{31} = S_2 \in GG_2 = P_{12}P_{23}$. Now we see P_{12}, P_{23}, and P_{31} are collinear, which means we have shown Desargue's Theorem!

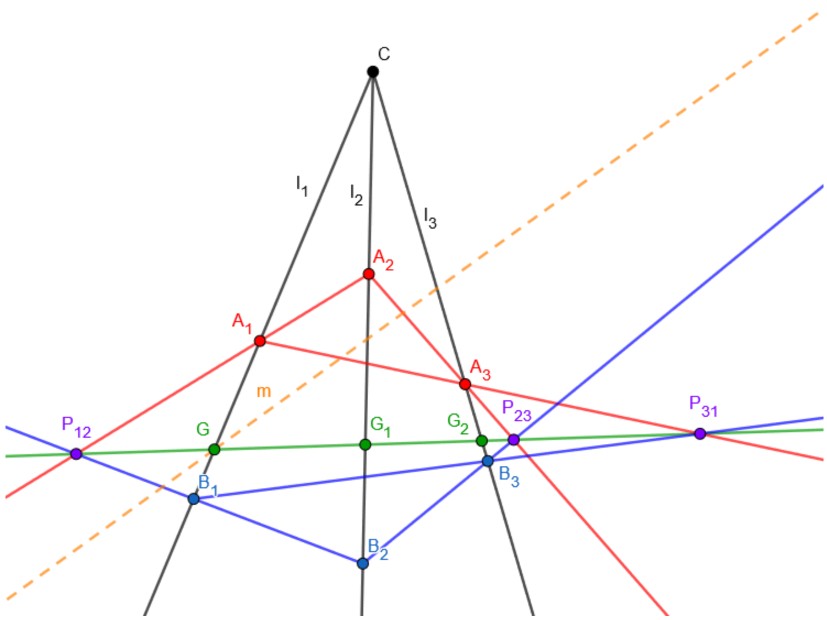

Figure 17: The first case

2. Next, we assume $C \notin P_{12}P_{23}$ and $P_{23} \in l_1$. Now, we define $G = P_{12}P_{23} \cap l_3$. Consider $\sigma_{P_{12}} : l_1 \longrightarrow l_2$ and $\sigma_{P_{23}} : l_2 \longrightarrow l_3$. As before, we cannot yet apply DPT.2, since l_1, l_2, and l_3 all go through C. Thus, instead, we consider a line m through G that's not equal to l_1 or $P_{12}P_{23}$.
   Similar to before, we take a look at perspectives $\mu_{P_{12}} : l_1 \longrightarrow m$ and $\nu_{P_{12}} : m \longrightarrow l_2$. To be allowed to define this, we need to argue P_{12} is on none of these lines. Now, it should be clear that $P_{12} \notin m$. Next, note that since $C \notin P_{12}P_{23}$, $l_1 \neq P_{12}P_{23}$. With $P_{23} \in l_1$, we have $P_{12} \notin l_1$. Last, $A_1 \notin l_2$ and $A_1 \in P_{12}A_2$, so $P_{12} \notin l_2$.
   Finally, we can say $\sigma_{P_{12}} = \nu_{P_{12}} \circ \mu_{P_{12}}$. Define $\sigma = \sigma_{P_{23}} \circ \sigma_{P_{12}}$. We now also have $\sigma = \sigma_{P_{23}} \circ \nu_{P_{12}} \circ \mu_{P_{12}}$.
   Now, since m, l_2, and l_3 aren't concurrent and since G is clearly a fixed point, we can apply DPT.1 to find that there is a point S_1 such that $\sigma_{S_1} = \sigma_{P_{23}} \circ \nu_{P_{12}}$. Similarly, we can also see that there is a point S_2 such that $\sigma_{S_2} = \sigma_{S_1} \circ \mu_{P_{12}} = \sigma_{P_{23}} \circ \nu_{P_{12}} \circ \mu_{P_{12}} = \sigma$.
   Our next goal is to find out where S_2 is. It's very easy to check that $\sigma_{S_2}(A_1) = A_3$ and $\sigma_{S_2}(B_1) = B_3$. Thus, we can immediately see $S_2 = A_1A_3 \cap B_1B_3 = P_{31}$. Finally, we set $G_1 = P_{12}P_{23} \cap l_2$. Once again, it's easy to check $\sigma_{S_2}(P_{23}) = (\sigma_{P_{23}} \circ \sigma_{P_{12}})(P_{23}) = \sigma_{P_{23}}(G_1) = G$. Thus, $P_{31} = S_2 \in GP_{23} = P_{12}P_{23}$, meaning P_{12}, P_{23}, and P_{31} are collinear.

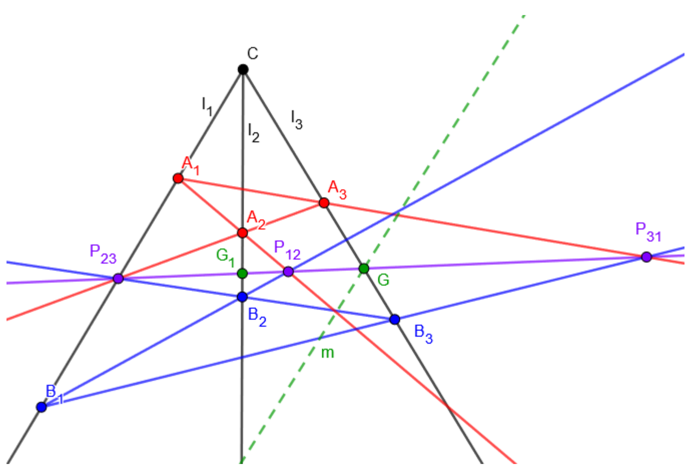

Figure 18: The second case

3. We have checked two cases for $C \notin P_{12}P_{23}$ and have proven both. Using similar methods, we can prove Desargue's Theorem for $C \notin P_{23}P_{31}$ or $C \notin P_{12}P_{31}$. Thus, the last case we must check is $C \in P_{12}23$ and $C \in P_{23}P_{31}$ and $C \in P_{12}P_{31}$. But then $P_{23} \in P_{12}C$ and $P_{31} \in P_{12}C$, meaning they are still collinear.

□

And with that, we have shown a major result in projective geometry. Pappus's Theorem implying Desargue's Theorem is not only a very interesting result, it is also majorly useful. Interestingly, however, it is not the only thing Pappus's Theorem is incredibly useful for, as we will see in the next section.

3.3 Fundamental Property

In the previous two sections, we have introduced and then used the concepts of perspectives and projective maps, but of course, we are not quite done with them yet. In this section, we will introduce a theorem known as the Fundamental Property of projective geometry. As may be expected, there is not just a single formulation for this theorem. In this section, we will showcase three (equivalent) ways of formulating it and then show they are all equivalent:

Fundamental Property. *The following are three equivalent formulations of this theorem:*

1. *If $p : l_1 \longrightarrow l_2$ is any projective map with $l_1 \neq l_2$ and $p(l_1 \cap l_2) = l_1 \cap l_2$, then p is a perspective.*
2. *If A_1, A_2, and A_3 are distinct points on l and for some projective $p : l \longrightarrow l$ we have $p(A_i) = A_i$ for $i \in \{1, 2, 3\}$, then $p = id_l$.*
3. *If A_1, A_2, and A_3 are distinct points on l_1 and B_1, B_2, and B_3 are distinct points on l_2 (with $l_1 \neq l_2$), then there exists a unique projective map $p : l_1 \longrightarrow l_2$ such that $p(A_i) = B_i$ for $i \in \{1, 2, 3\}$.*
4. *If A_1, A_2, A_3, B_1, B_2, and B_3 are distinct points on a line l, then there is a unique projective map $p : l \longrightarrow l$ such that $p(A_i) = B_i$ for $i \in \{1, 2, 3\}$.*

First, we will proof that these 4 statements are equivalent:

Proof. This proof will be presented in separate steps, each only looking at a single implication. The order may seem strange at first, but this is because we wish to begin with the simple proofs.

