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Dynamic Shrinkage Estimation of the High-Dimensional Minimum-Variance Portfolio

Taras Bodnar, Nestor Parolya, and Erik Thorsén

***Abstract*—In this paper, new results in random matrix theory are derived, which allow us to construct a shrinkage estimator of the global minimum variance (GMV) portfolio when the shrinkage target is a random object. More specifically, the shrinkage target is determined as the holding portfolio estimated from previous data. The theoretical findings are applied to develop theory for dynamic estimation of the GMV portfolio, where the new estimator of its weights is shrunk to the holding portfolio at each time of reconstruction. Both cases with and without overlapping samples are considered in the paper. The non-overlapping samples corresponds to the case when different data of the asset returns are used to construct the traditional estimator of the GMV portfolio weights and to determine the target portfolio, while the overlapping case allows intersections between the samples. The theoretical results are derived under weak assumptions imposed on the data-generating process. No specific distribution is assumed for the asset returns except from the assumption of finite $4 + \varepsilon, \varepsilon > 0$, moments. Also, the population covariance matrix with unbounded largest eigenvalue can be considered. The performance of new trading strategies is investigated via an extensive simulation. Finally, the theoretical findings are implemented in an empirical illustration based on the returns on stocks included in the S&P 500 index.**

***Index Terms*—Shrinkage estimator, high-dimensional covariance matrix, random matrix theory, minimum variance portfolio, parameter uncertainty, dynamic decision making.**

I. INTRODUCTION

GLOBAL minimum-variance (GMV) portfolio is the one of the mostly used investment strategies by both practitioners and researchers in finance. This portfolio possesses the smallest variance among all optimal portfolios obtained as solutions of Markowitz's mean-variance optimization problem (cf., Markowitz [1]). It solves the following problem

$$\mathbf{w}^\top \mathbf{\Sigma} \mathbf{w} \to min \quad \text{with} \quad \mathbf{w}^\top \mathbf{1}_p = 1, \tag{I.1}$$

where $\mathbf{w}$ denotes the vector of the portfolio weights which determines the structure of the investor portfolio, the symbol $\mathbf{1}_p$ stands for the p-dimensional vector of ones, and $\mathbf{\Sigma}$ is the covariance matrix of the p-dimensional vector of asset returns $\mathbf{y} = (y_1, \ldots, y_p)^\top$.

The solution of the optimization problem (I.1) is given by

$$\mathbf{w}_{GMV} = \frac{\mathbf{\Sigma}^{-1}\mathbf{1}_p}{\mathbf{1}_p^\top \mathbf{\Sigma}^{-1}\mathbf{1}_p}. \tag{I.2}$$

The weights of the GMV portfolio have several nice properties, which simplify its applicability in practice and, thus, make it a popular investment strategy. The weights of the GMV portfolio do not depend on the mean vector of the asset returns, which we will denote by $\boldsymbol{\mu}$ in the following. This is the only mean-variance optimal portfolio whose weights are independent of $\boldsymbol{\mu}$. Moreover, the GMV portfolio has a special location on the set of the mean-variance optimal portfolios, which is a parabola in the mean-variance space and is known as the efficient frontier (cf., Merton [2]). Its mean and variance determines the location of the vertex of this parabola (see, e.g, Kan and Smith [3], Bodnar and Schmid [4]).

The application of (I.2) requires the knowledge of $\mathbf{\Sigma}$ in practice, which is usually not provided. The covariance matrix $\mathbf{\Sigma}$ has to be estimated by using historical data of the asset returns, before the GMV portfolio can be constructed. The quality of the estimator of $\mathbf{\Sigma}$ has a large impact on the stochastic properties of holding the GMV portfolio and it leads to further uncertainty in the investor decision problem, known as the estimation uncertainty. The estimation uncertainty can have a great impact on the constructed portfolio which could be larger than the one induced by the model uncertainty included in the optimization problem (I.1). The effect becomes even stronger, when the portfolio dimension is comparable to the sample size used to estimate $\mathbf{\Sigma}$.

Traditionally, the covariance matrix is estimated by its sample counterpart given by

$$\mathbf{S}_n = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{y}_i - \bar{\mathbf{y}}_n)(\mathbf{y}_i - \bar{\mathbf{y}}_n)^\top$$
$$= \frac{1}{n-1}\mathbf{Y}_n\left(\mathbf{I}_n - \frac{1}{n}\mathbf{1}_n\mathbf{1}_n^\top\right)\mathbf{Y}_n^\top \tag{I.3}$$

with $\bar{\mathbf{y}}_n = \frac{1}{n}\sum_{i=1}^{n}\mathbf{y}_i$, where $\mathbf{y}_1, .., \mathbf{y}_n$ denotes the sample of asset returns and $\mathbf{Y}_n = (\mathbf{y}_1, .., \mathbf{y}_n)$ denotes the data matrix. The symbol $\mathbf{I}_n$ stands for the n-dimensional identity matrix. Then, the sample (also known) as the traditional estimator of $\mathbf{w}_{GMV}$

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is obtained as

$$\hat{\mathbf{w}}_S = \frac{\mathbf{S}_n^{-1}\mathbf{1}_p}{\mathbf{1}_p^\top \mathbf{S}_n^{-1}\mathbf{1}_p}. \tag{I.4}$$

The distributional properties of $\hat{\mathbf{w}}_S$ have extensively been studied in statistical and econometric literature. Jobson and Korkie [5] derive the asymptotic distribution of $\hat{\mathbf{w}}_S$ assuming that the asset returns are independent and normally distributed and the portfolio dimension is considerably smaller than the sample size. Okhrin and Schmid [6] obtain the exact distribution of the sample estimator of the GMV portfolio weights assuming normality, while Bodnar and Schmid [7] extend these results to elliptically contoured distribution and develop a statistical test theory on the GMV portfolio weights.

However, when the portfolio dimension is comparable to the sample size, the results derived under the classical asymptotic regime, that is when p is considerably smaller than n, can no longer be used. Moreover, the effect of dimensionality considerably influences the estimation of the covariance matrix needed to determine the weights of the GMV portfolio. Using the recent results of the random matrix theory several improved estimator for the weights of the GMV portfolio has been suggested when the portfolio dimension is comparable to the sample size (see, e.g., Rubio et al. [8], Yang et al. [9], Bodnar et al. [10], Ballal et al. [11], Bodnar et al. [12]), i.e., under the large-dimensional asymptotic regime (see, e.g., Bai and Silverstein [13]). The properties of high-dimensional optimal portfolio weights are also studied by Fan et al. [14], Hautsch et al. [15], Ao et al. [16], Kan et al. [17], Bodnar et al. [18], Cai et al. [19], Ding et al. [20], among others.

Shrinkage approach is one of the mostly used methods to construct an improved estimator for the weights of the GMV portfolio. Shrinkage-type estimators were first proposed by Stein [21] with the aim to reduce the estimation error present in the sample mean vector computed for a sample from a multivariate normal distribution. Recently, this procedure has also been applied in the construction of the improved estimators of the high-dimensional mean vector (cf, Chételat and Wells [22], Wang et al. [23], Bodnar et al. [24]), covariance matrix (see, e.g., Ledoit and Wolf [25], Ledoit and Wolf [26], Bodnar et al. [27]), inverse of the covariance matrix (see, e.g., Wang et al. [28], Bodnar et al. [29]), as well as of the optimal portfolio weights (see, Bodnar et al. [10], Bodnar et al. [12], Golosnoy and Okhrin [30], Frahm and Memmel [31], Ledoit and Wolf [32]). Robust estimators for the covariance matrix are suggested and applied to practical problems in Yang et al. [9], Tyler [33], Couillet and McKay [34], Sun et al. [35], Couillet et al. [36], Kammoun et al. [37], Elkhalil et al. [38], among others. Interval shrinkage estimators of optimal portfolio weights have recently been derived by Bodnar et al. [39], Bodnar et al. [40].

The shrinkage estimator for the weights of the GMV portfolio are obtained as a linear combination of the sample estimator $\hat{\mathbf{w}}_S$ and the target portfolio $\mathbf{b}$ with $\mathbf{b}^\top\mathbf{1} = 1$. The estimator is expressed as (see, Bodnar et al. [10])

$$\hat{\mathbf{w}}_{SH} = \hat{\psi}_n \hat{\mathbf{w}}_S + (1-\hat{\psi}_n)\mathbf{b} \tag{I.5}$$

where

$$\hat{\psi}_n = \frac{(1-c_n)\hat{R}_{\mathbf{b}}}{c_n + (1-c_n)\hat{R}_{\mathbf{b}}}, \tag{I.6}$$

with $\hat{R}_{\mathbf{b}} = (1-c_n)\mathbf{b}^\top\mathbf{S}_n\mathbf{b}\cdot\mathbf{1}_p^\top\mathbf{S}_n^{-1}\mathbf{1}_p - 1$ and $c_n = p/n$. The shrinkage estimator of the GMV portfolio weights is obtained by minimizing the out-of-sample variance. This leads to the expression of the optimal shrinkage intensity ψ_n, which depends on the unknown population covariance matrix $\mathbf{\Sigma}$. The methods of the random matrix theory are then used to derive a consistent estimator (I.6) of the optimal shrinkage intensity and to construct the resulting (bona-fide) estimator of the GMV portfolio weights as given in (I.5). Bodnar et al. [10] show that the shrinkage estimator outperforms the sample estimator of the GMV portfolio weights in terms of minimizing the out-of-sample portfolio variance and the difference becomes drastic when p approaches n. Moreover, the shrinkage estimator of the GMV portfolio weights (I.5) provides a simple and a promising procedure how the one-period portfolio choice problem based on minimizing the portfolio variance can be solved in practice.

Once an optimal portfolio is determined, an investor faces with the problem of optimal portfolio reallocation in the next period of time. One of the important decision to be made by the investor is to decide whether the holding portfolio is optimal or has to be adjusted (see, e.g., Bodnar [41]), while Golosnoy et al. [42] consider the exponential smoothing method to predict the weights of the GMV portfolio over some periods of time. In the current paper we contribute to the literature by developing a dynamic GMV portfolio based on the shrinkage approach. At each time point of the portfolio reconstruction the traditional estimator of the GMV portfolio weights is shrunk towards the weights of the holding portfolio, which by construction are the shrinkage estimator of the GMV portfolio from the previous period. The practical advantage of the new dynamic trading strategy is two-fold: (i) First, it diminishes the transaction costs required for the reconstruction of the holding portfolio; (ii) Second, it reduces the out-of-sample variance of the constructed GMV portfolio by applying the shrinkage approach in the estimation of the portfolio weights.

From the perspectives of statistical theory, we develop new results that allow us to use the shrinkage estimators with a random target. These estimators are obtained under weak conditions imposed on the data-generating process. In particular, only the existence of the fourth moments is needed without explicit specification of the probability distribution assumed for the asset returns. Moreover, no assumption about the boundedness of the largest eigenvalue of the population covariance matrix is imposed in the paper. In particular, it can be as large as in the factor models (see, e.g., Ding et al. [20], Fan et al. [43], Fan et al. [44]). We only require that the ratio of the variances of the target portfolio and the GMV portfolio is bounded.

To achieve the goal we provide the asymptotic limits of general bilinear forms of the product of inverses of two dependent random matrices. The dependence is arising from the dynamic structure of the considered stochastic model. This type of theoretical results are entirely new in the random matrix theory and allow the application of the overlapping samples in the determination of the target portfolio and in the construction of the traditional portfolio used in the specification of the shrinkage GMV portfolio. The derived dynamic trading strategies can also be applied when the sample size is smaller than the portfolio dimension. We also provide the expressions of the shrinkage estimators of the dynamic GMV portfolio weights, when a weighted estimator of the covariance matrix is used in both cases with non-overlapping and overlapping samples. Finally, the obtained findings are implemented in the R-package *DOSPortfolio* (see, Bodnar et al. [45]).

The existent in the literature results on the shrinkage estimation of optimal portfolio weights are obtained by considering single-period optimal portfolio choice problems. Moreover, the estimators of the optimal portfolio weights are derived by using improved estimators of the covariance matrix (Rubio et al. [8], Yang et al. [9], Ledoit and Wolf [32]) or by shrinking the sample estimator of the portfolio weights to a deterministic target portfolio (see, Bodnar et al. [10], Bodnar et al. [12], Golosnoy and Okhrin [30], Frahm and Memmel [31]). The new results derived in this paper extend previous studies by proposing dynamic trading strategies and allowing the shrinkage target to be data-driven and, consequently stochastic.

It is remarkable that the statistical methods developed in the paper can be linked to the approaches applied in statistical signal processing by noting that the GMV portfolio is related to the Capon or minimum variance spatial filter in signal processing literature (cf, Verdú [46] and Van Trees [47]). Rubio et al. [8] and Yang et al. [48] investigate the estimation risk of the high-dimensional minimum variance beamformer, while Li et al. [49] study its constrained versions. The applications of random matrix theory to signal processing and portfolio optimization are presented in Yang et al. [9], Bodnar et al. [39], Bodnar et al. [40], Feng and Palomar [50].