- **2.** $\Rightarrow$ **3.** Let l_1 and l_2 be lines ($l_1 \neq l_2$) with A_1, A_2, A_3, B_1, B_2, and B_3 as described in **3.** Let p_1 and p_2 be two projective maps such that $p_1(A_i) = p_2(A_i) = B_i$ for $i \in \{1, 2, 3\}$. We will show $p_1 = p_2$.
  Consider $p : l_1 \longrightarrow l_1$ with $p = p_2^{-1} \circ p_1$. Then we have $p(A_i) = p_2^{-1}(p_1(A_i)) = p_2^{-1}(B_i) = A_i$ for $i \in \{1, 2, 3\}$. By **2.**, we have $p = id_{l_1}$. Thus, $p_1^{-1} = p_2^{-1}$ and $p_1 = p_2$.

- **3.** $\Rightarrow$ **2.** Let l_1 be any line with points A_1, A_2, and A_3 on l_1. Define any line l_2 and points B_1, B_2, and B_3 and let $p : l_1 \longrightarrow l_2$ be such that $p(A_i) = B_i$ for $i \in \{1, 2, 3\}$. By **3.**, p is unique. Now, let $q : l_1 \longrightarrow l_1$ be such that $q(A_i) = A_i$ for $i \in \{1, 2, 3\}$. We will show $q = id_{l_1}$.
  Define $r = p \circ q$. Clearly, $r(A_i) = B_i$ for $i \in \{1, 2, 3\}$. Thus, by **3.**, $r = p$. But then $p \circ q = p \Rightarrow p^{-1} \circ p \circ q = p^{-1} \circ p \Rightarrow q = id_{l_1}$.

- **3.** $\Rightarrow$ **4.** Let l_1 be a line with 6 distinct points A_i and B_i for $i \in \{1, 2, 3\}$. Also let $p_1 : l \longrightarrow l$ and $p_2 : l \longrightarrow l$ such that $p_1(A_i) = p_2(A_i) = B_i$ for $i \in \{1, 2, 3\}$. We will show $p_1 = p_2$.
  Let l_2 be another line with three distinct points C_1, C_2, and C_3. By **3.**, there is a unique projective map $q : l_1 \longrightarrow l_2$ such that $q(A_i) = C_i$ for

$i \in \{1, 2, 3\}$, as well as a unique projective map $r : l_1 \longrightarrow l_2$ such that $r(B_i) = C_i$. Now, if we consider $r \circ p_1$ and $r \circ p_2$, we notice that for both of them, we have that they map A_i to B_i for $i \in \{1, 2, 3\}$. Thus, they must both equal q. But then $r \circ p_1 = r \circ p_2 \Rightarrow p_1 = p_2$.

- **4.** $\Rightarrow$ **2.** Let l be a line with 6 distinct points A_i and B_i for $i \in \{1, 2, 3\}$. In addition, let $p : l \longrightarrow l$ be a projective map such that $p(A_i) = A_i$ for $i \in \{1, 2, 3\}$. We will show $p = id_l$.
  By **4.**, there is a unique projective map $q : l \longrightarrow l$ such that $q(A_i) = B_i$ for $i \in \{1, 2, 3\}$. Now consider $r = q \circ p$. Clearly, $r(A_i) = B_i$ for $i \in \{1, 2, 3\}$. But then, since q is unique, $r = q$. From this, we clearly see $p \circ q = q \Rightarrow p = id_l$.

- **3.** $\Rightarrow$ **1.** Let l_1 and l_2 be lines and let $p : l_1 \longrightarrow l_2$ be a projective map with $p(l_1 \cap l_2) = l_1 \cap l_2$. Now, let A_1 and A_2 be distinct points on l_1 and define $B_1 = p(A_1)$ and $B_2 = p(A_2)$. Finally, let $A_3 = B_3 = l_1 \cap l_2$.
  We consider the perspective $\sigma_S : l_1 \longrightarrow l_2$ with $S = A_1B_1 \cap A_2B_2$. Clearly, σ_S is a projective map (of order 1) such that $\sigma_S(A_i) = B_i$ for $i \in \{1, 2, 3\}$. By **3.**, it is unique. However, note that p is also such a projective map. Thus, we can conclude that $p = \sigma_S$. Since p was chosen arbitrarily, we can conclude that we can find a point S for any projective map p.

- **1.** $\Rightarrow$ **3.** As this proof is more complicated, we will split it up into 4 cases. However, regardless of that, we will always consider two lines l_1 and l_2, with points A_1, A_2, and A_3 on l_1 and B_1, B_2, and B_3 on l_2.

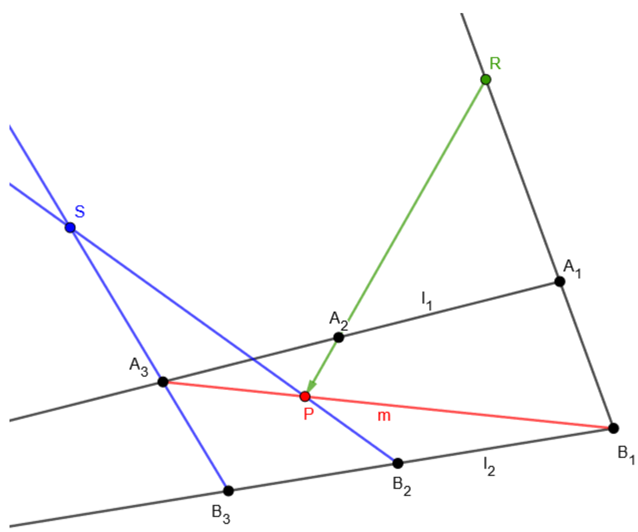

Figure 19: The first case

- *Case 1:* $A_i \neq l_1 \cap l_2$ *and* $B_i \neq l_1 \cap l_2$ *for* $i \in \{1, 2, 3\}$.
  Let $m = A_3B_1$ and let R be any point such that $R \in A_1B_1$. Consider $\sigma_R : l_1 \longrightarrow m$. Define $P = \sigma_R(A_2)$ and $S = A_3B_3 \cap PB_2$. We also consider $\sigma_S : m \longrightarrow l_2$ and $\sigma = \sigma_S \circ \sigma_R$. It is easily checked that $\sigma(A_i) = B_i$ for $i \in \{1, 2, 3\}$.
  Now, let $p : l_1 \longrightarrow l_2$ be a projective map such that $p(A_i) = B_i$ for $i \in \{1, 2, 3\}$. Let $\mu = \sigma_S^{-1} \circ p : l_1 \longrightarrow m$. Then we have $\mu(A_1) = B_1$, $\mu(A_2) = P$, and $\mu(A_3) = A_3 = l_1 \cap m$. Then, by **1.**, μ is a perspective, and it's easily checked that this must be σ_R. Thus $\sigma_S^{-1} \circ p = \mu = \sigma_R \Rightarrow p = \sigma_S \circ \sigma_R = \sigma$. Since p was chosen arbitrarily, we must have that σ is the only projective map with $\sigma(A_i) = B_i$ for $i \in \{1, 2, 3\}$.

- *Case 2:* $A_3 = l_1 \cap l_2$ *and* $B_i \neq l_1 \cap l_2$ *for* $i \in \{1, 2, 3\}$
  Define $m = A_1B_3$ and $R \in l_2$ with $R \neq A_3, B_3$. Consider $\sigma_R : l_1 \longrightarrow m$. Clearly, we have $\sigma_R(A_1) = A_1$ and $\sigma_R(A_3) = B_3$. We also define $P = \sigma_R(A_2)$ and $S = A_1B_1 \cap PB_2$. Now, consider $\sigma_S : m \longrightarrow l_2$. Let $\sigma = \sigma_S \circ \sigma_R : l_1 \longrightarrow l_2$ and see that $\sigma(A_i) = B_i$ for $i \in \{1, 2, 3\}$. As in the previous case, we let $p : l_1 \longrightarrow l_2$ be a projective map such that $p(A_i) = B_i$ for $i \in \{1, 2, 3\}$ and we look at $\mu = \sigma_S^{-1} \circ p : l_1 \longrightarrow m$. We find $\mu(A_1) = A_1 = l_1 \cap m$, $\mu(A_2) = P$, and $\mu(A_3) = B_3$. By **1.**, we know μ is a perspective, and so we quickly find $\mu = \sigma_R$. Using the same reasoning as before, we see $\sigma_S^{-1} \circ p = \sigma_R \Rightarrow p = \sigma$, meaning σ is unique.

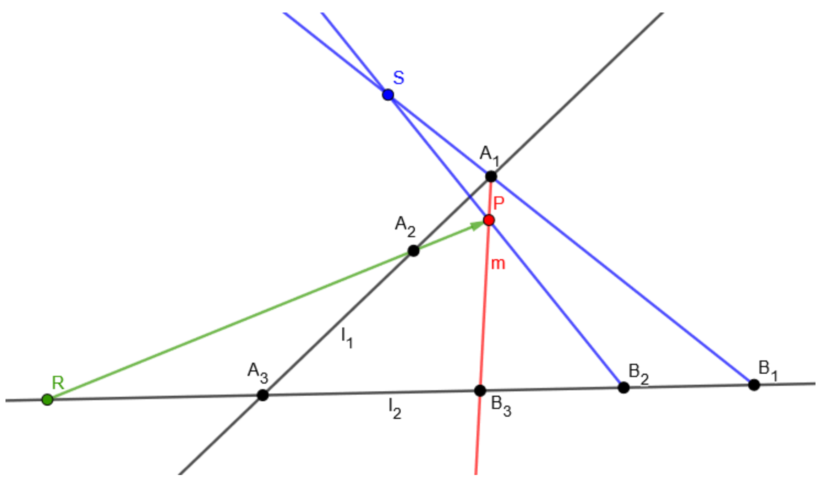

Figure 20: The second case

- *Case 3:* $A_i \neq l_1 \cap l_2$ *for* $i \in \{1, 2, 3\}$ *and* $B_3 = l_1 \cap l_2$
  This case is simple and analogous to the previous one; simply replace p with p^{-1}. Once you prove the inverse is unique, p itself must be unique too.

– *Case 4:* $A_3 = B_1 = l_1 \cap l_2$.
Define $m = A_1B_3$ and pick $R \in l_2$ such that $R \neq A_3, B_3$. Once again, we look at $\sigma_R : l_1 \longrightarrow m$ and $P = \sigma_R(A_2)$. Now let $S = l_1 \cap PB_2$ and consider $\sigma_S : m \to l_2$ and $\sigma = \sigma_S \circ \sigma_R$. It is easily checked that $\sigma(A_i) = B_i$ for $i \in \{1, 2, 3\}$.
Now let $p : l_1 \longrightarrow l_2$ be a projective map with $p(A_i) = B_i$ for $i \in \{1, 2, 3\}$. Just as before, we look at $\mu = \sigma_S^{-1} \circ p$ and conclude, by **1.**, that $\mu = \sigma_R$ and thus $p = \sigma$, meaning σ is unique. Also see figure on the next page.

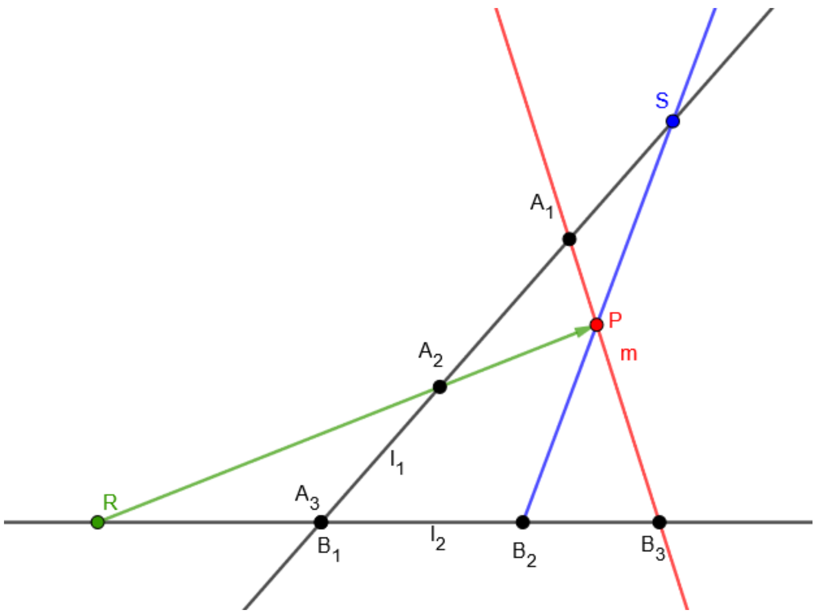

Figure 21: The fourth case

– *Case 5:* $A_3 = B_3 = l_1 \cap l_2$
This case is simple. Let $p : l_1 \longrightarrow l_2$ be such that $p(A_i) = B_i$ for $i \in \{1, 2, 3\}$. Then, clearly $p(l_1 \cap l_2) = p(A_3) = B_3 = l_1 \cap l_2$. Thus, p is a perspective $\sigma_S : l_1 \longrightarrow l_2$ for some S. We immediately see that there is only one S for which this holds: $S = A_1B_1 \cap A_2B_2$. Thus, $p = \sigma_S$ is unique.

By simply renumbering points, every other case can be proven now as well. Thus, **1.** $\Rightarrow$ **3.**.

Clearly, the four statements are now equivalent. □

From here on out, we will refer to the four statements as FP.1, FP.2, FP.3, and FP.4 respectively.
At the end of the previous section, it was stated that Pappus's Theorem would

prove important once more in this section. So far, however, we have seen nothing of the sort. As you may have guessed, we will know be moving on to a perhaps stunning result:

Theorem 3.5. *In a projective plane, the Fundamental Property holds if and only if Pappus's Theorem holds.*

Now, before we prove this, we will first prove two lemmas and a theorem that will greatly help us in our proof. Before we do so, however, we will be introducing a new notation for projective maps: $p = l \xrightarrow{O} m \xrightarrow{P} n$ means that for the lines l, m, and n and points O and P, we have $\sigma_O : l \longrightarrow m$ and $\sigma_P : m \longrightarrow n$, and $p = \sigma_P \circ \sigma_O$. Now, with that in mind, we introduce our lemmas:

Lemma 3.1. *Assume Pappus's Theorem holds. Let l, m, and n be non-concurrent lines and let O and P be points, with $O \notin l, m$ and $P \notin m, n$. Consider $p : l \xrightarrow{O} m \xrightarrow{P} n$. Then there exist lines m' and m'' with $l \cap m = l \cap m'$ and $m'' \cap n = m \cap n$, as well as points O and P such that we have $p = l \xrightarrow{O'} m' \xrightarrow{P} n$ and $p = l \xrightarrow{O} m'' \xrightarrow{P'} n$. In other words, we can replace m as well as either of the points.*