The rest of the paper is organized as follows. In Section II, the main theoretical findings of the paper are provided. The dynamic shrinkage estimator for the weights of the GMV portfolio is derived in the case of non-overlapping samples in Section II-A, while Section II-B presents the results in the overlapping case. Section III extends the results of Section II to the case when a weighted estimator of the sample covariance matrix is used. The performance of the new trading strategies is investigated in Section IV via an extensive simulation study, where the approaches are also compared to the existing ones. In Section V, the new approaches to estimate the GMV portfolio are implemented to the real data consisting of the returns on stocks included in S&P 500 index. Concluding remarks are given in Section VI, while the technical proofs are moved to the supplementary material.

II. Dynamic Estimation of GMV Portfolio

Throughout the paper we assume that the GMV portfolio is constructed at time point t_1 by using the sample of size n_1, and then the investor updates the constructed GMV portfolio as new information arrives on the capital market. The information set is presented as a sequence of asset returns taken between time point t_{i-1} and t_i for $i = 2, \ldots, T$. Between each pairs (t_{i-1}, t_i) it is assumed that n_i vectors of asset returns are available which are collected into the data matrix $\mathbf{Y}_{n_i}$ that is assumed to possess the following stochastic representation:

$$\mathbf{Y}_{n_i} = \boldsymbol{\mu}\mathbf{1}_{n_i}^\top + \boldsymbol{\Sigma}^{\frac{1}{2}}\mathbf{X}_{n_i}, \tag{II.1}$$

where $\mathbf{X}_{n_i}$ is a $p \times n_i$ matrix which consists of independent and identically distributed (i.i.d.) real random variables with zero mean and unit variance. Also, we assume that $\mathbf{Y}_{n_i}, i = 1, \ldots, T$, are independent random matrices and the entries of $\mathbf{X}_{n_i}$, $i = 1, \ldots, T$, possess the $4 + \varepsilon$, $\varepsilon > 0$, moments, while no specific distributional assumption is imposed on the element of $\mathbf{X}_{n_i}$. To this end, it is assumed that the smallest eigenvalue of the population covariance matrix $\boldsymbol{\Sigma}$ is uniformly bounded in p away from zero, which ensures that $\boldsymbol{\Sigma}^{-1}$ exists for all p and its smallest eigenvalue does not converge to zero even if $p \to \infty$.

We consider an investor who opts on the shrinkage estimation of the GMV portfolio weights in each period of time t_i. Namely, after constructing the shrinkage estimator of the GMV portfolio as defined in (I.5) at time point t_1, the investor updates the GMV portfolio weights by shrinking their sample estimator computed at each time point t_i to the holding GMV portfolio determined at time point t_{i-1}. Two estimation strategies are developed in this section, which are based on non-overlapping and overlapping samples, respectively. The first procedure can be related to the rolling window estimation but with probably different sample sizes. The main advantage here is that smaller sample sizes are used in the construction of the sample weights of the GMV portfolio and, thus, the extreme observation observed in the asset returns will sooner be detected. Such a strategy might be recommendable during the turbulent period on the capital market, since it allows a faster adjustment of the holding portfolio. In contrary, when the stable period on the capital market is present, then the investor would prefer to use all available information, which leads to the extended window estimation strategy. In this case the part of data used in the construction of the GMV portfolio has already been used to determine the currently holding portfolio to which the new estimator is shrunk and, consequently, we have the case with overlapping samples. Both situations require completely different techniques from random matrix theory to be developed in order to derive the stochastic properties of the estimation procedures, which are developed in the consequent two subsections.

A. Dynamic GMV Portfolio With Non-Overlapping Samples

Under the non-overlapping scenario, the investor uses the sample of asset returns collected in $\mathbf{Y}_{n_i}$ to construct the sample estimator of the GMV portfolio at each time point t_i expressed as

$$\hat{\mathbf{w}}_{S;n_i} = \frac{\mathbf{S}_{n_i}^{-1}\mathbf{1}_p}{\mathbf{1}_p^\top\mathbf{S}_{n_i}^{-1}\mathbf{1}_p} \tag{II.2}$$

where

$$\mathbf{S}_{n_i} = \frac{1}{n_i - 1}\mathbf{Y}_{n_i}\left(\mathbf{I}_{n_i} - \frac{1}{n_i}\mathbf{1}_{n_i}\mathbf{1}_{n_i}^\top\right)\mathbf{Y}_{n_i}^\top. \tag{II.3}$$

The shrinkage estimator of the GMV portfolio is then obtained at time point t_i by shrinking (II.2) to the weights of the holding portfolio, i.e., to the shrinkage estimator of the GMV portfolio $\hat{\mathbf{w}}_{SH;n_{i-1}}$ constructed in the previous period, by minimizing the loss function determined as the out-of-sample variance with respect to the shrinkage intensity ψ_{n_i} in the following way:

$$\min_{\psi_i} L_i(\psi_i) = \min_{\psi_i} \hat{\mathbf{w}}_{SH;i}^\top \mathbf{\Sigma} \hat{\mathbf{w}}_{SH;i} \tag{II.4}$$

with

$$\hat{\mathbf{w}}_{SH;n_i} = \psi_i \hat{\mathbf{w}}_{S;n_i} + (1-\psi_i)\hat{\mathbf{w}}_{SH;n_{i-1}}, \tag{II.5}$$

where $\hat{\mathbf{w}}_{SH;n_0} = \mathbf{b}$ is the shrinkage target used for the construction of the shrinkage estimator for the GMV portfolio weights at time point t_1.

Rewriting (II.4) we get

$$\begin{aligned} L_i(\psi_i) &= \psi_i^2 \hat{\mathbf{w}}_{S;n_i}^\top \mathbf{\Sigma} \hat{\mathbf{w}}_{S;n_i} \\ &\quad + 2\psi_i(1-\psi_i)\hat{\mathbf{w}}_{S;n_i}^\top \mathbf{\Sigma} \hat{\mathbf{w}}_{SH;n_{i-1}} \\ &\quad + (1-\psi_i)^2 \hat{\mathbf{w}}_{SH;n_{i-1}}^\top \mathbf{\Sigma} \hat{\mathbf{w}}_{SH;n_{i-1}}, \end{aligned} \tag{II.6}$$

which is minimized at

$$\psi_{n_i}^* = \frac{\hat{\mathbf{w}}_{SH;n_{i-1}}^\top \mathbf{\Sigma} \left(\hat{\mathbf{w}}_{SH;n_{i-1}} - \hat{\mathbf{w}}_{S;n_i}\right)}{\left(\hat{\mathbf{w}}_{SH;n_{i-1}} - \hat{\mathbf{w}}_{S;n_i}\right)^\top \mathbf{\Sigma} \left(\hat{\mathbf{w}}_{SH;n_{i-1}} - \hat{\mathbf{w}}_{S;n_i}\right)}. \tag{II.7}$$

The optimal shrinkage intensity $\psi_{n_i}^*$ depends on the unknown population covariance matrix $\mathbf{\Sigma}$ and, thus, it cannot be computed in practice. To derive its consistent estimator we proceed in two steps: (i) first, we find a deterministic asymptotic equivalent of $\psi_{n_i}^*$ and (ii) then, we estimate this univariate quantity consistently in the high-dimensional setting. In Theorem II.1, the asymptotic equivalent to $\psi_{n_i}^*$ is provided for each t_i, while its consistent estimator is given in the discussion after the theorem.

Theorem II.1: Let $\mathbf{Y}_{n_i}$ possess the stochastic representation as in (II.1) and let $\mathbf{b}$ be the deterministic shrinkage target for $i=1$. Assume that the relative loss of portfolio $\mathbf{b}$ given by

$$r_0 = \frac{V_{\mathbf{b}}}{V_{GMV}} - 1 = \mathbf{1}_p^\top \mathbf{\Sigma}^{-1} \mathbf{1}_p \mathbf{b}^\top \mathbf{\Sigma} \mathbf{b} - 1 \tag{II.8}$$

is uniformly bounded in p, where

$$V_{\mathbf{b}} = \mathbf{b}^\top \mathbf{\Sigma} \mathbf{b} \text{ and } V_{GMV} = \mathbf{w}_{GMV}^\top \mathbf{\Sigma} \mathbf{w}_{GMV} = \frac{1}{\mathbf{1}_p^\top \mathbf{\Sigma}^{-1} \mathbf{1}_p} \tag{II.9}$$

are the variances of the target portfolio $\mathbf{b}$ and of the population GMV portfolio, respectively. Then it holds that

$$\left|\psi_{n_i}^* - \psi_i^*\right| \overset{a.s.}{\to} 0 \quad \text{with} \quad \psi_i^* = \frac{(1-c_i)r_{i-1}}{(1-c_i)r_{i-1} + c_i} \tag{II.10}$$

for $p/n_i \to c_i \in (0,1)$ as $n \to \infty$ where r_i is the asymptotic equivalent of the relative loss $r_{\hat{\mathbf{w}}_{SH;n_i}} = \mathbf{1}_p^\top \mathbf{\Sigma}^{-1} \mathbf{1}_p \hat{\mathbf{w}}_{SH;n_i}^\top \mathbf{\Sigma} \hat{\mathbf{w}}_{SH;n_i} - 1$ of the portfolio with weights $\hat{\mathbf{w}}_{SH;n_i}$ given by

$$r_i = (\psi_i^*)^2 \frac{c_i}{1-c_i} + (1-\psi_i^*)^2 r_{i-1} \tag{II.11}$$

for $i = 1, \ldots, T$.

The proof of Theorem II.1 is given in the supplementary material. Its results provide a simple recursive algorithm how the shrinkage intensities have to be computed in practice. Independently of the number of portfolio reallocations, T, the only unknown quantity in the algorithm is the relative loss of the target portfolio $\mathbf{b}$ used in the construction of the shrinkage estimator for $i=1$. Using the sample $\mathbf{Y}_{n_1}$ its consistent estimator is given by

$$\hat{r}_0 = \left(1 - \frac{p}{n_1}\right) \mathbf{1}_p^\top \mathbf{S}_{n_1}^{-1} \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{n_1} \mathbf{b} - 1. \tag{II.12}$$

Then, the resulting (bona fide) shrinkage estimator of the GMV portfolio at time t_i is given by

$$\hat{\mathbf{w}}_{BF;n_i} = \hat{\psi}_i^* \hat{\mathbf{w}}_{S;n_i} + (1-\hat{\psi}_i^*)\hat{\mathbf{w}}_{BF;n_{i-1}} \tag{II.13}$$

where $\hat{\psi}_i^* = \frac{(n_i-p)\hat{r}_{i-1}}{(n_i-p)\hat{r}_{i-1}+p}$ and $\hat{r}_i$ is computed recursively by

$$\hat{r}_i = (\hat{\psi}_i^*)^2 \frac{p}{n_i - p} + (1-\hat{\psi}_i^*)^2 \hat{r}_{i-1} \tag{II.14}$$

with $\hat{r}_0$ as in (II.12) and $\hat{\mathbf{w}}_{BF;n_0} = \mathbf{b}$.

We conclude this section with several important remarks:

Remark II.2: The deterministic target portfolio $\mathbf{b}$ can also be replaced by the sample GMV portfolio computed by using data available before the sample $\mathbf{Y}_{n_1}$ is taken. If we denote these data by $\mathbf{Y}_{n_0}$, then the target weights $\mathbf{b}$ are replaced by

$$\hat{\mathbf{w}}_{S;n_0} = \frac{\mathbf{S}_{n_0}^{-1} \mathbf{1}_p}{\mathbf{1}_p^\top \mathbf{S}_{n_0}^{-1} \mathbf{1}_p} \tag{II.15}$$

In this case the relative loss r_0 does not longer depend on the population covariance matrix $\mathbf{\Sigma}$ and following the proof of Theorem II.1 it is given by

$$\tilde{r}_0 = \frac{c_0}{1-c_0} \approx \frac{p}{n_0 - p}.$$

As a result, the (bona fide) shrinkage estimator of the GMV portfolio weights is obtained as in (II.13) and (II.14) with $\hat{r}_0$ replaced by $\tilde{r}_0$ and $\hat{\mathbf{w}}_{BF;n_0} = \hat{\mathbf{w}}_{S;n_0}$. In a similar way other random targets can be employed into our model, e.g., nonlinear shrinkage Ledoit and Wolf [26], but then the asymptotics and estimation of r_0 become highly nontrivial and one needs to handle every of those targets separately. Because of the large number of possible target portfolios $\mathbf{b}$ we leave this interesting topic for the future research. For the sake of brevity concentrate ourselves on the naive equally weighted target $\mathbf{b} = \mathbf{1}_p/p$ in our simulation and empirical studies.