Proof. Let m' be a line through $l \cap m$ not equal to l or m. Clearly, $p = l \xrightarrow{O} m \xrightarrow{P} n = l \xrightarrow{O} m \xrightarrow{P} m' \xrightarrow{P} n$. Since Pappus's Theorem holds, so too does DPT.2, so we see that there is an O' such that $l \xrightarrow{O} m' = l \xrightarrow{O} m \xrightarrow{P} m'$. Thus, we get $p = l \xrightarrow{O'} m' \xrightarrow{P} n$.
Finding P' is done in similar fashion. □

Lemma 3.2. *Assume Pappus's Theorem holds. Let l, m, and n be non-concurrent lines and let $p = l \xrightarrow{O} m \xrightarrow{P} n$ be a projective map with $p(l \cap n) \neq l \cap n$. Then there exist points $O' \in n$ and $P' \in l$ such that $p = l \xrightarrow{O'} m' \xrightarrow{P'} n$ for some line m'. We will see $O' = OP \cap n$ and $P' = OP \cap l$.*

Proof. See figure 22 on the next page. Fix A and A' on l. Let $A \xrightarrow{O} B \xrightarrow{P} C$ and $A' \xrightarrow{O} B' \xrightarrow{P} C'$. Additionally, set $O' = OP \cap n$. Note that since $p(l \cap n) \neq l \cap n$, $l \cap n \notin OP$. Thus, $O' \neq l \cap n \Rightarrow O' \notin l$. Let $D = O'A \cap PC$, $D' = O'A \cap PC'$, and $m = DD'$. Consider the triangles ΔABD and $\Delta A'B'D'$ and realize that since Pappus's Theorem holds, so too does Desargue's Theorem and its dual theorem. Now, since $O = AB \cap A'B'$, $O' = AD \cap A'D'$, and $P = BD \cap B'D'$ are collinear, we have that $l = AA'$, $m = BB'$, and $m' = DD'$ are concurrent. So now, we have that m' goes through $l \cap m$. However, this means that if we find D via A, we can then see $m' = D(l \cap m)$. Thus, m' is only dependent on A. So, let us choose A'' on l and choose B'' and C'' in the same manner as before. Then if we choose $D'' = O'A'' \cap m'$, we will find $D''P \cap n = C''$, as was also the case for A'. Thus, we get $l \xrightarrow{O'} m' \xrightarrow{P} n = p$.
We can use a similar argument to find $P' \in n$. □

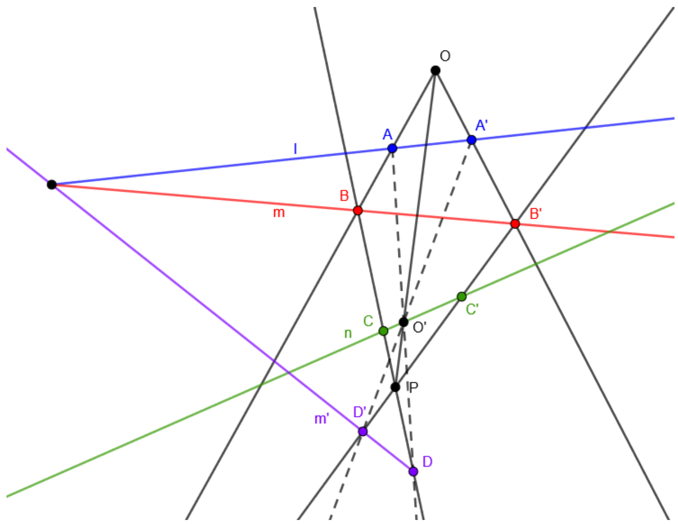

Figure 22: Lemma 3.2

Theorem 3.6. *If Pappus's Theorem holds, then any projective map can be written as a projective map of order 2. In other words, for any projective map* $p : l_1 \longrightarrow l_2$ *with* $l_1 \neq l_2$*, there exist points* O *and* P *such that* $p = l_1 \overset{O}{\longrightarrow} m \overset{P}{\longrightarrow} l_2$ *for some line* m*.*

Proof. We state that we only need to proof this for projective maps of order 3, as the rest follows from induction. In addition, we will show that every projective map of order 3 can be reduced to the general case $p : l \longrightarrow m \longrightarrow n \longrightarrow o$, with l, m, n, and o all distinct lines:

- **Case 1:** $p : l \longrightarrow m \longrightarrow n \longrightarrow m$ with l, m, and n distinct
  By lemma 3.1, we can find an m' distinct from l, m, and n such that $p = p' : l \longrightarrow m' \longrightarrow n \longrightarrow m$, which is the general case.

- **Case 2:** $p : l \longrightarrow m \longrightarrow l \longrightarrow o$ with l, m, and o distinct
  We use lemma 3.1 again. We find l' such that $p = p' : l \longrightarrow m \longrightarrow l' \longrightarrow o$, which is the general case.

- **Case 3:** $p : l \longrightarrow m \longrightarrow l \longrightarrow m$ with l and m distinct.
  Let O, P, and Q be the points such that $p = l \overset{O}{\longrightarrow} m \overset{P}{\longrightarrow} l \overset{Q}{\longrightarrow} m$. Let n be any line not equal to l or m such that $l \cap m \in n$. Clearly $p = l \overset{O}{\longrightarrow} m \overset{P}{\longrightarrow} n \overset{P}{\longrightarrow} l \overset{Q}{\longrightarrow} n$.
  Now, since Pappus's Theorem holds, so does DPT.2. Thus we can conclude that there is a point O' such that $l \overset{O'}{\longrightarrow} n = l \overset{O}{\longrightarrow} m \overset{P}{\longrightarrow} n$ and a point P' such that $n \overset{P'}{\longrightarrow} m = n \overset{P}{\longrightarrow} l \overset{Q}{\longrightarrow} n$. Then $p = l \overset{O}{\longrightarrow} m \overset{P}{\longrightarrow} n \overset{P}{\longrightarrow} l \overset{Q}{\longrightarrow} n = l \overset{O'}{\longrightarrow} n \overset{P'}{\longrightarrow} m$. This is a projective map of order 2.

Now we can move on to the general case.
Let $p : l \longrightarrow m \longrightarrow n \longrightarrow o$ with l, m, n, and o distinct lines. If $l \cap m = n \cap o$, the four lines are concurrent and we can use DPT.2 to quickly find that this is a perspective. Thus, assume $l \cap m \neq n \cap o$. Now, if $l \cap m \in o$, we can use lemma 3.1 to replace m with m'. In this way, we can now also assume $l \cap m \notin o$.
Finally, we use lemma 2 to get the following situation: $p = l \xrightarrow{P} m \xrightarrow{Q} n \xrightarrow{R} o$ with $Q \in O$, $R \in m$, and $l \cap m \notin o$.
Let $h = (m \cap l)(n \cap o)$. Note that since $Q \notin n$, $Q \neq n \cap o$ and thus $Q \notin h$. Let $A, A' \in l$ with $A, A' \xrightarrow{P} B, B' \xrightarrow{Q} C, C' \xrightarrow{R} D, D'$. In addition, we will consider $q = m \xrightarrow{Q} h$ with points $H, H' \in h$ such that $B, B' \xrightarrow{Q} H, H'$.
Since Pappus's Theorem holds, so too does Desargue's Theorem, and we will first apply it to the triangles ΔABH and $\Delta A'B'H'$. Clearly, AA', BB', and HH' are all collinear, so we find that $AB \cap A'B' = P$, $BH \cap B'H' = Q$, and $AH \cap A'H'$ are all collinear. If we call this last point M, we notice that M is equal to $PH \cap AH$. In other words, it only depends on our choice of A.
Next, we apply Desargue's Theorem to ΔCDH and $\Delta C'D'H'$. Again, we easily find the concurrent lines we need, so we get that $CD \cap C'D' = R$, $CH \cap C'H' = Q$, and $DH \cap D'H'$ are collinear. Just like before, we call this last point N and note that N only depends on our choice of A, as it is equal to $DH \cap QR$.

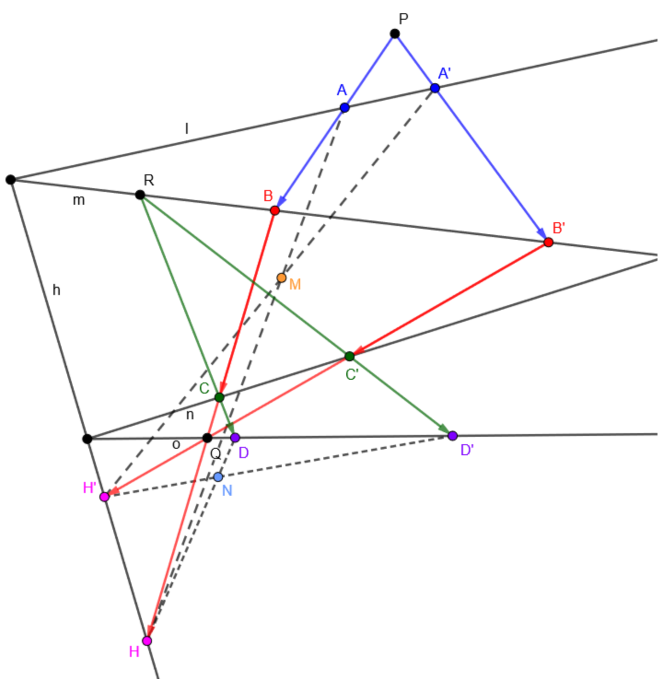

Figure 23: Theorem 3.6

Now that we have found that M and N only depend on A, we consider the projective map $p' = l \xrightarrow{M} h \xrightarrow{N} o$. Now we see $p'(A) = A \xrightarrow{M} H \xrightarrow{N} D$ and, for any choice of A', $p'(A') = A' \xrightarrow{M} H' \xrightarrow{N} D'$ (where $D' = p(A')$). In other words, $p = p'$. But clearly, p' is of order 2. Thus, if Pappus's Theorem holds, any projective map of order 3 is equal to some projective map of order 2.

A simple induction argument can be used to prove it for projective maps of greater orders:
Assume the statement holds for projective maps of order n. Let $p = l_1 \longrightarrow l_2 \longrightarrow \cdot \longrightarrow l_n \longrightarrow l_{n+1} \longrightarrow l_{n+2}$ be a projective map of order $n+1$. First assume $l_{n+1} \neq l_1$. Then we can reduce p to $p = l_1 \longrightarrow m \longrightarrow l_{n+1} \longrightarrow l_{n+2}$ by the assumption, and then use the same proof as before to reduce this to $p = l_1 \longrightarrow m' \longrightarrow l_{n+2}$.
Now we assume $l_{n+1} = l_1$ and $l_n \neq l_{n+2}$. Then we use lemma 1 to exchange l_{n+1} for some other line that is not equal to l_1, allowing us to reduce it once more.
Next, what if $l_{n+1} = l_1$ and $l_n = l_{n+2}$. Now if $l_{n-1} \neq l_{n+1}$, we can simply use lemma 3.1 to exchange l_n for some other line to achieve our result. And if $l_{n-1} = l_{n+1}$, we look at $q = l_{n+1} \longrightarrow l_n \longrightarrow l_{n+1} \longrightarrow l_n$. By the proof we used in Case 3 above, this is a perspective, meaning we can reduce p to order $n-1$, which we already know can be reduced to order 2. Thus, we are done and have proven any projective map $p : l_1 \longrightarrow l_2$ (with $l_1 \neq l_2$) can be written as a projective map of order 2. □

With that, we can finally move on to the proof of theorem 3.5, which has now been made simple:

Proof. As expected, we will split this up into two separate proves, as we are working with an 'if and only if'-statement.

- Assume the Fundamental Property holds. We will show Pappus's Theorem also holds.
  This is largely trivial. After all, clearly, FP.1 implies DPT.1, which is equivalent with Pappus's Theorem by theorem 3.2.

- Assume Pappus's Theorem holds. We will show the Fundamental Property also holds.
  We will prove FP.1. Let $p : l_1 \longrightarrow l_2$ (with $l_1 \neq l_2$) be a projective map such that $p(l_1 \cap l_2) = l_1 \cap l_2$. Now, since Pappus's Theorem holds, so too does theorem 3.6 and DPT. With the former, we can conclude that p is a projective map of order 2. But since $p(l_1 \cap l_2) = l_1 \cap l_2$, DPT tells us that p is now a perspective, which is what FP.1 states.

□

With that, we conclude this section, as we have now shown that Pappus's Theorem is equivalent to the Fundamental Property.

4 Harmonic Addition

In this shorter chapter, we will introduce a new notion, which we will then apply many different things we have learned to. It will function as a showcase of how we can use the different notions and theorems we have found to prove a variety of statements. To do so, we will first showcase its definition and some crucial properties that rely on theorems from chapter 2, then we move on to its relationship to the projective maps of chapter 3, and finally we consider how it functions under the Principle of Duality from chapter 1.

4.1 Definition and Properties

So first, let us introduce the harmonic additions as we will use them:

Definition 4.1. Let l be a line and let P_1, P_2, and Q_1 be distinct points on l. Let A_1 be any point not on l and let A_2 be a point on A_1Q_1, distinct from both A_1 and Q_1. Define $A_3 = P_1A_1 \cap P_2A_2$ and $A_4 = P_1A_2 \cap P_2A_1$. Finally, let $Q_2 = A_3A_4 \cap l$. We call Q_2 the *harmonic addition to* Q_1 *relative to* P_1 *and* P_2.

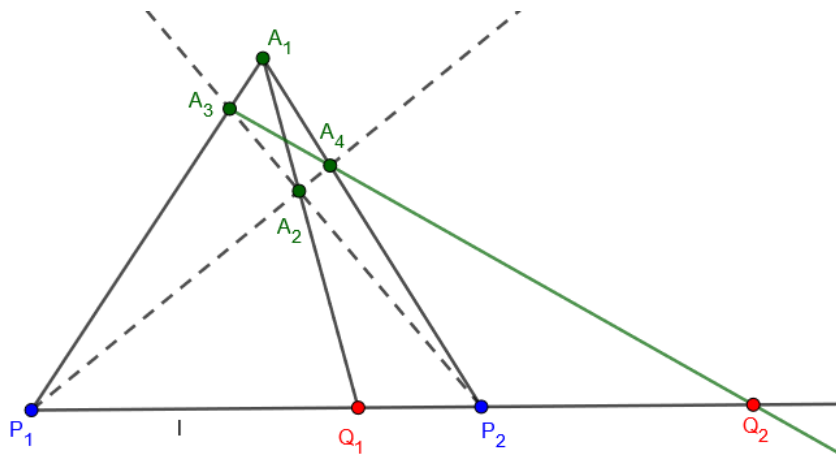

Figure 24: The harmonic addition

Now, at first glance, this definition doesn't seem very useful. After all, A_1 and A_2 are both arbitrarily chosen, meaning that we'd be able to get many different harmonic additions for the same Q_1, P_1, and P_2, right? Well, as it turns out, that is often not the case. Until we determine that, however, we will refer to Q_2 as determined via A_1 and A_2 as $Q_{A_1A_2}$. Note $Q_{A_1A_2} = Q_{A_2A_1}$. Now then, let us move on to a proof:

Theorem 4.1. *In any projective plane where Desargue's Theorem and Fano's Strong Axiom hold, the harmonic addition to* Q_1 *relative to* P_1 *and* P_2 *is independent of our choice of* A_1 *and* A_2*. In other words, it's unique.*

Proof. For this, we will consider two different constructions for Q_2, where we find it ones using A_1 and A_2, and once using B_1 and B_2 (which are chosen in the same way as A_1 and A_2 before) We will consider several different cases:

- **Case 1:** $A_1 = B_1$.
  See figure 25 on the next page. We simply apply Desargue's Theorem to

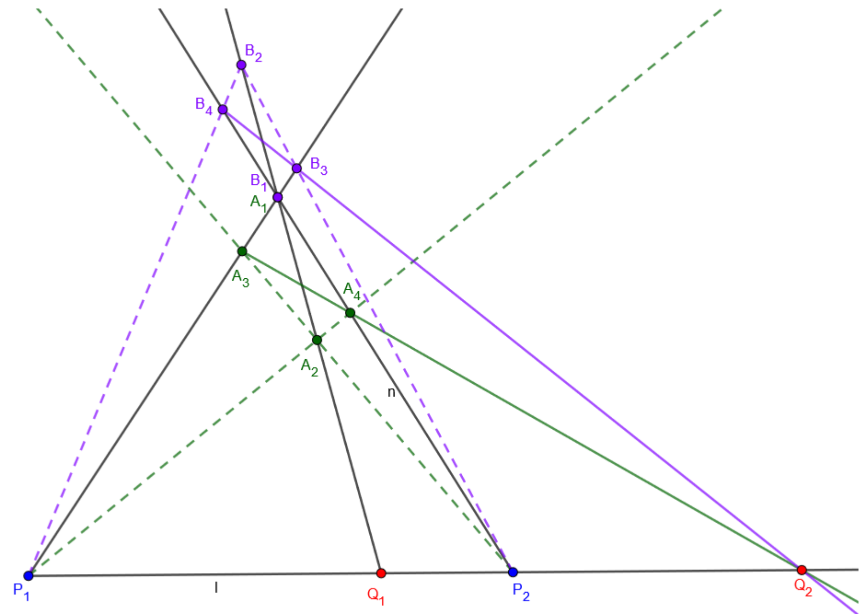

Figure 25: The first case

the triangles $\Delta A_2A_3A_4$ and $\Delta B_2B_3B_4$, as the relevant lines all meet in A_1. Now, we see that $P_1 = A_2A_4 \cap B_2B_4$, $P_2 = A_2A_3 \cap B_2B_3$, and $A_3A_4 \cap B_3B_4$ are collinear. But $P_1P_2 = l$, so $A_3A_4 \cap B_3B_4 = A_3A_4 \cap l = Q_{A_1A_2}$. Thus $Q_{B_1B_2} = B_3B_4 \cap l = Q_{A_1A_2}$.

- **Case 2:** $A_1A_2 = B_1B_2$ and $A_1 \neq B_1$.
Note that the roles of A_1 and A_2 are symmetrical. Thus, from the first case, we get $Q_{A_1A_2} = Q_{A_2A_1} = Q_{A_2B_1} = Q_{B_1A_2} = Q_{B_1B_2}$.

- **Case 3:** $m = A_1A_2 \neq B_1B_2 = n$.
For notational purposes, we state $Q_m = Q_{A_1A_2}$ and $Q_n = Q_{B_1B_2}$. We can do this as in the first two cases, we determined that Q_m is independent of where on m A_1 and A_2 are chosen. Thanks to this, we can also choose these points in ways that are convenient for us. So we pick $A_1 \in m$ arbitrarily and let $B_1 = P_1A_1 \cap n$. Next, choose $B_2 \in n$ arbitrarily and let $A_2 = P_2B_2 \cap m$.
Now we will apply Desargue's Theorem to $\Delta P_1A_2B_2$ and $\Delta P_2A_1B_1$, as the relevant lines meet in Q_1. We then conclude $P_1A_2 \cap P_2A_1 = A_4$, $A_2B_2 \cap A_1B_1 = A_3 = B_3$, and $P_1B_2 \cap P_2B_1 = B_4$ are collinear. Thus $A_3A_4 = B_3B_4$, giving us $Q_m = A_3A_4 \cap l = B_3B_4 \cap l = Q_n$.
Here, we use Fano's Strong Axiom to ensure $Q_1 \neq Q_m$. To prove this, we assume $Q_1 = Q_m$ by way of contradiction and consider the quadrilateral $P_1P_2A_3A_4$. Then by definition, the diagonal points are $P_1P_2 \cap A_3A_4 = Q_m = Q_1$, $P_1A_3 \cap P_2A_4 = A_1$ and $P_1A_4 \cap P_2A_3 = A_2$. However, we have $Q_1, A_1, A_2 \in m$, which is a contradiction to Fano's Strong Axiom. Thus, $Q_1 \neq Q_m$.

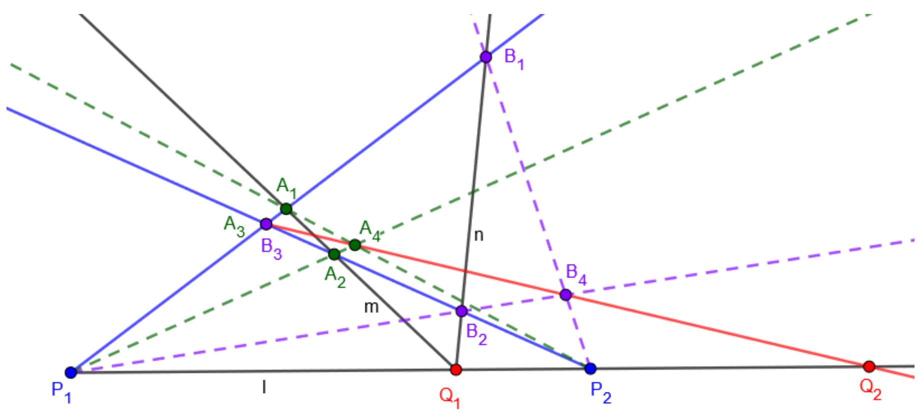

Figure 26: The third case

□

So, at least we now know we can speak of the harmonic addition to Q_1 relative to P_1 and P_2, as it is unique. However, there's two more properties that will make it even easier to talk about this definition:

Theorem 4.2. *If Q_2 is the harmonic addition to Q_1 relative to P_1 and P_2, then also Q_1 is the harmonic addition to Q_2 relative to P_1 and P_2. In other words, the relationship of 'harmonic addition' is symmetric.*

Proof. This is largely trivial. After all, if $Q_2 = Q_{A_1A_2}$ as in figure 24, then $Q_1 = Q_{A_3A_4}$, or the harmonic addition to Q_2 relative to P_1 and P_2, constructed via A_3 and A_4. □

Theorem 4.3. *In any projective plane where Desargue's Theorem and Fano's Strong Axiom hold, if Q_2 is the harmonic addition to Q_1 relative to P_1 and P_2, then P_2 is the harmonic addition to P_1 relative to Q_1 and Q_2.*