Remark II.3: The results of Theorem II.1 are derived under very week conditions which require the existent of $4+\varepsilon, \varepsilon > 0$, moments only. No structural assumption on $\mathbf{\Sigma}$ neither on $\mathbf{b}$ are imposed.

Remark II.4: Other consistent estimators for r_0 can be constructed. For instance, we can update our estimator at each time point t_i as soon as new data of the asset returns become available. Let $N_i = \sum_{j=1}^{i} n_j$ be the total number of asset return vectors available at time point t_i and let $\mathbf{Y}_{N_i}$ be the $p \times N_i$ matrix of the asset returns up to time N_i that is $\mathbf{Y}_{N_i} = (\mathbf{Y}_{n_1} \mathbf{Y}_{n_2} \ldots \mathbf{Y}_{n_i})$. Then, at time point t_i, a consistent estimator for r_0 is obtained by

$$\hat{r}_{0;i} = \left(1 - \frac{p}{N_i}\right) \mathbf{1}_p^\top \mathbf{S}_{N_i}^{-1} \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{N_i} \mathbf{b} - 1. \tag{II.16}$$

where $\mathbf{S}_{N_i}$ is the sample covariance matrix based on the data matrix $\mathbf{Y}_{N_i}$ as given in (I.3) with $n = N_i$. Then, the (bona fide)

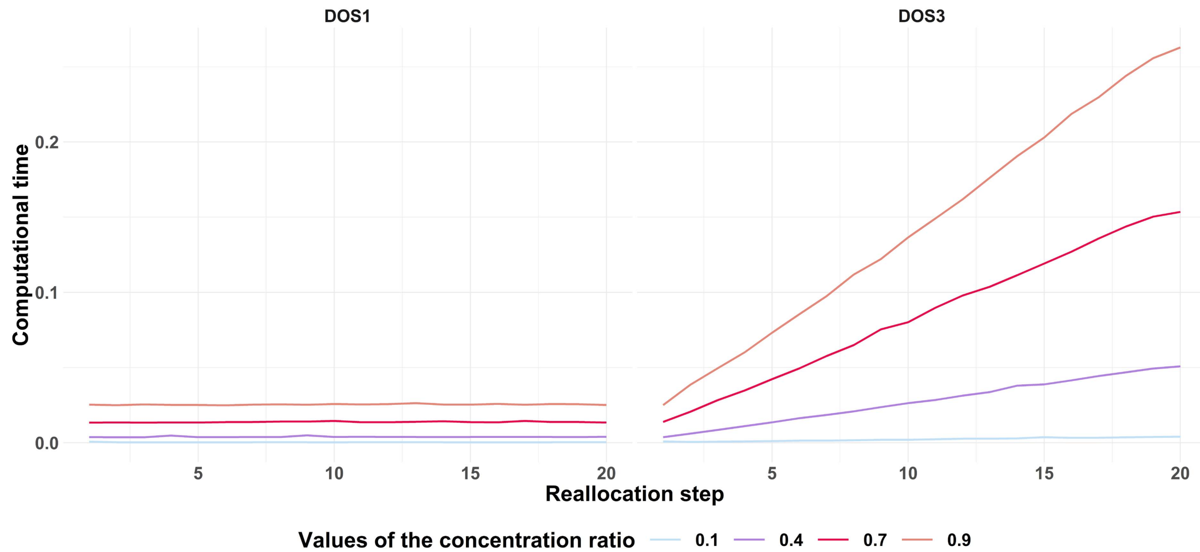

Fig. 1. The average computational time of dynamic GMV portfolios when the relative loss r_0 of the target portfolio $\mathbf{b}$ is estimated by $\hat{r}_0$ as in (II.12) labeled as **DOS1** and when the relative loss r_0 of the target portfolio $\mathbf{b}$ is re-estimated in each period as in (II.16) labeled as **DOS3**. The calculation of the computation time is based on 500 independent runs where $\mathbf{Y}_{n_i}$ is drawn from a t-distribution in each run with $n_i = 250, i = 1, 2, 3, \ldots 20$ for different values of the concentration ratio.

shrinkage estimator of the GMV portfolio weights is computed following (II.13) and (II.14) with $\hat{r}_0$ replaced by $\hat{r}_{0;i}$. Since larger dataset is used to estimate r_0, we expect that this approach will perform better as the one suggested in (II.12)–(II.14). On the other side, the new method is more time demanding, since the recursion in (II.14) has to be started from the beginning at each time t_i.

In Fig. 1 computation costs of two dynamic optimal shrinkage portfolios are depicted for several values of the concentration ratio. The results are obtained by drawing samples from a t-distribution as described in Section IV. As a target portfolio, the equally-weighted portfolio $\mathbf{b} = \mathbf{1}_p/p$ is used. The two plots show that the application of the trading strategy based on the re-estimated loss of the target portfolio can considerably increase the computation time, especially for a large value of the concentration ratio.

B. Dynamic GMV Portfolio With Overlapping Samples

In this section, we present the shrinkage estimator for the GMV portfolio which is constructed based on the overlapping samples. In Remark II.4 it is suggested to use all available data $\mathbf{Y}_{n_1}, \mathbf{Y}_{n_2}, \ldots, \mathbf{Y}_{n_i}$ up to time point t_i to determine a consistent estimator for the relative loss r_0 of portfolio $\mathbf{b}$. Here, we use the similar idea in the construction of the sample estimator of the GMV portfolio weights at time t_i. Such an approach possesses an advantage that we only require $n_1 > p$, while the other sample sizes $n_2, \ldots, n_T$ can also be smaller than the portfolio dimension p.

Using the notations N_i, $\mathbf{Y}_{N_i}$, and $\mathbf{S}_{N_i}$ introduced in Remark II.4, we define

$$\hat{\mathbf{w}}_{S;N_i} = \frac{\mathbf{S}_{N_i}^{-1}\mathbf{1}_p}{\mathbf{1}_p^\top \mathbf{S}_{N_i}^{-1}\mathbf{1}_p}. \quad \text{(II.17)}$$

as the sample estimator of the GMV portfolio weights based on data of the asset returns included in $\mathbf{Y}_{N_i}$. Substituting $\hat{\mathbf{w}}_{S;N_i}$ instead of $\hat{\mathbf{w}}_{S;n_i}$ in (II.5), the loss function $L_i(\psi_i)$ in (II.4) is maximized at

$$\Psi_{N_i}^* = \frac{\hat{\mathbf{w}}_{SH;N_{i-1}}^\top \boldsymbol{\Sigma}\left(\hat{\mathbf{w}}_{SH;N_{i-1}} - \hat{\mathbf{w}}_{S;N_i}\right)}{\left(\hat{\mathbf{w}}_{SH;N_{i-1}} - \hat{\mathbf{w}}_{S;N_i}\right)^\top \boldsymbol{\Sigma}\left(\hat{\mathbf{w}}_{SH;N_{i-1}} - \hat{\mathbf{w}}_{S;N_i}\right)}. \quad \text{(II.18)}$$

In Theorem II.5 we derive an iterative procedure for computing the deterministic equivalents to $\Psi_{N_i}^*$ for $i = 1, \ldots, T$. The proof of Theorem II.5 is given in the supplementary material.

Theorem II.5: Let $\mathbf{Y}_{n_i}$ possess the stochastic representation as in (II.1) and let $\mathbf{b}$ be the deterministic shrinkage target for $i = 1$. Assume that the relative loss of portfolio $\mathbf{b}$ given by $R_0 = \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}_p \mathbf{b}^\top \boldsymbol{\Sigma}\mathbf{b} - 1$ is uniformly bounded in p. Then it holds that

$$\left|\Psi_{N_i}^* - \Psi_i^*\right| \stackrel{a.s.}{\to} 0$$

for $p/N_j \to C_j \in (0, 1)$ as $N_j \to \infty$, $j = 1, \ldots, i$ and $i = 1, \ldots, T$ where

$$\Psi_i^* = \frac{(R_{i-1}+1) - K_i}{(R_{i-1}+1) + (1-C_i)^{-1} - 2K_i}, \quad \text{(II.19)}$$

$$K_i = \beta_{i-1;0}^* + \frac{1}{1-C_i}\sum_{j=1}^{i-1}\beta_{i-1;j}^*, \quad \text{(II.20)}$$

$$R_i = (\Psi_i^*)^2 \frac{C_i}{1-C_i} + (1-\Psi_i^*)^2 R_{i-1} + 2\Psi_i^*(1-\Psi_i^*)(K_i - 1), \quad \text{(II.21)}$$

with $\beta_{0;0}^* = 1$, $\beta_{i-1;i-1}^* = \Psi_{i-1}^*$ and

$$\beta_{i-1;k}^* = (1-\Psi_{i-1}^*)\beta_{i-2;k}^*, \quad \text{(II.22)}$$

for $k = 0, \ldots i-2$.

Similarly, to the case with non-overlapping samples, the recursive procedure derived in Theorem II.5 depends only on a univariate unobservable quantity R_0, which is the relative loss of the target portfolio $\mathbf{b}$ at time point t_1. Both approaches suggested in Section II-A can be used to construct a consistent estimator for R_0, and hence to obtain a (bona fide) estimator of the GMV portfolio weights. These procedures are the following:

- We estimate R_0 by
$$\hat{R}_0 = \hat{r}_0 = \left(1 - \frac{p}{N_1}\right) \mathbf{1}_p^\top \mathbf{S}_{N_1}^{-1} \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{N_1} \mathbf{b} - 1 \quad \text{(II.23)}$$
as in (II.12). In this case the estimator for R_0 is constructed by using the first sample $\mathbf{Y}_{N_1}$ only and the recursive procedure of Theorem II.5 is then used leading to the (bona fide) optimal shrinkage estimators for the weights at each time point t_i, $i \in 1, \ldots, T$ expressed as
$$\hat{\mathbf{w}}_{BF;N_i} = \hat{\Psi}_i^* \hat{\mathbf{w}}_{S;N_i} + (1 - \hat{\Psi}_i^*) \hat{\mathbf{w}}_{BF;N_{i-1}} \quad \text{(II.24)}$$
where $\hat{\Psi}_i^*$ is computed recursively as in Theorem II.5 with R_0 replaced by $\hat{R}_0$ and using the empirical counterpart for C_i given by $C_{N_i} = p/N_i$.
- At each time point i, we use all available information to estimate R_0, i.e.,
$$\hat{R}_{0;i} = \hat{r}_{0;i} = \left(1 - \frac{p}{N_i}\right) \mathbf{1}_p^\top \mathbf{S}_{N_i}^{-1} \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{N_i} \mathbf{b} - 1 \quad \text{(II.25)}$$
and recompute the recursion of Theorem II.5 at each time point t_i. Since a larger dataset is used to estimate R_0, better results are expected although the computation becomes more time demanding in the second case.

To this end, we note that the deterministic target portfolio $\mathbf{b}$ can be replaced by the sample GMV portfolio computed by using data $\mathbf{Y}_{n_0}$ available before the first sample $\mathbf{Y}_{N_1}$ as in (II.15) of Remark II.2. In this case we get
$$\widetilde{R}_0 = \frac{p}{n_0 - p},$$
which is used in the iterative computation of Theorem II.5 instead of R_0. Since no unknown quantities are present in the definition of $\widetilde{R}_0$, the iterative procedure of Theorem II.5 becomes deterministic.

C. Dynamic GMV Portfolio for Singular Sample Covariance Matrix

In this section we extend the results of Sections II-A and II-B to the case when the sample sizes in each block are smaller than the portfolio dimension p. In such a situation the sample covariance matrix $\mathbf{S}_{n_i}$ defined in (II.3) is singular and, consequently, its inverse does not exist for $i = 1, \ldots, T$. In practice, the classical inverse is replaced by a generalized inverse, for example, by the Moore-Penrose inverse $\mathbf{S}_{n_i}^+$, which is then used in the computation of portfolio weights. We recall that the Moore-Penrose inverse is uniquely determined and, for a matrix $\mathbf{A}$, it is defined as a matrix which fulfills the following four conditions: (i) $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$, (ii) $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$, (iii) $(\mathbf{A}^+\mathbf{A})^\top = \mathbf{A}^+\mathbf{A}$, and (iv) $(\mathbf{A}\mathbf{A}^+)^\top = \mathbf{A}\mathbf{A}^+$.