Proof. Let A_1, A_2, A_3, and A_4 be the points used to construct Q_2 as before. By Fano's Strong Axiom, $Q_1 \neq Q_2$. We will be constructing the harmonic addition to P_1 relative to Q_1 and Q_2 via A_1 and A_3, and show it must be equal to P_2. Consider the points $B_3 = Q_1A_1 \cap Q_2A_3$ and $B_4 = Q_1A_3 \cap Q_2A_1$. Clearly, $B_3B_4 \cap l$ (where l is the line on which P_1, P_2, Q_1, and Q_2 are) is the harmonic addition we are looking for.
Now, we apply Desargue's Theorem to the triangles $\Delta Q_2A_1A_4$ and $\Delta Q_1A_3A_2$, which we can do since the relevant lines are concurrent in P_1. Then we see that B_3, B_4, and $P_2 = A_1A_4 \cap A_3A_2$ are collinear. But then, since $P_2 \in l$, $P_2 = B_3B_4 \cap l$. As such, P_2 is the harmonic addition to P_1 relative to Q_1 and Q_2. □

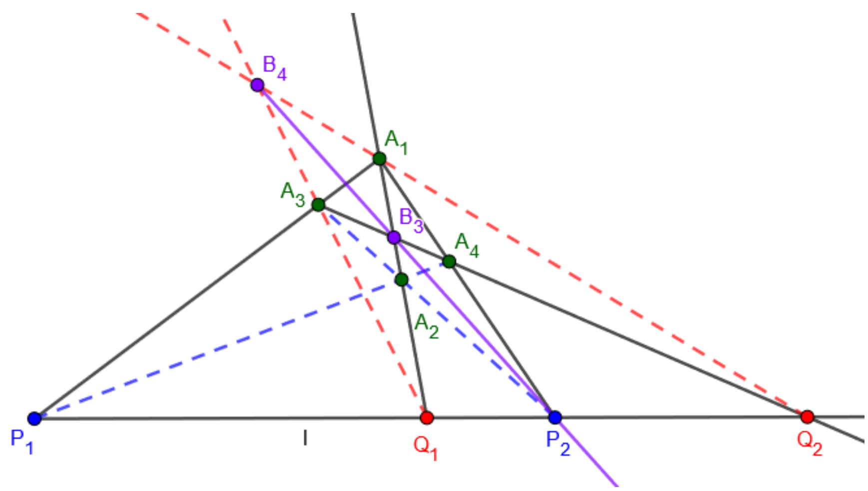

Figure 27: Theorem 4.3

Now that we have accomplished this, from here on out, as long as Desargue's Theorem and Fano's Strong Axiom hold, we will speak of the *harmonic pairs* P_1, P_2 and Q_1, Q_2, or we can say Q_1 *and* Q_2 *are harmonic to* P_1 *and* P_2.

4.2 Harmonic Pairs and Projective Maps

In this section, we will consider exactly how harmonic pairs act when we apply projective maps to them. As it turns out, they act quite nicely:

Theorem 4.4. *Assume Desargue's Theorem and Fano's Strong Axiom hold and let $p : l_1 \longrightarrow l_2$ (maybe $l_1 = l_2$) be a projective map. If $P_1, P_2, Q_1, Q_2 \in l_1$ and P_1 and P_2 are harmonic to Q_1 and Q_2, then $p(P_1)$ and $p(P_2)$ are harmonic to $p(Q_1)$ and $p(Q_2)$.*

Proof. Since projective maps are simply compositions of perspectives by definition, a trivial argument allows us to state that if we prove this for a perspective $\sigma_S : l \longrightarrow m$, we have proven it for all projective maps. We will split this up into two different cases:

- **Case 1:** $Q_1 = l \cap m$.
  Consider $P_1' = \sigma_S(P_1)$ and $P_2' = \sigma_S(P_2)$. We will first construct Q_2 via P_1' and P_2', so let $A_3 = P_1P_1' \cap P_2P_2' = S$ and $A_4 = P_1P_2' \cap P_2P_1'$. Then, clearly $Q_2 = SA_4 \cap l$.
  Next, we will construct $Q_2' \in m$ such it's the harmonic addition to Q_1 relative to P_1' and P_2'. To do so, we use P_1 and P_2 to construct it. Thus $B_3 = P_1'P_1 \cap P_2'P_2 = S$ and $B_4 = P_1'P_2 \cap P_2'P_1 = A_4$. Thus, $Q_2' = B_3B_4 \cap m = SA_4 \cap m$.
  But now we have $SQ_2 = SA_4 = SQ_2'$, so $\sigma_S(Q_2) = Q_2'$. Since σ_S was an arbitrarily chosen perspective, we have now shown it for any perspective. Also note that since all 4 points Q_1, Q_2, P_1, and P_2 play a similar role in their relationship as harmonic pairs, this case covers the situation that any one of them is equal to $l \cap m$.

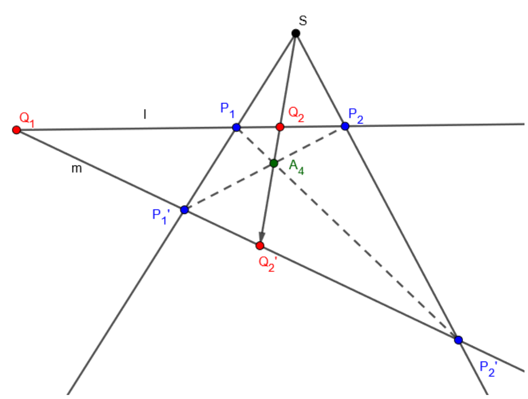

Figure 28: The first case

- **Case 2:** $Q_1, Q_2, P_1, P_2 \neq l \cap m$.
  Let P_1 and P_2 be harmonic to Q_1 and Q_2 and let $P_1' = \sigma_S(P_1)$, $P_2' =$

$\sigma_S(P_2)$, $Q_1' = \sigma_S(Q_1)$, and $Q_2' = \sigma_S(Q_2)$. We will show that Q_2' is the harmonic addition to Q_1' relative to P_1' and P_2'.
Let $n = Q_1Q_2'$ and consider the perspectives $\mu_S = l \longrightarrow n$ and $\nu_S : n \longrightarrow m$. Note that $\sigma_S = \nu_S \circ \mu_S$. Now, by case 1, $Q_2' = \mu_S$ is the harmonic addition to Q_1 relative to $\mu_S(P_1)$ and $\mu_S(P_2)$. And similarly by the first case, $Q_1' = \nu_S(\mu_S(Q_1)) = \sigma_S(Q_1)$ is the harmonic addition to $Q_2' = \nu_S(\mu_S(Q_2)) = \sigma_S(Q_2)$ relative to $P_1'\nu_S(\mu_S(P_1)) = \sigma_S(P_1)$ and $P_2' = \nu_S(\mu_S(P_2)) = \sigma_S(P_2)$. Thus, Q_1', Q_2' and P_1', P_2' are harmonic pairs.

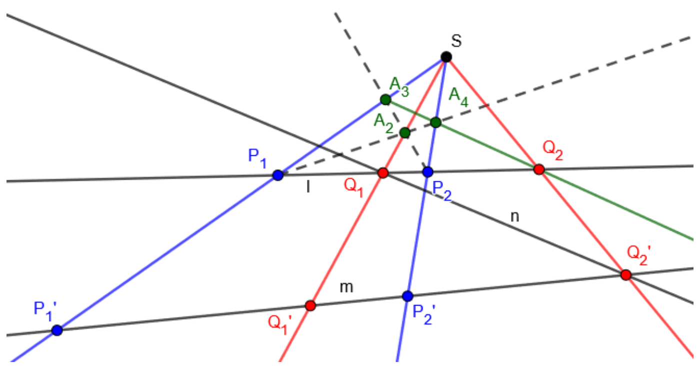

Figure 29: The second case

□

While this section may have been short, it hopefully showed off well enough how projective maps interact with other concepts in projective planes.