First, we present the results for the non-overlapping scenario. Using the Moore-Penrose inverse $\mathbf{S}_{n_i}$, the sample estimator of the GMV portfolio at each time point t_i is given by
$$\hat{\mathbf{w}}_{MPS;n_i} = \frac{\mathbf{S}_{n_i}^+ \mathbf{1}_p}{\mathbf{1}_p^\top \mathbf{S}_{n_i}^+ \mathbf{1}_p}. \quad \text{(II.26)}$$
Following the derivation of Section II-A, the minimization of the loss function leads to the shrinkage estimator of the GMV portfolio at time t_i expressed as
$$\hat{\mathbf{w}}_{MPSH;n_i} = \psi_{MP;n_i}^* \hat{\mathbf{w}}_{MPS;n_i} + (1 - \psi_{MP;n_i}^*) \hat{\mathbf{w}}_{MPSH;n_{i-1}}, \quad \text{(II.27)}$$
with $\hat{\mathbf{w}}_{MPSH;n_0} = \mathbf{b}$ and
$$\psi_{MP;n_i}^* = \frac{\hat{\mathbf{w}}_{MPSH;n_{i-1}}^\top \boldsymbol{\Sigma} \left(\hat{\mathbf{w}}_{MPSH;n_{i-1}} - \hat{\mathbf{w}}_{MPS;n_i}\right)}{\left(\hat{\mathbf{w}}_{MPSH;n_{i-1}} - \hat{\mathbf{w}}_{MPS;n_i}\right)^\top \boldsymbol{\Sigma} \left(\hat{\mathbf{w}}_{MPSH;n_{i-1}} - \hat{\mathbf{w}}_{MPS;n_i}\right)}. \quad \text{(II.28)}$$

In Theorem II.6 we derive the asymptotic equivalent to $\psi_{MP;n_i}^*$ whose consistent estimator will be later used in the construction of the shrinkage estimator at time t_i.

Theorem II.6: Let $\mathbf{Y}_{n_i}$ possess the stochastic representation as in (II.1) and let $\mathbf{b}$ be the deterministic shrinkage target for $i = 1$. Assume that the relative loss of portfolio $\mathbf{b}$ given by $r_0 = \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}_p \mathbf{b}^\top \boldsymbol{\Sigma} \mathbf{b} - 1$ is uniformly bounded in p. Then it holds that
$$\left|\psi_{MP;n_i}^* - \psi_{MP;i}^*\right| \xrightarrow{a.s.} 0$$
with
$$\psi_{MP;i}^* = \frac{(c_i - 1) r_{MP;i-1}}{(c_i - 1) r_{MP;i-1} + c_i + (c_i - 1)^2} \quad \text{(II.29)}$$
for $p/n_i \to c_i > 1$ as $n \to \infty$ where r_i is the asymptotic equivalent of the relative loss $r_{\hat{\mathbf{w}}_{MPSH;n_i}} = \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}_p \hat{\mathbf{w}}_{MPSH;n_i}^\top \boldsymbol{\Sigma} \hat{\mathbf{w}}_{MPSH;n_i} - 1$ of the portfolio with weights $\hat{\mathbf{w}}_{MPSH;n_i}$ given by
$$r_{MP;i} = (\psi_{MP;i}^*)^2 \frac{c_i^2 - c_i + 1}{c_i - 1} + (1 - \psi_{MP;i}^*)^2 r_{MP;i-1} \quad \text{(II.30)}$$
for $i = 1, \ldots, T$ with $r_{MP;0} = r_0$.

The proof of Theorem II.6 is given in the supplementary material. Similarly to the case when the portfolio dimension p is smaller than the sample size, the results of Theorem II.6 present a simple recursive algorithm for the computation of the shrinkage intensities with r_0 being the only unknown quantity which is consistently estimated by
$$\hat{r}_{MP;0} = \frac{p}{n_1} \left(\frac{p}{n_1} - 1\right) \mathbf{1}_p^\top \mathbf{S}_{n_1}^+ \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{n_1} \mathbf{b} - 1. \quad \text{(II.31)}$$
Then, the resulting (bona fide) shrinkage estimator of the GMV portfolio at time t_i is given by
$$\hat{\mathbf{w}}_{MPBF;n_i} = \hat{\psi}_{MP;i}^* \hat{\mathbf{w}}_{MPS;n_i} + (1 - \hat{\psi}_{MP;i}^*) \hat{\mathbf{w}}_{MPBF;n_{i-1}}, \quad \text{(II.32)}$$
where
$$\hat{\psi}_{MP;i}^* = \frac{n_i (p - n_i) \hat{r}_{MP;i-1}}{n_i (p - n_i) \hat{r}_{MP;i-1} + p n_i + (p - n_i)^2} \quad \text{(II.33)}$$

and

$$\hat{r}_{MP;i} = (\hat{\psi}^*_{MP;i})^2 \frac{p^2 - pn_i + n_i^2}{n_i(p-n_i)} + (1-\hat{\psi}^*_{MP;i})^2 \hat{r}_{MP;i-1} \tag{II.34}$$

with $\hat{r}_{MP;0}$ as in (II.31) and $\hat{\mathbf{w}}_{MPBF;n_0} = \mathbf{b}$.

The results of Theorem II.6 are derived by approximating the Moore-Penrose inverse with the reflexive inverse (see, e.g., Cook and Forzani [51]), which provides a good approximation of the Moore-Penrose inverse when $c_i \in (1,2)$ (see, Bodnar and Parolya [52]). In this case, the dynamic shrinkage estimator of the GMV portfolio (II.32)–(II.34) is expected to perform good in practice, while for larger values of c_i it should be used with caution. To this end, we note that the estimator $\hat{r}_{MP;0}$ for the relative loss of the target portfolio r_0 can be recomputed at each time t_i when a new sample becomes available. Given the sample of the asset returns $\mathbf{Y}_{N_i} = (\mathbf{Y}_{n_1}\, \mathbf{Y}_{n_2} \dots \mathbf{Y}_{n_i})$ at time point t_i, another consistent estimator for r_0 is given by

$$\hat{r}_{MP;0;i} = \begin{cases} \left(1 - \frac{p}{N_i}\right) \mathbf{1}_p^\top \mathbf{S}_{N_i}^{-1} \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{N_i} \mathbf{b} - 1 & p < N_i, \\ \frac{p}{N_i}\left(\frac{p}{N_i} - 1\right) \mathbf{1}_p^\top \mathbf{S}_{N_i}^{+} \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{N_i} \mathbf{b} - 1 & p > N_i. \end{cases}$$

Next, we present the dynamic shrinkage estimator of the GMV portfolio under the overlapping sample and singular sample covariance matrix. Following the previous discussion the results will be presented when c_i is below two, i.e., when the Moore-Penrose inverse of the sample covariance matrix can be good approximated by the reflexive inverse. Using the notations N_i, $\mathbf{Y}_{N_i}$, and $\mathbf{S}_{N_i}$ introduced in Remark II.4, we define

$$\hat{\mathbf{w}}_{MPS;N_i} = \frac{\mathbf{S}_{N_i}^{+}\mathbf{1}_p}{\mathbf{1}_p^\top \mathbf{S}_{N_i}^{+}\mathbf{1}_p}, \tag{II.35}$$

where the Moore-Penrose inverse of $\mathbf{S}_{N_i}$ becomes the ordinary inverse when $N_i > p$. Then, the dynamic shrinkage estimator of the GMV portfolio weights is given by

$$\hat{\mathbf{w}}_{MPSH;N_i} = \psi^*_{MP;N_i} \hat{\mathbf{w}}_{MPS;N_i} + (1-\psi^*_{MP;N_i}) \hat{\mathbf{w}}_{MPSH;N_{i-1}},$$

with $\hat{\mathbf{w}}_{MPSH;N_0} = \mathbf{b}$ and

$$\Psi^*_{MP;N_i} = \frac{\hat{\mathbf{w}}^\top_{MPSH;N_{i-1}} \boldsymbol{\Sigma} \left(\hat{\mathbf{w}}_{MPSH;N_{i-1}} - \hat{\mathbf{w}}_{MPS;N_i}\right)}{\left(\hat{\mathbf{w}}_{MPSH;N_{i-1}} - \hat{\mathbf{w}}_{MPS;N_i}\right)^\top \boldsymbol{\Sigma} \left(\hat{\mathbf{w}}_{MPSH;N_{i-1}} - \hat{\mathbf{w}}_{MPS;N_i}\right)}.$$

In Theorem II.7 we derive an iterative procedure for computing the deterministic equivalents to $\Psi^*_{MP;N_i}$ for $i = 1, \dots, T$. The proof of Theorem II.7 is given in the supplementary material.

Theorem II.7: Let $\mathbf{Y}_{n_i}$ possess the stochastic representation as in (II.1) and let $\mathbf{b}$ be the deterministic shrinkage target for $i = 1$. Assume that the relative loss of portfolio $\mathbf{b}$ given by $R_0 = \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}_p \mathbf{b}^\top \boldsymbol{\Sigma} \mathbf{b} - 1$ is uniformly bounded in p. Then it holds that

$$\left|\Psi^*_{MP;N_i} - \Psi^*_{MP;i}\right| \xrightarrow{a.s.} 0$$

for $p/N_j \to C_j$ as $N_j \to \infty$, $j = 1, \dots, i$ and $i = 1, \dots, T$ where

$$\Psi^*_{MP;i} = \frac{(R_{MP;i-1}+1) - K_{MP;i}}{(R_{MP;i-1}+1) + Q_{MP;i} - 2K_{MP;i}}, \tag{II.36}$$

$$R_{MP;i} = (\Psi^*_{MP;i})^2 (Q_{MP;i} - 1) \tag{II.37}$$

$$+ (1-\Psi^*_{MP;i})^2 R_{MP;i-1}$$

$$+ 2\Psi^*_{MP;i}(1-\Psi^*_{MP;i})(K_{MP;i}-1), \tag{II.38}$$

$$K_{MP;i} = \beta^*_{MP;i-1;0} + \sum_{j=1}^{i-1} \beta^*_{MP;i-1;j} D_{MP;j,i}, \tag{II.39}$$

with $\beta^*_{MP;0;0} = 1$, $\beta^*_{MP;i-1;i-1} = \Psi^*_{MP;i-1}$ and

$$\beta^*_{MP;i-1;k} = (1-\Psi^*_{MP;i-1})\beta^*_{MP;i-2;k}, \tag{II.40}$$

for $k = 0, \dots i-2$. Finally, $D_{MP;j,i}$ and $Q_{MP;i}$ are given by

$$D_{MP;j,i} = \begin{cases} \frac{C_j^2(1-C_i)}{(C_j - C_i)^2}, & j = 1, \\ (1-C_i)^{-1}, & j > 1. \end{cases} \tag{II.41}$$

and

$$Q_{MP;i} = \begin{cases} C_i^2 (C_i - 1)^{-1}, & i = 1, \\ (1-C_i)^{-1}, & i > 1. \end{cases} \tag{II.42}$$

Using the consistent estimator of R_0 expressed as

$$\hat{R}_{MP;0} = \hat{r}_{MP;0} = \frac{p}{N_1}\left(\frac{p}{N_1} - 1\right) \mathbf{1}_p^\top \mathbf{S}_{N_1}^{+} \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{N_1} \mathbf{b} - 1, \tag{II.43}$$

the (bona fide) shrinkage estimator of the GMV portfolio at time t_i is given by

$$\hat{\mathbf{w}}_{MPBF;N_i} = \hat{\Psi}^*_{MP;i} \hat{\mathbf{w}}_{MPS;N_i} + (1-\hat{\Psi}^*_{MP;i}) \hat{\mathbf{w}}_{MPBF;N_{i-1}},$$

where $\hat{\Psi}^*_{MP;i}$ is computed recursively, as provided in Theorem II.7 with R_0 replaced by $\hat{R}_{MP;0}$. To this end, we point out that similarly to the case with non-overlapping sample, the estimator of R_0 can be updated as soon as a new sample becomes available.

III. Dynamic GMV Portfolio With a Weighted Estimator of the Covariance Matrix

The sample estimator of the covariance matrix can considerably be impacted by the extreme asset returns and, as such, it is not robust to outliers. Robust estimators of the covariance matrix are considered in Tyler [33], Couillet and McKay [34], Sun et al. [35], Couillet et al. [36], Kammoun et al. [37], Maronna [53], among others, while Yang et al. [9] derive a shrinkage estimator of the GMV portfolio weights based on a robust estimator of the covariance matrix.

Following Rubio et al. [8], we consider a weighted estimator of the covariance matrix expressed as

$$\mathbf{S}_{\mathbf{W};n_i} = \frac{1}{n_i - 1} \mathbf{Y}_{n_i} \left(\mathbf{I}_{n_i} - \frac{1}{n_i} \mathbf{W}_{n_i} \mathbf{1}_{n_i} \mathbf{1}_{n_i}^\top\right) \times \mathbf{W}_{n_i} \left(\mathbf{I}_{n_i} - \frac{1}{n_i} \mathbf{1}_{n_i} \mathbf{1}_{n_i}^\top \mathbf{W}_{n_i}\right) \mathbf{Y}_{n_i}^\top, \tag{III.1}$$

where $\mathbf{W}_{n_i}$ is a deterministic diagonal matrix of weights with $\mathbf{1}_{n_i}^\top \mathbf{W}_{n_i} \mathbf{1}_{n_i} = n_i$. Then, under the non-overlapping scenario, the sample estimator of the GMV portfolio at each time t_i is given by

$$\hat{\mathbf{w}}_{WS;n_i} = \frac{\mathbf{S}_{\mathbf{W};n_i}^{-1} \mathbf{1}_p}{\mathbf{1}_p^\top \mathbf{S}_{\mathbf{W};n_i}^{-1} \mathbf{1}_p}. \tag{III.2}$$

Furthermore, the shrinkage estimator of the GMV portfolio is expressed as

$$\hat{\mathbf{w}}_{WSH;n_i} = \psi^*_{\mathbf{W};n_i} \hat{\mathbf{w}}_{WS;n_i} + (1 - \psi^*_{\mathbf{W};n_i}) \hat{\mathbf{w}}_{WSH;n_{i-1}}, \tag{III.3}$$

where $\hat{\mathbf{w}}_{WSH;n_0} = \mathbf{b}$ and

$$\psi^*_{\mathbf{W};n_i} = \frac{\hat{\mathbf{w}}_{WSH;n_{i-1}}^\top \boldsymbol{\Sigma} \left(\hat{\mathbf{w}}_{WSH;n_{i-1}} - \hat{\mathbf{w}}_{WS;n_i}\right)}{\left(\hat{\mathbf{w}}_{WSH;n_{i-1}} - \hat{\mathbf{w}}_{WS;n_i}\right)^\top \boldsymbol{\Sigma} \left(\hat{\mathbf{w}}_{WSH;n_{i-1}} - \hat{\mathbf{w}}_{WS;n_i}\right)}. \tag{III.4}$$

In Theorem III.1 we derive the asymptotic equivalents to $\psi^*_{\mathbf{W};n_i}$ for $i = 1, \ldots, T$.

Theorem III.1: Let $\mathbf{Y}_{n_i}$ possess the stochastic representation as in (II.1) and let $\mathbf{b}$ be the deterministic shrinkage target for $i = 1$. Assume that the relative loss of portfolio $\mathbf{b}$ given by $r_0 = \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}_p \mathbf{b}^\top \boldsymbol{\Sigma} \mathbf{b} - 1$ is uniformly bounded in p. Then it holds that

$$\left|\psi^*_{\mathbf{W};n_i} - \psi^*_{\mathbf{W};i}\right| \overset{a.s.}{\to} 0 \tag{III.5}$$

for $p/n_i \to c_i \in (0,1)$ as $n \to \infty$ with

$$\psi^*_{\mathbf{W};i} = \frac{u_{n_i}^2 r_{\mathbf{W};i-1}}{u_{n_i}^2 r_{\mathbf{W};i-1} + u'_{n_i} - u_{n_i}^2} \tag{III.6}$$

and $r_{\mathbf{W};i}$ the asymptotic equivalent of the relative loss $r_{\hat{\mathbf{w}}_{WSH;n_i}} = \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}_p \hat{\mathbf{w}}_{WSH;n_i}^\top \boldsymbol{\Sigma} \hat{\mathbf{w}}_{WSH;n_i} - 1$ of the portfolio with weights $\hat{\mathbf{w}}_{WSH;n_i}$ given by

$$r_{\mathbf{W};i} = (\psi^*_{\mathbf{W};i})^2 \frac{u'_{n_i} - u_{n_i}^2}{u_{n_i}^2} + (1 - \psi^*_{\mathbf{W};i})^2 r_{\mathbf{W};i-1} \tag{III.7}$$

for $i = 1, \ldots, T$ and $r_{\mathbf{W};0} = r_0$, where u_{n_i} is the solution of

$$1 - c_i = \frac{1}{n_i} \text{tr}[(\mathbf{I}_{n_i} + c_i u_{n_i} \mathbf{W}_{n_i})^{-1}]$$

and

$$u'_{n_i} = \frac{u_{n_i}}{\text{tr}[(\mathbf{I}_{n_i} + c_i u_{n_i} \mathbf{W}_{n_i})^{-2}]}.$$

The proof of Theorem III.1 is given in the supplementary material. To derive a consistent estimator for r_0 we need the following lemma whose proof is presented in the supplementary material.

Lemma III.2: Let $\mathbf{Y}_{n_i}$ possess the stochastic representation as in (II.1) and let $\mathbf{b}$ be the deterministic shrinkage target for $i = 1$. Then, it holds that

$$\left|\frac{\mathbf{b}^\top \mathbf{S}_{\mathbf{W};n_1} \mathbf{b}}{\mathbf{b}^\top \boldsymbol{\Sigma} \mathbf{b}} - 1\right| \overset{a.s.}{\to} 0, \tag{III.8}$$

for $p/n_1 \to c_1 \in (0,1)$ as $n_1 \to \infty$.

Using the proof of Theorem III.1 and the results of Lemma III.2 we get a consistent estimator of r_0 given by

$$\hat{r}_{\mathbf{W};0} = u_{n_1}^{-1} \mathbf{1}_p^\top \mathbf{S}_{\mathbf{W};n_1}^{-1} \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{\mathbf{W};n_1} \mathbf{b} - 1. \tag{III.9}$$

Then, the resulting recursive (bona fide) shrinkage estimator of the GMV portfolio at time t_i is expressed as

$$\hat{\mathbf{w}}_{WBF;n_i} = \hat{\psi}^*_{\mathbf{W};i} \hat{\mathbf{w}}_{WS;n_i} + (1 - \hat{\psi}^*_{\mathbf{W};i}) \hat{\mathbf{w}}_{WBF;n_{i-1}}, \tag{III.10}$$

where

$$\hat{\psi}^*_{\mathbf{W};i} = \frac{u_{n_i}^2 \hat{r}_{\mathbf{W};i-1}}{u_{n_i}^2 \hat{r}_{\mathbf{W};i-1} + u'_{n_i} - u_{n_i}^2} \tag{III.11}$$

and

$$\hat{r}_{\mathbf{W};i} = (\hat{\psi}^*_{\mathbf{W};i})^2 \frac{u'_{n_i} - u_{n_i}^2}{u_{n_i}^2} + (1 - \hat{\psi}^*_{\mathbf{W};i})^2 \hat{r}_{\mathbf{W};i-1} \tag{III.12}$$

with $\hat{r}_{\mathbf{W};0}$ as in (III.9) and $\hat{\mathbf{w}}_{WBF;n_0} = \mathbf{b}$.

Finally, using the sample of the asset returns $\mathbf{Y}_{N_i} = (\mathbf{Y}_{n_1}\, \mathbf{Y}_{n_2} \ldots \mathbf{Y}_{n_i})$ at time point t_i, another consistent estimator for r_0 is given by

$$\hat{r}_{\mathbf{W};0;i} = u_{N_i}^{-1} \mathbf{1}_p^\top \mathbf{S}_{\mathbf{W};N_i}^{-1} \mathbf{1}_p \mathbf{b}^\top \mathbf{S}_{\mathbf{W};N_i} \mathbf{b} - 1, \tag{III.13}$$

where u_{N_i} is the solution of the following equation

$$1 - C_i = \frac{1}{N_i} \text{tr}[(\mathbf{I}_{N_i} + C_i u_{N_i} \mathbf{W}_{N_i})^{-1}].$$

Next, we discuss the dynamic shrinkage estimator of the GMV portfolio under the overlapping sample when the weighted estimator of the covariance matrix is used. The sample estimator of the GMV portfolio based on the sample $\mathbf{Y}_{N_i}$ and the weighted estimator of the covariance matrix $\mathbf{S}_{\mathbf{W};N_i}$ is expressed as

$$\hat{\mathbf{w}}_{WS;N_i} = \frac{\mathbf{S}_{\mathbf{W};N_i}^{-1} \mathbf{1}_p}{\mathbf{1}_p^\top \mathbf{S}_{\mathbf{W};N_i}^{-1} \mathbf{1}_p}, \tag{III.14}$$

while the weights of the shrinkage portfolio are given by

$$\hat{\mathbf{w}}_{WSH;N_i} = \Psi^*_{\mathbf{W};N_i} \hat{\mathbf{w}}_{WS;N_i} + (1 - \Psi^*_{\mathbf{W};N_i}) \hat{\mathbf{w}}_{WSH;N_{i-1}}, \tag{III.15}$$

with

$$\Psi^*_{\mathbf{W};N_i} = \frac{\hat{\mathbf{w}}_{WSH;N_{i-1}}^\top \boldsymbol{\Sigma} \left(\hat{\mathbf{w}}_{WSH;N_{i-1}} - \hat{\mathbf{w}}_{WS;N_i}\right)}{\left(\hat{\mathbf{w}}_{WSH;N_{i-1}} - \hat{\mathbf{w}}_{WS;N_i}\right)^\top \boldsymbol{\Sigma} \left(\hat{\mathbf{w}}_{WSH;N_{i-1}} - \hat{\mathbf{w}}_{WS;N_i}\right)}. \tag{III.16}$$

In Theorem III.3 we derive an iterative procedure for computing the deterministic equivalents to $\Psi^*_{\mathbf{W};N_i}$ for $i = 1, \ldots, T$.

Theorem III.3: Let $\mathbf{Y}_{n_i}$ possess the stochastic representation as in (II.1) and let $\mathbf{b}$ be the deterministic shrinkage target for $i = 1$. Assume that the relative loss of portfolio $\mathbf{b}$ given by $R_0 = \mathbf{1}_p^\top \boldsymbol{\Sigma}^{-1} \mathbf{1}_p \mathbf{b}^\top \boldsymbol{\Sigma} \mathbf{b} - 1$ is uniformly bounded in p. Then it holds that

$$\left|\Psi^*_{\mathbf{W};N_i} - \Psi^*_{\mathbf{W};i}\right| \overset{a.s.}{\to} 0$$

for $p/N_j \to C_j \in (0,1)$ as $N_j \to \infty$, $j = 1,\ldots,i$ and $i = 1,\ldots,T$ where

$$\Psi^*_{\mathbf{W};i} = \frac{u^2_{N_i}(R_{\mathbf{W};i-1} + 1 - K_{\mathbf{W};i})}{u^2_{N_i}(R_{\mathbf{W};i-1} + 1 - 2K_{\mathbf{W};i}) + u'_{N_i}}, \tag{III.17}$$

$$R_{\mathbf{W};i} = (\Psi^*_{\mathbf{W};i})^2 \frac{u'_{N_i} - u^2_{N_i}}{u^2_{N_i}} + (1 - \Psi^*_{\mathbf{W};i})^2 R_{\mathbf{W};i-1} + 2\Psi^*_{\mathbf{W};i}(1 - \Psi^*_{\mathbf{W};i})(K_{\mathbf{W};i} - 1), \tag{III.18}$$

$$K_{\mathbf{W};i} = \beta^*_{\mathbf{W};i-1;0} + \sum_{j=1}^{i-1} \beta^*_{\mathbf{W};i-1;j} D_{\mathbf{W};j,i}, \tag{III.19}$$

with $\beta^*_{\mathbf{W};0;0} = 1$, $\beta^*_{\mathbf{W};i-1;i-1} = \Psi^*_{\mathbf{W};i-1}$ and

$$\beta^*_{\mathbf{W};i-1;k} = (1 - \Psi^*_{\mathbf{W};i-1})\beta^*_{\mathbf{W};i-2;k}, \tag{III.20}$$

for $k = 0,\ldots i-2$. Finally, $D_{\mathbf{W};j,i}$ is given by

$$D_{\mathbf{W};j,i} = Q^{-1}_{\mathbf{W};j,i} \frac{1}{N_i - N_j}\left(\frac{N_i}{u_{N_i}} - \frac{N_j}{u_{N_j}}\right) \tag{III.21}$$

and

$$Q_{\mathbf{W};j,i} = \frac{1}{N_i - N_j} \mathrm{tr}[\mathbf{W}_{n_{j+1}:n_i}(\mathbf{I}_{N_i-N_j} + C_i u_{N_i}\mathbf{W}_{n_{j+1}:n_i})^{-1}], \tag{III.22}$$

where $\mathbf{W}_{n_{j+1}:n_i} = diag(\mathbf{W}_{n_{j+1}},\ldots,\mathbf{W}_{n_i})$ and u_{N_i} is the solution of

$$1 - c_i = \frac{1}{N_i}\mathrm{tr}[(\mathbf{I}_{N_i} + C_i u_{N_i}\mathbf{W}_{N_i})^{-1}]$$

and

$$u'_{N_i} = \frac{u_{N_i}}{\mathrm{tr}[(\mathbf{I}_{N_i} + C_i u_{N_i}\mathbf{W}_{N_i})^{-2}]}.$$