4.3 Dual Harmonic Addition

Finally, we will look at one more concept, which will take us back to the Principle of Duality. After all, we have not yet considered what harmonic pairs of lines would be like, so let us do so now:

Definition 4.2. Let A be a point and let l_1, l_2, and m_1 be lines through A. Let n_1 be any line not through A and let n_2 be a line through $m_1 \cap n_1$, distinct from both m_1 and n_1. Define $n_3 = (l_1 \cap n_1)(l_2 \cap n_2)$ and $n_4 = (l_1 \cap n_2)(l_2 \cap n_1)$. We call $m_2 = (n_3 \cap n_4)A$ the *dual harmonic addition to* m_1 *relative to* l_1 *and* l_2.

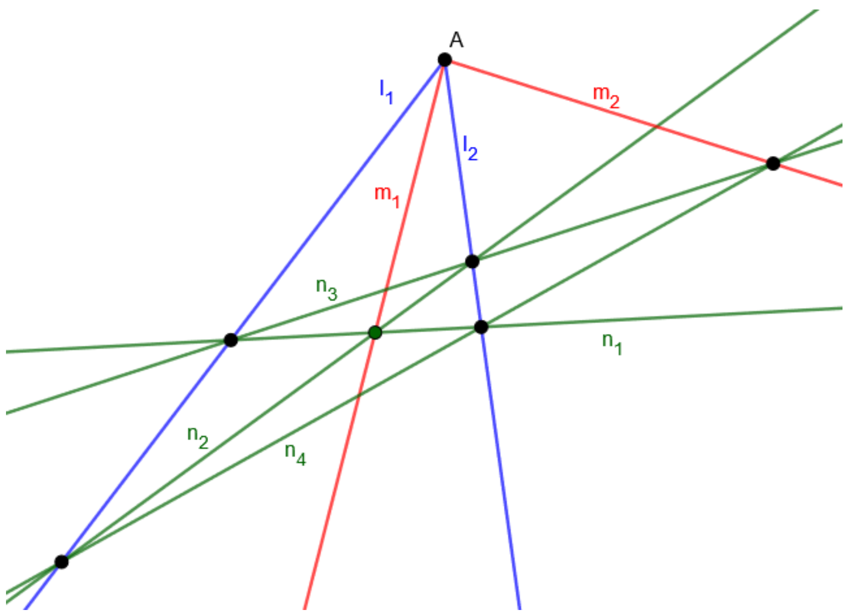

Figure 30: The dual harmonic addition

Up until this point, we have seen that theorems tend to imply their dual theorems, as is the case with Pappus's Theorem and Desargue's Theorem. As it turns out, a similar relationship holds when it comes to harmonic additions:

Theorem 4.5. *Assume Desargue's Theorem and Fano's Strong Axiom hold. Let* l_1, l_2, m_1, *and* m_2 *be concurrent in a point* A *and let* o *be a line not through* A. *Call* $P_1 = l_1 \cap o$, $P_2 = l_2 \cap o$, $Q_1 = m_1 \cap o$, *and* $Q_2 = m_2 \cap o$. *Then* P_1 *and* P_2 *are harmonic to* Q_1 *and* Q_2 *if and only if* m_2 *is the dual harmonic addition to* m_1 *relative to* l_1 *and* l_2.

Proof. We begin by recalling theorems 2.7 and 2.14, which allow us to use the dual theorems to Desargue's Theorem and Fano's Strong Axiom. Using this and by applying the principle of duality to theorem 3.7, we find that m_2 as the

dual harmonic addition to m_1 relative to l_1 and l_2 is unique.
Now then, we first assume we have all lines and points as described and assume P_1 and P_2 are harmonic to Q_1 and Q_2. We make a construction for Q_2 with $A_1 = A$ and A_2 any point on $AQ_1 = m_1$. Of course, A_3 and A_4 are found as before. We will show that $m_2 = AQ_2$ is the dual harmonic addition to m_1 relative to l_1 and l_2.
To do this, we will construct m_2 using $n_1 = P_1A_4$ and $n_2 = P_2A_3$. Then $n_3 = (l_1 \cap n_1)(l_2 \cap n_2) = P_1P_2 = o$ and $n_4 = (l_1 \cap n_2)(l_2 \cap n_1) = A_3A_4$. Then we easily see $n_3 \cap n_4 = o \cap A_3A_4 = Q_2$. By definition, we then see $m_2 = AQ_2$, which is what we were aiming to show.
Now, we assume m_2 is as required. By way of contradiction, we assume Q_2 isn't the harmonic addition to Q_1 relative to P_1 and P_2. Then we find this harmonic addition and name it Q_2'. By the proof we just gave, $m_2' AQ_2'$ is the harmonic addition to m_1 relative to l_1 and l_2. But we already had m_2 as this harmonic addition, and as we showed at the beginning of this proof, it is unique. This is a contradiction, so we must have Q_1 and Q_2 harmonic to P_1 and P_2. □

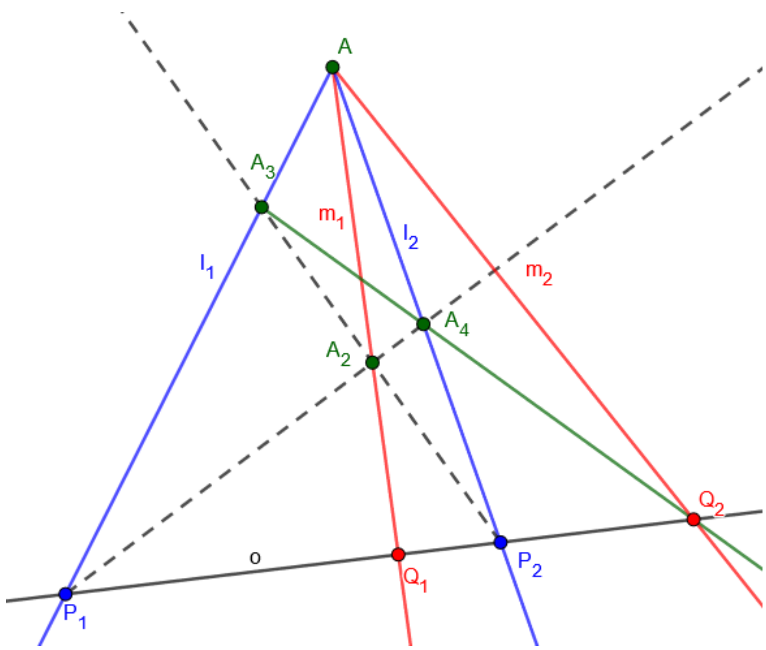

Figure 31: Theorem 4.5

Conclusion

Throughout this paper, we have seen a variety of interesting results, and practically all of them directly resulted from the lack of parallel lines in the projective plane. In all of our proofs, we got to simply assume that the point of intersection existed thanks to axiom 1.2, which resulted in theorems such as Pappus's Theorem, Desargue's Theorem, and even the existence of perspectives at all.
In addition, we saw the Principle of Duality arise, which showed us that to any projective plane there is what can be described as a parallel plane in which all the same rules hold, which is a fascinating result to think about.
Overall, it's important to consider that what we explored in this paper is but a small portion of the field of projective geometry. Many more advanced research includes Pappus's Theorem as a fourth axiom and then considers the implications that it has. Based on what we have found here, we know that the Fundamental Property then holds, but as of right now other results go outside of the scope of this paper. Rest assured, many interesting results are still to be discussed.

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