The proof of Theorem III.3 is given in the supplementary material. The application of the results of Theorem III.3 leads to the (bona fide) shrinkage estimator of the dynamic GMV portfolio given by

$$\hat{\mathbf{w}}_{WBF;N_i} = \hat{\Psi}^*_{\mathbf{W};i}\hat{\mathbf{w}}_{WS;N_i} + (1 - \hat{\Psi}^*_{\mathbf{W};i})\hat{\mathbf{w}}_{WBF;N_{i-1}}, \tag{III.23}$$

where $\hat{\Psi}^*_{\mathbf{W};i}$ are computed recursively using the recursion presented in Theorem III.3 and the following consistent estimator of R_0 expressed as

$$\hat{R}_{\mathbf{W};0} = \hat{r}_{\mathbf{W};0} = u^{-1}_{N_1}\mathbf{1}^\top_p\mathbf{S}^{-1}_{\mathbf{W};N_1}\mathbf{1}_p\mathbf{b}^\top\mathbf{S}_{\mathbf{W};N_1}\mathbf{b} - 1. \tag{III.24}$$

Finally, using the whole sample of the asset returns available up to time t_i another consistent estimator of the relative loss of the target portfolio can be constructed by

$$\hat{R}_{\mathbf{W};0;i} = u^{-1}_{N_i}\mathbf{1}^\top_p\mathbf{S}^{-1}_{\mathbf{W};N_i}\mathbf{1}_p\mathbf{b}^\top\mathbf{S}_{\mathbf{W};N_i}\mathbf{b} - 1. \tag{III.25}$$

IV. Finite-Sample Performance

A. Benchmark Strategies and the Setup of the Simulation Study

The suggested dynamic estimation strategies are compared to several benchmark strategies via en extensive simulation study in this section, while the results of the empirical illustration are provided in Section V. The performance of the following eight dynamic trading strategies will be established:

DOS1: Bona fide shrinkage estimator of the GMV portfolio (II.13) with (II.14) following Theorem II.1 where r_0 is estimated from the first sample as in (II.12);

DOS2: Bona fide shrinkage estimator of the GMV portfolio (II.24) where $\hat{\Psi}^*_i$ is computed recursively as in Theorem II.5 and R_0 is estimated from the first sample as in (II.23);

DOS3: Bona fide shrinkage estimator of the GMV portfolio (II.13) with (II.14) following Theorem II.1 where r_0 is recomputed as in (II.16) when a new sample becomes available;

DOS4: Bona fide shrinkage estimator of the GMV portfolio (II.24) where $\hat{\Psi}^*_i$ is computed recursively as in Theorem II.5 and R_0 is recomputed as in (II.25) when a new sample becomes available;

Sample: Sample estimator of the GMV portfolio computed at each time t_i, $i = 1,2,\ldots,T$, i.e., $\psi_i = 1$ in (II.5) for $i = 1,2,\ldots,T$;

EW: Target portfolio $\mathbf{b}$ used at each time t_i, $i = 1,2,\ldots,T$, i.e., $\psi_i = 0$ in (II.5) for $i = 1,2,\ldots,T$;

BPS18: One-period shrinkage estimator of the GMV portfolio (I.5) with (I.6) reconstructed at each time t_i, $i = 1,2,\ldots,T$;

LW17: Ledoit and Wolf [32] nonlinear shrinkage estimator of the GMV portfolio computed at each time t_i, $i = 1,2,\ldots,T$.

BAMA21: Ballal et al. [11] estimator of the GMV portfolio computed at each time t_i, $i = 1,2,\ldots,T$. Five-fold cross-validation is used to estimate the shrinkage coefficient with an equally-spaced grid on 0 to 9.

YCM15: Yang et al. [9] estimator of the GMV portfolio computed at each time t_i, $i = 1,2,\ldots,T$. An equally-spaced grid on [0,1] is used for the determination of the shrinkage coefficient.

The first four strategies are based on the theoretical results derived in Sections II-A and II-B where two different methods for constructing bona fide shrinkage estimators of the GMV portfolio weights are explored following the discussion after Theorems II.1 and II.5, respectively. The Strategies from **Sample** to **YCM15** are benchmark strategies, based on the traditional estimator of the GMV portfolio, on the target portfolio, and on different versions of one-period shrinkage estimators. It is interesting that the dynamic strategy based on holding the target portfolio $\mathbf{b}$ during the investment procedure can also be obtained as a special case of the proposed dynamic shrinkage approach. Namely, setting $\psi_1 = 0$ in (II.5) leads to $\hat{\mathbf{w}}_{SH;n_1} = \mathbf{b}$ at time t_1, which is then use as a shrinkage target at t_2. If we continue this procedure, then we get that $\hat{\mathbf{w}}_{SH;n_i} = \mathbf{b}$ for all $i = 1,\ldots,T$. The **LW17** strategy is a recent state-of-the-art method of Ledoit and Wolf [32] (see, also Ledoit and Wolf [54] for its efficient nonparametric estimation), which efficiently applies the nonlinear shrinkage estimator of the covariance matrix on the GMV portfolio weights. The **BAMA21** and **YCM15** strategies are based on the ridge estimator of the covariance matrix and on the robust Tyler M-estimator of the covariance matrix adopted to portfolio theory in Ballal et al. [11] and Yang et al. [9], respectively.

Since the GMV portfolio is the solution of the portfolio optimization problem with the aim to minimize the portfolio variance, the relative loss in the out-of-sample variance is used as a performance measure in this comparison study which for the portfolio with the estimated weights $\hat{\mathbf{w}}$ is expressed as:

$$\text{Relative loss}\,(\mathbf{w}) = \frac{\hat{\mathbf{w}}^\top \mathbf{\Sigma} \hat{\mathbf{w}} - V_{GMV}}{V_{GMV}} = \mathbf{1}^\top \mathbf{\Sigma}^{-1} \mathbf{1} \hat{\mathbf{w}}^\top \mathbf{\Sigma} \hat{\mathbf{w}} - 1, \quad \text{(IV.1)}$$

where we use the formula for the global minimum variance V_{GMV} given in (II.9).

In the simulation study, we will look at two investment horizons $T = 10$ and $T = 20$. For each segment we let $n_i = 250$, which would correspond to an investor who rebalances the holding portfolio on a yearly basis. The parameters of the listed below models are simulated according to $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)^\top$ with $\mu_i \sim U(-0.2, 0.2)$ and the covariance matrix $\mathbf{\Sigma}$ is configured such that 20% of the eigenvalues are equal to 0.2, 40% equal to one and 40% equal to 4, whereas the eigenvectors are generated from the Haar distribution. Following this simulation setup, $\mathbf{\Sigma}$ will have the same spectral distribution for all considered values of the concentration ratio c.

Four different stochastic models for the data-generating process will be considered, which are listed below:

Scenario 1: t-distribution The elements of $\mathbf{x}_t$ are drawn independently from t-distribution with 5 degrees of freedom, i.e., $x_{tj} \sim t(5)$ for $j = 1, \ldots, p$, while $\mathbf{y}_t$ is constructed according to (II.1).

Scenario 2: CAPM The vector of asset returns $\mathbf{y}_t$ is generated according to the CAPM (Capital Asset Pricing Model), i.e.,

$$\mathbf{y}_t = \boldsymbol{\mu} + \boldsymbol{\beta} z_t + \mathbf{\Sigma}^{1/2} \mathbf{x}_t,$$

with independently distributed $z_t \sim N(0, 1)$ and $\mathbf{x}_t \sim N_p(\mathbf{0}, \mathbf{I})$. The elements of vector $\boldsymbol{\beta}$ are drawn from the uniform distribution, that is $\beta_i \sim U(-1, 1)$ for $i = 1, \ldots, p$.

Scenario 3: CCC-GARCH model of Bollerslev [55] The asset returns are simulated according to

$$\mathbf{y}_t | \mathbf{\Sigma}_t \sim N_p(\boldsymbol{\mu}, \mathbf{\Sigma}_t)$$

where the conditional covariance matrix is specified by

$$\mathbf{\Sigma}_t = \mathbf{D}_t^{1/2} \mathbf{C} \mathbf{D}_t^{1/2} \text{ with } \mathbf{D}_t = \text{diag}(h_{1,t}, h_{2,t}, \ldots, h_{p,t}),$$

where

$$h_{j,t} = \alpha_{j,0} + \alpha_{j,1} (\mathbf{y}_{j,t-1} - \boldsymbol{\mu}_j)^2 + \beta_{j,1} h_{j,t-1},$$

for $j = 1, 2, \ldots, p$, and $t = 1, 2, \ldots, n_i, \tilde{\ } i = 1, \ldots, T$. The coefficients of the CCC model are sampled according to $\alpha_{j,1} \sim U(0, 0.1)$ and $\beta_{j,1} \sim U(0.6, 0.7)$ which implies that the stationarity conditions, $\alpha_{j,1} + \beta_{j,1} < 1$, are always fulfilled. The intercept $\alpha_{j,0}$ is thereafter chosen such that the unconditional covariance matrix is equal to $\mathbf{\Sigma}$.

Scenario 4: VARMA model The vector of asset returns $\mathbf{y}_t$ is simulated according to

$$\mathbf{y}_t = \boldsymbol{\mu} + \mathbf{\Gamma}(\mathbf{y}_{t-1} - \boldsymbol{\mu}) + \mathbf{\Sigma}^{1/2} \mathbf{x}_t, \text{ with } \mathbf{x}_t \sim N_p(\mathbf{0}, \mathbf{I})$$

for $t = 1, \ldots, n_i$, $i = 1, \ldots, T$, where $\mathbf{\Gamma} = \text{diag}(\gamma_1, \gamma_2, \ldots, \gamma_p)$ where $\gamma_i \sim U(-0.9, 0.9)$ for $i = 1, \ldots, p$.

Scenario 1 and **Scenario 2** fulfill the conditions imposed on the data-generating model in Section II. The application of both scenarios result in samples that consist of independent random vectors with finite $4 + \varepsilon$, $\varepsilon > 0$, moments. Furthermore, the covariance matrix possesses finite eigenvalues in **Scenario 1**, while it has an unbounded spectrum in **Scenario 2** (cf., Fan et al. [44]). On the other side, the samples obtained following **Scenario 3** and **Scenario 4** consist of dependent observations. In **Scenario 3** the random vector are uncorrelated although a non-linear dependence is present in the time series structure of the model, while the elements of the samples obtained from **Scenario 4** are strongly linearly dependent.

For each segment of the time partition we generate a new sample of $n = 250$ observations, which is applied in the computation of $\hat{\boldsymbol{\mu}}$, $\hat{\mathbf{S}}_{n_i}$, $\hat{\mathbf{S}}_{N_i}$, $\hat{\mathbf{w}}_{S;n_i}$, and $\hat{\mathbf{w}}_{SH;n_i}$. As a target portfolio in the **DOS1 to 4** and **BPS18** strategies, we use the equally-weighted portfolio with the weights $\mathbf{b} = \mathbf{1}_p/p$, while the remaining strategies do not require the specification of the target portfolio. The results of the simulation study are based on 1000 independent runs from which the average relative loss is computed for each scenario, strategy and several values of the concentration ratio c.

B. Performance of the Trading Strategies

Figs. 2–5 present the results of the simulation study for $i = \{5, 10, 15\}$ when $T = 20$. Interestingly, the computed average losses show a similar behaviour independently of the data-generating model used to draw the samples. Although **Scenarios 3 and 4** do not fulfill the assumptions imposed on the data-generating model in the derivation of the theoretical results, the differences in the behaviour of the ten trading strategies is not large. As such, one can conclude that the presence of non-linear dependence structure between the observation vectors or even strong linear dependence has only minor impact on the validity of the results derived in Theorems II.1 and II.5.

The best performance is obtained for the dynamic optimal shrinkage estimators **DOS1 to 4**. However, the ordering is not consistent. The differences between the computed values for the **DOS** strategies are not large. All dynamic estimation strategies considerably outperform the considered benchmark strategies, independently of the scenario used to generate samples. On the next place we often rank the nonlinear shrinkage estimator **LW17** and the ridge shrinkage **BAMA21**. These two strategies are very often close to each other and are followed by the single-period shrinkage estimator **PBS18** and the robust Tyler M-estimator **YCM15**. Finally, we note that the traditional **Sample** estimator performs better than the portfolio strategy based on the target portfolio, when the concentration ratio c is smaller than 0.75, while it produces extremely large values of relative losses, when c approaches one. This observation becomes even more prominent in case of the **Scenario 4**, where a strong autocorrelation was employed. Here, already for $c = 0.5$ the target portfolio starts outperforming the traditional estimator.

To this end, we conclude that the dynamic re-estimation of the relative loss of the target portfolio $\mathbf{b}$ shows a significant improvement when non-overlapping samples are used and the

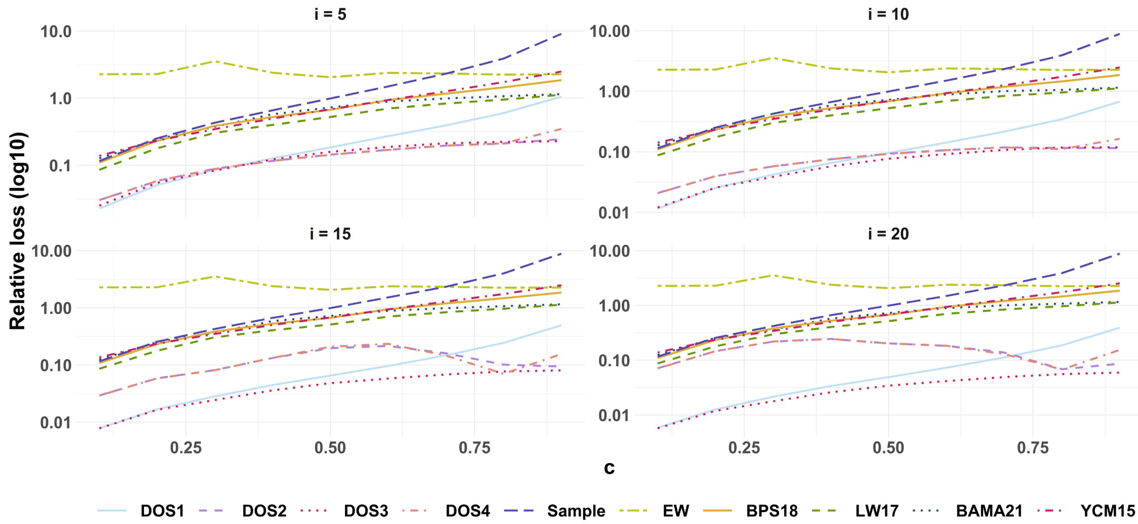

Fig. 2. Logarithm of relative losses for the different time steps i and investment horizon $T = 20$. Data were simulated following **Scenario 1** for different values of c.

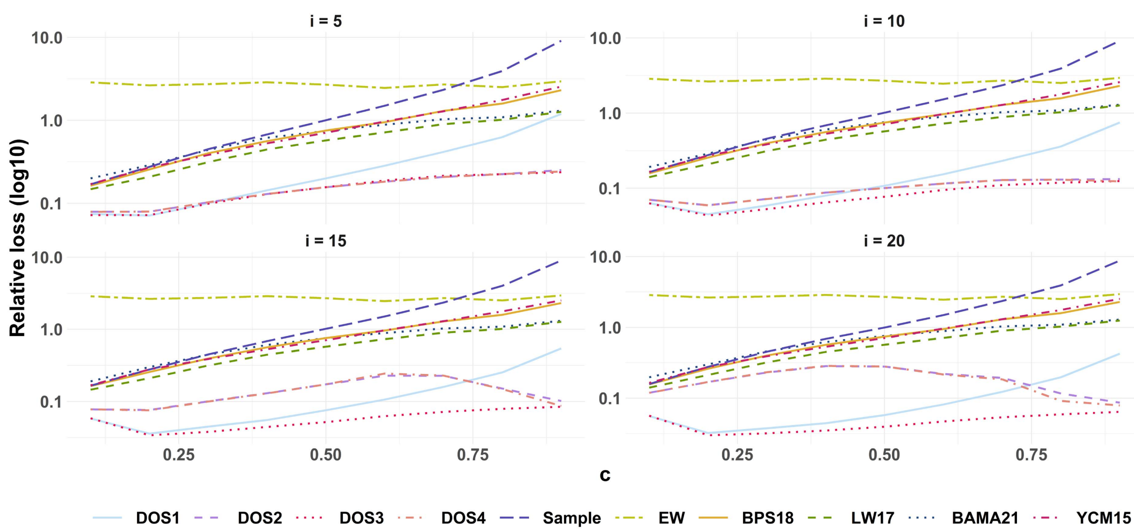

Fig. 3. Logarithm of relative losses for the different time steps i and investment horizon $T = 20$. Data were simulated following **Scenario 2** for different values of c.

concentration ratio c is relatively large. In contrast, the application of the dynamic re-estimation of the relative loss in the case of overlapping samples leads to the considerably large computation time (see, Fig. 1) without large improvements. Finally, the increase of the trading horizon T has only a minor impact on the plots presented in Figs. 2–5. The larger value of T slightly reduces the computed average relative losses in the case of **DOS1**, while they become a slightly larger for the **DOS3**, **DOS4** and the single-period shrinkage approaches.

V. Application to Stocks From S&P 500

In this section we will apply the suggested new approaches and the benchmark strategies presented in Section IV on daily market data. The computation is performed by using the R-package *DOSPortfolio* (see, Bodnar et al. [45]).

A. Data Description

We will use daily returns on 348 stocks included in the S&P 500 index from March 2011 up until March 2020. The stocks were chosen by the availability of their price data during the trading period. Two portfolios of size $p = 200$ (high-dimensional case) and $p = 150$ (low-dimensional case) are considered, where the stocks are chosen randomly from the 348 stocks included in the S&P 500 index. We set $c = 0.8$ or $c = 0.6$ and therefore, use $n_i = 250$ trading days for each year i.

B. Results of the Empirical Illustration

A consequence of the exponential weighting schemes to which the shrinkage estimators belong to, is that the portfolio structure changes by smaller increments. As a result, we expect that the portfolio turnover of the dynamic (estimated) GMV portfolios based on the introduced shrinkage approaches to be

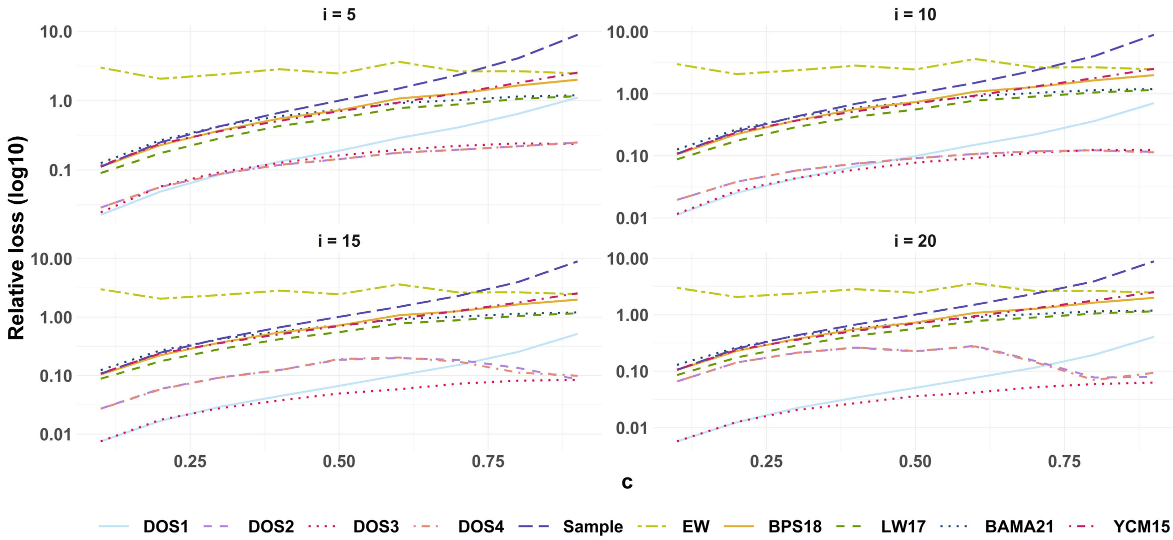

Fig. 4. Logarithm of relative losses for the different time steps i and investment horizon $T = 20$. Data were simulated following **Scenario 3** for different values of c.

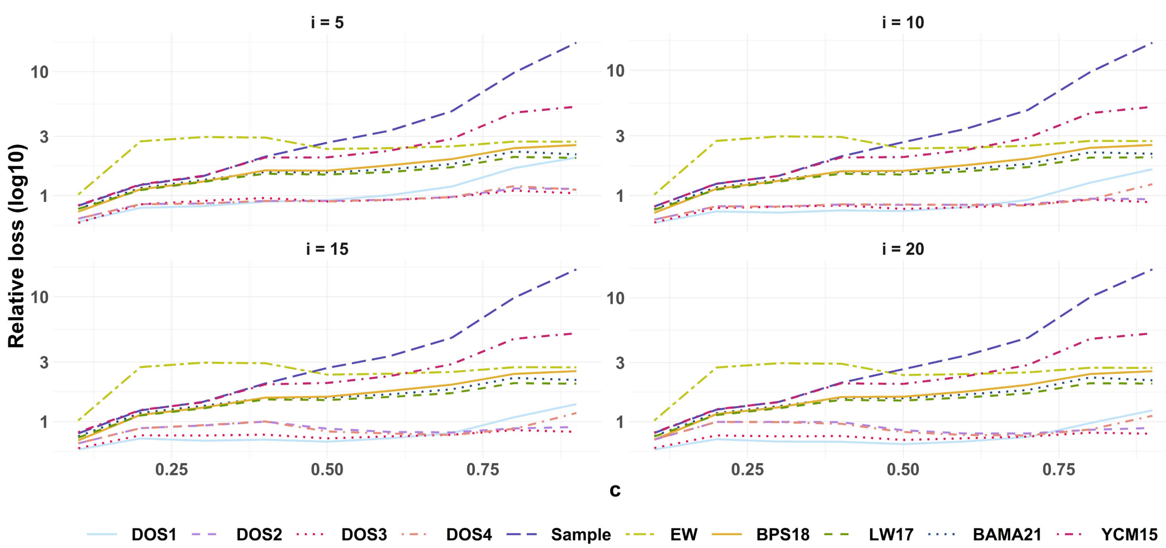

Fig. 5. Logarithm of relative losses for the different time steps i and investment horizon $T = 20$. Data were simulated following **Scenario 4** for different values of c.

smaller in comparison to the unconstrained strategy ($\psi_i = 1$), but to be larger in comparison to the static portfolio choice ($\psi_i = 0$). For each strategy k introduced in Section IV-A, let $\mathbf{w}^{(k)}$ denote the vector of the weights induced by the kth strategy and let $w_{i,j}^{(k)}$ stand for the weight for the j-th asset after the i-th portfolio rebalancing.

For each strategy k the turnover is defined by (see e.g. Golosnoy et al. [42])

$$\text{Turnover}^{(k)} = \frac{1}{T}\sum_{i=1}^{T} ||\mathbf{w}_i^{(k)} - \mathbf{w}_{i-1}^{(k)}||_1. \tag{V.1}$$

The turnover measures changes in weights and will therefore be connected to the cost of transitioning from one portfolio to another. This definition assumes that the transaction costs are constant for all assets and time periods. In practise, the amount of turnover will affect the development of wealth of the portfolio. Moreover, following Golosnoy et al. [42], we will compute the average absolute values of holding portfolio weights, the average minimum and maximum portfolio weights, the average sum of negative weights in the portfolio, and the average fraction of negative weights in the portfolio as further performance measures. These are given by

$$|\mathbf{w}^{(k)}| = \frac{1}{Tp}\sum_{i=1}^{T}\sum_{j=1}^{p} |w_{i,j}^{(k)}|, \tag{V.2}$$

$$\max \mathbf{w}^{(k)} = \frac{1}{T}\sum_{i=1}^{T}\left(\max_j w_{i,j}^{(k)}\right), \tag{V.3}$$

TABLE I
PERFORMANCE MEASURES OVER $T = 8$ PERIODS OF REBALANCING. THE STRATEGY WHICH IS MOST PERFORMANT (IN TERMS OF ITS MEASUREMENT) IS HIGHLIGHTED IN BOLD ON EACH ROW

Type	Size	DOS 1	DOS 2	DOS 3	DOS 4	Sample	EW	BPS18	LW17	BAMA21	YCM15
$\lvert\mathbf{w}^{(k)}\rvert$	150	0.0326	0.0328	0.0267	0.0320	0.0552	**0.0067**	0.0379	0.0233	**0.0067**	0.0380
	200	0.0352	0.0278	0.0238	0.0273	0.0676	**0.0050**	0.0336	0.0160	**0.0050**	0.0391
$\max \mathbf{w}^{(k)}$	150	0.3156	0.3675	0.3490	0.3657	0.5060	**0.0067**	0.3363	0.1557	0.0073	0.1730
	200	0.2996	0.2136	0.1932	0.2061	0.8373	**0.0050**	0.4190	0.0920	**0.0050**	0.1805
$\min \mathbf{w}^{(k)}$	150	-0.1183	-0.0924	-0.0873	-0.0911	-0.2977	0.0067	-0.1799	-0.0930	**0.0054**	-0.1388
	200	-0.2427	-0.1550	-0.1234	-0.1441	-0.4609	0.0050	-0.2086	-0.0591	**0.0049**	-0.1631
$\mathbf{w}_i^{(k)}\mathbb{1}(\mathbf{w}_i^{(k)} < 0)$	150	-1.9475	-1.9572	-1.5015	-1.9009	-3.6392		-2.3451	-1.2508	**-0.0002**	-2.3523
	200	-3.0226	-2.2811	-1.8754	-2.2273	-6.2566		-2.8624	-1.0974	**0**	-3.4108
$\mathbb{1}(\mathbf{w}_i^{(k)} < 0)$	150	0.4548	0.5059	0.4896	0.5081	0.4830		0.4452	0.4096	**0.0007**	0.4504
	200	0.4661	0.4678	0.4467	0.4678	0.4978		0.4406	0.3622	**0**	0.4611
σ^k	150	0.0067	0.0070	**0.0059**	0.0061	0.0103	0.0116	0.0096	0.0080	0.0114	0.0091
	200	0.0066	0.0053	**0.0049**	0.0057	0.0118	0.0110	0.0098	0.0080	0.0110	0.0086
$\bar{\mathbf{y}}_{\mathbf{w}}^k$	150	0.0003	**0.0004**	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003
	200	0.0003	0.0004	0.0003	0.0004	**0.0005**	0.0003	0.0004	0.0002	0.0003	0.0004
SR^k	150	0.0397	**0.0553**	0.0502	0.0462	0.0299	0.0221	0.0326	0.0434	0.0257	0.0343
	200	0.0513	0.0756	0.0723	**0.0806**	0.0433	0.0254	0.0413	0.0297	0.0255	0.0448
$\mathrm{Turnover}^{(k)}$	150	1.8073	1.4729	1.6521	1.1517	10.1881		6.6627	3.8570	**0.0460**	6.7022
	200	2.7539	2.6019	2.1868	2.1291	15.9824		7.5264	3.4566	**0.0023**	8.9914

$$\min \mathbf{w}^{(k)} = \frac{1}{T}\sum_{i=1}^{T}\left(\min_j w_{i,j}^{(k)}\right), \tag{V.4}$$

$$\mathbf{w}_i^{(k)}\mathbb{1}(\mathbf{w}_i^{(k)} < 0) = \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{p} w_{i,j}^{(k)}\mathbb{1}(w_{i,j}^{(k)} < 0), \tag{V.5}$$

$$\mathbb{1}(\mathbf{w}_i^{(k)} < 0) = \frac{1}{Tp}\sum_{i=1}^{T}\sum_{j=1}^{p} \mathbb{1}(w_{i,j}^{(k)} < 0). \tag{V.6}$$

Moreover, we also consider important classical portfolio performance measures: total excess portfolio return, out-of-sample variance and the average Sharpe ratio. The computed values of the introduced performance measures are summarized in Table I for each strategy over the entire period. The corresponding values for **EW** strategy are also included in the table but many of its entries are obviously equal to zero, and therefore omitted, since the weights are all equal.

For the first five performance measures, **BAMA21** together with the **EW** are the best. The former strategy often mimics the EW portfolios, with a slight difference in the average maximum weight where it takes slightly larger positions in comparison to the EW strategy. None of these weights are negative. The third, most performant strategy, is **LW17** as it takes the smallest positions, short the smallest amount of stocks and so forth. For these portfolios, the average maximum weights, mean of all shorted and the proportion shorted seem to decrease slightly with p. This can most likely be attributed to the nonlinear shrinkage of the eigenvalues, which seem to have a direct impact on the magnitude and direction of the weights. None of the other strategies has the same flexibility and can not compete with it, especially the **Sample** strategy.

The performance measures seen in the four last rows of Table I are based on the portfolio return. All strategies optimize for minimizing the portfolio variance and this should therefore be the primary measure of interest. In this setting, the **DOS3** is the best strategy since it obtains the smallest portfolio variance. This is consistent over both portfolio sizes. All strategies provide close to the same return for the portfolio size 150. The **DOS2** strategy appears to have a slight edge. It is not large, just a few percent relative to the second largest which is the LW17 strategy. For $p = 200$ the best strategy is the **Sample** in terms of return. However, it is one of the worst in terms of its Sharpe ratio. It takes on a lot of risk to create that return. The strategies with the highest Sharpe ratio is **DOS2** for the smaller portfolio size and **DOS4** for the larger.

The last performance measure is the turnover. As one can expect, the **Sample** strategy is worst, generating the largest turnover. It generated the largest return for $p = 200$ but it does so at a large cost. This classic strategy has the most flexibility. This flexibility often leads to unnecessary reconstruction of the holding portfolio. This becomes apparent in more extreme case when p/n is close to one. The smallest turnover is given by **BAMA21**. However, this strategy essentially copies the **EW** strategy with very little deviations resulting in relatively large variances as well as small Sharpe ratios. The **DOS** strategies provide second to smallest Turnover measurements. These strategies force small movements between reallocations. The **DOS3** and **DOS4** strategies change the perception of the initial relative loss. This implies that algorithm change its opinion of what the optimal weights should have been. In this application, **DOS3** and **DOS4** slightly decrease the turnover in comparison to their counterparts **DOS1** and **DOS2**. Note that the dynamic shrinkage approach does not optimize towards decreasing turnover but it is a consequence of enforcing small movements. This also implies that other

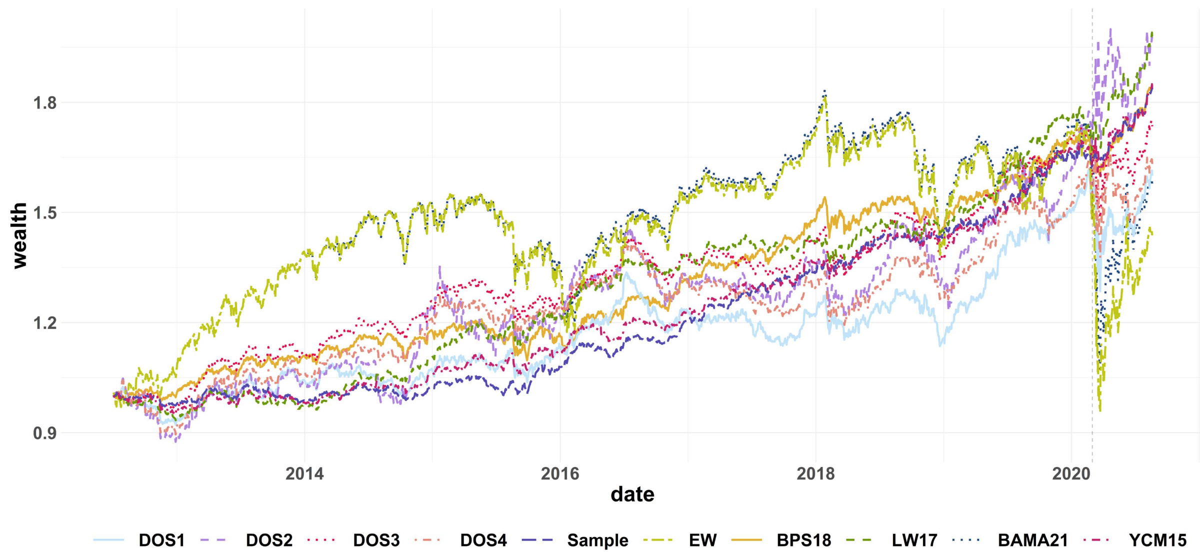

Fig. 6. Development of the investor wealth based on the dynamic trading strategies described in Section IV-A. In this figure the portfolio size is equal to 150.

strategies can be close or better than the dynamic shrinkage approach. The **LW17** strategy shows promise of this feature. When p increases the nonlinear shrinkage estimator becomes more stable (between reallocation periods) and decrease its turnover slightly.

The development of the investors wealth for the ten trading strategies introduced in Section IV-A over 10 years is depicted in Fig. 6. We limit the illustrations to the case of $p = 150$. The strategies are the same as those presented in Table I with $p = 150$, $p/n_i = 0.6$. The wealth is computed according to a buy-and-hold strategy until next reallocation period. That is, given that n_i days has passed, we use these to estimate the portfolio weights and then rebalance the holding portfolio to the new portfolio according to the different strategies. The wealth is accumulated on a daily basis which corresponds to the frequency of data used to construct the portfolio.

In Fig. 6 the worst strategy in terms of the final wealth is **EW**. It, together with **BAMA21** and **EW**, has large gains early in the portfolios lifetime but huge losses during COVID. Throughout the whole period they are very close to each other which one can also notice from Table I. The rest of the strategies are more or less close to each other with **DOS1** deviating in terms of having stagnated wealth during 2016–2019 and thereafter it manages to catch up to most of the other strategies. The strategy which accumulates most wealth is **LW17**. It results in a final wealth of 1.98 while **DOS2**, which had the highest average return, has a final return of 1.96. The **LW17** strategy is around a percent better and manages to be so from a higher risk profile, as indicated by the portfolio variance and the Sharpe ratio in I. This figure also illustrates why **DOS3** had the smallest variance in Table I. It varies very little in contrast to other strategies, even in turbulent times.

All portfolios are hit quite heavily by COVID in the early 2020, which is indicated by a dashed line on the 1st of March, 2020.[1] Some portfolios are quick to adapt to the event and other are not. However, all portfolios seem to experience a very sharp increase in wealth post COVID. In Table II we show the largest loss of the strategies throughout the whole period. We can see that **EW** ($p = 150$) and **LW17** ($p = 200$) suffers the largest loss. The **Sample** strategy suffers the least for both portfolio sizes. This is most likely due to the fact that this is one observation, and that it is very hard to describe the relationship between the largest sample loss and the sample portfolio variance. Second to best is the **YCM15** and thereafter the **DOS3**. These strategies seem to have worked well in the COVID crisis.

These results are in line with the previous empirical findings of Bodnar et al. [40] who document that the equally weighted portfolio performs well in the stable period on the capital market, but its performance is very bad during the turbulent periods. To conclude, all four of the proposed dynamic shrinkage strategies show impressively good performance over the state-of-the-art static portfolios especially in case when p becomes close to n_i.

[1]This is of course somewhat arbitrary since it is hard to specify a certain day that COVID hits the market.

VI. Summary

In many practical situation an investor after constructing an optimal portfolio faces the problem of the portfolio reallocation based on the new data which arrive on the capital market after the portfolio was built. We deal with challenging task in the current paper by developing several dynamic optimal shrinkage estimators for the weights of the GMV portfolio. In the derivation of the theoretical findings, new results in random matrix theory are deduced which allow us to obtained optimal shrinkage estimator in both important cases with and without overlapping samples. In the case of non-overlapping samples, the investor uses the data of asset returns after the last reconstruction of the portfolio, while the whole data might be used in the case of overlapping samples. It is remarkable that the two settings require different theoretical results in random matrix theory to be derived and they result in quite different optimal shrinkage intensities. Moreover, minor distributional assumptions are imposed on the data-generating process, like the existence of $4 + \varepsilon, \varepsilon > 0$, moments are required

TABLE II
LARGEST LOSS FOR THE DIFFERENT STRATEGIES OVER THE GIVEN PERIOD. BOLD VALUES CORRESPOND TO THE STRATEGIES WHICH HAVE THE LARGEST LOSSES

	DOS									
Size	1	2	3	4	Sample	EW	BPS18	LW17	BAMA21	YCM15
150	0.0769	0.0674	0.0456	0.0765	0.0117	**0.1362**	0.0269	0.0199	0.1205	0.0238
200	0.1036	0.0764	0.0386	0.0747	0.0076	0.1343	0.0297	**0.1560**	0.1337	0.0222

only. Also, the covariance matrix might have an unbounded spectrum.

The results of the simulation study show that the dynamic shrinkage procedures derived in the paper are robust against violations of the model assumptions. In particular, we conclude based on the results of the simulation study, that the performance of the suggested dynamic approach will not be strongly influenced when the asset returns are generated from a multivariate GARCH model and from a VARMA model. Although, both multivariate times series model assume that the asset returns are time dependent, it has only a minor influence on the suggested trading strategies. Finally, we apply the new approaches to real data of returns on stocks included in the S&P 500 index and compare them with several benchmark approaches, consisting of investing into the target portfolio, the sample GMV portfolio, and the single-period GMV portfolio. Several performance measures are considered and it is shown that the dynamic shrinkage portfolio constructed by using overlapping samples possesses the best performance in terms of the turnover and the development of the portfolio weights.

The dynamic strategies based non-overlapping sample are simple to implement and they provide drastically less turnover in comparison to the benchmark approaches. Although the approaches based on the overlapping estimators are harder to implement, they decrease the turnover by 50% in comparison to the corresponding non-overlapping strategies with no significant loss in wealth. Furthermore, they require that the sample size is larger than the portfolio dimension only when the portfolio is constructed for the first time, while the non-overlapping approaches need the sample size to be larger than the portfolio dimension by each reconstruction of the portfolio.

No portfolio is ever static. Making optimal transitions are therefore of great interest to any investor. These results provide a fully data-driven dynamic approaches how the GMV portfolio can be rebalanced. In many practical applications the investors might want to have more assets in their portfolios than the available sample size. This demands a special attention since the sample covariance matrix is singular in this case and its inverse does not exist any longer. We contribute to this challenging problem by deriving the dynamic shrinkage trading strategies when the Moore-Penrose inverse of the sample covariance matrix is used instead of the ordinary inverse. Besides that, we present the results for a weighted estimator of the covariance matrix.

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